# NOTES ON KNOT THEORY AND 3-MANIFOLDS

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Abstract. These are the notes of lectures given at the Allahabad low dimensional topology instructional conference on knot theory and 3-manifolds.

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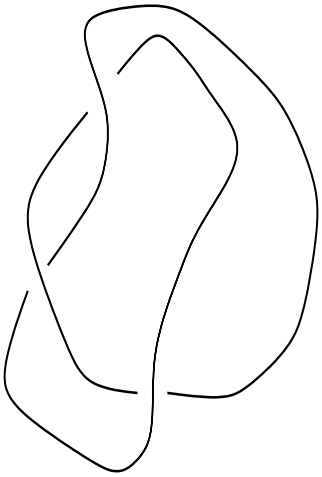
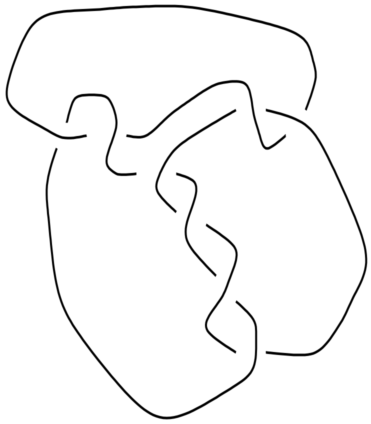
## 0. Introduction

Throughout this notes all spaces are ($PL$ or smooth) manifolds and all maps are $PL$ or smooth. Mainly we talk in dimension $\leq 3$ and in dimension $\leq 3$ it is known that any manifold has unique $PL$ or smooth structure. The prerequisites are basic Differential Topology and Algebraic Topology. Frequently we will be using the tubular neighborhood theorem, Hurewicz isomorphism theorem, Alexander-Schoenflies theorem. The standard references are [S], [Gr], [G], [GP] etc.

By a *knot* $k$ we mean the image of an embedding (PL or smooth) of the unit circle $\mathbb{S}^1$ in $\mathbb{S}^3$. Here recall $\mathbb{S}^n = \{\Sigma x_i^2 = 1 \mid (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}\}$ and $\mathbb{D}^n = \{\Sigma x_i^2 \leq 1 \mid (x_1, \cdots, x_n) \in \mathbb{R}^n\}$. The fundamental group of the complement of the knot is called the *knot group*. A *link* is finitely many knots disjointly embedded in the 3-sphere. Following are the few examples of knots and links:



A (2,3) torus knot



A link with 3 components

A *trivial knot* is the image of an embedding of $\mathbb{S}^1$ in $\mathbb{S}^3$ which can be extended to an embedding of the 2-disc $\mathbb{D}^2$. A consequence of Loop theorem of Papakyriakopolous is that a knot $k$ is trivial if and only if $\pi_1(\mathbb{S}^3 - k)$ is the infinite cyclic group. The fundamental group of non-trivial knots are very complicated. For an instance it is known that the fundamental group of any non-trivial knot complement contains any finitely generated non-abelian free group. On the other hand by Alexander duality the homology groups of any knot complement is isomorphic to the homology groups of the circle $\mathbb{S}^1$. Another very important and nontrivial fact about knot complement is that all the higher homotopy groups of any knot complement vanish, i.e., $\pi_i(\mathbb{S}^3 - k) = 0$ for all $i \geq 2$. We always suppress the base point when talking about homotopy groups because we only talk about connected spaces. Thus we see that to study a knot complement the only non-trivial invariant we have (so far) is its' fundamental group (which depends upon the isotopy class of the embedding of the knot). Two knots $k_1$ and $k_2$ are called *equivalent* if there is an orientation preserving diffeomorphism (or $PL$ homeomorphism) $f$ of the 3-sphere so that $f(k_1) = k_2$. If $k_1$ and $k_2$ are oriented knots then we require that $f$ preserves the orientation of $k_1$ onto $k_2$. Obviously if $k_1$ and $k_2$ are equivalent then their complements are diffeomorphic. The converse of this result is a difficult theorem in knot theory, recently proved by Gordon and Luecke ([GL]). This was open for almost from the beginning of this subject. Also the fundamental groups of two equivalent knot complements are isomorphic. However the converse is not true in general. But it is a standard fact that if the knot groups are isomorphic then the complements are homotopy equivalent. But the following is true:

"Consider small tubular neighborhood of $k_1$ and $k_2$ in two copies of $\mathbb{S}^3$ and remove the interior to get two compact 3-manifolds $M_1$ and $M_2$ with torus boundary. Let $\lambda_i$ and $\mu_i$ are longitude and meridian of the boundary $\partial M_i$ for $i = 1, 2$, i.e., $\lambda_i$ is an oriented simple closed curve parallel to $k_i$ in the tubular neighborhood of $k_i$ and $\mu_i$ is an oriented simple closed curve which bounds a disc in the tubular neighborhood which intersects $\lambda_i$ in a single point. Not that $[\lambda_i]$ and $[\mu_i]$ generate $\pi_1(\partial M_i)$. If $k_1$ and $k_2$ are oriented and equivalent then there is a diffeomorphism $h : M_1 \to M_2$ so that $h_*([\lambda_1]) = [\lambda_2]$ and $h_*([\mu_1]) = [\mu_2]$. The consequence of a nontrivial result of Waldhausen ([W]) says that if there is an isomorphism from $\pi_1(M_1)$ to $\pi_1(M_2)$ which sends longitude to longitude and meridian to meridian then there is a diffeomorphism from $M_1$ to $M_2$ inducing the isomorphism on the fundamental groups we started with."

In Section 2 we will define and prove the existence of a important notion called *Seifert surface* of a knot and use them to define the genus of a knot which in turn used to distinguish some knots. Also we use the genus of knot to give a prime factorisation of knots. Seifert surface with minimal genus will be used to characterize commutator subgroups of knot groups.

An important connection between links in $\mathbb{S}^3$ and general 3-manifolds is that any closed oriented 3-manifold can be obtained from the 3-sphere by removing a tubular neighborhood of each component of the link and sewing it back by a different diffeomorphism. We will give this proof (due to Lickorish) in Section 3.

Section 4 deals with Jones polynomial and some of its properties.

## 1. Some basic concepts and results on knots

First we recall the most fundamental theorem in 3-manifold topology, namely the Sphere theorem due to Papakyriakopolous. See [H]. The proof will be given in a separate lecture.

**The sphere theorem.** *Suppose that $M$ is an oriented 3-manifold and there is a map $\mathbb{S}^2 \to M$ that is not homotopic to a constant map (i.e., $\pi_2(M) \neq 0$). Then there is an embedding $\mathbb{S}^2 \to M$ that is not homotopic to a constant map.*

*Remark.* Note that the theorem does not claim that the embedding is homotopic to the map we started with.

An application of the sphere theorem is the following asphericity theorem for knot complements:

**The asphericity theorem for knot complement.** *Let $k \subset \mathbb{S}^3$ be a knot then $\pi_i(\mathbb{S}^3 - k) = 0$ for all $i \geq 2$, i.e., any map $\mathbb{S}^r \to \mathbb{S}^3 - k$ for $r \geq 2$ is homotopic to a constant map.*

*Proof.* At first lets check that $\pi_2(\mathbb{S}^3 - k) = 0$. Assume this is not the case, then there is a map $\mathbb{S}^2 \to \mathbb{S}^3 - k$ which is not homotopic to a constant map. By sphere theorem we get an embedding $f : \mathbb{S}^2 \to \mathbb{S}^3 - k$ which is not homotopic to a constant map. By Alexander-Schoenflies theorem $\mathbb{S}^3 - f(\mathbb{S}^2)$ has two components and the closure of each component is $PL$-homeomorphic to $\mathbb{D}^3$ with boundary $f(\mathbb{S}^2)$. As the knot $k$ is connected it lies inside on of these two 3-discs. The other 3-disc gives the contraction of the image $f(\mathbb{S}^2)$ in $\mathbb{S}^3 - k$ to a point - a contradiction.

Let $M = \mathbb{S}^3 - k$ and $\tilde{M}$ be its universal cover. It is standard by using lifting property of maps and homotopy for covering spaces that $\pi_i(M) \simeq \pi_i(\tilde{M})$ for all $i \geq 2$. Hence we get that $\pi_2(\tilde{M}) = 0$. Now since $\tilde{M}$ is simply connected by Hurewicz isomorphism theorem (Theorem 22.4, page 216 in [G]) $H_2(\tilde{M}) = 0$. Again by the same theorem we get that $H_3(\tilde{M}) \simeq \pi_3(\tilde{M})$. But $\tilde{M}$ is a noncompact 3-manifold and hence $H_i(\tilde{M}) = 0$ for $i \geq 3$ implying that $\pi_i(\tilde{M}) = 0$ for $i \geq 3$ by using Hurewicz theorem repeatedly. This proves the theorem. $\square$

*Remark 1.1.* Spaces $X$ with $\pi_i(X) = 0$ for $i \geq 2$ are called *aspherical*.

**Exercise 1.2** Let $X$ and $Y$ be two finite dimensional aspherical complexes and assume there is an isomorphism between their fundamental groups then this isomorphism is induced from a homotopy equivalence between $X$ and $Y$. A consequence of this result is that simply connected aspherical spaces are contractible.

**Exercise 1.3** A link $l$ in the 3-sphere is called *reducible* if there is an embedded 2-sphere $X \subset \mathbb{S}^3$ so that both the components of $\mathbb{S}^3 - X$ contains some component of the link. Otherwise the link is called *irreducible*. Show that any irreducible link complement in the 3-sphere is aspherical.

**Corollary 1.4.** *$\pi_1(\mathbb{S}^3 - k)$ is torsion free for any knot $k$ (and irreducible link).*

*Proof.* Suppose on the contrary that the fundamental group has a torsion element, say $\gamma$ of order $p$. Consider the covering $\tilde{M}$ of the complement with fundamental group isomorphic to $\mathbb{Z}_p$. As the knot complement and $\tilde{M}$ both have the same (contractible) universal cover, $\tilde{M}$ is also aspherical. But for aspherical spaces we have $H_i(\tilde{M}) \simeq H_i(\pi_1(\tilde{M}))$. Now as $\tilde{M}$ is finite dimensional the homology groups vanish after certain stage (in this case after $i = 2$). But the homology groups of $\mathbb{Z}_p$ are 0 for $i$ non-zero even and $\mathbb{Z}_p$ for $i$ odd (see [B]) - a contradiction. $\square$

**Lemma 1.5.** *The homology groups of a knot complement are isomorphic to the homology groups of the circle.*

*Proof.* Direct consequence of Alexander duality. Look at any standard book on Algebraic Topology for Alexander duality. $\square$

Thus we have a surjective homomorphism $\pi_1(\mathbb{S}^3 - k) \to \mathbb{Z}$ for any knot $k$ in the 3-sphere. This homomorphism is the abelianization homomorphism $\pi_1(\mathbb{S}^3 - k) \to H_1(\mathbb{S}^3 - k)$. Let $N(k)$ be an open tubular neighborhood of $k$ in the 3-sphere. Then using Mayer-Vietoris exact sequence for $\mathbb{S}^3 = (\mathbb{S}^3 - N(k)) \cup \overline{N(k)}$ it can be shown that $H_1(\mathbb{S}^3 - N(k))$ is generated by a simple closed curve on $\partial(\mathbb{S}^3 - N(k))$ which bounds a disc in $\overline{N(k)}$ meeting $k$ at only one point. This is a meridian of $\partial(\mathbb{S}^3 - N(k))$. (For a proof see page 11, Theorem 1.5 in [L1].)

Another important result in knot theory is the following:

**Theorem 1.6.** *Let $X$ be the exterior of a tubular neighborhood of a knot $k$ in the 3-sphere. If $k$ is not the trivial knot then $\pi_1(\partial X) \to \pi_1(X)$ is an injective homomorphism induced from the inclusion map.*

To prove the theorem we need the Loop theorem of Papakyriakopolous (see [H]) which we recall below:

**The Loop Theorem.** *Let $M$ be a 3-manifold with boundary $\partial M$ such that the inclusion-induced homomorphism $\pi_1(\partial M) \to \pi_1(M)$ is not injective. Then there exists an embedding of the 2-disc $e : (\mathbb{D}^2, \mathbb{S}^1) \to (M, \partial M)$ with $e^{-1}(\partial M) = \mathbb{S}^1$ and $e : \partial\mathbb{S}^1 \to \partial M$ is not homotopic to a constant map.*

*Proof of theorem 1.6.* Suppose the contrary, i.e., $\pi_1(\partial X) \to \pi_1(X)$ is not injective. Then by Loop theorem there is an embedding $e : (\mathbb{D}^2, \mathbb{S}^1) \to (X, \partial X)$ with $e|_{\partial\mathbb{D}^2}$ not null homotopic on $\partial X$ and $[e|_{\partial\mathbb{D}^2}] \in H_1(X)$ is null-homologous. But $H_1(X)$ is generated by a meridian on $\partial X$ and a longitude generates the kernel of the map $H_1(\partial X) \to H_1(X)$. Also a longitude is parallel to the knot $k$ in a tubular neighborhood (say $N$) of $k$. Thus $e(\partial\mathbb{D}^2)$ is parallel to $k$ in a tubular neighborhood of $k$. Now choose an annulus $A$ in $N$ with two boundary components $e(\partial\mathbb{D}^2)$ and $k$. Extend the embedding $e$ to $\mathbb{D}^2 \cup A \to \mathbb{S}^3$. The image of this disc is an embedded disc bounding the knot $k$ and hence $k$ is a trivial knot - a contradiction. □

**Corollary 1.7.** *A knot $k$ in $\mathbb{S}^3$ is trivial if and only if $\pi_1(\mathbb{S}^3 - k) \simeq \mathbb{Z}$.*

*Proof.* Exercise. □

## 2. Seifert surfaces and genus of knots

Now we come to a very important concept in knot theory, namely Seifert surface introduced by H. Seifert in 1934. Let $k$ be a knot in the 3-sphere. A *Seifert surface* is an embedded surface $S$ in the 3-sphere with one boundary component and $\partial S = k$. The minimal genus of all Seifert surfaces of the knot is called the *genus* of the knot and is denoted by $g(knot)$. The genus of a knot is 0 if and only if it is the trivial knot. Note that the genus of a knot is an invariant of the knot, i.e., two equivalent knots have the same genus. We proceed to prove the existence of a Seifert surface of a given knot.

**Theorem 2.1.** *Any knot in the 3-sphere is the boundary component of an embedded surface.*

*Proof.* At first we recall the definition of regular projection of knots. Consider the knot to be in the Euclidean 3-space $\mathbb{R}^3$. Consider a 2-plane $H$ in $\mathbb{R}^3$ which does not intersect the knot then project the knot on this plane $H$ from the point at infinity in the one point compactification of $\mathbb{R}^3$. A point on this projected curve is called a *multiple point* if its preimage contains more than one point on the knot. This projected curve on $H$ is called *regular* if the following are satisfied:

(1) there are only finitely many multiple points and all multiple points are double point, that is their inverse image contains only two points.
(2) outside the double points the projected curve is smooth (or $PL$).

In the space of all projections the regular ones are open and dense. So we have a plenty of collection of regular projections (Proposition 1.12 in [BZ]).

Now consider a knot $k$ and its regular projection on a plane. Also orient the knot. The double points looks as in Figure 1. By altering the projected curve as in Figure 2 we get a number of disjoint oriented simple closed curve called *Seifert cycles*. Choose an oriented 2-disc for each Seifert cycle and embed them in $\mathbb{R}^3$ so that there projection bound the Seifert cycle and induce the same orientation as that of the Seifert cycle. Now corresponding to each double point join the associated discs by twisted bands as shown in Figure 3. What we get is a compact surface embedded in the 3-space with $k$ as the boundary. $\square$

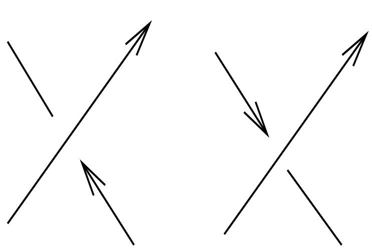


Figure 1

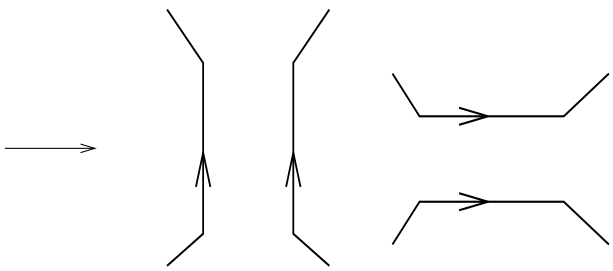


Figure 2

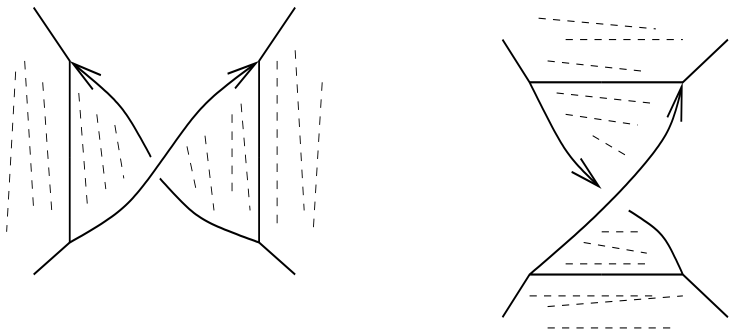


Figure 3

*Remark 2.2.* The genus of this Seifert surface is called the *canonical genus* of the knot associated to the regular projection. Here it is interesting to note that the canonical genus coincide with the genus of the knot in many cases. For more information on Seifert surfaces see [BZ].

**Exercise 2.3** Carry out the same procedure to find a Seifert surface for a link in $\mathbb{S}^3$.

We now define the addition of two oriented knots. Let $k_1$ and $k_2$ be two oriented knots in the 3-sphere with no linking or intersection between them. We can think of the 3-sphere as sum of two 3-discs glued along the boundary and the knot $k_1$ is contained in the interior of one disc and the other knot in the interior of the other 3-disc. Choose two points $p_1 \in k_1$ and $p_2 \in k_2$ and join them by an straight line (say $l$) which does not intersect the knots as in Figure 4.

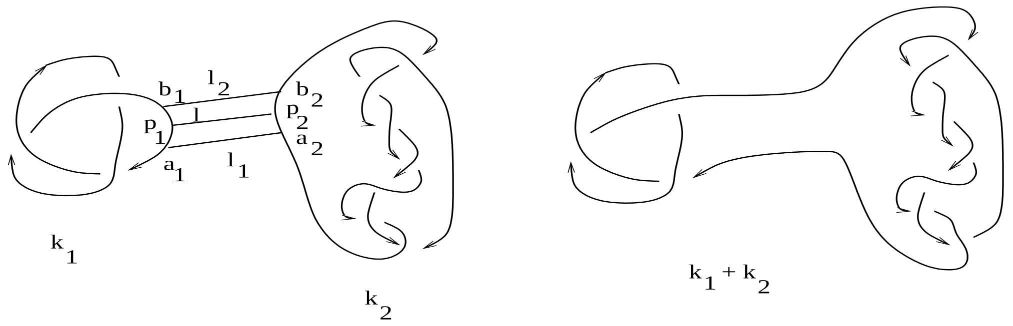



Figure 4

Choose an oriented small arc $[a_i, b_i]$ on each knot so that $p_i$ belong to some interior point of the arc $[a_i, b_i]$. Now join $a_1$ with $a_2$ and $b_1$ with $b_2$ by lines $l_1$ and $l_2$ (respectively) parallel to $l$. The knot $k = (k_1 - (a_1, b_1)) \cup (k_2 - (a_2, b_2)) \cup l_1 \cup l_2$ is called the sum of the knots $k_1$ and $k_2$ and is denoted by $k_1 + k_2$. A knot $k$ is called *prime* if $k = k_1 + k_2$ implies either $k_1$ or $k_2$ is the trivial knot.

**Exercise 2.4** Show that the above construction of sum of two knots is well-defined, i.e., the definition does not depend upon the choice of the points $p_i$ on the knot $k_i$.

**Exercise 2.5** Show that $\pi_1(\mathbb{S}^3 - (k_1 + k_2))$ is isomorphic to the generalized free product (see definition 2.10) $\pi_1(\mathbb{S}^3 - k_1) *_{\mathbb{Z}} \pi_1(\mathbb{S}^3 - k_2)$ where the inclusions $\mathbb{Z} \to \pi_1(\mathbb{S}^3 - k_i)$ are given by sending a generator of the infinite cyclic group to the meridian of the knot complement.

The following theorem shows that genus of a knot is additive with respect to sum operation on knots.

**Theorem 2.6.** *Let $k_1$ and $k_2$ be two knots in the 3-sphere then $g(k_1 + k_2) = g(k_1) + g(k_2)$.*

*Proof.* Let two knots $k_1$ and $k_2$ in the 3-sphere which are separated by a 2-sphere, i.e., they are contained in two disjoint embedded 3-balls. Consider Seifert surfaces $S_1$ and $S_2$ of the knots $k_1$ and $k_2$ which are contained in the two disjoint balls (this can be very easily managed). Now choose a point on the boundary component of $S_1$ and $S_2$ and join them by line and take a strip ($S$) neighborhood of this line so that its interior does not intersect the Seifert surfaces and intersects the knots on small interval around the points we started with. Now consider the surface $S_1 \cup S_2 \cup S$. The boundary of this Surface is $k_1 + k_2$ (to match the orientation of the knots may be we have to take a twisted strip) and hence is a Seifert surface of $k_1 + k_2$.

Obviously genus of this new surface is sum of the genera of $S_1$ and $S_2$. Thus we have proved $g(k_1 + k_2) \leq g(k_1) + g(k_2)$.

Now we come to the other inequality.

Let $S$ be a Seifert surface of the knot $k_1 + k_2$ and choose a sphere $X$ which intersects the knot $k_1 + k_2$ at two points $p$ and $q$ as in the definition of the sum of knots, so that $X$ separates the knots $k_1$ and $k_2$ before taking sum. Using general position argument we can assume that $S$ intersects $X$ transversally and hence $S \cap X$ is a disjoint collection of simple closed curves and an arc $\beta$ joining $p$ and $q$. Each of these simple closed curves bounds discs on the $X$. Choose the simple closed curve $C$ and a bounding disc $D$ which does not contain $\beta$ and whose interior does not intersect $S$. This can be done by choosing the innermost disc bounded by components of $S \cap X$ which does not contain $\beta$. Now we do surgery along $D$. That is consider a tubular neighborhood $N$ of $D$ which intersect $S$ along an annulus. We replace the surface $S$ by cutting it along $C$ and attaching two disc, parallel to $D$, along the two copies of $C$. We get a new Seifert surface of the knot $k_1 + k_2$. If $C$ did not separate $S$ then the new surface has genus one less than genus of $S$ which is a contradiction to the minimality of the genus of $S$. So $C$ separates $S$ and hence one of the component, say $S'$, contains the knot $k_1 + k_2$. $S'$ is again a Seifert surface of $k_1 + k_2$ which intersects $X$ in fewer number of simple closed curve. We repeat this process till we find a Seifert surface which intersect $X$ only along $\beta$. The points $p$ and $q$ separates the knot into two arcs $\alpha_1$ and $\alpha_2$ and $\alpha_1 \cup \beta$ and $\alpha_2 \cup \beta$ are nothing but copies of $k_1$ and $k_2$ respectively. Thus $X$ separates $S$ into two surfaces $S_1$ and $S_2$ which are Seifert surfaces of $k_1$ and $k_2$ respectively. Hence $g(k_1) + g(k_2) \leq g(k_1 + k_2)$. □

Some immediate applications of the above theorem are the following:

**Corollary 2.7.** *Any knot can be written as a sum of finitely many prime knots.*

*Proof.* Use induction argument on the genus of the knot. □

In fact the above decomposition of a knot into prime factors is unique up to permutation of the factors. See [L1].

**Corollary 2.8.** *There are infinitely many inequivalent knots provided there is one with genus $\geq 1$.*

*Proof.* Start with a knot $k$ of genus 1 (for example take the $(2, 3)$ torus knot) and take $k_n = k + k + \cdots + k$ ($n$-times). For different $n$ the knots $k_n$ are different. □

Another fact about Seifert surface is the following theorem.

**Theorem 2.9.** *Let $S$ be a Seifert surface of $k$ of minimal genus. Then the map $\pi_1(S) \to \pi_1(\mathbb{S}^3 - k)$ induced by inclusion is injective.*

*Proof.* Suppose the contrary is true, i.e., the kernel of the homomorphism $\pi_1(S) \to \pi_1(\mathbb{S}^3 - k)$ is nontrivial. Choose a simple closed curve $C \subset S$ which is not null-homotopic on $S$ but contracts in the knot complement. By Loop theorem there is an embedded disc $D$ in the complement of the knot whose interior does not intersect $S$ but $\partial D = C$. We do surgery along $D$ and replace the surface $S$ by a new surface $S'$. To be more precise we remove a annulus neighborhood of $C$ from $S$ and attach two discs, parallel to $D$, along the two copies of $C$. If $C$ does not separate $S$ then a computation of Euler characteristic shows that genus of $S'$ is one less than genus of $S$. Which is a contradiction to the minimality of $S$. If $C$ separates $S$ then again, because $C$ does not bound a disc on $S$, the component of $S'$ which contains the knot has genus strictly less than $S$. Again a contradiction. This proves the theorem. $\square$

In the remaining of the section we construct an infinite cyclic covering of a knot complement using Seifert surface and we use the Theorem 2.9 to find the structure of the commutator subgroup of a knot group.

**Definition 2.10.** Let $G_1$ and $G_2$ be two groups and $H$ be another group with injective homomorphisms $f_i : H \to G_i$, $i = 1, 2$. The *generalized free product* of $G_1$ and $G_2$ with amalgamation along $H$ is by definition the free product $G_1 * G_2$ modulo the normal subgroup generated by the subset $\{f_1(h)(f_2(h))^{-1} \mid h \in H\} \subset G_1 * G_2$ and is denoted by $G_1 *_H G_2$. This group depends on the two embeddings $f_1$ and $f_2$.

**Exercise 2.11** Show that the obvious maps $G_i \to G_1 *_H G_2$ are injective and the maps $H \to G_1 *_H G_2$ induced by $f_i$ are also injective.

By Seifert-Van-Kampen theorem if $X_1$ and $X_2$ are two complexes with a common subcomplex $X$ so that the inclusion maps $X \to X_i$ induce injective homomorphism on the fundamental group level then $\pi_1(X_1 \cup_X X_2) \simeq \pi_1(X_1) *_{\pi_1(X)} \pi_1(X_2)$.

Now we come back to knot complement. Let $k$ be a nontrivial knot in the 3-sphere and $S$ be a Seifert surface of $k$. As the surface $S$ is orientable the normal bundle of $S$ in $M = \mathbb{S}^3 - k$ is trivial and hence $M - S$ has two boundary components and each components is homeomorphic to $S$. Consider countably many copies $M_i$, $i \in \mathbb{Z}$ of $M - S$ and assume $S_i^1$ and $S_i^2$ be two components of the boundary of $M_i$. Let $f_i^j : S \to \partial M_i$ for $j = 1, 2$ be the two embeddings. Let us denote $f_i^j(S)$ by $S_i^j$. Construct the manifold $\tilde{M} = \bigcup M_i$ where the boundary component $S_i^2$ of $M_i$ is glued to the boundary component $S_{i+1}^1$ of $M_{i+1}$ by the homeomorphism $f_{i+1}^1 \circ (f_i^2)^{-1}$. $\tilde{M}$ is a connected noncompact 3-manifold with an obvious map $f : \tilde{M} \to M$. Sending a point $x \in M_i$ to the corresponding point on the $(i+1)$ the copy $M_{i+1}$ of $M$ gives a homeomorphism $t : \tilde{M} \to \tilde{M}$ so that $f \circ t = f$.

**Exercise 2.12** Show that $f$ is a covering projection of an infinite cyclic covering with the group of covering transformation generated by the homeomorphism $t$.

Choose a simple path $\gamma$ joining $x \in S_i^1$ and $t(x)$ then $f(\gamma)$ is a simple closed curve in $M$ homologous to a meridian of the knot and hence generate the first homology of the knot complement. Thus we have shown that $f_*(\pi_1(\tilde{M}))$ is the commutator subgroup of the fundamental group of the knot complement.

Now assume that the Seifert surface $S$ is of minimal genus. Then by Theorem 2.9 both the maps $f_i^j$, for $j = 1, 2$ induce injective homomorphism on the fundamental group level.

In the following theorem we get the structure of the commutator subgroup of a knot group.

**Theorem 2.13.** *Let $G$ be a knot group of a knot of genus $g$. Then one of the following holds:*

(1) *if $G'$ is finitely generated then $G'$ is isomorphic to a non-abelian free group on $2g$ generators.*
(2) *if $G'$ is not finitely generated then*

$$G' \simeq \ldots G_{-i} *_{H_{-i}} G_{-i+1} *_{H_{-i+1}} \ldots G_0 *_{H_0} G_1 *_{H_1} \ldots G_{i-1} *_{H_{i-1}} G_i \ldots$$

*where $H_n$ is isomorphic to a non-abelian free group of rank $2g$ and each $H_n$ is a proper subgroup of both $G_n$ and $G_{n+1}$ for $n \in \mathbb{Z}$.*

*Proof.* We prove only the case when $G'$ is finitely generated. Proof of case (2) can be found in [BZ].

Note from the above theorem that the fundamental group of the infinite cyclic covering of the knot complement is the commutator subgroup of the knot group. Also from the construction of the infinite cyclic covering and by the previous discussion on generalized free product it easily follows that $G'$ is isomorphic to the following iterated generalized free product:

$$G' \simeq \ldots G_{-i} *_{H_{-i}} G_{-i+1} *_{H_{-i+1}} \ldots G_0 *_{H_0} G_1 *_{H_1} \ldots G_{i-1} *_{H_{i-1}} G_i \ldots$$

where $G_n$ is isomorphic to the fundamental group of $\mathbb{S}^3 - (k \cup S)$ and $H_n \simeq \pi_1(S)$ for each $n \in \mathbb{Z}$.

Let $\mathcal{G}_i = G_{-i} *_{H_{-i}} G_{-i+1} *_{H_{-i+1}} \ldots G_0 *_{H_0} G_1 *_{H_1} \ldots G_{i-1} *_{H_{i-1}} G_i$. Now as $G'$ is finitely generated for some $i$ we have $\mathcal{G}_j = \mathcal{G}_{j+1}$ for all $j \geq i$. This shows that the injections $H_i \to G_{i+1}$ and $H_{-i} \to G_{-i}$ are isomorphism. From the construction of the infinite cyclic covering it follows that for each pair of integer $(i, j)$ there are commutative diagrams:

$$\begin{array}{ccc} H_i & \longrightarrow & H_j \\ \downarrow & & \downarrow \\ G_i & \longrightarrow & G_j \end{array} \qquad \begin{array}{ccc} H_i & \longrightarrow & H_j \\ \downarrow & & \downarrow \\ G_{i+1} & \longrightarrow & G_{j+1} \end{array}$$

Here the vertical maps are injective and the horizontal maps are isomorphism.

Hence the injective homomorphisms $H_n \to G_n$ and $H_n \to G_{n+1}$ are isomorphism for each $n \in \mathbb{Z}$. Thus $G'$ is isomorphic to $\pi_1(S)$ which is a non-abelian free group on $2g$ generators. $\square$

## 3. Dehn surgery and closed 3-manifolds

Let $k$ be a knot in the 3-sphere. By a *Dehn surgery* along $k$ we mean that remove a tubular neighborhood of $k$ (which is a solid torus) and glue it back by a different diffeomorphism along the torus boundary. If we start with a link $l$ then we can perform Dehn surgery along each component of the link by choosing small tubular neighborhood of each component of the link so that no two of these solid tori intersect.

In this section we prove the important fact that any closed 3-manifold can be obtained by Dehn surgery along a link in the 3-sphere. We follow the proof of Lickorish in [L]. An interesting consequence is that any closed 3-manifold is cobordant to the 3-sphere. By definition two manifolds are called *cobordant* if there is another manifold with two boundary components homeomorphic to the two manifolds we started with.

Let $S$ be a closed oriented piecewise linear 2-manifold. Here recall that any such manifold can be obtained from the standard sphere by attaching finitely many handles as in Figure 5.

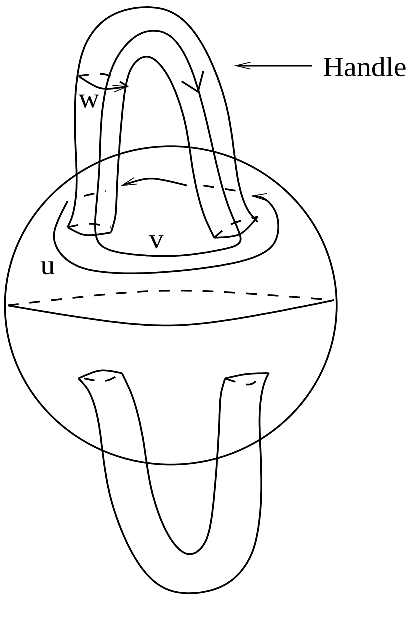



Figure 5

Let $N_S$ be the set of all homeomorphism of the surface $S$ which are isotopic to the identity. Two homeomorphism $f_0$ and $f_1$ of $S$ are called *isotopic* if there is a homeomorphism $F : S \times I \to S \times I$ such that $F(S \times t) = S \times t$ for each $t$ and

$F(s,0) = (f_0(s), 0)$ and $F(s,1) = (f_1(s), 1)$. Let $G_S$ be the group of all orientation preserving homeomorphism of $S$ under composition as the product operation.

**Exercise 3.1** Show that isotopy is an equivalence relation in $G_S$.

**Exercise 3.2** Show that $N_S$ is a normal subgroup of $G_S$ and $L_S = G_S/N_S$ is the isotopy classes of all orientation preserving homeomorphism of $S$. $L_S$ is called the *mapping class group* of the surface $S$.

**Exercise 3.3** Show that if $S = \mathbb{S}^1 \times \mathbb{S}^1$ then $L_S$ is isomorphic to $SL(2, \mathbb{Z})$, the group of all $2 \times 2$ determinant 1 matrices with integer entries.

**Exercise 3.4** In the following picture (Figure 6) show that there is a homeomorphism $n \in N_S$ which sends the first picture to the second one, i.e., $n$ sends $l$ to $m$.

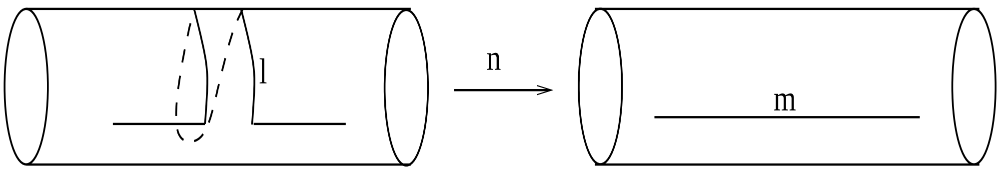



Figure 6

**Exercise 3.5** Given two paths $l$ and $m$ on the surface $S$ show that there is an $n \in N_S$ such that $n(l)$ intersects $m$ transversally and at finitely many points. This fact will be used throughout this section without mentioning. Also prove that physically pushing a curve on the surface (an example is Exercise 3.4) without making any self intersection is nothing but an application of a homeomorphism which is isotopic to the identity. This fact will be used frequently (sometimes with(out) mentioning) during this section.

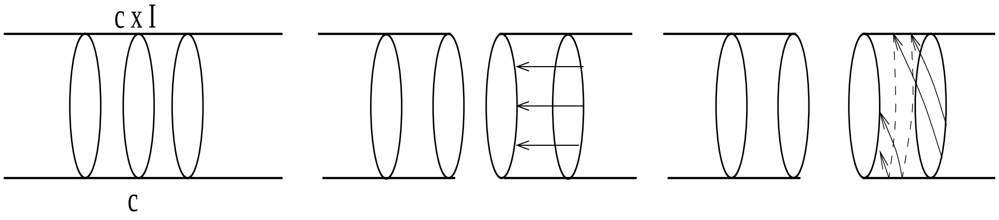



Figure 7

Let $c \subset S$ be a simple closed curve. Consider a tubular neighborhood $c \times I$ of $c$ in $S$. Cutting $S$ along $c$ we get a 2-manifold as in Figure 7. Twist one of the end by $2\pi$ and glue back again. More precisely $d_C$ is a homeomorphism given by: Let $A \subset S$ be an annular neighborhood of $C$ in $S$. Parametrise $A$ by $\mathbb{S}^1 \times [0,1]$. Then $d_C|_{S-A} = id$ and $d_C(e^{i\theta}, t) = (e^{i(\theta - 2\pi t)}, t)$ for each $(e^{i\theta}, t) \in \mathbb{S}^1 \times [0,1]$. This homeomorphism is popularly called a *Dehn twist* and is denoted by $d_c$. Two paths $p_1$ and $p_2$ contained in $S$ are said to be *equivalent* (we write $p_1 \sim_d p_2$) if there exists

$n \in N_S$ and Dehn twists $h_1, \cdots, h_m$ such that $nh_1h_2\cdots h_m p_1 = p_2$.
**Exercise 3.6** Check $\sim_d$ is an equivalence relation. Hint: Inverse of a Dehn twist is also a Dehn twist and $N_S$ is normal in $G_S$.

As in the paper ([L]) we first show that the mapping class group of any closed oriented surface is generated by Dehn twists. Then we use this important fact to prove the theorem.

**Theorem 3.7.** *Any orientation preserving piecewise linear homeomorphism $h$ of $S$ is isotopic to a product of Dehn twists.*

*Proof.* To prove the theorem we need the following three lemmas:

**Lemma 3.8.** *If $p_1$ and $p_2$ are simple closed curve on $S$ with only one point of intersection then $p_1 \sim_d p_2$.*

*Proof.* Let $x = p_1 \cap p_2$. Let $d_{p_2}$ be a Dehn twist along the curve $p_2$. Then $d_{p_2}p_1$ is a copy of $p_1$ broken at $x$ and a copy of $p_2$ inserted at the break. See Figure 8. Now let $p_3$ be a simple closed curve in a neighborhood of $p_1$ and parallel to it as in the picture. Consider the Dehn twist $d_{p_3}$ and apply over $d_{p_2}p_1$ what we get is shown in the third picture of Figure 8. Now apply the Exercise 3.4 to get an $n \in N_S$ which sends $d_{p_3}d_{p_2}p_1$ to $p_2$, i.e., $nd_{p_3}d_{p_2}p_1 = p_2$. □

**Corollary 3.9.** *If $p_1, p_2, \cdots, p_r$ are simple closed curves in $S$ such that for each $i$ $p_i$ intersects $p_{i+1}$ only at one point then $p_1 \sim_d p_r$.*

*Proof.* Exercise. □

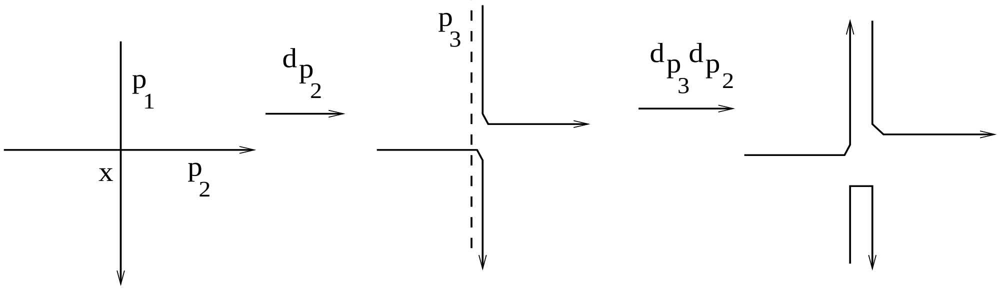



Figure 8

**Lemma 3.10.** *Let $p$ and $q$ be simple closed curves in $S$ and $N$ be any neighborhood of $q$ in $S$. Then there exists a path $p_*$ such that $p \sim_d p_*$, $p_* \cap (S - N) \subset p \cap (S - N)$, and $p_*$ either does not meet $q$ or meets $q$ precisely twice with zero algebraic intersection.*

*Remark.* Let $S$ be an oriented surface and $l$ and $m$ be two oriented (induced from the orientation of $S$) curves and they have transverse intersection. At a point of intersection choose a tangent vector $v_l$ to $l$ first then another one $v_m$ to $m$. If the pair $(v_l, v_m)$ induce the orientation of the surface at this point of intersection then associate +1 to this point otherwise −1. Then add up these numbers for all the points of intersections. This number is called the algebraic intersection number of the two curves $l$ and $m$.

*Proof of Lemma 3.10.* We prove the lemma by induction on the number of points of intersections of the two curves $p$ and $q$, say $r$. If $r = 1$ then by Lemma 3.8 $p \sim_d q$. Choose $n \in N_S$ so that $nq$ lies inside $N$ but do not intersect $q$. We take $p_*$ to be $nq$. If $r = 2$ and $p$ is oriented so that $p$ has mutually different direction at the points of intersection with respect to an orientation of $q$ then the lemma is trivial.

Suppose the lemma is true for all paths $p$ and $q$ with $r \leq k - 1$. So we start with two curves $p$ and $q$ with $r = k$. We orient $p$ and $q$. We divide the proof in two cases:

**Case 1** Let $x$ and $y$ be two consecutive points of intersection of $p$ and $q$ so that at $x$ and $y$ the curves are oriented in the same direction as in Figure 9.

Let $c$ be a simple closed curve starting at $A$ a point in $N$ and near $x$ and goes parallel along $p$ and come back to a point $B$ near $y$ and inside $N$ then connected to $A$ inside $N$ and intersecting $q$ and $p$ at one point each. (Here note that the segment $BA$ intersects $p$ because $S$ is orientable (so there is no Möbius band) and at the points $x$ and $y$ the curves are oriented in the same way). Now by Lemma 3.8 $p \sim_d c$ and $c$ and $q$ have fewer than $k$ points of intersection. It is easy to see that there is an $n \in N_S$ with the property $nc \cap (S - N) \subset p \cap (S - N)$ and that $nc \cap q$ has fewer than $k$ points. By the induction hypothesis there is a $p_*$ such that $p_* \sim_d nc \sim_d p$ and $p_* \cap (S - N) \subset nc \cap (S - N) \subset p \cap (S - N)$.

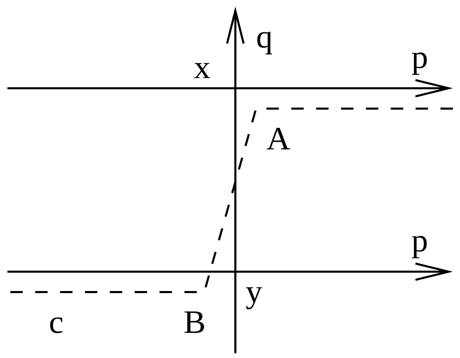



Figure 9

**Case 2** Suppose there are three consecutive points $x, y, z$ of intersection of $p$ and $q$ so that the orientation of the pair $(p, q)$ are alternating at these points. See Figure 10:

One of the segments $xz$ or $zx$ of $p$ contains $y$ and the other one does not. Assume $zx$ does not contain $y$. Let $C$ be the simple closed curve starting at $B$ in a neighborhood of $z$ (and inside $N$) and continue parallel in the direction of $p$ in a neighborhood of $p$ and reach $A$ a point near $x$ (and inside $N$) and then joined to

$B$ so that $AB$ lies inside $N$. $AB$ meets $p$ twice and $q$ once. Apply the Dehn twist $d_C$ on $p$ what we get is the first of the Figure 11.

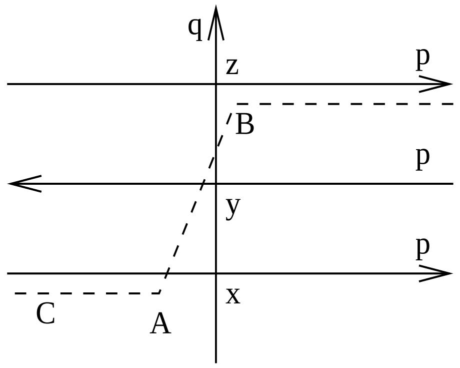



Figure 10

Choose an $n \in N_S$ so that $nd_Cp$ is as in the second picture in Figure 11, also one can choose $n$ such a way that $nd_Cp \cap (S - N) \subset p \cap (S - N)$ but $nd_Cp$ intersects $q$ twice less than $p$ intersects $q$. Now $p \sim_d nd_Cp$ and by the induction hypothesis $nd_Cp \sim_d p_*$ where $p_*$ is a path with the required property. This proves the lemma. $\square$

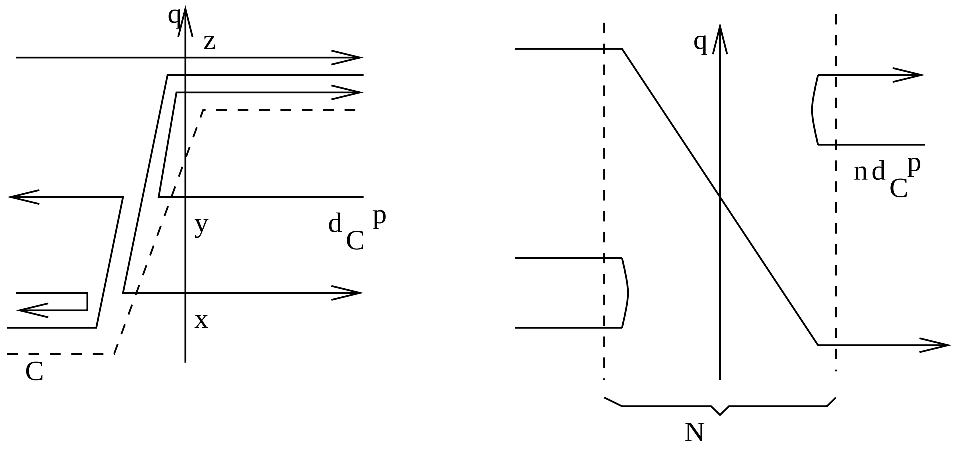



Figure 11

**Corollary 3.11.** *If $p, q_1, \cdots, q_r$ are simple closed curves on $S$ and $q_i \cap q_j = \emptyset$ for $i \neq j$, then there exist a path $p_*$ such that either $p_*$ does not meet $q_i$ or meets $q_i$ at two points with zero algebraic intersection and $p \sim_d p_*$.*

*Proof.* Choose neighborhoods $N_i$ of $q_i$ so that $N_i \cap N_j = \emptyset$ for $i \neq j$. Now apply Lemma 3.10 to $p$, $q_1$ and $N_1$ to get a path $p_1$ such that $p_1$ meets $q_1$ at most twice and $p_1 \sim_d p$. Now apply the lemma again to $p_1$ $q_2$ and $N_2$ to obtain $p_2$ so that $p_2$ meets $q_2$ at most twice and $p_2 \sim_d p_1 \sim_d p$. As $p_2 \cap (S - N_2) \subset p_1 \cap (S - N_2)$ and $q_1 \subset S - N_2$, $p_2$ meets $q_1$ at most twice. This process terminates in finding a $p_n$ such that $p \sim_d p_n$ and $p_n$ meets each $q_i$ at most twice. We choose $p_*$ to be $p_n$

unless $p_n$ meets one of the $q_j$ once, in that case we apply Lemma 3.8 and the proof of Lemma 3.10 to find a $p_*$ with the property $p_n \sim_d q_j$ and $q_j \sim_d p_*$ and $p_*$ meets no $q_j$. □

Now consider the typical picture of a surface represented as a sphere with handles as in Figure 5. We say that a path $p$ *does not meet* the handle if it does not intersect $w$ and a path not meeting a handle *goes through the handle* if it intersects $v$ an odd number of times.

**Lemma 3.12.** *If $p$ is a simple closed curve on $S$ then $p \sim_d p_*$, where $p_*$ does not meet any of the handles of $S$.*

*Proof.* Consider the simple closed curves $q_1, q_2, \cdots q_r$ on the surface such that they are either of the type $w$ or $u$ as in the picture. Corollary 3.11 says that $p \sim_d p_1$ where $p_1$ either does not meet $q_i$ or meets $q_i$ twice with zero algebraic intersection.

Now staring at the picture we see that if $p_1$ meets the handle then it cuts $u$ twist around the handle cutting $w$ twice only and then cuts $u$ and is away from the handle. It is easy to find an $n \in N_S$ so that $n$ is identity outside the handle in question and $np_1$ does not meet $w$. Such an $n$ can be found by noting that $p_1$ meets $w$ at two points and in opposite direction. Suppose $A$ and $B$ be the two points of intersection of $p_1$ and $w$ then consider the two simple closed curves $l$ and $m$ which are obtained by traveling from $A$ and following $p_1$ till we reach $B$ then walk along either of the two paths joining $B$ and $A$ by $w$. Note that the part $AB$ of $p_1$ wholly lie on a cylinder part of the handle (because of two points of intersection with zero algebraic intersection). All these observations are used to find a disc bounding either $l$ or $m$. Now it is easy to find $n$ with the required property. □

Now we are ready to prove Theorem 3.7:

Let $p_1, p_2, \cdots, p_r$ be paths, one around each handle of $S$, each $p_i$ going around the handle as does $w$.

*Induction hypothesis $\chi(t)$:* there exists a sequence of Dehn twists with product $s$ and an $n \in N_S$ such that $nshp_i = p_i$ for $i \leq t \leq r$.

$\chi(0)$ is obviously trivial. So assume that $\chi(t)$ is true. We proceed to check that $\chi(t+1)$ is true.

By Lemma 3.12, $nshp_{t+1} \sim_d p_*$ where $p_*$ meets none of the handles of $S$. Thus there is a product of Dehn twists $s_1$ and an $n_1 \in N_S$ so that $n_1 s_1 nshp_{t+1} = p_*$ and also $n_1$ and $s_1$ can be chosen so that $n_1 s_1 nshp_i = p_i$ for $i \leq t$. Now since $p_{t+1}$ is not homologous to zero in $S$, neither is $p_*$. As $p_*$ does not meet any of the handle of $S$ it lies on the sphere part of $S$. $p_*$ divides the sphere in two components, but as $p_*$ is not homologous to zero it does not bound on $S$ and hence $p_*$ does not separate the surface. So there is a handle which connects the two components. Hence $p_*$ must go through some of the handles of $S$. Since $p_{t+1}$ is not homologous to any of the $p_i$ (because the $p_1, \cdots, p_r$ are some of the generators of the first homology of the surface), neither is $p_*$ for $i \leq t$, so $p_*$ must go through a handle $H$, not containing a $p_i$ for $i \leq t$. Now consider the sequence of paths $p_*, l, m, p_{t+1}$ as in Figure 12.

Use Corollary 3.9 to conclude that $p_* \sim_d p_{t+1}$. Hence there is a product of Dehn twist $s_2$ and an $n_2 \in N_S$ such that $n_2 s_2 p_* = p_{t+1}$ and again we can choose $n_2$ and $s_2$ so that $n_2 s_2 p_i = p_i$ for $i \leq t$. Thus $n_2 s_2 n_1 s_1 n s h p_i = p_i$ for $i \leq t+1$ and then using normality of $N_S$ in $G_S$ we have $n_2 s_2 n_1 s_1 n s = n_3 s_2 s_2 s$ for some $n_3 \in N_S$.

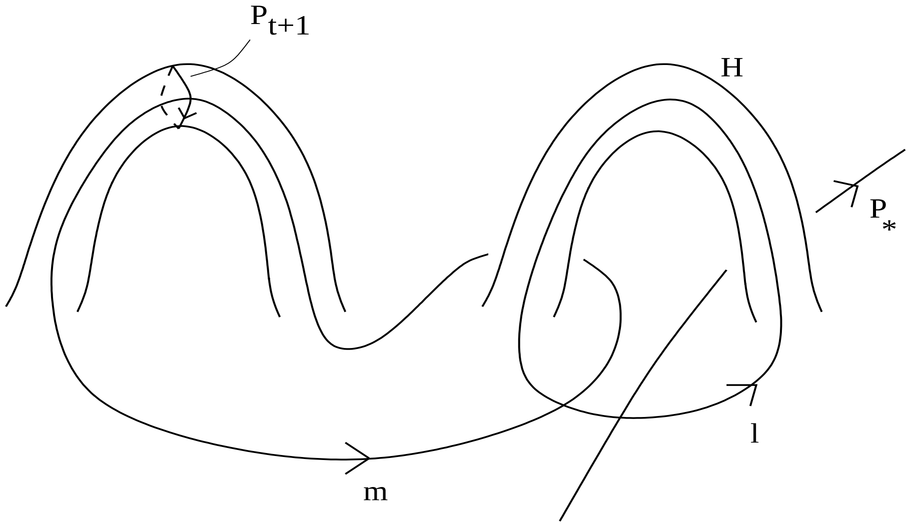



Figure 12

Thus we have shown by induction that $\chi(r)$ is true, i.e., there are $n$ and $s$ so that $nsh$ fixes each of $p_i$.

Now cutting $S$ along each $p_i$ we see that, since $nsh$ is orientation preserving, it is sufficient to prove the following statement $\tau(r)$:

$\tau(r)$: Let $D$ be a 2-disc after removing the interior of $r$ many disjoint collection of small discs from the interior of the disc. Let $f$ be a piecewise linear homeomorphism of $D$ which is identity when restricted to the boundary. Then there is a sequence of Dehn twist with product $s$ and an $n \in N_D$ which leave $D$ fixed on the boundary and such that $nsf$ is the identity on $D$.

We prove the statement by induction on $r$. For $r = 0$ any orientation preserving piecewise linear homeomorphism of the disc which is identity on the boundary is isotopic to the identity and it is identity on the boundary throughout the isotopy, see [Gu]. So assume that $\tau(r)$ is true and we proceed to show that $\tau(r+1)$ is true. So let $D$ be a disc with $r+1$ small disc removed from its interior and $f$ is a homeomorphism which is identity on the boundary of $D$. Let $p$ be a simple curve (without self intersection) joining a point $A$ on one boundary component of $D$ to a point $B$ on another boundary component. Isotope $f$ so that it is identity near a small neighborhood of both $A$ and $B$. Let $A_1, A_2, \cdots, A_{\alpha-1}, A_\alpha$ be points on $p$ so that $f$ is identity on the segments $AA_1$ and $A_r B$ and other $A_i$'s are points of intersections of $p$ and $fp$ in the order as shown in Figure 13.

Orient $p$ and choose the induced orientation on $fp$.

**Case 1** Suppose $fp$ is oriented in the same direction as $p$ at the points $A_1$ and $A_2$. Let $C_1$ be a closed curve as shown in the picture above starting near $A_1$ proceeding parallel to $fp$ to a point near $A_2$ and then back to its starting point in a neighborhood of $p$. Also $C_1$ cuts $fp$ only once. Now applying $d_{C_1}$ on $fp$ we get the second picture in Figure 13. Choose an element $n \in N_D$ such that $nd_{C_1}f$ is the identity on the segment $AA_2$ of $p$, and $nd_{C_1}fp$ intersects $p$ fewer times than $fp$.

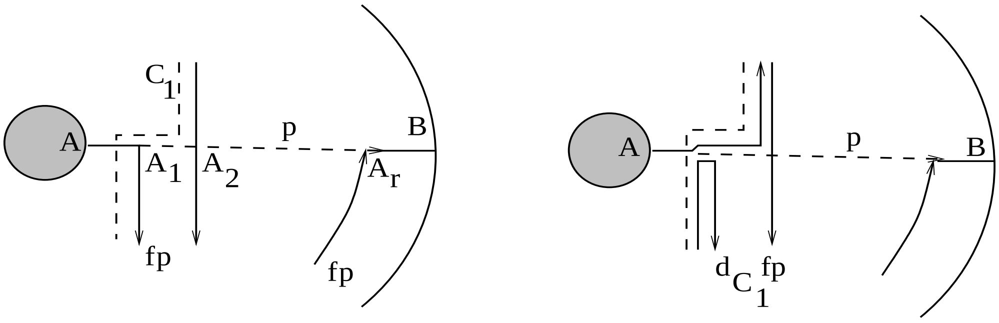



Figure 13

**Case 2** In this case assume that $fp$ is oriented in different direction at $A_1$ and $A_2$ with respect to $p$. Let $C_2$ be the path as shown in Figure 14. Then $d_{C_2}fp$ is as shown on the second picture in Figure 14.

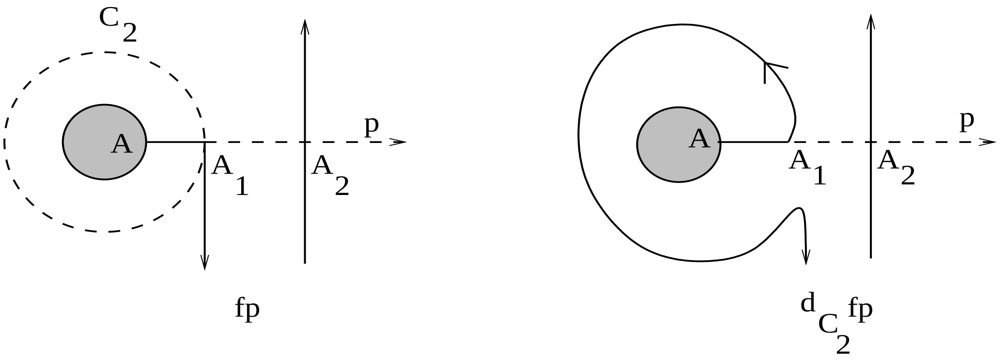



Figure 14

Here $d_{C_2}fp$ has the same direction at $A_1$ and $A_2$ with respect to the orientation of $p$. So we have reduced this case to Case 1.

Thus by a sequence of processes as Cases 1 and 2 we can find a product $s$ of Dehn twists and an $n \in N_D$ so that $nsf$ is the identity on the boundary of $D$ and on $p$. We can choose this $n$ to be isotopic to identity and is identity on the boundary of $D$ during the isotopy. Now cutting $D$ along $p$ we get a disc from which only $n$ discs have been removed. Hence the induction process proves the theorem. $\square$

**Theorem3.13.** *Any closed connected orientable* 3*-manifold* $M$ *is piecewise linearly homeomorphic to* $\mathbb{S}^3$ *from which a finite set of disjoint solid tori have been removed and sewn back in a different way.*

Before we give the proof of the theorem we recall some basic notions. A *handle body* of genus $g$ is an orientable 3-manifold that is a 3-ball with $g$ 1-handles attached. Another way of defining it is that it is an orientable 3-manifold with finitely many properly embedded 2-discs so that if we cut the manifold along these discs then we get a 3-ball. By proper embedding we mean an embedding which send boundary to boundary. Also it is known that the genus (of the boundary) determines a handle body up to homeomorphism. A *Heegard splitting* of a closed oriented connected 3-manifold $M$ is a pair of handle bodies $H_1$ and $H_2$ embedded in $M$ so that $H_1 \cup H_2 = M$ and $H_1 \cap H_2 = \partial H_1 = \partial H_2$. Note that $H_1$ and $H_2$ have the same genus, namely the genus of their boundary surface. In such a case it is called a Heegard splitting of genus $g$. Recall that given any $g$ there is a Heegard splitting of $\mathbb{S}^3$ of genus $g$.

We state the following basic fact whose proof can be found in any book on 3-manifolds ([H], [L1]).

**Lemma.** *Any closed connected orientable* 3*-manifold has a Heegard splitting.*

*Proof of theorem 3.13.* Consider a Heegard splitting of the manifold $M$, i.e., $M = H_1 \cup H_2$ and $H_1 \cap H_2 = \partial H_1 = \partial H_2$. And $\partial H_1$ and $\partial H_2$ are identified by a piecewise linear homeomorphism, say $f$. Also consider a homeomorphism $i : \partial H_1 \to \partial H_2$ so that identifying $H_1$ and $H_2$ by $i$ along their boundary gives $\mathbb{S}^3$. We choose $i$ so that $f^{-1}i$ is orientation preserving. By Theorem 3.7 there is a product $g$ of Dehn twists of $\partial H_1$ and an element $n \in N_{\partial H_1}$ such that $ng = f^{-1}i$; i.e., $fng = i$. To give the desired homeomorphism we need to find an extension of a Dehn twist on $\partial H_1$ to a modified $H_1$. The following exercise shows that this is not possible in general if we do not modify $H_1$:

**Exercise 3.14** Let $f : \partial H \to \partial H$ be a homeomorphism which extends to a homeomorphism of $H$ onto $H$, here $H$ is a handle body. Then $H \cup_f H$ is homeomorphic to $\mathbb{S}^3$.

In fact what we can prove that is the following:

**Claim** Given a Dehn twist of $\partial H_1$ there is a solid torus $T$ embedded in the interior of $H_1$ so that the Dehn twist extends to a homeomorphism of $\overline{H_1 - T}$ and hence the Dehn twist extends to $\overline{H_1 - T} \cup_\rho T$, where $\rho$ is a suitable diffeomorphism of $\partial(\overline{H_1 - T})$ to $\partial T$.

*Proof of claim.* Let $C$ be a simple closed curve on $\partial H_1$ and $d_C$ be the Dehn twist along it. As there is a neighborhood of $\partial H_1$ in $H_1$ homeomorphic to $\partial H_1 \times I$ we embed $(\mathbb{S}^1 \bigvee I) \times \mathbb{S}^1$ in $H_1$ so that $0 \times \mathbb{S}^1$ goes to $C$ and $(\mathbb{S}^1 \bigvee (0, 1]) \times \mathbb{S}^1$ goes in the interior of $H_1$. Here $\mathbb{S}^1 \bigvee I$ is obtained by identifying a point of $\mathbb{S}^1$ with $1 \in I$.

We cut $H_1$ along the embedded $(\mathbb{S}^1 \bigvee I) \times \mathbb{S}^1$ and twist it by $2\pi$ along the direction of the second coordinate, then glue back along $I \times \mathbb{S}^1$ only. What we have done is that we have cut out a suitable solid torus (say T) from the interior of $H_1$. Now obviously $d_C$ extends to a homeomorphism of $\overline{H_1 - T}$. $\square$

Now if we consider a product of Dehn twists then we can perform the same process as above so that the embedded solid tori are disjoint by choosing thiner and thiner solid tori. So a product of Dehn twists on $\partial H_1$ extends to the whole handlebody after removing a finitely many disjoint embedded solid tori from the interior of $H_1$.

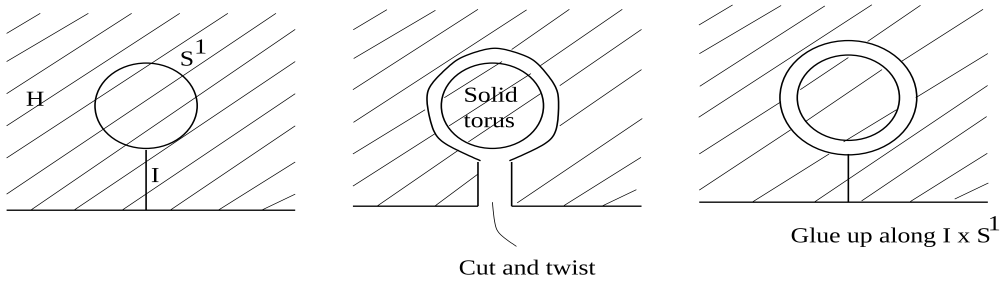



Figure 15

Since $\partial H_1$ has a neighborhood of the form $\partial H_1 \times I$ in $H_1$ and $n^{-1}$ is isotopic to the identity there is a homeomorphism $h$ of $H_1$ whose restriction to $\partial H_1$ is $n^{-1}$. Note that $g^{-1}$ is also a product of Dehn twists. Using the above procedure we remove finitely many disjoint solid tori from the interior of $hH_1$ to extend $g^{-1}$ from the boundary of $hH_1$ to the complement (say $K$) of these solid tori. Now consider the induced homeomorphism of the this extension of $g^{-1}$ to $K$ when restricted to the tori boundary components of $K$. We glue back some solid tori to $K$ along these tori boundary component by this restricted homeomorphism so that $g^{-1}$ extends to the whole of this new manifold, say $H_1'$. But $\partial H_1' = \partial H_1$ Now glue $H_1'$ to $H_2$ by $f$ to get back our original $M$. Notice that we have identified $x \in \partial H_1$ to $fng(x) = i(x)$. Recall now that $H_1 \cup_i H_2 = \mathbb{S}^3$. Thus we have proved that $M$ is obtained from $\mathbb{S}^3$ by removing some solid tori and gluing them back by (possibly) different homeomorphism. The core circles of these solid tori is our desired link. This prove the theorem. $\square$

*Remark 3.15.* Note that the choice of the homeomorphism $g$ (which is a product of Dehn twists) is not unique as there are relations in the mapping class group.

The following exercise is used in the proof of the above theorem.

**Exercise 3.16** Let $M_1$ and $M_2$ be two 3-manifolds with homeomorphic boundary. Let $f_1, f_2 : \partial M_1 \to \partial M_2$ be two isotopic homeomorphism. Show that $M_1 \cup_{f_1} M_2$ and $M_1 \cup_{f_2} M_2$ are homeomorphic.

**Exercise (difficult!!) 3.17** Show that in the above theorem we can choose the solid tori (which we removed and glued back) in such a way that their cores form an irreducible link (see Exercise 1.3) in the 3-sphere.

**Problem 3.18** Start with an irreducible link in the 3-sphere. Find out which Dehn surgery along the link produces aspherical 3-manifold. Recall that the link complement is aspherical. Obviously not all Dehn surgery produces aspherical 3-manifold. For example the trivial surgery (i.e., remove the solid tori neighborhood of the link and glue back in the same way) gives the 3-sphere itself which is not aspherical.

## 4. Jones polynomial of links and knots

The prerequisite for this section is Laurent polynomials and little bit of intuition.

A *link diagram* is a regular projection of a link in the 3-sphere on a plane. We recall the three types of Reidemeister moves of a link diagram. Move of type I sends the diagram to the diagram . Type II sends to and type III Reidemeister move sends the diagram to .

Let $\mathbb{Z}[A^{-1}, A]$ be the ring of Laurent polynomials in the indeterminate $A$.

The *Kauffman bracket* $< . >$ sends any diagram $D$ to an element in $\mathbb{Z}[A^{-1}, A]$ and is characterized by the following properties:

(1) $< \bigcirc >= 1$
(2) $< D \cup \bigcirc >= (-A^{-2} - A^2) < D >$
(3) $< \times >= A < )( > + A^{-1} < \asymp >$.

Here $\bigcirc$ is the diagram of the unknot and $D \cup \bigcirc$ is the disjoint union of the diagram $D$ and an unknot without an linking between them. The third formula relates three link diagram which differ only at one point as indicated. If we change a link diagram by an orientation preserving homeomorphism of the plane then the bracket polynomial does not change. We now see the effect of the Reidemeister moves on the bracket polynomial.

**Lemma 4.1.** *If a diagram is changed by a type I Reidemeister move then the bracket polynomial changes in the following way:*

$$< \ >= -A^3 < \ >, < \ >= -A^{-3} < \ >$$

*Proof.*

$$< \ >= A < \ > + < \ >$$
$$= (A(-A^{-2} - A^2) + A^{-1}) < \ > .$$

The second equation follows similarly. $\square$

Note that in (3) if we change the left side diagram by the otherway round crossing then on the right side we have to change $A$ by $A^{-1}$ and $A^{-1}$by $A$. In fact this is the effect by an rotation by $\pi/2$ angle of the plane. The above is used to find the bracket polynomial of knots and links. We give here a simple example:

$$< \quad > = A < \quad > + A^{-1} < \quad >$$

$$= (-A^4 - A^{-4}).$$

and

$$< \quad > = A < \quad > + A^{-1} < \quad >$$

$$= A(-A^4 - A^{-4}) + A^{-7} = A^{-7} - A^{-3} - A^5.$$

**Lemma 4.2.** *The bracket polynomial* $< D >$ *of a diagram* $D$ *does not change under the Reidemeister moves of type II and III. That is:*

$$(i) < \quad > = < \quad >, (ii) < \quad > = < \quad > .$$

*Proof.* $(i)$

$$< \quad > = A < \quad > + A^{-1} < \quad >$$

Now applying Lemma 4.1 and (3) we get:

$$-A^{-2} < \quad > + < \quad > + A^{-2} < \quad > .$$

$(ii)$

$$< \quad > = A < \quad > + A^{-1} < \quad >$$

Now apply $(i)$ twice on the last line to get the following:

$$= A < \quad > + A^{-1} < \quad >$$

$$= < \quad > .$$

$\square$

The *writhe w(D)* of a diagram of an oriented link is the sum of the signs of the crossings of $D$. Each crossing is give the sign $+1$ or $-1$ as in Figure 16.

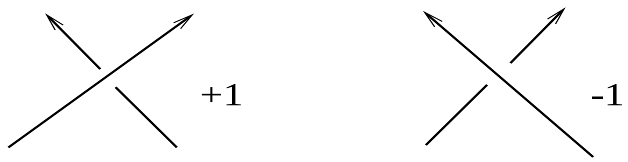



**Figure 16**

The writhe of a link diagram does not change under type II and III Reidemeister move but it changes by $+1$ or $-1$ under the type I move. The similarity of the effect of the Reidemeister moves on the bracket polynomial and writhe of a link diagram gives the following theorem:

**Theorem 4.3.** *Let $D$ be a diagram of an oriented link $L$. Then the expression $(-A)^{-3w(D)} < D >$ is an invariant of the link $L$.*

*Proof.* As any two diagrams of two equivalent links are equivalent under Reidemeister moves the theorem follows from Lemma 4.1, 4.2 and from the fact that the writhe changes by $+1$ or $-1$ under Reidemeister move of type I. $\square$

Now we are in a position to define the Jones polynomial of an oriented link.

**Definition 4.4.** The *Jones Polynomial* of an oriented link $L$ is the Laurent polynomial in $t^{1/2}$, with integer coefficients, defined by

$$V(L) = ((-A)^{-3w(D)} < D >)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}],$$

where $D$ is any oriented diagram for $L$.

**Exercise 4.5** From the definition it is not clear if $V(L)$ is a Laurent polynomial on the indeterminate $t^{1/2}$. Show that in fact this is the case. Also show that for links with odd number of components $V(L)$ is in fact a Laurent polynomial on $t$. Also check that $V(k) = 1$ when $k$ is the unknot.

Obviously if we change the orientation of each component of the link then the sign of the crossings do not change and hence the Jones polynomial also does not change. In particular for knots the Jones polynomial is independent of the orientation of the knot. But if we change the orientation of only one component, say $K$, of the link then the Jones polynomial changes in the following way $V(L^*) = t^{-3lk(K,L-K)}V(L)$, where $L^*$ is the link with the new orientation and $lk(K, L-K) = \Sigma_{L_i \subset L-K} lk(K, L_i)$. Here $lk(K_1, K_2)$ is half of the sum of the signs of crossing (one strand of $K_1$ and the other strand of $K_2$) in a diagram of the two component link $K_1 \cup K_2$.

**Problem 4.6** Investigate if there is a nontrivial knot $k$ such that $V(k) = 1$.

Now we prove the following proposition which characterize Jones polynomial.

**Proposition 4.7.** *The Jones polynomial invariant is a function*

$$V : \{\textit{Oriented links in } \mathbb{S}^3\} \longrightarrow \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

*such that*

(1) $V(\textit{unknot}) = 1$
(2) *whenever three oriented links $L_+$, $L_-$ and $L_0$ are the same except in the neighborhood of a point where they are as shown in Figure 17, then*

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0.$$

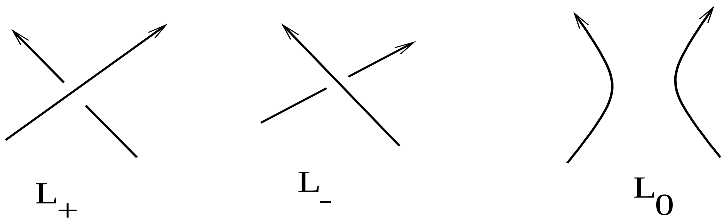



Figure 17

*Proof.* At first note that $w(L_+) - 1 = w(L_0) = w(L_-) + 1$. Now consider the two formulas below:

$$< \times >= A < )( > + A^{-1} < \asymp >,$$

$$< \times >= A^{-1} < )( > + A < \asymp > .$$

Multiplying the first equation by $A$ and the second by $A^{-1}$, and subtracting gives

$$A < \times > - A^{-1} < \times >= (A^2 - A^{-2}) < )( > .$$

Now by a little bit of calculation and noting the above remark on the writhe we get the following:

$$-A^4 V(L_+) + A^{-4} V(L_-) = (A^2 - A^{-2}) V(L_0).$$

Substituting $t^{1/2} = A^{-2}$ proves the proposition. $\square$

**Exercise 4.8** Show that if $L'$ is a link by adjoining an additional unknotted and unlinked knot with $L$ then $V(L') = (-t^{-1/2} - t^{1/2}) V(L)$.
**Exercise 4.9** Show that if $K_1$ and $K_2$ are two disjoint knots in $\mathbb{S}^3$ which are also unlinked then $V(K_1 + K_2) = V(K_1) V(K_2)$.

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