NOTES ON SIMPLE HOMOTOPY TYPES OF 4-MANIFOLDS

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Abstract. This is the note of a lecture given at the Allahabad low dimensional topology instructional conference on simple homotopy types of 4-manifolds.

1. Introduction

On the passage from homotopy equivalence to topological equivalence there lies an important notion of equivalence in topology; the *simple homotopy equivalence*. The celebrated s-cobordism theorem, the proof of generalized Poincare Conjecture are the most famous examples which circled around the notion of simple homotopy equivalence. Even in such a classical vast area of topology the most basic question is not yet answered; namely, if any homotopy equivalence between two finite aspherical complexes is homotopic to a simple homotopy equivalence, which is known as Whitehead's conjecture. There is an even stronger conjecture which asks; if any homotopy equivalence between two aspherical manifolds is homotopic to a homeomorphism. This is also known as Borel's conjecture. A result of Chapman says that any homeomorphism is a simple homotopy equivalence. So the first step to prove Borel's conjecture is to verify Whitehead's conjecture.

Now we come to the interesting complication in topology imposed by dimension. For example the techniques and tricks used in manifold topology of dimension ≤ 4 and in dimension ≥ 5 are completely different. For this reason there is a (clear!) division in manifold topology; one is called "low dimensional topology" and the other "high dimensional topology".

Dimension 2 is understood completely in the topological, piecewise linear or smooth category. Thurston's geometrization programme gives a conjectural picture in dimension 3. Also in dimension ≤ 3 any manifold supports a unique topological, piecewise linear or smooth structure. In dimension greater or equal to 5 an enormous literature exists; where there are enough machinery, language to attack a problem. The critical dimension is 4. In this dimension even the s-cobordism theorem is not yet known. So far this is proved for 4-manifolds with some restriction on the fundamental groups; namely, groups with subexponential growth. Also another interesting fact is that a 4-manifold can support infinitely many smooth structure. Even our familiar Euclidean 4-space $\mathbb{R}^4$ has infinitely many smooth structure.

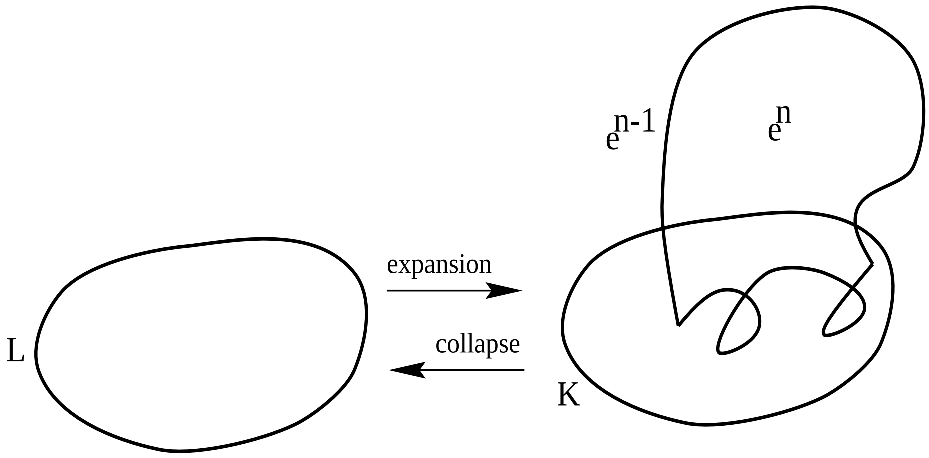
In this lecture we talk about a certain class of 4-manifolds and find how many simple homotopy types do they have upto s-cobordism. We will recall the definitions of basic notions and then state the theorems without proof. The references for the proofs of the theorems will be given.

2. Simple homotopy equivalence and Whitehead group

The reference for this section is [C]. Let K and K' be two finite CW-complexes. A homotopy equivalence $f : K \to K'$ is called a *simple homotopy equivalence* provided f is homotopic to a composition of maps of the following kind: $K = K_0 \to K_1 \to \cdots \to K_s = K'$ where the arrows are either an elementary expansion or collapse. A pair of CW-complexes (K, L) is called an *elementary collapse* (and we say K *collapses to* L or L *expands to* K) if the followings are satisfied:

1. $K = L \cup e^{n-1} \cup e^n$ where e^{n-1} and e^n are not in L (e^i denotes a i-cell),
2. there exist a ball pair $(\mathbb{D}^n, \mathbb{D}^{n-1})$ and a map $\phi : \mathbb{D}^n \to K$ such that
   - (a) ϕ is a characteristic map for e^n
   - (b) $\phi|_{\mathbb{D}^{n-1}}$ is a characteristic map for $\mathbb{D}^{n-1}$
   - (c) $\phi(P^{n-1}) \subset L^{n-1}$, where $P^{n-1} = \overline{\mathbb{D}^n - \mathbb{D}^{n-1}}$.

From the following figure the definition of an expansion (or collapse) will be clear.



Given any homotopy equivalence $f : K \to K'$, there is an obstruction $\tau(f)$ which lies in an abelian group $Wh(\pi_1(K'))$ (defined below) detecting if f is a simple homotopy equivalence.

Let π be a group. Let $GL_n(\mathbb{Z}[\pi])$ be the invertible $n \times n$ matrices with entries in $\mathbb{Z}[\pi]$ and $E_n(\mathbb{Z}[\pi])$ be the subgroup of elementary matrices, which is also the commutator subgroup of $GL_n(\mathbb{Z}[\pi])$. Define $GL(\mathbb{Z}[\pi]) = \lim_{n\to\infty} GL_n(\mathbb{Z}[\pi])$, $E(\mathbb{Z}[\pi]) = \lim_{n\to\infty} E_n(\mathbb{Z}[\pi])$ and $K_1(\mathbb{Z}[\pi]) = \frac{GL(\mathbb{Z}[\pi])}{E(\mathbb{Z}[\pi])}$. The *Whitehead group* $Wh(\pi)$ of π is by definition $\frac{K_1(\mathbb{Z}[\pi])}{<\pi, -\pi>}$.

We recall below a transparent geometric definition of Whitehead group which is naturally isomorphic to the above one. The proof of this isomorphism is very long and can be found in [C]. But we will give the map from this geometric definition to the algebraic definition above. But for computational purpose the algebraic definition is more useful.

Let K be a fixed finite CW-complex. Let $\mathcal{W}(K)$ be the collection of all pairs of finite complexes (L, K) so that K is a strong deformation retract of L. For any two objects $(L_1, K), (L_2, K) \in \mathcal{W}$ define $(L_1, K) \equiv (L_2, K)$ if and only if L_1 and L_2 are simple homotopical equivalent relative to the subcomplex K. Here by a relative simple homotopy equivalence we mean that in the definition of simple homotopy equivalence we do not collapse (or expand) any cells contained in K. Let $Wh(K) = \mathcal{W}/\equiv$. Let $[L_1, K]$ and $[L_2, K]$ be two classes in $Wh(K)$. Define $[L_1, K] \oplus [L_2, K] = [L_1 \cup_K L_2, K]$. Here $L_1 \cup_K L_2$ is the disjoint union of L_1 and L_2 identified along the common subcomplex K. This defines an abelian group structure on $Wh(K)$ called the Whitehead group of K.

Now let (L, K) be an element in $\mathcal{W}$. Consider the universal cover $(\tilde{L}, \tilde{K})$. It can be checked that the inclusion $\tilde{K} \subset \tilde{L}$ is a homotopy equivalence. Equip $(\tilde{L}, \tilde{K})$ with the CW-complex structure lifted from the CW-structure of (L, K). Let $C_*(\tilde{L}, \tilde{K})$ be the cellular chain complex of $(\tilde{L}, \tilde{K})$. The covering action of $\pi_1(L)$ on $(\tilde{L}, \tilde{K})$ induces an action on $C_*(\tilde{L}, \tilde{K})$ and makes it a chain complex of $\mathbb{Z}\pi_1(L)$ module. In fact it is a finitely generated free acyclic chain complex of $\mathbb{Z}\pi_1(L)$ module. Let d be the boundary map. One can find a contraction map δ of degree $+1$ of $C_*(\tilde{L}, \tilde{K})$ so that $d\delta + \delta d = id$ and $\delta^2 = 0$. Consider the module homomorphism $d + \delta : \bigoplus_{i=0}^{\infty} C_{2i+1}(\tilde{L}, \tilde{K}) \to \bigoplus_{i=0}^{\infty} C_{2i}(\tilde{L}, \tilde{K})$. It turns out that this homomorphism is an isomorphism of $\mathbb{Z}\pi_1(L)$ module. The image and range of this homomorphism are finitely generated free module with a preferred basis coming from the CW-structure on $(\tilde{L}, \tilde{K})$. We consider the matrix of this homomorphism $d + \delta$ which is an invertible matrix with entries in $\mathbb{Z}\pi_1(L)$ and hence lies in $GL_n(\mathbb{Z}[\pi_1(L)])$ for some n. We take the image of this matrix in $Wh(\pi_1(L))$. The proof that this map (say τ) is an isomorphism is given in [C].

Now consider a homotopy equivalence $f : K \to K'$ between two finite CW-complexes. Let M_f be the mapping cylinder of the map f. Consider the pair (M_f, K). As f is a homotopy equivalence it is easy to check that $(M_f, K) \in \mathcal{W}$. Now we recall that f is homotopic to a simple homotopy equivalence if and only if $\tau([M_f, K])$ is the trivial element in $Wh(\pi_1(K))$.

The Whitehead conjecture mentioned in the introduction, in fact, asks for even a stronger conclusion; namely $Wh(\pi) = 1$ for any finitely presented torsion free group. So far this has been checked for several classes of groups: for free abelian groups by Bass-Heller-Swan; for free nonabelian groups by Stallings; for the fundamental group of any complete nonpositively curved Riemannian manifold by Farrell and Jones; for the fundamental group of finite $CAT(0)$ complexes by B. Hu. Waldhausen proved that the Whitehead group of the fundamental group of any Haken manifold vanishes. Artin braid groups are a very important class of groups which is not yet known to belong to any class of groups mentioned above. In [AFR] it was shown that the Whitehead group of the pure Artin braid group vanish. Pure Artin group is a finite index subgroup of the Artin braid group with quotient S_n. In [FR] it is proved that in fact Whitehead group of any subgroup of the Artin braid group vanishes. Here we mention the two papers [AFR] and [FR] because some 3-manifold techniques and results were crucial to prove the vanishing theorems. Some results like Whitehead group of finite groups are also known: $Wh(F) = 0$, when F is a finite cyclic group of order $1, 2, 3, 4$ and 6 and $Wh(F)$ is infinite when F is any other finite cyclic group. Also $Wh(S_n) = 0$, here S_n is the symmetric group on n letters.

3. Simple homotopy types upto s-cobordism

We are in a position to state the s-cobordism theorem now.

s-cobordism theorem. *Let M_1 and M_2 be two compact manifolds of dimension ≥ 5 if they have empty boundary and of dimension ≥ 6 otherwise. Let W be another compact manifold with two boundary components M_1 and M_2. Assume that both the inclusions $M_i \to W$, $i = 1, 2$ are homotopy equivalences. If $\tau([W, M_1]) = 1$ in $Wh(\pi_1(M_1)$ then $W \simeq M_1 \times I$. In particular $M_1 \simeq M_2$.*

Remark 3.1. When $\partial W = M_1 \cup M_2$ and the inclusions $M_i \subset W$ are homotopy equivalence so that $\tau([W, M_1]) = 1$, then we say W is an s-cobordism between M_1 and M_2.

Here $\simeq$ is a homeomorphism, a piecewise linear homeomorphism or a diffeomorphism according as the manifolds are topological, piecewise linear or smooth respectively.

It is know that the smooth 4-dimensional s-cobordism theorem is not true. So from now onwards we only talk in the category of topological or piecewise linear manifolds.

Whether the s-cobordism theorem is true when the dimension of the manifold is 4 is not yet known completely. This has been proved only when the fundamental group of the 4-manifold is of subexponential growth. In [FQ] it was proved for elementary amenable fundamental groups. Later in [FT] the case of groups with subexponential growth was proved. A mistake in [FT] was corrected in [KQ].

We consider a large class $\mathcal{C}$ of 4-manifolds and show that the Whitehead group of the fundamental group of any member from this class is 0. Thus any homotopy equivalence is homotopic to a simple homotopy equivalence in this class. We would like to find the number of simple homotopy types of these manifolds up to homeomorphism. The techniques in surgery theory allows one to solve such a problem, first only up to s-cobordism and then show that any s-cobordism is the product.

As we still do not have the s-cobordism theorem in this dimension, we can only talk of simple homotopy types upto s-cobordism.

Now we define the class $\mathcal{C}$.

$\mathcal{C}$ is the class of compact orientable aspherical 4-manifolds so that for each $M \in \mathcal{C}$ there is a fiber bundle projection $M \to \mathbb{S}^1$ with irreducible fiber N and either M has nonempty boundary or it is closed and satisfies one of the following properties:

(1) $H^1(N, \mathbb{Z}) \neq 0$
(2) there is an embedded incompressible torus T in N so that $\pi_1(T)$ has square root closed image in $\pi_1(N)$
(3) N is a Seifert fibered space with base surface of genus ≥ 1
(4) N has more than 2 geometric pieces in its Jaco-Shalen and Johannson decomposition and the dual graph of this decomposition has a vertex which disconnects the graph and also the fundamental group of any edge emanating from this vertex is square root closed in the fundamental group of the target vertex
(5) N supports a hyperbolic metric.

Here recall that a subgroup H of a group G is called *square root closed* if for any $x \in G$, $x^2 \in H$ implies $x \in H$.

A 3-manifold is called *irreducible* if any embedded 2-sphere in it bounds an embedded 3-disc. An irreducible 3-manifold with nonempty boundary has nonvanishing first Betti number. Also any irreducible 3-manifold with nonvanishing first Betti number is Haken.

The dual graph of the JSJ-decomposition has vertices the pieces in the decomposition and edges are tori which are common boundary component of two pieces. By fundamental group of a vertex or an edge we mean the fundamental group of the associated spaces.

Lemma 3.2. *([R]) Let $M \in \mathcal{C}$. Then the Whitehead group of $\pi_1(M)$ is 0.*

Proof. See [R]. $\square$

Theorem 3.3. *([R]) Let $M \in \mathcal{C}$. Then there are only finitely many pairs (N_i, f_i), $i = 1, 2, \ldots, n$, where $f_i : N_i \to M$ is a homotopy equivalence for each i, so that for any pair (N, f) with $f : N \to M$ a homotopy equivalence there is an $i \in \{1, 2, \ldots, n\}$ and an s-cobordism W between N and N_i, a map $F : W \to M$ with $F|_N = f$ and $F|_{N_i} = f_i$.*

Proof. See [R]. $\square$

Remark 3.4. Here we recall that a particular case of the above theorem was proved in [H1] for a certain class of complex surfaces which fiber over $\mathbb{S}^1$. In [R] a correction was made in the proof of the theorem in [H1]. In [H2] it was proved that if a compact complex surface fiberes over $\mathbb{S}^1$ then the fiber is a Seifert fibered space.

Remark 3.5. Note that Borel's conjecture says that if the s-cobordism theorem is true in the class $\mathcal{C}$ then in the above theorem n will be equal to 1.

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