NOTE ON CARTAN-HADAMARD THEOREM FOR METRIC SPACES

Sayed K. Roushon

July 15, 2002

Abstract. This is the note of a lecture given at the Chennai Topological Methods in Group Theory workshop on the Cartan-Hadamard theorem for general metric spaces of nonpositive curvature.

1. Introduction

In this lecture we prove the Cartan-Hadamard theorem for general metric spaces. This theorem in the case of nonpositively curved Riemannian manifolds is well known. The theorem is due to Hadamard for 2-dimensional manifolds and to Cartan for manifolds of dimension higher than 2. The importance of the theorem lies in the fact that a simple local condition on the space has a global effect on the geometry and topology of the space.

For all the material behind the proof we follow Chapter II, Section 4 of the book "Metric spaces of nonpositive curvature" by Bridson and Haefliger.

2. Preliminaries

A *local geodesic* in a metric space (X, d) is a map $\alpha : [a, b] \to X$ such that for each $t \in [a, b]$ there is an $\epsilon > 0$ and $d(\alpha(t'), \alpha(t'')) = |t' - t''|$ for $|t-t'|+|t-t''| \leq \epsilon$.

A *geodesic space* is a metric space (X, d) if for all $x, y \in X$ there is a map $\alpha : [a, b] \to X$ with the property $\alpha(a) = x$, $\alpha(b) = y$ and $d(\alpha(t), \alpha(t')) = |t - t'|$ for all $t, t' \in [a, b]$. The metric of a geodesic metric space is called *convex* if any pair of geodesics $\alpha_1 : [0, a_1] \to X$ and $\alpha_2 : [0, a_2] \to X$ with the same initial point $\alpha_1(0) = \alpha_2(0)$ satisfy the inequality $d(\alpha_1(ta_1), \alpha_2(ta_2)) \leq td(\alpha_1(a_1), \alpha_2(a_2))$ for all $t \in [0, 1]$. In a complete convex space between any two points there is only one geodesic and geodesics vary continuously with end points. Recall that in a geodesic metric space by *geodesics vary continuously with end points* means if $\{x_n\}$ and $\{y_n\}$ are two sequences coverging to x and y respectively then the geodesics $c(x_n, y_n)$ converge to $c(x, y)$ uniformly. Here the notation $c(z, w)$ stands for the unique geodesic joining z and w. (X, d) is called *locally convex* if any point in X has a neighbourhood in which the induced metric is convex.

In a metric space (X,d) one can define the length of a path $\alpha : [a,b] \to X$ in the following way.

$$l(\alpha) = sup_{P_n} \Sigma_{k=1}^{n} \{d(\alpha(t_{k-1}), \alpha(t_k))\}$$

where the supremum is taken over all partition P_n of the interval $[a,b]$. Now given any two points x and y in the space X we define another distance function $d'(x,y) = inf\{l(\alpha)\}$ where the infimum is taken over all paths α joining x and y. Note that $d'(x,y)$ could be ∞. If $d'(.,.)$ is never infinity and $d(.,.) = d'(.,.)$ then the metric d is called an *inner* or *length metric* and the metric space (X,d) is sometimes called a *length metric space* or simply a *length space*. For an example consider the standard 2-sphere in the Euclidean 3-space and define the distance between any two points on the 2-sphere as the length of the straight line in the Euclidean space joining the two points. Then this metric on the sphere is not the length metric.

Let (X,d) be a metric space and assume that any pair of points which are close enough are joined by a geodesic. Then we say (X,d) is *nonpositively curved* if for any point $x_0 \in X$ there is a neighbourhood U of x_0 so that for any geodesic triangle $\Delta(p,q,r) \subset U$ and a triangle $\Delta'(p',q',r') \subset \mathbb{R}^2$ with the property that $d(p,q) = |p'-q'|$, $d(p,r) = |p'-r'|$ and $d(q,r) = |q'-r'|$ the following inequality is satisfied: $d(x,y) \leq |x'-y'|$ for any pair of points $x, y \in \Delta(p,q,r)$. Here for any $x \in \Delta(p,q,r)$ if $x \in [p,q]$ then by definition $x' \in [p',q'] \subset \Delta'(p',q',r')$, $d(p,x) = |p'-x'|$. Similar definition of x' if x lies on some other side of the geodesic triangle Δ. (X,d) is usually called $CAT(0)$ if any pair of points can be joined by a geodesic and the above inequality is true for any geodesic triangle in X. A simply connected $CAT(0)$ space is called *Hadamard space*. It is easy to check that a $CAT(0)$ space is convex and hence a nonpositively curved metric space is locally convex.

We need the following important lemma throughout the lecture.

Lemma 2.1. *Let (X,d) be a locally complete and locally convex metric space and $c : [0,1] \to X$ be a linearly parameterized local geodesic joining $x = c(0)$ and $y = c(1)$. Consider a real number $\epsilon > 0$ so that the closed balls $\overline{B}(c(t), 2\epsilon)$ are both complete and convex for each $t \in [0,1]$. Then the following are true.*

1. *let $x', y' \in X$ are such that $d(x,x') \leq \epsilon$ and $d(y,y') \leq \epsilon$ then there is a unique local geodesic c' joining x' and y' so that the function $t \mapsto d(c(t), c'(t))$ is convex.*
2. $l(c') \leq l(c) + d(x,x') + d(y,y')$

Proof. For a non-negative real number A consider the following statement.
P(A). For any $a, b \in [0,1]$ such that $|a-b| \leq A$ and for each p and q in X with $d(c(a),p) \leq \epsilon$ and $d(c(b),q) \leq \epsilon$ there is a local geodesic c' joining p and q so that the function $t \mapsto d(c(t), c'(t))$ is convex.

The fact that A is positive follows from the hypothesis that $\overline{B}(c(t), 2\epsilon)$ is convex for each t and from the following easy sublemma.

Sublemma 2.2. *Let X be a convex metric space and α and α' be two geodesics. Then the map $t \mapsto d(\alpha(t), \alpha'(t))$ is convex.*

Proof. As the space is convex there is a unique geodesic joining any pair of points. To prove the lemma join $\alpha(1)$ and $\alpha'(0)$ by the unique geodesic (say c), use convexity for the pair of geodesics (α_{-1}, c) and (c_{-1}, α') where $\alpha_{-1}(t) = \alpha(1-t)$ (and similarly for c_{-1}) and finally combine the two resulting inequalities. $\square$

We would like to show that if $\mathbf{P(A)}$ is true then $\mathbf{P(3A/2)}$ is also true. This will prove the existence of the geodesic c'.

Claim. $\mathbf{P(A)}$ implies $\mathbf{P(3A/2)}$.

Consider an interval $[a, b] \subset [0, 1]$ of length $3A/2$ and divide the interval in three equal parts at the points a' and b' so that $a < a' < b' < b$. Let p and q be two points with $d(a, p) \leq \epsilon$ and $d(b, q) \leq \epsilon$. By hypothesis there are local geodesics c'_1 from p to $c(b')$ and c''_1 from $c(a')$ to q with all the properties as stated in $\mathbf{P(A)}$. We follow the notations: $p_0 = c(a'), q_0 = c(b'), p_1 = c'_1(a'), q_1 = c''_1(b')$. By convexity we have $d(q_0, q_1) \leq \epsilon/2$ and $d(p_0, p_1) \leq \epsilon/2$. Again by hypothesis we have local geodesics c'_2 from p to q_1 and c''_2 from p_1 to q with all the properties as stated in $\mathbf{P(A)}$. We continue this way to get two sequences $\{p_n\}$ and $\{q_n\}$ with the property that $d(p_n, p_{n+1}) \leq \epsilon/2^{n+1}$ and $d(q_n, q_{n+1}) \leq \epsilon/2^{n+1}$. Also by convexity it follows that $d(c'_n(t), c'_{n+1}(t)) \leq td(q_n, q_{n+1}) \leq d(q_n, q_{n+1}) \leq \epsilon/2^{n+1}$. Similar inequality holds for $\{c''_n\}$. Thus we get two uniformly convergent sequences of local geodesics $\{c'_n\}$ and $\{c''_n\}$ converge to, say, c' and c''. Again using convexity of closed balls of radius 2ϵ one checks that $d(c'_n(t), c''_n(t))$ converges to 0 whenever $t \in [a', b']$ and hence $c'|_{[a',b']} = c''|_{[a',b']}$. Patching the two local geodesics c' and c'' we get a local geodesic joining p and q with all the required properties. Hence $\mathbf{P(3A/2)}$ is true.

Now we come to uniqueness of c'. Let c'' be another local geodesic satisfying the same properties as c'. Since the functions $t \mapsto d(c(t), c'(t))$ and $t \mapsto d(c(t), c''(t))$ are convex we get $d(c(t), c'(t)) \leq \epsilon$ and $d(c(t), c''(t)) \leq \epsilon$ for each t. Thus $d(c'(t), c''(t)) \leq 2\epsilon$. Since each closed ball of radius 2ϵ is convex the function $t \mapsto d(c'(t), c''(t))$ is locally convex and hence convex. Since c' and c'' have the same initial and final points we get that $c' = c''$.

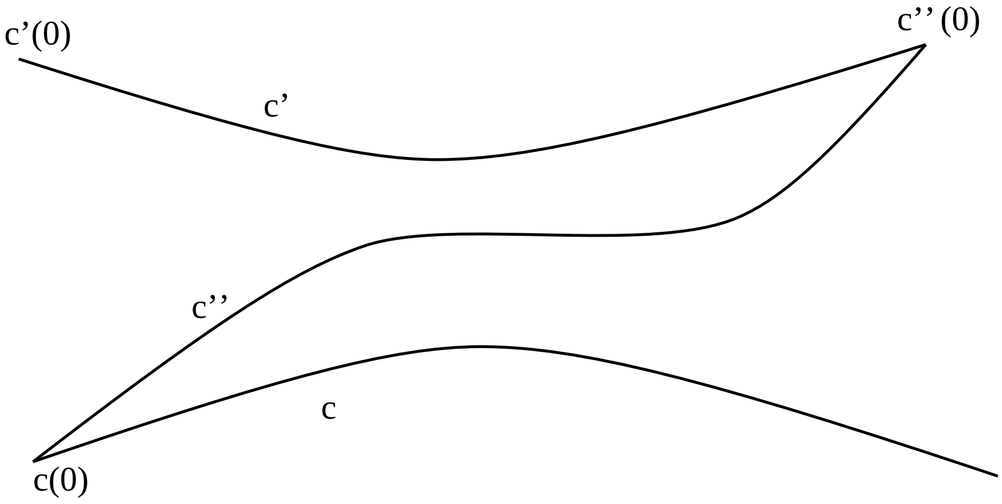
To prove (2) let us join $c'(1)$ and $c(0)$ by the unique local geodesic c'' (which exists by (1)). Note that for small t and for any local geodesic $\alpha : [0, 1] \to X$, $l(\alpha|_{[0,t]}) = tl(\alpha)$. Using convexity we get the following

$$tl(c'') = d(c''(1), c''(1-t)) \leq d(c(0), c(t)) + d(c''(1-t), c(t))$$

$$\leq tl(c) + td(c(1), c''(0)) = tl(c) + td(c(1), c'(1))$$

That is we get $l(c'') \leq l(c) + d(c(1), c'(1))$. Similarly we get $l(c') \leq l(c'') + d(c(0), c'(0))$. Combining the two inequalities we prove (2).

This proves our lemma. $\square$



3. Cartan Hadamard theorem and its proof

In few words the theorem says that in a locally convex complete metric space if all loops based at a point (say x_0) contracts to the point x_0 then the whole metric space contracts to the point x_0. In fact the space becomes globally convex. The theorem also says that any complete simply connected nonpositively curved metric space is $CAT(0)$.

The precise statement of the theorem is the following.

Cartan-Hadamard Theorem. *Let X be a complete connected metric space. If X is nonpositively curved, then the universal cover $\tilde{X}$ with the induced metric is a $CAT(0)$ space.*

In fact in this note we present the proof of another theorem from which the above theorem can be deduced by the Alexandorov patchwork procedure. In [BH] both these theorems together is called Cartan-Hadamard theorem.

Cartan-Hadamard Theorem. *Let X be a complete locally convex metric space, then the induced length metric on the universal cover $\tilde{X}$ of X is globally convex.*

A particular case of the above theorem is that between any two points in the universal cover there is a unique geodesic and geodesics vary continuously with the end points. Now fix a point $x_0 \in \tilde{X}$ and consider the map $F : \tilde{X} \times I \to \tilde{X}$ defined by $F(x,t) = \sigma_{x_0 x}(t)$ where $\sigma_{x_0 x}$ is the unique geodesic joining x_0 and x after a suitable reparametrization, that is $\sigma_{x_0 x}(0) = x_0, \sigma_{x_0 x}(1) = x$ and $d(\sigma_{x_0 x}(t), \sigma_{x_0 x}(t') = l(\sigma_{x_0 x})|t - t'|$. From the first line of this paragraph it follows that F is a continuous function. Also note that $F(x,1) = x$ for all $x \in \tilde{X}$ and $F(x,0) = x_0$. Hence $\tilde{X}$ is contractible. In the case of non-positively curved Riemannian manifold in fact it follows that the universal cover is diffeomorphic to the Euclidean n-space where n is the dimension of the manifold. More precisely the exponential map at any point becomes a covering projection making the tangent space (which is homeomorphic to $\mathbb{R}^n$) at that point the universal cover.

In the general situation of metric space we can define an exponential map from a suitable space to the metric space under consideration and deduce similar property as in the case of Riemannian manifold.

Definition 3.1. (**Exponential map**) Let (X, d) be a metric space with a base point $x_0 \in X$ and define $\tilde{X}_{x_0} = \{c : [0,1] \to X \mid c(0) = x_0,\ c$ is a linearly reparametrized local geodesic $\} \cup \{\tilde{x}_0,$ the constant map $\}$. Also we define a metric on $\tilde{X}_{x_0}$ by the following way.

$$d(c, c') := sup\{d(c(t), c'(t)) \mid t \in [0,1]\}$$

We denote the natural map $\tilde{X}_{x_0} \to X$ which sends c to $c(1)$ and $\tilde{x}_0$ to x_0 by *exp* and call it the *exponential map*.

We prove the following theorem.

Theorem 3.2. *Let X be a connected complete locally convex metric space. Fix $x_0 \in X$. Then the space X_{x_0} and the map exp have the following properties.*

(1) *X_{x_0} is contractible and exp is a local isometry*
(2) *X_{x_0} is complete and any point can be joined by a unique local geodesic to $\tilde{x}_0$*
(3) *exp is a covering map*
(4) *X_{x_0} is convex*

The corollary below is immediate.

Corollary 3.3. *(Cartan-Hadamard Theorem) Let X be a complete locally convex metric space, then the induced length metric on the universal cover $\tilde{X}$ of X is globally convex.*

Proof of Theorem 3.2. Consider the following map $F : X_{x_0} \times I \to X_{x_0}$ by $F(c, t) = c_t$ where $c_t(s) = c(st)$ for each $s \in [0, 1]$. Then $F(c, 0) = \tilde{x}_0$ and $F(c, 1) = c$. Also F is continuous. This proves the first assertion of (1).

Fix a $c \in X_{x_0}$. Recall that there is an $\epsilon > 0$ such that the ball $\overline{B}(c(1), 2\epsilon)$ is both complete and convex. Then using part (2) of Lemma 2.1 it follows that the restriction of *exp* on $B(c, \epsilon)$ is an isometry onto $B(c(1), \epsilon)$. Thus *exp* is a local isometry.

Now let $\{c_n\}$ be a Cauchy sequence in X_{x_0}. Then from the definition of the metric in X_{x_0} it follows that for each t the sequence $\{c_n(t)\}$ is a Cauchy sequence in X and hence converges to $c(t) \in X$ (say). Now using part (2) of Lemma 2.1 it follows that $l(c_n)$ is uniformly bounded. Hence the convergence $c_n \to c$ is uniform. Also it follows that c is a local geodesic. Thus we have shown that X_{x_0} is complete.

Since the map exp is a local isometry a path c in X_{x_0} is a local geodesic if and only if the path $exp \circ c$ is a local geodesic in X. Hence there is a bijection between local geodesic in X_{x_0} issuing from $\tilde{x}_0$ and local geodesics in X issuing from x_0. Thus for any point $c \in X_{x_0}$, $\tilde{x}_0$ and c are joined by the local geodesic $\tilde{c} : [0,1] \to X_{x_0}$ defined by $\tilde{c}(t) = c_t$ where $c_t(s) = c(ts)$.

For the proof of showing that exp is a covering map we refer the reader/audience to I-3.28 of the book.

We come to the proof that X_{x_0} is convex.

We have already shown that there is a unique local geodesic joining $\tilde{x}_0$ and any point of X_{x_0}. Now let $p, q \in X_{x_0}$. Since X_{x_0} is simply connected there is only one homotopy class of paths joining p and q. Hence under the projection exp this homotopy class goes to a single homotopy class (relative to end points), say $\mathcal{C}$ of paths in X. We claim that $\mathcal{C}$ contains a unique local geodesic. To prove this claim consider $exp_{exp(p)}$, the exponential map $X_{exp(p)} \to X$, and lift $\mathcal{C}$ under $exp_{exp(p)}$ to $X_{exp(p)}$ at the base point $\widetilde{exp(p)}$. Since $X_{exp(p)}$ is a universal cover each member of this lifted class will have the same end point. Hence by (2) there is a unique local geodesic in this lifted class. We project down this unique local geodesic under the map $exp_{exp(p)}$ to get a unique local geodesic in the class $\mathcal{C}$. Now lifting of this unique local geodesic in $\mathcal{C}$ to X_{x_0} with end points p and q is the unique local geodesic joining p and q. Let us denote this unique local geodesic by c_{pq}.

Note that by (1) of Lemma 2.1 local geodesics vary continuously with their end points. We now show that in fact c_{pq} is a geodesic for all p and q.

Recall that the universal cover of X gets a length metric induced from the metric of X. And since the length metric on the universal cover is unique (see I-3.25), the length metric on X_{x_0} is the same length metric induced from the metric on X.

To show that X_{x_0} is geodesic it is enough to show that for any rectifiable curve $\gamma : [0,1] \to X_{x_0}$ and every t we have $l(c_{\gamma(0)\gamma(t)}) \leq l(\gamma|_{[0,t]})$. Since X_{x_0} is locally convex there is a t' so that the above inequality holds for $t \leq t'$. It is clear that the subset A of points in $[0,1]$ for which the above inequality is true is a closed subset of $[0,1]$. We want to show that this subset is also open. Let t_0 be such that $l(c_{\gamma(0)\gamma(t)}) \leq l(\gamma|_{[0,t]})$ for all $t \leq t_0$. Then by (2) in Lemma 2.1 there is an ϵ for which

$$\begin{aligned} l(c_{\gamma(0)\gamma(t_0+\epsilon)}) &\leq l(c_{\gamma(0)\gamma(t_0)}) + d(\gamma|[t_0, t_0+\epsilon]) \\ &\leq l(\gamma|_{[0,t_0]}) + l(\gamma|_{[t_0,t_0+\epsilon]}) = l(\gamma|_{[0,t_0+\epsilon]}) \end{aligned}$$

This shows that A is both closed and open and hence the whole interval $[0,1]$.

We would like to show that for any three points $p, q_0, q_1 \in X_{x_0}$ the function $t \mapsto d(c_{pq_0}(t), c_{pq_1}(t))$ is convex or in other words $d(c_{pq_0}(t), c_{pq_1}(t)) \leq td(c_{pq_0}(1), c_{pq_1}(1))$ for every $t \in [0,1]$. By (1) of Lemma 2.1 we know that for a small ϵ

$$d(c_{pq_s}(t), c_{pq_{s+\epsilon}}(t)) \leq td(q_s, q_{s+\epsilon})$$

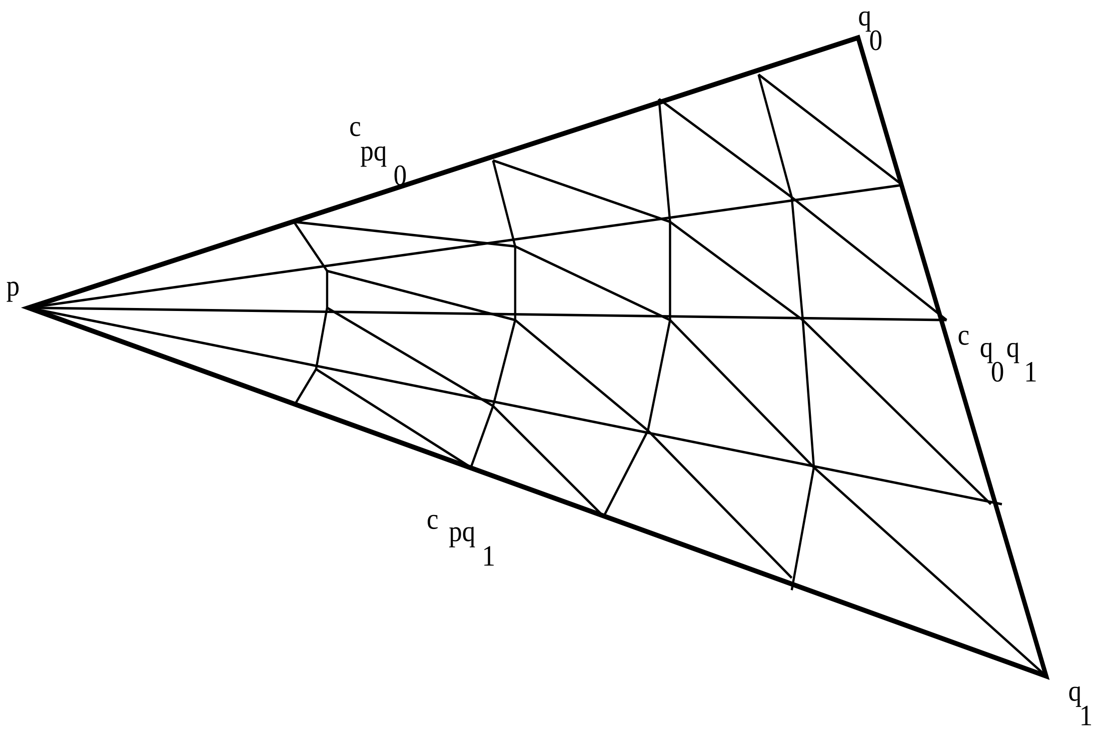
where $q_s = c_{pc_{q_0q_1}(s)}(1)$. Since the geodesics joining p and q_s vary continuously with q_s we get a partition of $0 = s_0 < s_1 < \cdots < s_n = 1$ of $[0,1]$ so that for each pair (s_i, s_{i+1}) the above inequality is satisfied if we replace s by s_i and $s+\epsilon$ by s_{i+1}. We add up all these n inequalities to get that $t \mapsto d(c_{pq_0}(t), c_{pq_1}(t))$ is convex for any three points on X_{x_0}.

This proves our theorem. $\square$

Now we come to the second Cartan Hadamard theorem.

Cartan-Hadamard Theorem. *Let X be a complete connected metric space. If X is nonpositively curved, then the universal cover $\tilde{X}$ with the induced metric is a $CAT(0)$ space.*

Proof. We change the notation of X_{x_0} to $\tilde{X}$. Note that as X is locally $CAT(0)$ it is locally convex. Hence by the above theorem $\tilde{X}$ is a convex metric space. Thus any two points of $\tilde{X}$ are joined by a unique geodesic and geodesics vary continuously with their end points. We want to show that for any geodesic triangle $\Delta(p, q_0, q_1)$ in $\tilde{X}$ the $CAT(0)$ inequality is satisfied. We will use the characterization of $CAT(0)$ inequality in terms of angle (see II-1.7(4) of the book) and the Alexandrov Lemma (see I-2.16 of the book) to prove the theorem.



Choose the continuous family of geodesics c_{pq_s} where $s \in [0,1]$ and $q_s = c_{q_0q_1}(s)$. Since $\tilde{X}$ is locally $CAT(0)$ and the above family of geodesics is continuous we can get two partitions $0 = s_0 < s_1 < \cdots < s_n = 1$ and $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ so that each of the geodesic triangles $\Delta(c_{pq_{s_i}}(t_j), c_{pq_{s_i}}(t_{j+1}), c_{pq_{s_{i+1}}}(t_{i+1})$ lie in a $CAT(0)$ ball of $\tilde{X}$. Repeated application of the Alexandrov's Lemma and characterization of $CAT(0)$ inequality in terms of angles proves the theorem.

□

3. Some topological consequences of the C-H theorem

We have already seen in the previous section that if X is a complete nonpositively curved metric space then the universal cover $\tilde{X}$ of X is uniquely geodesic, that is there is a unique geodesic joining any pair of points of $\tilde{X}$. Also as the covering map $\tilde{X} \to X$ is a local isometry there is a bijection between local geodesics in X with initial point $x_0 \in X$ and local geodesics in $\tilde{X}$ with initial point $\tilde{x}_0$ which lie over x_0. Consider a homotopy class β of loops based at x_0. If we lift the class β to the universal cover at the base point $\tilde{x}_0$ then we get the homotopy class of paths in $\tilde{X}$ joining $\tilde{x}_0$ and $\beta(x_0)$, here β is considered as a element of the fundamental group of X and the action of β on the universal cover is by Deck transformation. Thus there is a unique local geodesic in the homotopy class β.

Now recall from II-2.8 that an isometry of finite order of a $CAT(0)$ metric space must have a fixed point. On the other hand the fundamental group acts on the universal cover without fixed point. Also in the case of metric spaces the action on the universal cover is by isometries and hence by Cartan Hadamard theorem we get that a complete metric space of nonpositive curvature must have torsion free fundamental group.

Thus we have proved the following.

Theorem 3.1. *Let X be a complete nonpositively curved metric space and $x_0 \in X$ is a base point. Then,*

(1) *for any loop c in X based at x_0 the class $[c] \in \pi_1(X, x_0)$ contains a unique local geodesic*
(2) *$\pi_1(X, x_0)$ is torsion free*

Here we would like to point out another proof of (2). Suppose X is a complete nonpositively curved metric space and $H_i(X, \mathbb{Z})$ is trivial for large i. Assume on the contrary that $\pi_1(X, x_0)$ is not torsion free. If $\pi_1(X, x_0)$ has a torsion element, say α of order n then consider the covering $X_1 \to X$ of X corresponding to the conjugacy class of the finite subgroup $< \alpha >$ of $\pi_1(X, x_0)$. Then $\pi_1(X_1, x_1) \simeq < \alpha >$ and since X_1 also has a contractible universal cover $H_i(X_1, \mathbb{Z}) \simeq H_i(< \alpha >, \mathbb{Z})$ for each i. But we have $H_i(< \alpha >, \mathbb{Z}) \simeq \mathbb{Z}_n$ when $i = 0$ or odd and 0 otherwise. Hence $H_i(< \alpha >, \mathbb{Z})$ is not trivial for large i. On the other hand one can show from the hypothesis that $H_*(X, \mathbb{Z})$ is trivial for large i and $\tilde{X}$ is contractible that $H_i(X_1, \mathbb{Z})$ is trivial for large i. Which is a contradiction. Hence $\pi_1(X, x_0)$ is torsion free.

Here recall that when X is compact then it is homotopically equivalent to a finite dimensional simplicial complex (see I-7A.15) and hence in this particular case $H_*(X, \mathbb{Z})$ is finitely generated. A natural question arises at this point.

Question 3.2. *What are all the complete nonpositively curved metric spaces with finitely generated $H_*(X, \mathbb{Z})$?*

School of Mathematics, Tata Institute, Homi Bhabha Road, Mumbai 400 005, India.
http://www.math.tifr.res.in/~ roushon/madras.html
E-mail address: roushon@math.tifr.res.in