

# SUMS OF FRACTIONS AND FINITENESS OF MONODROMY

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ABSTRACT. We solve an elementary number theory problem on sums of fractional parts. We apply our result to deduce the finiteness of the image of certain monodromy representations.

## 1. INTRODUCTION

In this paper, we are concerned with an elementary number theoretic question on a sum of certain fractional parts. The simplest instance of this is when there are only three fractional parts involved. The classification of the three fractions satisfying the underlying number theoretic condition is known to be equivalent to Schwarz's classification of algebraic Euler-Gauss hypergeometric functions. We give a different proof of the Schwarz classification using elementary considerations. We then use the Schwarz classification to show that the number theoretic condition holds only sporadically when the number of fractional parts is four or five and does not hold at all when the number of fractional parts is more than six (see Theorem 1).

It turns out that the number theoretic condition is closely connected to the finiteness of certain monodromy groups. As a consequence of our main result on fractional parts, we classify when the image of certain specializations of the so called Gassner representation is finite. By linking these specializations with the monodromy representations associated to certain families of cyclic coverings of the projective line of the type considered by Deligne and Mostow (see [Del-Mos]), we recover results of Cohen and Wolfart [Coh-Wol] on finiteness of monodromy groups (see Theorem 8). Another corollary of the main result on fractional parts is the algebraicity of certain Lauricella  $F_D$ -type functions, also proved in [Coh-Wol] (see also [Ssk] and [Bod]).

We note that the methods used in previous works are different from those used in this paper. For instance, one of the proofs in [Coh-Wol] uses the more difficult classification of Deligne and Mostow [Del-Mos] of

those monodromy groups which are arithmetic subgroups of  $U(n-1, 1)$ . This classification is also described by Looijenga [Loo] (see also the table at the end of [Thu]). However, we do not use this classification, and directly classify the finite monodromy groups starting essentially from Schwarz's list.

We now go into some detail. First, we introduce some notation.

**Notation.** Let  $d \geq 2$  and  $n \geq 2$  be integers. Fix  $n+1$  integers  $1 \leq k_i \leq d-1$  such that the g.c.d. of  $d, k_1, \dots, k_{n+1}$  is 1. Given  $s$  in the multiplicative group  $(\mathbb{Z}/d\mathbb{Z})^*$  of units of  $\mathbb{Z}/d\mathbb{Z}$ , consider the numbers  $\mu_i(s)$  (denoted  $\mu_i = \{\frac{k_i}{d}\}$  when  $s = 1$ ) defined by

$$\mu_i(s) = \left\{ \frac{k_i s}{d} \right\},$$

where  $0 \leq \{x\} < 1$  denotes the fractional part of a real number  $x$ . We may write  $k_i s = q_i d + l_i$ , where  $1 \leq l_i \leq d-1$  and  $q_i$  is an integer; thus the remainder  $l_i$  has the property that  $\{\frac{k_i s}{d}\} = \frac{l_i}{d}$ . If we denote by  $[x]$  the integral part of  $x$ , then  $x = [x] + \{x\}$ . The number  $\mu_i(s) = \{\frac{k_i s}{d}\}$  depends only on the fraction  $\frac{k_i}{d} = \mu_i$ .

**Definition 1.** We say that the rational numbers  $\mu_1, \dots, \mu_{n+1}$  satisfy condition (1) if,

$$(1) \quad \forall s \in (\mathbb{Z}/d\mathbb{Z})^*,$$

$$\text{either } \sum_{i=1}^{n+1} \left\{ \frac{k_i s}{d} \right\} < 1 \quad \text{or} \quad \sum_{i=1}^{n+1} \left\{ -\frac{k_i s}{d} \right\} < 1.$$

In terms of the remainders  $l_i$  above, this means that either  $\sum l_i < d$  or else  $\sum (d - l_i) < d$ . We remark that a similar condition appears in [Loo, Thm. 4.3] (see also [Del-Mos, Prop. 12.7]).

The following theorem says that condition (1) on  $n, d, k_i$  is very stringent and holds in a very limited number of cases. Let us say that the tuple  $(\frac{k_1}{d}, \dots, \frac{k_{n+1}}{d})$  is *equivalent* to the tuple  $(\frac{l_1}{d}, \dots, \frac{l_{n+1}}{d})$ , if there exists  $t \in (\mathbb{Z}/d\mathbb{Z})^*$  such that, for all  $i$ , we have  $\{\frac{l_i}{d}\} = \{\frac{k_i t}{d}\}$ , up to a permutation of the indices. The validity of condition (1) depends only on the equivalence class of the tuple  $(\frac{k_1}{d}, \dots, \frac{k_{n+1}}{d})$ .

**Theorem 1.** *Suppose  $n, d, k_i$  are as in the foregoing and satisfy condition (1). Then*

$$n \leq 4.$$

Moreover, up to equivalence, the numbers  $\mu_1, \dots, \mu_{n+1}$  satisfy the conditions given below.

- (i) If  $n = 4$ , then  $\mu_i = \frac{1}{6}$ , for all  $i \leq n + 1 = 5$ .
- (ii) If  $n = 3$ , then there are only two cases:  $\mu_i = \frac{1}{6}$ , for all  $i \leq n + 1 = 4$ , or  $\mu_1 = \mu_2 = \mu_3 = \frac{1}{6}$  and  $\mu_4 = \frac{2}{6}$ .
- (iii) If  $n = 2$ , then either we may write  $\mu_i = \frac{k_i}{d}$  with  $d = 2m$ , for  $m \geq 1$ , and  $k_1 = k_2 = p$ ,  $k_3 = m - p$ , with  $1 \leq p \leq m - 1$  coprime to  $m$ , or else, the  $\mu_i = \frac{k_i}{d}$  lie in a finite list with  $d \leq 60$ .

*Remark.* Note that condition (1) is a purely number theoretic condition; the proof of the theorem, however, will depend on an analysis of certain finite subgroups of unitary groups generated by reflections.

The proof of Theorem 1 proceeds as follows. In Section 2, we prove the theorem for  $n = 2$ . We link condition (1) in the case  $n = 2$  to the finiteness of a certain subgroup of the unitary group of an explicit skew-Hermitian form (implicitly, this is the monodromy group of the Gauss hypergeometric function, but we do not use this). Then in Section 3, we use a bootstrapping argument to show that condition (1) holds in very few cases when  $n = 3$  and 4. Using this it is finally shown that condition (1) cannot hold for  $n \geq 5$ .

In Section 5, we show that a slightly modified version of condition (1), namely condition (6) in the text, is equivalent to the total anisotropy of an explicit skew-Hermitian form in  $n$  variables over the cyclotomic field  $E = \mathbb{Q}(e^{2\pi i \frac{1}{d}})$ . Conditions (1) and (6) coincide for  $n = 2$ , and we show that they are equivalent for general  $n$ . We deduce that the image of the Gassner representation at  $d$ -th roots of unity is finite if and only if condition (1) holds, providing us with the algebraicity results on monodromy groups mentioned above (see Theorem 8 in Section 4 below). As is (more or less) known, the finiteness of the image of the Gassner representation is equivalent to the algebraicity of the associated Lauricella  $F_D$ -functions and we list some of these results as corollaries in Section 6.

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about the material of the paper. That the existence of a totally definite Hermitian form is equivalent to finiteness of monodromy is already implicit in [CHL] and the authors thank E. Looijenga for mentioning this to one of them. The authors also thank the referee for providing us with several additional references.

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## 2. THE CASE $n = 2$ .

**2.1. Definition.** Let  $d, k_1, k_2, k_3$  be positive integers with g.c.d. equal to 1. We say that these integers satisfy condition (2) if,

$$\forall s \in (\mathbb{Z}/d\mathbb{Z})^*,$$

$$(2) \quad \text{either } \Sigma_s \stackrel{\text{def}}{=} \sum_{j=1}^3 \left\{ \frac{k_j s}{d} \right\} < 1 \quad \text{or} \quad \Sigma_{-s} = \sum_{j=1}^3 \left\{ -\frac{k_j s}{d} \right\} < 1.$$

*Remark.* [0] Condition (2) is just condition (1) for  $n = 2$ .

[1] Condition (2) depends only on the fractional parts  $\nu_j = \left\{ \frac{k_j}{d} \right\}$  of  $\frac{k_j}{d}$ , for  $j = 1, 2, 3$ , and not directly on the numbers  $(d, k_1, k_2, k_3)$ ; for example, condition (2) holds for  $(d, k_1, k_2, k_3)$  if and only if it holds for  $(d, k_1 + d, k_2, k_3)$ , etc.

[2] We may also permute the integers  $k_1, k_2, k_3$  without changing condition (2).

[3] If  $(d, k_1, k_2, k_3)$  is replaced by  $(d, k_1 t, k_2 t, k_3 t)$  for some integer  $t$  coprime to  $d$ , then condition (2) is unaltered.

[4] Since  $\{-x\} = 1 - \{x\}$ , for a real number  $x$ , condition (2) is equivalent to saying that either  $0 < \Sigma_s < 1$  or  $2 < \Sigma_s < 3$ , for each sum  $\Sigma_s$ . That is, the integral part of each sum  $\Sigma_s$  is either 0 or 2 (but not 1).

**2.2. Main result for triples.** We say that a triple of rational numbers  $(\nu_1, \nu_2, \nu_3)$  as above is equivalent to another such triple  $(\nu'_1, \nu'_2, \nu'_3)$  (for the same denominator  $d$ ), if there exists  $t \in (\mathbb{Z}/d\mathbb{Z})^*$  such that, after a permutation of the indices, we have  $\nu_j = \left\{ \frac{k_j}{d} \right\}$  and  $\nu'_j = \left\{ \frac{k_j t}{d} \right\}$ , for  $j = 1, 2, 3$ . By the remarks in the preceding subsection, if condition

(2) holds for one triple, then it holds for all equivalent triples.

For  $d$  and  $k_1, k_2, k_3$  as above, write

$$\begin{aligned}\lambda &= 1 - \left\{ \frac{k_1}{d} \right\} - \left\{ \frac{k_2}{d} \right\}, \\ \mu &= 1 - \left\{ \frac{k_1}{d} \right\} - \left\{ \frac{k_3}{d} \right\}, \\ \nu &= 1 - \left\{ \frac{k_2}{d} \right\} - \left\{ \frac{k_3}{d} \right\}.\end{aligned}$$

If  $(\frac{k_1}{d}, \frac{k_2}{d}, \frac{k_3}{d})$  satisfy condition (2), we may assume that  $0 < \lambda, \mu, \nu < 1$ .

**Theorem 2.** *(The case  $n = 2$ ) If  $(d, k_1, k_2, k_3)$  satisfy condition (2), then up to the foregoing equivalence, we have either*

$$\frac{k_1}{d} = \frac{k_2}{d} = \frac{p}{2m}, \quad \text{and} \quad \frac{k_3}{d} = \frac{m-p}{2m},$$

for some  $m \geq 1$  and some  $1 \leq p < m$  coprime to  $m$ , so that

$$\lambda = \frac{m-p}{m}, \quad \mu = \nu = \frac{1}{2}$$

(we refer to this as the “dihedral case”), or else

$$(\lambda, \mu, \nu) \in \text{the finite list in Table 1 below.}$$

*Remark.* Again, though the statement of the theorem is purely (elementary) number theoretic, the proof uses the finiteness of a certain group  $\Gamma$  in  $GL_2(\mathbb{C})$ . It would be interesting to find a purely number theoretic proof of the above theorem.

### 2.3. Relation of Condition (2) with a skew-Hermitian form.

**Notation.** Let  $E/F$  be a totally imaginary quadratic extension of a totally real number field. Then  $E = F[t]/(t^2 + \alpha)$  for some totally positive element  $\alpha$  in the real subfield  $F$ .  $E/F$  is called a CM extension. Denote by  $z \mapsto \bar{z}$  ( $\forall z \in E$ ) the action of the non-trivial element of the Galois group of  $E/F$ , induced by complex conjugation (under any embedding of  $E$  into  $\mathbb{C}$ ). Let  $h : E^n \times E^n \rightarrow E$ , with  $(x, y) \mapsto h(x, y)$ , be an  $F$ -bilinear form which is  $E$ -linear in the first variable  $x$  and such that for all  $x, y \in E^n$ ,  $h(y, x) = -\overline{h(x, y)}$ . Then  $h$  is called a skew-Hermitian form on  $E^n$ .

If we replace  $F$  by  $\mathbb{R}$  and  $E$  by  $\mathbb{C}$ , a skew-Hermitian form can still be defined and it is of the form  $h(x, y) = iH(x, y)$  where  $H$  is a Hermitian

form on  $\mathbb{C}^n$ .

We say that a skew-Hermitian form  $h$  on  $E^n$  is **anisotropic**, if  $h(x, x) = 0$ , for  $x \in E^n$ , implies that  $x = 0$ . Over  $\mathbb{C}/\mathbb{R}$ , a skew-Hermitian form  $h$  is anisotropic if and only if  $h = \pm iH$ , where  $H$  is Hermitian and positive definite. Furthermore, a diagonal skew-Hermitian form over  $\mathbb{C}/\mathbb{R}$  is anisotropic if and only if the diagonal entries are  $\lambda_1, \dots, \lambda_n$ , with  $\lambda_j \in i\mathbb{R} \setminus \{0\}$  being on the imaginary axis, and such that the successive ratios  $\lambda_{j+1}/\lambda_j$  are positive real numbers.

We say that a skew-Hermitian form  $h$  defined over  $E/F$  is **totally anisotropic** if it is anisotropic over  $\mathbb{C}/\mathbb{R}$ , for all embeddings of  $E$  into  $\mathbb{C}$ , or more precisely for all archimedean places of  $F$  into  $\mathbb{R}$ . Note that for a skew-Hermitian form  $h$  defined over  $E$ , anisotropy over  $\mathbb{C}$  implies anisotropy over  $E$ , but the converse does not hold.

Now let  $d$  and  $k_1, k_2, k_3$  be as above. Write  $x_j = e^{2\pi i \frac{k_j}{d}}$ , for  $j = 1, 2, 3$ . Let  $E = \mathbb{Q}(e^{\frac{2\pi i}{d}})$  be the  $d$ -th cyclotomic field, and let  $F = \mathbb{Q}(\cos(\frac{2\pi}{d}))$  be the maximal totally real subfield of  $E$ .

The matrix

$$h = \begin{pmatrix} \frac{1-x_1x_2}{(1-x_1)(1-x_2)} & -\frac{x_2}{1-x_2} \\ -\frac{1}{1-x_2} & \frac{1-x_2x_3}{(1-x_2)(1-x_3)} \end{pmatrix}$$

is easily seen to define a skew-Hermitian form over  $E/F$ , i.e.,  ${}^t h = -\bar{h}$ . The determinant  $\det(h)$  of  $h$  is also easily computed to be

$$\frac{1 - x_1x_2x_3}{(1-x_1)(1-x_2)(1-x_3)} = -\frac{1}{4} \cdot \frac{\sin(\frac{\pi(k_1+k_2+k_3)}{d})}{\sin(\frac{\pi k_1}{d}) \sin(\frac{\pi k_2}{d}) \sin(\frac{\pi k_3}{d})} \in F.$$

**Lemma 3.** *We have:*

- i) *The skew-Hermitian form  $h$  is totally anisotropic if and only if  $\det(h)$  is a totally negative element of  $F$ .*
- ii) *The numbers  $\frac{k_j}{d}$ , for  $j = 1, 2, 3$ , satisfy condition (2) if and only if the skew-Hermitian form  $h$  is totally anisotropic.*

*Proof.* Fix an embedding of  $E$  into  $\mathbb{C}$ . The Gram-Schmidt process says that under this embedding  $h$  is equivalent to the skew-Hermitian form  $h' = \begin{pmatrix} i\lambda_1 & 0 \\ 0 & i\lambda_2 \end{pmatrix}$  for some real numbers  $\lambda_1, \lambda_2$ . Moreover, the principal minors of  $h$  and  $h'$  are the same:  $\det(h) = \det(h')$  and  $i\lambda_1 = \frac{1-x_1x_2}{(1-x_1)(1-x_2)}$ .

The form  $h$  is anisotropic if and only if the equivalent form  $h'$  is anisotropic, and the latter holds if and only if the fraction  $\frac{\lambda_2}{\lambda_1}$  is positive. This fraction may also be written as

$$\frac{(i\lambda_1)(i\lambda_2)}{(i\lambda_1)^2} = \frac{\det(h)}{-\lambda_1^2}.$$

Thus  $h$  is anisotropic if and only if  $\det(h)$  is negative. This argument is independent of the embedding of the field  $E$  into  $\mathbb{C}$  and hence  $h$  is totally anisotropic if and only if its determinant is totally negative. This proves the first part of the lemma.

If  $t \in \mathbb{R} \setminus \mathbb{Z}$ , it is easily seen that the sign of  $\sin(\pi t)$  is  $(-1)^{[t]}$ , where  $[t]$  is the integral part of  $t$ . Therefore, by the paragraph preceding the statement of the lemma, the sign of the determinant of  $h$  is seen to be

$$-(-1)^{[\frac{k_1+k_2+k_3}{d}] - [\frac{k_1}{d}] - [\frac{k_2}{d}] - [\frac{k_3}{d}]}.$$

Now, for any three real numbers  $x, y, z$ , we have

$$[x + y + z] - [x] - [y] - [z] = [\{x\} + \{y\} + \{z\}].$$

Therefore, the sign of the determinant of  $h$  is  $-(-1)^{[\Sigma_1]}$  where  $\Sigma_1$  is the sum  $\sum_{j=1}^3 \{\frac{k_j}{d}\}$ . By condition (2) (see part [4] of the Remarks following the definition of (2)), the integral part of  $\Sigma_1$  is either 0 or 2 and hence the sign of the determinant of  $h$  is negative.

The same argument shows that the determinant of  $h_s$  is also negative, where  $h_s$  is the skew-Hermitian form which is obtained from  $h$  by changing  $x_j = e^{2\pi i \frac{k_j}{d}}$  to  $x_j^{(s)} = e^{2\pi i \frac{k_j s}{d}}$ . Here  $s \in (\mathbb{Z}/d\mathbb{Z})^*$  is viewed as an element ( $s$ ) of the Galois group of the cyclotomic extension  $E/\mathbb{Q}$ . The determinant of  $h_s$  is  $\det(h)^{(s)}$ , and is negative, whence  $\det(h)$  is totally negative. This proves the “only if” part of the second part of the Lemma.

The “if” part follows by retracing the proof of the “only if” part backwards.  $\square$

**2.4. Relation of the skew-Hermitian form  $h$  with a subgroup of  $U(h)$ .** Let  $\mathcal{O}_E, \mathcal{O}_F$  be the ring of integers of  $E$  and  $F$ . Suppose  $\Gamma \subset GL_2(\mathcal{O}_E)$  is the subgroup generated by the matrices

$$A = \begin{pmatrix} x_1 x_2 & 1 - x_1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ x_2(1 - x_3) & x_2 x_3 \end{pmatrix}.$$

It can be shown (for example, see [V], Lemmas 14 and 15 and Proposition 18), that  $\Gamma$  preserves the skew-Hermitian form  $h$  of the preceding subsection and that  $\Gamma$  acts irreducibly on  $E^2$  (the irreducibility is implied by the fact that the determinant of  $h$  is non-zero, since it is a nonzero multiple of  $1 - x_1x_2x_3$ . The number  $1 - x_1x_2x_3$  is nonzero since the sum  $\sum \frac{k_j}{d}$  is not an integer under the assumption (2)).

**Lemma 4.** *The group  $\Gamma$  is finite if and only if condition (2) holds for the numbers  $\frac{k_j}{d}$  ( $j = 1, 2, 3$ ).*

*Proof.* It is enough to show, because of Lemma 3, that  $\Gamma$  is finite if and only if  $h$  is totally anisotropic. This is proved in Lemma 11 below, for general  $n \geq 2$ .  $\square$

**2.5. The dihedral case.** Suppose that the finite group  $\Gamma \subset GL_2(\mathcal{O}_F)$  of the preceding subsection has the property that it has an abelian normal subgroup of index two. We then say that  $\Gamma$  is **dihedral**. Note that  $\Gamma$  is generated by two elements (namely  $A, B$ ).

**Lemma 5.**  *$\Gamma$  is dihedral if and only if two of the three elements  $A, B, C = AB$  have trace zero, i.e., if and only if two of the numbers  $x_1x_2, x_2x_3, x_3x_1$  are equal to  $-1$ .*

*Proof.* Suppose  $\Gamma$  is dihedral and  $N$  is an abelian normal subgroup of index two. Since  $\Gamma$  acts irreducibly on  $\mathbb{C}^2$ , it follows that  $\Gamma$  is not abelian, and hence there is an element  $g \notin N$  in  $\Gamma$ . Now  $N$  cannot consist of scalars. For otherwise the group generated by  $N$  and  $g$  would be abelian.

Let now  $g \notin N$  be arbitrary. Then  $g$  normalises (but does not centralise) the non-scalar abelian (and hence may be assumed to be diagonal) subgroup  $N$ . Therefore,  $g$  acts on  $N$  by the map switching the two diagonal entries of an element  $a \in N$ . Hence  $g$  is of the form  $tw$  where  $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $t$  is a diagonal matrix; hence the element  $g \notin N$  has trace zero.

Since  $\Gamma$  is generated by any two of the three matrices  $A, B, C = AB$ , it follows that two of these elements cannot lie in  $N$ ; therefore, two of the elements, say  $A$  and  $B$  have zero trace; this means that  $x_1x_2 + 1 = 0, x_2x_3 + 1 = 0$  (a small computation shows that  $\text{trace}(C) = (1 + x_1x_3)x_2$ ; hence  $\text{trace } C$  being zero implies that  $x_1x_3 = -1$ . Thus a similar statement holds if  $A, C$  do not lie in the subgroup  $N$ :  $x_1x_2 = x_1x_3 = -1$ ). This proves the lemma.  $\square$



The lemma means that the numbers  $\frac{k_1+k_3}{d} = \frac{1}{2}$  and  $\frac{k_2+k_3}{d} = \frac{1}{2}$  (say); suppose  $\frac{k_1+k_2}{d} = \frac{p}{m}$ , for some  $p$  coprime to  $m$ . Then it follows that  $\frac{k_1}{d} = \frac{k_2}{d} = \frac{p}{2m}$  and that  $\frac{k_3}{d} = \frac{m-p}{2m}$ . This is the first part of Theorem 2.

**2.6. Finite non-dihedral subgroups  $\Gamma$ .** It is well known that any irreducible non-dihedral finite subgroup of  $PGL_2(\mathbb{C})$  is the group of symmetries of one of the platonic solids (see, e.g., [Wey, App. A]). The following proposition is an immediate corollary, but can also be proved directly, for instance, by adapting the argument in [LT, Chap. 5, Sec. 4].

**Proposition 6.** *Suppose  $\Gamma \subset GL_2(\mathbb{C})$  is a finite non-dihedral irreducible subgroup with  $Z$  the centre of  $\Gamma$ . Then the order  $m$  of any element of the quotient  $\Gamma/Z$  does not exceed 5, i.e.,  $m = 1, 2, 3, 4, 5$ .*

We now return to the situation of Lemma 4. Consider  $x_j = e^{2\pi i \frac{k_j}{d}}$  ( $j = 1, 2, 3$ ). The irreducible finite subgroup  $\Gamma$  is generated by  $A = \begin{pmatrix} x_1x_2 & 1-x_1 \\ 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 \\ x_2(1-x_3) & x_2x_3 \end{pmatrix}$ . The image of  $\Gamma$  in  $PGL_2(\mathbb{C})$  contains the images  $A'$  and  $B'$  of  $A$  and  $B$  respectively; clearly the orders of  $A'$  and  $B'$  are respectively the orders of the roots of unity  $x_1x_2 = e^{2\pi i \frac{k_1+k_2}{d}}$  and  $x_2x_3 = e^{2\pi i \frac{k_2+k_3}{d}}$ .

A computation shows that the matrix

$$C = AB = \begin{pmatrix} x_2(1-x_3+x_1x_3) & x_2x_3(1-x_1) \\ x_2(1-x_3) & x_2x_3 \end{pmatrix}$$

has eigenvalues  $x_2$  and  $x_1x_2x_3$ ; clearly the order of the image of  $C$  in  $PGL_2(\mathbb{C})$  is the ratio of these eigenvalues  $\frac{x_1x_2x_3}{x_2} = x_3x_1 = e^{2\pi i \frac{k_3+k_1}{d}}$ . From the proposition follows the

**Corollary 1.** *If condition (2) holds, and  $\frac{k_1}{d}, \frac{k_2}{d}, \frac{k_3}{d}$  is not in the dihedral case, then the fractions  $\mu_1 = \frac{k_2+k_3}{d}, \mu_2 = \frac{k_3+k_1}{d}, \mu_3 = \frac{k_1+k_2}{d}$  are in the finite set  $S$  of fractions of the form  $\frac{t}{u}$  with  $t < u$  and  $u = 1, 2, 3, 4, 5$ .*

**2.7. A finite list.** Since the set  $S$  in Corollary 1 is finite, clearly the set of fractions  $\frac{k_1}{d}, \frac{k_2}{d}, \frac{k_3}{d}$  obtained from the set of  $\mu_1, \mu_2, \mu_3$  in  $S$  is also finite. Working up to permutation and up to the equivalence defined before, we may check that if  $\frac{k_1}{d}, \frac{k_2}{d}, \frac{k_3}{d}$  further satisfy condition (2), then the corresponding  $(\lambda, \mu, \nu)$  lie in the finite list in Table 1 below (we discard the dihedral cases with  $1 \leq p < m \leq 5$ ). This implies Theorem 2.  $\square$

Table 1: (Non-dihedral) Schwarz's List

$d$	$\mu_1$	$\mu_2$	$\mu_3$	$k_1/d$	$k_2/d$	$k_3/d$	$\lambda$	$\mu$	$\nu$	Wiki-row
12	2/3	2/3	1/2	1/4	1/4	5/12	1/2	1/3	1/3	2
6	2/3	2/3	1/3	1/6	1/6	1/2	2/3	1/3	1/3	3
30	2/3	2/3	3/5	3/10	3/10	11/30	2/5	1/3	1/3	7
60	2/3	3/5	1/2	13/60	17/60	23/60	1/2	2/5	1/3	14
30	2/3	3/5	2/5	1/6	7/30	13/30	3/5	2/5	1/3	15
24	3/4	2/3	1/2	5/24	7/24	11/24	1/2	1/3	1/4	4
12	3/4	3/4	1/3	1/6	1/6	7/12	2/3	1/4	1/4	5
10	3/5	3/5	3/5	3/10	3/10	3/10	2/5	2/5	2/5	11
60	4/5	2/3	1/2	11/60	19/60	29/60	1/2	1/3	1/5	6
30	4/5	2/3	1/3	1/10	7/30	17/30	2/3	1/3	1/5	12
15	4/5	2/3	2/5	2/15	4/15	8/15	3/5	1/3	1/5	10
20	4/5	3/5	1/2	3/20	7/20	9/20	1/2	2/5	1/5	9
30	4/5	4/5	1/3	1/6	1/6	19/30	2/3	1/5	1/5	8
10	4/5	4/5	1/5	1/10	1/10	7/10	4/5	1/5	1/5	13

All but the last column of Table 1 was generated using Pari-gp. The table is (the non-dihedral part of) Schwarz's well-known 1873 list [Sch], see Wikipedia: [https://en.wikipedia.org/wiki/Schwarz's\\_list](https://en.wikipedia.org/wiki/Schwarz's_list). The last column of Table 1 contains the row number of the corresponding entry in the Wikipedia table. Each of the 15 rows in that table is hit (the 1st row being the dihedral case).

Schwarz's list is known to classify the classical hypergeometric functions  ${}_2F_1$  that are algebraic. A classification of the higher algebraic hypergeometric functions (of Clausen-Thomae type) was given by Beukers and Heckman [Beu-Hec].

### 3. THE CASE $n \geq 3$

We now use a bootstrapping argument to prove the remaining parts of Theorem 1. We start with the following obvious lemma.

**Lemma 7.** *Let  $n \geq 3$ . Assume that the g.c.d. of  $d, k_1, k_2, \dots, k_{n+1}$  is equal to 1. If these integers satisfy condition (1), then so do the integers  $d, l_1, l_2, \dots, l_{m+1}$ , for all subsets  $\{l_i\}$  of cardinality  $m + 1$  of the  $\{k_j\}$ , for  $2 \leq m \leq n$ .*

We remark that however the g.c.d. of  $d, l_1, l_2, \dots, l_{m+1}$  may no longer necessarily be equal to 1.

3.1.  $n = 3$ . We use the lemma to treat the case  $n = 3$  using the result for  $n = 2$  proved in Theorem 2. A complete list of (non-dihedral) tuples  $(d, l_1, l_2, l_3)$  with the g.c.d. of  $d, l_1, l_2, l_3$  equal to 1 and satisfying condition (2) (let us call the corresponding triplet  $(l_1, l_2, l_3)$  primitive) is easily generated from Table 1, and is provided in Table 2 below.

Table 2: Primitive (non-dihedral) Schwarz triplets

$d$	$(l_1, l_2, l_3)$ with $l_1 \leq l_2 \leq l_3$
6	(1, 1, 1), (5, 5, 5) (1, 1, 3), (3, 5, 5)
10	(1, 1, 1), (3, 3, 3), (7, 7, 7), (9, 9, 9) (1, 3, 3), (3, 9, 9), (1, 1, 7), (7, 7, 9)
12	(1, 3, 5), (7, 9, 11) (each with multiplicity 2) (1, 2, 7), (5, 10, 11) (each with multiplicity 2) (1, 2, 2), (5, 10, 10), (2, 2, 7), (10, 10, 11) (1, 3, 3), (3, 3, 5), (7, 9, 9), (9, 9, 11)
15	(1, 2, 4), (2, 4, 8), (1, 4, 8), (7, 13, 14), (1, 2, 8), (7, 11, 14), (7, 11, 13), (11, 13, 14)
20	(1, 3, 7), (1, 3, 9), (1, 7, 9), (3, 7, 9), (11, 13, 17), (11, 13, 19), (11, 17, 19), (13, 17, 19)
24	(1, 5, 7), (1, 5, 11), (1, 7, 11), (5, 7, 11), (13, 17, 19), (13, 17, 23), (13, 19, 23), (17, 19, 23)
30	(1, 5, 5), (5, 5, 7), (11, 25, 25), (5, 5, 13), (17, 25, 25), (5, 5, 19), (23, 25, 25), (25, 25, 29) (3, 7, 17), (19, 21, 29), (1, 9, 11), (13, 23, 27) (each with multiplicity 2) (1, 9, 9), (3, 3, 7), (9, 9, 11), (13, 27, 27), (3, 3, 17), (19, 21, 21), (23, 27, 27), (21, 21, 29) (5, 7, 13), (1, 5, 19), (17, 23, 25), (11, 25, 29) (each with multiplicity 2)
60	(1, 11, 19), (7, 13, 17), (1, 11, 29), (7, 13, 23), (7, 17, 23), (1, 19, 29), (13, 17, 23), (11, 19, 29), (31, 41, 49), (37, 43, 47), (31, 41, 59), (37, 43, 53), (37, 47, 53), (31, 49, 59), (43, 47, 53), (41, 49, 59)

We remark that each line in Table 2 represents one  $(\mathbb{Z}/d\mathbb{Z})^*$ -orbit, and so has cardinality  $\varphi(d)$ , though for  $d = 12, 30$  some triplets occur with multiplicity 2, and for  $d = 60$  both lines form one orbit.

Now suppose that  $(d, k_1, k_2, k_3, k_4)$  satisfy condition (1). Assume that this tuple is primitive, i.e., the g.c.d. of  $d, k_1, k_2, k_3, k_4$  is equal to 1. Then, by the lemma, every sub-tuple  $(d, l'_1, l'_2, l'_3)$ , where the  $l'_i$  are obtained by discarding one  $k_j$ , also satisfies condition (2). Note that the g.c.d. of  $d, l'_1, l'_2, l'_3$  does not have to be 1. By Theorem 2, we see that  $(d, l'_1, l'_2, l'_3)$  is a positive integral multiple of some  $(d_1, l_1, l_2, l_3)$  occurring in Table 2 (or is dihedral), up to permutation.

Ignoring (momentarily) the tuples containing multiples of a dihedral Schwarz triplet, we see that  $d$  must be bounded by 120. Indeed, if say,

$$(3) \quad \begin{aligned} (d, k_1, k_2, k_3) &= a \cdot (d_1, l_1, l_2, l_3) \\ (d, k_1, k_2, k_4) &= b \cdot (d_2, m_1, m_2, m_4), \end{aligned}$$

for some tuples  $(d_1, l_1, l_2, l_3)$  and  $(d_2, m_1, m_2, m_4)$  in Table 2 (up to permutation), and for some positive integers  $a, b$ , then by primitivity, the g.c.d. of  $a, b$  has to be equal to 1, but  $d = ad_1 = bd_2$ , so  $a|d_2$ , so  $d|d_1d_2$ , so  $d$  divides the l.c.m. of all  $d$  occurring in Table 2, which is 120.

This reduces the problem of checking which primitive quadruples  $(d, k_1, k_2, k_3, k_4)$  satisfy condition (1) (for  $n = 3$ ) to a finite check. Table 3 lists all such primitive tuples which satisfy the property that every sub-tuple obtained by dropping exactly one of the  $k_j$  arises from Table 2, by possibly scaling up from a smaller denominator. (The fact that, for instance, the g.c.d. of  $a, b$  is 1 greatly reduces the number of smaller denominators that one has to consider.)

Table 3: Possible (non-dihedral) Schwarz 4-tuplets

$d$	$(k_1, k_2, k_3, k_4)$ with $k_1 \leq k_2 \leq k_3 \leq k_4$
6	(1, 1, 1, 1), (5, 5, 5, 5) (1, 1, 1, 3), (3, 5, 5, 5)
10	(1, 1, 1, 1), (3, 3, 3, 3), (7, 7, 7, 7), (9, 9, 9, 9) (1, 3, 3, 3), (3, 9, 9, 9), (1, 1, 1, 7), (7, 7, 7, 9)
12	(1, 2, 2, 7), (5, 10, 10, 11) (each with multiplicity 2) (1, 3, 3, 5), (7, 9, 9, 11) (each with multiplicity 2) (1, 2, 2, 2), (5, 10, 10, 10), (2, 2, 2, 7), (10, 10, 10, 11)
15	(1, 2, 4, 8), (7, 11, 13, 14) (each with multiplicity 4)
20	(1, 3, 7, 9), (11, 13, 17, 19) (each with multiplicity 4)
24	(1, 5, 7, 11), (13, 17, 19, 23) (each with multiplicity 4)
30	(1, 5, 5, 5), (5, 5, 5, 7), (11, 25, 25, 25), (5, 5, 5, 13), (17, 25, 25, 25), (5, 5, 5, 19), (23, 25, 25, 25), (25, 25, 25, 29) (1, 9, 9, 11), (3, 3, 7, 17), (19, 21, 21, 29), (13, 23, 27, 27) (each with multiplicity 2) (1, 9, 9, 9), (3, 3, 3, 7), (9, 9, 9, 11), (13, 27, 27, 27), (3, 3, 3, 17), (19, 21, 21, 21), (23, 27, 27, 27), (21, 21, 21, 29) (1, 5, 5, 19), (5, 5, 7, 13), (11, 25, 25, 29), (17, 23, 25, 25) (each with multiplicity 2)
60	(1, 11, 19, 29), (7, 13, 17, 23), (31, 41, 49, 59), (37, 43, 47, 53) (each with multiplicity 4)
120	No tuples

It is now straightforward to check that of these tuples, exactly two, namely (1, 1, 1, 1) and (5, 5, 5, 5), both for  $d = 6$ , satisfy condition (1).

Since these tuples are equivalent, we have proved one half of Theorem 1 (ii) (for  $n = 3$ ).

To treat the other half, we now assume that at least one of the sub-tuples of  $(d, k_1, k_2, k_3, k_4)$  is a multiple of a dihedral triplet. By rearranging the  $k_i$ , we may assume that the first sub-tuple in (3), is dihedral of the form:

$$(d, k_1, k_2, k_3) = a \cdot (d_1, l_1, l_2, l_3) = a \cdot (2m, p, p, m - p),$$

for some  $1 \leq p < m$ , with the g.c.d. of  $p, m$  equal to 1. Clearly the second tuple in (3) cannot be dihedral, for if

$$(d, k_1, k_2, k_4) = b \cdot (d_2, m_1, m_2, m_4) = b \cdot (2l, q, q, l - q),$$

for  $1 \leq q < l$ , with the g.c.d. of  $q, l$  equal to 1, then  $k_1 + k_3 = d/2 = k_1 + k_4$  so that  $k_4 = k_3 = a(m - p)$ . Then  $k_2/d + k_3/d = (ap/2ma) + a(m - p)/2ma = 1/2$ , and similarly  $k_1/d + k_4/d = 1/2$  so that condition (1) fails. (A similar argument applies if the second tuple  $(d, k_1, k_2, k_4)$  equals  $b \cdot (2l, q, l - q, q)$  instead.)

This means that the second tuple above is a multiple of a non-dihedral tuple  $(d_2, m_1, m_2, m_4)$ , occurring in the finite list in Table 2, up to permutation. As before, we have  $d = 2ma = bd_2$ , and since the g.c.d. of  $a, b$  is 1, we have  $a|d_2$  (and so is bounded) and  $b|2m$ . Moreover  $ap = k_1 = bm_1$  implies that  $b|p$ , and since the g.c.d. of  $p, m$  is 1, we see that  $b = 1, 2$ . However, the latter case cannot occur: if  $b = 2$  is even, then  $a$  is odd and  $p$  is even, so  $d_2 = 2ma/b = ma$  is odd (else  $m$  is even, so 2 divides the g.c.d. of  $d, k_1, k_2, k_3, k_4$ , contradicting primitivity). But then  $d_2$  must equal the only odd entry 15 in Table 2, which is impossible, since  $k_1 = ap = k_2$ , but all triplets for  $d = 15$  have distinct entries.

Thus  $b = 1$  and  $(d, k_1, k_2, k_3, k_4)$  has the shape:

$$(4) \quad (2ma = d_2, ap = m_1, ap = m_2, a(m - p), m_4).$$

Since  $d_2$  is bounded, both  $a$  and  $m$  divide  $d_2/2$ , and  $1 \leq p < m$  (with  $p$  is coprime to  $m$ ), clearly there are only finitely many possibilities for such tuples. Moreover, since  $m_1 = m_2$ , an inspection of Table 2 shows that  $d_2$  can only be one of 6, 10, 12, 30.

Table 4 below lists all possibilities for tuples having shape (4) above.

Table 4: Possible tuplets of the form (4)

$d$	$(k_1, k_2, k_3, k_4)$
6	(1, 1, 2, 1) (1, 1, 2, 3)
10	(1, 1, 4, 1) (1, 1, 4, 7) (3, 3, 2, 1) (3, 3, 2, 3)
12	(2, 2, 4, 1), (2, 2, 4, 7) (3, 3, 3, 1), (3, 3, 3, 5)
30	(3, 3, 17, 7), (3, 3, 12, 17), (9, 29, 6, 1), (9, 9, 6, 11) (5, 5, 10, 1), (5, 5, 10, 7), (5, 5, 10, 13), (5, 5, 10, 19)

A quick inspection now shows that of these possibilities only the tuple  $(1, 1, 2, 1)$  for  $d = 6$  satisfies (1), proving the second half of Theorem 1 (ii). This completes the proof of the case  $n = 3$ .

3.2.  $n = 4$ . This follows easily from Lemma 7 and the just established case  $n = 3$ . Indeed by the lemma, the only possible 5-tuplets of  $k_i$  would occur for  $d = 6$  and would be  $(1, 1, 1, 1, 1)$  or  $(1, 1, 1, 1, 2)$  up to equivalence. But only the former satisfies condition (1).

3.3.  $n = 5$ . The same argument shows that the only possible 6-tuplet of  $k_i$  is  $(1, 1, 1, 1, 1, 1)$  for  $d = 6$ . But one easily checks that this tuple fails to satisfy condition (1), so there are no 6-tuples satisfying (1).

3.4.  $n \geq 6$ . Finally, Lemma 7 shows that there are no tuplets for  $n \geq 6$  satisfying the condition (1) since we have just shown there is none for  $n = 5$ . This completes the proof of Theorem 1.  $\square$

#### 4. ON FINITENESS OF SOME MONODROMY

Consider the space  $S = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : z_i \neq z_j, \forall i \neq j\}$ , for  $n + 1 \geq 3$ . Let  $d \geq 2$  be an integer; fix integers  $k_1, k_2, \dots, k_{n+1}$  with  $1 \leq k_i \leq d - 1$  such that  $d\mathbb{Z} + \sum k_i\mathbb{Z} = \mathbb{Z}$ . The space of solutions  $(x, y)$  to the equation

$$y^d = (x - z_1)^{k_1}(x - z_2)^{k_2} \dots (x - z_{n+1})^{k_{n+1}}$$

with  $x \neq z_1, z_2, \dots, z_{n+1}$  and  $y \neq 0$  is the affine part of a smooth projective curve  $C = C_{d, k_i}$ . Write  $\mu_i = \frac{k_i}{d}$ ; our assumptions imply that  $\mu_i \in \mathbb{Q} \setminus \mathbb{Z}$ . Write, as in [Del-Mos], [Coh-Wol],  $\mu_\infty = 2 - \sum_{i=1}^{n+1} \mu_i$ ; the numbers  $\mu_i$  and  $\mu_\infty$  record the ramifications at the  $z_i$  and at  $\infty$ ; we

assume that  $\mu_\infty$  is also not integral so that the curve  $C$  is ramified at infinity as well.

The group  $G = \mathbb{Z}/d\mathbb{Z}$  acts on the curve  $C$ ; the action on the affine part is given by  $y \mapsto \omega y$  where  $\omega$  is a  $d$ -th root of unity. Consequently, the group  $G$  operates on the first cohomology of the curve  $C$  with rational coefficients; denote by  $M_d$  the direct sum of the cohomology over  $\mathbb{C}$ , on which a fixed generator of the group  $G$  operate by *some* primitive  $d$ -th root of unity.

The fundamental group of the space  $S$  acts (by monodromy) on the space  $M_d$ . It is well known that this fundamental group is the same as the pure braid group  $P_{n+1}$ . We classify the integers  $d, n$  and the numbers  $k_i$  for which the image of the fundamental group (the monodromy group) in  $\text{Aut}(M_d)$  is *finite*. This problem of finiteness has already been resolved by several authors ([Ssk], [Coh-Wol], [Bod], [Sch]), since this monodromy is the same as the monodromy of certain Appell-Lauricella hypergeometric functions. However, we believe our point of view is different: the explicit description of the monodromy in terms of the Gassner representation makes the proofs completely algebraic, and is formulated in terms of the definiteness of an explicit Hermitian form.

**Theorem 8.** *Suppose  $n, d, k_i$  are as in the preceding. Then the image of the monodromy representation in  $\text{Aut}(M_d)$  is finite if and only if condition (1) holds. Thus the monodromy on  $M_d$  is by a finite group if and only if*

$$n \leq 4.$$

*Moreover, up to equivalence, the numbers  $n, d, k_i$  satisfy the conditions given below.*

- (i) *If  $n = 4$ , then  $d = 6$  and  $k_i = 1$ , for all  $i \leq 5$ .*
- (ii) *If  $n = 3$ , then  $d = 6$  and  $k_1 = k_2 = k_3 = 1$  and  $k_4 = 1$  or  $2$ .*
- (iii) *If  $n = 2$ , then  $d = 2m$  and  $k_1 = k_2 = p$  and  $k_3 = m - p$ , or else  $d, k_i$  lie in a finite list, with  $d \leq 60$ .*

*Remark.* Again, note that condition (1) is a purely number theoretic condition; the proof of the theorem, however, depends on an analysis of certain finite subgroups of unitary groups generated by reflections.

We will prove Theorem 8 after some preliminaries on Hermitian forms and unitary groups generated by reflections.

### 5. SKEW-HERMITIAN FORMS

**Lemma 9.** *Suppose  $h$  is a skew-Hermitian form on  $\mathbb{C}^n$ . Then  $h$  is anisotropic if and only if the principal minors  $u_i$  have the property:  $\frac{u_{j+1}u_{j-1}}{u_j^2}$  is positive, for all  $j \geq 1$  (by convention  $u_0 = 1$ ).*

*Proof.* Suppose that  $h$  does not represent a zero; hence  $a_{11} \neq 0$ , where  $(a_{ij})$  is the matrix of  $h$  in the standard basis. By the Gram-Schmidt process, there exists an upper triangular unipotent matrix  $u \in GL_n(\mathbb{C})$  such that  ${}^t(\bar{u})hu = h'$  is diagonal, with diagonal entries  $\lambda_1, \dots, \lambda_n$  say. Now  $h$  is anisotropic if and only if the equivalent  $h'$  is anisotropic. The latter is anisotropic if and only if the successive ratios  $\beta_j = \lambda_{j+1}/\lambda_j$  are all positive.

Since  $u$  is a unipotent upper triangular matrix, the principal minors of  $h$  and  $h'$  are the same. Therefore,  $u_{j+1} = \lambda_1 \cdots \lambda_{j+1}$ . Consequently,  $h'$  is anisotropic if and only if for all  $j$ ,

$$\beta_j = (u_{j+1}/u_j)/(u_j/u_{j-1})$$

is positive. This is equivalent to  $\beta_j = \frac{u_{j+1}u_{j-1}}{u_j^2}$  being positive, for all  $j$ . Hence the lemma.  $\square$

Now suppose that  $E/F$  is a CM extension of number fields and that  $h$  is a skew-Hermitian form on  $E^n$ . Suppose that  $h$  does not represent a zero. Then the Gram-Schmidt process diagonalises  $h$ . Suppose the diagonal entries are  $\lambda_1, \dots, \lambda_n$ . Then we have

$$\lambda_1 \cdots \lambda_j = \det(h_j),$$

where  $\det(h_j)$  is the principal  $j \times j$  minor of  $h$ . Hence

$$\lambda_j = \det(h_j)/\det(h_{j-1}).$$

Write

$$\beta_j = \frac{\det(h_{j+1})\det(h_{j-1})}{\det(h_j)^2}.$$

From the previous lemma we obtain:

**Lemma 10.** *Suppose  $F \rightarrow \mathbb{R}$  is an embedding and  $E \otimes_F \mathbb{R} = \mathbb{C}$ . Then the skew-Hermitian form  $h$  is anisotropic in this embedding if and only if*

$$\beta_j > 0, \quad \forall j.$$



**Lemma 11.** *Suppose  $h$  is a skew-Hermitian form in  $n$  variables over a CM field  $E/F$ , and  $\Gamma \subset U(h)(\mathcal{O}_F)$  a subgroup which acts irreducibly on  $\mathbb{C}^n$ . Then  $\Gamma$  is finite if and only if  $h$  is totally anisotropic.*

*Proof.* Suppose  $\Gamma$  is finite. Fix any positive definite Hermitian form  $H$  on  $\mathbb{C}^n$ . Being a sum of positive definite forms, the average  $H'(x, y) = \sum_{\gamma \in \Gamma} H(\gamma x, \gamma y)$  is also positive definite and is  $\Gamma$ -invariant. Hence  $iH'$  is a  $\Gamma$ -invariant anisotropic skew-Hermitian form on  $\mathbb{C}^n$ . The irreducibility of the action of  $\Gamma$  implies, by Schur's lemma, that the invariant anisotropic skew-Hermitian form  $iH'$  is a scalar multiple of the form  $h$ , for any embedding of  $F$  into  $\mathbb{R}$ . Hence  $h$  is anisotropic over all embeddings of the field  $F$ .

Conversely, if  $h$  is anisotropic at all real places  $v$  of  $F$ , then  $ih$  is definite, for all  $v$ , and hence the group  $U(h)(F_v) \simeq U(ih)(F_v)$  is compact, for all  $v$ . Since  $U(h)(\mathcal{O}_F)$  is a discrete subgroup of  $U(h)(F \otimes_{\mathbb{Q}} \mathbb{R})$ , it follows that  $U(h)(\mathcal{O}_F)$  is finite, hence  $\Gamma$  is also finite.  $\square$

*Remark.* Let  $G = U(h)$ . A corollary of the proof is that if a finite subgroup of  $G(\mathcal{O}_F)$  acts irreducibly on  $\mathbb{C}^n$ , then  $G(\mathcal{O}_F)$  is finite.

**Notation.** Denote by  $R$  the Laurent polynomial ring  $\mathbb{Z}[X_1^{\pm 1}, \dots, X_{n+1}^{\pm 1}]$  in  $n + 1$  variables with  $\mathbb{Z}$ -coefficients. The map  $X_i \mapsto X_i^{-1}$ , for all  $i$ , induces an involution of order two on the ring  $R$ . Denote by  $R^n$  the standard free  $R$  module of rank  $n$  with standard basis  $\varepsilon_i$ , for  $1 \leq i \leq n$ . Define the skew-Hermitian form  $h = (h_{ij})_{1 \leq i, j \leq n}$  by the formulae  $h(\varepsilon_i, \varepsilon_j) = 0$  if  $|i - j| \geq 2$  and

$$h(\varepsilon_i, \varepsilon_i) = \frac{1 - X_i X_{i+1}}{(1 - X_i)(1 - X_{i+1})}, \quad h(\varepsilon_i, \varepsilon_{i+1}) = -\frac{1}{1 - X_{i+1}}.$$

**Lemma 12.** *The form  $h$  does not represent a zero in  $R^n$ . Moreover, for each  $j$ , the principal  $j \times j$  minor is given by*

$$u_j = \frac{1 - X_1 \cdots X_{j+1}}{(1 - X_1) \cdots (1 - X_{j+1})}.$$

*Proof.* In [V], the determinant of  $h$  was computed to be

$$\frac{1 - X_1 X_2 \cdots X_{n+1}}{(1 - X_1) \cdots (1 - X_{n+1})}.$$

Taking  $n = j$ , we get the formula for the determinant of the  $j \times j$  principal minor. Take  $X_j = e^{2\pi i \theta_j}$  to be transcendental with  $\theta_j \in \mathbb{R}$  positive and close to 0. Put  $\Sigma_{j-1} = \theta_1 + \cdots + \theta_j$  (the sum of the first  $j$  terms). Using the equality

$$(5) \quad 1 - e^{2\pi i \theta} = e^{\pi i \theta}((-2i) \sin(\pi \theta)),$$

and that  $\sin(\theta)$  is close to  $\theta$ , for  $\theta$  small (and positive), we see that the numbers

$$\beta_j = \frac{u_{j+1}u_{j-1}}{u_j^2} = \frac{\sin(\pi\Sigma_{j+1})\sin(\pi\Sigma_{j-1})\sin(\pi\theta_{j+1})}{\sin(\pi\Sigma_j)^2\sin(\pi\theta_{j+2})}$$

are positive, for all  $j$ . By Lemma 9 it follows that  $h$  is anisotropic.  $\square$

The map  $X_i \mapsto t_i = e^{2\pi\sqrt{-1}\frac{k_i s}{d}}$  maps  $R$  onto the ring  $\mathcal{O}_E$  of integers in the  $d$ -th cyclotomic extension  $E = \mathbb{Q}(e^{2\pi i/d})$ . Let  $F = \mathbb{Q}(\cos(\frac{2\pi}{d}))$  be the maximal totally real subfield of  $E$ . We then get a skew-Hermitian form on  $E^n$  induced from  $h$ .

Define, for each  $j \leq n-1$ , the numbers

$$\nu_j = \nu_j(s) = \left\{ \sum_{i=1}^j \frac{k_i s}{d} \right\}.$$

Denote by (6) the conditions satisfied by the numbers  $n, d, k_i$ :

$$(6) \quad \begin{aligned} \varepsilon_j &= \varepsilon_j(s) \stackrel{\text{def}}{=} (-1)^{[\nu_j(s) + \mu_{j+1}(s) + \mu_{j+2}(s)]} = 1, \\ \forall s \in (\mathbb{Z}/d\mathbb{Z})^* \quad \text{and} \quad \forall j \quad \text{with} \quad 1 \leq j \leq n-1. \end{aligned}$$

**Lemma 13.** *The group  $G(\mathcal{O}_F) = U(h)(\mathcal{O}_F)$  is finite if and only if  $(n, d, k_i)$  satisfy condition (6).*

*Proof.* Consider the ‘‘Gassner representation’’  $G(X) : P_{n+1} \rightarrow U(h, R)$  [V] (recall that the ring  $R$  is the Laurent polynomial ring in the variables  $X_j : 1 \leq j \leq n+1$  with integer coefficients). Specializing  $X_j$  to  $x_j = e^{2\pi i \frac{k_j}{d}} = e^{2\pi i \mu_j}$  we obtain a representation  $\rho_d$  of the pure braid group  $P_{n+1}$ . The image of  $\rho_d$  is contained in the group  $G(\mathcal{O}_F)$  (e.g., p. 26, paragraph before Theorem 16, of [V]). We have assumed that  $\sum_{j=1}^{n+1} \mu_j (= 2 - \mu_\infty)$  is not an integer, so  $\prod x_j \neq 1$ . Therefore, by Proposition 19 of [V],  $G(\mathcal{O}_F)$  acts irreducibly. By Lemma 11,  $G(\mathcal{O}_F)$  is finite if and only if  $h$  is totally anisotropic. We must then prove that the condition of the anisotropy of  $h$  is equivalent to condition (6).

Let  $\det(h_j)$  be the  $j \times j$  principal minor of the form  $h$  obtained by specializing to the  $t_i$ , for some fixed  $s \in (\mathbb{Z}/d\mathbb{Z})^*$ . By Lemma 12, the determinant of  $h_j$  is

$$\frac{1 - t_1 t_2 \cdots t_{j+1}}{(1 - t_1) \cdots (1 - t_{j+1})}.$$

It is easily seen, in view of (5), that this determinant is

$$\det(h_j) = \frac{i^j \sin\left(\frac{\pi(k_1+\dots+k_{j+1})s}{d}\right)}{2^j \prod_{i=1}^{j+1} \sin\left(\frac{\pi k_i s}{d}\right)}.$$

As in the proof of Lemma 12, we have

$$\beta_j = \frac{\det(h_{j+1}) \det(h_{j-1})}{\det(h_j)^2} = \left( \frac{\sin\left(\frac{\pi(k_1+\dots+k_{j+2})s}{d}\right) \sin\left(\frac{\pi(k_1+\dots+k_j)s}{d}\right)}{\sin^2\left(\pi \frac{(k_1+\dots+k_{j+1})s}{d}\right)} \right) \frac{\sin\left(\frac{\pi k_{j+1}s}{d}\right)}{\sin\left(\frac{\pi k_{j+2}s}{d}\right)}.$$

If  $x$  is not an integer then the sign of  $\sin(\pi x)$  is simply the number  $(-1)^{[x]}$ . Therefore, the sign of  $\beta_j$  is

$$(-1)^{\left[\frac{(k_1+\dots+k_{j+2})s}{d}\right] - \left[\frac{(k_1+\dots+k_j)s}{d}\right] - \left[\frac{k_{j+1}s}{d}\right] - \left[\frac{k_{j+2}s}{d}\right]}.$$

Since, for all  $x, y, z \in \mathbb{R}$ , we have

$$[x + y + z] - [x] - [y] - [z] = [\{x\} + \{y\} + \{z\}],$$

it follows that the sign of  $\beta_j$  is just the number  $\varepsilon_j(s)$ . Hence, by Lemma 10, the anisotropy of  $h$  is equivalent to the condition that  $\varepsilon_j(s) = 1$ , for all  $j$ . This is exactly condition (6).  $\square$

**Lemma 14.** *Condition (6) is equivalent to condition (1).*

*Proof.* Assume that condition (6) holds. Applying the condition with  $j = 1$ , we must have  $\{\frac{k_1 s}{d}\} + \{\frac{k_2 s}{d}\} \neq 1$ , for all  $s \in (\mathbb{Z}/d\mathbb{Z})^*$ . For each  $\bar{s}$  in the quotient group  $(\mathbb{Z}/d\mathbb{Z})^*/\{\pm 1\}$ , there are two representatives  $s$  and  $d-s$  in the group  $(\mathbb{Z}/d\mathbb{Z})^*$  of units mapping to  $\bar{s}$ . We consider the numbers  $a = a_s = [\{\frac{k_1 s}{d}\} + \{\frac{k_2 s}{d}\}]$  and  $b = b_{d-s} = [\{\frac{k_1(d-s)}{d}\} + \{\frac{k_2(d-s)}{d}\}]$ . Since the fractional part of  $x$  does not change if  $x$  is replaced by  $x+m$ , for  $m \in \mathbb{Z}$ , it follows that  $b = [\{\frac{-k_1 s}{d}\} + \{\frac{-k_2 s}{d}\}] = [(1 - \{\frac{k_1 s}{d}\}) + (1 - \{\frac{k_2 s}{d}\})] = 2 - (a + 1) = 1 - a$ . Thus one of the numbers  $a, b$  is zero since  $0 \leq a, b \leq 1$ . We choose the representative  $s$  so that  $a = a_s = 0$ .

We now prove by induction on  $j \leq n$  that, for this choice of  $s$ , the integral part of  $\alpha_{j+1} \stackrel{def}{=} \{\frac{k_1 s}{d}\} + \dots + \{\frac{k_{j+1} s}{d}\}$  is zero. The case  $j = 1$  was just treated. Applying this with  $j = n$  will then prove that condition (6) implies condition (1).

We now claim that condition (6) is equivalent to

$$[\alpha_{j+1}] \equiv [\alpha_{j-1}] \pmod{2},$$

for  $2 \leq j \leq n$ . Indeed, if  $m \leq \alpha_{j-1} < m+1$ , for some integer  $m \geq 0$ , then  $\nu_{j-1} = \alpha_{j-1} - m$ . Thus,  $\nu_{j-1} + \mu_j + \mu_{j+1}$  lies in  $(0, 1)$  or  $(2, 3)$  if and only if  $\alpha_{j+1} = \alpha_{j-1} + \mu_j + \mu_{j+1}$  lies in  $(m, m+1)$  or  $(m+2, m+3)$ , so

that its integral part is either the same as, or 2 more than, the integral part  $m$  of  $\alpha_{j-1}$ .

By induction, we may assume that  $[\alpha_k] = 0$  for all  $k \leq j$ ; therefore, by the above congruence, we have  $[\alpha_{j+1}] \equiv 0 \pmod{2}$ .

On the other hand,  $[\alpha_{j+1}] = [\alpha_j] + [\{\alpha_j\} + \{\frac{k_{j+1}s}{d}\}]$ . Since by induction,  $[\alpha_j] = 0$ , it follows that  $[\alpha_{j+1}] = [\{\alpha_j\} + \{\frac{k_{j+1}s}{d}\}]$ . Being the integral part of a sum of two numbers in the closed open interval  $[0, 1)$ , the latter is at most one and hence  $(0 \leq) [\alpha_{j+1}] \leq 1$ . The conclusion of the preceding paragraph now implies that  $[\alpha_{j+1}] = 0$ , completing the induction step. Hence condition (1) follows.

Conversely, if condition (1) holds, then all the numbers  $[\alpha_j]$  are zero, and hence the numbers  $\varepsilon_j(s)$  are all 1. This is condition (6).  $\square$

We can now prove Theorem 8.

*Proof.* Since, by the assumption on  $\mu_\infty$ , we have  $\sum \mu_j \notin \mathbb{Z}$ , it follows by Proposition 19 of [V], that  $\rho_d$  is irreducible; since (again by [V], Corollary 3) the monodromy representation  $M_d$  is a quotient of  $\rho_d$ , it follows that  $M_d$  is the representation  $\rho_d$ .

The monodromy representation (being the specialised Gassner representation  $\rho_d$ ) has image in  $U(\mathfrak{h})(\mathcal{O}_F)$  where  $\mathfrak{h}$  is as above (this is in subsection 4.1 of [V]). By Lemma 11, the image is finite if and only if  $U(\mathfrak{h})(\mathcal{O}_F)$  is finite. So, by Lemma 13, the image is finite if and only if condition (6) holds. By the above lemma, this is equivalent to condition (1).  $\square$

## 6. ALGEBRAIC LAURICELLA FUNCTIONS

We list some corollaries to Theorem 8. We assume as before, that  $\mu_j = \frac{k_j}{d}$  and  $\mu_\infty$  are rational and not integral. The corollaries below follow from the finiteness of monodromy and the observation that the Lauricella  $F_D$ -functions are the period integrals associated to homology classes in the curve  $C$  whose affine part is given by  $y^d = (x - z_1)^{k_1} \cdots (x - z_{n+1})^{k_{n+1}}$ , with  $x \neq z_1, \dots, z_{n+1}$  and  $y \neq 0$ .

**Corollary 2.** *The Lauricella  $F_D$ -function*

$$F_D(z_1, \dots, z_{n+1}) = \int_{z_i}^{z_j} \frac{dx}{y},$$

for  $i \neq j$ , is an algebraic function of the variables  $z_1, \dots, z_{n+1}$  if and only if condition (1) holds for the numbers  $(n, d, k_i)$ .

The following corollary is to be read up to equivalence of the  $\mu_j$  as defined in the Introduction.

**Corollary 3.** *If  $n + 1 \geq 6$ , then the function  $F_D(z_1, \dots, z_{n+1})$  is not algebraic.*

*If  $n + 1 = 5$ , then  $F_D$  is algebraic if and only if  $d = 6$  and all the  $k_i$  are equal to 1.*

*If  $n + 1 = 4$ , then  $F_D$  is algebraic if and only if  $d = 6$  and all the  $k_i$  are equal to 1; or else, all but one of the  $k_i$  are equal to 1 and one of the  $k_i = 2$ .*

*If  $n + 1 = 3$ , and if  $F_D$  is algebraic, then  $d = 2m$  and  $k_1 = k_2 = p$  and  $k_3 = m - p$ , or else  $d, k_i$  lie in a finite list, with  $d \leq 60$ .*

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