On uniform large Galois images for modular abelian varieties

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Abstract

We formulate a question regarding uniform versions of "large Galois image properties" for modular abelian varieties of higher dimension, generalizing the well-known case of elliptic curves. We then answer our question affirmatively in the exceptional image case, and provide lower estimates for uniform bounds in the remaining cases.

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1 Introduction

Let A be a simple abelian variety of dimension d over $\mathbb{Q}$, without complex multiplication (CM) over $\overline{\mathbb{Q}}$. The images of the Galois representations

$$\rho_{A,p}\colon G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(T_p(A))$$

defined by the p-adic Tate modules of A, for p running through the set of primes, are expected to be "generically big". The most famous instance of such a statement is probably Serre's result on elliptic curves [31], which shows that, given a non-CM elliptic curve A (over any number field) and large enough p relative to A, the image $\rho_{A,p}(G_{\mathbb{Q}})$ is all of $\mathrm{GL}(T_p(A)) \simeq \mathrm{GL}_2(\mathbb{Z}_p)$.

When A is a higher dimensional abelian variety, endowed with a polarization e, the existence of the Weil pairing implies that the image of $\rho_{A,p}$ lands in $\mathrm{GSp}(T_p(A), e)$. For A such that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \otimes \mathbb{Q} = \mathbb{Q}$ Serre partly extended his result for elliptic curves: when d is odd, or $d = 2$ or 6, he showed that $\rho_{A,p}(G_{\mathbb{Q}})$ equals $\mathrm{GSp}_{2d}(\mathbb{Z}_p)$, for p large enough relative to A (see [33], Théorème 3).

The orthogonal situation, that is, when A has as many $\mathbb{Q}$-endomorphisms as possible, has also been worked out. Recall a (simple) abelian variety A over $\mathbb{Q}$ is said to be *of* GL_2*-type* if $\mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$ is a number field E of (maximal) degree $[E : \mathbb{Q}] = \dim(A)$. A result of Ribet (cf. [28], Theorem 4.4) implies that this is equivalent to A being *modular*, i.e., a quotient of some modular jacobian $J_1(N)$. This result was conditional on Serre's modularity conjecture which has

since been proved by Khare-Wintenberger and Kisin (see [14] and the references therein). Then, as the name indicates, the image of the representation

$$\rho_{A,p}\colon G_{\mathbb{Q}} \to \mathrm{GL}(T_p(A))$$

lands in $\mathrm{GL}_2(E \otimes \mathbb{Q}_p)$, and the image inside this last group is still expected to be "generically big". Indeed, the unavoidable constraint on the image given by the prescribed form of the determinant was shown by Momose, Papier and Ribet to be, asymptotically, the main one:

Theorem 1.1 *([25], Theorem 3.1) Let $f = \sum a_n(f)q^n$ be a non-CM newform of level $N \geq 1$ and weight $k \geq 2$. Let $E = \mathbb{Q}(a_n(f))$ be the number field generated by its Fourier coefficients. There are a subfield F of E, the fixed field of the automorphisms of E corresponding to the extra twists of f (so that $F = E$ when f has no extra twists), and an abelian number field K, cut out by the Dirichlet characters corresponding to the extra twists of f, such that the following holds: for large enough p relative to f, the image of the restriction to $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ of the Shimura-Deligne representation $\rho_{f,p}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ lands in $\mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$, and is equal to*

$$\{u \in \mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p) \text{ such that } \det(u) \in \mathbb{Z}_p^{*(k-1)}\}.$$

We content ourselves here with this somewhat imprecise statement in order to avoid recalling too many definitions, and we further note that [25], Theorem 4.1 describes the image $\rho_{f,p}(G_{\mathbb{Q}})$ completely. Specializing the above result in weight $k = 2$ yields the desired "big image result" for simple non-CM abelian varieties A of GL_2-type.

Now, a natural question is whether or not there are *uniform* versions for these large image theorems. Even more, one might first ask what are the correct questions in the abelian varieties setting. Going back again to the case of elliptic curves, Serre asked if one can find an absolute constant C (that is, independent of the non-CM rational elliptic curve A) such that, for $p > C$, the representation $\rho_{A,p}$ (or, equivalently, the residual representation $\overline{\rho}_{A,p}$) is surjective (see [31], p. 299 or [32], p. 199).

The following classification due to Dickson on finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$ (cf. e.g., [10], Satz 8.27) allows one to break up Serre's question further: up to conjugacy every finite subgroup H of $\mathrm{PGL}_2(\overline{\mathbb{F}}_p)$, for p an odd prime, is isomorphic to either:

(a) an *exceptional group*, that is a permutation group isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$;

(b) a *Borel subgroup* (that is, a finite subgroup of upper triangular matrices);

(c) a *dihedral group* D_r, for some $r \in \mathbb{N}$ not divisible by p;

(d) $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ or $\mathrm{PGL}_2(\mathbb{F}_{p^r})$, for some $r \in \mathbb{N}$.

For elliptic curves A, when considering the image of $\overline{\rho}_{A,p} \subset \mathrm{GL}_2(\mathbb{F}_p)$ (and not its projectivization $\mathbb{P}\overline{\rho}_{A,p}$), it is natural to divide case (c) into two further subcases, depending on whether the image belongs to the normalizer of a *split* or *nonsplit* Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$. More generally, this distinction occurs as soon as there is a natural *finite* field of coefficients $\mathbb{F}_q$ for our representation (as will be the case for us): Cartan subgroups in the general case are by definition maximal subtori of $\mathrm{GL}_n(\mathbb{F}_q)$, and when $n = 2$ they can only be either split (that is, isomorphic to $\mathbb{F}_q^* \times \mathbb{F}_q^*$) or nonsplit ($\simeq \mathbb{F}_{q^2}^*$).

Serre's question for elliptic curves therefore boils down to asking for absolute lower bounds on primes p such that, for p larger that these bounds, the projective images of $\overline{\rho}_{A,p}$, as A varies over all non-CM elliptic curves over $\mathbb{Q}$, are not contained in groups of the first three types above: namely, exceptional, Borel, or normalizer of split or nonsplit Cartan subgroups. The exceptional cases (a) are relatively easy to rule out for elliptic curves over arbitrary number fields (see [17], p. 36). For elliptic curves over $\mathbb{Q}$ it is known that such absolute upper bounds also exist in cases (b) ([18]), and (c) when the Cartan group is split ([2]), but we still do not know if there are rational non-CM elliptic curves with normalizer of nonsplit Cartan structure modulo arbitrarily large p.

For a higher-dimensional A which is (and, from this point on, will always be) a (simple) abelian variety of GL_2-type over $\mathbb{Q}$ without complex multiplication, set $E := \mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$, and consider the residual representation

$$\overline{\rho}_{A,\mathfrak{P}} := \rho_{A,p} \mod \mathfrak{P} \colon G_{\mathbb{Q}} \to \mathrm{GL}(A[\mathfrak{P}]) = \mathrm{GL}_2(\mathbb{F}_{p^n}), \tag{1}$$

for $\mathfrak{P}$ a prime of E above p, with residue field $\mathbb{F}_{p^n} = \mathbb{F}_{\mathfrak{P}} := \mathcal{O}_{E,\mathfrak{P}}/\mathfrak{P}$. Using Dickson's theorem, we will say a simple and non-CM abelian variety A of GL_2-type has *big image mod p* if the projectivization $\mathbb{P}\overline{\rho}_{A,\mathfrak{P}}$ of the above $\overline{\rho}_{A,\mathfrak{P}}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ for all primes $\mathfrak{P}$ of E of characteristic p. Theorem 1.1 shows that each A as above has big image mod p, for p large enough *relative to A*.

Can this dependence on A be removed? Keeping in mind that even the simplest situation of rational elliptic curves still remains not completely understood, we might nevertheless consider the general case, and the motivation for this note is a very preliminary study of what could happen for higher dimensional abelian varieties of GL_2-type. In this setting, as far as we are aware, virtually nothing is known regarding uniform results. As we will see below, if uniform bounds are to be expected they must at least depend on the dimension d of the abelian varieties. We feel the most natural statement is the following.

Question 1.2 *Does there exist a function $B(d)$ for $d \in \mathbb{N}$ such that, if A is a rational simple non-CM abelian variety of GL_2-type with dimension less than or equal to d, then for all places $\mathfrak{P}$ of $E := \mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$, the image of $\mathbb{P}\overline{\rho}_{A,\mathfrak{P}}$ contains $\mathrm{PSL}_2(\mathbb{F}_p)$ if $\mathrm{char}(\mathfrak{P}) =: p > B(d)$?*

As above, it helps to divide Question 1.2 according to Dickson's theorem. Here, as alluded to above, we speak of split or nonsplit Cartan depending on whether the implicit maximal torus is split or nonsplit over the base field $\mathbb{F}_{\mathfrak{P}}$. In the sequel we write B for "Borel", E for "exceptional", SP for "normalizer of split Cartan" and NSP for "normalizer of non-split Cartan".

Question 1.3 *For* $* = \mathrm{B}, \mathrm{E}, \mathrm{SP}$ *or* NSP*, does there exist a function* $B_*(d)$ *for* $d \in \mathbb{N}$*, such that if* A *is a simple rational non-CM abelian variety of* GL_2*-type with dimension less than or equal to* d*, then* $\mathbb{P}\overline{\rho}_{A,\mathfrak{P}}$ *cannot have image contained in a subgroup of type* $*$ *as soon as* $\mathrm{char}(\mathfrak{P}) =: p > B_*(d)$*?*

In cases SP and NSP, these questions can be reinterpreted in terms of Hecke algebras (see Remark 4.4 in the last section).

In this paper we first give an effective positive answer to Question 1.3 in the easiest case of exceptional groups (we have not tried to obtain sharp bounds here). We then give negative results (i.e., lower bounds for the expected $B_*(d)$) in the three other cases. The Borel case can be deduced in a straightforward way from Ribet's famous work [23] on the converse to Herbrand's criterion. For the dihedral case, we estimate how other classical results of Ribet on level raising ([24], or [26] Theorem 7.3) can be used. We thus produce families of varieties with controlled dimension. But there we also exhibit a more efficient technique: we show that Hida theory gives further information towards Question 1.3 in the dihedral case. This last technique gives sharper results (than level raising) on the quantitative side, but it does not seem to distinguish easily between the split and nonsplit subcases.

Putting all this together yields the following theorem.

Theorem 1.4 *We have:*

(a) **(Exceptional subgroups).** *Assume* A *is a rational simple abelian variety of* GL_2*-type without complex multiplication of dimension* d*, endowed with a Galois structure of exceptional type modulo some prime* p*. Then* p *is bounded above in terms of* d*, and more precisely, for* d *large enough one has*

$$p \leq d^2 \cdot 3^{4d}.$$

(b) **(Borel subgroups).** *There is an infinite sequence of prime numbers* p *for each of which there is a rational simple abelian variety of* GL_2*-type without complex multiplication endowed with a Borel structure modulo* p*, whose dimension* d *satisfies*

$$d \leq \frac{(p-5)(p-7)}{24}.$$

(c) **(Dihedral subgroups, I).** *For each sufficiently large prime* $p \equiv 1 \mod 4$ *(respectively* $p \equiv 3 \mod 4$*), there is a rational simple abelian variety of* GL_2*-type without complex multiplication, endowed with a normalizer-of-split-Cartan structure modulo* p *(respectively, a normalizer-of-nonsplit-Cartan structure modulo* p*), whose dimension* d *satisfies*

$$d \leq C \cdot p^{11/2}$$

for some absolute constant C*.*

(d) **(Dihedral subgroups, II).** *For each large enough prime $p \equiv 3 \mod 4$ there is a rational simple abelian variety of GL_2-type without complex multiplication endowed with a projectively dihedral structure modulo p, whose dimension d satisfies*

$$d \leq \frac{(p-5)(p-7)}{24}.$$

Theorem 1.4 has the following immediate consequence for the bound in Question 1.2:

Corollary 1.5 *If a uniform bound $B(d)$ exists in Question 1.2, then it is of order not less than $O(\sqrt{d})$.*

Remark 1.6 It would be interesting to see if the results proved here carry over to the case of modular forms of arbitrary weight. For these one might need to have some control on the variation of the degree of Hecke fields in a Hida (or Coleman) family, perhaps along the lines of [9].

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2 Exceptional case: Nebentypus and monodromy

We first prove part (a) of Theorem 1.4, which is actually a generalization of a remark of Serre for the case of elliptic curves (see [17], p. 36). Our first approach is geometric, though later we give a more automorphic proof in a special case. For a more computational approach, see [15].

2.1 A geometric approach

With notation as in Theorem 1.4, let A be a d-dimensional abelian variety corresponding to (the Galois orbit of) a weight 2 newform $f(q) = \sum_{n>0} a_n q^n$ of some level N. Though this is not necessary, we remark that, up to replacing A by an isogenous variety, one may assume that $\mathrm{End}_{\mathbb{Q}}(A) = \mathcal{O}_E$ is the full ring of integers of the Hecke field $E := \mathbb{Q}(a_n)$, with the standard notation. Let $\mathfrak{P}$ be a prime of $\mathcal{O}_E$ above p. Let K be a (totally ramified) extension of $\mathbb{Q}_p$ over which A acquires semistable (i.e., possibly good) reduction, with ring of integers $\mathcal{O}_K$. Denote by G_K the absolute Galois group of K and I_K its inertia subgroup. It is well-known that $e := [K : \mathbb{Q}_p]$ can be bounded in terms of d only. Indeed, by the Galois criterion for semistable reduction ([6], Théorème IX.3.6 and Proposition IX.3.5; see also [35], Theorem 1), A acquires good (respectively, bad semistable) reduction over any extension field such that the image of inertia

at p on the ℓ-adic Tate module $T_\ell(A)$ is trivial (respectively, unipotent of degree 2), for any $\ell \neq p$. If an element γ in $\mathrm{Aut}(T_\ell(A))$ is torsion (respectively, has some power which is unipotent of degree two) and is trivial mod $\ell > 2$, one readily checks that $\gamma = 1$ (respectively, γ is unipotent of degree two). It follows that one can choose a K as desired inside $\mathbb{Q}_p(A[\ell])$. For $p > 3$, we thus have $e \leq \mathrm{card}(\mathrm{GL}_2(\mathcal{O}_E \otimes \mathbb{F}_3)) < 3^{4d}$. This bound is known to be far from sharp - a (non-CM) elliptic curve over $\mathbb{Q}$ acquires semistable reduction over a number field of degree dividing 6, and similarly, better bounds for jacobian varieties can be found in [16], Proposition 4 (γ).

Let us still denote by A what will actually be its Néron model over $\mathcal{O}_K$. The $\mathfrak{P}$-adic Tate module $A_\mathfrak{P} := \varprojlim_n A[\mathfrak{P}^n]$ is $\mathcal{O}_\mathfrak{P}$-free of rank 2 (where $\mathcal{O}_\mathfrak{P}$ is the completion of the Hecke ring $\mathcal{O}_E$ at $\mathfrak{P}$), and similarly for $A[\mathfrak{P}] = A_\mathfrak{P} \otimes \mathcal{O}_\mathfrak{P}/\mathfrak{P}$ over $\mathcal{O}_\mathfrak{P}/\mathfrak{P}$. We define $A^0[\mathfrak{P}]$ to be the connected part of the latter: this can be seen as a $\mathbb{F}_\mathfrak{P}$-*vector space scheme*, for $\mathbb{F}_\mathfrak{P} = \mathcal{O}_\mathfrak{P}/\mathfrak{P}$, finite and flat over $\mathcal{O}_K$, or as a certain subspace in the p-torsion of the corresponding formal group. It follows from Raynaud's classification of group schemes of type $(p, \ldots, p)$ that the tame inertia subgroup of G_K acts on the semi-simplification $A^0[\mathfrak{P}]^{\mathrm{ss}}$ of $A^0[\mathfrak{P}]$ via sums of products of fundamental characters raised to powers in the range $\{0, \ldots, e\}$ ([22], Corollaire 3.4.4). As we are dealing with 2-dimensional representations, these fundamental characters are actually of level (0 or) 1 or 2 ([34], Proposition 1). We now distinguish the following four cases: either

(i) $A^0[\mathfrak{P}]$ has dimension 0 over $\mathbb{F}_\mathfrak{P}$, or

(ii) $A^0[\mathfrak{P}]$ has dimension 1 over $\mathbb{F}_\mathfrak{P}$, or

(iii) $A^0[\mathfrak{P}]$ has dimension 2 over $\mathbb{F}_\mathfrak{P}$ and the tame inertia acts on $A^0[\mathfrak{P}]^{\mathrm{ss}}$ via fundamental characters of level 2, or

(iv) $A^0[\mathfrak{P}]$ has dimension 2 over $\mathbb{F}_\mathfrak{P}$ and the tame inertia acts on $A^0[\mathfrak{P}]^{\mathrm{ss}}$ via fundamental characters of level (0 or) 1.

Assume we are in case (i). Let χ be the Nebentypus of the newform corresponding to A, let ω be the Teichmüller lift of the cyclotomic character and I_p an inertia subgroup at p in $G_\mathbb{Q}$, extending I_K. Define $\bar{\rho}_{A,\mathfrak{P}}$ as in (1). As A is modular, one knowns that $\det(\bar{\rho}_{A,\mathfrak{P}}) = \chi\omega \mod \mathfrak{P}$. On the other hand, it follows from our hypothesis (i) above that $\bar{\rho}_{A,\mathfrak{P}}|_{I_K}$ has trivial semi-simplification, so that $(\chi\omega|_{I_p})^e = 1 \mod \mathfrak{P}$.

This implies that χ has order divisible by $(p-1)/\gcd(e, p-1)$. Now the field $E := \mathrm{End}_\mathbb{Q}(A) \otimes \mathbb{Q}$ contains the values of χ, so

$$\varphi((p-1)/\gcd(e, p-1)) \leq [E : \mathbb{Q}] = \dim(A) = d$$

(where φ is Euler's totient function). We know that

$$\liminf \frac{\varphi(n)\log(\log(n))}{n} = e^{-\gamma}$$

for $\gamma = 0.577...$ the Euler-Mascheroni's constant (see e.g., [37], Théorème 5.6), or more simply that

$$\varphi(n) > C_\varepsilon \cdot n^{1-\varepsilon}$$

for any $\varepsilon > 0$ and $C_\varepsilon > 0$ depending on ε, so that

$$p \leq c_\varepsilon \cdot d^{1+\varepsilon} 3^{4d}$$

for some $c_\varepsilon > 0$ depending on ε. This concludes the proof that our case (i) does not occur for sufficiently large p relative to d.

Assume now we are in case (ii). By Raynaud's theorem mentioned above, the tame inertia of G_K at p acts on $A^0[\mathfrak{P}]$ via some power ω^a of the fundamental character of level 1 so that:

$$\bar{\rho}_{A,\mathfrak{P}} \mid_{I_K} \simeq \begin{pmatrix} \omega^a & * \\ 0 & 1 \end{pmatrix}$$

with $1 \leq a \leq e$. This has projective image containing a cyclic group of order $(p-1)/\gcd(a, p-1) \geq (p-1)/e$, which cannot be included in an exceptional subgroup if $p > 5e+1$ (recall that exceptional subgroups have no element of order larger than 5). Using our estimate for e we obtain $p \leq 5 \cdot 3^{4d} + 1$, whence our claim in case (ii).

The same argument works for case (iii). Here Raynaud's result implies that the tame inertia of G_K acts on $A[\mathfrak{P}]$ $(=A^0[\mathfrak{P}])$ via fundamental characters of level 2 raised to powers less than e. As we are dealing with 2-dimensional representations we know more precisely that this inertia action is via the diagonal matrix with coefficients ω_2^a and ω_2^{pa}, with $1 \leq a \leq e$ and $\{\omega_2, \omega_2^p\}$ the conjugate pair of fundamental characters of level 2, which have order $p^2 - 1$ on I_p (see [34], Proposition 1). Therefore $\bar{\rho}_{A,\mathfrak{P}} \mid_{I_K}$ has projective image a cyclic group of order $(p+1)/\gcd(a, p+1) \geq (p+1)/e$, and we now conclude that $p \leq 5 \cdot 3^{4d} - 1$.

Assume finally we are in case (iv). Here we prove that, if $p > e + 1$, the action of the local Galois group G_K on $A[\mathfrak{P}]$ is not diagonal: it therefore has a p-part, the same is true projectively and this implies the image cannot be exceptional.

Let indeed ϵ be the ramification index of $\mathfrak{P}$ in the endomorphism ring $\mathcal{O}_\mathfrak{P}$, and π a uniformizing parameter of $\mathcal{O}_\mathfrak{P}$: one has $\mathfrak{P} = \pi\mathcal{O}_\mathfrak{P}$ and $p\mathcal{O}_\mathfrak{P} = \pi^\epsilon \mathcal{O}_\mathfrak{P}$. Let also $r = [\mathbb{F}_\mathfrak{P} : \mathbb{F}_p]$, for $\mathbb{F}_\mathfrak{P} := \mathcal{O}_\mathfrak{P}/\mathfrak{P}$, be the residual degree. Let $\mathcal{A}$ be the formal group associated with the $\mathfrak{P}$-adic Tate module $A_\mathfrak{P}$, and call δ its dimension (which is the $\overline{\mathbb{F}}_p$-dimension of its cotangent space over $\overline{\mathbb{F}}_p$, or simply the "number of variables" of $\mathcal{A}$). One has $\delta = [\mathcal{O}_\mathfrak{P} : \mathbb{Z}_p] = \epsilon r$. The uniformizer π can be seen as an element of $\mathrm{End}_{\mathcal{O}_K}(\mathcal{A})$. The action of G_K commutes with multiplication by π, so there is a Galois-compatible filtration:

$$0 \subseteq A[\mathfrak{P}] \subseteq A[\mathfrak{P}^2] \subseteq \cdots \subseteq A[\mathfrak{P}^{\epsilon-1}] \subseteq A[\mathfrak{P}^\epsilon]$$

which can be identified with

$$0 \subseteq \pi^{\epsilon-1}\mathcal{A}[p] \subseteq \pi^{\epsilon-2}\mathcal{A}[p] \subseteq \cdots \subseteq \pi\mathcal{A}[p] \subseteq \mathcal{A}[p]$$

and each subquotient of the above filtration is isomorphic, as a G_K-module, to $A[\mathfrak{P}]$. One therefore sees that, whereas $\mathcal{A}$ has dimension δ, the group scheme $A[\mathfrak{P}]$ (which has rank p^{2r}) has a cotangent space (over $\overline{\mathbb{F}}_p$) of dimension $r = \delta/\epsilon$ ("r variables").

By [22], Théorème 3.3.3, one knows that, if $p > e+1$, any (finite flat) group scheme of type $(p, \ldots, p)$ over $\mathcal{O}_K$ is uniquely determined by its generic fiber. Therefore, assuming that the local Galois action is diagonal on $A[\mathfrak{P}]$, the latter splits uniquely into the direct sum $A_1 \oplus A_2$ of two finite flat group schemes of rank p^r over $\mathcal{O}_K$ (and not only K), each of which is an $\mathbb{F}_{\mathfrak{P}}$-vector scheme of rank one. Considering the geometric special fibers we claim that each $A_j \times_{\mathcal{O}_K} \overline{\mathbb{F}}_p$ is isomorphic to

$$\mathrm{Spec}(\overline{\mathbb{F}}_p[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)). \tag{2}$$

It indeed follows from [22] that one can take equations for A_j (over $\mathcal{O}_K$) of the shape $X_i^p = \delta_i X_{i+1}$, for some $\delta_i \in \mathcal{O}_K$ whose valuation verifies $0 \leq v(\delta_i) \leq e$ (cf. loc. cit., Corollaire 1.5.1 and p. 266). Moreover, Théorème 3.4.1 of [22] tells us that the tame inertia acts on A_j via

$$\psi_1^{v(\delta_r)} \cdot \psi_2^{v(\delta_1)} \cdot \ldots \cdot \psi_r^{v(\delta_{r-1})},$$

with notations of loc. cit.: $\psi_i := \psi_1^{p^{i-1}}$, and $\{\psi_i\}_{1 \leq i \leq r}$ is the set of conjugate fundamental characters of level r. On the other hand, our running hypothesis (iv) implies that G_K acts on the A_j via some power ω^a of the fundamental character of level 1, which can be expressed in terms of the fundamental characters of level r as:

$$\omega = \psi_1^{1+p+p^2+\cdots+p^{r-1}} = \psi_1 \cdots \psi_r.$$

Assuming $e < p-1$, we therefore see that the $v(\delta_i)$ are all equal to a (as $0 \leq v(\delta_i) \leq e$). Moreover $a > 0$ (otherwise the equations $X_i^p = \delta_i X_{i+1}$ would show A_j is isomorphic over $\overline{\mathbb{F}}_p$ to $\mathrm{Spec}(\overline{\mathbb{F}}_p[X]/(X^{p^r} - cX))$, for some $c \neq 0$ in $\overline{\mathbb{F}}_p$; the A_j would therefore be étale over $\mathcal{O}_K$, a contradiction). So $v(\delta_i) > 0$ and Raynaud's equations $X_i^p = \delta_i X_{i+1}$ do give our claim (2) above.

But this is not compatible with what we know about $A[\mathfrak{P}] \times_{\mathcal{O}_K} \overline{\mathbb{F}}_p$. For instance, (2) implies all nilpotent functions on $A_1 \oplus A_2$ are killed by raising them to the pth power, so if $A[\mathfrak{P}] \times_{\mathcal{O}_K} \overline{\mathbb{F}}_p$ was split, it would in turn be of shape $\mathrm{Spec}(\mathcal{R})$, with $\mathcal{R} \otimes_{\mathcal{O}_K} \overline{\mathbb{F}}_p$ a quotient of $\overline{\mathbb{F}}_p[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$ (recall $A[\mathfrak{P}]$ has r variables). This would be a contradiction with the fact that $A[\mathfrak{P}]$ has rank p^{2r}.

This proves our claim that $A[\mathfrak{P}]$ is nonsplit as a scheme, from which it follows that the local Galois projective image has order greater or equal to p. Using once more that exceptional subgroups have no element of order larger than 5, this concludes the study of case (iv), and therefore the proof of Theorem 1.4 (a).

Remark 2.1 It might also help to briefly recall how things work in the technically simpler setting of elliptic curves (over arbitrary number fields). Case (i)

does not occur, as $A^0[p]$ is never 0 in this case. Cases (ii) and (iii) can actually be copied with no change. For case (iv), instead of specializing the method above (see, for instance, [21], proof of Lemma 1.3), one can use Serre's study of the one-parameter formal group defined by an elliptic curve ([31], paragraph 1.9 and 1.10). If the elliptic curve in question is supersingular but the tame inertia acts via powers of the fundamental character of level 1, it indeed follows from loc. cit. that the relevant Newton polygon is broken, and there are points in the p-torsion of the corresponding formal group which have valuation with denominator divisible by p (see [31], p. 272). This precisely means that the Galois extension cut out by the p-torsion of the elliptic curve has degree divisible by p. Therefore the Galois action is non-diagonal, and the projectivization of its image again has a non-trivial p-part. This approach might be generalizable to higher-dimensional abelian varieties and formal groups, up to some more technicalities.

We also note that an elliptic curve over a number field F acquires semistable reduction over an extension of F of degree dividing 12. Therefore, the associated mod p representations is not projectively exceptional as soon as $p > 60[F : \mathbb{Q}] + 1$: see [17], p. 36. (There Mazur asserts that for elliptic curves over $\mathbb{Q}$, $p > 13$ would even do.)

2.2 An automorphic approach, when $\mathrm{val}_p(N) \leq 1$

For forms of weight 2 and conductor having p-adic valuation at most 1, one can give a purely automorphic proof of part (a) of Theorem 1.4. This proof has the virtue of appealing to more modern technology, but does not cover all cases, since the mod p reductions of forms with high powers of p in the level are not yet fully known.

Suppose $f \in S_2(\Gamma_1(N))$. Assume that the power of p dividing N is at most 1. We show that $\bar{\rho}_f$ cannot have exceptional projective image, if the dimension of the corresponding abelian variety is bounded, for p sufficiently large depending on the dimension.

First assume that N is prime to p. We show that for p sufficiently large, $\bar{\rho}_f$ on I_p has large projective image irrespective of the dimension of the underlying abelian variety. Indeed, since N is prime to p, then (e.g., see [29]) the Serre weight of $\bar{\rho}_f$ is 2. If f is ordinary at p, then it is well-known that on I_p, $\bar{\rho}_f$ is of the form

$$\begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}$$

which has projective image a group of order divisible by $p-1$. On the other hand, if f is non-ordinary at p, then by Fontaine's theorem ([4], Theorem 2.6) (which applies, since the Serre weight k satisfies $2 \leq k \leq p+1$), $\bar{\rho}_f$ on I_p has the form

$$\begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{pmatrix}$$

which has projective image a cyclic group of order $p+1$. If the global projective image of $\bar{\rho}_f$ is of exceptional type, then, in either case, this cyclic group cannot have cardinality larger than 5, so $p \geq 7$ cannot occur.

Now suppose that p exactly divides N. Assume $\bar{\rho}_f$ has exceptional projective image. If there is no Nebentypus at p, then f is of Steinberg type at p, and in particular ordinary, so the projective image on I_p is again of order divisible by $p-1$, and so $p \leq 5$. So we may assume that the Nebentypus at p is ω^j with $1 \leq j \leq p-2$. Then, by Proposition 6.18 of [30], $\bar{\rho}_f$ on I_p has one of the following three shapes:

$$\begin{cases} \begin{pmatrix} \omega^{j+1} & * \\ 0 & 1 \end{pmatrix} & \text{if } v_p(a_p) = 0, \\ \begin{pmatrix} \omega & * \\ 0 & \omega^j \end{pmatrix} & \text{if } v_p(a_p) = 1, \\ \begin{pmatrix} \omega_2^{j+1} & 0 \\ 0 & \omega_2^{p(j+1)} \end{pmatrix} & \text{if } 0 < v_p(a_p) < 1. \end{cases}$$

In the first case we see that the order of ω^{j+1} must be smaller than 5, so that the order of ω^j is $\geq \frac{p-1}{5}$. In particular $d \geq \varphi(\frac{p-1}{5})$. Thus if $p > O(d^{1+\varepsilon})$, then the exceptional case does not occur. In the second case, the projective image of I_p on the diagonal is the image of ω^{j-1} and a similar argument applies. In the last case, the projective image of I_p is the image of $\theta := \omega_2^{(p-1)(j+1)}$. We claim that if this has order at most 5, then ω^j has order at least $\frac{p-1}{3}$, so that we are again done, except in one case which we treat separately below. Indeed, the order of θ is $\frac{p+1}{g}$ where g is the greatest common divisor of $j+1$ and $p+1$. If the order of θ is at most 5, then $\frac{p+1}{5} \leq g \leq \frac{p+1}{2}$. Writing $j+1 = mg$ and $p+1 = ng$, for some $1 \leq m < n \leq 5$ (note $j+1 < p+1$), we have $j+1 = \frac{m(p+1)}{n}$ for $2 \leq n \leq 5$ and $1 \leq m < n$, with $(m,n) = 1$. An easy check shows that for these finitely many values of j, the greatest common divisor of j and $p-1$ is at most 3, so that the order of ω^j is at least $\frac{p-1}{3}$, as desired, except when $j+1 = \frac{p+1}{2}$, in which case θ is quadratic, but ω^j is also quadratic.

We claim, however that this last subcase cannot occur for sufficiently large p. To see this suppose that there is an exceptional type form in $S_2(\Gamma_0(Mp), \chi)$, where M is prime to p and the p-part χ_p of χ is quadratic. Then consider the Teichmüller lift of the associated mod p representation (this exists since for $p > 5$ the mod p representation has order prime to p, since it is an extension of an exceptional group by a subgroup of scalars of order $p^n - 1$, for some n). We obtain an odd finite image representation into $\mathrm{G}L_2(W)$, with W is the ring of Witt vectors of the residue field, which by the recent proof of Artin's conjecture (which in turn follows from the proof due to Khare-Wintenberger/Kisin of Serre's conjecture [14], and Khare's proof that Serre implies Artin [13]), we know comes from a form in $S_1(\Gamma_0(Mp), \chi')$. Comparing determinants mod p we see that the p-part of χ' must be $\omega^{(p+1)/2}$. But an elementary argument (see

[1], Proposition 5.1) shows that the Nebentypus at p of an exceptional weight 1 form which is tamely ramified at p must be of bounded order, which is clearly impossible if p is large.

This completes the proof.

3 Borel subgroups and irregular primes

We check part (b) of Theorem 1.4. We know from the work of K. L. Jensen ([12]; or [38], Theorem 5.17) that there are an infinite number of irregular primes, that is, primes p dividing the order of the class group C_p of the cyclotomic field $\mathbb{Q}(\mu_p)$. Kummer proved that they are exactly the primes dividing the numerator of some Bernoulli number B_k with k even, $2 \leq k \leq p-3$, and Herbrand showed more precisely that if p divides the order of the ω^{1-k}-isotypic component $C_p(\omega^{1-k})$ of C_p mod p, then p divides B_k (where ω is as before the cyclotomic character). In his celebrated and seminal paper [23] Ribet proved the converse to Herbrand's criterion, and to that end he showed that when $p|B_k$, there is a newform f in $S_2^{\mathrm{new}}(\Gamma_0(p), \omega^{k-2})$ whose associated abelian variety A_f has a p-isogeny. We claim these abelian varieties are not of CM type. One way to see this goes as follows. Assuming f is CM, level considerations show that the associated quadratic imaginary field K has discriminant $-p$. As p ramifies in K, the classical theory of complex multiplication implies f is supersingular above p. Now Ribet's representation has semi-simplification $1 \oplus \omega^{k-1}$, with $2 \leq k \leq p-3$ (see [23], Proposition 4.2). Therefore A_f has ($\mathbb{Z}_p$-étale subgroups of) p-torsion points, and cannot be supersingular at p. Invoking finally the fact that

$$\dim(J_1(p)) = (p-5)(p-7)/24$$

(this follows for instance from [36], Chapter 2) we conclude the proof.

Remark 3.1 Of course Ribet's representation shows the existence of p-torsion points, not only a p-isogeny, on A_f.

Remark 3.2 If we can choose k and p such that ω^{k-2} has order tending to infinity, then, the above construction gives an infinite sequence of abelian varieties of GL_2-type of dimension tending to ∞, which are residually mod p of Borel type, with p tending to ∞.

4 Dihedral cases

4.1 Using level raising

We prove part (c) of Theorem 1.4.

Proposition 4.1 *There exists an absolute constant C such that, for each prime $p \equiv 3 \mod 4$ (respectively, $p \equiv 1 \mod 4$), there is a non-CM abelian variety with a normalizer of nonsplit Cartan (respectively, split Cartan) Galois structure mod p and dimension $d \leq C \cdot p^{5.5}$.*

Proof Let D be the discriminant of an imaginary quadratic number field K with ring of integers $\mathcal{O}_K$. Let A be a simple $\mathbb{Q}$-abelian variety of GL_2-type having complex multiplication by $\mathcal{O}_K$ (take for instance the Weil restriction to $\mathbb{Q}$ of the Galois conjugacy class of elliptic curves over the Hilbert class field H_K of K, having CM by $\mathcal{O}_K$) and let $f = \sum_n a_n(f)q^n$ be one of the conjugate CM newforms associated with A, with conductor N_D. Assume for simplicity that f has trivial Nebentypus. Let p be a prime which does not divide N_D. For ℓ another prime such that $\ell \equiv -1 \mod p$ and ℓ remains inert in K one has $a_\ell(f) = 0 \equiv (\ell+1) \mod p$, so Ribet's theorem ([27], Theorem 1) shows that p is a congruence prime between $f \in S_2^{\mathrm{new}}(\Gamma_0(N_D))$ and some $g \in S_2^{\mathrm{new}}(\Gamma_0(\ell N_D))$. Let B be the abelian variety over $\mathbb{Q}$ associated with g. Having semi-stable bad reduction at ℓ, it cannot have complex multiplication. The dimension of B is bounded above by $\dim(S_2^{\mathrm{new}}(\Gamma_0(\ell N_D)))$, which is of shape $\lambda \ell N_D + o(\ell N_D)$ for some $\lambda \in \mathbb{Q}$ ([36], Chapter 2). Now Linnik's theorem, in the improved explicit version proved by Heath-Brown, shows that one can take $\ell \leq c \cdot p^{5.5}$, for some constant c depending only on D (see e.g., [11], Theorem 18.1, and the references therein). Fix for instance $D = -4$ in the above, and take for A the elliptic curve over $\mathbb{Q}$ with j-invariant 1728, conductor 2^6, having multiplication by the Gaussian integers $\mathbb{Z}[i]$. The condition $p \equiv 3 \mod 4$ (respectively, $p \equiv 1 \mod 4$) ensures B has a normalizer-of-*nonsplit*-Cartan Galois structure mod p (respectively, normalizer-of-*split*-Cartan Galois structure), and our statement follows. $\square$

Remark 4.2 We understand from [11], Chapter 18 that the exponent 5.5 used here in Linnik's theorem is conjecturally improvable, but not under 2. We will see in the next section that some elementary Hida theory allows us to produce examples of abelian varieties with projectively dihedral Galois structure mod p, and dimension bounded above by $O(p^2)$. On the other hand, these techniques do not allow us to distinguish easily between the split and nonsplit cases.

4.2 Using Hida families

We first recall a few standard facts on Λ-adic modular forms. Fix p an odd prime number, and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (which will allow us to think of elements of $\overline{\mathbb{Q}}$ as living in $\overline{\mathbb{Q}}_p$). Set $\Lambda = \mathbb{Z}_p[[X]]$ and let L be the ring of integers of a finite extension of $\mathrm{Frac}(\Lambda)$. An *arithmetic point* P_{k,ζ_r} is a morphism $L \to \overline{\mathbb{Q}}_p$ of $\mathbb{Z}_p$-algebras extending the algebra homomorphism $\Lambda \to \overline{\mathbb{Q}}_p$ induced by $X \mapsto \zeta_r(1+p)^{k-1} - 1$, with $2 \leq k \in \mathbb{N}$ and ζ_r a primitive p^{r-1}-th root of unity, $r \geq 1$. If $N = N_0 p$ with $\gcd(N_0, p) = 1$, let $\psi\colon (\mathbb{Z}/N_0 p\mathbb{Z})^* \to \overline{\mathbb{Q}}^*$ be a Dirichlet character. Let $\chi_{\zeta_r}\colon (\mathbb{Z}/p^r\mathbb{Z})^* \to \overline{\mathbb{Q}}^*$ be the character which, under the decomposition

$$(\mathbb{Z}/p^r\mathbb{Z})^* \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^{r-1}\mathbb{Z},$$

maps the first factor to 1 and the generator $(1+p)$ of the second factor to ζ_r.

A Λ-adic cusp form of tame character ψ is a formal series

$$F(X,q) = \sum_{n\geq 0} a_n(X) q^n \in L[[X]]$$

such that the specialization f_{k,ζ_r} at any arithmetic point P_{k,ζ_r} belongs to the space of modular forms $S_k(\Gamma_0(Np^r), \psi\omega^{1-k}\chi_{\zeta_r})$. (This is the weight normalization adapted to "deformations of weight 1".) Such a form is said to be a Λ-adic newform if all its arithmetic specializations are p-stablilized N_0-newforms. A fundamental theorem of Hida [7] asserts that one can attach to such an eigenform F a representation

$$\rho_F \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathrm{Frac}(L))$$

which can be seen as a family $P_{k,\zeta_r}(\rho_F) = \rho_{P_{k,\zeta_r}(F)} = \rho_{f_{k,\zeta_r}}$ of representations interpolating those associated by Eichler-Shimura and Deligne to the classical eigenforms f_{k,ζ_r} at arithmetic points. The weight 1 specializations do give rise to Galois representations too, but they might or might not correspond to classical modular forms via Deligne-Serre theory. One checks that the restriction to an inertia group I_p at p is of shape

$$\rho_{F|I_p} \simeq \begin{pmatrix} \psi\kappa & * \\ 0 & 1 \end{pmatrix}$$

where κ is the character $\kappa \colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \to \Lambda^*$ which maps the topological generator $1+p$ of $1+p\mathbb{Z}_p \simeq \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ to $(1+X) \in \Lambda^*$. The mod p representations $\overline{\rho}_{f_{k,\zeta_r}} := \rho_{f_{k,\zeta_r}} \mod p$ are all isomorphic when irreducible (and, in any case, have the same semi-simplification).

Let us finally prove part (d) of Theorem 1.4. Let p be a prime number equal to $3 \mod 4$, so that $\mathbb{Q}(\sqrt{-p})$, whose ring of integers we denote by $\mathcal{O}_{\mathbb{Q}(\sqrt{-p})}$, has discriminant $-p$. Let $\alpha_{-p} := \left(\frac{-p}{\cdot}\right)$ be the corresponding quadratic character. The class number formula shows that $h(\mathcal{O}_{\mathbb{Q}(\sqrt{-p})})$ is prime to p, as it is bounded above by $\frac{p-1}{2}$. The Brauer-Siegel theorem in the case of imaginary quadratic fields actually yields that, for any $\varepsilon > 0$,

$$p^{1/2-\varepsilon} < h(\mathcal{O}_{\mathbb{Q}(\sqrt{-p})}) < p^{1/2+\varepsilon}$$

if p is large enough, so that in particular $h(\mathcal{O}_{\mathbb{Q}(\sqrt{-p})})$ is non-trivial (and, again, prime to p) for large enough p. Choose

$$\Psi \colon \mathrm{Gal}(H(\mathbb{Q}(\sqrt{-p}))/\mathbb{Q}(\sqrt{-p})) \to \overline{\mathbb{Q}}^* \subseteq \overline{\mathbb{Q}}_p^*$$

a non-trivial (hence not self-conjugate, cf. Remark 4.3) character, where $H(\mathbb{Q}(\sqrt{-p}))$ is the Hilbert class field of $\mathbb{Q}(\sqrt{-p})$. Let

$$f_\Psi(q) = \sum_{\mathfrak{A} \subset \mathcal{O}_{\mathbb{Q}(\sqrt{-p})}} \Psi(\mathfrak{A}) q^{N(\mathfrak{A})}$$

be the theta series associated with Ψ. This is a (classical) ordinary eigenform of level p, weight 1 and Nebentypus α_{-p} (cf. e.g., [8], paragraph 7.6). The associated Galois representation ρ_{f_Ψ} is $\mathrm{Ind}_{\mathbb{Q}(\sqrt{-p})}^{\mathbb{Q}}(\Psi)$, whose image is included in the normalizer of a Cartan subgroup, but not the Cartan itself (and as Ψ has prime-to-p order, the same is true for $\overline{\rho}_{f_\Psi} = \rho_{f_\Psi} \mod p$).

A result of Wiles ([39], Theorem 3), generalizing Hida theory for arithmetic points, says that such classical eigenforms of weight 1 can also be embedded in Λ-adic eigenfamilies. Let F be one such form passing through f_Ψ. This F does not have complex multiplication (i.e., no arithmetic member has complex multiplication), for similar reasons as in paragraph 3. Indeed, assuming F has CM, a look at the ramification shows that the CM field would have to be $\mathbb{Q}(\sqrt{-p})$, in which p ramifies: so the weight 2 members of the family would be supersingular, whereas arithmetic specialization of a Hida family are ordinary. Let $\psi\omega^{1-k}\chi_{\zeta_r}$ be the decomposition of the Nebentypus at P_{k,ζ_r}, using the same notations as in the beginning of this paragraph. The tame level N_0 of F is 1 and ψ is some power ω^a of the Teichmüller character ω. Together with the fact that f_Ψ has Nebentypus α_{-p}, we see that $a = (p-1)/2$, which implies that $P_{2,1}(F)$ is a newform in $S_2(\Gamma_0(p), \omega^{(p-3)/2}) \subseteq S_2(\Gamma_1(p))$. We have therefore built some rational simple abelian variety A of GL_2-type, endowed with a nontrivial normalizer-of-Cartan structure mod p, which is isogenous to a quotient of $J_1(p)$. The shape of the Nebentypus and the known dimension of $J_1(p)$ give the announced bounds

$$\varphi\left(\frac{p-1}{2}\right) \leq \dim(A) \leq \frac{(p-5)(p-7)}{24}. \qquad \square$$

Remark 4.3 Note that dihedral groups are ambiguously defined in the case when the projective image is the Klein 4-group $(\mathbb{Z}/2\mathbb{Z})^2$, when there are three possible choices for the cyclic subgroup, but this is not the case for the Galois group built here. Indeed, if it were, one would have two quadratic subextensions of $\mathrm{Gal}(\mathbb{Q}(A[p])/\mathbb{Q})$ apart from $\mathbb{Q}(\sqrt{-p})$. But the only allowed ramification locus for these number fields is p, a contradiction. Of course, this yields a somewhat artificial proof that $h(\mathcal{O}_{\mathbb{Q}(\sqrt{-p})})$ is odd when $p \equiv 3 \mod 4$, a classical fact (see e.g., [3], Théorème 4 on p. 388).

Remark 4.4 Let A be a GL_2-type abelian variety as in Question 1.3, endowed with a projectively dihedral Galois structure mod $\mathfrak{P}$. Then $\overline{\rho}_{A,\mathfrak{P}}$ is the induced representation $\mathrm{Ind}_K^{\mathbb{Q}}(\overline{\psi})$ of some character $\overline{\psi}$ with values in a finite field $\mathbb{F}$. Let $\psi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{Q}}^*$ be the character deduced from $\overline{\psi}$ by Teichmüller lift. If K is an imaginary quadratic field (as is necessarily the case, for instance, for nonsplit dihedral structures when A is an elliptic curve, as one checks by looking at the image of a complex conjugation) then ψ gives rise to a CM abelian variety A_ψ, whose induced representation $\rho_{A_\psi,\mathfrak{P}}$ is $\mathrm{Ind}_K^{\mathbb{Q}}(\psi)$. It is known since Hecke that A_ψ is modular and, as noticed in the Introduction, it follows from Ribet and Khare-Wintenberger/Kisin that the same is true for A. Denoting by N and N_ψ the conductors of A and A_ψ respectively, one checks that $N_\psi | N$, so both abelian varieties can be seen as irreducible components in the spectrum $\mathrm{Spec}(\mathbb{T}_{\Gamma_1(N)})$ of the full Hecke algebra $\mathbb{T}_{\Gamma_1(N)}$ for $\Gamma_1(N)$ in weight 2, and

by construction these components intersect in characteristic p (which therefore is a *congruence prime* for the cuspforms f_A and f_{A_ψ} associated to A and A_ψ).

Question 1.3 then specializes to: if a non-CM irreducible component of some $\mathrm{Spec}(\mathbb{T}_{\Gamma_1(N)})$ intersects a CM one in characteristic p, is it true that the degree d of the former component has to be such that $B_{\mathrm{SP}}(d)$ or $B_{\mathrm{NSP}}(d)$ are larger than p? We note in passing that sharp bounds for congruence primes of irreducible components of degree one (that is, elliptic curves) are closely related to deep problems such as the modular degree conjecture, or the *abc* conjecture (see e.g., [5] or [19]). Of course, the connectedness of Hecke's algebra spectra in weight 2 ("spaghetti principle", see [17], Proposition 10.6) shows that some intersection between CM and non-CM components has to occur - but *quantifying the primes of fusion* is the hard part of the story. We understand that experimental data seem to indicate that a large part of fusion occurs in characteristic 2 (cf., e.g., the remark on page 11 of [20]. A typical drawing of the situation can be found in [29], pp. 40-41).

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