Values of L-Functions and Periods of Integrals

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In this article, I present a conjecture (1.8, 2.8) relating the values of certain L-functions at certain integer points to the periods of integrals.

The L-functions we consider here are those of motives - a word to which we shall not attach a precise meaning. They include, notably, Artin L-functions, L-functions attached to algebraic Hecke characters (= Grossencharacter of type A_0), and those attached to primitive holomorphic modular forms on the Poincare upper half plane (=new forms; we will consider all the L-functions, L_k, attached to the symmetric power, Sym^k, of the corresponding l-adic representation).

This article owes its existence to D. Zagier : for his insistence on a conjecture, and for the experimental confirmation that he has provided for the L-functions L_3 and L_4 attached to $\Delta = \sum \tau(n) q^n$ (see [**18**]). It is this confirmation that gave me the necessary confidence to verify that the conjecture is compatible with the results of Shimura [**13**] on the values of L-functions of algebraic Hecke characters.

0. Motives

The reader need only consult this paragraph when necessary. We review some of the formalism of motives due to Grothendieck. For the proofs, I refer the reader to [**8**].

0.1. Grothendieck's definition of *motives over a field k* is the following:

(a) Let $\mathcal{V}(k)$ be the category of smooth projective varieties over k. We construct an additive category $\mathcal{M}'(k)$, for which the groups $\mathrm{Hom}(M, N)$ are vector spaces over $\mathbf{Q}$, equipped with

α. A tensor product $\otimes$, which is associative, commutative and distributive with respect to the addition of objects ("associative" and "commutative" are not properties of the functor $\otimes$, but assumptions subject to certain compatibility restrictions - cf. Saavedra [**19**]);

β. A contravariant functor H^*, of $\mathcal{V}(k)$ in $\mathcal{M}'(k)$, bijective on the objects, taking disjoint sums to sums, and products to tensor products (assuming we are given the isomorphism of functors $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$, compatible with associativity and commutativity).

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The essential point is to define $\mathrm{Hom}(H^*(X), H^*(Y))$. For the definition, utilized by Grothendieck, of this group as a group of correspondences between X and Y, see 0.6.

(b) Recall (SGA 4 IV 7.5) that an additive category is *Karoubian* if every projector (= idempotent endomorphism) arises from a direct sum decomposition, and every additive category has a *Karoubian envelope*, obtained by formally adjoining the images of projectors. The category $\mathcal{M}_{eff}(k)$ *of effective motives over* k is the Karoubian envelope of $\mathcal{M}'(k)$.

(c) We define the *Tate motive* $\mathbf{Z}(-1)$ as an appropriate direct factor $H^2(\mathbf{P}^1)$ of $H^*(\mathbf{P}^1)$. One can verify that the symmetry (deduced from the commutativity of $\otimes$): $\mathbf{Z}(-1)\otimes\mathbf{Z}(-1) \to \mathbf{Z}(-1)\otimes\mathbf{Z}(-1)$ is the identity map, and that the functor $M \to M \otimes \mathbf{Z}(-1)$ is fully faithful.

(d) The category $\mathcal{M}(k)$ of motives over k is constructed from $\mathcal{M}_{eff}(k)$ by inverting the functor $M \to M \otimes \mathbf{Z}(-1)$.

We write (n) for the $(-n)^{th}$ iterate of the auto-equivalence $M \to M \otimes \mathbf{Z}(-1)$. The category $\mathcal{M}(k)$ admits $\mathcal{M}_{eff}(k)$ as a full sub-category, and each object of $\mathcal{M}(k)$ is of the form $M(n)$ for M in $\mathcal{M}_{eff}(k)$ and n an integer.

By definition, if F is an additive functor of $\mathcal{M}'(k)$ in a Karoubian category $\mathcal{A}$, it extends to $\mathcal{M}_{eff}(k)$. If $\mathcal{A}$ has an auto equivalence $A \to A(-1)$, and F extends to an isomorphism of functors $F(M(-1)) = F(M)(-1)$, then it extends to $\mathcal{M}(k)$.

EXAMPLE 0.1.1 If k' is an extension of k, and F the functor $H^*(X) \to H^*(X \otimes_k k')$ of $\mathcal{M}'(k)$ in $\mathcal{M}(k')$, one obtains the *extension of scalars* functor of $\mathcal{M}(k)$ in $\mathcal{M}(k')$. If k' is a finite extension of k, we can use Grothendieck's restriction of scalars functor

$$\coprod_{k'/k} : \mathcal{V}(k') \to \mathcal{V}(k) : (X \to Spec(k')) \longmapsto (X \to Spec(k') \to Spec(k)).$$

One can now extend it to $F : H^*(X) \to H^*(\coprod_{k'/k} X)$ from which we obtain the *restriction of scalars* functor $R_{k'/k} : \mathcal{M}(k') \to \mathcal{M}(k)$.

EXAMPLE 0.1.2. Let $\mathcal{H}$ be a "cohomology theory" with values in a Karoubian category $\mathcal{A}$, functorial on the morphisms in $\mathcal{M}'(k)$. The functor $\mathcal{H}$ extends to $\mathcal{M}_{eff}(k)$. If $\mathcal{A}$ has a tensor product for which $\mathcal{H}$ satisfies the Kunneth formula, and tensoring with $\mathcal{H}(\mathbf{Z}(-1))$ is an auto-equivalence of $\mathcal{A}$, it extends to $\mathcal{M}(k)$. This extension is the functor "*realization of a motive in the theory* $\mathcal{H}$".

We write (n) for the auto-equivalence of $\mathcal{A}$ given by the $(-n)^{th}$ iterate of the tensor product with $\mathcal{H}(\mathbf{Z}(-1))$. For the determination of $\mathcal{H}(\mathbf{Z}(-1))$ for various theories $\mathcal{H}$, See 3.1.

0.2. We will use the following realizations:

0.2.1. *Betti realization* H_B. Corresponding to $k = \mathbf{C}$, $\mathcal{A}$ = vector spaces over $\mathbf{Q}$, $\mathcal{H}$ = rational cohomology: $X \mapsto H^*(X(\mathbf{C}), \mathbf{Q})$;

0.2.2. *de Rham realization* H_{DR}. Corresponding to k of characteristic 0, $\mathcal{A}$ = vector spaces over k, $\mathcal{H}$ = de Rham cohomology: $X \mapsto H^*(X, \Omega^*_X)$;

0.2.3. *l-adic realization* H_l. Corresponding to k algebraically closed, of characteristic $\neq l$, $\mathcal{A}$ = vector spaces over $\mathbf{Q}_l$, $\mathcal{H}$ = l-adic cohomology: $X \mapsto H^*(X, \mathbf{Q}_l)$.

And their variants:

0.2.4. *Hodge realization.* Here k is an algebraic closure of $\mathbf{R}$ and $\mathcal{A}$ the category of vector spaces over $\mathbf{Q}$, whose complexification $V \otimes k$ has a bi-grading $V \otimes k = \oplus V^{p,q}$ such that $V^{q,p}$ is the complex conjugate of $V^{p,q}$. For a cohomology theory we take the functor $X \mapsto H^*(X(k), \mathbf{Q})$ equipped with the bi-grading of its complexification $X \to H^*(X(k), k)$ provided by Hodge theory. The Betti realization is the underlying realization of the Hodge realization.

0.2.5. For arbitrary k, and σ a complex embedding of k, we write $H_\sigma(M)$ for the Betti realization of the motive over $\mathbf{C}$ constructed from M via the extension of scalars $\sigma : k \to \mathbf{C}$. We denote complex conjugation by c. From the transport of structure we obtain the isomorphism $F_\infty : H_\sigma(M) \xrightarrow{\sim} H_{c\sigma}(M)$, and $F_\infty \otimes c$ takes H_σ^{pq} to $H_{c\sigma}^{pq}$. For real σ, F_∞ is an involution of $H_\sigma(M)$, whose scalar extension $F_\infty \otimes 1$ exchanges H_σ^{pq} and H_σ^{qp}.

The *Hodge σ-realization* is $H_\sigma(M)$, equipped with the Hodge decomposition and, if σ is real, with the involution F_∞ as well. For $k = \mathbf{Q}$, or $k = \mathbf{R}$ and σ the identity embedding, we replace the subscript σ by B. For $k = \mathbf{R}$, and $M = H^*(X)$, the involution F_∞ of $H_B^*(M) = H^*(X(\mathbf{C}), \mathbf{Q})$ is the involution induced by complex conjugation $F_\infty : X(\mathbf{C}) \to X(\mathbf{C})$.

0.2.6. *de Rham realization.* The de Rham cohomology has a natural filtration, the *Hodge filtration*, which is the limit of the hyper-cohomology spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p) \Rightarrow H_{DR}^{p+q}(X).$$

Hence, we get a filtration F on $H_{DR}(M)$.

0.2.7 *l-adic realization.* If X is a variety over a field k, $\bar{k}$ the algebraic closure of k, then the Galois group $\mathrm{Gal}(\bar{k}/k)$ acts by transport of structure on $H^*(X_{\bar{k}}, \mathbf{Q}_l)$. If M is a motive over k, and if we define $H_l(M)$ as the l-adic realization of the motive over $\bar{k}$, that one constructs via extension of scalars, then this provides an action of $\mathrm{Gal}(\bar{k}/k)$ on $H_l(M)$.

0.2.8 *Finite adelic realization.* For k algebraically closed, of characteristic 0, one may consider the l-adic cohomologies simultaneously via the adelic cohomology

$$H^*(X, \mathbf{A}^f) = (\prod H^*(X, \mathbf{Z}_l)) \otimes_Z \mathbf{Q}.$$

We can collect the variants 0.2.7 and 0.2.8.

0.2.9. In the examples 0.2.1, 0.2.2, 0.2.3, and their variants, we can replace the category $\mathcal{A}$ by $\mathcal{A}^*$, the category of the graded objects of $\mathcal{A}$, utilizing the natural grading of H^*. In order that the Kunneth formula provides an isomorphism of functors $\mathcal{H}(M \otimes N) \xrightarrow{\sim} \mathcal{H}(M) \otimes \mathcal{H}(N)$, compatible with associativity and commutativity, it is necessary to assume the commutativity in $\mathcal{A}^*$ as being that given by Koszul's rule.

0.3. For $k = \mathbf{C}$ the comparison theorem between classical cohomology and étale cohomology provides an isomorphism $H^*(X(\mathbf{C}), \mathbf{Q}) \otimes \mathbf{Q}_l \to H^*(X, \mathbf{Q}_l)$. If X is defined over $\mathbf{R}$, this isomorphism transforms F_∞ (0.2.5) into the action of complex conjugation (0.2.7). Similarly, for a motive we have $H_B(M) \otimes \mathbf{Q}_l \xrightarrow{\sim} H_l(M)$.

0.4 For $k = \mathbf{C}$, the de Rham complex, holomorphic on X^{an}, is a resolution of the constant sheaf $\mathbf{C}$. By GAGA, we therefore have an isomorphism

$$H^*(X(\mathbf{C}), \mathbf{Q}) \otimes \mathbf{C} = H^*(X(\mathbf{C}), \mathbf{C}) \xrightarrow{\sim} \mathcal{H}^*(X^{an}, \Omega^{*an}) \xleftarrow{\sim} H^*(X, \Omega^*).$$

For a motive, we obtain an isomorphism $H_B(M) \otimes \mathbf{C} \to H_{DR}(M)$ (compatible with the Hodge filtration 0.2.4: $F^p = \oplus_{p' \geq p} V^{p'q'}$ and 0.2.6).

More generally let M be a motive over a field k, and let σ be a complex embedding of k. Applying the above to the motive over $\mathbf{C}$ constructed from M by extension of scalars, we obtain an isomorphism

$$I : H_\sigma(M) \otimes \mathbf{C} \xrightarrow{\sim} H_{DR}(M) \otimes_{k,\sigma} \mathbf{C}. \tag{0.4.1}$$

Considering the case $k = \mathbf{Q}$, we find two natural $\mathbf{Q}$-structures on the complex cohomological realization of M : one $H_B(M)$, attached to the description of cohomology in terms of cycles, and the other $H_{DR}(M)$, attached to the description in terms of algebraic differential forms.

0.5 It is clear what becomes of these realizations and compatibilities when we extend the field of scalars. For the restriction of scalars, the results are as follows. Let k' be a finite extension of k, M' a motive over k', and $M = R_{k'/k}(M')$.

$$H_\sigma(M) = \oplus_\tau H_\tau(M') \tag{0.5.1}$$

where the sum is over $J(\sigma)$, the collection of complex embeddings of k' that extend σ. This isomorphism is compatible with the Hodge decompositions, and with F_∞.

$$H_{DR}(M) = H_{DR}(M') \qquad \text{(restriction of scalars from } k' \text{ to } k) \tag{0.5.2}$$

This isomorphism is compatible with the Hodge filtration.

$$H_l(M) = \mathrm{Ind}\ H_l(M') \qquad \text{(induced representation of } \mathrm{Gal}(\bar{k}/k') \text{ to } \mathrm{Gal}(\bar{k}/k)). \tag{0.5.3}$$

(0.5.4) Via the isomorphism $k' \otimes_{k,\sigma} \mathbf{C} = \mathbf{C}^{J(\sigma)}$, and the induced isomorphism $H_{DR}(M') \otimes_{k,\sigma} \mathbf{C} = \oplus_{\tau \in J(\sigma)} H_{DR}(M') \otimes_{k',\tau} \mathbf{C}$, the isomorphism (0.4.1) for M and σ is the sum of the isomorphisms (0.4.1) for M' and τ ($\tau \in J(\sigma)$):

$$\begin{array}{ccc} H_\sigma(M) \otimes \mathbf{C} & \xrightarrow{(0.4.1)} & H_{DR}(M) \otimes_{k,\sigma} \mathbf{C} \\ \Big\| (0.5.1) & & \Big\| (0.5.2) \\ \oplus_\tau H_\tau(M') \otimes \mathbf{C} & \xrightarrow{(0.4.1)} & \oplus_\tau H_{DR}(M') \otimes_{k',\tau} \mathbf{C}. \end{array}$$

0.6 For X smooth and projective over k, we let $Z^d(X)$ denote the vector space over $\mathbf{Q}$ with basis the set of irreducible closed sub-schemes of X of codimension d, and we let $Z^d_R(X)$ denote its quotient by an equivalence relation R. For k of characteristic 0, one of Grothendieck's definitions of motives is obtained by setting (for X and Y connected) $\mathrm{Hom}(H^*(Y), H^*(X)) = Z_R^{dim(Y)}(X \times Y)$, R being cohomological equivalence.

As far as characteristic p is concerned, we point out two difficulties: we don't know if cohomological equivalence - in the l-adic cohomology for $l \neq p$ - is independent of l, and the definition of the class of a cycle poses some problems in crystalline cohomology.

0.7 Let X be a smooth, projective complex variety. A *Hodge cycle* of codimension d on X is an element of $H^{2d}(X(\mathbf{C}), \mathbf{Q})$ of type (d, d), or equivalently, an element of $H^{2d}(X(\mathbf{C}), \mathbf{Q})(d)$ of type $(0, 0)$.

Let k be an algebraically closed field and X a smooth projective variety over k. Then set

$$H^{2d}(X, k \times \mathbf{A}^f)(d) = H^{2d}_{DR}(X)(d) \times H^{2d}(X, \mathbf{A}^f)(d).$$

For $k = \mathbf{C}$, the image of a Hodge cycle in

$$H^{2d}(X, k \times \mathbf{A}^f)(d) = H^{2d}(X(\mathbf{C}), \mathbf{Q})(d) \otimes (k \times \mathbf{A}^f)$$

will still be called a *Hodge cycle.* For k algebraically closed, and embeddable in the field of complex numbers, an *absolute Hodge cycle* of codimension d on X is an element of $H^{2d}(X, k \times \mathbf{A}^f)(d)$, such that for every complex embedding σ of k, its image in $H^{2d}(X \otimes_k \mathbf{C}, \mathbf{C} \times \mathbf{A}^f)(d)$ is a Hodge cycle. One can verify

PROPOSITION 0.8 (i) *The vector space over* $\mathbf{Q}$ *of absolute Hodge cycles* $Z^d_{ha}(X)$ *is invariant under the extension of scalars of* k *to an algebraically closed field* k' *(also embeddable in the field of complex numbers).*

(ii) *For* k *algebraically closed of characteristic 0, and* X *defined over an algebraically closed subfield* k_0 *of* k*, admitting a complex embedding* : $X = X_0 \otimes_{k_0} k$, *let*

$$Z^d_{ha}(X) = Z^d_{ha}(X_0) \subset H^{2d}(X_0, k_0 \times \mathbf{A}^f)(d) \subset H^{2d}(X, k \times \mathbf{A}^f)(d).$$

By (i), *this definition does not depend on the choices of* X_0 *and* k_0.

(iii) *For* X *defined over a subfield* k_0 *of* k : $X = X_0 \otimes_{k_0} k$, *the group* $\mathrm{Aut}(k/k_0)$ *acting on* $H^{2d}(X, k \times \mathbf{A}^f)(d)$ *stabilizes* $Z^d_{ha}(X)$. *It acts on* $Z^d_{ha}(X)$ *through a finite quotient, corresponding to a finite extension* k'_0 *of* k_0. *We set* $Z^d_{ha}(X_0) = Z^d_{ha}(X)^{\mathrm{Aut}(k/k_0)}$.

0.9 A useful notion of a motive is obtained by setting (for X and Y connected)

$$\mathrm{Hom}\ (H^*(Y), H^*(X)) = Z^{dim(Y)}_{ha}(X \times Y).$$

The Kunneth components of the diagonal of $X \times X$ are absolute Hodge. This allows us to decompose $H^*(X)$ as a sum of motives $H^i(X)$, to equip the category of motives with a grading (with $H^i(X)$ of weight i) and to modify the commutativity constraint for $\otimes$ as in [**19**, VI, 4.2.1.4]. This being done, one can verify that the $\otimes$-category of motives over k is Tannakian, and is isomorphic to the category of representations of a proalgebraic reductive group.

For $k = \mathbf{C}$, the following conjecture, weaker than that of Hodge, is equivalent to saying that the functor "Hodge realization" is an equivalence of the category of motives 0.9 with the category of the direct factor Hodge structures of the cohomology of an algebraic variety (or constructed from the Tate twist of such direct factors).

Hope 0.10. Every Hodge cycle is an absolute Hodge cycle.

If X is an abelian variety with complex multiplication by an imaginary quadratic field K, with $\mathrm{Lie}(X)$ free over $k \otimes K$, the methods of B. Gross [**7**] prove that certain non trivial Hodge cycles are absolute Hodge. More generally, I have verified 0.10 for abelian varieties. I have also verified that the $\otimes$-category of motives (0.9) generated by the $H^*(X)$, for X an abelian variety, contains the motives $H^*(X)$, for X a K3 surface or a Fermat hypersurface.

0.11. The principal weakness of the definition of a motive 0.9 is that it does not lend itself to reduction mod p. We do not know if a motive over a number field F (as in 0.9) provides a compatible system of l-adic representations of $\mathrm{Gal}(\bar{F}/F)$.

0.12. We will use the word "motive" loosely without worrying about the framework of motives considered by Grothendieck. What is essential for us are the realizations $\mathcal{H}(M)$, for the theories $\mathcal{H}$ considered in 0.2, and to have for these groups the same formalism as for $\mathcal{H}^*(X)$.

1. Statement of the Conjecture (Rational Case)

1.1. Let M be a motive over $\mathbf{Q}$. We assume that the l-adic realizations of M, $H_l(M)$, form a strictly compatible system of l-adic representations, in the sense of Serre [**11**, I.11]. That is to say : there exists a finite set S of prime numbers, so that each $H_l(M)$ is unramified outside of $S \cup \{l\}$, and that, denoting the geometric Frobenius at p (the inverse of the Frobenius substitution ϕ_p) by F_p, the polynomial $\det(1-F_pt, H_l(M)) \in \mathbf{Q}_l[t]$ with $p \notin S \cup \{l\}$ has rational coefficients and is independent of l. We let $Z_p(M,t)$ be its inverse, and set $L_p(M,s) = Z_p(M,p^{-s})$.

The Dirichlet series with rational coefficients given by the Euler product $L_S(M,s) = \prod_{p \notin S} L_p(M,s)$ converges for $\mathcal{R}(s)$ large. For arbitrary s, we define $L_S(M,s)$ via an analytical continuation (which we hope exists).

Our goal is to state a conjecture giving the values of $L_S(M,s)$ at certain integers, up to multiplication by a rational number. Since p^{-s} is rational, for s an integer, the choice of S is unimportant - except that enlarging S may produce unwanted zeros.

1.2 To write the proper conjectural functional equation of these L-functions, we need to complete the Euler product by specifying the local factors at $p \in S$, and at infinity. The definition of the local factors at $p \in S$ requires an additional hypothesis, which we suppose has been verified:

(1.2.1) Let p be a prime number, $D_p \subset \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ the decomposition group at p, $I_p \subset D_p$ the inertia subgroup at p, and $F_p \in D_p$ a geometric Frobenius. The polynomial $\det(1 - F_pt, H_l(M)^{I_p}) \in \mathbf{Q}_l[t]$ with $l \neq p$ has rational coefficients, and is independent of l.

We set $Z_p(M,t) = \det(1 - F_pt, H_l(M)^{I_p})^{-1} \in \mathbf{Q}(t)$, and $L_p(M,s) = Z_p(M,p^{-s})$. We define

$$L(M,s) = \prod_p L_p(M,s). \tag{1.2.2}$$

The factor at infinity $L_\infty(M,s)$ (essentially a product of Γ functions) depends on the Hodge realization of M - in fact only on the isomorphism class of the complex vector space $H_B(M) \otimes \mathbf{C}$, equipped with the Hodge decomposition and the involution F_∞. The exact definition is given in 5.2.

Setting $\Lambda(M,s) = L_\infty(M,s)L(M,s)$, the conjectural functional equation can be written as

$$\Lambda(M,s) = \epsilon(M,s)\Lambda(\check{M}, 1-s) \tag{1.2.3}$$

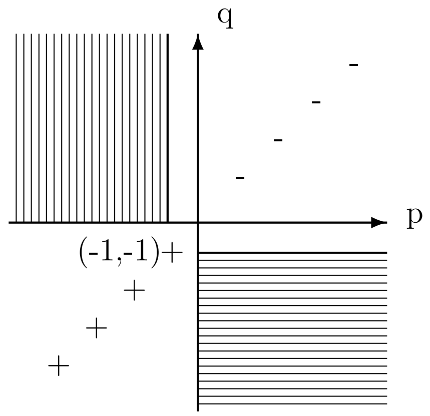
where $\check{M}$ is the dual of M (with realizations the duals of the realizations of M) and where $\epsilon(M,s)$, considered as a function of s, is the product of a constant and an exponential function. The exact definition is given in 5.2. It depends on an additional hypothesis.

DEFINITION 1.3 *An integer n is critical for M if neither $L_\infty(M,s)$ nor $L_\infty(\check{M}, 1-s)$ has a pole at $s = n$.*

Our goal is to conjecture the value of $L(M,n)$, for n critical, up to multiplication by a rational number. We have $L(M(n),s) = L(M,n+s)$ (3.1.2), and the same for L_∞. This allows us to consider only $L(M) =_{def} L(M,0)$. We say that M is *critical* if 0 is critical for M. One may verify that for M to be critical, it is necessary and sufficient that the Hodge numbers $h_{pq} =_{def} \dim H^{pq}(M)$ of M should satisfy the following conditions:

1. For all (p,q) with $p \neq q$, $h_{pq} \neq 0$ only when (p,q) lies in the shaded part of the diagram below

2. The action of F_∞ on H^{pp} is trivial if $p < 0$ and -1 if $p \geq 0$.



Suppose M is homogeneous of weight w, so that $h^{pq} = 0$ for $p+q \neq w$. Set $\mathcal{R}(M) = -w/2$. It follows from Weil's conjecture that, for S sufficiently large, the Dirichlet series $L_S(M,s)$ converges absolutely for $\mathcal{R}(M) + \mathcal{R}(s) > 1$. For $\mathcal{R}(M) + \mathcal{R}(s) = 1$, which is the border of the half plane of convergence, we conjecture that

(a) $L_S(M,s)$ does not vanish, and that

(b) $L_S(M,s)$ is holomorphic, except if M is of even weight $-2n$, and contains $\mathbf{Z}(n)$ as a factor : in this case we expect a pole at $s = 1-n$ (a non-critical value).

The analogy with the function field case, and known cases, leads us to believe that the local factors $L_p(M,s)$ (p arbitrary, including ∞) only have poles for $\mathcal{R}(M) + \mathcal{R}(s) \leq 0$. If this is the case, then (a), (b) and the conjectural functional equation (1.2.3) imply that $L(M) \neq 0$ or ∞, for M critical and $\mathcal{R}(M) \neq \frac{1}{2}$. For $\mathcal{R}(M) = \frac{1}{2}$, $L(M)$ vanishes sometimes. Our conjecture is therefore empty.

PROPOSITION 1.4. *Let M be a motive over $\mathbf{R}$. Via the isomorphism* (0.4.1)

$$H_B(M) \otimes \mathbf{C} \xrightarrow{\sim} H_{DR}(M) \otimes_{\mathbf{R}} \mathbf{C},$$

$H_{DR}(M)$ can be identified with the subspace of $H_B(M) \otimes \mathbf{C}$ fixed by $c \to F_\infty \bar{c}$.

Let us take $M = H^*(X)$. Complex conjugation on $H_{DR}(M)_{\mathbf{C}} = H^*(X_{\mathbf{C}}, \Omega^*)$ is induced by the functoriality of the anti-linear automorphism F_∞ of the scheme $X_{\mathbf{C}}$. Using the map I in (0.4) we identify complex conjugation with the involution induced by the automorphism $(F_\infty, \bar{F}_\infty^*)$ of $(X(\mathbf{C}), \Omega^*)$, then with the composition of $\bar{F}_\infty^* : H^*(X(\mathbf{C}), \mathbf{C}) \to H^*(X(\mathbf{C}), \mathbf{C})$ and with complex conjugation on the coefficients. This verifies 1.4.

1.5. For M a motive over $\mathbf{R}$, we let $H_B^+(M)$ (respectively $H_B^-(M)$) be the subspace of $H_B(M)$ fixed by F_∞ (respectively $F_\infty = -1$). Let $d(M) = \dim\, H_B(M)$ and $d^\pm(M) = \dim\, H_B^\pm(M)$. For M a motive over k, and σ a complex embedding of k that factors over $\mathbf{R}$, we let $H_\sigma^\pm(M)$, $d_\sigma^\pm(M)$ and $d(M)$ be the corresponding objects for the motive over $\mathbf{R}$ constructed from M by σ. For $k = \mathbf{Q}$, we omit the σ.

The following corollary follows immediately from 1.4, and from the fact that F_∞ exchanges H^{pq} and H^{qp}.

COROLLARY 1.6. *Let M be a motive over $\mathbf{R}$. For the real structure $H_{DR}(M)$ of $H_B(M) \otimes \mathbf{C}$, the sub-spaces H^{pq} are defined over $\mathbf{R}$; the sub-space $H_B^+(M) \otimes \mathbf{R}$ is real,* and *$H_B^-(M) \otimes \mathbf{R}$ is purely imaginary.*

1.7. In the final part of this paragraph, we only consider motives over $\mathbf{Q}$; unless otherwise stated, we suppose that they are homogeneous. If their weight w is even, we also assume that F_∞ acts on H^{pp} (with $w = 2p$) like the scalar 1 or -1. This hypothesis has been verified for M critical. Since F_∞ exchanges H^{pq} and H^{qp}, we know that for the dimensions $d^+(M)$ and $d^-(M)$, one is equal to $\sum_{p>q} h^{pq}$, and the other to $\sum_{p\geq q} h^{pq}$. In particular these dimensions are equal to those of subspaces F^+ and F^- appearing in the Hodge filtration. We set $H_{DR}^\pm(M) = H_{DR}(M)/F^\mp$, and we have again $\dim\, H_{DR}^\pm(M) = d^\pm(M)$

Since F_∞ exchanges H^{pq} and H^{qp}, we deduce that the following composition of maps

$$I^\pm : H_B^\pm(M)_{\mathbf{C}} \to H_B(M)_{\mathbf{C}} \xrightarrow{I} H_{DR}(M)_{\mathbf{C}} \to H_{DR}^\pm(M)_{\mathbf{C}} \tag{1.7.1}$$

are isomorphisms. We set

$$c^\pm(M) = \det\, (I^\pm), \tag{1.7.2}$$

$$\delta(M) = \det\, (I), \tag{1.7.3}$$

the determinants being calculated in rational bases of $H_B^\pm(M)$ and $H_{DR}^\pm$ (respectively H_B and H_{DR}). The definition of $\delta(M)$ does not require the hypotheses on M.

By 1.6, I^+ is real, i.e. induces $I^+ : H_B^+(M)_{\mathbf{R}} \to H_{DR}^+(M)_{\mathbf{R}}$, and likewise I^- is purely imaginary. Therefore the numbers $c^+(M), i^{d^-(M)}c^-(M)$ and $i^{d^-(M)}\delta(M)$ are real and non-zero. Up to multiplication by a rational number, these numbers only depend on M.

Classically the *periods* of M are $< \omega, c >$ where $\omega \in H_{DR}(M)$ and $c \in H_B(M)^\vee$. For example, if X is an algebraic variety over $\mathbf{Q}$, ω an n-form on X defined over $\mathbf{Q}$ and c an n-cycle on $X(\mathbf{C})$, then $< \omega, c >= \int_c \omega$ is a period of $H^n(X)$. Let us express $c^\pm(M)$ in terms of periods. The dual of $H_{DR}^\pm(M)$ is the subspace $F^\pm$ of $H_{DR}(M^\vee)$, where $M^\vee$ is the dual motive of M. If the chosen basis for $H_{DR}^\pm(M)$ has for dual basis the basis (ω_i) of $F^\pm(H_{DR}(M^\vee)$, and (c_i) is the chosen basis for $H_B^\pm(M)$, then the matrix of $I^\pm$ is $< \omega_i, c_j >$ and $c^\pm(M) = \det\, (< \omega_i, c_j >)$.

Conjecture 1.8. *If M is critical, $L(M)$ is a rational multiple of $c^+(M)$.*

2. Statement of the Conjecture (General Case)

2.1. We have considered L-functions given by Dirichlet series with rational coefficients. To improve on this, we now consider motives with coefficients over a number field.

Here are two equivalent methods for constructing the category of motives over k, with coefficients in a number field E, from the category of motives over k. We must provide a valid construction for all additive Karoubian categories (0.1(b)) in which the $\mathrm{Hom}(X, Y)$ are vector spaces over $\mathbf{Q}$.

A. A motive over k, with coefficients in E, is a motive M over k equipped with the structure of an E-module: $E \to \mathrm{End}(M)$.

B. The category $\mathcal{M}_{k,E}$ of motives over k with coefficients in E, is the Karoubian envelope (cf. 0.1(b)) of the category of motives over k - where we denote an object M in $\mathcal{M}_{k,E}$ by M_E, and the morphisms are given by $\mathrm{Hom}(X_E, Y_E) = \mathrm{Hom}(X, Y) \otimes E$.

Passage from B *to* A. For a motive X, and V a finite dimensional vector space over $\mathbf{Q}$, we let $X \otimes V$ denote the motive, isomorphic to a sum of $\dim(V)$ copies of X, characterized by $\mathrm{Hom}(Y, X \otimes V) = \mathrm{Hom}(Y, X) \otimes V$ (or by $\mathrm{Hom}(X \otimes V, Y) = \mathrm{Hom}(V, \mathrm{Hom}(X, Y))$). We pass from B to A by associating to M_E the motive $M \otimes E$, with its natural E-module structure.

Passage from A *to* B. If M has an E-module structure, we recover two E-module structures on M_E: one coming from M, and the other by virtue of being an object in $\mathcal{M}_{k,E}$. The object of $\mathcal{M}_{k,E}$ corresponding to M is the largest direct factor of M_E on which the structures coincide. In more detail: the algebra $E \otimes E$ is a product of fields, among which there is a copy of E in which $1 \otimes x$ and $x \otimes 1$, both project as x. The corresponding idempotent e acts on M_E (which is an $E \otimes E$-module) and its image is the object of $\mathcal{M}_{k,E}$ that corresponds to M.

Motives with coefficients in E are most often given in form A. Form B has the advantage of making sense for E not of finite rank over $\mathbf{Q}$. The latter form is useful for understanding the tensor formalism: one may define the tensor product and dual, for motives with coefficients in E, by their functoriality and the formulas $X_E \otimes_E Y_E = (X \otimes Y)_E$ and $(X_E)^\vee = (X^\vee)_E$. In language A, $X \otimes_E Y$ is the largest direct summand of $X \otimes Y$ for which the two E-module structures of $X \otimes Y$ coincide, and X is the usual dual of $X^\vee$, equipped with the transposed E-module structure. If we apply these remarks to the category of vector spaces over $\mathbf{Q}$, rather than that of motives, we find the isomorphism of the F-dual of a vector space over E with its $\mathbf{Q}$-dual, given by

$$\omega \mapsto \text{the form } \mathrm{Tr}_{E/\mathbf{Q}}(< \omega, v >).$$

In 0.1.1 we defined some restriction and extension of scalars functors of the field of scalars k. They transform motives with coefficients in E into other motives with coefficients in E. We also have at our disposal the restriction and extension of coefficients functors: let F be a finite extension of E:

Extension of coefficients. In language A, it is $X \mapsto X \otimes_E F$; in language B it is $X_E \mapsto X_F$.

Restriction of coefficients. In language A, we restrict the F-module structure to E.

The reader should take care not to confuse the roles of k and E. A typical example we should remember is that of H_1 of an abelian variety over k, with complex multiplication by an order in E. In terms of the abelian variety - up to an isogeny - the above functors become: extension of the base field, Weil's restriction of scalars, construction of $\otimes_E F$, which multiplies the dimension by $[F : E]$, and restriction to E of the F-module structure.

2.2 Let M be a motive over $\mathbf{Q}$, with coefficients in a number field E. For each prime number l, the l-adic realization $H_l(M)$ of M is a module over the l-adic completion E_l of E. This completion is the product of the completions E_λ, where the λ are prime ideals lying over l, from which we have a decomposition of $H_l(M)$ as a product of E_λ-modules $H_\lambda(M)$.

For the $H_\lambda(M)$ we conjecture a compatibility analogous to 1.2.1; if it is verified, we can define, for each complex embedding σ of E a Dirichlet series with coefficients in σE, converging for $\mathcal{R}(s)$ sufficiently large:

$$L(\sigma, M, s) = \prod_p L_p(\sigma, M, s), \quad \text{where}$$

(2.2.1) $$L_p(\sigma, M, s) = \sigma Z_p(M, p^{-s}) \quad \text{with} \quad Z_p \in E(t) \subset E_\lambda(t) \quad \text{given by}$$

$$Z_p(M, t) = \det(1 - F_p t, H_\lambda(M)^{I_p})^{-1} \quad \text{for } \lambda \not| p.$$

As σ varies these Dirichlet series differ from one another by conjugation of the coefficients. We will regard the system of $L(\sigma, M, s)$ as a single function $L^*(M, s)$ with values in the $\mathbf{C}$-algebra $E \otimes \mathbf{C}$, which we identify with $\mathbf{C}^{\mathrm{Hom}(E,\mathbf{C})}$ via

(2.2.2) $$E \otimes \mathbf{C} \xrightarrow{\sim} \mathbf{C}^{\mathrm{Hom}(E,\mathbf{C})} : e \otimes z \mapsto (z \cdot \sigma(e))_\sigma.$$

This function can also be defined directly by an Euler product. We hope, as in 1.1, that it has an analytical continuation to all $s \in \mathbf{C}$.

It is necessary to complete the Euler product $L(\sigma, M, s)$ with a factor at infinity $L_\infty(\sigma, M, s)$ which depends on the Hodge realization of M. Setting $\Lambda(\sigma, M, s) = L_\infty(\sigma, M, s)L(\sigma, M, s)$, the conjectural functional equation of the L-function can the be written as

(2.2.3) $$\Lambda(\sigma, M, s) = \epsilon(\sigma, M, s)\Lambda(\sigma, M^\vee, 1 - s),$$

where $\epsilon(\sigma, M, s)$, as a function of s, is the product of a constant and an exponential function. The definitions of L_∞ and ϵ are given in 5.2. As above, we regard the system of Λ and ϵ, as σ varies, as functions Λ^* and ϵ^* with values in $E \otimes \mathbf{C}$.

It follows from 2.5 below that $L_\infty(\sigma, M, s)$ is independent of σ, and that the function L_∞, for the motive $R_{E/\mathbf{Q}}M$ constructed from M by restriction of scalars of the field of coefficients (2.1) is the $[E : \mathbf{Q}]^{th}$ power of $L_\infty(\sigma, M, s)$. This justifies the

PROPOSITION-DEFINITION 2.3 *Let M be a motive over $\mathbf{Q}$ with coefficients in E. An integer n is critical for M if the following equivalent conditions hold*

(i) *The integer n is critical for $R_{E/\mathbf{Q}}M$;*

(ii) *Neither $L_\infty(\sigma, M, s)$ nor $L_\infty(\sigma, \check{M}(1), s)$ has a pole at $s = n$.*

We say that M is critical *if 0 is critical for M.*

Our goal is to conjecture the value of $L^*(M) =_{def} L^*(M, 0)$, for M critical, up to multiplication by an element of E. In other words, we would like to simultaneously conjecture the values of $L(\sigma, M) =_{def} L(\sigma, M, 0)$ up to multiplication by a system of numbers $\sigma(e)$, $e \in E$.

2.4. The realization $H_B(M)$ of M in rational cohomology is equipped with an E-vector space structure. Its dimension is the *rank* of M over E. The involution F_∞ is E-linear, therefore the $+$ and $-$ eigenspaces are E-subspaces. We denote their dimensions by $d^+(M)$ and $d^-(M)$.

The complexification of $H_B(M)$ is a free $E\otimes\mathbf{C}$-module. Identifying $E\otimes\mathbf{C}$ with $\mathbf{C}^{\mathrm{Hom}(E,\mathbf{C})}$ (2.2.2), we have the decomposition

$$H_B(M)\otimes\mathbf{C}=\bigoplus_\sigma H_B(\sigma,M),$$

with

$$H_B(\sigma,M)=(H_B(M)\otimes\mathbf{C})\otimes_{E\otimes\mathbf{C},\sigma}\mathbf{C},$$

or

$$H_B(\sigma,M)=H_B(M)\otimes_{E,\sigma}\mathbf{C}.$$

Since the $H_B^{pq}(M)$ of the Hodge decomposition are stable under E, each $H_B(\sigma,M)$ inherits a Hodge decomposition $H_B(\sigma,M)=\oplus H_B^{pq}(\sigma,M)$. The involution F_∞ permutes the H^{pq} and H^{qp}, in particular it stabilizes H^{pp}, dividing it into $+$ and $-$ parts. We write $h^{pq}(\sigma,M)$ for the dimension of $H_B^{pq}(\sigma,M)$ and $h^{pp\pm}(\sigma,M)$ for the dimension of $H_B^{pp\pm}(\sigma,M)$. The following proposition permits us to omit σ from the notation.

PROPOSITION 2.5 *The numbers $h^{pq}(\sigma,M)$ and $h^{pp\pm}(\sigma,M)$ are independent of σ.*

We may suppose, and do suppose, that M is homogeneous. In this case, $H^{pq}(M)$ can be identified with the complexification of the E-vector space $\mathrm{Gr}_F^p(H_{DR}(M))$: it is a free $E\otimes\mathbf{C}$-module and the first assertion follows. For the second we observe that $h^{pp\pm}(\sigma,M)$ is the excess of $d^\pm(M)$ over $\sum_{p>q}h^{pq}(\sigma,M)$.

2.6. Let M be a motive over $\mathbf{Q}$ with coefficients in E. For the rest of this section, unless stated otherwise, we suppose that $R_{E/\mathbf{Q}}M$ verifies the hypotheses of 1.7. This time the spaces $F^\pm$ and $H_{DR}^\pm$ are vector spaces over E. The isomorphisms (0.4.1) and (1.7.1)

$$I:H_B(M)\otimes\mathbf{C}\to H_{DR}(M)\otimes\mathbf{C}, \tag{2.6.1}$$

$$I^\pm:H_B^\pm(M)\otimes\mathbf{C}\to H_{DR}^\pm(M)\otimes\mathbf{C}, \tag{2.6.2}$$

are isomorphisms of $E\otimes\mathbf{C}$-modules between the complexifications of vector spaces over E. We set

$$c^\pm(M)=\det\,(I^\pm)\in(E\otimes\mathbf{C})^*,$$

$$\delta(M)=\det\,(I)\in(E\otimes\mathbf{C})^*,$$

the determinants being calculated in some E-rational bases of $H_B^\pm(M)$ and $H_{DR}^\pm(M)$ (respectively $H_B(M)$ and $H_{DR}(M)$). The definition of $\delta(M)$ does not require the hypotheses made on M. Up to multiplication by an element of E^*, these numbers depend only on M. It follows anew from 1.6 that $c^+(M)$, $i^{d^-(M)}c^-(M)$ and $i^{d^-(M)}\delta(M)$ are in $(E\otimes\mathbf{R})^*$

Conjecture 2.7. *Let $\mathcal{R}(M)=-\frac{1}{2}w$. If M is critical,*

(i) *$L(\sigma, M, s)$ never has a pole at $s = 0$, and can only vanish at $s = 0$ when $\mathcal{R}(M) = 1/2$.*
(ii) *The multiplicity of the zero of $L(\sigma, M, s)$ at $s = 0$ is independent of σ.*

For (i), I refer you to the discussion at the end of 1.3. That (ii) should seem reasonable was suggested to me by B. Gross.

Conjecture 2.8. *For M critical and $L(\sigma, M) \neq 0$, $L^*(M)$ is the product of $c^+(M)$ and an element of E^*.*

REMARK 2.9. A motive M over a number field k, with coefficients in E, also defines a function $L^*(M, s)$. These functions are covered by our conjecture, in view of the identity $L^*(M, s) = L^*(R_{k/\mathbf{Q}}M, s)$, where $R_{k/\mathbf{Q}}$ is the restriction of scalars from k to $\mathbf{Q}$ (see (0.5.3)).

The functions $L(\sigma, M, s)$ are Euler products indexed by the finite places of k. It is necessary to complete these with the factors at infinity $L_\nu(\sigma, M, s)$, indexed by the infinite places, the definitions of which are given in 5.2. In general, they depend on σ. Only $L_\infty(\sigma, R_{k/\mathbf{Q}}M, s)$, the product of all the $L_\nu(\sigma, M, s)$ for ν infinite, is independent of σ.

REMARK 2.10. Let F be an extension of E, ι the structure morphism of E in F and $\iota_{\mathbf{C}}$ its complexification: $E \otimes \mathbf{C} \hookrightarrow F \otimes \mathbf{C}$. We have $L^*(M \otimes_E F, s) = \iota_{\mathbf{C}} L^*(M, s)$ and $c^\pm(M \otimes_E F) = \iota_{\mathbf{C}} c^\pm(M)$. Therefore, the conjecture is compatible with the extension of the field of coefficients. For a Galois extension F of E, with Galois group G, Hilbert's Theorem 90, $H^1(G, F^*) = 0$ assures us that $((F \otimes \mathbf{C})^*/F^*)^G = (E \otimes \mathbf{C})^*/E^*$: so the conjecture is invariant under the extension of the field of coefficients.

REMARK 2.11. If E is an extension of F, we have $L^*(R_{E/F}M, s) = N_{E/F}L^*(M, s)$. Since the periods $c^\pm$ verify the same identity, the conjecture is compatible with the restriction of coefficients.

REMARK 2.12. Let D be a division algebra over E of rank d^2. A motive M over $\mathbf{Q}$, having the structure of a D-module, and of rank n over D, defines a Dirichlet series with coefficients in E, whose Euler factors are almost all of degree nd: for λ a finite place of E, $H_\lambda(M)$ is a free module over the completion $D_\lambda = D \otimes_E E_\lambda$, and we repeat the definition (2.2.1) by setting

$$Z_p(M, t) = \det \operatorname{red}(1 - F_p t, H_\lambda(M)^{I_p})^{-1}$$

(if D_λ is an algebra of matrices over E_λ, and e an indecomposable idempotent, the reduced determinant of an endomorphism A of a D_λ-module H is the determinant, calculated over E, of the restriction of A to eH; the definition in the general case proceeds by descent).

The liberty which 2.10 gives us, to extend the field of coefficients, makes it possible to include these L-functions in Conjecture 2.8: choosing an extension $\iota : E \hookrightarrow F$ of E that neutralizes D, and an indecomposable idempotent e of $D \otimes_E F$ we have $\iota_{\mathbf{C}} L^*(M, s) = L^*(e(M \otimes_E F), s)$.

We may also define $c^+(M)$ directly, as is explained in 2.13 below.

The rest of this section will not be used in the rest of the article.

2.13. Let D be a simple algebra over a field E (for instance an algebra of Azumaya over a ring...). For the tensor formalism, it is convenient to regard D-modules as "fake E-vector spaces":

(a) Given a vector space V over E reduces to being given, for every étale extension F of E, an F-vector space V_F, and some compatible isomorphisms $V_G = V_F \otimes_F G$ for G an extension of F. We take $V_F = V \otimes_E F$. By descent it suffices to only consider the V_F for F sufficiently large.

(b) Let W be a D-module. For every étale extension F of E, and every F-isomorphism $D \otimes F \sim \mathrm{End}_F(L)$, with L free, we set $W_{F,L} = \mathrm{Hom}_{D\otimes F}(L, W \otimes F)$ (the tensor products being over E). We have a $D \otimes F$-module module isomorphism $W \otimes_E F = L \otimes_F W_{F,L}$.

If F is sufficiently big to neutralize D, L is unique up to a non-unique isomorphism; the lack of uniqueness is due to homotheties, that act trivially on $\mathrm{End}(L)$. This is why the given $W_{F,L}$ is not of type (a); it is a "fake vector space over E".

Let (W^α) be a family of D-modules, and T a tensor operation. If the homotheties of L act trivially on $T(W^\alpha_{F,L})$, the F-vector space $T(W^\alpha_{F,L})$ is independent of the choice of L and we obtain a system of type (a), from which we get a vector space $T(W^\alpha)$ over E.

EXAMPLE. If W' and W'' are two D-modules of rank $n \cdot [D : E]^{1/2}$ over E, we can take $T = \mathrm{Hom}(\bigwedge^n W'_{F,L}, \bigwedge^n W''_{F,L})$; we obtain a vector space $\delta(W', W'')$ of rank 1 over E, and every homomorphism $f : W' \to W''$ has a reduced determinant $\det \mathrm{red}(f) \in \delta(W', W'')$.

To define $c^+(M)$, we apply this construction to the D-modules $H^+_B(M)$ and to $H^+_{DR}(M)$. We set $\delta = \delta(H^+_B(M), H^+_{DR}(M))$. The reduced determinant of the isomorphism of $D \otimes \mathbf{C}$-modules $I^+ : H^+_B(M)_{\mathbf{C}} \xrightarrow{\sim} H^+_{DR}(M)_{\mathbf{C}}$ is in the free of rank 1 $E \otimes \mathbf{C}$-module $\delta \otimes \mathbf{C}$. We set $\det \mathrm{red}(I^+) = c^+(M) \cdot e$ for e a basis vector of δ.

3. Example : The ζ function.

3.1. To understand the various realizations of the Tate motive $\mathbf{Z}(1)$, it is easiest to write $\mathbf{Z}(1) = H_1(\mathbf{G}_m)$. Since the multiplicative group is not an algebraic variety, this can not be considered in Grothedieck's framework, which insists that we define $\mathbf{Z}(1)$ as the dual of the direct factor $H^2(\mathbf{P}^1)$ of $H^*(\mathbf{P}^1)$.

The realization in $\mathbf{Z}_l$-cohomology of $\mathbf{Z}(1)$ is the Tate module $T_l(\mathbf{G}_m)$ of $\mathbf{G}_m$

$$\mathbf{Z}_l(1) = \text{ proj lim } \mu_{l^n}.$$

In the Hodge realization, we have $H_B(\mathbf{Z}(1)) = H_1(\mathbf{C}^*)$ is isomorphic to $\mathbf{Z}$ (and likewise to $\mathbf{Q}$ in rational cohomology). This group is pure, of type $(-1 - 1)$ and $F_\infty = -1$.

In the de Rham realization $H_{DR}(\mathbf{Z}(1))$ is the dual of $H^1_{DR}(\mathbf{G}_m)$, and is isomorphic to $\mathbf{Q}$, with generator the class of dz/z. The unique period of $H^1(\mathbf{G}_m)$ is

$$\oint \frac{dz}{z} = 2\pi i \tag{3.1.1}$$

The arithmetic Frobenius ϕ_p (for $p \neq l$) acts on $\mathbf{Z}_l(1)$ via multiplication by p. Therefore the geometric Frobenius acts via multiplication by p^{-1}. This justifies the identity cited in 1.3

$$L(M(n), s) = L(M, n+s). \tag{3.1.2}$$

Since F_∞ acts as $(-1)^n$ on $H_B(\mathbf{Z}(n))$, $H_B^\epsilon(\mathbf{Z}(n))$ vanishes for $\epsilon = -(-1)^n$, and $c^\epsilon(\mathbf{Z}(n)) = 1$. For $\epsilon = (-1)^n$, it follows from (3.1.1) that $c^\epsilon(\mathbf{Z}(n)) = \delta(\mathbf{Z}(n)) = (2\pi i)^n$:

$$c^\epsilon(\mathbf{Z}(n)) = (2\pi i)^n \quad \text{for } \epsilon = (-1)^n, \tag{3.1.3}$$

$$c^\epsilon(\mathbf{Z}(n)) = 1, \quad \delta(\mathbf{Z}(n)) = (2\pi i)^n \quad \text{for } \epsilon = -(-1)^n.$$

3.2. The function $\zeta(s)$ is the L-function attached to the unit motive $\mathbf{Z}(0) = H^*(\text{Point})$. The critical integers for $\mathbf{Z}(0)$ are the even integers > 0 and the odd integers ≤ 0. Because of the pole of $\zeta(s)$ at $s = 1$, 0 is not a trivial zero and it seems reasonable to include 0 among the critical integers. Equation (3.1.3) and the known values of $\zeta(n) = L(\mathbf{Z}(n))$ for n critical satisfy 1.8: $\zeta(n)$ is rational if n is odd and ≤ 0, and is a rational multiple of $(2\pi i)^n$ if n is even and ≥ 0.

4. Compatibility with the Birch and Swinnerton-Dyer Conjecture

4.1. Let A be an abelian variety over $\mathbf{Q}$, of dimension d. The conjecture of Birch and Swinnerton-Dyer **[15]** states:

(a) $L(H^1(A), 1)$ is non-zero if and only if $A(\mathbf{Q})$ is finite.

(b) Let ω be a generator of $H^0(A, \Omega^d)$. Then $L(H^1(A), 1)$ is the product of $\int_{A(\mathbf{R})} |\,\omega\,|$ and a rational number.

The motive $H^1(A)(1)$ is isomorphic to the dual $H_1(A)$ of $H^1(A)$: this is a restatement of the existence of a polarization, self-dual of $H^1(A)$ with values in $\mathbf{Z}(-1)$. By 1.7, $c^+(H^1(A)(1))$ is therefore calculated as follows: if $\omega_1, \cdots, \omega_d$ is a basis of $H^0(A, \Omega^1) = F^+H^1_{DR}(A)$, and $e_1, \cdots, e_d$ is a basis of $H_1(A(\mathbf{C}), \mathbf{Q})^+$, we have

$$c^+(H^1(A)(1)) = \det < \omega_i, e_j > . \tag{4.1.1}$$

Designating a representative cycle by e_i again, we have $< \omega_i, e_j >= \int_{e_j} \omega_i$.

Let (e_i) be a $\mathbf{Z}$-basis of $H_1(A(\mathbf{R})^\circ, \mathbf{Z}) \subset H_1(A(\mathbf{C}), \mathbf{Z})$. The Pontryagin product of the e_i is represented by the d-cycle $A(\mathbf{R})^\circ$ in $A(\mathbf{C})$, with an appropriate orientation, and, if ω is the exterior product of the ω_i, the determinant (4.1.1) is the integral $\int_{A(R)^\circ} \omega$. We have

$$\int_{A(\mathbf{R})} |\,\omega\,| = [A(\mathbf{R}) : A(\mathbf{R})^\circ] \left| \int_{A(\mathbf{R})^\circ} \omega \right|,$$

and 1.8 for $H^1(A)(1)$ is therefore equivalent to 4.1(b) above.

4.2. The Birch and Swinnerton-Dyer conjecture gives the exact value of $L(H^1(A), 1)$; the description of the rational factor in 4.1(b) for the motive $H^1(A)(1)$ is as follows:

(a) The realization of $M = H_1(A)(1)$ is equivalent to that of A up to an isogeny. It is necessary to begin by first choosing A. This reduces to choosing an integral lattice in $H_B(M)$, whose l-adifications are stable under the action of $\mathrm{Gal}(^-\mathbf{Q}/\mathbf{Q})$.

(b) Then one chooses ω, for example as the exterior product of elements of a basis of an integral lattice in $H_{DR}(M)$. This ω determines the period, $c^+(M)$, and for each prime p, a rational factor $c_p(M)$. The $c_p(M)$ are almost all equal to 1, and the formula for the product ensures that $c^+(M) \cdot \prod_p c_p(M)$ is independent of ω.

(c) Another rational number, $h(M)$, is defined in terms of the cohomological invariants of A; the rational number we seek is $h(M) \cdot \prod_p c_p(M)^{-1}$.

The invariance of the conjecture up to isogeny is however a nontrivial theorem.

4.3. To generalize 4.1(a) to an arbitrary motive M of weight -1, we must use an analog of $A(\mathbf{Q})$. The group $A(\mathbf{Q})$ can be interpreted as the group of extensions of $\mathbf{Z}(0)$ by $H_1(A)$, in the category of 1-motives [**6**, §10] over $\mathbf{Q}$. This suggests we consider the group of extensions of $\mathbf{Z}(0)$ by M, in a category of mixed motives, similar to mixed Hodge structures, but we do not even have a conjectural definition of such a category.

I will only observe that, in all the usual cohomology theories, a cycle Y of dimension d, cohomologous to 0, in a proper, smooth algebraic variety X, determines a torsor on $H^{2d-1}(X)(d)$: we obtain an exact sequence of cohomology

$$0 \to H^{2d-1}(X)(d) \to H^{2d-1}(X - Y)(d) \xrightarrow{\partial} H^{2d}_Y(X)(d) \to H^{2d}(X)(d),$$

Y defines a cohomology class $\mathrm{cl}(Y) \in H^{2d}_Y(X)(d)$, with vanishing image in $H^{2d}(X)(d)$, and we take $\partial^{-1}\mathrm{cl}(Y)$. This construction corresponds to that which associates a divisor of degree 0 on a curve to a point of its Jacobian.

5. Compatibility with the Functional Equation

Let E be a number field. In $(E \otimes \mathbf{C})^*$, we write $\sim$ for the equivalence relation defined by the subgroup E^*.

PROPOSITION 5.1. *Let M be a motive over $\mathbf{Q}$, with coefficients in E. We suppose it verifies the hypotheses of* 1.7. *Then we have*

$$c^+(M) \sim (2\pi i)^{-d^-(M)} \cdot \delta(M) \cdot c^+(\check{M}(1)).$$

For a free module L of rank n over a commutative ring A (A will be E or $E \otimes \mathbf{C}$), we set $\det L = \bigwedge^n L$. By localization, we extend this definition to the case where L is only projective of finite type (this generalization is not essential for the proof of 5.1). We have canonical isomorphisms

(5.1.1) $$\det(L^\vee) = \det(L)^{-1}$$

and for each direct factor P of L,

(5.1.2) $$\det(L) = \det(P) \cdot \det(L/P)$$

(here $\cdot$ means tensor product and $^{-1}$ means dual). There are some problems of sign here, that we may resolve by considering $\det(L)$ as a graded invertible module, placed in degree the rank of L. Our final results are modulo $\sim$, and so there is no cause for worry.

For X and Y of the same rank, we set $\delta(X,Y) = \operatorname{Hom}(\det X, \det Y) = \det(X)^{-1} \cdot \det(Y)$. The *determinant* of $f : X \to Y$ is in $\delta(X,Y)$. We obtain from (5.1.1) and (5.1.2) the isomorphisms

$$\delta(X,Y) = \delta(Y^\vee, X^\vee) \tag{5.1.3}$$

and, for $F \subset X$ and $G \subset Y$,

$$\delta(X,Y) = \delta(F, Y/G) \cdot \delta(G, X/F)^{-1}. \tag{5.1.4}$$

LEMMA 5.1.5. *Via the isomorphism* (5.1.3), *we have* $\det(u) = \det({}^t u)$.

LEMMA 5.1.6. *Let F and G be direct factors of X and Y. We suppose that the isomorphism $f : X \to Y$ induces isomorphisms $f_F : F \to Y/G$ and $f_G^{-1} : G \to X/F$. Via the isomorphism* (5.1.4), *we have* $\det(F) = \det(f_F)\,\det(f_G^{-1})^{-1}$.

The verification of these lemmas is left to the reader. For 5.1.6, we only note that, for $G^1 = f^{-1}(G)$ and $F^1 = f(F)$, we have $X = F \oplus G^1$, $Y = G \oplus F^1$, and that f exchanges F and F^1, G and G^1.

We apply Lemma 5.1.6 to the complexifications of $H_B^+(M) \subset H_B(M)$ and of $F^- \subset H_{DR}(M)$. We find that, via the isomorphism (5.1.4), the determinant of $I : H_B(M) \otimes \mathbf{C} \to H_{DR}(M) \otimes \mathbf{C}$ is the product of the determinant of $I^+ : H_B^+(M) \otimes \mathbf{C} \to H_{DR}^+(M)/F^- \otimes \mathbf{C}$ and the inverse of the determinant of the morphism induced by the inverse of I:

$$J^- : F^- \otimes \mathbf{C} \to (H_B(M)/H_B^+(M)) \otimes \mathbf{C}.$$

The morphism J^- is the transpose of the morphism I^- for the dual motive $\check{M}$ of M. Applying 5.1.5, and using E-bases for the spaces $\delta(X,Y)$, we obtain finally

$$\delta(M) \sim c^+(M) \cdot c^-(\check{M})^{-1}. \tag{5.1.7}$$

We obtain 5.1, by applying the following formula (a consequence of 3.1) to the motive $\check{M}$:

$$c^\pm(M) = (2\pi i)^{-d^\pm(M)\cdot n} c^{\pm(-1)^n}(M(n)). \tag{5.1.8}$$

For later use, we also note the analogous formula

$$\delta(M) = (2\pi i)^{-d(M)\cdot n} \delta(M(n)). \tag{5.1.9}$$

5.2. We recall the exact form of the conjectural functional equation of L-functions attached to motives (**[12], [4]**). We consider the general case of a motive over a number field k, with coefficients in a field E, having a complex embedding σ (cf. 2.9). First the general form:

(a) For each place ν of k, we define a local factor $L_\nu(\sigma, M, s)$. For ν finite, the definition of $L_\nu(\sigma, M, s) = \sigma Z_\nu(M, N\nu^{-s})$ (with $Z_\nu(M, t) \in E(t)$) depends on a compatibility hypothesis analogous to (1.2.1). We have $Z_\nu(M, t) = \det(1 - F_\nu t, H_\lambda(M)^{I_\nu})^{-1}$. For ν infinite, induced by the complex place τ, we obtain $L_\nu(\sigma, M, s)$ by decomposing $H_\tau(M) \otimes_{E,\sigma} \mathbf{C}$ as a direct sum of subspaces minimally stable under the projectors that give the Hodge decomposition, and by F_∞ for ν real, and then associating to each the "Γ factor" from Table (5.3), and taking their product. For ν complex, the minimal subspaces are of dimension 1, of type (p, q). For ν real, they are of dimension 2, of type $\{(p, q), (q, p)\}$ (with $p \neq q$), and of dimension 1, of type (p, p), with $F_\infty = \pm 1$. We let $\Lambda(\sigma, M, s)$ denote the product of the $L_\nu(\sigma, M, s)$.

(b) Let ψ be a non trivial character of the group $(\mathbf{A} \otimes k)/k$ of adele classes of k, and let ψ_ν be the component at ν. For each place ν of k, let dx_ν be a Haar measure on k_ν. We suppose that for almost all ν, dx_ν gives the integers of k_ν measure 1, and that the product $\otimes dx_\nu$ of the dx_ν is the Tamagawa measure, giving the group of adele classes measure 1.

We define the local constants $\epsilon_\nu(\sigma, M, s, \psi_\nu, dx_\nu)$, almost all equal to 1, to be, as a function of s, the product of a constant and an exponential factor. Let $\epsilon(\sigma, M, s)$ denote their product (it is independent of ψ and the dx_ν).

(c) The conjectural functional equation is

$$\Lambda(\sigma, M, s) = \epsilon(\sigma, M, s)\Lambda(\sigma, \check{M}, 1 - s).$$

To define the ϵ_ν an additional compatibility hypothesis between the $H_\lambda(M)$ is required. It permits us to associate to ν, σ, and M an isomorphism class of complex representations of the Weil group $W(\bar{k}_\nu/k_\nu)$, for ν infinite, and of the thickened Weil group $'W(\bar{k}_\nu/k_\nu)$, for ν finite. We then take the ϵ as in [**4**, 8.12.4], with $t = p^{-s}$.

For ν infinite, this reduces to decomposing $H_\tau(M) \otimes_{E,\sigma} \mathbf{C}$ as in (a), associating to each subspace of the decomposition a factor ϵ'_ν, and to taking their product. The table of the ϵ'_ν, for a particular choice of ψ_ν and of the dx_ν, is given in 5.3.

For ν finite, we begin by restricting the representations $H_\lambda(M)$ to a decomposition group $\mathrm{Gal}(\bar{k}_\nu/k_\nu) \subset \mathrm{Gal}(\bar{k}/k)$, and then to the Weil group $W(\bar{k}_\nu/k_\nu)$. Applying [**4**, 8.3, 8.4], we construct from $H_\lambda(M)$ an isomorphism class of representations ρ_λ of $'W(\bar{k}_\nu/k_n u)$ over E_λ. It is possible here, and useful, to replace it by its F-semi-simplification [**4**, 8.6]. As λ varies we insist that these representations be compatible, i.e. if we extend the scalars of E_λ to $\mathbf{C}$, by $\tilde{\sigma} : E_\lambda \to \mathbf{C}$, extending σ, the isomorphism class of the representation so obtained is independent of λ and of σ. This is the isomorphism class we seek.

We recall, for the reader, the essential ingredients of [**4**]. A representation of $'W$ is given by a representation ρ of the Weil Group in $GL(V)$, and by a nilpotent endomorphism N of V. In terms of the local constant [**4**, 4.1] of ρ, that of (ρ, N) is given by

$$\epsilon((\rho, N), s, \psi, dx) = \epsilon(\rho \otimes \omega_s, \psi, dx) \cdot \det(-FN\nu^{-s}, V^{\rho(I)}/\ker(N)^{\rho(I)}).$$

REMARK 5.2.1. The function of s

$$\epsilon((\rho, N), s, \psi, dx)\; L((\rho, N)^\vee, 1 - s)\; L((\rho, N), s)^{-1}$$

is the same for (ρ, N) and for $(\rho, 0)$. This permits us to state the conjectural functional equation of the L-functions by assuming only a compatibility between the semi-simplifications of the restrictions of the $H_\lambda(M)$ to a decomposition group.

5.3. In the table below, we have given the local factors, and the constants, associated to different types of minimal sub-spaces of the Hodge realization. For the constants, we have assumed ψ_ν and the measure dx_ν are chosen as following:

ν real : $\psi_\nu(x) = \exp(2\pi i x)$, measure $= dx$,

ν complex : $\psi_\nu(z) = \exp(2\pi i Tr_{\mathbf{C}/\mathbf{R}}(z))$, measure $= |dz \wedge d\bar{z}|$, or if $z = x + iy$, $\exp(4\pi i x)$ and $2dxdy$ respectively.

We use the notation $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbf{C}}(s) = 2 \cdot (2\pi)^{-s}\Gamma(s)$.

Place	Type	Γ Factor	Constant
Complex	(p, q) or (q, p), $\quad p \le q$	$\Gamma_{\mathbf{C}}(s - p)$	i^{q-p}
Real	$\{(p, q),\ (q, p)\}$, $\quad p < q$	$\Gamma_{\mathbf{C}}(s - p)$	i^{q-p+1}
	(p, p), $F_\infty = (-1)^{p+\epsilon}$, $\epsilon = 0$ or 1	$\Gamma_{\mathbf{R}}(s + \epsilon - p)$	i^ϵ

The case that interests us is when $k = \mathbf{Q}$. In this case we can take $\psi_\infty(x) = \exp(2\pi i x)$, $\psi_p(x) = \exp(-2\pi i x)$ (via the isomorphism $\mathbf{Q}_p/\mathbf{Z}_p =$ the p-primary part of $\mathbf{Q}/\mathbf{Z}$), $dx_\infty =$ the Lebesgue measure dx, and $dx_p =$ the Haar measure on $\mathbf{Q}_p$ giving $\mathbf{Z}_p$ measure 1.

PROPOSITION 5.4. *If M is critical of weight w, we have, modulo a rational number independent of σ,*

$$L_\infty(\sigma, \check{M}(1))\, L_\infty(\sigma, M)^{-1} \sim (2\pi)^{-d^-(M)} \cdot (2\pi)^{-wd(M)/2}.$$

By 2.5, the $L_\infty(\sigma, \ )$ are independent of σ. This permits us to only verify 5.4 for the case in which σ is fixed. The formula is compatible with the substitution $M \mapsto \check{M}(1)$: $\check{M}(1)$ has weight $-2-w$, its d^- is $d^+(M)$ and abbreviating $d(M)$ and $d^\pm(M)$ by d and $d^\pm$ respectively, we have $d = d^+ + d^-$ and

$$\Big(-d^- - \frac{wd}{2}\Big) + \Big(-d^+ - \frac{(-2-w)d}{2}\Big) = 0.$$

This allows us to only verify 5.4 for $w \ge -1$.

(5.4.1) For s an integer, we have, modulo $\mathbf{Q}^*$,

$$\begin{array}{lcl} \Gamma_{\mathbf{R}}(s) & \sim & (2\pi)^{-s/2} \text{ for } s \text{ even } > 0, \\ \Gamma_{\mathbf{R}}(s) & \sim & (2\pi)^{(1-s)/2} \text{ for } s \text{ odd}, \\ \Gamma_{\mathbf{C}}(s) & \sim & (2\pi)^{-s} \text{ for } s > 0. \end{array}$$

For $w \ge -1$, the power of 2π in the contribution of each subspace of $H_B(M) \otimes \mathbf{C}$, as in 5.2(a), is therefore given by

	$L_\infty(M)$	$L_\infty(M^\vee(1))$	$L_\infty(M^\vee(1))L_\infty(M)$
$\{(pq),(qp)\}, p \le q$	p	$-1-p$	$-1-w$
$(pp), p$ even $\ge 0, F_\infty = -1$	$p/2$	$-1-p/2$	$-1-w/2$
$(pp), p$ odd $\ge 0, F_\infty = -1$	$(1+p)/2$	$(-1-p)/2$	$-1-w/2$

The proposition follows immediately.

Set det $M = \bigwedge^{d(M)} M$ (exterior power over E).

PROPOSITION 5.5. $\epsilon^*(M) \sim \epsilon^*(\det M)$.

For the dx_ν chosen as suggested in 5.4, we will prove, more precisely, the equivalences

$$\epsilon^*_\nu(M, \psi_\nu, dx_\nu) \sim \epsilon^*_\nu(\det M, \psi_\nu, dx_\nu). \tag{5.5.1}$$

Setting $\eta_\nu(\sigma, M, \psi_\nu) = \epsilon_\nu(\sigma, M, \psi_\nu, dx_\nu)\epsilon_\nu^{-1}(\sigma, \det M, \psi_\nu, dx_\nu)$, this is equivalent to

$$\tau\eta_\nu(\sigma, M, \psi_\nu) = \eta_\nu(\tau\sigma, M, \psi_\nu) \tag{5.5.2}$$

for every automorphism τ of $\mathbf{C}$.

It follows from [**4**, 5.4] that, if $a \in \mathbf{Q}^*_\nu$ has absolute value 1, and if we set $(\psi_\nu \cdot a)(x) = \psi_\nu(ax)$, we have

$$\eta_\nu(\sigma, M, \psi_\nu) = \eta_\nu(\sigma, M, \psi_\nu \cdot a) \quad (\text{for } \|a\|_\nu = 1). \tag{5.5.3}$$

For τ an automorphism of $\mathbf{C}$, and ν finite, $\tau\psi_\nu$ is of the form $\psi_\nu \cdot a$, with $\|a\|_\nu = 1$; and similarly $\bar{\psi}_\infty = \psi_\infty \cdot (-1)$.

For ν finite, the definition of ϵ_ν is purely algebraic, whence $\tau\eta_\nu(\sigma, M, \psi_\nu) = \eta_\nu(\tau\sigma, M, \tau\psi_\nu)$, and (5.5.2) follows from (5.5.3).

For $\nu = \infty$, we have again $\bar{\eta}_\infty(\sigma, M, \bar{\psi}_\infty) = \eta_\infty(\bar{\sigma}, M, \bar{\psi}_\infty) = \eta_\infty(\bar{\sigma}, M, \psi_\infty)$; if we take ψ_∞ as suggested in 5.3, η_∞ is a power of i independent of σ, from which it follows $\eta_\infty(\sigma, M, \psi_\infty) = \pm 1$, independent of σ, and this verifies (5.5.2).

THEOREM 5.6. *Modulo Conjecture* 6.6 *on the nature of motives of rank* 1, *Conjecture* 2.8 *is compatible with the conjectural functional equation of the L-functions : we have*

$$L^*_\infty(M)c^+(M) \sim \epsilon^*(M)L^*_\infty(\check{M}(1))c^+(\check{M}(1)).$$

By 5.1, 5.3, and 5.5 this formula is equivalent to

$$(2\pi i)^{-d^-(M)} \cdot \delta(M) \sim (2\pi)^{-d^-(M)} \cdot (2\pi)^{-wd(M)/2} \cdot \epsilon^*(\det M).$$

We set $D =$ det M and $\epsilon = d^-(D)$. We have $\delta(M) = \delta(D)$, $d^-(M) \equiv \epsilon \,(\text{mod } 2)$ and $wd(M)$ is the weight $w(D)$ of D, so that the formula is now equivalent to

$$\epsilon^*(D) \sim (2\pi)^{w(D)/2} \cdot i^\epsilon \cdot \delta(D). \tag{5.6.1}$$

In 6.5 we will prove that (5.6.1) holds for a class of motives of rank 1 that conjecturally (see 6.6) includes all motives of rank 1.

6. Example : Artin L-functions

DEFINITION 6.1. *The category of* Artin motives *is the Karoubian envelope of the dual of the category of objects consisting of varieties over* $\mathbf{Q}$ *of dimension* 0, *and morphisms consisting of the correspondences defined over* $\mathbf{Q}$.

By definition, every variety of dimension 0, X, defines an Artin motive $H(X)$, where the functor H is a contravariant, fully faithful functor

$$H : (\text{varieties of dim 0, correspondences}) \to (\text{Artin motives})$$

and every Artin motive is a direct factor of an $H(X)$.

6.2. Let us make this definition explicit. Let $\bar{\mathbf{Q}}$ be an algebraic closure of $\mathbf{Q}$, and G the Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. A variety of dimension 0 is the spectrum of a finite product A of number fields, and Galois theory (in Grothendieck's formulation) says that the functor

$$X = \mathrm{spec}(A) \mapsto X(\bar{\mathbf{Q}}) = \mathrm{Hom}(A, \bar{\mathbf{Q}}) :$$

(category of varieties of dimension 0 over $\mathbf{Q}$, and morphisms of schemes) $\to$ (category of finite sets on which G acts continuously) is an equivalence of categories. The inverse functor is $I \mapsto$ Spec of the ring of G-invariant functions of I into $\bar{\mathbf{Q}}$

A *correspondence* of a variety X of dimension 0, in another, Y, is a formal linear combination of connected components of $X \times Y$ with coefficients in $\mathbf{Q}$. The mapping

$$\sum a_i Z_i \mapsto \sum a_i (\text{characteristic function of } Z_i(\bar{\mathbf{Q}}) \subset (X \times Y)(\bar{\mathbf{Q}}))$$

identifies correspondences and G-invariant functions, with rational values, on $(X \times Y)(\bar{\mathbf{Q}}) = X(\bar{\mathbf{Q}}) \times Y(\bar{\mathbf{Q}})$, and identifies the composition of correspondences with matrix products.

Let us write H for the contravariant functor $X \mapsto$ the vector space $\mathbf{Q}^{X(\bar{\mathbf{Q}})}$, equipped with the natural G action; (correspondence $F : X \to Y$) $\mapsto$ the morphism $F^* : \mathbf{Q}^{Y(\bar{\mathbf{Q}})} \to \mathbf{Q}^{X(\bar{\mathbf{Q}})}$ with matrix ${}^tF$. It is fully faithful, and identifies the category of Artin motives with the category of rational representations of G.

6.3. In this model, if $\bar{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, the functor "Betti realization" H_B is the functor "underlying vector space". The Hodge structure is pure of type $(0,0)$, and the involution F_∞ is the action of complex conjugation $F_\infty \in G$. Indeed, we have an isomorphism, functorial on correspondences,

$$H^*(X(\mathbf{C}), \mathbf{Q}) = \mathbf{Q}^{X(\mathbf{C})} = \mathbf{Q}^{X(\bar{\mathbf{Q}})} = H(X).$$

The functor "l-adic realization" H_l is the functor $H_l(V) = V \otimes \mathbf{Q}_l$. In fact we have an isomorphism, functorial on correspondences,

$$H^*(X(\bar{\mathbf{Q}}), \mathbf{Q}_l) = \mathbf{Q}_l^{X(\bar{\mathbf{Q}})} = \mathbf{Q}^{X(\bar{\mathbf{Q}})} \otimes \mathbf{Q}_l.$$

Similarly we calculate the de Rham realization. For $X = \mathrm{Spec}(A)$, we have $H^*_{DR}(X) = A$. Letting $A = \mathrm{Hom}_G(X(\bar{\mathbf{Q}}), \bar{\mathbf{Q}}) = (\mathbf{Q}^{X(\bar{\mathbf{Q}})} \otimes \bar{\mathbf{Q}})^G$, we obtain

(6.3.1) $$H_{DR}(V) = (V \otimes \bar{\mathbf{Q}})^G.$$

Formula (6.3.1) realizes $H_{DR}(V)$ as a subspace of $V \otimes \bar{\mathbf{Q}}$. This subspace is a $\mathbf{Q}$-structure: we have $(V \otimes \bar{\mathbf{Q}})^G \otimes \bar{\mathbf{Q}} \tilde{\to} V \otimes \bar{\mathbf{Q}}$. Therefore, after extending scalars to $\bar{\mathbf{Q}}$, $H_B(V)$ and $H_{DR}(V)$ are canonically isomorphic. Extending scalars further to $\mathbf{C}$, we obtain the isomorphism (0.4.1) (verified for $V = H(X)$).

6.4. Let E be a finite extension of $\mathbf{Q}$. The *category of Artin motives with coefficients in* E is the category constructed from the category of Artin motives as in 2.1. We will identify this with the category of finite dimensional E-vector spaces, having a G-action.

Artin motives with coefficients in E, of rank 1 over E, correspond to the characters $\epsilon : G \to E^*$. We are going to calculate their periods. So let $\epsilon : G \to E^*$ and let f be the conductor of ϵ. ϵ factors through a character, again denoted by ϵ, of the quotient $(\mathbf{Z}/f\mathbf{Z})^* = \mathrm{Gal}(\mathbf{Q}(\exp(2\pi i/f))/\mathbf{Q})$ of G. We write $[\epsilon]$ for the vector space E_s, of dimension 1 over E, on which G acts via ϵ. The Gauss sum

$$g = \sum \epsilon(u) \otimes \exp(2\pi iu/f) \in [\epsilon] \otimes \bar{\mathbf{Q}}$$

is non-zero, and G-invariant : it is a basis, over E, of $H_{DR}([\epsilon])$. The determinant of $I : H_B([\epsilon]) \otimes \mathbf{C} \tilde{\to} H_{DR}([\epsilon]) \otimes \mathbf{C}$, calculated in the bases 1 and g, is g^{-1}. Hence, for every complex embedding σ of E, we have $\sigma g \cdot \bar{\sigma} g = f$, a rational number independent of σ, from which we obtain

$$\delta([\epsilon]) \sim \sum \epsilon^{-1}(u) \otimes \exp(-2\pi iu/f) \in (E \otimes \mathbf{C})^*.$$

PROPOSITION 6.5. *Let D be the motive $[\epsilon](n)$. It is a motive over $\mathbf{Q}$, with coefficients in E, of rank* 1 *and weight $-2n$. Set $\epsilon(-1) = (-1)^\eta$, with $\eta = 0$* or 1. We have

$$\epsilon^*(D) \sim (2\pi)^{-n} i^{\eta-n} \delta(D).$$

We know that the constant of the functional equation of the Dirichlet L-function $L(\sigma, [\epsilon], s) = \sum \sigma\epsilon^{-1}(n) \cdot n^{-s}$ (σ is a complex embedding of E) is given by

$$\epsilon(\sigma, [\epsilon], s) = i^\eta \cdot f^s \cdot \sum \sigma\epsilon(u)^{-1} \exp(-2\pi iu/f).$$

Furthermore, by (5.1.9), we have $\delta(D) = (2\pi i)^n \delta([\epsilon])$. We conclude by applying (6.4.1) and by noting that for s an integer ($s = n$), f^n is rational and independent of σ.

This proposition verifies (5.6.1) for the motives $[\epsilon](n)$. To complete the proof of 5.6, it only remains to state the

Conjecture 6.6. *Every motive over $\mathbf{Q}$, with coefficients in E and of rank* 1 *is of the form $[\epsilon](n)$, for ϵ a character of $G = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ with values in the roots of unity of E and n an integer.*

PROPOSITION 6.7. *Conjecture* 2.8 *is true for Artin L-functions.*

For Artin motives, we dispose of the functional equation of the L-functions. First note the determinant of an Artin motive, with a Tate twist, is as predicted in 6.6. Therefore the arguments in sections 5 and 6 show the compatibility of 2.8 with the functional equation, and so it suffices to prove 2.8 for the motives $V(n)$, for V an Artin motive, and n an integer

≤ 0. If $V(n)$ is critical, F_∞ acts on V by multiplication by $-(-1)^n$, and $c^+ = 1$ (cf. 3.1). So it remains to prove that, for every embedding σ of E into $\mathbf{C}$, and every automorphism τ of $\mathbf{C}$, we have $\tau L(\sigma, V(n)) = L(\tau\sigma, V(n))$.

We deduce this from results of Siegel [**14**]. See [**2**, 1.2].

7. L-functions attached to Modular Forms

7.1 Set $q = e^{2\pi i z}$, and let $f = \sum a_n q^n$ be a primitive holomorphic cusp form (new form) of weight $k \geq 2$, conductor N, and character ϵ. The Dirichlet series $\sum a_n n^{-s}$ admits an Euler product, with local factors at $p \not| N$ equal to $(1 - a_p p^{-s} + \epsilon(p) p^{k-1} p^{-2s})^{-1}$.

Let E be the sub-field of $\mathbf{C}$ generated by the a_n. The form f gives rise to a motive $M(f)$ of rank 2, with coefficients in E, with Hodge type {(k - 1, 0) (0, k - 1)}, determinant $[\epsilon^{-1}](1-k)$ (notation as in 6.4), and L-function given by the Dirichlet series $\sum a_n n^{-s}$.

I have not tried to define $M(f)$ as being a motive in Grothendieck's sense. One difficulty is that $M(f)$ appears naturally as a direct factor in the cohomology of a non-compact variety, or as a direct factor in the cohomology of a complete modular curve, with coefficients in the direct image of a locally constant sheaf (rather, a local system of motives!) defined in the complement of a neighbourhood of the boundary of this modular curve. This avoids Grothendieck's formalism, but permits us to define the realizations of the motive $M(f)$.

7.2. First, let us assume that $k = 2$, and that ϵ is trivial. Let X be the Poincaré half-plane, N the conductor of f, $\Gamma_0(N)$ the sub-group of SL(2,$\mathbf{Z}$) consisting of all matrices whose reduction mod N is of the form $\binom{*\,*}{0\,*}$, and set $\omega_f = \sum a_n q^n \cdot dq/q = \sum a_n q^n \cdot 2\pi i dz$. The form ω_f is a holomorphic differential form on the completed curve $\bar{M}(\Gamma_0(N))$ of $M(\Gamma_0(N)) = X/\Gamma_0(N)$. It is an eigenvector of the Hecke operators:

$$T_n^* \omega_f = a_n \omega_f \quad \text{(for } n \text{ relatively prime to } N\text{)} \tag{7.2.1}$$

and is characterized up to a factor by (7.2.1). This is a consequence of the strong multiplicity 1 theorem and the theory of primitive forms; the reader can, if he or she prefers, complete (7.2.1) by an analogous condition for n not prime to N, and do the same below as well: the assertion then becomes elementary, because condition (7.2.1) completely determines (up to a factor) the Taylor development of ω_f at the point $i\infty$.

Let us shorten $M(\Gamma_0(N))$ to M and $\bar{M}(\Gamma_0(N))$ to $\bar{M}$. These curves have a natural $\mathbf{Q}$-structure, for which the Hecke operators are defined over $\mathbf{Q}$. The form ω_f is defined over E, and its conjugate under an automorphism σ of $\mathbf{C}$ is $\omega_{\sigma f}$ (apply σ to the coefficients).

Define the motive $M(f)$ as being, in the language 2.1(B), the sub-motive of $H^1(\bar{M})_E$ kernel of the endomorphisms $T_n^* - a_n$. If we only want to consider the kernels of projectors, replace $T_n^* - a_n$ by $P(T_n - a_n\Delta)^*$ for P a suitable polynomial. We know that $M(f)$ has the properties stated in 7.1.

In each of the cohomology theories $\mathcal{H}$ that interest us, the map

$$\mathcal{H}_c^1(M) \longrightarrow \mathcal{H}^1(\bar{M}) \tag{7.2.2}$$

is surjective, and the system of eigenvalues of the T_n^* on the kernel do not appear in $\mathcal{H}^1(\bar{M})$. The $\mathcal{H}$-realization of $M(f)$ is therefore again the common kernel of the $T_n^* - a_n$ in $\mathcal{H}_c^1(M) \otimes E$.

The cohomology group $H^1_c(M, \mathbf{Q})$ has mixed Hodge structure of type $\{(0,0),(0,1),(1,0)\}$, and quotient $H^1(\bar{M}, \mathbf{Q})$ of type $\{(0,1),(1,0)\}$. In particular (7.2.2) induces an isomorphism on the subspaces F^1 of the Hodge filtration; every holomorphic differential form ω on $\bar{M}$ thus defines a cohomology class on M with proper support. One may see this directly : if $\mathcal{I} \subset \mathcal{O}$ is the ideal of points, $\mathbf{H}^*_c(M)$, in de Rham cohomology, is the hypercohomology on $\bar{M}$ of the complex $\mathcal{I} \to \Omega^1$ (analytically, this complex is a resolution of the constant sheaf $\mathbf{C}$ on M, extended by 0 on $\bar{M}$), and the $\mathbf{H}^1$?? $H^\circ(\bar{M}, \Omega^1)$.

To simplify matters, assume that $E = \mathbf{Q}$ and calculate $c^+(M(f)(1))$. The motive $M(f)(1)$ is the dual of $M(f)$ and c^+ is therefore a period of $M(f)$ (1.7): it is necessary to integrate ω_f against a rational cohomology class of $\bar{M}$, fixed by F_∞. Realizing $M(f)$ in $\mathcal{H}^1_c(M)$, one sees that ω_f can then be integrated against a cohomology class without support in M, fixed by F_∞. All the non-zero integrals of this type will be commensurable.

To calculate F_∞, it is useful to write M as a quotient of $X^\pm = \mathbf{C} - \mathbf{R}$ by the subgroup of $\mathrm{GL}(2, \mathbf{Z})$ consisting of the matrices having reduction mod N of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. Hence complex conjugation is induced by $z \to \bar{z}$, and the image of $i\mathbf{R}^+$ in $M(\mathbf{C})$ is a cycle without support fixed by F_∞. The formula $L(M(f), 1) = -\int_0^{i\infty} \omega_f$ justifies 1.8 for $M(f)(1)$: the second expression is either 0 or $\sim c^+(M(f)(1))$.

REMARK 7.3. We also have

$$c^\pm(M(f)(1)) \sim \int_{a/b}^{i\infty} \omega_f \pm \int_{-a/b}^{i\infty} \omega_f.$$

7.4. For arbitrary ϵ (and E), it is necessary to replace $\Gamma_0(N)$ by $\Gamma_1(N)$: the subgroup of $\mathrm{SL}(2, \mathbf{Z})$ consisting of the matrices having reduction mod N of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. As above, to calculate F_∞, it is more convenient to work with $X^\pm$ and the subgroup of $\mathrm{GL}(2, \mathbf{Z})$ consisting of the matrices having reduction mod N of type $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$. Otherwise, a dualization appears, hidden in 7.2 by the symmetry of the operators T_n. For a convenient definition of T_n, we have

(a) $M(f)$ is the common kernel of the $T^*_n - a_n$ in $H^1(M)$, $\bar{M} = X/\Gamma_1(N)$.

(b) We have ${}^tT^*_n \omega_f = a_n \omega_f$, and $T^*_n \omega_f = \bar{a}_n \omega_f$ (note the relation $\bar{a}_n = \epsilon(n)^{-1} a_n$), in such a way that ω_f is in the de Rham realization of $M(f) = M(f)^\vee(-1)$.

We find that $c^+(M(f)(1)) \in (E \otimes \mathbf{C})^*/E^* \in \mathbf{C}^{*\mathrm{Hom}(E,\mathbf{C})}/E^*$ is given by the system of periods

$$c^+(M(f)(1)) \sim \Big(\int_0^{i\infty} \omega_{\sigma f} \Big)_\sigma,$$

if the latter expression is non-zero. We note that if any one of the integrals is zero, then they all are zero. This is the case if and only if the system of eigenvalues a_n of the T_n does not appear in the part of the homology without support generated by the cycle $i\mathbf{R}^+$ and its transformations by the T_n. This justifies 2.8, and partially justifies 2.7, for $M(f)(1)$.

REMARK 7.5. The system of eigenvalues of the T_n in $\mathrm{Ker}(\mathcal{H}^1_c(M) \longrightarrow \mathcal{H}^1(\bar{M}))$ are linked to Eisenstein series, hence those that appear in the $\mathcal{H}^1(\bar{M})$ are linked to parabolic forms. This is why these sets are disjoint. It follows that the mixed Hodge structure of $H^1_c(M, \mathbf{Q})$ is the sum of Hodge structures : the extension of $H^1(\bar{M}, \mathbf{Q})$ by $\mathrm{Ker}(H^1_c(M, \mathbf{Q}) \to H^1(\bar{M}, \mathbf{Q}))$

splits. This, generalized to the case of any congruence subgroup of $\mathrm{SL}(2,\mathbf{Z})$, is equivalent to the theorem of Manin [**9**] which states that the difference between two points is always of finite order in the Jacobian (cf. [**6**, 10.3.4. 10.3.8, 10.1.3]). The proof we have given does not differ otherwise in substance from Manin's.

7.6. If the weight k is arbitrary, it is necessary to replace the cohomology of $\bar{M}$ by the cohomology of $\bar{M}$ with coefficients in a convenient sheaf. To describe what happens, I will replace M and $\bar{M}$ by M_n and $\bar{M}_n$ (respectively) relative to the congruence group $\Gamma(n)$, with n a multiple of N and ≥ 3. Afterwards we return to M and $\bar{M}$ by taking invariants by an appropriate finite group. Let $g : E \to M_n$ be the universal elliptic curve and j the inclusion of M_n in $\bar{M}_n$. The cohomology group to consider is $H^1(\bar{M}_n, j_*\mathrm{Sym}^{k-2}(R^1g_*\mathbf{Q}))$. The realizations of $M(f)$ are direct factors of this group, calculated in the corresponding cohomology theory. As before, we may realize them in $H^1_c(M_n, \mathrm{Sym}^{k-2}(R^1g_*\mathbf{Q}))$. To calculate $c^\pm$ of the dual of $M(f)$, it is necessary then to integrate the class in H^1_c defined by f against the homology classes without support of M_n with coefficients in the local dual system of $\mathrm{Sym}^{k-2}(R^1g_*\mathbf{Q})$. If we take as the cycle the image of $i\mathbf{R}^+$, equipped with different sections of the dual (a basis), we obtain Eichler's integrals

$$\int_0^{i\infty} f(q)\cdot\frac{dq}{q}\cdot(2\pi i z)^l \qquad (0\leq l\leq k-2)$$

(even periods for even l, odd for l odd) - and we know how to write $L(M(f),n)$ for n critical in terms of these periods.

PROPOSITION 7.7. *Let M be a motive of rank* 2, *with coefficients in E, of Hodge type* $\{(a,b),(b,a)\}$ *with* $a\neq b$. *Then,* $d^\pm\mathrm{Sym}^nM$ *and* $c^\pm\mathrm{Sym}^nM$ *are given by the following formulas*:

(1) *If* $n=2l+1$, *then* $d^\pm=l+1$, *and*

$$c^\pm\mathrm{Sym}^nM = c^\pm(M)^{(l+1)(l+2)/2}c^\mp(M)^{l(l+1)/2}\delta(M)^{l(l+1)/2};$$

(2) *If* $n=2l$, *then* $d^+=l+1$, $d^-=l$ *and*

$$c^+\mathrm{Sym}^nM = (c^+(M)c^-(M))^{l(l+1)/2}\delta(M)^{l(l+1)/2},$$

$$c^-\mathrm{Sym}^nM = (c^+(M)c^-(M))^{l(l+1)/2}\delta(M)^{l(l-1)/2}.$$

This is a question of simple linear algebra, that we will treat using "dimensional analysis".

(a) $H_B(M)$ is a vector space of rank 2 over E, having an involution F_∞ of the form $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ in an appropriate basis e^+, e^-. The $d^\pm\mathrm{Sym}^nM$ are the dimensions of the $+$ and $-$ parts of the n^{th} symmetric power, with respective bases $\{e^{+^n}, e^{+^{n-2}}e^{-^2},\cdots\}$ and $\{e^{+^{n-1}}e^-, e^{+^{n-3}}e^{-^3},\cdots\}$ - from which the result follows.

(b) Let ω,η be a dual basis of $H_{DR}(M)$, such that ω annihilates $F^+H_{DR}(M)$. The subspace $F^\pm$ of $H_{DR}(\mathrm{Sym}^nM)^\vee\sim\mathrm{Sym}^nH_{DR}(M)^\vee$ then has basis $\omega^n,\omega^{n-1}\eta,\cdots,\omega^{n-d^\pm+1}\eta^{d^\pm-1}$, and

$$(7.7.1)^+ \qquad c^+\mathrm{Sym}^nM = <\omega^n\wedge(\omega^{n-1}\eta)\wedge\cdots,\ e^{+^n}\wedge(e^{+^{n-2}}e^{-^2})\wedge\cdots>,$$

$$(7.7.1)^- \qquad c^-\mathrm{Sym}^n M = < \omega^n \wedge (\omega^{n-1}\eta) \wedge \cdots, \; ({e^+}^{n-1}e^-) \wedge ({e^+}^{n-3}{e^-}^3) \wedge \cdots > .$$

(c) Set $V = H_B(M) \otimes \mathbf{C} \sim H_{DR}(M) \otimes \mathbf{C}$; this is an $E \otimes \mathbf{C}$-module. The formulas above show that $c^\pm \mathrm{Sym}^n M$ only depends on the $E \otimes \mathbf{C}$-module V, on the choice of its basis e^+, e^-, on $\omega \in V^*$, and on the image $\bar\eta$ of η in $V^*/ < \omega >$: the $d^\pm$-vector in the left of the scalar product (7.7.1) does not change if we replace η by $\eta + \lambda\omega$. Otherwise, $< \omega, e^+ >$ and $< \omega, e^- >$ are invertible, and $\bar\eta$ is a basis of $V^*/ < \omega >$. The system $(V, e^+, e^-, \omega, \bar\eta)$ is therefore described up to isomorphism by the quantities $c^+ = < \omega, e^+ >$, $c^- = < \omega, e^- >$ and $\delta = < \omega \wedge \eta, e^+ \wedge e^- >$, in $(E \otimes \mathbf{C})^*$. Replace e^+, e^-, ω, and $\bar\eta$ by λe^+, μe^-, ω, and $\nu\bar\eta/\lambda\mu$, and replace c^+, c^- and δ by λc^+, μc^-, and $\nu\delta$. For $c^+ = c^- = \delta = 1$, the right half of $(7.7.1)^\pm$ is in $\mathbf{Q}^*$. Therefore, in the general case, $c^\pm \mathrm{Sym}^n M$ is a rational multiple of the product of $c^+(M)$, $c^-(M)$ and $\delta(M)/c^+(M)c^-(M)$ raised to the respective powers corresponding to the degrees of e^+, e^- and η in (7.7.1). We are spared half of the calculation if we note that replacing F_∞ by $-F_\infty$, exchanges e^+ and e^-, hence c^+ and c^-, respects δ, and exchanges $(7.7.1)^\pm$ for n odd, and preserves them for n even.
If $n = 2l + 1$ then $d^+ = d^- = l + 1$, and

$$\begin{aligned}
\deg \eta \text{ in } (7.7.1)^\pm &= 0 + 1 + ... + l = l(l+1)/2, \\
\deg e^+ \text{ in } (7.7.1)^+ &= (2l+1) + (2l-1) + \cdots + 1 = (l+1)^2 \\
&= \deg e^- \text{ in } (7.7.1)^-, \\
\deg e^+ \text{ in } (7.7.1)^- &= 2l + (2l-2) + \cdots + 0 = l(l+1) \\
&= \deg e^- \text{ in } (7.7.1)^+;
\end{aligned}$$

If $n = 2l$ then $d^+ = l + 1$, $d^- = l$, and

$$\begin{aligned}
\deg \eta \text{ in } (7.7.1)^+ &= 0 + 1 + \cdots + l = l(l+1)/2, \\
\deg \eta \text{ in } (7.7.1)^- &= 0 + 1 + \cdots + l = l(l-1)/2, \\
\deg e^\pm \text{ in } (7.7.1)^+ &= 2l + (2l-2) + \cdots + 0 = l(l+1), \\
\deg e^\pm \text{ in } (7.7.1)^- &= (2l-1) + (2l-3) + \cdots + 1 = l^2.
\end{aligned}$$

7.8. This proposition provides a conjecture for the values at the critical integers of $L(\mathrm{Sym}^n M(f), s)$. We have:

$$(7.8.1) \qquad L(\sigma, M(f), s) = \sum \sigma a_n n^{-s} = \prod_p L_p(\sigma, M(f), s)$$

where for almost every p

$$L_p(M(f), s) = (1 - a_p p^{-s} + \epsilon(p) p^{k-1-2s})^{-1} = ((1 - \alpha'_p p^{-s})(1 - \alpha''_p p^{-s}))^{-1},$$

where ϵ is a Dirichlet character. We have also $\bigwedge^2 M(f) = [\epsilon^{-1}](1-k)$, from which it follows that

$$(7.8.2) \qquad \delta(M(f)) \sim (2\pi i)^{1-k} \cdot \sum \epsilon(u) \otimes \exp\,(-2\pi i u/F),$$

if ϵ has conductor F, and

$$(7.8.3) \qquad L^*(M(f), m) \sim (2\pi i)^m c^\pm(M(f)), \qquad \pm = (-1)^m, \text{ for } 1 \leq m \leq k-1.$$

Now

$$L(\sigma, \mathrm{Sym}^n M(f), s) = \prod_p L_p(\sigma, \mathrm{Sym}^n M(f), s) \tag{7.8.4}$$

where for almost every p

$$L_p(\mathrm{Sym}^n M(f), s)^{-1} = \prod_{i=0}^{n} (1 - (\alpha'_p)^i (\alpha''_p)^{n-i} p^{-s}).$$

Conjecturally, for m critical, we have

$$L(\mathrm{Sym}^n M(f), m) \sim (2\pi i)^{m d^{\pm} \mathrm{Sym}^n M(f)} \cdot c^{\pm} \mathrm{Sym}^n M(f), \quad \pm = (-1)^m,$$

where $c^{\pm}$ and $d^{\pm}$ are given in terms of $c^{\pm}(M)$ and $\delta(M)$ (characterized by (7.8.2), (7.8.3)) in the formulae 7.7.

For numerical evidence in favor of this conjecture, see [**18**].

8. Algebraic Hecke Characters

We will use the same notation as in [**3**, paragraph 5], where the reader will find the essential definitions relating to algebraic Hecke characters (= grössencharaktere of type A_0).

Conjecture 8.1. *Let k and E be two finite extensions of $\mathbf{Q}$, and χ an algebraic Hecke character of k with values in E.*

(i) There exists a motive $M(\chi)$ with coefficients in E, of rank one over E, such that, for every place λ of E, the λ-adic representation $H_\lambda M(\chi)$ is that defined by χ: the geometric Frobenius at $\mathcal{P}$ prime to the conductor of χ and to the residue characteristic l of λ acts by multiplication by $\chi(\mathcal{P})$.

(ii) This motive is characterized up to isomorphism by this property.

(iii) Every motive of rank 1 is of the form $M(\chi)$.

(iv) Decompose $k \otimes E$ into a product of fields: $k \otimes E = \prod K_i$, and write the algebraic part $\chi_{\mathrm{alg}} : k^* \to E^*$ of χ in the form $\chi_{\mathrm{alg}}(x) = \prod N_{K_i/E}(x)^{n_i}$. The decomposition of $k \otimes E$ induces a decomposition of $H_{\mathrm{DR}}(M(\chi))$ into the $H_{\mathrm{DR}}(M(\chi))_i = H_{\mathrm{DR}}(M(\chi)) \otimes_{k \otimes E} K_i$; with this notation, the Hodge filtration is the filtration by the $\bigoplus_{n_i \geq p} H_{\mathrm{DR}}(M(\chi))_i$.

The uniqueness property 8.1(ii) imposes the following formalism on the $M(\chi)$

$$M(\chi'\chi'') \sim M(\chi') \otimes M(\chi''). \tag{8.1.1}$$

(8.1.2) If $\iota : E \longrightarrow E'$ is a finite extension of E, $M(\iota\chi)$ is obtained from $M(\chi)$ by extension of scalars from E to E'.

(8.1.3) If k' is a finite extension of k, $M(\chi \circ N_{k'/k})$ is obtained from $M(\chi)$ by extension of scalars from k to k'.

REMARK 8.2. Set the notation:

c = complex conjugation, $\bar{\mathbf{Q}}$ = the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, $S = \mathrm{Hom}(k, \bar{\mathbf{Q}}) = \mathrm{Hom}(k, \mathbf{C})$, $J = \mathrm{Hom}(E, \bar{\mathbf{Q}}) = \mathrm{Hom}(E, \mathbf{C})$. The decomposition of $k \otimes E$ into the K_i corresponds to the partition of $S \times J$ into the $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-orbits.

Every algebraic homomorphism $\eta : k^* \to E^*$ can be written in the form $\prod N_{K_i/E}(x)^{n_i}$. If the $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-orbit of $(\sigma, \tau) \in S \times I$ corresponds to K_i, i.e., if $\sigma \otimes \tau : k \otimes E \to \mathbf{C}$ factors through K_i, we will write $n(\eta; \sigma, \tau)$, or simply $n(\sigma, \tau)$, for the integer n_i. The function $n(\sigma, \tau)$ is constant on the $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-orbits. If η is the algebraic part of a Hecke character χ, we also have

(8.2.1) The integer $w = n(\eta; \sigma, \tau) + n(\eta; c\sigma, \tau)$ is independent of σ and τ.

It is the *weight* of χ (and of $M(\chi)$). We will sometimes write $n(\chi; \sigma, \tau)$ for $n(\chi_{\mathrm{alg}}; \sigma, \tau)$.

Conversely, a homomorphism η satisfying (8.2.1) is almost of the form χ_{alg}, for a suitable Hecke character χ:

(a) one of its powers is of that form;

(b) there is a finite extension $\iota : E \to E'$ of E such that $\iota\eta$ is of that form;

(c) there is a finite extension k' of k such that $\eta \circ N_{k'/k}$ is of that form.

The rule 8.1(iv) allows us to obtain the Hodge bigradation of $H_\sigma M(\chi)$ from its E-module structure: the direct factor $H_\sigma M(\chi) \otimes_{E,\tau} \mathbf{C}$ of $H_\sigma M(\chi) \otimes \mathbf{C}$ is of Hodge type (p, q), with $p = n(\chi; \sigma, \tau)$ and $q = w - p = n(\chi; \sigma, c\tau)$.

EXAMPLE 8.3. Let A be an abelian variety over k, with complex multiplication by E. Suppose $H_1(A)$ is of rank 1 over E (CM-type). It follows from Shimura-Taniyama theory that $H_1(A)$ satisfies the condition 8.1(i), for an algebraic Hecke character χ of k with values in E, and that the algebraic part χ_{alg} of χ is defined over the $k \otimes E$-module $\mathrm{Lie}(A)$:

$$\chi_{\mathrm{alg}}(x) = \det{}_E(x \otimes 1, \mathrm{Lie}(A))^{-1}.$$

EXAMPLE 8.4. Let k be the field of n^{th} roots of unity, let V be the Fermat hypersurface with projective equation $\sum_{i=0}^{m} {X_i}^n = 0$ and let M be the motive "primitive cohomology with dimension half that of V". Let G be the quotient of μ_n^{m+1} by its diagonal subgroup. This group acts on V by $(\alpha_i) * (X_i) = (\alpha_i X_i)$, and on M by transport of structure. Decomposing M by means of the decomposition of the group algebra $\mathbf{Q}[G]$ into a product of fields, we obtain motives satisfying 8.1(i) for suitable algebraic Hecke characters: those introduced by Weil in his study of Jacobi sums.

EXAMPLE 8.5. The motive $\mathbf{Z}(-1)$ satisfies 8.1(i) for χ = the norm. For χ of finite order, a suitable Artin motive satisfies 8.1(i).

8.6. For every $\sigma \in S$, $H_\sigma(M(\chi))$ is of rank 1 over E. Choose a basis e_σ of each $H_\sigma(M(\chi))$. Beyond its E-module structure, the sum of the $H_\sigma(M(\chi)) \otimes \mathbf{C}$ has a natural $k \otimes \mathbf{C} = \mathbf{C}^S$-module structure — in sum, a $k \otimes E \otimes \mathbf{C}$-module structure, free of rank 1, for which $e = \sum e_\sigma$ is a basis.

8.7. The De Rham realization $H_{\rm DR}(M(\chi))$ is a $k \otimes E$-module, free of rank 1. Let ω be a basis of it. The sum of the $I_\sigma : H_\sigma(M(\chi)) \otimes \mathbf{C} \xrightarrow{\sim} H_{\rm DR}(M(\chi)) \otimes_{k,\sigma} \mathbf{C}$ is the $k \otimes E \otimes \mathbf{C}$-module isomorphism (0.4)

$$I : \oplus_\sigma H_\sigma(M(\chi)) \otimes \mathbf{C} \xrightarrow{\sim} H_{\rm DR}(M(\chi)) \otimes_{\mathbf{Q}} \mathbf{C}.$$

Let ω be a basis of the $k \otimes E$-module $H_{\rm DR}(M(\chi))$, and set $p'(\chi) = \omega/I(e) \in (k \otimes E \otimes \mathbf{C})^*$. This *period* depends on χ, ω and e. Taken modulo $(k \otimes E)^*$ and E^{*S}, it depends only on χ. We will write $p'(\chi; \sigma, \tau)$ — or simply $p'(\sigma, \tau)$ — for the component with index (σ, τ) in the image under the isomorphism $k \otimes E \otimes \mathbf{C} \xrightarrow{\sim} \mathbf{C}^{S \times J}$.

8.8. Let A be as in 8.3 and let's calculate the periods $p'(\sigma, \tau)$, modulo $\bar{\mathbf{Q}}^*$, for the motive $H^1(A)$. We will start by extending scalars from k to $\bar{\mathbf{Q}}^*$, by means of σ. If $n(\sigma, \tau) = 1$, there exists a holomorphic 1-form ω defined over $\bar{\mathbf{Q}}$ such that $u^*\omega = \tau(u)\omega$ for $u \in E$, and, for $Z \in H_1(A(\mathbf{C}))$, we have $p'(\sigma, \tau) \sim \int_Z \omega$. We can pass from there to the general case by means of the formula $p'(\sigma, \tau) \cdot p'(\sigma, c\tau) \sim 2\pi i$.

8.9. Conjecture 8.1(ii) asserts in particular that if two motives satisfy the condition of 8.1(i), they have the same period p. For the motives as in 8.4, the periods are expressed in terms of values of the Γ-function and, working mod $\bar{\mathbf{Q}}^*$, 8.1 suggests the formation of the conjecture of B. Gross [**7**] relating certain periods with products of values of the Γ-function.

Comparing 8.4 and 8.5 leads to Conjectures 8.11 and 8.13 below. The result stated after 0.10 implies their proofs.

8.10. Let N be an integer, $k = \mathbf{Q}(\exp(2\pi i/N))$, $\mathcal{P}$ a prime ideal of k, prime to N, $k_\mathcal{P}$ the residue field and $q = N\mathcal{P} = |k_\mathcal{P}|$. We will write t for the inverse of reduction mod $\mathcal{P}$, from the $N^{\rm th}$ roots of 1 in $k_\mathcal{P}$ to those of k.

For $a \in N^{-1}\mathbf{Z}/\mathbf{Z}, a \neq 0$, consider the Gauss sum $g(\mathcal{P}, a, \Psi) = -\sum t(x^{-a(q-1)})\Psi(x)$. The sum is extended to $k_\mathcal{P}^*$ and $\Psi : k_\mathcal{P} \to \mathbf{C}^*$ is a nontrivial additive character.

Let $\boldsymbol{a} = \sum N(a)\delta_a$ in the free abelian group with basis $N^{-1}\mathbf{Z}/\mathbf{Z} - \{0\}$. If $\sum n(a)a = 0$, the product of the $g(\mathcal{P}, a, \Psi)^{n(a)}$ is independent of Ψ and so we set $g(\mathcal{P}, \boldsymbol{a}) = g(\mathcal{P}, a, \Psi)^{n(a)}$. Weil [**16**] has shown that, as a function of $\mathcal{P}$, $g(\mathcal{P}, \boldsymbol{a})$ is an algebraic Hecke character χ_a of k with values in k. Write $\langle a \rangle$ for the representative of a between 0 and 1 in $\mathbf{Z}/N$. If $\boldsymbol{a} = \sum n(a)\delta_a$ satisfies

$(*)$ For every $u \in (\mathbf{Z}/N)^*$, we have $\sum n(a)\langle ua \rangle = 0$,

then by Weil's calculation of the algebraic part of χ_a, $\chi_{\boldsymbol{a}}$ is of finite order; we will still write χ_a for the character of $\mathrm{Gal}(\bar{\mathbf{Q}}/k)$ equal to $\chi_a(\mathcal{P})$ on the geometric Frobenius in $\mathcal{P}$.

Set $\Gamma(\boldsymbol{a}) = \Gamma(\langle a \rangle)^{n(a)}$. One of the first parts of the conjecture: the algebraicity of $\Gamma(\boldsymbol{a})$ when $\boldsymbol{a}$ satisfies $(*)$, has been proven by Koblitz and Ogus: see the appendix. More precisely, we wish to obtain:

Conjecture 8.11. If $\boldsymbol{a}$ satisfies $(*)$, we have $\sigma\Gamma(\boldsymbol{a}) = \chi_a(\sigma) \cdot \Gamma(\boldsymbol{a})$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/k)$.

If $\boldsymbol{a}$ is invariant under a subgroup H of $(\mathbf{Z}/N)^*$, we can make 8.11 more precise by replacing $\mathrm{Gal}(\bar{\mathbf{Q}}/k)$ by $\mathrm{Gal}(\bar{\mathbf{Q}}/k^H)$, and by using SGA $4^{1/2}$ 6.5 to define a character of this group, with values in the roots of unity of k^H. This is what we do below.

8.12. Let H be a subgroup of $(\mathbf{Z}/N)^*$, $\mathcal{P}$ a prime ideal of k^H, prime to N, κ its residue field and $\Psi : \kappa \to \mathbf{C}^*$ a nontrivial additive character. For $a \in N^{-1}\mathbf{Z}/\mathbf{Z} - \{0\}$, let $\mathcal{P}_{a,i}$ be the prime ideals of $k^H(\exp(2\pi i a))$ lying above $\mathcal{P}$. If κ' is the residue field of $\mathcal{P}_{a,i}$, and if $|\kappa'| = q'$, we write $g(\mathcal{P}_{a,i}, a, \Psi)$ for the Gauss sum $-\sum t(x^{-a(q'-1)})\Psi(\mathrm{Tr}_{\kappa'/\kappa} x)$ (the sum extended to κ'^*). The product $g(\mathcal{P}, a, \Psi) = \prod_i g(\mathcal{P}_{a,i}, a, \Psi)$ depends only on the orbit of a under H.

Let $\boldsymbol{a} = \sum n(a)\delta_a$, invariant under H, and satisfying $\sum n(a)a = 0$. For every orbit 0 of H in $N^{-1}\mathbf{Z}/\mathbf{Z} - \{0\}$, we write $n(0)$ for the constant value of $n(a)$ on 0. We set

$$g(\mathcal{P}, \boldsymbol{a}) = \prod_{a \bmod H} g(\mathcal{P}, a, \Psi)^{n(a)}.$$

As a function of $\mathcal{P}$, $g(\mathcal{P}, \boldsymbol{a})$ is an algebraic Hecke character χ_a of k^H with values in k^H. To prove this, by additivity we may assume $n(a) \geq 0$ for all a. We then apply [**3**, 6.5] for $F = k^H, \bar{F} = \bar{\mathbf{Q}}, k =$ our k, and $I =$ a disjoint sum of copies of orbits $H : n(0)$ copies of 0; for $i \in I$, with image a/N in $N^{-1}\mathbf{Z}/\mathbf{Z}$, we let λ_i be the composition $\hat{\mathbf{Z}}(1)_{\bar{F}} \to (\mathbf{Z}/N)(1)_{\bar{F}} = \mu_N(k) \stackrel{x^a}{\to} k^*$; the Hecke character thus obtained is the product of χ_a with the character "signature of the permutation representation of H on I". One particular case of this result already appears in Weil [**17**]. The character χ_a in 8.10 is the composition of χ_a above with the norm N_{k/k^H}.

Conjecture 8.13. *Let $\mathcal{P}$ be a prime ideal of k^H, prime to N, and $F_{\mathcal{P}}$ a geometric Frobenius of $\mathcal{P}$. If $\boldsymbol{a}$, invariant under H, satisfies* $(*)$, *we have*

$$F_{\mathcal{P}}\Gamma(\boldsymbol{a}) = g(\mathcal{P}, \boldsymbol{a}) \cdot \Gamma(\boldsymbol{a}).$$

In particular, if the group of roots of unity of k^H is of order N', we have $\Gamma(\boldsymbol{a})^{N'} \in k^H$.

From 8.13 we can derive the following, apparently more general, variation. We replace the condition $(*)$ by

$$(*') \qquad \sum n(a)\langle ua\rangle = k \text{ is an integer independent of } u \in (\mathbf{Z}/N)^*.$$

The Hecke character $N\mathcal{P}^{-k}\cdot g(\mathcal{P}, \boldsymbol{a})$ is therefore of finite order. We identify it with a character χ of $\mathrm{Gal}(\bar{\mathbf{Q}}/k^H)$, and we hope to have

$$\sigma((2\pi i)^{-k}\Gamma(\boldsymbol{a})) = \chi(\sigma) \cdot ((2\pi i)^{-k}\Gamma(\boldsymbol{a})).$$

8.14. If k is a totally imaginary quadratic field extension of a totally real field, or, as we shall say, a field of CM-type, Shimura [**13**] has determined the critical values of L-functions of algebraic Hecke characters of k, up to multiplication by an algebraic number. He expresses them in terms of periods of abelian varieties of CM-type, with complex multiplication *by* k. In the rest of the section, we will show that his theorem is compatible with Conjecture 2.8, which expresses them in terms of periods of motives *over* k, of rank 1.

8.15. Let χ and $M(\chi)$ be as in 8.1. Exclude the case 8.5, and suppose that $R_{k/\mathbf{Q}}M(\chi)$ satisfies 1.7. The field k is therefore totally imaginary, and the Hodge numbers h^{pp} are zero: with the notation of 8.1 and 8.2, no n_i is equal to $w/2$. Our first job is to calculate $c^+R_{k/\mathbf{Q}}M(\chi)$ in terms of periods $p'(\chi;\sigma,\tau)$. Recall that it is the determinant of the $E\otimes\mathbf{C}$-module isomorphism

$$I^+ : H_B^+R_{k/\mathbf{Q}}M(\chi)\otimes\mathbf{C} \xrightarrow{\sim} H_{\mathrm{DR}}R_{k/\mathbf{Q}}M(\chi)\otimes\mathbf{C},$$

calculated in bases defined over E.

Choose the e_σ in 8.6 such that $F_\infty e_\sigma = e_{c\sigma}$. Therefore the $e_\sigma \pm e_{c\sigma}$ form a basis of $H_B^{\pm}R_{k/\mathbf{Q}}M(\chi) \subset H_BR_{k/\mathbf{Q}}M(\chi) = \oplus_{\sigma\in S}H_\sigma(M(\chi))$. We note in passing that $d^+ = d^- = \frac{1}{2}[k:\mathbf{Q}]$. Let $\bar{S}$ be the quotient of S by $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$. To calculate $c^+ = \det(I^+)$, we will use the basis $(e_\sigma + e_{c\sigma})$ of H_B^+. It is indexed by $\bar{S}$.

By 8.1(iv), the Hodge filtration of $H_{\mathrm{DR}}R_{k/\mathbf{Q}}M(\chi) = H_{\mathrm{DR}}(M(\chi))$ depends on its $k\otimes E$-module structure: if we write $(k\otimes E)^+$ for the direct factor of $k\otimes E$ consisting of the product of the K_i such that $n_i < w/2$, the quotient $H_{\mathrm{DR}}^+R_{k/\mathbf{Q}}M(\chi)$ of $H_{\mathrm{DR}}R_{k/\mathbf{Q}}M(\chi)$ is the corresponding direct factor:

$$H_{\mathrm{DR}}^+R_{k/\mathbf{Q}}M(\chi) = H_{\mathrm{DR}}(M(\chi))\otimes_{k\otimes E}(k\otimes E)^+.$$

Let ω be as in 8.7, and use the $k\otimes E\otimes\mathbf{C} = \mathbf{C}^{S\times J}$-module structure of $H_{\mathrm{DR}}(M(\chi))$ to decompose ω : $\omega = \sum\omega_{\sigma,\tau}$. By definition, we have $I(e_\sigma) = \sum_\tau p'(\sigma,\tau)^{-1}\omega_{\sigma,\tau}$, and therefore

$$I^+(e_\sigma + e_{c\sigma}) = \sum_{n(\sigma,\tau)<w/2} p'(\sigma,\tau)^{-1}\omega_{\sigma,\tau} + \sum_{n(\sigma,\tau)>w/2} p'(c\sigma,\tau)^{-1}\omega_{c\sigma,\tau};$$

it is the sum, indexed by $\tau\in J$, of terms equal to $p'(\sigma,\tau)^{-1}\omega_{\sigma,\tau}$ for $n(\sigma,\tau)<w/2$, and equal to $p'(c\sigma,\tau)^{-1}\omega_{c\sigma,\tau}$ for $n(\sigma,\tau)>w/2$. For $\bar\sigma\in\bar{S}$, with representative σ, set

$$\omega_{\bar\sigma} = \sum_{n(\sigma,\tau)<w/2}\omega_{\sigma,\tau} + \sum_{n(c\sigma,\tau)<w/2}\omega_{c\sigma,\tau}.$$

The $I^+(e_\sigma + e_{c\sigma})$ are multiples of the $\omega_{\bar\sigma}$ by elements of $E\otimes\mathbf{C}$; the $\omega_{\bar\sigma}$ therefore form a basis of H_{DR}^+. In the bases $e_\sigma + e_{c\sigma}$ and $\omega_{\bar\sigma}$, the matrix of I^+ is diagonal; its determinant $\det'(I^+)\in(E\times\mathbf{C})^* = \mathbf{C}^{*J}$ has for its coordinates

$$\det{}'(I^+)_\tau = \prod_{n(\sigma,\tau)<w/2} p'(\sigma,\tau)^{-1}.$$

Let the functions $E^{\bar{S}}\to E^S : 1_{\bar\sigma}\to 1_\sigma + 1_{c\sigma}$, for $\bar\sigma$ the image of σ, $E^S\otimes\mathbf{C}\xrightarrow{\sim} k\otimes E\otimes\mathbf{C}$, be derived from the isomorphism of $k\otimes\mathbf{C}$ with $\mathbf{C}^S$, and the projection of $k\otimes E$ to $(k\otimes E)^+$. Composing the functions, we obtain an $E\otimes\mathbf{C}$-module isomorphism:

$$E^{\bar{S}}\otimes\mathbf{C}\xrightarrow{\sim}(k\otimes E)^+\otimes\mathbf{C}.$$

We will write its determinant as $D(\chi)$, calculated on each side in a basis defined over E. Identifying H_{DR} with $k\otimes E$ by means of the basis ω, we see that this is the determinant of the identity function of $H_{\mathrm{DR}}^+\otimes\mathbf{C}$, calculated on the left side in the basis $\omega_{\bar\sigma}$, and on the right side in a basis defined over E. Writing $D_\tau(\chi)$ for its components in $\mathbf{C}^J$, we find, for $c^+ = \det'(I^+) \cdot D(\chi)$, the following formula.

PROPOSITION 8.16. *We have*

$$c^+ R_{k/\mathbf{Q}} M(\chi) = \left(\prod_{n(\sigma,\tau)<w/2} p'(\sigma,\tau)^{-1} \cdot D_\tau(\chi) \right)_{\tau\in J}.$$

REMARK 8.17. Suppose k is of CM-type, a quadratic extension of k_0, totally real. The quotient $\bar{S}$ of S is therefore identified with the set of complex embeddings of k_0, and the diagram

$$\begin{array}{ccccc} E^{\bar{S}} \otimes \mathbf{C} & \xrightarrow{\sim} & k_0 \otimes E \otimes \mathbf{C} & & \\ \downarrow & & \downarrow & & \\ E^S \otimes \mathbf{C} & \longrightarrow & k \otimes E \otimes \mathbf{C} & \longrightarrow & (k\otimes E)^+ \otimes \mathbf{C} \end{array}$$

is commutative. The composition function $k_0 \otimes E \to (k \otimes E)^+$ is therefore an isomorphism, and $D(\chi)$ is still the determinant of $E^{\bar{S}} \otimes \mathbf{C} \xrightarrow{\sim} k_0 \otimes E \otimes \mathbf{C}$, derived by extension of scalars from $\mathbf{Q}$ to E from the isomorphism $C^{\bar{S}} \xrightarrow{\sim} k_0 \otimes \mathbf{C}$. This provides us with a representative for $D(\chi)$ in $(\mathbf{Q}\otimes\mathbf{C})^* = \mathbf{C}^* \subset (E\otimes\mathbf{C})^*$, well defined mod E^*, namely, the determinant of the inverse of the matrix (σa), for $\sigma \in \bar{S}$ and a running through a basis of k_0 over $\mathbf{Q}$. The isomorphism $\mathbf{C}^{\bar{S}} \xrightarrow{\sim} k_0 \otimes \mathbf{C}$ transforms the quadratic form $\sum {x_i}^2$ into the form $\mathrm{Tr}(xy)$. This allows us to identify $(\det(\sigma a))^2$ with the discriminant of k_0:

$$D(\chi) \sim \text{ the square root of the discriminant of } k_0.$$

8.18. Write $p''(\chi;\sigma,\tau)$ for the image of $p'(\chi;\sigma,\tau)$ in $\mathbf{C}^*/\bar{\mathbf{Q}}^*$. It depends only on χ, σ and τ. If an algebraic homomorphism $\eta : k^* \to E^*$ satisfies (8.2.1), one of its powers is the algebraic part of a Hecke character: $\eta^N = \chi_{\text{alg}}$. Also, if ${\chi_{\text{alg}}}' = {\chi_{\text{alg}}}''$, χ' and χ'' only differ by a character of finite order and ${\chi'}^M = {\chi''}^M$ for a suitable M. We obtain from (8.1.1) that $p''(\chi^M;\sigma,\tau) = p''(\chi;\sigma,\tau)^M$, and this allows us to state without ambiguity

$$p(\eta;\sigma,\tau) = p(\chi;\sigma,\tau)^{1/N}, \text{ for } \eta^N = \chi_{\text{alg}}.$$

These periods obey the following formalism:

(8.18.1) $p(\eta'\eta'';\sigma,\tau) = p(\eta';\sigma,\tau)p(\eta'';\sigma,\tau)$.

(8.18.2) $p(\eta;\sigma,\tau)$ does not change when we replace E by an extension E' of E, and τ by one of its extensions to E'.

(8.18.3) $p(\eta;\sigma,\tau)$ does not change when we replace k by an extension k' of k, σ by one of its extensions to k', and χ by $\chi \circ N_{k'/k}$.

(8.18.4) If α is an automorphism of k, and β is an automorphism of E, we have $p(\eta;\sigma,\tau) = p(\beta\eta\alpha^{-1}, \sigma\alpha^{-1}, \tau\beta^{-1})$.

(8.18.5) The complex conjugate of $p(\eta;\sigma,\tau)$ is $p(\eta;\bar{\sigma},\bar{\tau})$.

(8.18.6) For $k = F = \mathbf{Q}$, $p(\mathrm{Id};\,\mathrm{Id},\,\mathrm{Id}) = 2\pi i$.

The formulas (8.15.1) to (8.15.3) are consequences of (8.1.1) to (8.1.3); (8.18.4) and (8.18.5) can be seen by transport of structure, and (8.18.6) is a consequence of (8.5).

8.19. Let $\eta : k^* \to E^*$ be a homomorphism satisfying (8.2.1). We also assume that $n(\eta;\sigma,\tau)$ is never $w/2$, which allows us to define $(k\otimes E)^+$ as in 8.15. Define $\eta^* : E^* \to k^*$ by

$$\eta^*(y) = \det_k(1\otimes y, (k\otimes E)^+).$$

This homomorphism still satisfies (8.2.1), and

$$\begin{aligned} n(\eta^*;\sigma,\tau) &= 1 \text{ if } n(\eta;\sigma,\tau) < w/2,\\ &= 0 \text{ if } n(\eta;\sigma,\tau) > w/2. \end{aligned}$$

If η^* is the algebraic part of a Hecke character χ^*, $M(\chi^*)$ is the H^1 of an abelian variety over E, with complex multiplication by k, whose Lie algebra is isomorphic to $(k\otimes E)^+$, as a $k\otimes E$-module.

PROPOSITION 8.20. *With the hypotheses and notation of* 8.19, *Let E be a subfield of* $\mathbf{C}$, *and write* 1 *for the inclusion of E in* $\mathbf{C}$. *We have*

$$\prod_{n(\sigma,1)<w/2} p(\eta;\sigma,1) = \prod_\sigma p(\eta^*;1,\sigma)^{n(\eta;\sigma,1)}.$$

If $\iota : E \to E' \subset \mathbf{C}$ is a finite extension of E, the left-hand side does not change when we replace η by $\iota\eta$ (8.18.2). We have $(\iota\eta)^* = \eta^*\circ N_{E'/E}$ and, by (8.18.3), the right-hand side does not change either.

If $\iota : k \to k'$ is a finite extension of k of degree d, the left-hand side is taken to the power d when we replace k by k', and η by $\eta\circ N_{k'/k}$: a complex embedding of k is induced by d embeddings of k', and we apply (8.18.3). The same is true for the right-hand side, by (8.18.2) and the equality $(\eta\circ N_{k'/k})^* = \iota\eta^*$.

These compatibilities lead us to suppose that E is Galois and that k is isomorphic to E. For each isomorphism ω of k with E, set $n(\omega) = n(\eta; 1\circ\omega,\omega)$. Writing the group of homomorphisms of k^* in E^* additively, we have $\eta = \sum n(\omega)\omega$. Since $p(\eta;1\circ\omega,1) = p(\eta\circ\omega^{-1};1,1)$ (8.15.4), we have

$$\text{(8.20.1)}\qquad \prod_{n(\sigma,1)<w/2} p(\eta;\sigma,1) = \prod_{n(\omega)<w/2} p(\eta\circ\omega^{-1};1,1) = p\left(\sum_{n(\omega)<w/2} \eta\circ\omega^{-1};1,1\right).$$

On the other hand,

$$\begin{aligned} \sum_{n(\omega)<w/2} \eta\circ\omega^{-1} &= \sum_{n(\omega_1)<w/2;\,\omega_2} n(\omega_2)\omega_2\circ{\omega_1}^{-1} = \sum_{\omega_2} n(\omega_2) \sum_{n(\omega_1)<w/2} \omega_2\circ{\omega_1}^{-1}\\ &= \sum_{\omega_2} n(\omega_2)\omega_2\circ\eta^*. \end{aligned}$$

This allows us to continue (8.20.1) by

$$= \prod_\omega p(\omega\circ\eta^*;1,1)^{n(\omega)} = \prod_\omega p(\eta^*;1,1\circ\omega)^{n(\omega)} = \prod_\sigma p(\eta^*;1,\sigma)^{n(\eta;\sigma,1)},$$

(applying 8.15.4 again), which proves 8.20.

8.21. Combining 8.16 and 8.20, we find the following expression, mod $\bar{\mathbf{Q}}^*$, for the component with index 1 of $c^+R_{k/\mathbf{Q}}M(\chi) \in (E \otimes \mathbf{C})^*/E^* = \mathbf{C}^{*J}/E^*$,

$$c_1{}^*R_{k/\mathbf{Q}}M(\chi) \sim \prod_\sigma p(\chi^*_{\text{alg}}; 1, \sigma)^{n(\chi;\sigma,1)}.$$

If χ (i.e., $M(\chi)$) is critical, Conjecture 2.8 then asserts that

$$L(1 \circ \chi, 0) \sim \prod_\sigma p(\chi^*_{\text{alg}}; 1, \sigma)^{-n(\chi;\sigma,1)} \qquad (\text{mod } \bar{\mathbf{Q}}^*).$$

For E large enough, χ^*_{alg} is the algebraic part of a Hecke character χ^*, and the periods can be interpreted as periods of abelian integrals (8.19), (8.3). The final result is that which Shimura proved for k of CM-type, or abelian over a field of CM-type (with a restriction on the weight).

REMARK 8.22. If $\eta : k^* \to E^*$ satisfies (8.2.1), there exist subfields k' of k and $\iota : E' \to E$ of E, either of CM-type, or equal to $\mathbf{Q}$, and a factorization $\eta = \iota\eta' N_{k'/k}$. We then have $p(\eta; \sigma, \tau) = p(\eta'; \sigma|k', \tau|k')$.

If k and E are now of CM-type (or $\mathbf{Q}$), still writing c for their complex conjugation, we have $c\sigma = \sigma c$, $c\tau = \tau c$, and $\eta c = c\eta$, so

$$p(\eta; \sigma, \tau)^- = p(\eta; c\sigma, c\tau) = p(\eta; \sigma c, \tau c) = p(c\eta c^{-1}; \sigma, \tau) = p(\eta; \sigma, \tau) :$$

the periods, a priori in $\mathbf{C}^*/\bar{\mathbf{Q}}^*$, are real, i.e., in $\mathbf{R}^*/(\bar{\mathbf{Q}}^* \cap \mathbf{R}^*)$.

REMARK 8.23. Let G be the Galois group of the union of extensions of CM-type of $\mathbf{Q}$ in $\bar{\mathbf{Q}} \subset \mathbf{C}$, and $c \in G$ complex conjugation. It is a central element of G. If ϕ is a locally constant function on G with integer values, we will set $\phi^*(x) = \phi(xc)$. Suppose that $\phi + \phi^*$ is constant. Let G_1 be a finite quotient of G such that ϕ factors by a function ϕ_1 on G_1, and k is the corresponding field. The hypothesis above implies that the endomorphism $\sum \phi_1(\sigma)\sigma$ of k^* satisfies (8.2.1). The period $p(\sum \phi_1(\sigma)\sigma; 1, 1)$ does not depend on the choice of G_1; we set $P(\phi) = p(\sum \phi_1(\sigma)\sigma; 1, 1)$. The functional P is a homomorphism in $\mathbf{C}^*/\bar{\mathbf{Q}}^*$ of the group of locally constant functions with integer values on G which satisfy $\phi + \phi^* =$ constant.

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