NOTES ON SEIFERT FIBERED SPACE

Sayed K. Roushon

Abstract. These are the notes of lectures given at the Allahabad low dimensional topology instructional conference on Seifert fibered space.

0. Introduction

The most basic class of 3-manifolds are Seifert fibered manifolds. Conjecturally any 3-manifold has a decomposition into pieces which are either Seifert fibered or hyperbolic.

In this note we recall preliminaries on Seifert fibered manifolds and some standard theorems. Mainly from the book "3-manifolds" by Hempel. But some results from other sources are also included. Another advanced reference on Seifert fibered spaces is [JS].

We assume that classification and standard properties of 2-manifolds are known to the reader. Also basic group theory, differential topology and algebraic topology are assumed (some references are given in the other notes "Knot theory and 3-manifolds").

1. Seifert fibered spaces in general

The main reference for this section is Chapter 12 of [H]. But here we try to include some details explanation of the material there.

A 3-manifold M is called a *Seifert fibered space* if M is the union of a pairwise disjoint collection $\{C_\alpha\}$ of simple closed curves (called *fiberes*) such that for each α there is a closed neighborhood V of C_α in M which is a solid torus and a covering map $p : \mathbb{D}^2 \times \mathbb{S}^1 \to V$ satisfying:

(1) p maps each $x \times \mathbb{S}^1$ $(x \in \mathbb{D}^2)$ to some C_β
(2) $p^{-1}(C_\alpha)$ is connected and
(3) the group of covering translation is generated by $\tau_{m,n}$ for some pair of relatively prime integers (m, n) where

$$\tau_{m,n}(re^{i\theta}, e^{i\phi}) = (re^{i(\theta+2m\pi/n)}, e^{i(\phi+2\pi/n)}).$$

Typeset by $\mathcal{AMS}$-TEX

If $|n| = 1$ then p is an embedding and C_α is called a *regular fiber*.

If $|n| \geq 1$ then $p(0 \times \mathbb{S}^1) = C_\alpha$ and for $x \in \mathbb{D}^2 - 0$, $p|_{x\times\mathbb{S}^1}$ is a homeomorphism onto some C_β. C_β intersects $p(\mathbb{D}^2 \times 1)$ n-times and spirals m-times around C_α. That is if $\pi_1(\partial V)$ is generated by λ and μ, where λ generates $\pi_1(V)$ and μ bounds a disc in V (λ and μ are called the *longitude* and *meridian* of ∂V respectively) then in such a situation for $\alpha \neq \beta$, C_β represents the element $\lambda^m\mu^n$ in $\pi_1(V - C_\alpha) = \pi_1(\partial V)$. In this case we say C_α is a *singular fiber* of type (m, n). If M is compact then as all the singular fiberes are isolated in the Seifert fibration of M there are only finitely many singular fiber in M. Now we identify each fiber to a point and let S be the quotient space and $p : M \to S$ be the quotient map. If C_α is a regular fiber then a fiber neighbourhood of it goes to a disc neighbourhood of its image under p. Also if C_α is a singular fiber then also the image is a disc but with a distinguished point. We associate the fiber type (m, n) of C_α with this distinguished point. This shows that the space S is actually a compact 2-manifold. And M has nonempty boundary if and only if so does S. S is called the *Seifert surface* of the Seifert fibre space M. Also sometimes called *base surface*.

Example 1.1 The 3-sphere $\mathbb{S}^3$ has many Seifert fibration. In fact given any pair of coprime integer (p, q) there is a Seifert fibration of $\mathbb{S}^3$ with two singular fibers of type (p, q) and (q, p) and base surface the 2-sphere. Start with (p, q) and consider the solid torus $X_1 = \mathbb{D}^2 \times \mathbb{S}^1$ with a Seifert fibration of type (p, q), that is it is foliated by simple closed curves whose regular fibers gives the homology class $pl + qm$ in $H_1(\mathbb{S}^1 \times \mathbb{S}^1)$, here $l = \mathbb{S}^1 \times 1$ and $m = 1 \times \mathbb{S}^1$. The circle $0 \times \mathbb{S}^1$ is the singular fiber. Now consider a copy of the same solid torus X_2 with a fibration of type (q, p). Now glue X_1 and X_2 along the boundary by the map $(x, y) \mapsto (y, x)$. It is obvious that the Seifert fibration of X_1 and the Seifert fibration of X_2 patch together. What we get is $\mathbb{S}^3$ with a Seifert fibration with two singular fibers and obviously the base surface is the 2-sphere.

Example 1.2 Now consider a pair of coprime integers (p, q) and the associated Seifert fibration on the 3-sphere as described in the previous example. Note that in any Seifert fibered space if we remove the interior of a fibered (solid torus) neighborhood of a fiber then again we get a Seifert fibered space. Now in the above example consider the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ in the 3-sphere which is (p, q) fibered. Let C be a regular fiber in the interior of this solid torus. This is nothing but a (p, q) torus knot in the 3-sphere. We remove the interior of a fibered neighborhood of this fiber. We get the torus knot complement. Thus the torus knot complements are a class of examples of Seifert fibered spaces. In this case the base surface is a disc.

Another set of examples of Seifert fibered spaces are given in the following Lemma.

Lemma 1.3. *Let M be a compact Seifert fibered space and $\tilde{M}$ be a finite sheeted* *covering of M. Then $\tilde{M}$ is also Seifert fibered.*

Proof. As the covering is finite sheeted inverse image of any fiber of M is a disjoint union of embedded circles in $\tilde{M}$ and inverse image of all fiberes clearly gives a foliation of $\tilde{M}$ into a pairwise disjoint collection of embedded simple closed curves. The other properties are easily checked by some simple covering space theory. □

A 3-manifold M is called *irreducible* if any embedded 2-sphere in it bounds an embedded 3-ball. By Sphere theorem any irreducible 3-manifold has $\pi_2 = 0$. M is called *prime* if $M = M_1 \# M_2$ implies either M_1 or M_2 is homeomorphic to the 3-sphere. Obviously an irreducible 3-manifold is prime. The otherway implication is also true except for $\mathbb{S}^2$ bundles over $\mathbb{S}^1$ which is prime but not irreducible. Also note that $\mathbb{S}^2 \times \mathbb{S}^1$ is a Seifert fibered space with no singular fiber and the base surface is the 2-sphere. But the following is always true.

Lemma 1.4. *Any compact Seifert fibered space with nonempty boundary is irreducible.*

Proof. As the manifold has nonempty boundary it is easy to find properly embedded 1-manifold on the base surface whose inverse image is a properly embedded incompressible surface in the manifold and whose complement is a disjoint union of fibered solid torus. The manifold is obtained by identifying incompressible annulus on the boundaries of these solid tori.

Now let S be an embedded sphere in the manifold. We isotope S in M so that it meets all the incompressible annuli (discussed above) in the interior and along circles. As all these circles bound discs on S (and hence in M) and all the annuli are incompressible these circles bound discs on these annuli also. Consider a circle γ which bound a disc D on one of these annuli (say A) and interior of D does not intersect S. Choose a regular neighborhood N of D in M which meets the sphere along an annulus and meets A only along γ. Now we perform surgery along D using N. We continue these process till we get rid of all the intersecting circles. What we get is a disjoint collection of spheres $\mathcal{A}_i$ whose components are embedded in the interior of fibered solid tori we discussed before. And S is obtained by removing small discs from two components of the spheres and joining them by tube. Obviously each $\mathcal{A}_i$ bounds a ball in the solid torus. These informations can be used to find a ball inside M which is bounded by S. □

In the following we describe a presentation of the fundamental group of a compact Seifert fibered manifold. For simplicity assume there are at least one singular fiber in M. Let $p_1, p_2, \dots, p_r$ be the image of the singular fibers of M on the surface S. Choose small pairwise disjoint open disc neighborhoods B_i of each of the point p_i. Let $S^* = S - \cup B_i$ and $M^* = M - \cup p^{-1}(B_i)$. $p^{-1}(B_i)$ are fibered solid torus neighborhood of the singular fibers $p^{-1}(p_i)$. We have $M^* = p^{-1}(S^*)$ and the restriction of p to M^* is a smooth fiber bundle projection over the base S^*. Also note that as there is at least one singular fiber in the Seifert fibration of M, $\partial S^* \neq \emptyset$. Also since $H_2(S^*, \mathbb{Z}) = 0$ by standard obstruction theory there is a section $s : S^* \to M^*$ to the fiber bundle projection p.[1] $p^{-1}(\partial B_i)$ is a torus boundary of M^* and $\pi_1(p^{-1}(\partial B_i))$ has a natural basis (t_i, c_i), where t_i is a regular fiber on $p^{-1}(\partial B_i)$ and c_i is the boundary circle $s(\partial B_i)$ of the section of the fiber bundle. We orient all the regular fiber so as to become freely homotopic in M^*. Also $\pi_1(p^{-1}(\partial B_i))$ has another basis (λ_i, μ_i) considering it as boundary of the solid torus $p^{-1}(B_i)$, where (λ_i, μ_i) are standard longitude and meridian of $\pi_1(\partial p^{-1}(B_i))$ so that $t_i = \mu_i^{m_i}\lambda_i^{n_i}$. Here we are assuming that the singular fiber $p^{-1}(p_i)$ is of type (m_i, n_i). Now there is a pair of coprime integers (r_i, s_i) so that $c_i = \mu_i^{r_i}\lambda_i^{s_i}$ and $m_i s_i - n_i r_i = 1$ (this is clear because the map $\mu_i \to c_i, \lambda_i \to t_i$ is an isomorphism of $\pi_1(p^{-1}(\partial B_i))$. We get:

$$\mu_i = \mu_i^{m_i s_i - n_i r_i} = (\mu_i^{m_i})^{s_i}(\mu_i^{r_i})^{-n_i} = (t_i\lambda_i^{-n_i})^{s_i}(c_i\lambda_i^{-s_i})^{-n_i} = t_i^{s_i}c_i^{-n_i}.$$

By Seifert-Van-Kampen theorem a presentation (in terms of generator and relations) of $\pi_1(M^*)$ can be used to give a presentation of $\pi_1(M)$ by including the relations $t_i^{s_i}c_i^{-n_i} = 1$ as μ_i bounds a disc in M, namely the meridinal disc of $p^{-1}(B_i)$.

Recall that we have a split short exact sequence of groups:

$$1 \longrightarrow < t > \longrightarrow \pi_1(M^*) \longrightarrow \pi_1(S^*) \longrightarrow 1.$$

Here t represents a regular fiber of the Seifert fibration of M. Thus we have $\pi_1(M^*) \simeq < t > \rtimes \pi_1(S^*)$. The action of $\pi_1(S^*)$ on $< t >$ comes from the monodromy action of the fiber bundle projection $M^* \to S^*$. Thus $\pi_1(M^*) =< \pi_1(S^*), t \mid gtg^{-1} = t^{\epsilon_g} >$. Where $\epsilon_g = +1$ or -1 according as the action of g on the fiber preserves or reverses the orientation of the fiber. Here we identify g with $s_*(g)$ in the above presentation. Hence we get the following:

$$\pi_1(M) =< \pi_1(S^*), t \mid gtg^{-1} = t^{\epsilon_g}, t_i^{s_i} = c_i^{-n_i}, i = 1, 2, \ldots, r \mathrm{and} g \in \pi_1(S^*) > .$$

Thus we have proved the following theorem:

Theorem 1.5. *Let M be a compact Seifert fibered manifold with Seifert surface S so that S has genus g and k boundary components then $\pi_1(M)$ has the presentation* (1) *or* (2) *below according as S is orientable or not:*

(1) $< a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r, d_1, \ldots, d_k, t \mid a_i t a_i^{-1} = t^{\epsilon_i}, b_i t b_i^{-1} = t^{\delta_i}, c_i t c_i^{-1}$
$= t^{\eta_i}, d_i t d_i^{-1} = t^{\theta_i}, c_i^{n_i} = t^{s_i}, c_r = [a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_{r-1} d_1 \cdots d_k >$

[1] The section also can be constructed in the following way. Note first that we can cut the surface S^* successively along properly embedded intervals and finally we land up on a disc. Now as the intervals are contractible we have section to the bundle when restricted to this intervals. Finally we reach the disc on the boundary of which we have defined a section. Now use a trivialization of the bundle over this disc to extend this section to the whole of the disc. We patch back several parts of the boundary of the disc to get back S^* and obviously we get a section to the bundle $M^* \to S^*$.

(2) $< a_1, \dots, a_g, c_1, \dots, c_r, d_1, \dots, d_k, t \mid a_i t a_i^{-1} = t^{\epsilon_i}, c_i t c_i^{-1} = t^{\eta_i}, d_i t d_i^{-1} = t^{\theta_i}, c_i^{n_i} = t^{s_i}, c_r = a_1^2 \cdots a_g^2 c_1 \cdots c_{r-1} d_1 \cdots d_k >$

Here $\epsilon_i, \delta_i, \eta_i, \theta_i$ are either $+1$ or -1 and t is represented by any regular fiber.

Remark 1.6. If S is orientable then M is orientable if and only if $\epsilon_i = \delta_i = \eta_i = \theta_i = 1$.

Now we come to a very important class of examples of Seifert fibered manifolds, namely certain class of surface bundles over the unit circle. Let M be a compact surface bundle over the circle with fiber diffeomorphic to F. That is, there is a diffeomorphism $f : F \to F$ so that $M = M_f = F \times [0,1]/\sim$ where $(x, 0) \sim (f(x), 1)$ for $x \in F$. Fix $x \in F$ and consider the set $S_x = \cup_{k=0}^{\infty} f^k(x) \times [0,1]/\sim \subset M$. Then either S_x is a line or a simple closed curve according as $f^k(x) \neq x$ for all k or $f^k(x) = x$ for some k respectively. If f is of finite order then M_f (also known as *mapping torus* of f) is foliated by circles. We will prove that in this case the mapping torus is in fact a Seifert fibered manifold.

Two homeomorphism f_0 and f_1 of F are called *homotopic* if there is a map $f : F \times I \to F \times I$ so that $f|_{F\times 0} = f_0$ and $f|_{F\times 1} = f_1$. And they are called *isotopic* if in addition f is a homeomorphism with $f|_{F\times t}$ are homeomorphism onto $F \times t$ for each t. Note that isotopy implies homotopy. But in the case of compact surfaces the reverse implication is also true, that is any two homotopic homeomorphisms of a compact surface are isotopic (see [E]).

Exercise 1.7 Show that if f_0 and f_1 are two isotopic homeomorphism of a compact surface F then the two mapping tori M_{f_0} and M_{f_1} are fiber preservingly homeomorphism.

Let $Out(\pi_1(F))$ be the quotient group of the automorphism group of $\pi_1(F)$ with the normal subgroup of all inner automorphism. If f_0 and f_1 are homotopic then they induce the same element in $Out(\pi_1(F))$ and also the converse is true. Thus the mapping torus M_f is determined by the outer automorphism class of the homeomorphism f.

We also know that $\pi_1(M_f) = \pi_1(F) \rtimes_{f_*} < t >= \pi_1(F) * < t > /N$ where N is the normal subgroup generated by the set $\{t\alpha t^{-1} f_*(\alpha^{-1}) \ : \ \alpha \in \pi_1(F)\}$, here $f_* : \pi_1(F) \to \pi_1(F)$ is induced by f and t is the image of a generator of $\mathbb{Z}$ in $\pi_1(M_f)$ under a section to the surjective map $\pi_1(M_f) \to \mathbb{Z}$ induced by the fiber bundle projection. Also t is represented by a simple closed curve in M which intersects each fiber transversally at one point. Each element of $\pi_1(M)$ has a unique representation αt^n, $\alpha \in \pi_1(F)$ and the multiplication between two elements is given by $(\alpha t^n)(\beta t^m) = \alpha f_*^n(\beta) t^{m+n}$.

A properly embedded (i.e., an embedding which sends boundary to boundary) oriented surface S in an oriented 3-manifold is called *incompressible* if the inclusion map induces an injective homomorphism $\pi_1(S) \to \pi_1(M)$. A compact oriented 3-manifold is called *Haken* if it is irreducible and contains a properly embedded oriented incompressible surface S with $\pi_1(S) \neq 1$.

We prove the following theorem (Chapter 12 [H]):

Theorem 1.8. *Let M be a Haken manifold and N be an infinite cyclic normal subgroup of $\pi_1(M)$. Then* (1) *M is a Seifert fibered space and N is a subgroup of the group generated by any regular fiber or* (2) *there are two twisted I-bundles M_1 and M_2 over nonorientable surfaces so that $M = M_1 \cup_{\partial M_1 = \partial M_2} M_2$ or $\pi_1(M_i) = \mathbb{Z}_2$.*

Remark 1.9. If M is irreducible, $H^1(M, \mathbb{Z}) \neq 0$ and $\pi_1(M)$ has an infinite cyclic normal subgroup then the hypothesis of the theorem is satisfied.

Remark 1.10. Here note that any Haken 3-manifold has torsion free fundamental group. This is well known and also can be proved by induction on the number of incompressible surfaces in the manifold. Here we recall that if we start cutting the manifold along incompressible surfaces then eventually we get only 3-discs. This is a similar situation as in the case of compact surface.

Throughout the remaining section we will be using the following lemma due to Epstein [E].

Lemma 1.11. *Let S be a compact surface and $f_0, f_1 : \mathbb{S}^1 \to S$ are two embeddings which are homotopic and they are not null homotopic then there is an isotopy $g : S \times I \to S \times I$ so that $g_0 = id$ and $g_1 \circ f_0 = f_1$.*

Proof of theorem 1.8. If $\partial M \neq \emptyset$ then we can assume that all the boundary component are of genus greater than or equal to 1. For if some component is the 2-sphere then by irreducibility $M = \mathbb{D}^3$ which is a contradiction as the fundamental group contains an infinite cyclic subgroup. Now we leave the following as an exercise, which is a simple computation of Euler characteristic and using Mayer-Vietoris sequence to the double of M:
Exercise 1.12 If M is orientable, irreducible, $\partial M \neq \emptyset$ and genus of ∂M is greater or equal to 1 then $H^1(M, \mathbb{Z}) \neq 0$.

We state the following lemma whose proof can be found in [H], page 62. An embedded surface F in M is called *separating* (or *nonseparating*) if $M - F$ is disconnected (or connected).

Lemma 1.13. *If M is as in the theorem and $\partial M \neq \emptyset$ then there is a 2-sided nonseparating incompressible surface properly embedded in M.*

Now assume either $\partial M \neq \emptyset$ or M is closed and contains an incompressible surface F which is nonseparating, that is $M - F$ is connected. (Now as F is nonseparating there is a simple closed curve in M which intersect F only at one point. Now a homological argument shows that $H^1(M, \mathbb{Z}) \neq 0$ in this case also.)

Two cases occur:
Case 1 N is not a subgroup of $\pi_1(F)$.

Cut M along F and take the closure. Let M' be the resulting manifold. M' has two boundary components (say F^1 and F^2) each homeomorphic to F and there is a map $q : M' \to M$ which identifies F^1 with F^2 and is a homeomorphism from $M' - F^1 \cup F^2$ onto $M - F$. Also let q^j for $j = 1, 2$ be the restrictions of q to F^j which is a homeomorphism onto F respectively. Consider countably many copies M_i of M' and denote by F_i^j and q_i^j the copies of the corresponding data. Take the union $\cup M_i$ and identify F_i^2 with F_{i+1}^1 by the homeomorphism $(q_{i+1}^1)^{-1} q_i^2$ and denote the resulting manifold by $\tilde{M}$. Denote by F_i the identified surfaces F_i^2 and F_{i+1}^1. We get a covering projection map $p : \tilde{M} \to M$ with infinite cyclic group of covering transformation generated by $\tau : \tilde{M} \to \tilde{M}$ where $\tau(M_i) = M_{i+1}$ and $\tau(F_i) = F_{i+1}$. Also this homeomorphism corresponds to the loop in M which intersects F transversally at a single point x_0 and any lifting of this loop is a path in M_i from a point $\tilde{x}_{i-1} \in F_{i-1}$ to $\tilde{x}_i \in F_i$.

We show that M_0 is homeomorphic to $F_0 \times I$ (by a homeomorphism h, say) so that the following diagrams are commutative:

$$\begin{tikzcd} F_0 \arrow[r, "q_0^1"] \arrow[dr, "* \times 0"] & M_0 \arrow[d, "h"] \\ & F_0 \times 0 \subset F_0 \times I \end{tikzcd} \qquad \begin{tikzcd} F_1 \arrow[r, "q_0^2"] \arrow[dr, "* \times 1"] & M_0 \arrow[d, "h"] \\ & F_0 \times 1 \subset F_0 \times I \end{tikzcd}$$

Suppose this is not the case, then by the following theorem (a particular case of Theorem 10.2 in [H]) none of the injections $\pi_1(F_i) \to \pi_1(M_i)$ and $\pi_1(F_i) \to \pi_1(M_{i-1})$ is an epimorphism.

Theorem. *Let M be a compact, oriented, irreducible 3-manifold with a submanifold $F \subset \partial M$ ($F \neq \mathbb{D}^2$) so that the inclusion map induces an isomorphism $\pi_1(F) \to \pi_1(M)$. Then M is homeomorphic to $F \times I$ taking F to $F \times 0$.*

Hence $\pi_1(F_0)$ is of infinite index in $\pi_1(\tilde{M})$. Write $\tilde{M}$ in the following way: $\tilde{M} = \tilde{M}_- \cup_{F_0} \tilde{M}_+$ where $\tilde{M}_- = \cup_{i<0} M_i$ and $\tilde{M}_+ = \cup_{i \geq 0} M_i$. Then by Seifert-Van-Kampen theorem $\pi_1(\tilde{M}) = \pi_1(\tilde{M}_-) *_{\pi_1(F_0)} \pi_1(\tilde{M}_+)$.

We leave the following as an exercise:

Exercise 1.14 Let N be an infinite cyclic normal subgroup of an amalgamated free product $G_1 *_H G_2$ then either $N \subset H$ or H has index two in each of G_1 and G_2. Hint: Use normal form of element of an amalgamated free product. Here we recall that if we fix coset representative of H in G_1 and G_2 then any element $g \in G$ can be written uniquely in the form; $g = hg^1 g^2 \cdots g^n$ where $h \in H$ and g^i's are coset representatives of H in G_1 and G_2 alternatively. This is called the *normal form.*

Now as $\pi_1(F_0)$ is of infinite index in $\pi_1(\tilde{M})$ and by hypothesis N is not a subgroup of $\pi_1(F)$ we have N is not a subgroup of $p_*(\pi_1(\tilde{M}))$.

We have $\pi_1(M) \simeq p^*(\pi_1(\tilde{M})) \rtimes_\tau \mathbb{Z}$. Let yt^r be a generator of N where $y \in p_*(\pi_1(\tilde{M}))$. Then $r \neq 0$. Since N is normal in $\pi_1(M)$ for any $z \in p_*(\pi_1(\tilde{M}))$ and $s \in \mathbb{Z}$ we have one of the followings:

$$(zt^s)(yt^r)(zt^s)^{-1} = yt^r,\ (zt^s)(yt^r)(zt^s)^{-1} = (yt^r)^{-1}.$$

Since $p_*(\pi_1(\tilde{M}))$ is normal in $\pi_1(M)$ a simple calculation shows that the second possibility implies that $t^{2r} \in p_*(\pi_1(\tilde{M}))$, which is impossible. Thus we have shown that N is central in $\pi_1(M)$.

Now for any m, $t^{-m}(yt^r)t^m = (t^{-m}yt^m)t^r \in N$ and $y_1 = t^{-m}yt^m \in p_*\pi_1(\tilde{M})$. By choosing m large enough there is a loop α in $\tilde{M}_-$ based at $\tilde{x}_0$ such that $p_*([\alpha]) = y_1$. Let β be any loop in M_0 based at $\tilde{x}_0$ and γ be a path in $M_0 \cup \cdots \cup M_{r-1}$ from $\tilde{x}_0$ to $\tilde{x}_r$ such that $[p \circ \gamma] = t^r$ and consider the loop $\theta = \beta^{-1}(\alpha \cdot \gamma)(\tau^r \circ \beta)(\alpha \cdot \gamma)^{-1}$. Now $p_*([\theta]) = [p_*[\beta]^{-1}, y_1 t^r]$. But $y_1 t^r \in N$ and N is central and p_* is injective; so $[\theta] = 1$. Now as θ intersects M_r only along $\tau^r \circ \beta$ and θ is null-homotopic in $\tilde{M}$ we can use the incompressibility of F_r to show that $\tau^r \circ \beta$ is homotopic to a loop in F_r. As the choice of β was arbitrary we have shown that the map $\pi_1(F_0) \to \pi_1(M_0)$ is surjective - a contradiction. Hence M_0 is homeomorphic to $F_0 \times I$ sending F_0 to $F_0 \times 0$ and thus M is a fiber bundle over the circle with fiber diffeomorphic to F.

We already know that N is central in $\pi_1(M) = \pi_1(F) \rtimes \mathbb{Z}$ and is generated by yt^r, $r \neq 0$. Let ϕ be the monodromy diffeomorphism of the fiber bundle projection $M \to \mathbb{S}^1$. For any $\beta \in \pi_1(F)$ we have $(\beta)(yt^r) = (yt^r)(\beta)$. Rewriting both sides in terms of the product structure in the semidirect product we get $\beta y t^r = y\phi^*(\beta)t^r$ and hence $\phi^*(\beta) = y^{-1}\beta y$. Thus ϕ_* is an inner automorphism. Now by a theorem of Nielsen ([N2]) ϕ is homotopic and hence isotopic to a finite order homeomorphism, say ϕ_1, of F. Now by Exercise 1.7 M is (fiber preservingly) homeomorphic to a fiber bundle projection with finite order monodromy. Denote this manifold again by M. From previous discussion M is foliated by circles. Chose a circle $C \subset M$. Then $C \cap F$ has only finitely many points. Choose a point $p \in C \cap F$. Choose a small disc neighborhood D of p in F so that the discs $D, \phi_1(D), \ldots, \phi_1^{m-1}(D)$ are disjoint and $\phi_1^m(D) = D$. Again we refer to a theorem of Nielsen ([N1]) to deduce that $\phi_1^m|_D$ is a rotation about p by an angle $2\pi n/m$ where n is the order of ϕ_1. This clearly gives a Seifert fibration of M. Note that the fiber C is regular if $\#(C \cap F) =$ order of ϕ_1.

Here is an interesting exercise:

Exercise 1.15 Let $\phi : S \to S$ be a diffeomorphism of a closed orientable surface S of order n. Show that the set $\mathcal{A} = \{x \in S \mid \phi^k(x) = x \text{ for } k < n\}$ is finite. Also show that the fiberes of the mapping torus M of the map ϕ which pass through points of $\mathcal{A}$ are exactly the singular fiberes of M in the Seifert fibration of M we described above.

At this point the following exercise is relevant:

Exercise 1.16 Any diffeomorphism of $\mathbb{R}^2$ is homotopic to a product of rotations and translations. And in particular, if the diffeomorphism fixes a point then it is homotopic to a rotation about that point.

Next we have to check that N lies in the subgroup generated by any regular fiber. We leave the following as an exercise.

Exercise 1.17 Let $M \to \mathbb{S}^1$ be a fiber bundle with finite order monodromy ϕ. Show that a regular fiber represents an element in $\pi_1(M) = \pi_1(F) \rtimes_\phi \mathbb{Z}$ of the form γt^n, when $\gamma \in \pi_1(F)$ and n is the order of ϕ and this element is central in $\pi_1(M)$.

So the regular fiber is generated by γt^n where order of ϕ_1 is n.

Next assume that the center of $\pi_1(M)$ is infinite cyclic. We claim that this is also generated by γt^n. Otherwise we will have a central element of the form δt^r where r divides n. But then again we can show by the same method as above that ϕ_*^r is an inner automorphism and hence ϕ_1 is homotopic to a homeomorphism of order r. Thus we have shown that M has a Seifert fibration so that N is in the subgroup generated by a regular fiber.

If the center is not infinite cyclic then it is easy to check that $Z(\pi_1(M)) \cap \pi_1(F) \neq 1$. Here $Z(G)$ denotes the center of the group G. And hence F is either an annulus or the torus.

Exercise 1.18 Show that if the center of the fundamental group of a compact orientable 2-manifold is nontrivial then the surface is either the annulus or the torus.

When F is the annulus $M = \mathbb{S}^1 \times \mathbb{S}^1 \times I$ or the twisted I-bundle over the Klein bottle. But the fundamental group of the later has center infinite cyclic as it is homotopically equivalent to the Klein bottle. And the former can be given a Seifert fibered structure so that N is generated by a regular fiber: for take a simple closed curve C on the torus of type (p, q), where p and q are coprime so that N lies in the infinite cyclic group generated by (p, q) in $\mathbb{Z} \times \mathbb{Z}$. Take all the parallels of C on the torus and the curves of the type $C \times t$ for $t \in I$. This gives a Seifert fibration on $\mathbb{S}^1 \times \mathbb{S}^1 \times I$ with the required property.

Now assume that F is the torus. It is easy to check that ϕ_{1*} fixes every element of $Z(\pi_1(M)) \cap \pi_1(F)$. We recall the following lemma. An easy proof is given in [H].

Lemma 1.19. *Let $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ be a periodic automorphism then ϕ and the associated mapping torus is one of the following type (think of $\mathbb{S}^1$ as the additive group of reals modulo 1):*

(1) $\phi = 1$ *and* $M_\phi = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. $\pi_1(M_\phi) = \{\alpha, \beta, t \mid [\alpha, \beta] = [\alpha, t] = [\beta, t] = 1\} = H_1(M_\phi)$. *$M_\phi$ is orientable.*

(2) $\phi(x, y) = (-x, -y)$, $\phi^2 = 1$. $\pi_1(M_\phi) = \{\alpha, \beta, t \mid [\alpha, \beta] = 1, t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \beta^{-1}\}$. *$M_\phi$ is orientable.*

(3) $\phi(x, y) = (x, -y)$, $\phi^3 = 1$. $\pi_1(M_\phi) = \{\alpha, \beta, t \mid [\alpha, \beta] = 1, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \beta^{-1}\}$. *$M_\phi$ is nonorientable.*

(4) $\phi(x,y) = (x+y,-y)$, $\phi^4 = 1$. $\pi_1(M_\phi) = \{\alpha,\beta,t \mid [\alpha,\beta] = 1, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \alpha\beta^{-1}\}$. *$M_\phi$ is nonorientable.*
(5) $\phi(x,y) = (-y,x-y)$, $\phi^5 = 1$. $\pi_1(M_\phi) = \{\alpha,\beta,t \mid [\alpha,\beta] = 1, t\alpha t^{-1} = \beta, t\beta t^{-1} = \alpha^{-1}\beta^{-1}\}$. *$M_\phi$ is orientable.*
(6) $\phi(x,y) = (y,-x)$, $\phi^6 = 1$. $\pi_1(M_\phi) = \{\alpha,\beta,t \mid [\alpha,\beta] = 1, t\alpha t^{-1} = \beta^{-1}, t\beta t^{-1} = \alpha\}$. *$M_\phi$ is orientable.*
(7) $\phi(x,y) = (-y,x+y)$, $\phi^7 = 1$. $\pi_1(M_\phi) = \{\alpha,\beta,t \mid [\alpha,\beta] = 1, t\alpha t^{-1} = \beta, t\beta t^{-1} = \alpha^{-1}\beta\}$. *$M_\phi$ is orientable.*

In the above classification some easy calculation shows that ϕ_* fixes an element in $Z(\pi_1(M)) \cap \pi_1(F)$ only in case (1). And hence $M = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. We leave this as an exercise to check that in this case there is a Seifert fibration so that N lies in the cyclic subgroup generated by a regular fiber.

Case 2: N is a subgroup of $\pi_1(F)$. Now since F has an infinite cyclic normal subgroup $\chi(F) = 0$. And hence F is either an annulus or a torus. Remember that we are still in the case when F is nonseparating in M. So cut M along F we get a connected manifold $M_1 = M - F$ with two components of the boundary, say F^1 and F^2, homeomorphic to F. Since M_1 has nonempty boundary it again has a nonseparating 2-sided properly embedded incompressible surface, say F_1 in M_1. Note that $N < \pi_1(M_1)$. If N is not contained in $\pi_1(F_1)$ then we can apply Case (1) to deduce that M_1 is Seifert fibered. Now as M_1 is a Seifert fibered space it has either incompressible boundary or it is a solid torus. In the later case F^1 and F^2 are annuli and they are disjoint and hence a fibration on $F^1 \cup F^2$ can be extended to a fibration of the solid torus. Thus when we glue back F^1 and F^2 we get a Seifert fibration for M with all the required properties. Nest assume that M_1 has incompressible boundary with two specific boundary components F^1 and F^2 which are tori. Note that we have N as a subgroup of the infinite cyclic group generated by a regular fiber also by hypothesis N is contained in both $\pi_1(F^1)$ and $\pi_1(F^2)$. Let h be a simple closed curve representing a generator of N and f_1 and f_2 are two regular fibers on F^1 and F^2 respectively. Thus we have two integers n_1 and n_2 so that $f_1^{n_1}$ and $f_2^{n_2}$ are homotopic (and hence isotopic) to h. Now it is easy to check that both f_1 and f_2 are isotopic to a simple closed curve of type (p,q) for some coprime integers p and q. Thus we can isotope the Seifert fibration of M_1 near F^1 and F^2 so that when we identify F^1 and F^2 to get M, the Seifert fibration obviously extends to an one on M.

Let us now summarize what we have proved so far assuming that the 2-sided incompressible surface F in M is nonseparating:

Statement 1 If M is a Haken manifold and N is an infinite cyclic normal subgroup of M and there is an incompressible surface F in M so that N is not contained in $\pi_1(F)$ then M is a Seifert fibered space with N contained in the subgroup generated by a regular fiber.

Statement 2 M and F are same as above but N is contained in $\pi_1(F)$ and $M - N(F)$ is a Seifert fibered space, where $N(F)$ is a regular neighborhood of F, then M is a Seifert fibered space with N contained in the subgroup generated by a regular fiber.

As M has a nonseparating 2-sided incompressible surface and any irreducible manifold with nonempty boundary has such a surface we get a sequence: $M = M_0 \supset M_1 \supset \cdots \supset M_n \supset \cdots$ and nonseparating incompressible surfaces $F_i \subset M_i$ such that $M_{i+1} = \overline{M_i - N(F_i)}$, $0 \leq i < n$ where $N(F_i)$ is a regular neighborhood of F_i. By [W1] or [H1] the sequence stops at a stage when some M_n is a 3-disc. And hence M_{n-1} is a solid torus. Now the induction process starts and using Statement 1 and 2 we arrive at the conclusion of the theorem in this case.

We complete the proof of the theorem by showing that if the incompressible surface is separating then either we come back to the previous case or option (2) in the theorem occur.

So assume that F is a separating 2-sided incompressible surface in the closed manifold M (recall that the nonempty boundary case is already treated before). Remove a open regular neighborhood of F from M. Then we get two manifolds M_1 and M_2 each has boundary F_1 and F_2 homeomorphic to F. Note that we have $\pi_1(M) \simeq \pi_1(M_1) *_{\pi_1(F)} \pi_1(M_2)$. If N is not contained in $\pi_1(M)$ then by Exercise 1.14 $\pi_1(F)$ has index 2 in both $\pi_1(M_1)$ and $\pi_1(M_2)$. In this case we appeal to the Theorem 10.5 in [H] to get (2) in the theorem. Now if N is a subgroup of $\pi_1(F)$ then by the previous case M_1 and M_2 are both Seifert fibered space with incompressible (tori) boundary and N is contained in the subgroup generated by a regular fiber. Now as N is contained in both $\pi_1(F_1)$ and $\pi_1(F_2)$ an easy argument, as we did in the previous case, shows that after an isotopy the Seifert fibration of M_1 and M_2 can be extended to one on M. $\square$

References

[**E**] D.B.A. Epstein, *Curves on* 2*-manifolds and isotopies*, Acta Math. **115** (1966), 83–107.

[**H**] John Hempel, 3*-manifolds, Ann. of Math. Studies, No. 86*, Princeton University press, Princeton, N.J., 1976.

[**H1**] ———, *Über das Homöomorphieproblem der* 3*Mannigfaltigkeiten*, Math. Z. **80** (1962), 89–120.

[bf JS] William H. Jaco and Peter B. Shalen, *Seifert fibered spaces in* 3*-manifolds*, Mem. Amer. Math. Soc. **220** (1979).

[**N1**] J. Nielsen, *Die Struktur periodischer Transformationen von Flächen*, Math.-fys. Meddelelser Kgl. Danske Vidensk. Selsk **XV** (1937), 1–75.

[**N2**] ———, *Abbildungsklassen endlicher Ordnung*, Acta Math. **75** (1942), 23–115.

[**W**] F. Waldhausen, *On irreducible* 3*-manifolds which are sufficiently large*, Ann. of Math. **87** (1968), 56–88.

[**W1**] ———, *Gruppen mit Zentrum und* 3*-dimensionale Mannigfaltigkeiten*, Topology **6** (1967), 505–517.

School of Mathematics, Tata Institute, Homi Bhabha Road, Mumbai 400 005, India.
http://www.math.tifr.res.in/~ roushon/allahabad.html

E-mail address: roushon@math.tifr.res.in