NON-ADMISSIBLE IRREDUCIBLE REPRESENTATIONS OF p-ADIC GL_n IN CHARACTERISTIC p

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ABSTRACT. Let $p > 3$ and F be a non-archimedean local field with residue field a proper finite extension of $\mathbb{F}_p$. We construct smooth absolutely irreducible non-admissible representations of $\mathrm{GL}_2(F)$ defined over the residue field of F extending the earlier results of the authors for F unramified over $\mathbb{Q}_p$. This construction uses the theory of diagrams of Breuil and Paškūnas. By parabolic induction, we obtain smooth absolutely irreducible non-admissible representations of $\mathrm{GL}_n(F)$ for $n > 2$.

1. INTRODUCTION

Let p be a prime number. This note concerns the smooth representation theory of (connected) p-adic reductive groups over coefficient fields of characteristic p initiated in [BL94]. This theory has its origins in the study of congruences between automorphic forms and plays an important role in the mod p Langlands program proposed by Breuil [Bre03]. In our context, smooth means that the stabilizers of vectors are open subgroups. Spaces of automorphic forms provide natural sources of smooth representations which are also *admissible*, i.e., the space of vectors invariant under any compact open subgroup is finite-dimensional. Over characteristic 0 fields, building upon Harish-Chandra's work [HC70], Jacquet [Jac75] and Bernstein [Ber74] showed that any irreducible (or finite length) smooth representation of a p-adic reductive group is automatically admissible by reducing to the supercuspidal case. Vignéras extended this result to base fields of positive characteristic different from p [Vig96]. The proofs use Haar measures which do not exist in characteristic p. Nevertheless, [AHHV17, Question 1] asked whether a similar result holds in characteristic p. It is not hard to see that smooth irreducible representations of p-adic reductive groups which are anisotropic modulo center are finite-dimensional. Berger showed that any irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over an algebraically closed field of characteristic p admits a central character [Ber12]. Barthel-Livné and Breuil classified the irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over an algebraically closed field of characteristic p with central character [BL94, Bre03] and a direct computation shows that each such representation is admissible. Together these results imply that any absolutely irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over a field of characteristic p is admissible. Recently, the authors [Le19, GS20] used the theory of diagrams developed by Breuil and Paškūnas [Pas04, BP12] to construct absolutely irreducible smooth representations of $\mathrm{GL}_2(F)$ in characteristic p which are *not* admissible

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when F is a proper finite unramified extension of $\mathbb{Q}_p$ and $p > 2$ (see also [GS22]). This naturally leads one to ask which p-adic reductive groups admit irreducible non-admissible representations. Here, we focus on the case of $\mathrm{GL}_n(F)$.

Theorem 1.1. *Let $p > 3$ and $n \geq 2$. Let F be a non-archimedean local field with residue field a proper finite extension of $\mathbb{F}_p$. Then there is an absolutely irreducible non-admissible smooth representation of $\mathrm{GL}_n(F)$ defined over the residue field of F.*

The hypothesis in Theorem 1.1 that the residue field of F is not $\mathbb{F}_p$ cannot be entirely removed given the results of Berger, Barthel-Livné, and Breuil above (see also Remark 3.5). Following the methods of [Le19], we also have a counterexample to a Schur-type lemma for irreducible representations of $\mathrm{GL}_2(F)$.

Theorem 1.2. *Let $p > 3$ and F be a non-archimedean local field with residue field a proper finite extension of $\mathbb{F}_p$. Then there is an irreducible smooth representation of $\mathrm{GL}_2(F)$ over the residue field of F whose endomorphism algebra contains an algebraically closed field.*

We prove Theorem 1.1 by first constructing smooth absolutely irreducible non-admissible representations for $\mathrm{GL}_2(F)$. The construction is uniform and provides a new construction in the cases when F is an unramified extension of $\mathbb{Q}_p$. By parabolically inducing non-admissible irreducible representations of $\mathrm{GL}_2(F)$, we obtain such representations of $\mathrm{GL}_n(F)$ for $n > 2$. The proof of the irreducibility of induced representations uses Herzig's comparison isomorphism between compact and parabolic inductions. We remark that the non-admissible irreducible representations constructed here have a central character. The ones for $\mathrm{GL}_2(F)$ are necessarily supersingular by the classification of Barthel-Livné. The ones for $\mathrm{GL}_n(F)$ with $n > 2$ are, by contrast, not supersingular.

The reason for restricting to unramified extensions of $\mathbb{Q}_p$ in our earlier works is that we used some of the results of [BP12] relying on delicate Witt vector computations to prove the irreducibility. Recently, one of us [She22] introduced cyclic modules to circumvent the irreducibility arguments of [BP12] and construct infinitely many supercuspidal representations of $\mathrm{GL}_2(F)$ with fixed central character under the assumptions in Theorem 1.1. Our construction of an irreducible non-admissible representation of $\mathrm{GL}_2(F)$ involves splicing two cyclic modules together. The resulting diagram is quite different from the diagrams appearing in [BP12, Le19, GS20], namely the $\mathrm{GL}_2(\mathcal{O}_F)$-subrepresentation generated by a pro-p Iwahori fixed eigenvector can have reducible socle. This construction was inspired by similar features of the mod p cohomology of $U(3)$ arithmetic manifolds (see [LLHLM20]). Finally, one of the motivations for our construction is a recent conjecture of Emerton, Gee, Hellmann, and Zhu [EGH, Conjecture 2.4.3] stating that there should exist a fully faithful functor from the category of smooth representations of $\mathrm{GL}_n(F)$ to the category of quasicoherent sheaves on an appropriate moduli stack of Langlands parameters. The existence of irreducible non-admissible smooth $\mathrm{GL}_n(F)$-representations should have an interpretation in terms of the geometry of this moduli stack. We hope to return to this in future work.

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Notation and convention. Let $p > 3$ be a prime number. Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the finite field $\mathbb{F}_{p^f}$ of size p^f. Fix an embedding $\mathbb{F}_{p^f} \hookrightarrow \overline{\mathbb{F}}_p$. Let F be a non-archimedean local field of residual characteristic p and residue degree $f > 1$. Let $\mathcal{O}_F \subseteq F$ be the valuation ring with a uniformizer ϖ. Throughout the note, except for the last part, we work with the group $\mathrm{GL}_2(F)$. Let $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathcal{O}_F)$, $\Gamma = \mathrm{GL}_2(\mathbb{F}_{p^f})$, and Z be the center of G. Let B and U be the subgroups of Γ consisting of the upper triangular matrices and the upper triangular unipotent matrices respectively. Let I and $I(1)$ be the preimages of B and U respectively under the reduction modulo ϖ map $K \twoheadrightarrow \Gamma$. The subgroups I and $I(1)$ of K are the Iwahori and the pro-p Iwahori subgroup of K respectively. The normalizer N of I in G is a subgroup generated by I and $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$. Note that N is also the normalizer of $I(1)$ in G. Let $K(1)$ denote the kernel of the map $K \twoheadrightarrow \Gamma$, i.e., the first principal congruence subgroup of K. Unless stated otherwise, all representations considered in this note are on $\overline{\mathbb{F}}_p$-vector spaces.

A *weight* is an irreducible representation of Γ. Any weight is of the form of

$$\left(\bigotimes_{j=0}^{f-1} \mathrm{Sym}^{r_j} \overline{\mathbb{F}}_p^2 \circ \Phi^j\right) \otimes \det{}^m$$

for some integers $0 \le r_0, \ldots, r_{f-1} \le p-1$ and $0 \le m \le p^f - 2$, where $\Phi : \Gamma \to \Gamma$ is the automorphism induced by the Frobenius map $\alpha \mapsto \alpha^p$ on $\mathbb{F}_{p^f}$ and $\det : \Gamma \to \mathbb{F}_{p^f}^\times$ is the determinant character. We denote such a weight by $\boldsymbol{r} \otimes \det^m$ where $\boldsymbol{r}$ is the f-tuple $(r_0, \ldots, r_{f-1})$ of integers. Let $\sigma = \boldsymbol{r} \otimes \det^m$ be a weight; its subspace σ^U of U-fixed vectors is 1-dimensional and stable under the action of B because B normalizes U. The resulting B-character, denoted by $\chi(\sigma)$, sends $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B$ to $a^r(ad)^m$ where $r = \sum_{j=0}^{f-1} r_j p^j$. Any B-character valued in $\overline{\mathbb{F}}_p^\times$ factors through the quotient B/U which is identified with the subgroup of diagonal matrices in B by the section $B/U \to B$, $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) U \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$. For a B-character χ, let χ^s be the inflation to B of the conjugation-by-s character $t \mapsto \chi(sts^{-1})$ on B/U where $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We say that a weight is *generic* if it is not equal to $(0, 0, \ldots, 0) \otimes \det^m$ or $(p-1, p-1, \ldots, p-1) \otimes \det^m$ for any m. The map $\sigma \mapsto \chi(\sigma)$ gives a bijection from the set of generic weights to the set of B-characters χ such that $\chi \neq \chi^s$. If σ is a generic weight, let us denote by $\sigma^{[s]}$ the generic weight corresponding to the character $\chi(\sigma)^s$. For $\sigma = \boldsymbol{r} \otimes \det^m$, we have $\sigma^{[s]} = (p-1-r_0, \ldots, p-1-r_{f-1}) \otimes \det^{m+r}$. For a B-representation V and a character χ, we denote by V^χ the χ-isotypic component of V. We refer the reader to [BL94, §1] for all non-trivial assertions in this paragraph.

Given two weights σ and τ, let $E(\sigma,\tau)$ be the unique non-split Γ-extension

$$0 \longrightarrow \sigma \longrightarrow E(\sigma,\tau) \longrightarrow \tau \longrightarrow 0$$

of τ by σ whenever it exists [BP12, Corollary 5.6]. A finite-dimensional representation of Γ is said to be *multiplicity-free* if the multiset of its Jordan-Hölder factors is multiplicity-free. For any group H, the socle and the cosocle of an H-representation π are denoted by $\mathrm{soc}_H\pi$ and $\mathrm{cosoc}_H\pi$ respectively.

Note that a weight is a smooth irreducible representation of K (resp. of KZ) and a B-character is a smooth I-character (resp. IZ-character) via the map $K \twoheadrightarrow \Gamma$ (resp. $KZ \twoheadrightarrow \Gamma$). In fact, the weights exhaust all smooth irreducible representations of K (resp. of KZ such that ϖ acts trivially). In the last section, we also talk of $\mathrm{M}(\mathcal{O}_F)$-weights for a Levi subgroup $\mathrm{M} \subseteq \mathrm{GL}_n$ which mean smooth irreducible representations of $\mathrm{M}(\mathcal{O}_F)$.

2. The spliced module

We recall some notation from [She22, §1] that is used in this section. Let $(\mathbb{Z} \pm x)^f$ be the set of f-tuples of linear polynomials in x having integral coefficients with leading coefficient ± 1. For $\boldsymbol{\lambda} = (\lambda_0(x), \dots, \lambda_{f-1}(x))$ and $\boldsymbol{\lambda}' = (\lambda'_0(x), \dots, \lambda'_{f-1}(x)) \in (\mathbb{Z} \pm x)^f$, let

$$\boldsymbol{\lambda} \circ \boldsymbol{\lambda}' := (\lambda_0(\lambda'_0(x)), \dots, \lambda_{f-1}(\lambda'_{f-1}(x))) \in (\mathbb{Z} \pm x)^f.$$

Let $\boldsymbol{\mu} \in (\mathbb{Z} \pm x)^f$ be the f-tuple of polynomials defined by

$$\begin{aligned} \mu_0(x) &:= x - 1, \\ \mu_1(x) &:= p - 2 - x, \\ \mu_j(x) &:= p - 1 - x \text{ for } 2 \le j \le f-1. \end{aligned}$$

When $f = 2$, the condition $2 \le j \le f-1$ is empty and $\boldsymbol{\mu} = (\mu_0(x), \mu_1(x)) = (x-1, p-2-x)$. Let $g \in S_f$ denote the cyclic permutation $(123\dots f)$, and let

$$\boldsymbol{\mu}^{(k)} := g^{k-1}\boldsymbol{\mu} \circ g^{k-2}\boldsymbol{\mu} \circ \dots \circ g\boldsymbol{\mu} \circ \boldsymbol{\mu} \text{ for all } 1 \le k \le l,$$

where l is equal to f (resp. $2f$) if f is odd (resp. even). We let $\boldsymbol{\mu}^{(0)} = (x, x, \dots, x)$. It follows from the definition of $\boldsymbol{\mu}^{(k)}$ that, for $1 \le k \le l$,

$$\mu_j^{(k)}(x) = \begin{cases} \mu_j^{(k-1)}(x) - 1 & \text{if } j \equiv 1-k \mod f, \\ p - 2 - \mu_j^{(k-1)}(x) & \text{if } j \equiv 2-k \mod f, \\ p - 1 - \mu_j^{(k-1)}(x) & \text{otherwise.} \end{cases} \tag{2.1}$$

Recall from [She22, Lemma 1.4 (1)] that $\boldsymbol{\mu}^{(l)} = \boldsymbol{\mu}^{(0)} = (x, x, \dots, x)$. We assign to $\boldsymbol{\mu}^{(k)}$ an element $\boldsymbol{m}^{(k)} \in (\mathbb{Z}/2\mathbb{Z})^f$ according to the rule that its j-th entry $m_j^{(k)}$ is 0 if and only if the sign of x in $\mu_j^{(k)}(x)$ is $+$.

Lemma 2.2. *(1) For all $1 \le k \le l$, $\boldsymbol{m}^{(k)} = g^k \boldsymbol{m}^{(l-k)}$.*

(2) For $1 \le k_1, k_2 \le l-1$ and $k_1 \ne k_2$, $\boldsymbol{m}^{(k_1)}$ and $\boldsymbol{m}^{(k_2)}$ are (cyclic) permutations of each other if and only if $k_2 = l - k_1$.

(3) For $1 \le k \le l-1$, $k \neq \frac{l}{2}$ if f is even, $\boldsymbol{m}^{(k)}$ is not equal to any of its non-trivial cyclic permutations.

Proof. (1) By definition, $\boldsymbol{m}^{(k)} = \sum_{i=0}^{k-1} g^i \boldsymbol{m}^{(1)}$. Since $\boldsymbol{m}^{(l)} = (0,0,\dots,0)$, we have

$$\sum_{i=0}^{l-1} g^i \boldsymbol{m}^{(1)} = (0,0,\dots,0).$$

Thus,

$$\sum_{i=0}^{k-1} g^i \boldsymbol{m}^{(1)} + g^k \sum_{i=0}^{l-k-1} g^i \boldsymbol{m}^{(1)} = (0,0,\dots,0), \text{ i.e., } \boldsymbol{m}^{(k)} + g^k \boldsymbol{m}^{(l-k)} = (0,0,\dots,0).$$

Since an element of $(\mathbb{Z}/2\mathbb{Z})^f$ is equal to its additive inverse, (1) follows.

(2) If $\boldsymbol{m}^{(k_1)}$ and $\boldsymbol{m}^{(k_2)}$ are (cyclic) permutations of each other for $1 \le k_1, k_2 \le l-1$, then the tuples $\boldsymbol{m}^{(k_1)}$ and $\boldsymbol{m}^{(k_2)}$ have the same number of 0's. When f is odd (resp. even), the number of 0's in $\boldsymbol{m}^{(k)}$ for odd k equals k (resp. k if $k \le \frac{l}{2}$ and $l-k$ if $k > \frac{l}{2}$), and the number of 0's in $\boldsymbol{m}^{(k)}$ for even k equals $l-k$ (resp. $\frac{l}{2} - k$ if $k \le \frac{l}{2}$ and $k - \frac{l}{2}$ if $k > \frac{l}{2}$). Hence, it follows that if $\boldsymbol{m}^{(k_1)}$ and $\boldsymbol{m}^{(k_2)}$ are (cyclic) permutations of each other, then either $k_1 = k_2$ or $k_2 = l - k_1$. This proves the forward implication. The converse statement follows from (1).

(3) By (1), it is enough to show (3) for $1 \le k \le f-1$. Now, (3) follows from the observation that for $1 \le k \le f-1$, the tuple $\boldsymbol{m}^{(k)}$ is a cyclic permutation of a tuple of the form k 0's followed by $(f-k)$ 1's (resp. $(f-k)$ 0's followed by k 1's) for odd (resp. even) k. □

Lemma 2.3. *$\{\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \dots, \boldsymbol{\mu}^{(l-1)}, \boldsymbol{\mu}^{(l)}, g\boldsymbol{\mu}^{(1)}, g\boldsymbol{\mu}^{(2)}, \dots, g\boldsymbol{\mu}^{(l-1)}\}$ is a set of distinct f-tuples in $(\mathbb{Z} \pm x)^f$.*

Proof. By [She22, Lemma 1.4 (2)], it is enough to prove that $\boldsymbol{\mu}^{(k_1)} \neq g\boldsymbol{\mu}^{(k_2)}$ for $1 \le k_1, k_2 \le l-1$. If $\boldsymbol{\mu}^{(k_1)} = g\boldsymbol{\mu}^{(k_2)}$ for some $1 \le k_1, k_2 \le l-1$, then we have $\boldsymbol{m}^{(k_1)} = g\boldsymbol{m}^{(k_2)}$ for the corresponding elements in $(\mathbb{Z}/2\mathbb{Z})^f$. We now find all the pairs (k_1, k_2) satisfying $\boldsymbol{m}^{(k_1)} = g\boldsymbol{m}^{(k_2)}$. If $k_1 = k_2 = k$, then $\boldsymbol{m}^{(k)} = g\boldsymbol{m}^{(k)}$. By Lemma 2.2 (3), it follows that f is even and $k = f = \frac{l}{2}$. If $k_1 \neq k_2$, we use Lemma 2.2 (1) and (2) to find that $\boldsymbol{m}^{(l-k_1)} = g^{k_1-1}\boldsymbol{m}^{(l-k_1)}$. By Lemma 2.2 (3), g^{k_1-1} must be the identity permutation. This gives $k_1 = 1$ (resp. $k_1 = 1$ or $\frac{l}{2}+1$) for odd (resp. even) f. Therefore, the pairs (k_1, k_2) satisfying $\boldsymbol{m}^{(k_1)} = g\boldsymbol{m}^{(k_2)}$ are

(1) $(1, l-1)$ if f is odd,
(2) $(1, l-1)$, $(\frac{l}{2}+1, \frac{l}{2}-1)$, $(\frac{l}{2}, \frac{l}{2})$ if f is even.

In Case (1), one checks using (2.1) that $\mu_0^{(1)}(x) = x-1 \neq x+1 = \mu_1^{(l-1)}(x)$. Thus $\boldsymbol{\mu}^{(1)} \neq g\boldsymbol{\mu}^{(l-1)}$. In Case (2), one checks using (2.1) again that $\mu_0^{(1)}(x) = x-1 \neq x+1 = \mu_1^{(l-1)}(x)$ in the first subcase, $\mu_1^{(\frac{l}{2}+1)}(x) = x+1 \neq x-1 = \mu_2^{(\frac{l}{2}-1)}(x)$ in the second subcase, and $\mu_0^{(\frac{l}{2})}(x) = p-1-x \neq p-3-x = \mu_1^{(\frac{l}{2})}(x)$ in the third subcase. □

For $\boldsymbol{\lambda} = (\lambda_0(x), \dots, \lambda_{f-1}(x)) \in (\mathbb{Z} \pm x)^f$ and $\boldsymbol{r} \in \mathbb{Z}^f$,

$$\boldsymbol{\lambda}(\boldsymbol{r}) := (\lambda_0(r_0), \lambda_1(r_1), \dots, \lambda_{f-1}(r_{f-1})) \in \mathbb{Z}^f.$$

Recall the linear polynomial $e(\boldsymbol{\lambda}) \in \mathbb{Z}[x_0, x_1, \dots, x_{f-1}]$ associated to $\boldsymbol{\lambda} \in (\mathbb{Z} \pm x)^f$ in [BP12, §2]:

$$e(\boldsymbol{\lambda})(x_0, \dots, x_{f-1}) := \begin{cases} \frac{1}{2}\left(\sum\limits_{j=0}^{f-1} p^j(x_j - \lambda_j(x_j))\right) & \text{if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1}-1\}, \\ \frac{1}{2}\left(p^f - 1 + \sum\limits_{j=0}^{f-1} p^j(x_j - \lambda_j(x_j))\right) & \text{otherwise.} \end{cases}$$

Now let $\boldsymbol{r} = (r_0, r_1, \dots, r_{f-1}) \in \mathbb{Z}^f$ such that $1 \leq r_j \leq p-3$ for all j, and consider the following generic weights of Γ

$$\sigma_k := \boldsymbol{\mu}^{(k)}(\boldsymbol{r}) \otimes \det{}^{e_k(\boldsymbol{r})} \text{ for all } 0 \leq k \leq l,$$

where

$$e_0(\boldsymbol{r}) := 0 \text{ and } e_k(\boldsymbol{r}) := \sum_{j=0}^{k-1} e(g^j\boldsymbol{\mu})(\boldsymbol{\mu}^{(j)}(\boldsymbol{r})) \text{ for all } 1 \leq k \leq l.$$

It is shown in [She22, Lemma 1.4 and Theorem 1.6] that $\sigma_l = \sigma_0 = \boldsymbol{r}$, $E(\sigma_k, \sigma_{k-1}^{[s]})$ exists for all $1 \leq k \leq l$, and $E(\sigma_k, \sigma_{k-1}^{[s]})^U = \chi(\sigma_k) \oplus \chi(\sigma_{k-1})^s$ for all $1 \leq k \leq l$. In other words, $C := \bigoplus_{k=1}^l E(\sigma_k, \sigma_{k-1}^{[s]})$ is a *cyclic module* of Γ (see [She22, Definition 1.1]). Permuting the f-tuples of σ_k's by the application of $g \in S_f$, we obtain another cyclic module of Γ. Indeed, let

$$\sigma'_k := (g\boldsymbol{\mu}^{(k)})(\boldsymbol{r}) \otimes \det{}^{e'_k(\boldsymbol{r})} \text{ for all } 0 \leq k \leq l,$$

where

$$e'_0(\boldsymbol{r}) := 0 \text{ and } e'_k(\boldsymbol{r}) := \sum_{j=0}^{k-1} e(g^{j+1}\boldsymbol{\mu})((g\boldsymbol{\mu}^{(j)})(\boldsymbol{r})) \text{ for all } 1 \leq k \leq l.$$

Lemma 2.4. *For all $1 \leq k \leq l$, $E(\sigma'_k, \sigma'^{[s]}_{k-1})$ exists, and $C' := \bigoplus_{k=1}^l E(\sigma'_k, \sigma'^{[s]}_{k-1})$ is a multiplicity-free cyclic module of Γ.*

Proof. The arguments similar to those in the proof of [She22, Lemma 1.4 (3)] show that the integer $e'_l(\boldsymbol{r})$ is independent of $\boldsymbol{r}$ and is 0 modulo $p^f - 1$. Thus $\sigma'_l = \sigma'_0 = \boldsymbol{r}$. Now the first graded piece $\mathrm{gr}^1_{\mathrm{cosoc}}(\mathrm{Ind}_B^\Gamma \chi(\sigma'_{k-1})^s)$ of the cosocle filtration of $\mathrm{Ind}_B^\Gamma \chi(\sigma'_{k-1})^s$ is

$$\bigoplus_{i=0}^{f-1} (g^i\boldsymbol{\mu})((g\boldsymbol{\mu}^{(k-1)})(\boldsymbol{r})) \otimes \det{}^{(g^i\boldsymbol{\mu})((g\boldsymbol{\mu}^{(k-1)})(\boldsymbol{r}))} \det{}^{e'_{k-1}(\boldsymbol{r})}.$$

So, $g\boldsymbol{\mu}^{(k)} = g^k\boldsymbol{\mu} \circ g\boldsymbol{\mu}^{(k-1)}$ implies that $\sigma'_k \subseteq \mathrm{gr}^1_{\mathrm{cosoc}}(\mathrm{Ind}_B^\Gamma \chi(\sigma'_{k-1})^s)$ for all $1 \leq k \leq l$. As a result, $E(\sigma'_k, \sigma'^{[s]}_{k-1})$ exists for all k, and $E(\sigma'_k, \sigma'^{[s]}_{k-1})^U = \chi(\sigma'_k) \oplus \chi(\sigma'_{k-1})^s$. As the f-tuples $\{g\boldsymbol{\mu}^{(1)}, g\boldsymbol{\mu}^{(2)}, \dots, g\boldsymbol{\mu}^{(l)}\}$ are all distinct, it follows that C' is a cyclic module. The multiplicity-freeness of C implies that C' is also multiplicity-free. $\square$

Let $\sigma := \sigma_l = \sigma'_l$ and $\sigma^{[s]} := \sigma_l^{[s]} = \sigma'^{[s]}_l$. Note that σ (resp. $\sigma^{[s]}$) occurs with multiplicity two in the socle (resp. cosocle) of $C \oplus C'$ while all the other socle (resp. cosocle) weights occur with multiplicity one by Lemma 2.3. We construct a certain subquotient of $C \oplus C'$ by splicing C and C' together along σ and $\sigma^{[s]}$. The resulting spliced module will have multiplicity-free socle and cosocle.

Let ι_σ and $\iota_{\sigma^{[s]}}$ be the compositions

$$\sigma \overset{\Delta}{\hookrightarrow} \sigma \oplus \sigma \hookrightarrow \mathrm{soc}_\Gamma\,(C \oplus C') \quad \text{and} \quad \sigma^{[s]} \overset{\Delta}{\hookrightarrow} \sigma^{[s]} \oplus \sigma^{[s]} \hookrightarrow \mathrm{cosoc}_\Gamma\,(C \oplus C') \quad \text{respectively,}$$

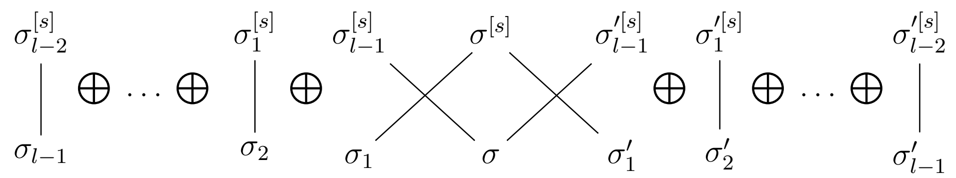
where the first map Δ in both is the diagonal embedding and the second map in both is the natural inclusion. As the cyclic modules C and C' are individually multiplicity-free (Lemma 2.4), $\sigma \notin \mathrm{cosoc}_\Gamma\,(C \oplus C')$ and $\sigma^{[s]} \notin \mathrm{soc}_\Gamma\,(C \oplus C')$. Thus, one has the following short exact sequence of Γ-modules

$$0 \longrightarrow \sigma \oplus \left(\bigoplus_{k=1}^{l-1} \sigma_k \oplus \sigma'_k \right) \longrightarrow \frac{C \oplus C'}{\iota_\sigma(\sigma)} \longrightarrow \mathrm{cosoc}_\Gamma\,(C \oplus C') \longrightarrow 0.$$

Define the spliced module D_0 to be the submodule of $\frac{C \oplus C'}{\iota_\sigma(\sigma)}$ that sits in the following short exact sequence

$$0 \longrightarrow \sigma \oplus \left(\bigoplus_{k=1}^{l-1} \sigma_k \oplus \sigma'_k \right) \longrightarrow D_0 \longrightarrow \iota_{\sigma^{[s]}}(\sigma^{[s]}) \oplus \left(\bigoplus_{k=1}^{l-1} \sigma_k^{[s]} \oplus \sigma'^{[s]}_k \right) \longrightarrow 0. \tag{2.5}$$

The Hasse diagram of the cosocle filtration of D_0 looks as follows:



Notice that D_0 is a direct sum of $2(l-2)$ non-split extensions and two indecomposable modules of length 3 shown in the middle of the above diagram. Of these two indecomposable modules, let us denote the one with socle σ by $M(\sigma)$ and the other one with cosocle $\sigma^{[s]}$ by $M(\sigma^{[s]})$. The module $M(\sigma)$ is a quotient of $E(\sigma, \sigma_{l-1}^{[s]}) \oplus E(\sigma, \sigma'^{[s]}_{l-1})$ such that the natural surjection $E(\sigma, \sigma_{l-1}^{[s]}) \oplus E(\sigma, \sigma'^{[s]}_{l-1}) \twoheadrightarrow M(\sigma)$ restricted to individual extensions is an isomorphism. Similarly, the module $M(\sigma^{[s]})$ is a submodule of $E(\sigma_1, \sigma^{[s]}) \oplus E(\sigma'_1, \sigma^{[s]})$ such that the natural maps $\frac{M(\sigma^{[s]})}{M(\sigma^{[s]}) \cap E(\sigma_1, \sigma^{[s]})} \to E(\sigma'_1, \sigma^{[s]})$ and $\frac{M(\sigma^{[s]})}{M(\sigma^{[s]}) \cap E(\sigma'_1, \sigma^{[s]})} \to E(\sigma_1, \sigma^{[s]})$ are isomorphisms.

Remark 2.6. Though the socle and the cosocle of D_0 are multiplicity-free by construction, D_0 need not be multiplicity-free. For example, when $f=2$, the weight $(p-2-r_0, r_1+1)\otimes \det^{r_0+p(p-1)}$ occurs in the socle of C as well as in the cosocle of C'.

Let $D_1 := D_0^U$, $S_1 := (\mathrm{soc}_\Gamma D_0)^U$, and $Q_1 := (\mathrm{cosoc}_\Gamma D_0)^U$. The B-representations S_1 and Q_1 are multiplicity-free, i.e., for a B-character χ, we have $\dim_{\overline{\mathbb{F}}_p} S_1^\chi \leq 1$ and $\dim_{\overline{\mathbb{F}}_p} Q_1^\chi \leq 1$.

Lemma 2.7. *As B-representations,*

$$D_1 = S_1 \oplus Q_1 = \chi(\sigma)\oplus\chi(\sigma)^s \oplus \left(\bigoplus_{k=1}^{l-1} \chi(\sigma_k)\oplus\chi(\sigma_k')\oplus\chi(\sigma_k)^s\oplus\chi(\sigma_k')^s\right).$$

Thus, for a B-character χ, $\dim_{\overline{\mathbb{F}}_p} S_1^\chi = 1$ if and only if $\dim_{\overline{\mathbb{F}}_p} Q_1^{\chi^s} = 1$.

Proof. The second part follows from the first part and the discussion before the lemma. The first part is equivalent to the claim that $\dim_{\overline{\mathbb{F}}_p} D_1 = 4l-2$ because D_0, by definition, has length $4l-2$ (2.5). Note that $\dim_{\overline{\mathbb{F}}_p}(C\oplus C')^U = 4l$ implies that $\dim_{\overline{\mathbb{F}}_p}\left(\frac{C\oplus C'}{\iota_\sigma(\sigma)}\right)^U \geq 4l-1$. However, the Γ-module $\frac{C\oplus C'}{\iota_\sigma(\sigma)}$ has length $4l-1$. Hence $\dim_{\overline{\mathbb{F}}_p}\left(\frac{C\oplus C'}{\iota_\sigma(\sigma)}\right)^U = 4l-1$. Since D_0 sits in the short exact sequence

$$0 \longrightarrow D_0 \longrightarrow \frac{C\oplus C'}{\iota_\sigma(\sigma)} \longrightarrow \sigma^{[s]} \longrightarrow 0$$

and the functor of U-invariants is left exact, we have

$$\dim_{\overline{\mathbb{F}}_p} D_1 + \dim_{\overline{\mathbb{F}}_p} \mathrm{Im}\left(\left(\frac{C\oplus C'}{\iota_\sigma(\sigma)}\right)^U \to \left(\sigma^{[s]}\right)^U\right) = \dim_{\overline{\mathbb{F}}_p}\left(\frac{C\oplus C'}{\iota_\sigma(\sigma)}\right)^U = 4l-1.$$

As $\dim_{\overline{\mathbb{F}}_p} \mathrm{Im}\left(\left(\frac{C\oplus C'}{\iota_\sigma(\sigma)}\right)^U \to \left(\sigma^{[s]}\right)^U\right) \leq 1$ and $\dim_{\overline{\mathbb{F}}_p} D_1 \leq 4l-2$, the claim follows. $\square$

Remark 2.8. We remark that one can work with any two cyclic modules of Γ arising from two different cyclic permutations of $\boldsymbol{\mu}$ to form a spliced module D_0 (see [She22, Remark 1.7]).

Remark 2.9. Recently, M. Schein [Sch23] constructed interesting *cyclic* diagrams built out of principal series to construct irreducible admissible supercuspidal representations of G with K-socles compatible with Serre's weight conjecture in the ramified setting.

3. Infinite-dimensional irreducible diagram

To construct diagrams in the sense of [BP12, §9], equip the spliced module D_0 with a smooth KZ-action via $KZ \twoheadrightarrow \Gamma$ such that ϖ acts trivially. Equip D_1 with a smooth N-action by defining the action of Π to be a linear automorphism of order 2 that maps S_1^χ to $Q_1^{\chi^s}$ for all I-characters χ such that $S_1^\chi \neq 0$ (see Lemma 2.7). This gives rise to a basic 0-diagram (D_0, D_1, can) where $\mathrm{can} : D_1 \hookrightarrow D_0$ is the canonical inclusion (see [BP12, Introduction, page 3] for the definition of a basic 0-diagram). It is easy to see that the diagram (D_0, D_1, can) is irreducible.

Let $D_0(\infty) := \bigoplus_{i\in\mathbb{Z}} D_0(i)$ be the smooth KZ-representation with component-wise KZ-action, where there is a fixed isomorphism $D_0(i) \cong D_0$ of KZ-representations for every $i \in \mathbb{Z}$. Following [Le19], we denote the natural inclusion $D_0 \xrightarrow{\sim} D_0(i) \hookrightarrow D_0(\infty)$ by ι_i, and write $v_i := \iota_i(v)$ for $v \in D_0$ for every $i \in \mathbb{Z}$. Let $D_1(\infty) := D_0(\infty)^{I(1)} \cong \bigoplus_{i\in\mathbb{Z}}(S_1 \oplus Q_1)$. We define a Π-action on $D_1(\infty)$ as follows. Let $\lambda = (\lambda_i) \in \prod_{i\in\mathbb{Z}} \overline{\mathbb{F}}_p^\times$. For all integers $i \in \mathbb{Z}$, define

$$\Pi v_i := \begin{cases} \lambda_i(\Pi v)_i & \text{if } v \in S_1^{\chi(\sigma)}, \\ (\Pi v)_{i-1} & \text{if } v \in S_1^{\chi(\sigma_1)}, \\ (\Pi v)_{i+1} & \text{if } v \in S_1^{\chi(\sigma_1')}, \\ (\Pi v)_i & \text{if } v \in S_1^{\chi} \text{ for } \chi \in \{\chi(\sigma_2), \dots, \chi(\sigma_{l-1}), \chi(\sigma_2'), \dots, \chi(\sigma_{l-1}')\}. \end{cases}$$

This uniquely determines a smooth N-action on $D_1(\infty)$ such that $\varpi = \Pi^2$ acts trivially on it. Thus we get a basic 0-diagram $D(\lambda) := (D_0(\infty), D_1(\infty), \mathrm{can})$ with the above actions where can is the canonical inclusion $D_1(\infty) \hookrightarrow D_0(\infty)$.

Proposition 3.1. *If $\lambda_i \neq \pm\lambda_0$ for all $i \neq 0$, then the basic 0-diagram $D(\lambda)$ is irreducible.*

Proof. Let $W \subseteq D_0(\infty)$ be a non-zero KZ-subrepresentation such that Π stabilizes $W^{I(1)}$. The claim is $W = D_0(\infty)$. We have $\mathrm{Hom}_K(\tau, W) \neq 0$ for some $\tau \in \mathrm{soc}_K D_0$. We first consider the case $\tau = \sigma$.

There exists a non-zero $(c_i) \in \bigoplus_{i\in\mathbb{Z}} \overline{\mathbb{F}}_p$ such that

$$\left(\sum_i c_i\iota_i\right)(\sigma) \subseteq W.$$

We pick (c_i) with $\#(c_i) := \#\{i \in \mathbb{Z} : c_i \neq 0\}$ minimal. We first show that $\#(c_i) = 1$. The Π-action on $(\sum_i c_i\iota_i)(S_1^{\chi(\sigma)})$ gives $(\sum_i \lambda_i c_i\iota_i)(Q_1^{\chi(\sigma)^s}) \subseteq W^{I(1)}$ which implies that $(\sum_i \lambda_i c_i\iota_i)(M(\sigma^{[s]})) \subseteq W$ because $M(\sigma^{[s]})$ is indecomposable. Hence

$$\left(\sum_i \lambda_i c_i\iota_i\right)(\sigma_1 \oplus \sigma_1') \subseteq W. \tag{3.2}$$

Now the Π-action on $(\sum_i \lambda_i c_i\iota_i)(S_1^{\chi(\sigma_1)})$ and $(\sum_i \lambda_i c_i\iota_i)(S_1^{\chi(\sigma_1')})$ gives respectively

$$\left(\sum_i \lambda_i c_i\iota_{i-1}\right)(Q_1^{\chi(\sigma_1)^s}) \subseteq W^{I(1)} \text{ and } \left(\sum_i \lambda_i c_i\iota_{i+1}\right)(Q_1^{\chi(\sigma_1')^s}) \subseteq W^{I(1)}.$$

Hence

$$\left(\sum_i \lambda_i c_i\iota_{i-1}\right)(E(\sigma_2, \sigma_1^{[s]})) \subseteq W \text{ and } \left(\sum_i \lambda_i c_i\iota_{i+1}\right)(E(\sigma_2', \sigma_1'^{[s]})) \subseteq W.$$

The cyclicity of the Π-action on I-characters of C and C' then gives respectively

$$\left(\sum_i \lambda_i c_i \iota_{i-1}\right)(E(\sigma_k, \sigma_{k-1}^{[s]})) \subseteq W \text{ and } \left(\sum_i \lambda_i c_i \iota_{i+1}\right)(E(\sigma'_k, \sigma'^{[s]}_{k-1})) \subseteq W \tag{3.3}$$

for all $2 \le k \le l$. Therefore

$$\left(\sum_i \lambda_i c_i \iota_{i-1}\right)(\sigma) \subseteq W \text{ and } \left(\sum_i \lambda_i c_i \iota_{i+1}\right)(\sigma) \subseteq W. \tag{3.4}$$

Thus, by increasing or decreasing the index i if needed, we may assume $c_0 \neq 0$. Now, repeating the above argument for $\left(\sum_i \lambda_i c_i \iota_{i-1}\right)(\sigma) \subseteq W$, we obtain

$$\left(\sum_i \lambda_i^2 c_i \iota_{i-2}\right)(\sigma) \subseteq W \text{ and } \left(\sum_i \lambda_i^2 c_i \iota_i\right)(\sigma) \subseteq W.$$

Note that $\left(\sum_i \lambda_0^2 c_i \iota_i\right)(\sigma) \subseteq W$. So it follows that $\left(\sum_i (\lambda_i^2 - \lambda_0^2) c_i \iota_i\right)(\sigma) \subseteq W$. Write $c'_i := (\lambda_i^2 - \lambda_0^2) c_i$ so that $\left(\sum_i c'_i \iota_i\right)(\sigma) \subseteq W$. If $\#(c_i) > 1$, then the hypothesis on (λ_i) contradicts the minimality of (c_i) because $\#(c'_i) = \#(c_i) - 1$. Therefore $\iota_0(\sigma) \subseteq W$.

Now we repeat the above argument for $\iota_0(\sigma) \subseteq W$ to show that $\iota_0(D_0) \subseteq W$. Indeed, the Π-action on $\iota_0(S_1^{\chi(\sigma)})$ gives

$$\iota_0(M(\sigma^{[s]})) \subseteq W.$$

By (3.4), we have

$$\iota_{-1}(\sigma) \subseteq W \text{ and } \iota_1(\sigma) \subseteq W.$$

Using (3.2) for the above inclusions, we obtain

$$\iota_1(\sigma_1) \subseteq W \text{ and } \iota_{-1}(\sigma'_1) \subseteq W,$$

and then using (3.3), we get

$$\iota_0(E(\sigma_k, \sigma_{k-1}^{[s]})) \subseteq W \text{ and } \iota_0(E(\sigma'_k, \sigma'^{[s]}_{k-1})) \subseteq W$$

for all $2 \le k \le l$. Together with the inclusion $\iota_0(M(\sigma^{[s]})) \subseteq W$, this gives

$$\iota_0(D_0) \subseteq W.$$

Repeat the argument for $\iota_{-1}(\sigma) \subseteq W$ and $\iota_1(\sigma) \subseteq W$ to obtain $\bigoplus_{i=0,\pm 1} \iota_i(D_0) \subseteq W$, and so on. This process eventually gives $\bigoplus_{i\in\mathbb{Z}} \iota_i(D_0) = D_0(\infty) \subseteq W$.

If $\mathrm{Hom}_K(\tau, W) \neq 0$ for $\tau \neq \sigma$, then using the cyclicity of the Π-action as above, we reduce to the case $\mathrm{Hom}_K(\sigma, W) \neq 0$. □

Remark 3.5. The main idea here to construct an infinite-dimensional irreducible diagram is to arrange the Π-action on the infinite sum of a spliced module so that the cycling on one loop increases the index and the cycling on the other decreases the index. This construction does not work for $\mathrm{GL}_2(F)$ when F has residue degree 1 because the cyclic modules of $\mathrm{GL}_2(\mathbb{F}_p)$ are principal series representations, i.e., extensions of the form $E(\tau, \tau^{[s]})$, and principal series are too small to form spliced modules with two loops.

4. Proofs of main theorems

Proof of Theorem 1.1 for $n = 2$. We first construct a desired representation π of $G = \mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$. The construction is similar to that of [Le19, Theorem 3.1] or [GS20, Theorem 1]. Let Ω be the smooth injective K-envelope of D_0 equipped with the KZ-action such that ϖ acts trivially. The smooth injective I-envelope $\mathrm{inj}_I D_1$ of D_1 is an I-direct summand of Ω. Let e denote the projection of Ω onto $\mathrm{inj}_I D_1$. There is a unique N-action on $\mathrm{inj}_I D_1$ compatible with that of I and compatible with the action of N on D_1. By [BP12], Lemma 9.6, there is a non-canonical N-action on $(1-e)(\Omega)$ extending the given I-action. This gives an N-action on Ω whose restriction to IZ is compatible with the action coming from KZ on Ω.

Let $D(\lambda) = (D_0(\infty), D_1(\infty), \mathrm{can})$ be an irreducible infinite-dimensional diagram from Proposition 3.1. Let $\Omega(\infty) := \bigoplus_{i\in\mathbb{Z}} \Omega(i)$ with component-wise KZ-action where there is a fixed isomorphism $\Omega(i) \cong \Omega$ of KZ-representations for every $i \in \mathbb{Z}$. As before, denote the natural inclusion $\Omega \xrightarrow{\sim} \Omega(i) \hookrightarrow \Omega(\infty)$ by ι_i, and write $v_i := \iota_i(v)$ for $v \in \Omega$. Let Ω_χ denote the smooth injective I-envelope of an I-character χ. We have

$$e(\Omega) = \mathrm{inj}_I D_1 = \mathrm{inj}_I S_1 \oplus \mathrm{inj}_I Q_1 = \bigoplus \Omega_{S_1^\chi} \oplus \Omega_{Q_1^{\chi^s}}.$$

If $v \in (1-e)(\Omega)$, we define $\Pi v_i := (\Pi v)_i$ for all integers i. Otherwise, we define

$$\Pi v_i := \begin{cases} \lambda_i (\Pi v)_i & \text{if } v \in \Omega_{S_1^{\chi(\sigma)}}, \\ (\Pi v)_{i-1} & \text{if } v \in \Omega_{S_1^{\chi(\sigma_1)}}, \\ (\Pi v)_{i+1} & \text{if } v \in \Omega_{S_1^{\chi(\sigma_1')}}, \\ (\Pi v)_i & \text{if } v \in \Omega_{S_1^\chi} \text{ for } \chi \in \{\chi(\sigma_2), \ldots, \chi(\sigma_{l-1}), \chi(\sigma_2'), \ldots, \chi(\sigma_{l-1}')\}. \end{cases}$$

By demanding that Π^2 acts trivially, this defines a smooth N-action on $\Omega(\infty)$ which is compatible with the N-action on $D_1(\infty)$, and whose restriction to IZ is compatible with the action coming from KZ on $\Omega(\infty)$. By [Pas04, Corollary 5.5.5], there is a smooth G-action on $\Omega(\infty)$. Take π to be the G-representation generated by $D_0(\infty)$ inside $\Omega(\infty)$. The smooth representation π has a property that $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0(\infty)$. Since $D(\lambda)$ is irreducible and $\mathrm{soc}_K D_0(\infty)$ is infinite-dimensional, it follows that π is irreducible and non-admissible.

Note that the spliced module D_0, the diagram $D(\lambda)$, and the module $\Omega(\infty)$ are all defined over the residue field $\mathbb{F}_{p^f}$ of F. Hence, if $(\lambda_i) \in \prod_{i\in\mathbb{Z}} \mathbb{F}_{p^f}^\times$, then the G-representation π has a model over π_0 over $\mathbb{F}_{p^f}$ that is absolutely irreducible and non-admissible. This gives Theorem 1.1 for $n = 2$. □

Proof of Theorem 1.2. Let π be a non-admissible irreducible representation of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$ constructed in the proof of Theorem 1.1 for $n = 2$. If we take $(\lambda_i) \in \prod_{i\in\mathbb{Z}} \overline{\mathbb{F}}_p^\times$ so that the $\mathbb{F}_{p^f}$-span of $\{\lambda_i^2\}$ is $\overline{\mathbb{F}}_p$, then the restriction of scalars of π to $\mathbb{F}_{p^f}$ is irreducible and its endomorphism algebra contains $\overline{\mathbb{F}}_p$, cf. the proof of [Le19, Theorem 1.2]. □

Proof of Theorem 1.1 for $n > 2$. Let $\mathrm{P} = \mathrm{MN}$ be the standard parabolic subgroup of GL_n with Levi subgroup $\mathrm{M} = \mathrm{GL}_2 \times (\mathrm{GL}_1)^{n-2}$. Let $\overline{\mathrm{P}} = \mathrm{M}\overline{\mathrm{N}}$ be the opposite parabolic subgroup. Let ρ be a non-admissible irreducible representation of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$ constructed in the proof of Theorem 1.1 for $n = 2$, and let χ be a character of $(F^\times)^{n-2}$. Consider the smooth irreducible non-admissible representation $\rho \otimes \chi$ of $\mathrm{M}(F)$, and let

$$\pi = \mathrm{Ind}_{\overline{\mathrm{P}}(F)}^{\mathrm{GL}_n(F)}(\rho \otimes \chi)$$

be the parabolically induced representation of $\mathrm{GL}_n(F)$. It is clear that π is non-admissible because

$$\pi^{K(1)} = \left(\mathrm{Ind}_{\overline{\mathrm{P}}(\mathcal{O}_F)}^{\mathrm{GL}_n(\mathcal{O}_F)}(\rho \otimes \chi)\right)^{K(1)} = \mathrm{Ind}_{\overline{\mathrm{P}}(\mathbb{F}_{p^f})}^{\mathrm{GL}_n(\mathbb{F}_{p^f})}\left((\rho \otimes \chi)^{M(1)}\right)$$

and the latter is not finite-dimensional. Here, $K(1) = \mathrm{Ker}(\mathrm{GL}_n(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F}_{p^f}))$ and $M(1) = \mathrm{Ker}(\mathrm{M}(\mathcal{O}_F) \twoheadrightarrow \mathrm{M}(\mathbb{F}_{p^f}))$.

Recall that for a Levi subgroup $\mathrm{L} \subseteq \mathrm{GL}_n$, an $\mathrm{L}(\mathcal{O}_F)$-weight is, by definition, a smooth irreducible representation of $\mathrm{L}(\mathcal{O}_F)$. The endomorphism algebra $\mathrm{End}_{\mathrm{L}(F)}(\text{c-Ind}_{\mathrm{L}(\mathcal{O}_F)}^{\mathrm{L}(F)}\tau)$ of the compactly induced representation $\text{c-Ind}_{\mathrm{L}(\mathcal{O}_F)}^{\mathrm{L}(F)}\tau$ of an $\mathrm{L}(\mathcal{O}_F)$-weight τ is called the spherical algebra of $\mathrm{L}(F)$ and is denoted by $\mathcal{H}_{\mathrm{L}(F)}(\tau)$. For a smooth representation V of $\mathrm{L}(F)$, an $\mathrm{L}(\mathcal{O}_F)$-weight of V simply means a smooth irreducible $\mathrm{L}(\mathcal{O}_F)$-subrepresentation of V.

Lemma 4.1. *If every $\mathrm{GL}_n(\mathcal{O}_F)$-weight of π is M-regular (in the sense of* [Her11, Definition 2.4]*), then π is irreducible.*

Proof. Let τ be a (non-zero) $\mathrm{GL}_n(\mathcal{O}_F)$-weight of π. We will show that τ generates π as a $\mathrm{GL}_n(F)$-representation. By Frobenius reciprocity, the canonical inclusion $\tau \hookrightarrow \pi\big|_{\mathrm{GL}_n(\mathcal{O}_F)}$ corresponds to an injection $\tau^{\mathrm{N}(\mathbb{F}_{p^f})} \hookrightarrow (\rho \otimes \chi)\big|_{M(\mathcal{O}_F)}$ which makes $\tau^{\mathrm{N}(\mathbb{F}_{p^f})}$ into an $\mathrm{M}(\mathcal{O}_F)$-weight of $\rho \otimes \chi$, cf. [Her11, Lemma 2.3 and (2.13)]. Let $\tau_\rho := \tau^{\mathrm{N}(\mathbb{F}_{p^f})}\big|_{\mathrm{GL}_2(\mathcal{O}_F)}$ and $\chi_0 := \chi\big|_{(\mathcal{O}_F^\times)^{n-2}}$ so that $\tau \cong \tau_\rho \otimes \chi_0$. The spherical Hecke algebra $\mathcal{H}_{\mathrm{M}(F)}(\tau^{\mathrm{N}(\mathbb{F}_{p^f})})$ of $\mathrm{M}(F)$ is isomorphic to the tensor product $\mathcal{H}_{\mathrm{GL}_2(F)}(\tau_\rho) \otimes \mathcal{H}_{(F^\times)^{n-2}}(\chi_0)$ of the spherical Hecke algebras of $\mathrm{GL}_2(F)$ and $(F^\times)^{n-2}$. The algebra $\mathcal{H}_{\mathrm{GL}_2(F)}(\tau_\rho)$ is commutative by [BL94, Proposition 8 (1)] and the algebra $\mathcal{H}_{(F^\times)^{n-2}}(\chi_0)$ is commutative by [HV15, §2.10]. Hence, the algebra $\mathcal{H}_{\mathrm{M}(F)}(\tau^{\mathrm{N}(\mathbb{F}_{p^f})})$ is commutative. Under Frobenius reciprocity, the injection $\tau^{\mathrm{N}(\mathbb{F}_{p^f})} \hookrightarrow (\rho \otimes \chi)\big|_{M(\mathcal{O}_F)}$ corresponds to a map $f \in \mathrm{Hom}_{\mathrm{M}(F)}\left(\text{c-Ind}_{\mathrm{M}(\mathcal{O}_F)}^{\mathrm{M}(F)}\tau^{\mathrm{N}(\mathbb{F}_{p^f})}, \rho \otimes \chi\right)$. We claim that f is an eigenvector for the action of $\mathcal{H}_{\mathrm{M}(F)}(\tau^{\mathrm{N}(\mathbb{F}_{p^f})})$ on $\mathrm{Hom}_{\mathrm{M}(F)}\left(\text{c-Ind}_{\mathrm{M}(\mathcal{O}_F)}^{\mathrm{M}(F)}\tau^{\mathrm{N}(\mathbb{F}_{p^f})}, \rho \otimes \chi\right)$. Indeed, the restriction of the injection $\tau^{\mathrm{N}(\mathbb{F}_{p^f})} \hookrightarrow (\rho \otimes \chi)\big|_{M(\mathcal{O}_F)}$ to $\mathrm{GL}_2(\mathcal{O}_F)$ gives a map $f_\rho \in \mathrm{Hom}_{\mathrm{GL}_2(F)}\left(\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_F)}^{\mathrm{GL}_2(F)}\tau_\rho, \rho\right)$. It is enough to show that f_ρ is an eigenvector for the action of $\mathcal{H}_{\mathrm{GL}_2(F)}(\tau_\rho)$ on $\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_F)}^{\mathrm{GL}_2(F)}\tau_\rho, \rho\right)$. The Hecke algebra $\mathcal{H}_{\mathrm{GL}_2(F)}(\tau_\rho)$ is isomorphic to the polynomial algebra $\overline{\mathbb{F}}_p[S^{\pm 1}, T]$ where the Hecke operators S and T correspond to the characteristic functions supported on $\mathrm{GL}_2(\mathcal{O}_F)\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)\mathrm{GL}_2(\mathcal{O}_F)$ and

$\mathrm{GL}_2(\mathcal{O}_F)\left(\begin{smallmatrix}1&0\\0&\varpi\end{smallmatrix}\right)\mathrm{GL}_2(\mathcal{O}_F)$ respectively. Since ρ has central character, f_ρ is an eigenvector for the operator S. We now show that $T\cdot f_\rho = f_\rho\circ T = 0$. By [Sch23, Lemma 2.1], $f_\rho(T(\tau_\rho))$ is contained in a K-subrepresentation W of ρ generated by Πv for a non-zero $v\in\tau_\rho^{I(1)}$. As ρ is constructed from a spliced module, W has length at most 3 (see the Hasse diagram). On the other hand, W naturally receives a surjection from $\mathrm{Ind}_I^K\chi(\tau_\rho)^s$ which is multiplicity-free of length at least 4 (as $f>1$) and has socle isomorphic to τ_ρ, cf. [BP12, Theorem 2.4]. Therefore τ_ρ is not a Jordan-Hölder factor of W. Hence $f_\rho(T(\tau_\rho))=0$. As f_ρ and T are G-equivariant, $T\cdot f=0$ on $\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_F)}^{\mathrm{GL}_2(F)}\tau_\rho$. This finishes the proof of the claim.

The set of eigenvalues of f gives a character $\psi:\mathcal{H}_{\mathrm{M}(F)}(\tau^{\mathrm{N}(\mathbb{F}_{p^f})})\to\overline{\mathbb{F}}_p$ and a surjective map

$$\text{c-Ind}_{\mathrm{M}(\mathcal{O}_F)}^{\mathrm{M}(F)}\tau^{\mathrm{N}(\mathbb{F}_{p^f})}\otimes_{\mathcal{H}_{\mathrm{M}(F)}(\tau^{\mathrm{N}(\mathbb{F}_{p^f})}),\psi}\overline{\mathbb{F}}_p\twoheadrightarrow\rho\otimes\chi \tag{4.2}$$

of $\mathrm{M}(F)$-representations. Further, as τ is M-regular, there is a natural isomorphism

$$\text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}\tau\otimes_{\mathcal{H}_{\mathrm{GL}_n(F)}(\tau),\psi}\overline{\mathbb{F}}_p\xrightarrow{\sim}\mathrm{Ind}_{\overline{\mathrm{P}}(F)}^{\mathrm{GL}_n(F)}\left(\text{c-Ind}_{\mathrm{M}(\mathcal{O}_F)}^{\mathrm{M}(F)}\tau^{\mathrm{N}(\mathbb{F}_{p^f})}\otimes_{\mathcal{H}_{\mathrm{M}(F)}(\tau^{\mathrm{N}(\mathbb{F}_{p^f})}),\psi}\overline{\mathbb{F}}_p\right) \tag{4.3}$$

of $\mathrm{GL}_n(F)$-representations by [Her11, Theorem 3.1]. Therefore, (4.2) and (4.3) together give a surjective map

$$\text{c-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}\tau\otimes_{\mathcal{H}_{\mathrm{GL}_n(F)}(\tau),\psi}\overline{\mathbb{F}}_p\twoheadrightarrow\pi \tag{4.4}$$

of $\mathrm{GL}_n(F)$-representations because $\mathrm{Ind}_{\overline{\mathrm{P}}(F)}^{\mathrm{GL}_n(F)}$ is exact. Since τ generates the left-hand side of (4.4) as a $\mathrm{GL}_n(F)$-representation, it also generates π as a $\mathrm{GL}_n(F)$-representation.

Now, if $\pi'\subseteq\pi$ is a non-zero subrepresentation, then π' contains a (non-zero) $\mathrm{GL}_n(\mathcal{O}_F)$-weight. By the previous paragraph, this weight generates π as a $\mathrm{GL}_n(F)$-representation. Hence $\pi'=\pi$. ☐

Lemma 4.5. *There exists a smooth character χ of $(F^\times)^{n-2}$ such that $\pi=\mathrm{Ind}_{\overline{\mathrm{P}}(F)}^{\mathrm{GL}_n(F)}(\rho\otimes\chi)$ is irreducible.*

Proof. We use the notation $F(a_1,a_2,\dots,a_n)$ in [Her09, §3.3] to denote weights. By Lemma 4.1, it suffices to show that there exists a smooth character χ of $(F^\times)^{n-2}$ such that every $\mathrm{GL}_n(\mathcal{O}_F)$-weight of π is M-regular. We pick $0\le a,b<p^f-1$ such that $a\neq b$ and a is different from all the determinant powers of weights in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_F)}\rho$. Such an a exists because there are at most $4f-1$ distinct weights in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_F)}\rho=\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_F)}D_0(\infty)$, and $p^f-1>4f-1$ for $p>3$ and $f>1$. Consider the alternating tensor product $\chi_0=F(a)\otimes F(b)\otimes F(a)\dots$ of $F(a)$ and $F(b)$ as a character of $(\mathcal{O}_F^\times)^{n-2}$, and let χ be a character of $(F^\times)^{n-2}$ such that $\chi|_{(\mathcal{O}_F^\times)^{n-2}}=\chi_0$. We claim that χ works. Indeed, let $\tau=F(a_1,\dots,a_n)$ be a $\mathrm{GL}_n(\mathcal{O}_F)$-weight of π with $p^f-1\ge a_i-a_{i+1}\ge 0$ for all i. Note that τ is M-regular if and only if $a_2,a_3,\dots,a_n$ are distinct, cf. the paragraph after [Her11, Definition 2.4]. Since $\tau^{\mathrm{N}(\mathbb{F}_{p^f})}=F(a_1,a_2)\otimes F(a_3)\otimes\ldots\otimes F(a_n)$ is an $\mathrm{M}(\mathcal{O}_F)$-weight of $\rho\otimes\chi$, we find that a_2 modulo p^f-1 is the determinant power of a weight in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_F)}\rho$, and for $i \geq 3$, $a_i \equiv a \mod p^f - 1$ (resp. $b \mod p^f - 1$) if i is odd (resp. even). By the construction of χ, we have $a_i \not\equiv a_{i+1} \mod p^f - 1$ for all $2 \leq i \leq n-1$. As the sequence $a_2, a_3, \ldots, a_n$ is decreasing, this implies that $a_i \neq a_j$ for all $2 \leq i, j \leq n$ and $i \neq j$. $\square$

We now take χ as in the proof of Lemma 4.5. Then it follows from Lemma 4.5 that $\mathrm{GL}_n(F)$ admits a smooth irreducible non-admissible representation $\pi = \mathrm{Ind}_{\overline{\mathrm{P}}(F)}^{\mathrm{GL}_n(F)}(\rho \otimes \chi)$ over $\overline{\mathbb{F}}_p$. As explained in the proof of Theorem 1.1 for $n = 2$, the $\mathrm{GL}_2(F)$-representation ρ can be chosen to have a model ρ_0 over $\mathbb{F}_{p^f}$. Then

$$\pi_0 = \mathrm{Ind}_{\overline{\mathrm{P}}(F)}^{\mathrm{GL}_n(F)}(\rho_0 \otimes \chi)$$

is a model of π over $\mathbb{F}_{p^f}$ because $\mathrm{Ind}_{\overline{\mathrm{P}}(F)}^{\mathrm{GL}_n(F)}$ commutes with scalar extension [HV19, Proposition III.12 (i)]. It is clear that π_0 is absolutely irreducible and non-admissible. $\square$

Remark 4.6. We remark that the methods of this note to construct non-admissible irreducible representations also apply to other connected split reductive groups $\mathbb{G}$ whenever $\mathbb{G}$ has GL_2 as a Levi factor, e.g., GSp_4 or G_2.

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