Local Galois Representations and Applications

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Overview

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   - with possible applications to a generalization of Serre's uniform boundedness conjecture,
2. of weight 1
   - with possible applications to counting exotic weight 1 forms.

Cusp forms and Galois representations

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a (primitive) cusp form of

weight $k \geq 2$,

level $N \geq 1$, and

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Let

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$$

be the Galois representation attached to f and p, satisfying

$$\text{trace}\, \rho_f(\text{Frob}_\ell) = a_\ell \quad \text{and} \quad \det \rho_f(\text{Frob}_\ell) = \epsilon(\ell)\ell^{k-1},$$

for all primes $\ell \nmid Np$.

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One expects the full case to be the generic case.

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Conjecture (Serre)

There exists an absolute constant C such that for all primes $p > C$, the image of

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This is now proven in all but one case, by Serre, Mazur and Bilu-Parent.

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One way to prove this is to work **locally**: show that the image of the local residual Galois representation at p is big!

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We shall also assume that $\epsilon = 1$ on G_p, for simplicity.

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Finally, let

$$\mu_\lambda : G_p \to \bar{\mathbb{F}}_p^\times$$

be the **unramified character** of G_p taking Frob_p^{-1} to $\lambda \in \bar{\mathbb{F}}_p^\times$.

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Question: Which case occurs & for what values of λ, a, b, c?

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However, there are some partial results...

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Assume $v > 0$ is positive. Then the p-**adic** local rep.

$$\rho_f|_{G_p} \simeq V_{k,a_p},$$

where V_{k,a_p} is a **crystalline** representation of G_p.

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- **Small weights** (Fontaine-Edixhoven)

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- **Large slopes** (Berger-Li-Zhu)

$$v > \left\lfloor \frac{k-2}{p-1} \right\rfloor \text{ or } \{v = \infty\} \implies \bar{V}_{k,a_p} \simeq \operatorname{ind} \omega_2^{k-1}.$$

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is a well-defined element of $\bar{\mathbb{F}}_p$, in which case

$$\bar{V}^{ss}_{k,a_p} \simeq \mu_\lambda \cdot \omega \oplus \mu_{\lambda^{-1}} \cdot \omega,$$

where λ is a root of the equation above.

Slope 1 and beyond

We have extended this result to the case of slopes

- $v = 1$, in S. Bhattacharya, E. Ghate, S. Rozensztajn, *Preprint* (2016)

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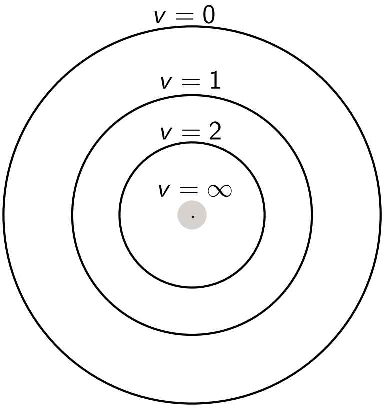

p-adic unit disk

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We assume that $p \geq 5$, though sometimes allow $p = 3$.

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Let $p \geq 5$, let $k \geq 2p+2$ and let $r = k-2 \equiv b \mod (p-1)$.

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where

$$u(a_p) := \frac{b}{b-r}\left(\frac{a_p}{p} - \binom{r}{2}\frac{p}{a_p}\right).$$

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   NB: $2 = \lceil \alpha \rceil$, for $\alpha \in (1,2)$. Is this local constancy a reflection of the Gouvêa-Mazur conjecture?

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5. **Trichotomy:** The trichotomy for $v = 1$ and $b = 2$ is analogous to the **dichotomy** for $v = \frac{1}{2}$ and $a = 1$.

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Galois side: G_p-reps **Automorphic side**: G-reps

$$\begin{array}{ccc} V_{k,a_p} & \xrightarrow{p-adic\ LLC} & B(V_{k,a_p}) \\ \downarrow & & \downarrow \\ \bar{V}^{ss}_{k,a_p} & \xrightarrow{mod\ p\ LLC} & LL(\bar{V}^{ss}_{k,a_p}) \simeq \overline{B(V_{k,a_p})}^{ss}. \end{array}$$

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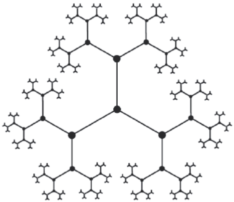

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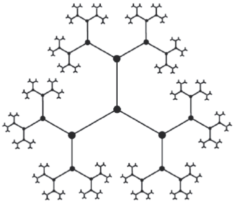

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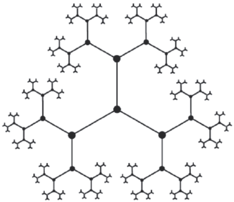

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$$KZ\backslash(G \times V) \to KZ\backslash G = X$$

defines a **local system** on X.

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We need to study its kernel using T, as follows.

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allowing us to compute $\overline{B(V_{k,a_p})}$, so $\bar{V}_{k,a_p}^{ss}$ by the mod p LLC. □

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However, sometimes the extension class lies in the 2-**dimn'l**

$$\mathrm{H}^1(G_p, \bar{\mathbb{F}}_p(\omega)) = \frac{\mathbb{Q}_p^\times}{(\mathbb{Q}_p^\times)^p} \otimes \bar{\mathbb{F}}_p.$$

This happens exactly when $\bar{V}_{k,a_p}$ is an **extension of** 1 **by** ω, up to a twist.

Peu vs très ramifiée extensions

Definition (Serre)

Such an extension is **peu ramifiée** if it lies on the line

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Question: Can one distinguish between peu and très?

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Rmk: In part 1b), the extra condition that $r \not\equiv 2 \mod p$ can partly be removed. We we can also prove 1b) when $r \equiv 2 \mod p$, if we assume $u(a_p) - \varepsilon$ is a uniformizer of $\mathbb{Q}_p(a_p)$, or if $\mathbb{Q}_p(a_p)/\mathbb{Q}_p$ is unramified.

Picture for $v = 1$, $b = 2$, $t = v(r-2) > 0$: $\bar{V}_{k,a_p}$ irreducible (grey and green), $\bar{V}_{k,a_p}$ reducible (yellow) with peu ramifiée (orange) and très ramifiée (blue):

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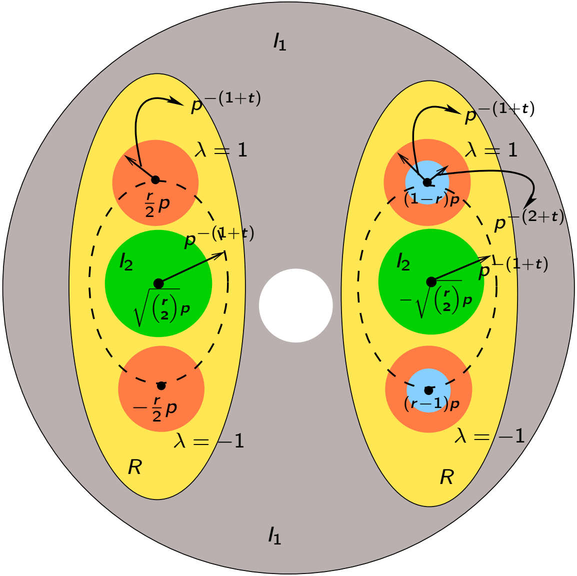

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whenever it is a non-split extension of ω by ω^2.

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We are especially interested in counting exotic forms (last three types).

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We use the fact that the following sets are closely related:

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Rmk: Minimal forms have cube-free conductors.

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In case 1, twisting by $\chi_p = 1, \omega^{-a}$ preserves $N_p = p$, whereas in case 2, twisting by any $\chi_p = \omega^b$ preserves $N_p = p^2$. $\square$

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- We write $V_4 \subset S_4$ for the Klein 4-group and $D_4 \subset S_4$ for the dihedral group with 8 elements.

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corresponding to all possible choices of I_p and G_p in S_4.

There are 10 possibilities.

For each ramification type we list the power of p appearing in

- D_K, the absolute discriminant of K, and,
- N_f, the conductor of $\tilde{\rho}_f$, that is, N_p.

In the third last column we list congruence conditions on p.

In the last column we list the number of twists by characters of p-power conductor preserving minimality.

Table: Conductor vs Discriminant

I_p	G_p	ram. of p in K	D_K	N_p	$p \equiv$	good	twists
(12)	I_p	$1^2 1 1$	p	p		yes	2
(12)	(12), (34)	$1^2 2$	p	p^2		yes	$p-1$
(12)(34)	I_p	$1^2 1^2$	p^2	p			2
(13)(24)	(1234)	2^2	p^2	p			2
(12)(34)	V_4	2^2	p^2	p^2		yes	$p-1$
(12)(34)	(12), (34)	$1^2 1^2$	p^2	p^2		yes	$p-1$
(123)	I_p	$1^3 1$	p^2	p	1 (3)		2
(123)	S_3	$1^3 1$	p^2	p^2	2 (3)	yes	$p-1$
(1234)	I_p	1^4	p^3	p	1 (4)		2
(1234)	D_4	1^4	p^3	p^2	3 (4)		$p-1$

Octahedral Ramification Table

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I_p	G_p	ram. of p in K	D_K	N_p	$p \equiv$	good	twists
(12)	I_p	$1^2 11$	p	p		yes	2
(12)	(12), (34)	$1^2 2$	p	p^2		yes	$p-1$
(12)(34)	I_p	$1^2 1^2$	p^2	p			2
(13)(24)	(1234)	2^2	p^2	p			2
(12)(34)	V_4	2^2	p^2	p^2		yes	$p-1$
(12)(34)	(12), (34)	$1^2 1^2$	p^2	p^2		yes	$p-1$
(123)	I_p	$1^3 1$	p^2	p	1 (3)		2
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Octahedral Ramification Table

In 5 rows, the power of p in D_K is at most N_p.

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Octahedral Ramification Table

In 5 rows, the power of p in D_K is at most N_p. These cases are 'good' for us, since one can then hope to show

$$\sum_{f\,:\,N_f<X} 1 \text{ is bounded by (a constant times) } \sum_{K\,:\,|D_K|<X} 1,$$

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Octahedral Ramification Table

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$\sum_{f \,:\, N_f < X} 1$ is bounded by (a constant times) $\sum_{K \,:\, |D_K| < X} 1$,

and estimate the right hand sum by Bhargava.

Concluding arguments: octahedral forms of prime level

In prime level, it turns out D_K is either p or p^3, so only the first and ninth line of the table apply.

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Question: Can one use the table to get instead good *lower* bounds on the number of octahedral forms (of minimal level)?

Thank you