Critical Values of the Twisted Tensor L-function in the Imaginary Quadratic Case

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1 Introduction

The twisted tensor L-function of f, which we denote by $G(s,f)$, is a certain Dirichlet series associated to a quadratic extension of number fields K/F, and a cuspidal automorphic function f over K. It was introduced in [1] by Asai, following previous work of Shimura, in the case when f is a Hilbert modular cusp form over a real quadratic extension K of $\mathbf{Q}$.

In the past twenty odd years, this L-function has been considered more generally: for instance [11] and [12] deal with quadratic extensions of totally real fields, [17] with imaginary quadratic extensions of $\mathbf{Q}$, and [3], [4] and [14] with general quadratic extensions of number fields. All these papers have been primarily concerned with establishing analytic properties of $G(s,f)$ analogous to those in [1], such as meromorphic continuation to the entire complex plane, location and finiteness of the number of poles, and functional equation.

The aim of this paper is to prove a rationality result for $G(s,f)$ in the imaginary quadratic setting. If K is an imaginary quadratic field, and f a cusp form associated to K, we establish that there is a 'period' $\Omega_j(f)$ such that

$$\frac{G(j,f)}{\Omega_j(f)} \in E$$

for a finite set of integers j depending on the weight of f. Here E is a finite extension of the number field generated by the Fourier coefficient of f. Moreover, the period $\Omega_j(f)$ is the product of a fixed (probably transcendental) constant $\Omega(f)$, a power of $2\pi i$ depending on j and a certain Gauss sum.

If we assume that $G(s,f)$ is motivic, we may interpret our result in the framework of Deligne's famous conjectures on motivic L-functions, outlined in [2]. Indeed, assuming that there is a rank 2 motive associated to f defined over K, the integers j above are the critical integers (in the sense of [2]) in the right half of the critical strip of the appropriately defined motive, whose L-function is $G(s,f)$. Moreover, the general shape of the period $\Omega_j(f)$ is predicted by this conjectural framework (Proposition 3 and Remarks 3 and 4).

Unfortunately, since it is only conjectured that $G(s, f)$ is motivic, we cannot realize $\Omega_j(f)$ as a motivic period. Rather, we follow an alternative program of Hida outlined in [10], to define a period independently of the motivic setting. Indeed, in [10], Hida has used his period to establish rationality results for other L-functions: for the standard L-function associated to a cusp form over an arbitrary number field, as well as for the Rankin product L-function of two such forms. Our proof in the 'twisted tensor case' follows very closely the methods of that paper.

Let us now briefly summarize the method of proof. The cusp form f may be realized as a tuple of functions on $\mathcal{H}$, the symmetric space (upper half hyperbolic three space) associated to K, each of which satisfies an automorphy condition with respect to a congruence subgroup of K. Since $\mathbf{Q}$ has class number one, for the purposes of our computations it turns out that it suffices to work with any one of these functions; we pick one, for which the corresponding congruence subgroup is $\Gamma_0(\mathfrak{N})$ ($\mathfrak{N}$ is a fixed ideal of K, the level), and continue to denote this function by f. Now $f \in \mathcal{S}_{\boldsymbol{n}}(\Gamma_0(\mathfrak{N}), \chi_{\mathfrak{N}}^{-1})$, the corresponding space of cusp forms with 'nebentypus' character $\chi_{\mathfrak{N}}^{-1}$. Via the Eichler-Shimura-Harder isomorphism

$$\delta : \mathcal{S}_{\boldsymbol{n}}(\Gamma_0(\mathfrak{N}), \chi_{\mathfrak{N}}^{-1}) \longrightarrow \mathrm{H}^1_{cusp}(\Gamma_0(\mathfrak{N})\backslash\mathcal{H}, \widetilde{L(\boldsymbol{n}, \mathbf{C})})$$

we may realize f as a differential 1-form, $\delta(f)$, on the locally symmetric threefold $\Gamma_0(\mathfrak{N})\backslash\mathcal{H}$. Note that $\delta(f)$ takes values in the sheaf $\widetilde{L(\boldsymbol{n}, \mathbf{C})}$ constructed (up to a twist) from the irreducible $SL_2(K)$-module $L(\boldsymbol{n}, \mathbf{C}) = Sym^n(\mathbf{C}^2) \otimes Sym^n(\mathbf{C}^2)$, with $SL_2(K)$ acting on the two factors via the two embeddings of K into $\mathbf{C}$.

Imitating the (now) standard method of Asai and Shimura (in [1]), we represent $G(s, f)$ as the Rankin-Selberg convolution of $\delta(f)\big|_H$ with an elliptic modular Eisenstein series. Here $\delta(f)\big|_H$ denotes the restriction of $\delta(f)$ to the elliptic modular twofold corresponding to the inclusion $\mathbf{Q} \subset K$. Since $L(\boldsymbol{n}, \mathbf{C})$ is no longer irreducible as an $SL_2(\mathbf{Q})$-module, $\delta(f)\big|_H$ decomposes accordingly as a sum of differential forms indexed by the irreducible $SL_2(\mathbf{Q})$-factors. Actually it is these forms that we integrate, and so in reality we obtain a finite collection of integral expressions for $G(s, f)$ (see Section 6.3, Equation 29).

By specializing these expressions at $s = 0$, we may interpret the resulting special values of $G(s, f)$ in terms of the perfect pairing between ordinary and compactly supported de Rham cohomology, that is, in terms of the Poincaré duality pairing. There is a natural E-rational structure on these complex cohomology groups. This rational structure differs, of course, from that coming from the Whittaker model (Fourier expansion for f), via the Eichler-Shimura-Harder map δ. Following Hida, we measure the difference in these rational structures via the period $\Omega(f) \in \mathbf{C}^\times$ defined as follows:

$$\delta(f) = \Omega(f)\, \eta(f),$$

where $\eta(f)$ is an E-rational cohomology class. Then, since (the class of) the

Eisenstein series at $s = 0$ is E-rational, we are led to the rationality result we seek (Theorem 1).

There is an interesting computational complication that we must tackle along the way. In reality, each of the integral expressions contains an algebraic *sum* of Gamma factors. It is not *a priori* obvious that this sum does not vanish at $s = 0$, and that, for this reason, in the proof of Theorem 1, we are not dividing by zero! Though it seems quite difficult to establish this non-vanishing by direct computation we are nonetheless able to do this indirectly thanks to a suggestion of Hida (see Proposition 5).

The methods of this paper readily generalize to the CM case: that is, we are able to obtain a rationality result for the twisted tensor L-function attached to a cusp form over a CM field. Though the idea of proof is the same, the calculations are more complicated. The details will appear in [5]. We should also point out that that an algebraicity result for $G(s, f)$ had been obtained by Shimura much earlier in the totally real case, but only for those Hilbert modular forms f coming from the Jacquet-Langlands correspondence (see [15]).

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Notation

From now on K will be an imaginary quadratic field, say $K = \mathbf{Q}(\sqrt{-D})$, $D > 0$, with $-D$ the discriminant of K. Let $\mathcal{O}$ be the ring of integers of K. Fix an ideal $\mathfrak{N} \subset \mathcal{O}$, and let $N > 0$ denote the generator of the ideal $\mathfrak{N} \cap \mathbf{Z}$.

We denote the two embeddings of K into $\mathbf{C}$ by i and c. Let $\boldsymbol{n} = n_i i + n_c c$ and $\boldsymbol{v} = v_i i + v_c c$ be formal sums in $\mathbf{Z}[\{i, c\}]$. We assume n_i, $n_c \geq 0$.

Let $W(n^*, \mathbf{C})$ be the space of homogeneous polynomials of degree $n^* = n_i + n_c + 2$ over $\mathbf{C}$ in two variables $\boldsymbol{s} = \binom{S}{T}$. In general we will consider functions $f : X \to W(n^*, \mathbf{C})$, for various spaces X, in which case we will sometimes write $f(x, \boldsymbol{s})$ or $f(x, \binom{S}{T})$, to emphasize the dependence of f on the variables $\boldsymbol{s} = \binom{S}{T}$.

2 Cusp forms associated to K

We consider modular forms on the adèle group $G_{\boldsymbol{A}} = GL_2(K_{\boldsymbol{A}})$, where $K_{\boldsymbol{A}}$ is the adèle ring of K.

Identify the center of $G_{\boldsymbol{A}}$ with the idele group $K_{\boldsymbol{A}}^{\times}$. Let $\chi : K^{\times}\backslash K_{\boldsymbol{A}}^{\times} \to \mathbf{C}^{\times}$ be a Hecke character, whose conductor divides $\mathfrak{N}$, and whose infinity type is $-\boldsymbol{n} - 2\boldsymbol{v}$.

Let $\widehat{\mathcal{O}} = \prod_{\wp\, finite} \mathcal{O}_{\wp}$. Fix $U_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\widehat{\mathcal{O}}) \mid c \equiv 0 \bmod \mathfrak{N}\widehat{\mathcal{O}} \}$, a compact open subgroup of the finite part of $G_{\boldsymbol{A}}$. Note $\chi|_{\widehat{\mathcal{O}}^{\times}}$ may be considered as a character on $(\mathcal{O}/\mathfrak{N}\mathcal{O})^{\times}$, and we will denote this character by $\chi_{\mathfrak{N}}$. For $u_f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(\mathfrak{N})$ we set $\chi_{\mathfrak{N}}(u_f) = \chi_{\mathfrak{N}}(d) = \prod_{\wp \mid \mathfrak{N}} \chi_{\wp}(d_{\wp})$.

Definition 1 *A cusp form (of weight $(\boldsymbol{n}, \boldsymbol{v})$, level $\mathfrak{N}$, and central action χ) is a function $f : G_{\boldsymbol{A}} \to W(n^*, \mathbf{C})$ satisfying the following properties*

1. $f(\gamma g,\, \boldsymbol{s}) = f(g,\, \boldsymbol{s})$ *for all* $\gamma \in GL_2(K)$
2. $f(zg,\, \boldsymbol{s}) = \chi(z) f(g,\, \boldsymbol{s})$ *for all* $z \in K_{\boldsymbol{A}}^{\times}$
3. $f(gu,\, \boldsymbol{s}) = \chi_{\mathfrak{N}}(u_f) f(g, u_{\infty}\left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right))$ *for* $u = u_f \cdot u_{\infty} \in U_0(\mathfrak{N}) \cdot SU_2(\mathbf{C})$
4. *f is an eigenfunction of the operators D_{σ}, for $\sigma = i, c$:*

$$D_{\sigma} f = (n_{\sigma}^2/2 + n_{\sigma}) f,$$

where $D_{\sigma}/4$ denotes a component of the Casimir operator in the Lie algebra $sl_2(\mathbf{C}) \otimes_{\mathbf{R}} \mathbf{C}$ (see [9], Section 1.3), and where we consider $f(g_f g_{\infty},\, \boldsymbol{s})$ as a function of $g_{\infty} \in GL_2(\mathbf{C})$

5. *f satisfies the cuspidal condition*

$$\int_{U(\mathbf{Q})\backslash U(\boldsymbol{A})} f(vg,\, \boldsymbol{s})\, du = 0$$

for all $g \in G_{\boldsymbol{A}}$, where

$$U(\mathbf{Q}) = \{ v = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \mid u \in K \},$$

$$U(\boldsymbol{A}) = \{ v = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \mid u \in K_{\boldsymbol{A}} \}$$

and du is the Lebesgue measure on $K_{\boldsymbol{A}}$.

Let us denote the space of such forms by $\boldsymbol{S}_{(\boldsymbol{n},\boldsymbol{v})}(\mathfrak{N}, \chi)$. Any such form f spans the space for a cuspidal automorphic representation of $G_{\boldsymbol{A}}$ whose infinity component is

$$\pi_{\infty}\left(z^{-(n_i+v_i+3/2)}\, \bar{z}^{-(v_c+1/2)},\, z^{-(n_c+v_c+3/2)}\, \bar{z}^{-(v_i+1/2)} \right).$$

Further since we are dealing with cusp forms we may assume (see [10], Section 2.5, Corollary 2.2) that $n_i = n_c$, so from now on we simply denote this common value by n. In particular $n^* = 2n + 2$.

2.1 Fourier expansions

If $f : G_{\boldsymbol{A}} \to W(n^*, \mathbf{C})$ is a cusp form as above, then f has Fourier expansion (cf. [10], Theorem 6.1):

$$f\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) = |y|_K \sum_{\xi \in K^\times} c\,(\xi y d, f)\, W(\xi y_\infty)\, \mathbf{e}_K(\xi x).$$

Here

- $|\;|_K$ is the usual idele character of $K_{\boldsymbol{A}}^\times$ trivial on $K^\times$
- d is an idele such that $d\mathcal{O} = \vartheta$, the different of K
- The Fourier coefficient $c\,(\quad, f)$ may be considered as a function on the fractional ideals of K that vanishes outside the integral ideals
- $W : \mathbf{C}^\times \to W(n^*, \mathbf{C})$ is the Whittaker function

  $$W(s) = \sum_{\alpha=0}^{n^*} \binom{n^*}{\alpha} \frac{1}{s^{v_i}\bar{s}^{v_c}} \left(\frac{s}{i|s|}\right)^{n+1-\alpha} K_{\alpha-(n+1)}(4\pi|s|)\, S^{n^*-\alpha} T^\alpha$$

  where $K_\alpha(x)$ is (a modified Bessel function which is) a solution to the differential equation

  $$\frac{d^2 K_\alpha}{dx^2} + \frac{1}{x}\frac{dK_\alpha}{dx} - \left(1 + \frac{\alpha^2}{x^2}\right) K_\alpha = 0$$

  having asymptotic behaviour $K_\alpha(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$ as $x \to \infty$
- $\mathbf{e}_K$ is an additive character of $K_{\boldsymbol{A}}$, trivial on K, defined as follows

  $$\mathbf{e}_K = \prod_{\wp} (\mathbf{e}_p \circ \mathrm{tr}_{K_\wp/\mathbf{Q}_p}) \cdot (\mathbf{e}_\infty \circ \mathrm{tr}_{\mathbf{C}/\mathbf{R}}),$$

  where $\mathbf{e}_p(\sum_j c_j p^j) = e^{-2\pi i \sum_{j<0} c_j p^j}$ and $\mathbf{e}_\infty(r) = e^{2\pi i r}$.

2.2 Hyperbolic three space $\mathcal{H}$

Since we will need to work with certain functions and differential forms associated to f defined on hyperbolic upper half three space

$$\mathcal{H} = \left\{ \left(\begin{smallmatrix} x & -y \\ y & \bar{x} \end{smallmatrix} \right) \middle|\; x \in \mathbf{C},\ y \in \mathbf{R},\ y > 0 \right\},$$

we discuss some of its properties here (cf. [9], Section 1.1).

Note $SL_2(\mathbf{C})$ acts transitively on $\mathcal{H}$, via

$$\gamma \cdot z = (\rho(a)\, z + \rho(b))\, (\rho(c)\, z + \rho(d))^{-1}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{C})$, $z = \left(\begin{smallmatrix} x & -y \\ y & \bar{x} \end{smallmatrix}\right) \in \mathcal{H}$ and $\rho(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & \bar{t} \end{smallmatrix}\right)$. Fix a point $\epsilon = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{H}$. Then the stabilizer of ϵ is $SU_2(\mathbf{C})$ and so we may identify the symmetric spaces

$$SL_2(\mathbf{C}) \big/ SU_2(\mathbf{C}) \simeq \mathcal{H}.$$

Note that the Poincaré upper half plane H naturally sits in $\mathcal{H}$ via

$$x + iy \hookrightarrow \left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right).$$

The above action of $SL_2(\mathbf{C})$ on $\mathcal{H}$ extends the standard action of $SL_2(\mathbf{R})$ on H via fractional linear transformations.

Let

$$\pi : SL_2(\mathbf{C}) \longrightarrow SL_2(\mathbf{C}) \big/ SU_2(\mathbf{C}) \simeq \mathcal{H}$$

denote the canonical projection. It $T_\epsilon\mathcal{H}$ denotes the tangent space of $\mathcal{H}$ at $\epsilon \in \mathcal{H}$, then the differential of π at the identity $I \in SL_2(\mathbf{C})$ induces a surjection $d_I\pi : \mathfrak{sl}_2(\mathbf{C}) \to T_\epsilon\mathcal{H}$, whose kernel is $\mathfrak{su}_2(\mathbf{C})$. On the other hand we have the decomposition

$$\mathfrak{sl}_2(\mathbf{C}) = \mathfrak{su}_2(\mathbf{C}) \oplus \mathfrak{p}$$

where $\mathfrak{p}$ is the Lie algebra of the subgroup of upper triangular matrices in $SL_2(\mathbf{C})$ with real diagonal entries. Thus we may canonically identify $\mathfrak{p}$ with $T_\epsilon\mathcal{H}$. Under this identification the basis $P = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $R = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $S = \left(\begin{smallmatrix} 0 & i \\ 0 & 0 \end{smallmatrix}\right)$ of $\mathfrak{p}$ generate the vector fields $\frac{1}{y}\frac{\partial}{\partial y}$, $\frac{1}{y}\frac{\partial}{\partial r}$, and $\frac{1}{y}\frac{\partial}{\partial s}$ on $\mathcal{H}$, under translation by $g = \frac{1}{\sqrt{y}}\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(\mathbf{C})$. Here $x = r + is$. Furthermore, P, $Q = 1/2(R - iS)$ and $\overline{Q} = 1/2(R + iS)$ form a basis for the complexified Lie algebra $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p} \otimes_{\mathbf{R}} \mathbf{C}$ and generate the vector fields $\frac{1}{y}\frac{\partial}{\partial y}$, $\frac{1}{y}\frac{\partial}{\partial x}$, and $\frac{1}{y}\frac{\partial}{\partial \bar{x}}$.

A computation shows that the adjoint action of $SU_2(\mathbf{C})$ on $\mathfrak{p}_{\mathbf{C}}$ (which coincides with the natural action of $d_\epsilon u \otimes 1$ on $T_\epsilon(\mathcal{H}) \otimes \mathbf{C}$ for $u \in SU_2(\mathbf{C})$) is nothing but the symmetric square of the standard representation of $GL_2(\mathbf{C})$ on $\mathbf{C}^2$. More precisely, let $\rho_m = Sym^m(\mathbf{C}^2)$ be the mth symmetric tensor representation of the standard representation of $GL_2(\mathbf{C})$ on $\mathbf{C}^2$. Thus

$$\rho_m\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)\left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)^m = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)\right)^m,$$

where

$$\left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)^m = (S^m, S^{m-1}T, \ldots, ST^{m-1}, T^m)^t.$$

Then, in the *ordered basis* Q, $-P$, $-\overline{Q}$ of $\mathfrak{p}_{\mathbf{C}}$, we have

$$Ad(u) \otimes 1 = \rho_2(u), \tag{1}$$

for all $u \in SU_2(\mathbf{C})$.

Let us translate this into the dual setting. Note that $dx, dy, d\bar{x}$ form a 'basis' (over $C^\infty(\mathcal{H})$, the space of C^∞ functions on $\mathcal{H}$), for $\Omega^1(\mathcal{H})$, the space of C^∞ global 1-forms on $\mathcal{H}$. There is a left action of $SL_2(\mathbf{C})$ on $\Omega^1(\mathcal{H})$ induced by pull back: for $\gamma \in SL_2(\mathbf{C})$ and $\omega \in \Omega^1(\mathcal{H})$ set $\gamma \cdot \omega = \omega \circ \gamma^{-1}$ where

$$\begin{aligned}(f_1(z)dx + f_2(z)dy + f_3(z)d\bar{x}) \circ \gamma \qquad &= \\ f_1(z)d(x \circ \gamma) + f_2(z)d(y \circ \gamma) + f_3(z)d(\bar{x} \circ \gamma).&\end{aligned} \tag{2}$$

Evaluation at the origin $\epsilon \in \mathcal{H}$ induces a map

$$\Omega^1(\mathcal{H}) \to T_\epsilon(\mathcal{H})^*, \qquad \omega \mapsto \omega_\epsilon$$

and this map is SU_2-equivariant. Let $L(2, \mathbf{C})$ denote the space of homogeneous polynomials over $\mathbf{C}$ of degree two in $\boldsymbol{a} = \binom{A}{B}$. Under the action $g \cdot P(\binom{A}{B})) = P(g^\iota \binom{A}{B}))$ (where $g^\iota = \det(g)g^{-1}$ is the adjoint of g), $L(2, \mathbf{C})$ is a model for the representation dual to ρ_2. Moreover, in view of (1) above, the map from $T_\epsilon(\mathcal{H})^* \otimes \mathbf{C}$ to $L(2, \mathbf{C})$ defined by

$$dx_\epsilon \mapsto A^2, \quad -dy_\epsilon \mapsto AB, \quad -d\bar{x}_\epsilon \mapsto B^2 \tag{3}$$

is an isomorphism of $SU_2(\mathbf{C})$-modules. Composing the evaluation map with the map (3) above, we get an $SU_2(\mathbf{C})$-map

$$\Omega^1(\mathcal{H}) \to L(2, \mathbf{C}). \tag{4}$$

We introduce the automorphy factor $j(\gamma, z) = \rho(c)\, z + \rho(d)$. Let $\Theta(\gamma) = \begin{pmatrix} \rho(a) & \rho(b) \\ \rho(c) & \rho(d) \end{pmatrix}$. Then since $\Theta(\gamma) \begin{pmatrix} z \\ Id_2 \end{pmatrix} = \begin{pmatrix} \gamma z \\ Id_2 \end{pmatrix} j(\gamma, z)$, we indeed see that

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z)\, j(\gamma_2, z),$$

for all $\gamma_1, \gamma_2 \in SL_2(\mathbf{C})$.

Using the automorphy factor we may describe the behaviour of differential forms on $\mathcal{H}$ under pull back. Indeed if $\gamma \in SL_2(\mathbf{C})$, a lengthy, but elementary, computation yields (cf. [10], Equation 2.4)

$$\begin{pmatrix} d\,(x \circ \gamma) \\ -d\,(y \circ \gamma) \\ -d\,(\bar{x} \circ \gamma) \end{pmatrix} = \rho_2({}^t j(\gamma, z))^{-1} \begin{pmatrix} dx \\ -dy \\ -d\bar{x} \end{pmatrix}. \tag{5}$$

The pull back action of $\gamma \in SL_2(\mathbf{C})$ described in (2) above, induces via the map (4), a map on $L(2, C)$ which, in view of (5), may easily be checked to be given by

$$P(\tbinom{A}{B})) \mapsto P({}^t j(\gamma^{-1}, \epsilon)^{-1} \tbinom{A}{B})). \tag{6}$$

We will use this fact later in the construction of the Eichler-Shimura-Harder isomorphism.

2.3 Relation between forms on $GL_2(K_{\boldsymbol{A}})$ and $\mathcal{H}$

Let h denote the class number of K. Starting with an 'adèlic' cusp form $f \in \boldsymbol{S}_{(\boldsymbol{n},\boldsymbol{v})}(\mathfrak{N}, \chi)$, we show how to associate h 'cusp forms' F_i on $GL_2(\mathbf{C})$, and thus, h 'cusp forms' f_i on $\mathcal{H}$, to f.

The strong approximation theorem gives us the decomposition

$$GL_2(K_{\boldsymbol{A}}) = \coprod_{i=1}^{h} GL_2(K)\, t_i\, U_0(\mathfrak{N})\, GL_2(\mathbf{C})$$

where $t_i = \left(\begin{smallmatrix} a_i & 0 \\ 0 & 1 \end{smallmatrix}\right)$, for certain finite ideles a_i. We may assume $a_1 = 1$. Set $\mathfrak{a}_i = a_i\mathcal{O}$. For $\mathfrak{a}$ a fractional ideal of $\mathcal{O}$, we define the discrete subgroup $\Gamma_{\mathfrak{a}}$ of $SL_2(K)$ via

$$\Gamma_{\mathfrak{a}} = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \,\middle|\, a, d \in \mathcal{O},\ b \in \mathfrak{a},\ c \in \mathfrak{a}^{-1}\mathfrak{N},\ ad - bc = 1 \right\}.$$

We see that $SL_2(K) \cap t_k\, U_0(\mathfrak{N})\, t_k^{-1}\, GL_2(\mathbf{C}) = \Gamma_{\mathfrak{a}_i}$.

Define for $i = 1, \ldots, h$,

$$\begin{array}{rcl} F_i : GL_2(\mathbf{C}) & \longrightarrow & W(n^*, \mathbf{C}) \\ F_i(g) & = & f(t_i g). \end{array}$$

Each of the F_i is a 'cusp form' on $GL_2(\mathbf{C})$, and determines in turn a 'cusp form' f_i on $\mathcal{H}$ as follows. Define $f_i : \mathcal{H} \to W(n^*, \mathbf{C})$ by

$$f_i(z, \left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)) = F_i(g,\, j(g, \epsilon)^t \left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)),$$

where $g \in SL_2(\mathbf{C})$ is chosen such that $g \cdot \epsilon = z$. It is a routine matter to check that f_i is well defined and that it satisfies the automorphy condition

$$f_i(\gamma z,\, \boldsymbol{s}) \quad = \quad \chi_{\mathfrak{N}}^{-1}(d)\, f_i(z,\, j(\gamma, z)^t\, \boldsymbol{s}),$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathfrak{a}_i}$. Thus $f_i \in \mathcal{S}_{\boldsymbol{n}}(\Gamma_{\mathfrak{a}_i}, \chi_{\mathfrak{N}}^{-1})$, the space of cusp forms on $\mathcal{H}$ satisfying the above automorphy condition.

The Fourier expansion of f also descends to $\mathcal{H}$. In fact one may check each f_i has Fourier expansion

$$\begin{aligned} f_i\left(\left(\begin{smallmatrix} x & -y \\ y & \bar{x} \end{smallmatrix}\right), \left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)\right) \quad = \quad & |a_i|_K\, y \sum_{\alpha=0}^{n^*} \binom{n^*}{\alpha} \Bigg[\sum_{\xi \in K^\times} c\,(\xi a_i d) \\ & \left(\frac{\xi}{i|\xi|}\right)^{n+1-\alpha} \frac{1}{\xi^{v_i}\bar{\xi}^{v_c}}\, K_{\alpha-n-1}(4\pi y|\xi|)\, \mathbf{e}_K(\xi x) \Bigg] S^{n^*-\alpha} T^{\alpha} \end{aligned} \tag{7}$$

We will use the above expansion when $i = 1$.

3 What is the Twisted Tensor L-function?

3.1 Definitions

Let $f \in \boldsymbol{S}_{(\boldsymbol{n},\boldsymbol{v})}(\mathfrak{N}, \chi)$ be as in Definition 1. In addition, we will hereafter assume that f is a normalized primitive form. In this section we recall the definition of the twisted tensor L-function of f, and the 'twisted' twisted tensor L-function of f (the tongue-twisting terminology is regretted, but the latter L-function will not play a big role in this paper, even though a rationality result for it should be provable using the techniques of this paper).

Let

$$\psi = \chi|_{\mathbf{Q}} \cdot |\ |_{\mathbf{Q}}^{2n+2v_i+2v_c} : \mathbf{Q}^{\times} \backslash \mathbf{Q}_{\boldsymbol{A}}^{\times} \longrightarrow \mathbf{C}^{\times},$$

where $|\ |_{\mathbf{Q}}$ is the usual idele character of $\mathbf{Q}_{\boldsymbol{A}}^{\times}$. We will regard ψ as a Dirichlet character

$$\psi_N : (\mathbf{Z}/NZ)^{\times} \to \mathbf{C}^{\times},$$

though we note that $L_N(s,\, \psi) = L_N(s,\, \psi_N^{-1})$. We will also need the quadratic character (i.e. the Legendre symbol) associated to K:

$$\chi_D = \left(\frac{-D}{\ }\right) : (\mathbf{Z}/DZ)^{\times} \to \{\pm 1\}.$$

Recall that

$$\chi_D(p) = \begin{cases} 1 & \text{if } p \text{ splits in } K, \\ -1 & \text{if } p \text{ is inert in } K, \\ 0 & \text{if } p|D, \text{ i.e. if } p \text{ ramifies in } K. \end{cases}$$

Definition 2 *The standard L-function associated to f is*

$$D(s, f) = \sum_{\mathfrak{m} \subset \mathcal{O}} \frac{c\,(\mathfrak{m}, f)}{N(\mathfrak{m})^s}.$$

In analogy with [1] we also make the following definitions:

Definition 3 *The twisted tensor L-function of f is*

$$G(s, f) = L_N(2s - 2n - 2v_i - 2v_c - 2,\ \psi_N^{-1}) \cdot \sum_{m=1}^{\infty} \frac{c\,(m, f)}{m^s}.$$

Note that the twisted tensor L-function is essentially a 'sub' L-function of $D(s, f)$, obtained by restricting the summation to (integral) ideals coming from $\mathbf{Q}$. The choice of nomenclature 'twisted tensor' is more natural in the context of automorphic representations (see for instance [3]) and here we use this name for lack of a better alternative. However, in the literature $G(s, f)$ is sometimes referred to as an L-function of Asai type.

Definition 4 *The 'twisted' twisted tensor L-function of f is*

$$\begin{aligned} G(s,f,\chi_D) &= L_N(2s-2n-2v_i-2v_c-2,\ \psi_N^{-1})\cdot \\ &\quad \prod_{p|D}(1+\chi(\wp)p^{-s+1})\cdot\sum_{m=1}^{\infty}\frac{c\,(m,f)\chi_D(m)}{m^s}. \end{aligned}$$

Definition 5 *The Rankin product L-function of f (and f^c) is*

$$H(s,f) = L_{\mathfrak{N}}(2s-2,\ \chi\chi^c)\cdot\sum_{\mathfrak{m}\subset\mathcal{O}}\frac{c\,(\mathfrak{m},f)\,c\,(\bar{\mathfrak{m}},f)}{N(\mathfrak{m})^s}.$$

Here f^c denotes the common eigenform of the Hecke operators with eigenvalues $c\,(\mathfrak{m},f^c)=c\,(\bar{\mathfrak{m}},f)$, and χ^c is the character defined by $\chi^c(\wp)=\chi(\bar{\wp})$.

The Hecke L-functions appearing in the definitions above ensure that each of the above L-functions has a 'good' Euler product expansion.

3.2 Euler product expansions

Each of the above L-functions has a well known Euler product expansion (see [1], [11], [15], [17]) which we list for the sake of completeness, as well as for the purposes of our (conjectural) interpretation of these L-functions as motivic L-functions in Section 4. For simplicity we write $c(\mathfrak{m})$ for $c\,(\mathfrak{m},f)$. Recall that $D(s,f)=$

$$\prod_{\wp|\mathfrak{N}}\frac{1}{1-c(\wp)N(\wp)^{-s}}\cdot\prod_{\wp\nmid\mathfrak{N}}\frac{1}{1-c(\wp)N(\wp)^{-s}+\chi(\wp)N(\wp)^{1-2s}}.$$

Let $\alpha_\wp$ and $\beta_\wp$ denote the reciprocal roots of the polynomial

$$1-c(\wp)X+\chi(\wp)\,N(\wp)X^2.$$

A routine computation (see [6], Chapter 3.2) shows that the (reciprocal of the) p-Euler factor of $H(s,f)$ is given by $\frac{1}{H_p(s,f)}=$

$$\begin{cases} (1-\alpha_\wp\alpha_{\bar\wp}p^{-s})^2(1-\alpha_\wp\beta_{\bar\wp}p^{-s})^2(1-\beta_\wp\alpha_{\bar\wp}p^{-s})^2(1-\beta_\wp\beta_{\bar\wp}p^{-s})^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } p=\wp\bar\wp, \\ (1-\alpha_\wp^2p^{-2s})(1-\chi(\wp)p^{-2s+2})^2(1-\beta_\wp^2p^{-2s}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } p=\wp, \\ (1-\alpha_\wp^2p^{-s})(1-\chi(\wp)p^{-s+1})^2(1-\beta_\wp^2p^{-s}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } p=\wp^2. \end{cases}$$

Lemma 1 $L_N(2s-2,\ \chi|_{\mathbf{Q}})=L_N(2s-2n-2v_i-2v_c-2,\ \psi_N^{-1})$.

Proof

$$\begin{aligned}
L_N(2s-2,\,\chi|_{\mathbf{Q}}) &= L_N(2s-2,\,\psi\cdot|\;|_{\mathbf{Q}}^{-2n-2v_i-2v_c}) \\
&= \prod_{p\not\,\mid N}\frac{1}{1-\psi(\tilde{\omega_p})\,|\tilde{\omega_p}|_{\mathbf{Q_A}}^{-2n-2v_i-2v_c}p^{-(2s-2)}} \\
&= \prod_{p\not\,\mid N}\frac{1}{1-\psi(\tilde{\omega_p})p^{-(-2n-2v_i-2v_c+2s-2)}} \\
&= L_N(2s-2n-2v_i-2v_c-2,\,\psi) \\
&= L_N(2s-2n-2v_i-2v_c-2,\,\psi_N^{-1}). \qquad \square
\end{aligned}$$

Using Lemma 1, one may similarly compute that the (reciprocal of the) p-Euler factor of $G(s,f)$ is given by $\frac{1}{G_p(s,f)}=$

$$\begin{cases}
(1-\alpha_\wp\alpha_{\bar\wp}p^{-s})(1-\alpha_\wp\beta_{\bar\wp}p^{-s})(1-\beta_\wp\alpha_{\bar\wp}p^{-s})(1-\beta_\wp\beta_{\bar\wp}p^{-s}) & \\
 & \text{if } p=\wp\bar\wp, \\
(1-\alpha_\wp p^{-s})(1-\chi|_{\mathbf{Q}}(p)p^{-2s+2})(1-\beta_\wp p^{-s}) & \\
 & \text{if } p=\wp, \\
(1-\alpha_\wp^2 p^{-s})(1-\chi(\wp)p^{-s+1})(1-\beta_\wp^2 p^{-s}) & \\
 & \text{if } p=\wp^2.
\end{cases}$$

Finally, another use of Lemma 1, shows that the (reciprocal of the) p-Euler factor for $G(s,f,\chi_D)$ is given by $\frac{1}{G_p(s,f,\chi_D)}=$

$$\begin{cases}
(1-\alpha_\wp\alpha_{\bar\wp}p^{-s})(1-\alpha_\wp\beta_{\bar\wp}p^{-s})(1-\beta_\wp\alpha_{\bar\wp}p^{-s})(1-\beta_\wp\beta_{\bar\wp}p^{-s}) & \\
 & \text{if } p=\wp\bar\wp, \\
(1+\alpha_\wp p^{-s})(1-\chi|_{\mathbf{Q}}(p)p^{-2s+2})(1+\beta_\wp p^{-s}) & \\
 & \text{if } p=\wp, \\
(1-\chi(\wp)p^{-s+1}) & \\
 & \text{if } p=\wp^2.
\end{cases}$$

Remark 1 Following Asai, we note the splitting formula (cf. [1], Theorem 3)

$$H(s,f)=G(s,f)\cdot G(s,f,\chi_D).$$

This follows immediately from the Euler product expansions above.

4 Motivic Interpretation of $G(s,f)$

It is natural to expect that each of the L-functions in Section 3 is motivic, in the sense of [2]. In this section we make this conjectural identification concrete. In particular, this allows us to compute the 'critical strip' for the twisted tensor L-function of f and to state Deligne's conjecture describing its behaviour at the critical integers.

4.1 Motives

Let f be as in Section 3. Following [10], we hypothesize the existence of a rank 2 motive M over K, with coefficients in E, a finite extension of $\mathbf{Q}(f)$ (the number field generated by the Fourier coefficients of f), and weight $w = n + 1 + v_i + v_c$. We have the following realizations of M (see [2])

- Betti realization: $H_\sigma(M)$, where $\sigma = i, c$. Each is a two-dimensional E-vector space.
- de Rham realization: $H_{DR}(M)$, a two-dimensional $E \otimes K$-module.
- ℓ-adic realization: $H_\ell(M)$. This is a two-dimensional $E \otimes \mathbf{Q}_\ell$-module equipped with a continuous action of $\mathrm{Gal}(\bar{\mathbf{Q}}/K)$, say ρ. We expect ρ to be unramified outside $\mathfrak{N}\ell$, and we also expect the identity

$$\begin{aligned} D_\wp(s, f) &= (1 - \alpha_\wp N(\wp)^{-s})^{-1}(1 - \beta_\wp N(\wp)^{-s})^{-1} \\ &= \det(1 - \rho(\mathrm{Frob}_\wp)N(\wp)^{-s}, H_\ell(M)^{I_\wp})^{-1} \end{aligned} \tag{8}$$

  to hold for all $\wp$. In particular we expect that $D(s, f) = L(s, M)$.

We have the Hodge decompositions (see [10], Section 8)

$$\begin{aligned} H_i(M) \otimes \mathbf{C} &= H_i^{(n+1+v_i, v_c)}(M) \oplus H_i^{(v_i, n+1+v_c)}(M) \\ &= (E \otimes \mathbf{C})\, x \oplus (E \otimes \mathbf{C})\, y \\ H_c(M) \otimes \mathbf{C} &= H_c^{(n+1+v_c, v_i)}(M) \oplus H_c^{(v_c, n+1+v_i)}(M) \\ &= (E \otimes \mathbf{C})\, y^c \oplus (E \otimes \mathbf{C})\, x^c. \end{aligned}$$

If $F_\infty : H_\sigma(M) \to H_{\sigma c}(M)$ is the isomorphism induced functorially from the action of complex conjugation, then $F_\infty \otimes 1$ takes x to x^c and y to y^c.

Similarly, for the conjugate form f^c, we hypothesize the existence of the rank 2 motive M^c over K, conjugate to M. It is also expected to have weight w, coefficients in E, and realizations similar to the ones above. In fact we may identify

$$H_i(M^c) = H_c(M) \text{ and } H_c(M^c) = H_i(M). \tag{9}$$

In this case the Hodge decompositions are

$$\begin{aligned} H_i(M^c) \otimes \mathbf{C} &= H_i^{(n+1+v_c, v_i)}(M^c) \oplus H_i^{(v_c, n+1+v_i)}(M^c) \\ &= (E \otimes \mathbf{C})\, a \oplus (E \otimes \mathbf{C})\, b \\ H_c(M^c) \otimes \mathbf{C} &= H_c^{(n+1+v_i, v_c)}(M^c) \oplus H_c^{(v_i, n+1+v_c)}(M^c) \\ &= (E \otimes \mathbf{C})\, b^c \oplus (E \otimes \mathbf{C})\, a^c \end{aligned}$$

with $F_\infty \otimes 1$ taking a to a^c and b to b^c. Because of (9), we identify $a = y^c$, $b = x^c$, $a^c = x$ and $b^c = y$.

Further if $\rho^c : \mathrm{Gal}(\bar{\mathbf{Q}}/K) \longrightarrow \mathrm{Aut}(H_\ell(M^c))$ denotes the corresponding ℓ-adic representation of M^c, then we may identify $H_\ell(M^c)$ with $H_\ell(M)$ and ρ^c with the conjugate action of ρ, namely

$$\rho^c(h) = \rho(chc^{-1}). \tag{10}$$

Let us now consider the motive $M \otimes M^c$, a rank 4 motive over K, of weight $2w = 2n + 2 + 2v_i + 2v_c$, with coefficients in E, obtained by tensoring the above two motives. This has realizations

- Betti: $H_\sigma(M \otimes M^c) = H_\sigma(M) \otimes_E H_\sigma(M^c)$, for $\sigma = i, c$
- de Rham : $H_{DR}(M \otimes M^c) = H_{DR}(M) \otimes_{E \otimes K} H_{DR}(M^c)$
- ℓ-adic : $H_\ell(M \otimes M^c) = H_\ell(M) \otimes_{E \otimes \mathbf{Q}_\ell} H_\ell(M^c)$, with Galois action $\rho \otimes \rho^c$: $\mathrm{Gal}(\bar{\mathbf{Q}}/K) \longrightarrow \mathrm{Aut}(V \otimes V^c)$, where $V = V^c = H_\ell(M)$, but the action on V^c is as in (10).

Except for finitely many $\wp$ we have

$$\begin{aligned} (\rho \otimes \rho^c)(\mathrm{Frob}_\wp) &= \rho(\mathrm{Frob}_\wp) \otimes \rho(\mathrm{Frob}_{\bar\wp}) \\ &= \begin{pmatrix} \alpha_\wp & 0 \\ 0 & \beta_\wp \end{pmatrix} \otimes \begin{pmatrix} \alpha_{\bar\wp} & 0 \\ 0 & \beta_{\bar\wp} \end{pmatrix} \end{aligned}$$

so that

$$\det(1 - (\rho \otimes \rho^c)(\mathrm{Frob}_\wp) N(\wp)^{-s})^{-1} = H_\wp(s, f).$$

This yields the identification

$$H(s, f) = L(s, M \otimes M^c).$$

Let us now realize $G(s, f)$ and $G(s, f, \chi_D)$ as motivic L-functions. We first extend the representation $\rho \otimes \rho^c$ in two different ways. Set $G = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, $H = \mathrm{Gal}(\bar{\mathbf{Q}}/K)$. We have the exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow < c > \longrightarrow 1$$

where c denotes the nontrivial automorphism of $\mathrm{Gal}(K/\mathbf{Q})$. We have also been denoting c to be 'complex conjugation,' regarding it an automorphism of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ with $c^2 = 1$. In our computations, it will be convenient sometimes to think of c more generally as an arbitrary lift of the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$ to G, in which case we simply have $c^2 \in H$, say $c^2 = h_0$.

Now define $R_\pm : G \longrightarrow \mathrm{Aut}(V \otimes V^c)$ via

$$\begin{aligned} R_\pm(h)(v_1 \otimes v_2) &= hv_1 \otimes chc^{-1}v_2 \quad \text{if } h \in H, \\ R_\pm(c)(v_1 \otimes v_2) &= \pm v_2 \otimes h_0 v_1. \end{aligned}$$

Here we are omitting ρ from the notation. One can easily check that $R_\pm : G \longrightarrow \mathrm{Aut}(V \otimes V^c)$ are well defined homomorphisms, such that $R_\pm|_H = \rho \otimes \rho^c$. Furthermore, R_+ (respectively R_-) is independent of the choice of $c \in G$ up to isomorphism. Regarding $\chi_D = \left(\frac{-D}{\cdot}\right)$ as a character of G (trivial on H) we see that $R_- = R_+ \otimes \chi_D$.

Having defined two extensions of $\rho \otimes \rho^c$ to G, we also expect that the motive $M \otimes M^c$ descends to $\mathbf{Q}$ in two corresponding ways, which we denote by $(M \otimes M^c)_+$ and $(M \otimes M^c)_-$. These are expected to be rank 4 motives over $\mathbf{Q}$, of weight $2w$, with coefficients in E, defined with respect to the descent data

$$\begin{array}{lcll} i_+ : M \otimes M^c & \longrightarrow & M^c \otimes M & \quad u \otimes v \mapsto v \otimes u \\ i_- : M \otimes M^c & \longrightarrow & M^c \otimes M & \quad u \otimes v \mapsto -v \otimes u. \end{array}$$

Thus, $(M \otimes M^c)_+$ has realizations

- Betti: $H_B((M \otimes M^c)_+) = H_i(M) \otimes H_i(M^c) \stackrel{i_\pm}{=} H_i(M^c) \otimes H_i(M) = H_c(M) \otimes H_c(M^c)$
- de Rham: $H_{DR}((M \otimes M^c)_+) = (H_{DR}(M) \otimes H_{DR}(M^c))^{\mathrm{Gal}(K/\mathbf{Q})}$ for a suitable action of $\mathrm{Gal}(K/\mathbf{Q})$ on the differentials over K.
- ℓ-adic: $H_\ell((M \otimes M^c)_+) = H_\ell(M) \otimes_{E \otimes \mathbf{Q}_\ell} H_\ell(M^c) \stackrel{i_\pm}{=} H_\ell(M^c) \otimes H_\ell(M)$, with $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-action given by R_+.

We have analogous realizations for $(M \otimes M^c)_-$. The Hodge decomposition of $(M \otimes M^c)_\pm$ is

$$\begin{aligned} & H_B((M \otimes M^c)_\pm) \otimes \mathbf{C} \\ & \quad = H_B^{(2n+2+v_i+v_c, v_i+v_c)} \oplus H_B^{(n+1+v_i+v_c, n+1+v_i+v_c)} \\ & \qquad \oplus H_B^{(n+1+v_i+v_c, n+1+v_i+v_c)} \oplus H_B^{(v_i+v_c, 2n+2+v_i+v_c)} \\ & \quad = (E \otimes \mathbf{C})\,(x \otimes a) \oplus (E \otimes \mathbf{C})\,(x \otimes b) \\ & \qquad \oplus (E \otimes \mathbf{C})\,(y \otimes a) \oplus (E \otimes \mathbf{C})\,(y \otimes b). \end{aligned}$$

Note that the (p,p)-space is two dimensional. One may compute the action of $F_\infty \otimes 1$ on this four dimensional space as follows:

$$\begin{aligned} x \otimes a \mapsto x^c \otimes a^c = b \otimes y \mapsto \pm y \otimes b \\ x \otimes b \mapsto x^c \otimes b^c = b \otimes x \mapsto \pm x \otimes b \\ y \otimes a \mapsto y^c \otimes a^c = a \otimes y \mapsto \pm y \otimes a \\ y \otimes b \mapsto y^c \otimes b^c = a \otimes x \mapsto \pm x \otimes a \end{aligned}$$

where the last map is $i_\pm$. Thus we see that for $(M \otimes M^c)_+$, $F_\infty \otimes 1$ acts by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on the (p,p)-part, whereas for $(M \otimes M^c)_-$, it acts by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Proposition 1
$$G(s,f) = L(s,(M\otimes M^c)_+)$$
$$G(s,f,\chi_D) = L(s,(M\otimes M^c)_-).$$

Proof The proof is based on comparing Euler factors at p.

<u>Case $p=\wp\bar\wp$</u> Pick eigenbases for $\mathrm{Frob}_\wp$ and $\mathrm{Frob}_{\bar\wp}$ respectively. Say $\mathrm{Frob}_\wp e_1 = \alpha_\wp e_1$, $\mathrm{Frob}_\wp e_2 = \beta_\wp e_2$; $\mathrm{Frob}_{\bar\wp} e_3 = \alpha_{\bar\wp} e_3$, $\mathrm{Frob}_{\bar\wp} e_4 = \beta_{\bar\wp} e_4$. Then since we may assume that $\mathrm{Frob}_p = \mathrm{Frob}_\wp \in H$ a simple computation shows (in the basis $e_1\otimes e_3, e_1\otimes e_4, e_2\otimes e_3, e_2\otimes e_4$)

$$R_\pm(\mathrm{Frob}_p) = \begin{pmatrix} \alpha_\wp\alpha_{\bar\wp} & & & \\ & \alpha_\wp\beta_{\bar\wp} & & \\ & & \beta_\wp\alpha_{\bar\wp} & \\ & & & \beta_\wp\beta_{\bar\wp} \end{pmatrix}.$$

Thus $\det(1-R_\pm(\mathrm{Frob}_p)p^{-s})^{-1} =$

$$(1-\alpha_\wp\alpha_{\bar\wp}p^{-s})(1-\alpha_\wp\beta_{\bar\wp}p^{-s})(1-\beta_\wp\alpha_{\bar\wp}p^{-s})(1-\beta_\wp\beta_{\bar\wp}p^{-s}).$$

<u>Case $p=\wp$</u> First assume that $\mathrm{Frob}_p^2 = \mathrm{Frob}_\wp$. Since we are interested in computing characteristic polynomials only, we may assume that $\mathrm{Frob}_p = c$ and $\mathrm{Frob}_\wp = h_0 = c^2$. Pick an eigenbasis of $\mathrm{Frob}_\wp$. Say $\mathrm{Frob}_\wp e_1 = \alpha_\wp e_1$ and $\mathrm{Frob}_\wp e_2 = \beta_\wp e_2$. Then again it is easy to compute that (in the basis $e_1\otimes e_1, e_1\otimes e_2, e_2\otimes e_1, e_2\otimes e_2$)

$$\begin{aligned} R_\pm(\mathrm{Frob}_p) = R_\pm(c) &= \begin{pmatrix} \pm\alpha_\wp & & & \\ & & \pm\beta_\wp & \\ & \pm\alpha_\wp & & \\ & & & \pm\beta_\wp \end{pmatrix} \\ &\sim \begin{pmatrix} \pm\alpha_\wp & & & \\ & \pm\sqrt{\alpha_\wp\beta_\wp} & & \\ & & \mp\sqrt{\alpha_\wp\beta_\wp} & \\ & & & \pm\beta_\wp \end{pmatrix}. \end{aligned}$$

Thus $\det(1-R_\pm(\mathrm{Frob}_p)p^{-s})^{-1} =$

$$(1\mp\alpha_\wp p^{-s})(1-\alpha_\wp\beta_\wp p^{-2s})(1\mp\beta_\wp p^{-s}).$$

<u>Case $p=\wp^2$</u> First assume $\mathrm{Frob}_p = \mathrm{Frob}_\wp \in H$. Then we may assume that $c^2 = h_0 = 1$. Again pick an eigenbasis for $\mathrm{Frob}_\wp$. Say $\mathrm{Frob}_\wp e_1 = \alpha_\wp e_1$ and $\mathrm{Frob}_\wp e_2 = \beta_\wp e_2$. Then (in the basis $e_1\otimes e_1, e_1\otimes e_2, e_2\otimes e_1, e_2\otimes e_2$) we have

$$R_\pm(\mathrm{Frob}_p) = \begin{pmatrix} \alpha_\wp^2 & & & \\ & \alpha_\wp\beta_\wp & & \\ & & \alpha_\wp\beta_\wp & \\ & & & \beta_\wp^2 \end{pmatrix}.$$

Further the action of c is given by

$$R_\pm(c) = \begin{pmatrix} \pm1 & & & \\ & & \pm1 & \\ & \pm1 & & \\ & & & \pm1 \end{pmatrix}.$$

We pick an eigenbasis for c, namely $e_1 \otimes e_1$, $\frac{e_1\otimes e_2+e_2\otimes e_1}{2}$, $e_2 \otimes e_2$, $\frac{e_1\otimes e_2-e_2\otimes e_1}{2}$ In this basis

$$R_{\pm}(c) = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \pm 1 & \\ & & & \mp 1 \end{pmatrix}.$$

Since $(V \otimes V)^{I_p} = (V \otimes V)^c$ is a 3 (respectively 1) dimensional space, we see that in the basis $e_1 \otimes e_1$, $\frac{e_1\otimes e_2+e_2\otimes e_1}{2}$, $e_2 \otimes e_2$ (respectively $\frac{e_1\otimes e_2-e_2\otimes e_1}{2}$)

$$R_+(\mathrm{Frob}_p)|_{(V\otimes V)^c} = \begin{pmatrix} \alpha_\wp^2 & & \\ & \alpha_\wp \beta_\wp & \\ & & \beta_\wp^2 \end{pmatrix}$$

(respectively

$$R_-(\mathrm{Frob}_p)|_{(V\otimes V)^c} = \alpha_\wp \beta_\wp).$$

Hence

$$\det(1 - R_+(\mathrm{Frob}_p)p^{-s})^{-1} = (1 - \alpha_\wp^2 p^{-s})(1 - \alpha_\wp \beta_\wp p^{-s})(1 - \beta_\wp^2 p^{-s})$$

(respectively

$$\det(1 - R_-(\mathrm{Frob}_p)p^{-s})^{-1} = (1 - \alpha_\wp \beta_\wp p^{-s})). \qquad \square$$

Finally note that since $R_- = R_+ \otimes \chi_D$, we can recover Asai's splitting formula in Remark 1 because of the easily proved

Proposition 2 *Let R_1 and R_2 denote extensions of a representation R of H, to G, with the property that $R_2 = R_1 \otimes \chi_D$. Then $L(s, R) = L(s, R_1) \cdot L(s, R_2)$.*

4.2 Critical values

Let us compute the critical strips of $G(s, f)$ and $G(s, f, \chi_D)$. We already saw in the last section, that F_∞ acts by a scalar on the (p, p) part (+1 or -1). This is fortunate, for it is a prerequisite for the existence of critical values!

Recall, the integer j is critical for $L(s, (M \otimes M^c)_\pm) \iff 0$ is critical for $L(s, (M \otimes M^c)_\pm(j))$. The Hodge numbers of $(M \otimes M^c)_\pm(j)$ are

$$\begin{gathered} (2n + 2 + v_i + v_c - j, v_i + v_c - j), \\ (n + 1 + v_i + v_c - j, n + 1 + v_i + v_c - j), \\ (n + 1 + v_i + v_c - j, n + 1 + v_i + v_c - j), \\ (v_i + v_c - j, 2n + 2 + v_i + v_c - j). \end{gathered}$$

Following the recipe in Section 5 of [2], we see that j is critical for $(M \otimes M^c)_\pm \iff$

1. $2n + 2 + v_i + v_c - j \geq 0$ and $v_i + v_c - j \leq -1$.
   Thus $v_i + v_c + 1 \leq j \leq 2n + 2 + v_i + v_c$.

2. The action of F_∞ on the (p,p) spaces, which is given in the two cases by $\left(\begin{smallmatrix} \pm(-1)^j & 0 \\ 0 & \pm(-1)^j \end{smallmatrix}\right)$, should be

$$\begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{if } n+1+v_i+v_c-j<0, \\ \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) & \text{if } n+1+v_i+v_c-j\geq 0. \end{cases}$$

We conclude that the critical strip for $(M\otimes M^c)_+$ is

$$\{j\in[v_i+v_c+1,n+1+v_i+v_c]\mid j \text{ is odd }\} \quad \cup$$
$$\{j\in(n+1+v_i+v_c,2n+2+v_i+v_c]\mid j \text{ is even }\}$$

whereas for $(M\otimes M^c)_-$ it is

$$\{j\in[v_i+v_c+1,n+1+v_i+v_c]\mid j \text{ is even}\} \quad \cup$$
$$\{j\in(n+1+v_i+v_c,2n+2+v_i+v_c]\mid j \text{ is odd }\}.$$

Note that since the parity of the critical values is reversed in the two cases, the 'product' L-function $H(s,f)=L(s,M\otimes M^c)$ does *not* have any critical values. Of course one could check this directly by looking at the Gamma factors!

4.3 Deligne's conjecture

Recall that there is a comparison isomorphism

$$I: H_B((M\otimes M^c)_+)\otimes\mathbf{C}\xrightarrow{\sim}H_{DR}((M\otimes M^c)_+)\otimes\mathbf{C}.$$

Let $H_B^\pm((M\otimes M^c)_+)=\ker\left[F_\infty\mp 1: H_B((M\otimes M^c)_+)\to H_B((M\otimes M^c)_+)\right]$. Let $F^\pm$ denote the part of the Hodge filtration on $H_{DR}((M\otimes M^c)_+)$ as in [2], and set $H_{DR}^\pm((M\otimes M^c)_+)=H_{DR}((M\otimes M^c)_+)/F^\mp$. Then we have the induced maps

$$I^\pm: H_B^\pm\otimes\mathbf{C}\hookrightarrow H_B\otimes\mathbf{C}\xrightarrow{I}H_{DR}\otimes\mathbf{C}\to H_{DR}^\pm\otimes\mathbf{C}.$$

Let

$$\begin{aligned} c^\pm((M\otimes M^c)_+) &= \det(I^\pm)\in(E\otimes\mathbf{C})^* \\ \delta((M\otimes M^c)_+) &= \det(I)\in(E\otimes\mathbf{C})^* \end{aligned}$$

where the determinants are calculated in E-rational bases of $H_B^\pm$ and $H_{DR}^\pm$. Then Deligne conjectures that for every critical integer j,

$$\frac{L(j,(M\otimes M^c)_+)}{c^+((M\otimes M^c)_+(j))}\in E\otimes 1\subset E\otimes\mathbf{C}.$$

Here $c^+((M\otimes M^c)_+(j))$ is defined as above with the motive $(M\otimes M^c)_+$ replaced with $(M\otimes M^c)_+(j)$. In any case we have the formula (see [2], Equation (5.1.8))

$$c^+((M\otimes M^c)_+(j)) = \begin{cases} c^+((M\otimes M^c)_+)\cdot(2\pi i)^{3j} & \text{if } j \text{ is even,} \\ c^-((M\otimes M^c)_+)\cdot(2\pi i)^{j} & \text{if } j \text{ is odd.} \end{cases} \tag{11}$$

It is interesting, but perhaps not surprising, that we may express $c^+((M\otimes M^c)_+)$ in terms of the periods of the motive $N = Res_{K/\mathbf{Q}}M$. Indeed, recall that the latter is a rank 2 motive over $\mathbf{Q}$, having coefficients in E, and realizations

- Betti: $H_B(N) = H_i(M)\oplus H_c(M)$
- de Rham : $H_{DR}(N) = H_{DR}(M)$ regarded as an $E\otimes\mathbf{Q}$-module
- ℓ-adic : $H_\ell(N) = \mathrm{Ind}_H^G\, H_\ell(M)$.

As above, we also have the periods $c^+(N)$, $\delta(N)$ and $c^-(N)$.

Proposition 3 *We have the period relations*

$$\begin{aligned} c^+((M\otimes M^c)_+) &\doteq c^+(N)\,\delta(N), \\ \delta((M\otimes M^c)_+) &\doteq \delta(N)^2, \\ c^-((M\otimes M^c)_+) &\doteq c^+(N), \end{aligned}$$

where $\doteq$ denotes equality up to multiplication by an algebraic number.

Proof We have already picked bases for the complexifications of the Betti realizations $H_i(M)$ and $H_c(M)$. Consequently, $H_B(N)\otimes\mathbf{C}$ has x, y, x^c, y^c as a basis over $E\otimes\mathbf{C}$. We note that a basis for $H_B^+(N)$ is $(x+x^c)/2$, $(y+y^c)/2$.

Say $H_{DR}(M)$ is spanned by ω_1, ω_2, and $H_{DR}(M^c)$ by ω_1^c, ω_2^c - both as $E\otimes K$-modules. Then, we may assume that $H_{DR}(N)$ is spanned by ω_1, ω_2, ω_1^c, ω_2^c as an $E\otimes\mathbf{Q}$-module. Further, if $F^-H_{DR}(M)$ is spanned by ω_1, and $F^-H_{DR}(M^c)$ is spanned by ω_1^c, then $F^-H_{DR}(N)$ is spanned by ω_1, ω_1^c, and so $H_{DR}^+(N)$ is spanned by ω_2, ω_2^c. Now say

$$\begin{aligned} x &\mapsto a_{11}\omega_1 + a_{12}\omega_2 \qquad & x^c &\mapsto b_{11}\omega_1^c + b_{12}\omega_2^c \\ y &\mapsto a_{21}\omega_1 + a_{22}\omega_2 \qquad & y^c &\mapsto b_{21}\omega_1^c + b_{22}\omega_2^c. \end{aligned}$$

We see immediately that

$$c^+(N) = \frac{1}{4}\begin{vmatrix} a_{12} & a_{22} \\ b_{12} & b_{22} \end{vmatrix} \text{ and } \delta(N) = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}\begin{vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{vmatrix}.$$

On the other hand, for $(M\otimes M^c)_+$, a basis of H_B^+ is $x\otimes x^c$, $y\otimes y^c$, $(x\otimes y^c + y\otimes x^c)/2$. To write a basis for H_{DR}^+, we note first that $H_{DR}(M\otimes M^c)$ has basis $\omega_1 \otimes \omega_1^c$, $\omega_1 \otimes \omega_2^c$, $\omega_2 \otimes \omega_1^c$, $\omega_2 \otimes \omega_2^c$. Further the action of $c \in \mathrm{Gal}(K/\mathbf{Q})$ on this space is given by

$$a \otimes b \mapsto a^c \otimes b^c \overset{i_+}{\mapsto} b^c \otimes a^c.$$

Since $H_{DR}((M \otimes M^c)_+)$ is identified, with the Galois invariants under this action, it is spanned by $\omega_1 \otimes \omega_1^c$, $(\omega_1 \otimes \omega_2^c + \omega_2 \otimes \omega_1^c)/2$, $\sqrt{-D}(\omega_1 \otimes \omega_2^c - \omega_2 \otimes \omega_1^c)/2$, $\omega_2 \otimes \omega_2^c$. Further $F^- H_{DR}((M \otimes M^c)_+)$ is one dimensional and spanned by $\omega_1 \otimes \omega_1^c$. Thus $H^+_{DR}((M \otimes M^c)_+)$ is spanned by the last three vectors. In these bases we compute that $c^+((M \otimes M^c)_+)$

$$\begin{aligned}
&= \begin{vmatrix} a_{11}b_{12} + a_{12}b_{11} & a_{21}b_{22} + a_{22}b_{21} & \frac{a_{11}b_{22}+a_{21}b_{12}+a_{12}b_{21}+a_{22}b_{11}}{2} \\ a_{11}b_{12} - a_{12}b_{11} & a_{21}b_{22} - a_{22}b_{21} & \frac{a_{11}b_{22}+a_{21}b_{12}-a_{12}b_{21}-a_{22}b_{11}}{2\sqrt{-D}} \\ a_{12}b_{12} & a_{22}b_{22} & \frac{a_{12}b_{22}+a_{22}b_{12}}{2} \end{vmatrix} \\
&= \frac{-1}{\sqrt{-D}} \begin{vmatrix} a_{12} & a_{22} \\ b_{12} & b_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = \frac{-4}{\sqrt{-D}} c^+(N)\delta(N).
\end{aligned}$$

A similar argument works for $\delta((M \otimes M^c)_+)$. Indeed in the above mentioned bases we get $\delta((M \otimes M^c)_+)$

$$\begin{aligned}
&= \begin{vmatrix} a_{11}b_{11} & a_{11}b_{21} & a_{21}b_{11} & a_{21}b_{21} \\ a_{11}b_{12} + a_{12}b_{11} & a_{11}b_{22} + a_{12}b_{21} & a_{21}b_{12} + a_{22}b_{11} & a_{21}b_{22} + a_{22}b_{21} \\ \frac{a_{11}b_{12} - a_{12}b_{11}}{\sqrt{-D}} & \frac{a_{11}b_{22} - a_{12}b_{21}}{\sqrt{-D}} & \frac{a_{21}b_{12} - a_{22}b_{11}}{\sqrt{-D}} & \frac{a_{21}b_{22} - a_{22}b_{21}}{\sqrt{-D}} \\ a_{12}b_{12} & a_{12}b_{22} & a_{22}b_{12} & a_{22}b_{22} \end{vmatrix} \\
&= \frac{-2}{\sqrt{-D}} \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}^2 \begin{vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{vmatrix}^2 = \frac{-2}{\sqrt{-D}} \delta(N)^2.
\end{aligned}$$

Finally note that $H_B^-((M \otimes M^c)_+)$ is spanned by $(x \otimes y^c - y \otimes x^c)/2$, and $H^-_{DR}((M \otimes M^c)_+)$ is spanned by $\omega \otimes \omega^c$. So

$$c^-((M \otimes M^c)_+) = \frac{1}{2} \begin{vmatrix} a_{12} & a_{22} \\ b_{12} & b_{22} \end{vmatrix} = \frac{1}{2} c^+(N). \qquad \square$$

By including discriminant factors, one should be able to obtain the exact period relations up to multiplication by an element in E, but we do not pursue this point here.

5 The Connection with Differential Forms

In this section we explain how to realize cusp forms over K as harmonic differential forms on quotients of hyperbolic three space. The calculation is based on identifying de Rham cohomology with relative Lie algebra cohomology (see [7], Section 3). Here we only provide an explicit procedure for constructing these differential forms as outlined in Section 2.5 of [10].

5.1 The Eichler-Shimura-Harder isomorphism

Let $L(\boldsymbol{n},A)$ denote the space of homogeneous polynomials of degree n in $\boldsymbol{x} = \binom{X}{Y}$ and degree n in $\overline{\boldsymbol{x}} = \binom{\overline{X}}{\overline{Y}}$, with coefficients in some $K(\chi_{\mathfrak{N}})$-algebra A. We will make $L(\boldsymbol{n},A)$ into a $\Gamma_{\mathfrak{a}_i}$-module via

$$\gamma \cdot P(\boldsymbol{x}, \overline{\boldsymbol{x}}) = \chi_{\mathfrak{N}}^{-1}(d_\gamma)\, P(\gamma^\iota \boldsymbol{x}, \overline{\gamma}^\iota \overline{\boldsymbol{x}}). \tag{12}$$

We give $L(\boldsymbol{n},A)$ the discrete topology, and denote $\widetilde{L(\boldsymbol{n},A)}$ to be the sheaf of locally constant sections of the projection

$$\Gamma_{\mathfrak{a}_i}\backslash\mathcal{H} \times L(\boldsymbol{n},A) \to \Gamma_{\mathfrak{a}_i}\backslash\mathcal{H}.$$

Over K, there are two isomorphisms, which generalize the Eichler-Shimura isomorphism in the elliptic modular case, and which in turn are special cases of the isomorphisms for GL_2 over general number fields relating cusp forms to C^∞ harmonic differential forms (described in Section 3 of [7]). We denote these by

$$\delta_q : \mathcal{S}_{\boldsymbol{n}}(\Gamma_{\mathfrak{a}_i}, \chi_{\mathfrak{N}}^{-1}) \xrightarrow{\sim} \mathrm{H}^q_{cusp}(\Gamma_{\mathfrak{a}_i}\backslash\mathcal{H}, \widetilde{L(\boldsymbol{n},\mathbf{C})}),$$

with $q = 1,\,2$. There is an action of the Hecke algebra on both sides, and the δ_q are Hecke equivariant. In this paper we will only be concerned with the first isomorphism, that is we realize cusp forms over K as differential *1-forms.* For simplicity we will write δ for δ_1.

Now let f be as in Definition 1. Let $f_i \in \mathcal{S}_{\boldsymbol{n}}(\Gamma_{\mathfrak{a}_i}, \chi_{\mathfrak{N}}^{-1})$ (resp. F_i), for $i = 1, 2, \ldots, h$, be the cusp forms defined on $\mathcal{H}$ (resp. $GL_2(\mathbf{C})$) as in Section 2.3. Let us describe how to construct $\delta(f_i)$ explicitly. Let $F = F_i|_{SL_2(\mathbf{C})}$ (note we are dropping the subscript i from the notation). By the Clebsch-Gordan formula there is an $SU_2(\mathbf{C})$-injection

$$\Phi : W(n^*, \mathbf{C}) \longrightarrow L(\boldsymbol{n},\mathbf{C}) \otimes L(2,\mathbf{C}),$$

and $\delta(f_i)$ is given by

$$\delta(f_i)(g) = g \cdot (\Phi \circ F(g)). \tag{13}$$

Here the action of g on $L(\boldsymbol{n},\mathbf{C})$ is as in (12) above, whereas the action of g on $L(2,\mathbf{C})$ is given by (6) of Section 2.2. Here (see the discussion in Section 2.2) we have replaced $\Omega^1(\mathcal{H})$ with $L(2,\mathbf{C})$, the space of homogeneous polynomials of degree two in $\boldsymbol{a} = \binom{A}{B}$, and we have replaced the pull back action on $\Omega^1(\mathcal{H})$, by the induced action (6) on $L(2,\mathbf{C})$. Thus ultimately, we must replace (A^2, AB, B^2) by $(dx, -dy, -d\bar{x})$. Note that, since Φ is $SU_2(\mathbf{C})$-equivariant, we have

$$\begin{aligned}\delta(f_i)(gu) &= gu \cdot (\Phi \circ F(gu, \tbinom{S}{T}))\\ &= gu \cdot (\Phi \circ F(g, u\tbinom{S}{T})) = gu \cdot (\Phi \circ u^{-1} \cdot F(g, \tbinom{S}{T})) = \delta(f_i)(g)\end{aligned} \tag{14}$$

for all $u \in SU_2(\mathbf{C})$. Thus $\delta(f_i)$ can be thought of as a differential 1-form on $\mathcal{H}$. Moreover, a similar computation shows that for $\gamma \in \Gamma_{\mathfrak{a}_i}$

$$\gamma^* \, \delta(f_i) = \gamma \cdot \delta(f_i).$$

Thus $\delta(f_i)$ takes values in the sheaf $\widetilde{L(\boldsymbol{n}, \mathbf{C})}$.

It is possible to make $\Phi \circ F$ completely explicit. To this end introduce the auxiliary variables $\boldsymbol{u} = \binom{U}{V}$. Set

$$\boldsymbol{Q} \;=\; \left(\binom{n^*}{\alpha} (-1)^{n^*-\alpha} \; U^{\alpha} V^{n^*-\alpha} \right)_{\alpha=0,1,\dots,n^*}.$$

Let $\boldsymbol{\psi} \;=\; \boldsymbol{\psi}\,(\boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}) \;=\; (\psi_0, \psi_1, \dots, \psi_{n^*})^t$ where the $\psi_i = \psi_i(\boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a})$ are homogeneous polynomial of degree n in $\boldsymbol{x} = \binom{X}{Y}$, degree n in $\overline{\boldsymbol{x}} = \binom{\overline{X}}{\overline{Y}}$ and degree 2 in $\boldsymbol{a} = \binom{A}{B}$ defined by

$$(XV - YU)^{\,n}\, (\overline{X}U + \overline{Y}V)^{\,n}\, (AV - BU)^{\,2} = \; \boldsymbol{Q} \cdot \; \boldsymbol{\psi}\,. \tag{15}$$

These polynomials have the special property (cf. [10], (2.8b)) :

$$\boldsymbol{\psi}\,(u\,\boldsymbol{x}, \overline{u}\,\overline{\boldsymbol{x}}, u\,\boldsymbol{a}) = \rho_{n^*}(u) \cdot \; \boldsymbol{\psi}\,(\boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}) \tag{16}$$

for all $u \in SU_2(\mathbf{C})$.

Recall F takes values in $W(n^*, \mathbf{C})$, and so we let F^{α} be the 'components' of F, namely $F(g,\, \boldsymbol{s}) = \sum_{\alpha=0}^{n^*} F^{\alpha}(g)\, S^{n^*-\alpha} T^{\alpha}$. Since $W(n^*, \mathbf{C})$ is a model for the dual of ρ_{n^*} (= the symmetric $n^{*\text{th}}$ power representation of $GL_2(\mathbf{C})$ on $\mathbf{C}^2$) we get

$$F(gu,\, \boldsymbol{s}) = F(g, u\, \boldsymbol{s}) = F(g,\, \boldsymbol{s}) \cdot \rho_{n^*}(u) \tag{17}$$

for all u in $SU_2(\mathbf{C})$.

Now define $\Phi \circ F = F' : SL_2(\mathbf{C}) \to L(\boldsymbol{n}, \mathbf{C}) \otimes L(2, \mathbf{C})$ by

$$F'(g,\, \boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}) = (F^0(g), F^1(g), \dots, F^{n^*}(g)) \cdot \; \boldsymbol{\psi}\,(\boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}).$$

Then, as predicted by the $SU_2(\mathbf{C})$-equivariance of Φ,

$$\begin{aligned} F'(gu,\, \boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}) \quad &= \quad (F^0(gu), F^1(gu), \dots, F^{n^*}(gu)) \cdot \; \boldsymbol{\psi}\,(\boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}) \\ &\overset{(17)}{=} \quad (F^0(g), F^1(g), \dots, F^{n^*}(g)) \cdot \rho_{n^*}(u) \cdot \; \boldsymbol{\psi}\,(\boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}) \\ &\overset{(16)}{=} \quad (F^0(g), F^1(g), \dots, F^{n^*}(g)) \cdot \; \boldsymbol{\psi}\,(u\,\boldsymbol{x},\, \overline{u}\,\overline{\boldsymbol{x}},\, u\,\boldsymbol{a}) \\ &= \quad F'(g,\, u\,\boldsymbol{x},\, \overline{u}\,\overline{\boldsymbol{x}},\, u\,\boldsymbol{a}). \end{aligned} \tag{18}$$

That is, for $u \in SU_2(\mathbf{C})$

$$(\Phi \circ F)(gu) = u^{-1} \cdot (\Phi \circ F)(g). \tag{19}$$

Finally define $F'' : SL_2(\mathbf{C}) \to L(\boldsymbol{n}, \mathbf{C}) \otimes L(2, \mathbf{C})$ by

$$F''(g,\, \boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}) = F'(g,\, g^{\iota}\, \boldsymbol{x},\, \overline{g}^{\iota}\, \overline{\boldsymbol{x}},\, {}^t j(g^{-1}, \epsilon)^{-1}\, \boldsymbol{a}).$$

Then, noting $j(u, \epsilon)^t = u^{-1}$ if $u \in SU_2(\mathbf{C})$, we have

$$\begin{aligned} F''(gu,\, \boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}) &= F'(gu,\, u^{-1} g^{\iota}\, \boldsymbol{x},\, \overline{u}^{-1} \overline{g}^{\iota}\, \overline{\boldsymbol{x}},\, u^{-1}\, {}^t j(g^{-1}, \epsilon)^{-1}\, \boldsymbol{a}) \\ &\overset{(18)}{=} F'(g,\, g^{\iota}\, \boldsymbol{x},\, \overline{g}^{\iota}\, \overline{\boldsymbol{x}},\, {}^t j(g^{-1}, \epsilon)^{-1}\, \boldsymbol{a}) \\ &= F''(g,\, \boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}). \end{aligned}$$

So, in confirmation of (14), F'' is invariant on the right by $SU_2(\mathbf{C})$, and so is really defined on $\mathcal{H}$. Thus, in summary we have

Definition 6 *$\delta(f_i)$ is the $L(\boldsymbol{n}, \mathbf{C})$-valued differential form on $\mathcal{H}$ obtained from F'' by replacing (A^2, AB, B^2) by $(dx, -dy, -d\bar{x})$. More specifically, if $g\epsilon = z$, then*

$$\delta(f_i)(z) = (F^0(g), F^1(g), \ldots, F^{n^*}(g)) \cdot\ \boldsymbol{\psi}\, (g^{\iota}\, \boldsymbol{x},\, \overline{g}^{\iota}\, \overline{\boldsymbol{x}},\, {}^t j(g^{-1}, \epsilon)^{-1}\, \boldsymbol{a})$$

where (A^2, AB, B^2) is replaced by $(dx, -dy, -d\bar{x})$.

5.2 Computing $\widetilde{\delta(f)}\Big|_H$

As usual fix f be as in Definition 1, and let f_1 and F_1 be the first components of f as in Section 2.3. To simplify notation, we call these f and F respectively, but we emphasize these are now functions on $\mathcal{H}$, respectively $GL_2(\mathbf{C})$. In this section we compute $\delta(f)(= \delta(f_1))$ explicitly using the method sketched in the previous section. Note that since we are assuming $i = 1$ and $a_1 = 1$, we have $\mathfrak{a}_1 = \mathcal{O}$ and $\Gamma_{\mathfrak{a}_1} = \Gamma_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \,\middle|\, a, b, d \in \mathcal{O},\ c \in \mathfrak{N},\ ad - bc = 1 \right\}$. Hence $\Gamma_{\mathfrak{a}_1} \cap GL_2^+(\mathbf{Q}) = \Gamma_0(N)$.

We will simplify our computation of $\delta(f)$ in two ways. Note that the Poincaré upper half plane H embeds in a natural way into $\mathcal{H}$ via $z = x + iy \hookrightarrow \left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix} \right)$. Since we are really interested in computing $\delta(f)\big|_H$, we will assume from the outset that $dx = d\bar{x}$.

Secondly, we really want to compute $\left(\begin{smallmatrix} 1 & -x \\ 0 & 1 \end{smallmatrix} \right) \cdot \delta(f)\big|_H$. Since this amounts to setting $x = 0$ in $\delta(f)\big|_H$, we assume this is true from the beginning, and actually compute this 'modified' differential form, which we call $\widetilde{\delta(f)}\big|_H$.

Let us begin the computation. Using the definition of $\boldsymbol{\psi}$ in (15), we see that for $\alpha = 0, 1, \ldots, n^* = 2n + 2$,

$$\psi_\alpha(\boldsymbol{x},\, \overline{\boldsymbol{x}},\, \boldsymbol{a}) = (-1)^\alpha \frac{A^2 c_\alpha - 2AB c_{\alpha-1} + B^2 c_{\alpha-2}}{\binom{2n+2}{\alpha}}$$

where

$$c_\alpha(X,Y,\overline{X},\overline{Y}) = \sum_{\substack{j,k=0\\ n-(j-k)=\alpha}}^{n} (-1)^k \binom{n}{k}\binom{n}{j} X^{n-k}\,\overline{X}^{n-j}\,Y^k\,\overline{Y}^j \tag{20}$$

Now let $g = \frac{1}{\sqrt{y}}\left(\begin{smallmatrix} y & x\\ 0 & 1\end{smallmatrix}\right) \in SL_2(\mathbf{C})$, with $x, y \in \mathbf{R}$. Then $g\epsilon = z$, $g^\iota = \overline{g}^\iota = \frac{1}{\sqrt{y}}\left(\begin{smallmatrix} 1 & -x\\ 0 & y\end{smallmatrix}\right)$, and $j(g^{-1},\epsilon)^{-1} = j(g,\epsilon) = \frac{1}{\sqrt{y}}\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Let f^α (resp. F^α) be the components of f (resp. F) as a $W(n^*,\mathbf{C})$-valued function on $\mathcal{H}$ (resp. $GL_2(\mathbf{C})$). Since $f(z,\left(\begin{smallmatrix} S\\ T\end{smallmatrix}\right)) = F(g, j(g,\epsilon)^t\left(\begin{smallmatrix} S\\ T\end{smallmatrix}\right))$ we get $f^\alpha(z) = (\frac{1}{\sqrt{y}})^{2n+2}F^\alpha(g)$. By Definition 6, we have

$$\begin{aligned} \delta(f)\big|_H &= \sum_{\alpha=0}^{2n+2} F^\alpha(g)\,\psi_\alpha\Big(\frac{1}{\sqrt{y}}\left(\begin{smallmatrix} 1 & -x\\ 0 & y\end{smallmatrix}\right)\left(\begin{smallmatrix} X\\ Y\end{smallmatrix}\right), \frac{1}{\sqrt{y}}\left(\begin{smallmatrix} 1 & -x\\ 0 & y\end{smallmatrix}\right)\left(\begin{smallmatrix} \overline{X}\\ \overline{Y}\end{smallmatrix}\right), \frac{1}{\sqrt{y}}\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} A\\ B\end{smallmatrix}\right)\Big) \\ &= \sum_{\alpha=0}^{2n+2} \sqrt{y}^{2n+2} f^\alpha(z)\,\psi_\alpha\Big(\frac{1}{\sqrt{y}}X, \sqrt{y}\,Y, \frac{1}{\sqrt{y}}\overline{X}, \sqrt{y}\,\overline{Y}, \frac{1}{\sqrt{y}}A, \frac{1}{\sqrt{y}}B\Big) \end{aligned}$$

where we have already set $x = 0$. Simplifying this we get

$$\widetilde{\delta(f)}\big|_H = \sum_{\alpha=0}^{2n+2} f^\alpha(z)\,\psi_\alpha(X, yY, \overline{X}, y\overline{Y}, A, B)$$

with (A^2, AB, B^2) replaced by $(dx, -dy, -dx)$. Thus,

$$\begin{aligned} \widetilde{\delta(f)}\big|_H = \Bigg[\sum_{\alpha=0}^{2n+2} \frac{(-1)^\alpha f^\alpha(z)}{\binom{2n+2}{\alpha}}\Big(c_\alpha(X,yY,\overline{X},y\overline{Y}) - c_{\alpha-2}(X,yY,\overline{X},y\overline{Y})\Big)\Bigg]dx \\ +2\Bigg[\sum_{\alpha=0}^{2n+2} \frac{(-1)^\alpha f^\alpha(z)}{\binom{2n+2}{\alpha}} c_{\alpha-1}(X,yY,\overline{X},y\overline{Y})\Bigg]dy. \end{aligned}$$

It follows from (20) that

$$c_{2n-\alpha}(X,Y,\overline{X},\overline{Y}) = (-1)^{n-\alpha}c_\alpha(\overline{X},\overline{Y},X,Y). \tag{21}$$

Using (21) we may rewrite

$$\begin{aligned} &\widetilde{\delta(f)}\big|_H = \\ &\Bigg[\Bigg(\sum_{\alpha=0}^{n} \frac{(-1)^\alpha f^\alpha(z)}{\binom{2n+2}{\alpha}}\big(c_\alpha(X,yY,\overline{X},y\overline{Y}) - c_{\alpha-2}(X,yY,\overline{X},y\overline{Y})\big) \\ &\quad + \frac{(-1)^{n+1} f^{2n+2-\alpha}(z)}{\binom{2n+2}{\alpha}}\big(c_\alpha(\overline{X},y\overline{Y},X,yY) - c_{\alpha-2}(\overline{X},y\overline{Y},X,yY)\big)\Bigg) \\ &\quad + \frac{(-1)^{n+1} f^{n+1}(z)}{\binom{2n+2}{n+1}}\big(c_{n+1}(X,yY,\overline{X},y\overline{Y}) - c_{n-1}(X,yY,\overline{X},y\overline{Y})\big)\Bigg]dx \end{aligned}$$

$$+2\left[\sum_{\alpha=0}^{n}\left(\frac{(-1)^{\alpha}f^{\alpha}(z)}{\binom{2n+2}{\alpha}}\,c_{\alpha-1}(X,yY,\overline{X},y\overline{Y})\right.\right.$$
$$+\;\frac{(-1)^{n+1}f^{2n+2-\alpha}(z)}{\binom{2n+2}{\alpha}}\,c_{\alpha-1}(\overline{X},y\overline{Y},X,yY)\Bigg)$$
$$\left.+\;\frac{(-1)^{n+1}f^{n+1}(z)}{\binom{2n+2}{n+1}}\,c_{n}(X,yY,\overline{X},y\overline{Y})\right]dy.$$

This gives us an explicit expression for $\widetilde{\delta(f)}\big|_H \in \mathrm{H}^1_{cusp}(\Gamma_0(N)\backslash H, \widetilde{L(\boldsymbol{n},\mathbf{C})})$.

Since we have restricted to H, $L(\boldsymbol{n},\mathbf{C})$ is no longer an irreducible $\Gamma_0(N)$-module and in general we have

Lemma 2 *As $SL_2(\mathbf{Z})$-modules we have the decomposition*

$$\begin{aligned} L(\boldsymbol{n},A) &\cong \bigoplus_{m=0}^{n} L(2n-2m,A) \\ P(\boldsymbol{x},\,\overline{\boldsymbol{x}}) &\mapsto \bigoplus_{m=0}^{n}\frac{1}{m!^2}\left(\frac{\partial^2}{\partial X\partial\overline{Y}}-\frac{\partial^2}{\partial\overline{X}\partial Y}\right)^m P(\boldsymbol{x},\,\overline{\boldsymbol{x}})\,\Bigg|_{\substack{\overline{X}=X\\ \overline{Y}=Y}}. \end{aligned}$$

Proof See [10], Section 11. $\square$

Lemma 2 induces the decomposition

$$\begin{aligned} \mathrm{H}^1_{cusp}(\Gamma_0(N)\backslash H, \widetilde{L(\boldsymbol{n},\mathbf{C})}) &\cong \bigoplus_{m=0}^{n}\mathrm{H}^1_{cusp}(\Gamma_0(N)\backslash H, \widetilde{L(2n-2m,\mathbf{C})}) \\ \widetilde{\delta(f)}\big|_H &= \bigoplus_{m=0}^{n}\widetilde{\delta_{2n-2m}}(f). \end{aligned}$$

Let us now compute $\widetilde{\delta_{2n-2m}}(f)$. Set

$$\nabla=\left(\frac{\partial^2}{\partial X\partial\overline{Y}}-\frac{\partial^2}{\partial\overline{X}\partial Y}\right).$$

Then using the relation

$$\nabla^m c_\alpha(X,Y,\overline{X},\overline{Y})\,\Bigg|_{\substack{\overline{X}=X\\ \overline{Y}=Y}} = (-1)^m\nabla^m c_\alpha(\overline{X},\overline{Y},X,Y)\,\Bigg|_{\substack{\overline{X}=X\\ \overline{Y}=Y}} \tag{22}$$

we see that

$$\widetilde{\delta_{2n-2m}}(f)(x+iy)=$$
$$\sum_{\alpha=0}^{n+1}(-1)^\alpha\,g^\alpha(z)\,\frac{1}{m!^2}\,\nabla^m\Big(c_\alpha(X,yY,\overline{X},y\overline{Y})-c_{\alpha-2}(X,yY,\overline{X},y\overline{Y})\Big)_{\substack{\overline{X}=X\\ \overline{Y}=Y}}dx$$
$$+2\sum_{\alpha=0}^{n+1}(-1)^\alpha\,g^\alpha(z)\,\frac{1}{m!^2}\,\nabla^m\Big(c_{\alpha-1}(X,yY,\overline{X},y\overline{Y})\Big)_{\substack{\overline{X}=X\\ \overline{Y}=Y}}dy,$$

where

$$g^{\alpha}(z) = \begin{cases} \frac{f^{\alpha}(z) + (-1)^{n+1-\alpha+m}\, f^{2n+2-\alpha}(z)}{\binom{2n+2}{\alpha}} & \text{if } \alpha = 0, 1, \ldots, n; \\ \frac{f^{n+1}(z)}{\binom{2n+2}{n+1}} & \text{if } \alpha = n+1. \end{cases} \tag{23}$$

The following Lemma is useful:

Lemma 3

$$\frac{1}{m!^2} \nabla^m c_\alpha(X, Y, \overline{X}, \overline{Y}) \,\Big|_{\substack{\overline{X}=X \\ \overline{Y}=Y}} = \binom{n}{m}^2 \sum_{p=0}^{m} (-1)^p \binom{m}{p} \sum_{\substack{j,k=0 \\ n-(j-k)=\alpha}}^{n} (-1)^k \binom{n-m}{k-p} \binom{n-m}{j-(m-p)} X^{2n-(j+k)-m} Y^{j+k-m}.$$

Proof

$$\begin{aligned}
\frac{1}{m!^2} \nabla^m c_\alpha(X, Y, \overline{X}, \overline{Y}) \,\Big|_{\substack{\overline{X}=X \\ \overline{Y}=Y}}
&= \frac{1}{m!^2} \sum_{p=0}^{m} (-1)^p \binom{m}{p} \left(\frac{\partial^{2m}}{\partial X^{m-p} \partial \overline{X}^p \partial Y^p \partial \overline{Y}^{m-p}} \right) \cdot c_\alpha(X, Y, \overline{X}, \overline{Y}) \,\Big|_{\substack{\overline{X}=X \\ \overline{Y}=Y}} \\
&= \frac{1}{m!^2} \sum_{p=0}^{m} (-1)^p \binom{m}{p} \sum_{\substack{j,k=0 \\ n-(j-k)=\alpha}}^{n} (-1)^k \binom{n}{k} \binom{n}{j} \\
&\qquad \frac{(n-k)!}{(n-k-(m-p))!} \frac{(n-j)!}{(n-j-p)!} \frac{k!}{(k-p)!} \frac{j!}{(j-(m-p))!} \\
&\qquad X^{n-k-(m-p)} \overline{X}^{n-j-p} Y^{k-p} \overline{Y}^{j-(m-p)} \,\Big|_{\substack{\overline{X}=X \\ \overline{Y}=Y}} \\
&= \binom{n}{m}^2 \sum_{p=0}^{m} (-1)^p \binom{m}{p} \sum_{\substack{j,k=0 \\ n-(j-k)=\alpha}}^{n} (-1)^k \binom{n-m}{k-p} \binom{n-m}{j-(m-p)} \\
&\qquad X^{2n-(j+k)-m} Y^{j+k-m}. \qquad \square
\end{aligned}$$

By Lemma 3 we may finally write

$$\widetilde{\delta_{2n-2m}}(f)\,(x+iy) = \sum_{l=0}^{2n-2m} (A_l\, dx + 2B_l\, dy)\, y^{2n-m-l} X^l Y^{2n-2m-l} \tag{24}$$

where

$$A_l = \sum_{\alpha=0}^{n+1} (-1)^\alpha\, g^\alpha(z)\, a\,(m, l, \alpha) \tag{25}$$

$$B_l = \sum_{\alpha=0}^{n+1} (-1)^\alpha\, g^\alpha(z)\, b\,(m, l, \alpha), \tag{26}$$

and where $a\,(m,l,\alpha)$ and $b\,(m,l,\alpha)$ are constants given by

$$a\,(m,l,\alpha)=\binom{n}{m}^2(-1)^{\frac{n+\alpha-l-m}{2}}\sum_{p=0}^{m}(-1)^p\binom{m}{p}$$
$$\left[\binom{n-m}{\frac{n-m-l+\alpha}{2}-p}\binom{n-m}{\frac{3n-3m-l-\alpha}{2}+p}+\right.$$
$$\left.\binom{n-m}{\frac{n-m-l+\alpha-2}{2}-p}\binom{n-m}{\frac{3n-3m-l-\alpha+2}{2}+p}\right],$$
$$b\,(m,l,\alpha)=\binom{n}{m}^2(-1)^{\frac{n+\alpha-1-l-m}{2}}\sum_{p=0}^{m}(-1)^p\binom{m}{p}$$
$$\left[\binom{n-m}{\frac{n-m-l+\alpha-1}{2}-p}\binom{n-m}{\frac{3n-3m-l-\alpha+1}{2}+p}\right].$$

6 An Integral Expression for $G(s,f)$

In this section we will derive an integral expression for $G(s,f)$. The essential idea will be to integrate $\delta_{2n-2m}(f)\wedge E_{2n-2m+2}$, for a suitable elliptic modular Eisenstein series $E_{2n-2m+2}$, over $D=\Gamma_0(N)\backslash H$. This integral expression will be used in the next section to establish the rationality result. Though we do not touch upon this here, we may also use the integral expression to establish the meromorphic continuation and functional equation of $G(s,f)$ (cf. [6], Section 7.2).

6.1 Poincaré duality

Since the 1-forms $\delta_{2n-2m}(f)$ and $E_{2n-2m+2}$ are both $L(2n-2m,\mathbf{C})$-valued, to evaluate the integral we introduce a pairing by means of which we may regard their wedge product as a scalar-valued 2-form on $\Gamma_0(N)\backslash H$.

Lemma 4 *Let A be a $\mathbf{Q}$-algebra, and let $L(n,A)$ denote the space of homogeneous polynomials of degree n in (X,Y) with coefficients in A. Then*

$$\begin{array}{rcl} L(n,A)\otimes L(n,A) & \stackrel{<,>}{\to} & A \\ <\sum_{l=0}^{n}a_lX^{n-l}Y^l\,,\sum_{l=0}^{n}b_lX^{n-l}Y^l> & = & \sum_{l=0}^{n}(-1)^l\,a_l\,b_{n-l}\binom{n}{l}^{-1} \end{array}$$

is an $SL_2(\mathbf{Z})$-invariant pairing, i.e. $<\gamma\cdot P,\gamma\cdot Q>=<P,Q>$ for all $P,Q\in L(n,A)$ and all $\gamma\in SL_2(\mathbf{Z})$. If $A=\mathbf{C}$, then we may replace $SL_2(\mathbf{Z})$ by $SL_2(\mathbf{C})$ in the statement of the Lemma.

Proof See [10], Equation 3.1b. □

The above pairing induces a pairing (Poincaré duality) on cohomology. We have

Lemma 5

$$H^1_c(\Gamma_0(N)\backslash H, \widetilde{L(n,A)}) \otimes H^1(\Gamma_0(N)\backslash H, \widetilde{L(n,A)}) \xrightarrow{\cup}$$
$$H^2_c(\Gamma_0(N)\backslash H, \widetilde{L(n,A)} \otimes \widetilde{L(n,A)}) \xrightarrow{<\,,\,>} H^2_c(\Gamma_0(N)\backslash H, \tilde{A})$$

is a perfect duality, where the first map is cup product (wedge product).

Proof See [10], Equation 5.3. $\square$

We continue to denote this pairing by $<\,,\,>$.

6.2 Eisenstein series

Let Γ_∞ denote the stabilizer of ∞ in $\Gamma_0(N)$. Set $\omega = (X - zY)^{2n-2m} dz \in \mathrm{H}^1(\Gamma_\infty\backslash H, \widetilde{L(2n-2m, \mathbf{Z})})$. We define the Eisenstein differential form $E_{2n-2m+2}$ by

$$E_{2n-2m+2}(s,z) = \sum_{\Gamma_\infty\backslash\Gamma_0(N)} \psi_N^{-1}(\gamma)\,\gamma^{-1}\cdot\gamma^*(\omega\, y^s).$$

One may check that

$$E_{2n-2m+2}(s,z) = \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_\infty\backslash\Gamma_0(N)} \frac{\psi_N^{-1}(d)}{(cz+d)^{2n-2m+2}|cz+d|^{2s}} \cdot y^s\,\omega.$$

The following proposition establishes the rationality of $E_{2n-2m+2}(s,z)$ at $s = 0$.

Proposition 4 *Say $m \neq n$ or ψ_N^{-1} is not trivial. Then*

$$E_{2n-2m+2}(0,z) \in H^1(\Gamma_0(N)\backslash H, \widetilde{L(2n-2m, \mathbf{Q}(\psi_N)})).$$

If $m = n$ and ψ_N^{-1} is trivial, then we still have

$$E_2(0,z) - pE_2(0,pz) \in H^1(\Gamma_0(N)\backslash H, \widetilde{L(2n-2m, \mathbf{Q}(\psi_N)})).$$

Proof See [10], Section 10. $\square$

It is also convenient to introduce the 'completed' Eisenstein differential form $E^*_{2n-2m+2}(s,z)$, especially for the purposes of establishing the functional equation of $G(s,f)$ (for which see [6], Section 7.2)

$$E^*_{2n-2m+2}(s,z) = \pi^{-(s+2n-2m+2)}\,\Gamma(s+2n-2m+2)\times$$
$$2\,L_N(2s+2n-2m+2,\,\psi_N^{-1})\,E_{2n-2m+2}(s,z) \qquad (27)$$

A simple computation allows us to rewrite (cf. [13], Equation 7.2.62)

$$E^*_{2n-2m+2}(s,z) = \pi^{-(s+2n-2m+2)}\,\Gamma(s+2n-2m+2)\times \sum_{(0,0)\neq(c,d)\,\in\,\mathbf{Z}^2} \frac{\psi_N^{-1}(d)}{(cNz+d)^{2n-2m+2}|cNz+d|^{2s}}\cdot y^s\,\omega.$$

6.3 Integral expression

We are now ready to begin integrating. First note

$$\begin{aligned}
&\int\!\!\int_D < \delta_{2n-2m}(f), E_{2n-2m+2} > \\
&\quad = \int\!\!\int_D < \left(\begin{smallmatrix}1 & -x\\ 0 & 1\end{smallmatrix}\right)\cdot\delta_{2n-2m}(f),\ \left(\begin{smallmatrix}1 & -x\\ 0 & 1\end{smallmatrix}\right)\cdot E_{2n-2m+2} > \\
&\quad = \int\!\!\int_D < \widetilde{\delta_{2n-2m}}(f),\ \widetilde{E_{2n-2m+2}} >
\end{aligned}$$

and a standard unwinding argument shows this last integral

$$= \int_0^\infty\!\!\int_0^1 < \widetilde{\delta_{2n-2m}}(f),\ \tilde{\omega}\,y^s > .$$

Here $\tilde{\omega} = (X-iyY)^{2n-2m}\,dz$ since the tilde amounts to setting $x=0$. Using the expression (24) for $\widetilde{\delta_{2n-2m}}(f)$ and the pairing of Lemma 5 we get

$$\begin{aligned}
&\int_0^\infty\!\!\int_0^1 < \widetilde{\delta_{2n-2m}}(f),\ \tilde{\omega}\,y^s > \\
&\quad = \int_0^\infty\!\!\int_0^1 \sum_{l=0}^{2n-2m} (A_l dx + 2B_l dy)\,y^{2n-m-l}\wedge(iy)^l\,y^s\,(dx+idy) \\
&\quad = \int_0^\infty\!\!\int_0^1 \sum_{l=0}^{2n-2m} i^{l+1}A_l\,y^{2n-m+s}\,dxdy \\
&\qquad\quad -2\int_0^\infty\!\!\int_0^1 \sum_{l=0}^{2n-2m} i^l B_l\,y^{2n-m+s}\,dxdy
\end{aligned}$$

Let us call the first integral in the right hand side of the last equation above $\boldsymbol{I_1}$ and the second $\boldsymbol{I_2}$. We evaluate $\boldsymbol{I_1}$ now. By (25) we have

$$\begin{aligned}
\boldsymbol{I_1} &= \int_0^\infty\!\!\int_0^1 \sum_{l=0}^{2n-2m} i^{l+1}A_l\,y^{2n-m+s}\,dxdy \\
&= \sum_{l=0}^{2n-2m} i^{l+1}\sum_{\alpha=0}^{n+1}(-1)^\alpha\,a\,(m,l,\alpha)\int_0^\infty\!\!\int_0^1 g^\alpha(z)\,y^{2n-m+s}\,dxdy.
\end{aligned}$$

Using (7) with $i = 1$ and $a_1 = 1$, the Fourier expansion of the α^{th} component of $f = f_1$ is given by $f^\alpha(x+iy) =$

$$y \binom{2n+2}{\alpha} \left[\sum_{\xi \in K^\times} c\,(\xi d) \left(\frac{\xi}{i|\xi|} \right)^{n+1-\alpha} \frac{1}{\xi^{v_i} \bar{\xi}^{v_c}} \, K_{\alpha-n-1}(4\pi y|\xi|)\, \mathbf{e}_K(\xi x) \right]. \quad (28)$$

Then, using (23) and (28) above we get

$$\begin{aligned} \boldsymbol{I_1} = & \sum_{l=0}^{2n-2m} i^{l+1} \sum_{\alpha=0}^{n} (-1)^\alpha \, a\,(m,l,\alpha) \int_0^\infty \sum_{\xi \in K^\times} c\,(\xi d)\, \frac{1}{\xi^{v_i} \bar{\xi}^{v_c}}\, y^{2n+1-m+s} \\ & \quad \left(\left(\frac{\xi}{i|\xi|} \right)^{n+1-\alpha} K_{\alpha-n-1}(4\pi y|\xi|) \; + \right. \\ & \quad \left. (-1)^{n+m+1-\alpha} \left(\frac{i|\xi|}{\xi} \right)^{n+1-\alpha} K_{n+1-\alpha}(4\pi y|\xi|) \right) dy \int_0^1 \mathbf{e}_K(\xi x)\, dx \\ & + \sum_{l=0}^{2n-2m} i^{l+1}\, (-1)^{n+1}\, a\,(m,l,n+1) \\ & \quad \int_0^\infty \sum_{\xi \in K^\times} c\,(\xi d)\, \frac{1}{\xi^{v_i} \bar{\xi}^{v_c}}\, y^{2n+1-m+s}\, K_0(4\pi y|\xi|)\, dy \int_0^1 \mathbf{e}_K(\xi x)\, dx. \end{aligned}$$

Lemma 6 *Let ξ be such that $c\,(\xi d) \neq 0$. Then*

$$\int_0^1 \mathbf{e}_K(\xi x)\, dx = \begin{cases} 1, & \textit{if } \xi = \frac{k}{\sqrt{-D}} \textit{ for some } k \neq 0 \in \mathbf{Z}; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof This is easily checked noting that $c\,(\ )$ vanishes outside integral ideals and that the different of K is $\vartheta = d\mathcal{O} = (\sqrt{-D})$ (or see [6], Lemma 9). $\square$

By Lemma 6 we see that as regards the integral in x, we may restrict our summation to $\xi = \frac{k}{\sqrt{-D}}$ as k varies through all non-negative integers. Hence

$$\begin{aligned} \boldsymbol{I_1} = & \sum_{l=0}^{2n-2m} i^{l+1} \sum_{\alpha=0}^{n} (-1)^\alpha a\,(m,l,\alpha) i^{v_i - v_c} \sqrt{D}^{v_i+v_c} \sum_{k \neq 0} c\,(k) \frac{1}{k^{v_i+v_c}} \left(\frac{-k}{|k|} \right)^{n+1-\alpha} \\ & \int_0^\infty y^{2n+1-m+s} \left[K_{\alpha-n-1}\left(\frac{4\pi y|k|}{\sqrt{D}} \right) + (-1)^{n+m+1-\alpha} K_{n+1-\alpha}\left(\frac{4\pi y|k|}{\sqrt{D}} \right) \right] dy \\ & + \sum_{l=0}^{2n-2m} i^{l+1}\, (-1)^{n+1}\, a\,(m,l,n+1)\, i^{v_i-v_c}\, \sqrt{D}^{v_i+v_c} \sum_{k \neq 0} c\,(k)\, \frac{1}{k^{v_i+v_c}} \\ & \int_0^\infty y^{2n+1-m+s}\, K_0\left(\frac{4\pi y|k|}{\sqrt{D}} \right) dy. \end{aligned}$$

Lemma 7

$$\int_0^\infty K_\nu(ay)\, y^{\mu-1}\, dy = 2^{\mu-2}\, a^{-\mu}\, \Gamma\big(\frac{\mu+\nu}{2}\big)\, \Gamma\big(\frac{\mu-\nu}{2}\big)$$

Proof See, for instance, [16], Chapter 13.21, Equation 8. $\square$

Because of the $\pm$ sign in the first y integral, Lemma 7 shows that the integrals of the two Bessel functions cancel each other unless $\alpha \equiv n+m+1$. Hence setting $s' = 2n+2-m+v_i+v_c+s$, we have

$$\begin{aligned} \boldsymbol{I_1} = {} & \frac{(-1)^{n+1}\, i^{v_i-v_c}\, \sqrt{D}^{s'}}{2\,(2\pi)^{2n+2-m+s}} \sum_{l=0}^{2n-2m} i^{l+1} \sum_{\substack{\alpha=0 \\ \alpha\equiv n+1+m\ (2)}}^{n} (-1)^m\, a\,(m,l,\alpha) \\ & \sum_{0\neq k\,\in\,\mathbf{Z}} c\,(k) \frac{1}{k^{v_i+v_c}} \left(\frac{-k}{|k|}\right)^{n+1-\alpha} \frac{1}{|k|^{2n+2-m+s}} \\ & \Gamma\left(\frac{n+1-m+\alpha+s}{2}\right) \Gamma\left(\frac{3n+3-m-\alpha+s}{2}\right) \\ & + \frac{(-1)^{n+1}\, i^{v_i-v_c}\, \sqrt{D}^{s'}}{4\,(2\pi)^{2n+2-m+s}} \sum_{l=0}^{2n-2m} i^{l+1}\, a\,(m,l,n+1) \\ & \sum_{0\neq k\,\in\,\mathbf{Z}} c\,(k)\, \frac{1}{k^{v_i+v_c}} \frac{1}{|k|^{2n+2-m+s}}\, \Gamma\left(\frac{2n+2-m+s}{2}\right)^2 \end{aligned}$$

Note that the first sum in k vanishes unless $m \equiv v_i + v_c \pmod 2$, whereas the second vanishes unless $v_i + v_c \equiv 0 \pmod 2$. Thus, incorporating the $\alpha = n+1$ summand into the sum on α, we get

$$\begin{aligned} \boldsymbol{I_1} = {} & \frac{(-1)^{n+1}\, i^{v_i-v_c}\, \sqrt{D}^{s'}}{(2\pi)^{2n+2-m+s}} \sum_{k=1}^{\infty} \frac{c\,(k)}{k^{s'}} \sum_{l=0}^{2n-2m} i^{l+1} \sum_{\substack{\alpha=0 \\ \alpha\equiv n+1+m\ (2)}}^{n+1} a\,(m,l,\alpha) \\ & \Gamma\left(\frac{n+1-m+\alpha+s}{2}\right) \Gamma\left(\frac{3n+3-m-\alpha+s}{2}\right). \end{aligned}$$

There is really an extra factor of $1/2$ in the $\alpha = n+1$ term which we will adjust for in due course.

Lastly, note that since the definition of $a\,(m,l,\alpha)$ involved a $(-1)^{\frac{n+\alpha-m-l}{2}}$ term we get another parity condition, namely $l \equiv n+\alpha-m \pmod 2$. Combining this with the first parity condition on α yields $l \equiv 1 \pmod 2$. Thus we finally get

$$\boldsymbol{I_1} = \frac{(-1)^{n+1}\sqrt{D}^{s'}\, i^{v_i - v_c}}{(2\pi)^{2n+2-m+s}} \sum_{k=1}^{\infty} \frac{c\,(k)}{k^{s'}} \sum_{\substack{l=0\\ odd}}^{2n-2m} i^{l+1} \sum_{\substack{\alpha=0\\ \alpha\equiv n+1+m\,(2)}}^{n+1} a\,(m,l,\alpha)$$
$$\Gamma\left(\frac{n+1-m+\alpha+s}{2}\right)\Gamma\left(\frac{3n+3-m-\alpha+s}{2}\right)$$

A similar analysis for $\boldsymbol{I_2}$ (which we do not write down!) yields

$$\begin{aligned}\boldsymbol{I_2} &= -2\int_0^{\infty}\int_0^1 \sum_{l=0}^{2n-2m} i^l B_l\, y^{2n-m+s}\, dxdy\\ &= \frac{-2\,(-1)^{n+1}\sqrt{D}^{s'}\, i^{v_i - v_c}}{(2\pi)^{2n+2-m+s}} \sum_{k=1}^{\infty} \frac{c\,(k)}{k^{s'}} \sum_{\substack{l=0\\ even}}^{2n-2m} i^{l} \sum_{\substack{\alpha=0\\ \alpha\equiv n+1+m\,(2)}}^{n+1} b\,(m,l,\alpha)\\ &\qquad \Gamma\left(\frac{n+1-m+\alpha+s}{2}\right)\Gamma\left(\frac{3n+3-m-\alpha+s}{2}\right)\end{aligned}$$

where again we assume that $m \equiv v_i + v_c \pmod 2$, and there is an extra factor of $1/2$ in the $\alpha = n+1$ summand.

Combining these two expressions yields

$$\int_0^{\infty}\int_0^1 <\widetilde{\delta_{2n-2m}}(f),\, \tilde{\omega}\, y^s> = \frac{\sqrt{D}^{s'}\, i^{v_i - v_c}}{(2\pi)^{2n+2-m+s}} \sum_{k=1}^{\infty} \frac{c\,(k)}{k^{s'}}\, G'_{\infty}(s,f)$$

where $G'_{\infty}(s,f)$ is a sum of Gamma factors given by $G'_{\infty}(s,f) =$

$$\sum_{\substack{\alpha=0\\ \alpha\equiv n+1+m\,(2)}}^{2n+2} c\,(m,\alpha)\,\Gamma\left(\frac{n+1-m+\alpha+s}{2}\right)\Gamma\left(\frac{3n+3-m-\alpha+s}{2}\right).$$

We have changed the upper index of sum on α from $n+1$ to $2n+2$. This seems more natural since it takes into account the missing factor of $1/2$ in the $\alpha = n+1$ summand. Thus $c\,(m,\alpha)$ is the constant

$$\begin{aligned} c\,(m,\alpha) &= \frac{(-1)^{n+1}}{2} \sum_{\substack{l=0\\ odd}}^{2n-2m} i^{l+1}\, a\,(m,l,\alpha) - 2\sum_{\substack{l=0\\ even}}^{2n-2m} i^l\, b\,(m,l,\alpha)\\ &= \frac{(-1)^{n+1}}{2} \sum_{\substack{l=0\\ even}}^{2n-2m} i^l\,(a\,(m,l-1,\alpha) - 2b\,(m,l,\alpha)).\end{aligned}$$

Now note (see Definition 3) that

$$L_N(2s+2n-2m+2,\, \psi_N^{-1}) \cdot \sum_{k=1}^{\infty} \frac{c\,(k)}{k^{s'}} = G(s',f).$$

Hence multiplying the integral expression above by

$$2\,\pi^{-(s+2n-2m+2)}\,L_N(2s+2n-2m+2,\,\psi_N^{-1})\,\Gamma\,(s+2n-2m+2),$$

and using (27), we get an integral expression for the twisted tensor L-function of f, namely

$$\int\!\!\int_D <\delta_{2n-2m}(f), E^*_{2n-2m+2}(s)> = \frac{2\,\sqrt{D}^{s'}\;i^{v_i-v_c}}{\pi^{s+2n-2m+2}\,(2\pi)^{2n+2-m+s}}\,G(s',f)\,G_\infty(s,f) \qquad (29)$$

where $s' = 2n+2-m+v_i+v_c+s$, and

$$G_\infty(s,f) = \Gamma\,(s+2n-2m+2)\,G'_\infty(s,f). \qquad (30)$$

We emphasize (29) is valid only if $m \equiv v_i + v_c \pmod 2$.

6.4 Simplifying $G_\infty(s,f)$

Since $G_\infty(s,f)$ is a 'sum' of a product of Γ-functions, and since we are interested in the non-vanishing of $G_\infty(s,f)$ at certain values of s, it behooves us to simplify $G_\infty(s,f)$, writing it as a 'simple product' of Γ-functions. This is unfortunately a daunting task. Nonetheless, we offer the following

Conjecture 1

$$G_\infty(s,f) = c_{n,m}\cdot P_{n,m}(s)\cdot\Gamma(s+2n-m+2)\cdot\Gamma\left(\frac{s+n+1-m+\epsilon}{2}\right)^2$$

where

$$\epsilon = \begin{cases} 0 & \textit{if } n+m+1 \textit{ is even} \\ 1 & \textit{if } n+m+1 \textit{ is odd,}\end{cases}$$

$$\begin{aligned} P_{n,m}(s) \quad = \quad & (s+1)(s+3)(s+5)\cdots(s+n-m-\epsilon)\ \times \\ & \Big(\frac{s}{2}+n-m\Big)\Big(\frac{s}{2}+n-m-1\Big)\cdots\Big(\frac{s}{2}+n-m-\frac{n-m-1-\epsilon}{2}\Big)\end{aligned}$$

is a polynomial in s, and $c_{n,m}$ is the constant

$$\frac{(-1)^m\cdot n!^2}{m!^2\cdot P_{n,m}(0)\cdot(\frac{n-m-1+\epsilon}{2})!^2}.$$

(When $n = m$ we take $c_{n,m} = (-1)^n$ and $P_{n,m}(s) = 1$).

We can prove this conjecture for a variety of cases: for instance for 'small' n and m, say for $0 \le m \le n \le 6$, and for some special general cases such as $m = n-1$ and $m = n$ (see [6], Chapter 6.4). This is why we are convinced about the truth of this conjecture and expect a general proof to be just a matter of being competent in dealing with identities involving the Γ-function.

Fortunately, for our current purposes, we do not need the full strength of this conjecture. We are interested only in the non-vanishing of $G_\infty(0, f)$. While this follows immediately from the Conjecture, noting that $P_{n,m}(0) \neq 0$ and that the Gamma function does not vanish, we provide instead an ingenious alternate method for showing this non-vanishing, indicated to us by Prof. Hida.

Proposition 5 *If $\psi_N^{-1} \neq 1$ or $m \neq n$, then $G_\infty(0, f) \neq 0$.*

Proof We will construct another integral expression for $G(s, f)$ that is much simpler than (and actually nothing but just a special case of) the expression (29). Since we can show that the Gamma factors in this new integral expression do not vanish at $s = 0$, the Proposition will follow once we show that the two integral expressions coincide at $s = 0$.

Let us denote $E_{2n-2m+2}$, the Eisenstein series of Section 6.2, by $E(\omega, s)$. More generally, for a differential form $\eta \in \mathrm{H}^k(\Gamma_\infty \backslash H, L(\widetilde{2n-2m}, \mathbf{Z}))$, we set

$$E(\eta, s) = \sum_{\Gamma_\infty \backslash \Gamma_0(N)} \psi_N^{-1}(\gamma)\, \gamma^{-1} \cdot \gamma^*(\eta\, y^s).$$

For $\phi \in \mathrm{H}^0(\Gamma_\infty \backslash H, L(\widetilde{2n-2m}, \mathbf{Z}))$, note that

$$d(\phi y^s) = y^s d\phi + s\phi y^{s-1} dy$$

implies that

$$dE(\phi, s) = E(d\phi, s) + sE(\phi y^{-1} dy, s).$$

This is valid for all s with $\mathrm{Re}(s)$ sufficiently large, and thus also for the meromorphic continuations of the three Eisenstein series above. In particular, if $E(\phi y^{-1} dy, s)$ is finite at $s = 0$ then we get

$$dE(\phi, 0) = E(d\phi, 0). \tag{31}$$

Now say ϕ is given by

$$\phi = \frac{(X - zY)^{2n-2m+1} - (X - xY)^{2n-2m+1}}{Y}.$$

Then since

$$\frac{1}{2n-2m+1}\, d\phi = (X - xY)^{2n-2m} dx - (X - zY)^{2n-2m} dz,$$

(31) above, together with the Claim below, implies that

$$E((X-zY)^{2n-2m}dz,0) \text{ is cohomologous to } E((X-xY)^{2n-2m}dx,0). \tag{32}$$

Claim If $\psi_N^{-1} \neq 1$ or $m \neq n$ then $E(\phi y^{-1}dy, s)$ is finite at $s=0$.
Proof of Claim Let $S = X - zY$ and $T = X - \bar{z}Y$. Then

$$\begin{aligned}
\phi &= 2iy\Big(\frac{S^{2n-2m+1} - (\frac{S+T}{2})^{2n-2m+1}}{T-S}\Big) \\
&= -2^{-2n-2m} iy\Big(\frac{(2S)^{2n-2m+1} - (S+T)^{2n-2m+1}}{2S-(S+T)}\Big) \\
&= -2^{-2n-2m} iy \sum_{j=0}^{2n-2m} (2S)^{2n-2m-j}(S+T)^j \\
&= -iy \sum_{j=0}^{2n-2m} \sum_{k=0}^{j} 2^{-j}\binom{j}{k} S^{2n-2m-k}T^k
\end{aligned}$$

Hence

$$\phi y^{-1}dy = -i \sum_{j=0}^{2n-2m} c_j (X-zY)^j (X-\bar{z}Y)^{2n-2m-j} d(z-\bar{z})$$

for some non-zero constants c_j. In particular

$$E(\phi y^{-1}dy, s) = \sum_{j=0}^{2n-2m} c_j E((X-zY)^j(X-\bar{z}Y)^{2n-2m-j}d(z-\bar{z}), s).$$

This shows that the finiteness of $E(\phi y^{-1}dy, s)$ at $s=0$ will follow from the finiteness of $E(\omega_1, s)$ and $E(\omega_2, s)$ at $s=0$, where $\omega_1 = (X-zY)^j(X-\bar{z}Y)^{2n-2m-j}dz$ and $\omega_2 = (X-zY)^j(X-\bar{z}Y)^{2n-2m-j}d\bar{z}$. One may check that

$$2\,L_N(2s+2n-2m+2,\,\psi_N^{-1}) \cdot E(\omega_1, s) = \sum_{(0,0)\neq(c,d)\,\in\,\mathbf{Z}^2} \frac{\psi_N^{-1}(d)}{(cNz+d)^{-2n+2m+2j+2}\,|cNz+d|^{2(s+2n-2m-j)}} \cdot y^s\omega_1,$$

and that

$$2\,L_N(2s+2n-2m+2,\,\psi_N^{-1}) \cdot E(\omega_2, s) = \sum_{(0,0)\neq(c,d)\,\in\,\mathbf{Z}^2} \frac{\psi_N^{-1}(d)}{(cNz+d)^{-2n+2m+2j-2}\,|cNz+d|^{2(s+2n-2m-j+2)}} \cdot y^s\omega_2.$$

Note that $L_N(2n-2m+2,\,\psi_N^{-1})$ is finite and non-zero. Moreover when ψ_N^{-1} is non-trivial, then both the last Eisenstein series are entire (cf. [13], Corollary

7.2.10 (1)). Thus we need only check the finiteness of these sums when $\psi_N^{-1} = 1$. In this case note that, the first series has a pole only when the weight $-2n+2m+2j+2=0$ and the evaluation point $2n-2m-j=1$ (cf. [13], Corollary 7.2.11). This is only possible if $m=n$. A similar analysis shows that the second series is again finite if ψ_N^{-1} is non-trivial or $m \neq n$. This ends the proof of the Claim.

If $s=0$, then (32) implies

$$\begin{aligned}
&\int\!\!\int_D < \delta_{2n-2m}(f), E(\omega, s) > \\
&\quad = \int\!\!\int_D < \delta_{2n-2m}(f), E((X-xY)^{2n-2m}dx, s) > \\
&\quad = \int\!\!\int_D < \widetilde{\delta_{2n-2m}}(f),\ E(X^{2n-2m}dx, s) > \\
&\quad = \int_0^\infty\!\!\int_0^1 < \widetilde{\delta_{2n-2m}}(f),\ X^{2n-2m}\, y^s dx > .
\end{aligned}$$

This last integral we have already computed, for it is just the $l=0$ summand of $\boldsymbol{I_2}$. We get

$$\begin{aligned}
\int\!\!\int_D < \delta_{2n-2m}(f), E(\omega, s) > &= -2\int_0^\infty\!\!\int_0^1 B_0\, y^{2n-m+s}\, dxdy \\
&= \frac{\sqrt{D}^{s'}\ i^{v_i - v_c}}{(2\pi)^{2n+2-m+s}} \sum_{k=1}^{\infty} \frac{c\,(k)}{k^{s'}}\, G''_\infty(s,f) \quad (33)
\end{aligned}$$

where

$$\begin{aligned}
G''_\infty(s,f) &= \frac{(-1)^{n+1}}{2} \sum_{\substack{\alpha=0 \\ \alpha\equiv n+1+m\ (2)}}^{2n+2} -2b\,(m,0,\alpha) \\
&\qquad \Gamma\Big(\frac{n+1-m+\alpha+s}{2}\Big)\,\Gamma\Big(\frac{3n+3-m-\alpha+s}{2}\Big) \\
&= (-1)^{n+1}\binom{n}{m}^2 \sum_{\substack{\alpha=0 \\ \alpha\equiv n+1+m\ (2)}}^{2n+2} (-1)^{\frac{n+\alpha-m+1}{2}} \sum_{p=0}^{m} (-1)^p \binom{m}{p} \\
&\qquad \binom{n-m}{\frac{n-m+\alpha-1}{2}-p}\binom{n-m}{\frac{3n-3m-\alpha+1}{2}+p} \\
&\qquad \Gamma\Big(\frac{n+1-m+\alpha+s}{2}\Big)\,\Gamma\Big(\frac{3n+3-m-\alpha+s}{2}\Big).
\end{aligned}$$

Making the change of variables $k = \frac{n+\alpha-m-1}{2} - p$ we see that

$$G''_\infty(s,f) = (-1)^n \binom{n}{m}^2 \sum_{\substack{\alpha=0 \\ \alpha\equiv n+1+m\ (2)}}^{2n+2} \sum_{p=0}^{m} (-1)^k \binom{m}{p}\binom{n-m}{k}\binom{n-m}{2n-2m-k}$$
$$\Gamma\left(\frac{n+1-m+\alpha+s}{2}\right)\Gamma\left(\frac{3n+3-m-\alpha+s}{2}\right).$$

Since the only term that survives is the $k = n-m$ term, we see that $p = \frac{-n+m-1+\alpha}{2}$. Hence $G''_\infty(s,f) =$

$$(-1)^m \binom{n}{m}^2 \sum_{p=0}^{m} \binom{m}{p} \Gamma\left(\frac{s}{2}+n-m+1+p\right)\Gamma\left(\frac{s}{2}-p+n+1\right).$$

Now we make the change of variables $s \mapsto s+2m$ and use the well known identity (see, for instance, [10], page 505)

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \sum_{p=0}^{m} \binom{m}{p} \frac{\Gamma(x+m-p)\Gamma(y+p)}{\Gamma(x+y+m)},$$

to get

$$G''_\infty(s,f) \quad = \quad (-1)^m \binom{n}{m}^2 \frac{\Gamma\left(\frac{s}{2}+n-m+1\right)^2 \Gamma(s+2n-m+2)}{\Gamma(s+2n-2m+2)}.$$

Clearly $G''_\infty(0,f) \neq 0$ and so, by (33), $G'_\infty(0,f) \neq 0$ also. But then by (30) $G_\infty(0,f) \neq 0$ as well. $\square$

7 Rationality result

Let λ be the Hecke algebra character corresponding to f. Thus $T(\mathfrak{m})f = \lambda(T(\mathfrak{m}))f = c(\mathfrak{m},f)f$, for all the Hecke operators $T(\mathfrak{m})$. Let $E = \mathbf{Q}(f, \chi_{\mathfrak{N}})$. Then, since the following modules are free of rank 1 over E (see [10], Section 8), we may write

$$\begin{aligned} \mathcal{S}_{\boldsymbol{n}}(\Gamma_0(\mathfrak{N}), \chi_{\mathfrak{N}}^{-1})[\lambda] &= E \cdot f_1 \\ \mathrm{H}^1_{cusp}(\Gamma_0(\mathfrak{N})\backslash\mathcal{H}, \widetilde{L(\boldsymbol{n},E)})[\lambda] &= E \cdot \eta(f_1) \end{aligned}$$

where $\eta(f_1)$ is a rational Hecke eigen-differential form.

Following Hida, we define a period $\Omega_j(f)$ as follows. Using the isomorphism δ we define $\Omega(f)$ via $\delta(f_1) = \Omega(f)\cdot\eta(f_1)$. Note that the value of $\Omega(f)$ depends on the choice of basis $\eta(f_1)$, but up to multiplication by an element of E, is independent of this choice.

We now restrict to H. By Lemma 2 we may write $\eta(f_1) = \oplus_{m=0}^{n} \eta_{2n-2m}(f_1)$, where $\eta_{2n-2m}(f_1)$ is a rational form $\in \mathrm{H}^1_{cusp}(\Gamma_0(N)\backslash H, L(2n-2m, E))$. Thus

$$\delta_{2n-2m}(f_1) = \Omega(f) \cdot \eta_{2n-2m}(f_1).$$

Set

$$\Omega_j(f) \quad \stackrel{\text{def}}{=} \quad \frac{\Omega(f) \cdot G(\psi_N^{-1}) \cdot (2\pi i)^{3j-2w}}{(2\pi i)^{v_i+v_c}}.$$

Theorem 1 *Say ψ_N^{-1} is primitive, and non-trivial. Then for all even integers*

$$j \in (n+1+v_i+v_c, 2n+2+v_i+v_c],$$

that is, for all critical integers j in the right half of the critical strip,

$$\frac{G(j,f)}{\Omega_j(f)} \in E.$$

Proof The proof follows by analyzing the integral expressions (29) for $G(s,f)$ for $m = 0, 1, 2, \ldots, n$, evaluated at $s = 0$. First note

$$\begin{aligned} &\int\!\!\int_D < \delta_{2n-2m}(f), E^*_{2n-2m+2}(0) > \\ &= \; 2\,\pi^{-(2n-2m+2)}\, L_N(2n-2m+2,\, \psi_N^{-1})\, \Gamma(2n-2m+2)\, \Omega(f) \cdot \\ &\qquad\qquad \int\!\!\int_D < \eta_{2n-2m},\, E_{2n-2m+2}(0) > . \end{aligned} \tag{34}$$

Since ψ_N^{-1} is even, it is well known (see for instance [8], Section 4.2, Theorem 1)

$$L_N(2n-2m+2,\, \psi_N^{-1}) = G(\psi_N^{-1})\,(2\pi)^{2n-2m+2}$$

up to multiplication by an element of $\mathbf{Q}(\psi_N^{-1}) \subset E$.

Also, Proposition 4 $\Rightarrow E_{2n-2m+2}(0)$ is E-rational. We would like to use the pairing of Lemma 5 to claim that the last integral in (34) is an element of E. However $\eta_{2n-2m}(f_1) \in \mathrm{H}^1_{cusp}(\Gamma_0(N)\backslash H, L(2n-2m, E))$, so is not compactly supported. We get around this as follows.

We replace $\Gamma_0(N)\backslash H$ by its (homotopically equivalent) Borel-Serre compactification. This is a manifold with boundary, with the boundary consisting of finitely many copies of the circle S^1, indexed by (equivalence classes of) cusps. For each cusp t we define a function (locally, on an open set U_t)

$$\phi_t(z) = \int_t^z \eta_{2n-2m}(f_1),$$

noting that $d\phi_t = \eta_{2n-2m}(f_1)$ on U_t. We may assume that the U_t are mutually disjoint and that the complement of $\cup_t U_t$ is compact. Now pick any C^∞ bump function ϵ_t satisfying

$$\epsilon_t(z) = \begin{cases} 1 & \text{for } z \in U_t, \\ 0 & \text{for } z \text{ outside another open set } V_t \text{ containing } U_t. \end{cases}$$

Set $\phi = \sum_t \epsilon_t \phi_t$, and let $\omega(f_1) = \eta_{2n-2m}(f_1) - d\phi$. Then $\omega(f_1)$ is a compactly supported closed 1-form. Further the class of $\omega(f_1)$ as an element of $\mathrm{H}^1_c(\Gamma_0(N)\backslash H, \widetilde{L(2n-2m, E)})$ is well defined since it is independent of the choice of the ϵ_t. Indeed, if $\omega(f_1)' = \eta_{2n-2m}(f_1) - d\phi'$ for some other ϕ', then $\omega(f_1)' - \omega(f_1) = d(\phi - \phi')$. And since $\phi - \phi' = 0$ on each U_t, $\phi - \phi'$ is compactly supported.

Now note

$$\int\!\!\int_D < \eta_{2n-2m}(f_1),\ E_{2n-2m+2}(0) > =$$
$$\int\!\!\int_D < \omega(f_1),\ E_{2n-2m+2}(0) > \ + \int\!\!\int_D < d\phi,\ E_{2n-2m+2}(0) > .$$

The first integral $\in E$ by Lemma 5. The second integral vanishes. Indeed, since $E_{2n-2m+2}(0)$ is a closed form, the second integral

$$= \int\!\!\int_D d(\phi \wedge E_{2n-2m+2}(0)) = \int_{\partial D} \phi \wedge E_{2n-2m+2}(0)$$

by Stoke's Theorem. The last integral vanishes since ϕ is 'exponentially decreasing' near the cusps.

In sum, we obtain,

$$\int\!\!\int_D < \delta_{2n-2m}(f), E^*_{2n-2m+2}(0) >$$
$$\doteq \Omega(f)\, G(\psi_N^{-1})\, (2\pi)^{2n-2m+2}\, \pi^{-(2n-2m+2)},$$

where the $\doteq$ means up to multiplication by an element of E.

On the other hand, when $s = 0$, $s' = 2n - m + 2 + v_i + v_c = j$ say. Moreover, since (29) is only valid when $m \equiv v_i + v_c \pmod 2$, we see that j is even, and so is a critical integer of $G(s, f)$ in the right half of the critical strip. In fact, as m varies between 0 and n, j varies through all critical integers in the right half of the critical strip. By Proposition 5, $G_\infty(0, f) \neq 0$. Thus, since j is even

$$\begin{aligned} G(j, f) &\doteq \frac{\Omega(f)\, G(\psi_N^{-1})\, (2\pi)^{2n-2m+2}\, \pi^{-(2n+2m+2)} \cdot (2\pi)^{2n-m+2}\, \pi^{2n+2m+2}}{i^{v_i - v_c} \sqrt{D}^{\,j}} \\ &\doteq \frac{\Omega(f)\, G(\psi_N^{-1})\, (2\pi i)^{3j-2w}}{(2\pi i)^{v_i + v_c}} \\ &= \Omega_j(f) \qquad \square \end{aligned}$$

Remark 2 The hypothesis of non-trivialness on ψ_N^{-1} could probably be removed. By Proposition 4, at $m = n$ we could use instead an integral expression with $E_2(s,z)$ replaced by $E_2(s,z) - pE_2(s,pz)$ for a suitable prime p.

Remark 3 The period we obtain is compatible with the period relations of Section 4.3. First note that

$$\Omega_j(f) = \Omega_0(f) \cdot (2\pi i)^{3j}$$

for all even critical integers j. This was predicted by (11). In fact, in this case, we may 'identify'

$$c^+((M \otimes M^c)_+) = \Omega_0(f) = \frac{\Omega(f)\, G(\psi_N^{-1})}{(2\pi i)^{v_i+v_c}\,(2\pi i)^{2w}}.$$

Moreover, by the results of [10], we see that we can identify

$$c^+(N) = c^+(Res_{K/\mathbf{Q}}M) = \frac{\Omega(f)}{(2\pi i)^{v_i+v_c}},$$
$$\delta(N) = \delta(Res_{K/\mathbf{Q}}M) = \sqrt{-D}\,G(\psi_N^{-1})\,(2\pi i)^{-2w}.$$

Thus we 'recover' the first period relation of Proposition 3.

Remark 4 To prove an algebraicity result for the odd critical integers (in the left half of the critical strip), one could possibly evaluate the integral expressions at $s = 1 - (2n - 2m + 2)$ (where the Eisenstein series is again rational) and then proceed as above. We have not worked out the complete details of this calculation. Alternatively, one may use the functional equation (derived in [6], Section 7.2, in the case when $w = n+1+v_i+v_c$ is odd, $\mathfrak{N} = N\mathcal{O}$ is an extended ideal, and ψ_N^{-1} is invertible, following methods of [17]) to get information about these remaining critical values. For instance, in the level one case with $n+1+v_i+v_c$ odd one may compute that

$$G(j,f) = \frac{\Omega(f)\,(2\pi i)^j}{(2\pi i)^{v_i+v_c}\sqrt{-D}}$$

for all odd critical integers j. This is again compatible with (11), and as above we may 'identify'

$$c^-((M \otimes M^c)_+) = \frac{\Omega(f)}{(2\pi i)^{v_i+v_c}\sqrt{-D}}.$$

Thus we 'recover' the last period relation of Proposition 3.

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