p-ADIC ASAI L-FUNCTIONS ATTACHED TO BIANCHI CUSP FORMS

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ABSTRACT. We establish a rationality result for the twisted Asai L-values attached to a Bianchi cusp form and construct distributions interpolating these L-values. Using the method of abstract Kummer congruences, we then outline the main steps needed to show that these distributions come from a measure.

1. PRELIMINARIES

Let F be an imaginary quadratic field with ring of integers $\mathcal{O}_F$. Write $F = \mathbb{Q}(\sqrt{-D})$ with $D > 0$ and $-D$ the discriminant of F. Let $S_{\underline{k}}(\mathfrak{n})$ denote the space of Bianchi cusp forms of weight $\underline{k} = (k, k)$, $k \geq 2$, and level $\mathfrak{n}$ and central character with trivial finite part and infinity type $(2-k, 2-k)$. Let $\mathbf{f} \in S_{\underline{k}}(\mathfrak{n})$ be a normalized eigenform and let $c(\mathfrak{m}, \mathbf{f})$ be the Fourier coefficients of $\mathbf{f}$, for any integral ideal $\mathfrak{m} \subset \mathcal{O}_F$. The eigenform $\mathbf{f}$ corresponds to a tuple $(f_1, \ldots, f_h)$ of classical Bianchi cusp forms, where h is the class number of F. We take $f = f_1 \in S_{\underline{k}}(\Gamma_0(\mathfrak{n}))$ and only focus on this since the Asai L-function depends only on f_1.

Let $c(r)$, for $r \geq 1$, denote $c((r), f)$. Define the Asai L-function of f by the formula

$$G(s, f) = L_N(2s - 2k + 2, \mathbb{1}) \sum_{r=1}^{\infty} \frac{c(r)}{r^s},$$

where N is the positive generator of the ideal $\mathfrak{n} \cap \mathbb{Z}$ and $L_N(s, \mathbb{1})$ is the L-function attached to the trivial character modulo N. The special values of this function are investigated in [Gha99]. A generalization to cusp forms defined over CM fields can be found in [Gh99b].

Let $p \in \mathbb{Z}$ be an odd prime integer that is relatively prime to N and that is also unramified in F. Let $\chi : (\mathbb{Z}/p^j\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$ be a Dirichlet character with conductor dividing p^j. Define the twisted Asai L-function of f by the formula

$$G(s, \chi, f) = L_N(2s - 2k + 2, \chi^2) \sum_{r=1}^{\infty} \frac{c(r)\chi(r)}{r^s}.$$

This has an Euler product expansion

$$G(s, \chi, f) = \prod_l G_l(s, \chi, f),$$

where the local L-functions at all but finitely many primes l are described as follows. Let $l \neq p$ be an integer prime, not dividing N. For any $\mathfrak{l}|l$, let $\alpha_1(\mathfrak{l})$ and $\alpha_2(\mathfrak{l})$ denote the reciprocal roots of the Hecke polynomial of f at $\mathfrak{l}$: $1 - c(\mathfrak{l}, f)X + \mathrm{Nm}(\mathfrak{l})^{k-1}X^2$. Then

$$\frac{1}{G_l(s, \chi, f)} = \begin{cases} \prod_{i,j}(1 - \chi(l)\alpha_i(\mathfrak{l})\alpha_j(\bar{\mathfrak{l}})l^{-s}) & \text{if } l = \bar{\mathfrak{l}}\mathfrak{l}, \\ (1 - \chi(l)\alpha_1(\mathfrak{l})l^{-s})(1 - \chi^2(l)l^{-2s+2k-2})(1 - \chi(l)\alpha_2(\mathfrak{l})l^{-s}) & \text{if } l = \mathfrak{l} \text{ is inert,} \\ (1 - \chi(l)\alpha_1^2(\mathfrak{l})l^{-s})(1 - \chi(l)l^{-s+k-1})(1 - \chi(l)\alpha_2^2(\mathfrak{l})l^{-s}) & \text{if } l = \mathfrak{l}^2 \text{ is ramified.} \end{cases}$$

We want to find 'periods' and prove that the special values of the twisted Asai L-functions are algebraic after dividing by these periods. We also want to p-adically interpolate the special values of $G(s, \chi, f)$ as χ varies over characters of p-power conductor.

2. COMPLEX VALUED DISTRIBUTIONS

Following Panchishkin, we now construct a complex valued distribution that is related to the twisted Asai L-function. This section basically follows Coates–Perrin-Riou [CP89] and Courtieu–Panchishkin [CP04, §1.6].

The function $G(s, f)$ has an Euler product formula

$$G(s, f) = \prod_l G_l(s, f) = \sum_{r=1}^{\infty} \frac{d(r)}{r^s},$$

and hence satisfies the hypothesis in the above references. We now assume that our fixed prime p splits as $\mathfrak{p}\bar{\mathfrak{p}}$ in F. A similar argument will also work for p inert. Then the local Euler factor at p is of the form $G_p(s, f) = F(p^{-s})^{-1}$ where

$$F(X) = (1 - \alpha_1(\mathfrak{p})\alpha_1(\bar{\mathfrak{p}})X)(1 - \alpha_1(\mathfrak{p})\alpha_2(\bar{\mathfrak{p}})X)(1 - \alpha_2(\mathfrak{p})\alpha_1(\bar{\mathfrak{p}})X)(1 - \alpha_2(\mathfrak{p})\alpha_2(\bar{\mathfrak{p}})X).$$

In what follows we shall assume that f is totally ordinary at p. Hence we may assume, by possibly switching the subscripts $i = 1, 2$, that the inverse root $\kappa := \alpha_1(\mathfrak{p})\alpha_1(\bar{\mathfrak{p}})$ of the polynomial $F(X)$ is a p-adic unit. Also define a polynomial $H(X)$ as

$$H(X) = (1 - \alpha_1(\mathfrak{p})\alpha_2(\bar{\mathfrak{p}})X)(1 - \alpha_2(\mathfrak{p})\alpha_1(\bar{\mathfrak{p}})X)(1 - \alpha_2(\mathfrak{p})\alpha_2(\bar{\mathfrak{p}})X).$$

Let $B_0 = 1$ and define B_1, B_2 and B_3 such that

$$H(X) = 1 + B_1 X + B_2 X^2 + B_3 X^3.$$

Let $\chi : (\mathbb{Z}/p^j\mathbb{Z})^\times \to \mathbb{C}^\times$ be a character with conductor $C_\chi = p^{j_\chi}$. We want to define a complex valued distribution that interpolates the values of the twisted L-function:

$$G(s, \chi, f) = \sum_{r=1}^{\infty} \frac{d(r)\chi(r)}{r^s}.$$

Define functions $P_s : \mathbb{Q} \to \mathbb{C}$ by the formula

$$P_s(b) = \sum_{r=1}^{\infty} \frac{d(r)e^{2\pi i r b}}{r^s}$$

which converges absolutely for $\Re(s)$ sufficiently large. Define a distribution $\tilde{\mu}$ on $\mathbb{Z}_p^\times$ by the formula

$$\tilde{\mu}_s(a + p^j\mathbb{Z}_p) = \frac{p^{j(s-1)}}{\kappa^j} \sum_{i=0}^{3} B_i P_s(ap^i/p^j)p^{-is}.$$

We need to check that this satisfies the distribution relations. We will do this by showing that

$$\sum_{a \bmod p^j} \chi(a)\tilde{\mu}_s(a + p^j\mathbb{Z}_p) \tag{1}$$

is independent of j as long as $j \geq j_\chi$. For any character χ and integer M, define the generalized Gauss sum

$$G_{M,p^j} = \sum_{a \bmod p^j} \chi(a)e^{2\pi i a M/p^j}.$$

It can be verified that

$$G_{M,p^j} = \begin{cases} p^{j-j_\chi} G(\chi)\bar{\chi}(M/p^{j-j_\chi}) & \text{if } p^{j-j_\chi}|M \\ 0 & \text{otherwise,} \end{cases}$$

where $G(\chi) = G_{1,p^{j_\chi}}$ is the Gauss sum of χ.

From the definition, we can write the quantity in equation (1) as

$$\frac{p^{j(s-1)}}{\kappa^j} \sum_{a \mod p^j} \chi(a) \sum_{i=0}^{3} B_i p^{-is} \sum_{r=1}^{\infty} d(r) e^{2\pi i a p^i r/p^j} r^{-s}$$

$$= \frac{p^{j(s-1)}}{\kappa^j} \sum_{i,r} B_i p^{-is} d(r) r^{-s} G_{p^i r,p^j}$$

$$(2) \qquad = \frac{p^{js-j_\chi}}{\kappa^j} G(\chi) \sum_{i,r} B_i p^{-is} d(r) r^{-s} \bar{\chi}\left(\frac{p^i r}{p^{j-j_\chi}}\right) \mathbb{1}_{\mathbb{Z}}\left(\frac{p^i r}{p^{j-j_\chi}}\right).$$

Here $\mathbb{1}_{\mathbb{Z}}$ is the characteristic function of integers. It appears since the only terms that contribute to the sum are those with $p^{j-j_\chi}|p^i r$ (this follows from the above formula for $G_{p^i r,p^j}$). Now write each $r = r_1 r_2$, where r_1 is the p-power part and r_2 is the away from p-part of r respectively. We know that

$$\bar{\chi}\left(\frac{p^i r}{p^{j-j_\chi}}\right) = \bar{\chi}(r_2)\bar{\chi}\left(\frac{p^i r_1}{p^{j-j_\chi}}\right), \quad \text{and}$$

$$\mathbb{1}_{\mathbb{Z}}\left(\frac{p^i r}{p^{j-j_\chi}}\right) = \mathbb{1}_{\mathbb{Z}}\left(\frac{p^i r_1}{p^{j-j_\chi}}\right).$$

Using these in equation (2), we get the expression

$$(3) \qquad \frac{p^{js-j_\chi}}{\kappa^j} G(\chi) \left(\sum_{r_2} \bar{\chi}(r_2) d(r_2) r_2^{-s}\right) \sum_{i,r_1} B_i p^{-is} d(r_1) r_1^{-s} \bar{\chi}\left(\frac{p^i r_1}{p^{j-j_\chi}}\right) \mathbb{1}_{\mathbb{Z}}\left(\frac{p^i r_1}{p^{j-j_\chi}}\right).$$

Note that here we have also used the fact that $d(r) = d(r_1)d(r_2)$.

We also know that

$$\sum_{r_1} d(r_1) r_1^{-s} = F(p^{-s})^{-1},$$

where the sum is taking over all powers of p. Moreover, we also have

$$\left(\sum_i B_i p^{-is}\right) F(p^{-s})^{-1} = H(p^{-s}) F(p^{-s})^{-1} = (1 - \kappa p^{-s})^{-1}.$$

Hence

$$\sum_{i,r_1} B_i p^{-is} d(r_1) r_1^{-s} = \sum_{r_3} \kappa^{\operatorname{ord}_p r_3} r_3^{-s},$$

where the r_3 varies over all powers of p. We also have the relation

$$\kappa^{\operatorname{ord}_p r_3} = \sum_{r_3 = p^i r_1} B_i d(r_1).$$

Hence setting $r_3 = p^i r_1$, we see that the only terms that contribute to the sum in equation (3) are those p-powers r_3 of the form $p^{j-j_\chi} r_4$ for some p-power r_4. Also note that as r_2 varies over all positive integers prime to p, we get

$$\sum_{r_2} \bar{\chi}(r_2) d(r_2) r_2^{-s} = G(s, \bar{\chi}, f).$$

We remark that if χ is the trivial character of $\mathbb{Z}_p^\times$, then $G(s, \bar{\chi}, f)$ is just the p-deprived Asai L-function $G_p(s, f)^{-1}G(s, f)$, where $G_p(s, f)$ is the local Euler factor at p, since the function on $\mathbb{Z}$ induced by the trivial character χ is taken to vanish on $p\mathbb{Z}$. In any case, we can rewrite equation (3) as

$$\frac{p^{js-j_\chi}}{\kappa^j}G(\chi)G(s, \bar{\chi}, f)\sum_{r_3}\kappa^{\mathrm{ord}_p r_3}r_3^{-s}\bar{\chi}\left(\frac{r_3}{p^{j-j_\chi}}\right)\mathbb{1}_{\mathbb{Z}}\left(\frac{r_3}{p^{j-j_\chi}}\right)$$

$$= \frac{p^{js-j_\chi}}{\kappa^j}G(\chi)G(s, \bar{\chi}, f)\sum_{r_4}\kappa^{j-j_\chi}\kappa^{\mathrm{ord}_p r_4}p^{-s(j-j_\chi)}r_4^{-s}\bar{\chi}(r_4)$$

$$= \frac{p^{j_\chi(s-1)}}{\kappa^{j_\chi}}G(\chi)G(s, \bar{\chi}, f)\sum_{r_4}\kappa^{\mathrm{ord}_p r_4}r_4^{-s}\bar{\chi}(r_4)$$

$$= \frac{p^{j_\chi(s-1)}}{\kappa^{j_\chi}}G(\chi)G(s, \bar{\chi}, f),$$

since $\bar{\chi}(r_4) = 0$, unless $r_4 = 1$, since by convention all Dirichlet characters of $\mathbb{Z}_p^\times$, including the trivial character, are thought of as functions on $\mathbb{Z}$ by requiring that they vanish on $p\mathbb{Z}$. This simultaneously checks the distribution relations and establishes the relationship

$$(4) \qquad \int \chi \, d\tilde{\mu}_s = \frac{p^{j_\chi(s-1)}}{\kappa^{j_\chi}}G(\chi)G(s, \bar{\chi}, f)$$

between these measures and twisted Asai L-values.

We remark here that these calculations hold only for $s \in \mathbb{C}$ where $G(s, f)$ is absolutely convergent. And this is known for all s such that $\Re(s) > k + 1$, in view of the Hecke bound $c(\mathfrak{l}, f) = O(\mathrm{Nm}(\mathfrak{l})^{k/2})$, for all but finitely many primes $\mathfrak{l}$ of F.

In order to construct a measure, we need to show that this is a bounded distribution (after possibly dividing by some periods). We now modify the distribution $\tilde{\mu}_s$ to construct the distribution

$$\mu_s(a + p^j\mathbb{Z}_p) = \tilde{\mu}_s(a + p^j\mathbb{Z}_p) + \tilde{\mu}_s(-a + p^j\mathbb{Z}_p).$$

The distribution relations for μ_s follows from those of $\tilde{\mu}_s$. Moreover

$$\int \chi \, d\mu_s = \begin{cases} 2\int \chi \, d\tilde{\mu}_s & \text{if } \chi \text{ is even,} \\ 0 & \text{if } \chi \text{ is odd.} \end{cases}$$

In the next section, we shall prove that the values of the distribution $\int \chi \, d\mu_s$ in (4), for specific values of s, are rational, after dividing by some periods. In Section 6, we conjecture that these values are even p-adically bounded. We then conjecture that these values satisfy the so called abstract Kummer congruences, and hence come from a measure.

3. Rationality result for twisted Asai L-values

Let $n = k - 2$ and set $\underline{n} = (n, n)$. For any $\mathcal{O}_F$-algebra A, let $L(\underline{n}, A)$ denote the set of polynomials in 4 variables $(X, Y, \overline{X}, \overline{Y})$ with coefficients in A, which are homogeneous of degree n in (X, Y) and homogeneous of degree n in $(\overline{X}, \overline{Y})$. We define an action of $\mathrm{SL}_2(A)$ on this set by

$$\gamma \cdot P(X, Y, \overline{X}, \overline{Y}) = P(\gamma^\iota(X, Y)^t, \bar{\gamma}^\iota(\overline{X}, \overline{Y})^t)$$

$$= P(dX - bY, -cX + aY, \bar{d}\,\overline{X} - \bar{b}\,\overline{Y}, -\bar{c}\overline{X} + \bar{a}\overline{Y}),$$

for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(A)$ and where $\gamma^\iota = \det(\gamma)\gamma^{-1}$ is the adjoint matrix of γ. Let $\Gamma_0(\mathfrak{n})$ and $\Gamma_1(\mathfrak{n})$ be the usual congruence subgroups of $\mathrm{SL}_2(\mathcal{O}_F)$ with respect to the ideal $\mathfrak{n}$. Let

$$\mathcal{H} = \{(z,t) \mid z \in \mathbb{C} \text{ and } t \in \mathbb{R} \text{ with } t > 0\}$$

be the hyperbolic upper half-space in $\mathbb{R}^3$. There is an action of $\mathrm{SL}_2(\mathcal{O}_F)$ on $\mathcal{H}$ which is induced by identifying $\mathcal{H}$ with $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/[\mathrm{SU}_2(\mathbb{C}) \cdot \mathbb{C}^\times]$. The last identification is given by a transitive action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{H}$ defined via $g \mapsto g \cdot \epsilon$, for $g \in \mathrm{SL}_2(\mathbb{C})$ and $\epsilon = (0,1) \in \mathcal{H}$. We will let $\mathcal{L}(\underline{n}, A)$ denote the system of local coefficients associated to $L(\underline{n}, A)$. So $\mathcal{L}(\underline{n}, A)$ is the sheaf of locally constant sections of the projection

$$\Gamma_0(\mathfrak{n})\backslash(\mathcal{H} \times L(\underline{n}, A)) \longrightarrow \Gamma_0(\mathfrak{n})\backslash\mathcal{H}.$$

Analogous to the Eichler-Shimura isomorphism for classical cusp forms, there are isomorphisms

$$\delta_q : S_{\underline{k}}(\Gamma_0(\mathfrak{n})) \longrightarrow H^q_{cusp}(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(\underline{n}, \mathbb{C})),$$

for $q = 1, 2$. Here the cohomology on the right is cuspidal cohomology with local coefficients. We take $\delta = \delta_1$ since we are interested in 1-forms.

Let us now describe the image of $\delta(f)$ under this map. Let $\gamma \in \mathrm{SL}_2(\mathbb{C})$ and $\underline{z} = (z,t) \in \mathcal{H}$. After identifying $\underline{z}$ with the matrix $\begin{bmatrix} z & -t \\ t & \bar{z} \end{bmatrix}$, recall that the action of γ on $\underline{z}$ is by

$$\gamma \cdot \underline{z} = [\rho(a)\underline{z} + \rho(b)][\rho(c)\underline{z} + \rho(d)]^{-1},$$

where $\rho(\alpha) = \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix}$. Define the automorphy factor $j(\gamma, \underline{z}) = \rho(c)\underline{z} + \rho(d) \in \mathrm{GL}_2(\mathbb{C})$. Let $L(2n+2, \mathbb{C})$ denote the space of homogeneous polynomials of degree $2n+2$ in two variables (S, T) and coefficients in $\mathbb{C}$. We will consider $L(2n+2, \mathbb{C})$ with a left action of $\mathrm{SL}_2(\mathbb{C})$.

Recall that f is a function $\mathcal{H} \to L(2n+2, \mathbb{C})$ that satisfies the transformation property

$$f(\gamma\underline{z}, (S,T)^t) = f(\underline{z}, {}^t j(\gamma, \underline{z})(S,T)^t),$$

for $\gamma \in \Gamma_0(\mathfrak{n})$. There is a related 'cusp form' $F : \mathrm{SL}_2(\mathbb{C}) \to L(2n+2, \mathbb{C})$ on $\mathrm{SL}_2(\mathbb{C})$ which is defined by the formula

$$f(\underline{z}, (S,T)^t) = F(g, {}^t j(g, \epsilon)(S,T)^t),$$

where $g \in \mathrm{SL}_2(\mathbb{C})$ is chosen such that $g \cdot \epsilon = \underline{z}$.

By Clebsch-Gordon, there is an $\mathrm{SU}_2(\mathbb{C})$-equivariant homomorphism

$$\Phi : L(2n+2, \mathbb{C}) \hookrightarrow L(\underline{n}, \mathbb{C}) \otimes L(2, \mathbb{C}).$$

Then $\delta(f)$ can explicitly be described as [Gha99, (13)]

$$(5) \qquad \qquad \delta(f)(g) = g \cdot (\Phi \circ F(g)), \quad \forall g \in \mathrm{SL}_2(\mathbb{C}).$$

Note that here the action of g on $L(\underline{n}, \mathbb{C})$ is as described above. But the action of $\mathrm{SL}_2(\mathbb{C})$ on $L(2, \mathbb{C})$ is identified with the natural action of $\mathrm{SL}_2(\mathbb{C})$ on $\Omega^1(\mathcal{H}) = \mathbb{C}dz \oplus \mathbb{C}dt \oplus \mathbb{C}d\bar{z}$ (see [Gha99, (6)]). The identification of $L(2, \mathbb{C})$ with $\Omega^1(\mathcal{H})$ is given by sending $A^2 \mapsto dz, AB \mapsto -dt$ and $B^2 \mapsto -d\bar{z}$. With this identification, we view $\delta(f)$ as a $L(\underline{n}, \mathbb{C})$-valued differential form. It is also invariant under the action of $\mathrm{SU}_2(\mathbb{C})$ [Gha99, (14)], so it descends to a vector valued 1-form on $\mathcal{H}$. Moreover, if $\gamma \in \Gamma_0(\mathfrak{n})$, then

$$\delta(f)(\gamma\underline{z}) = \gamma \cdot (\delta(f)(\underline{z})).$$

So $\delta(f)$ descends to an element of $H^1_{cusp}(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(\underline{n}, \mathbb{C}))$.

We make this formula more explicit. Let U and V be auxiliary variables and define the following homogeneous polynomial of degree $2n + 2$ by

$$Q = \left(\binom{2n+2}{\alpha} (-1)^{2n+2-\alpha} U^\alpha V^{2n+2-\alpha} \right)_{\alpha=0,\ldots,2n+2}.$$

Define $\psi(X, Y, \overline{X}, \overline{Y}, A, B) = [\psi_0(X, Y, \overline{X}, \overline{Y}, A, B), \ldots, \psi_{2n+2}(X, Y, \overline{X}, \overline{Y}, A, B)]^t$ by the formula

$$(XV - YU)^n (\overline{X}U + \overline{Y}V)^n (AV - BU)^2 = Q \cdot \psi,$$

where each ψ_i is a polynomial that is homogeneous of degree n in (X, Y), homogeneous of degree n in $(\overline{X}, \overline{Y})$ and homogeneous of degree 2 in (A, B).

Let $\underline{z} \in \mathcal{H}$, then since $\mathrm{SL}_2(\mathbb{C})$ acts transitively on $\mathcal{H}$, there is a $g \in \mathrm{SL}_2(\mathbb{C})$ such that $\underline{z} = g \cdot \epsilon$ where $\epsilon = (0, 1) \in \mathcal{H}$. Let $F^\alpha : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}$, for $\alpha = 0, \ldots, 2n + 2$, be the components of the function $F : \mathrm{SL}_2(\mathbb{C}) \to L(2n + 2, \mathbb{C})$. Then

$$(\Phi \circ F)(g) = [F^0(g), \ldots, F^{2n+2}(g)] \cdot \psi(X, Y, \overline{X}, \overline{Y}, A, B)$$

and

$$\delta(f)(\underline{z}) = g \cdot (\Phi \circ F(g)) = [F^0(g), \ldots, F^{2n+2}(g)] \cdot \psi(g^\iota(X, Y)^t, \bar{g}^\iota(\overline{X}, \overline{Y})^t, {}^t j(g^{-1}, \epsilon)^{-1}(A, B)^t),$$

where A^2, AB, B^2 are replaced by $dz, -dt, -d\bar{z}$.

Now, let $\beta \in F \subset \mathbb{C}$ and let T_β denote the translation map $\mathcal{H} \to \mathcal{H}$ given by sending $\underline{z} = (z, t) \mapsto (z + \beta, t)$. When we view $\mathcal{H}$ as a quotient space of $\mathrm{SL}_2(\mathbb{C})$, this map is induced by sending the coset $g\mathrm{SU}_2(\mathbb{C}) \mapsto \gamma_\beta g\mathrm{SU}_2(\mathbb{C})$, where

$$\gamma_\beta := \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}.$$

Let $\Gamma_0^\beta(\mathfrak{n}) := \gamma_\beta^{-1} \Gamma_0(\mathfrak{n}) \gamma_\beta \subset \mathrm{SL}_2(F)$. Notice that if $\Gamma_0^\beta(\mathfrak{n}) g' \mathrm{SU}_2(\mathbb{C}) = \Gamma_0^\beta(\mathfrak{n}) g \mathrm{SU}_2(\mathbb{C})$, then $\Gamma_0(\mathfrak{n}) \gamma_\beta g' \mathrm{SU}_2(\mathbb{C}) = \Gamma_0(\mathfrak{n}) \gamma_\beta g \mathrm{SU}_2(\mathbb{C})$. Thus the translation map T_β induces a well-defined map

$$\Gamma_0^\beta(\mathfrak{n}) \backslash \mathcal{H} \xrightarrow{T_\beta} \Gamma_0(\mathfrak{n}) \backslash \mathcal{H},$$

which we again denote by T_β.

We now recall some basic facts about functoriality of cohomology with local coefficients. For $i = 1, 2$, let X_i be topological spaces with universal covers $\tilde{X}_i$ and fundamental groups Γ_i (after fixing some base points). Let M_i be local coefficient systems on X_i, i.e., each M_i is an abelian group with an action of the fundamental group Γ_i. Let $\phi : X_1 \to X_2$ be a map between the spaces, it induces a map $\phi_* : \Gamma_1 \to \Gamma_2$ on the fundamental groups. A map between the coefficient systems $\tilde{\phi} : M_2 \to M_1$ is said to be compatible with ϕ if it satisfies

$$\gamma_1 \tilde{\phi}(m_2) = \tilde{\phi}(\phi_*(\gamma_1) m_2), \quad \forall m_2 \in M_2 \text{ and } \gamma_1 \in \Gamma_1.$$

In other words, $\tilde{\phi}$ must be a map between representations when M_2 is viewed as a representation of Γ_1 via the map ϕ_*. For any compatible pair $(\phi, \tilde{\phi})$, there exists an induced map

$$\phi^* : H^q(X_2, M_2) \to H^q(X_1, M_1)$$

at the level of cohomology. This map is constructed as follows. Let $S_*(\tilde{X}_i)$ denote the singular complex of the universal covers. There is a natural action on the right by Γ_i via deck transformations. Given a singular q-simplex $\sigma : \Delta^q \to \tilde{X}_i$ and $g \in \Gamma_i$, we convert this right action into a left action by setting $g \cdot \sigma = \sigma \cdot g^{-1}$. The cohomology groups with local coefficients are given by the homology of the complex $\mathrm{Hom}_{\mathbb{Z}\Gamma_i}(S_*(\tilde{X}_i), M_i)$. The map ϕ^* is induced by the following map on the complexes (which we again denote by ϕ^*)

$$\phi^* : \mathrm{Hom}_{\mathbb{Z}\Gamma_2}(S_q(\tilde{X}_2), M_2) \to \mathrm{Hom}_{\mathbb{Z}\Gamma_1}(S_q(\tilde{X}_1), M_1).$$

Given a cochain $C \in \mathrm{Hom}_{\mathbb{Z}\Gamma_2}(S_q(\tilde{X}_2), M_2)$ and $\tau : \Delta^q \to \tilde{X}_1 \in S_q(\tilde{X}_1)$, define

$$(6) \qquad \phi^*(C)(\tau) = \tilde{\phi}(C(\phi \circ \tau)),$$

where we continue to denote by $\phi : \tilde{X}_1 \to \tilde{X}_2$ the unique lift of $\phi : X_1 \to X_2$ to the universal covers. This construction is independent of the base points chosen in the beginning.

Let us now apply this to our situation with $X_1 = \Gamma_0^\beta(\mathfrak{n})\backslash\mathcal{H}$ and $X_2 = \Gamma_0(\mathfrak{n})\backslash\mathcal{H}$. In this case, $\Gamma_1 = \Gamma_0^\beta(\mathfrak{n})$ and $\Gamma_2 = \Gamma_0(\mathfrak{n})$ and they act on $M_1 = M_2 = L(n, \mathbb{C})$ since they are subgroups of $\mathrm{SL}_2(F)$. We take the map ϕ to be the translation map T_β which induces, at the level of fundamental groups, the map $(T_\beta)_* : \Gamma_0^\beta(\mathfrak{n}) \to \Gamma_0(\mathfrak{n})$ which sends $\gamma_\beta^{-1}\gamma\gamma_\beta \mapsto \gamma$. It is an easy check that the map $\tilde{T}_\beta : M_2 \to M_1$ sending $P \mapsto \gamma_\beta^{-1}P$, for $P \in L(n, \mathbb{C})$, is compatible with T_β. By the discussion above this induces a map at the level of cohomology

$$T_\beta^* : H^q(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(n, \mathbb{C})) \to H^q(\Gamma_0^\beta(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(n, \mathbb{C})).$$

When $q = 1$, what is the image of the element $\delta(f) \in H^1(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(n, \mathbb{C}))$? After translating the above map T_β^* in terms of vector valued differential forms, we get that

$$T_\beta^*(\delta(f))(\underline{z}) \overset{(6)}{=} \gamma_\beta^{-1}\delta(f)(\gamma_\beta \underline{z}) \overset{(5)}{=} \gamma_\beta^{-1}\gamma_\beta g \cdot (\Phi \circ F(\gamma_\beta g)) = g \cdot (\Phi \circ F(\gamma_\beta g))$$
$$= [F^0(\gamma_\beta g), \dots, F^{2n+2}(\gamma_\beta g)] \cdot \psi(g^\iota(X, Y)^t, \bar{g}^\iota(\overline{X}, \overline{Y})^t, {}^t j(g^{-1}, \epsilon)^{-1}(A, B)^t).$$

Here $\underline{z} \in \mathcal{H}$ and we take $g \in \mathrm{SL}_2(\mathbb{C})$ such that $g\underline{z} = \epsilon$, and A^2, AB, B^2 are to be replaced by $dz, -dt, -d\bar{z}$.

Following [Gha99, §5.2], we now want to compute the restriction $T_\beta^*(\delta(f))|_\mathbb{H}$ where $\mathbb{H} = \{x + it \mid x, t \in \mathbb{R} \text{ and } t > 0\}$ is the usual upper half-plane which is embedded into the hyperbolic 3-space $\mathcal{H}$ as

$$x + it \mapsto \begin{bmatrix} x & -t \\ t & x \end{bmatrix}.$$

As in *loc. cit.*, we make the following two simplifications. Firstly, since we wish to compute this differential form on $\mathbb{H}$, we set $dz = d\bar{z}$ in our computations. Secondly, we only need to calculate the differential form $\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \cdot (T_\beta^*(\delta(f))|_\mathbb{H})$, so we set $x = 0$ in ψ and only calculate the modified differential form which we denote by $\widetilde{T_\beta^*(\delta(f))}|_\mathbb{H}$. Note that the components ψ_α of ψ, for $\alpha = 0, \dots, 2n + 2$, are given by

$$\psi_\alpha(X, Y, \overline{X}, \overline{Y}, A, B) = (-1)^\alpha \frac{A^2 c_\alpha - 2AB c_{\alpha-1} + B^2 c_{\alpha-2}}{\binom{2n+2}{\alpha}},$$

where

$$c_\alpha(X, Y, \overline{X}, \overline{Y}) = \sum_{\substack{j,k=0 \\ n=\alpha+j-k}}^n (-1)^k \binom{n}{j}\binom{n}{k} X^{n-k} Y^k \overline{X}^{n-j} \overline{Y}^j.$$

For $x, t \in \mathbb{R}$ and $t > 0$, let $g = \frac{1}{\sqrt{t}}\begin{bmatrix} t & x \\ 0 & 1 \end{bmatrix}$. Then $g \cdot \epsilon = (x, t) \in \mathbb{H} \subset \mathcal{H}$ and $g^\iota = \bar{g}^\iota = \frac{1}{\sqrt{t}}\begin{bmatrix} 1 & -x \\ 0 & t \end{bmatrix}$. Moreover, $j(g^{-1}, \epsilon)^{-1} = j(g, \epsilon) = \frac{1}{\sqrt{t}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let f^α, for $\alpha = 0, \dots, 2n + 2$, be the components of $f : \mathcal{H} \to L(2n + 2, \mathbb{C})$. The precise relationship between f^α and F^α is given by

$$f^\alpha(\underline{z}) = \frac{1}{\sqrt{t}^{2n+2}} F^\alpha(g).$$

Note that if $\underline{z} = (z, t)$, then $T_\beta(\underline{z}) = (z + \beta, t)$ does not affect the t coordinate. Hence

$$f^\alpha(T_\beta \underline{z}) = \frac{1}{\sqrt{t}^{2n+2}} F^\alpha(\gamma_\beta g).$$

Using this and the pullback formula, we get

$$T_\beta^*\widetilde{(\delta(f))}|_{\mathbb{H}} = \sum_\alpha^{2n+2} \sqrt{t}^{2n+2} f^\alpha(T_\beta \underline{z}) \psi_\alpha \left(\frac{1}{\sqrt{t}} X, \sqrt{t}\, Y, \frac{1}{\sqrt{t}} \overline{X}, \sqrt{t}\, \overline{Y}, \frac{1}{\sqrt{t}} A, \frac{1}{\sqrt{t}} B \right)$$

$$= \sum_\alpha^{2n+2} f^\alpha(T_\beta \underline{z}) \psi_\alpha(X, tY, \overline{X}, t\overline{Y}, A, B),$$

where we replace (A^2, AB, B^2) by $(dx, -dt, -dx)$. We have now constructed an element $T_\beta^*\widetilde{(\delta(f))}|_{\mathbb{H}} \in H^1_{cusp}(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(\underline{n}, \mathbb{C}))$ where $\Gamma_0^\beta(N) := \Gamma_0^\beta(\mathfrak{n}) \cap \Gamma_0(N) = \Gamma_0^\beta(\mathfrak{n}) \cap \mathrm{SL}_2(\mathbb{Z})$, since in the latter matrix group, the lower left entries are divisible by N. As in [Gha99, see below Lemma 2], we have a decomposition of this cohomology group as

$$(7) \qquad H^1_{cusp}(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(\underline{n}, \mathbb{C})) \xrightarrow{\sim} \bigoplus_{m=0}^{n} H^1_{cusp}(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(2n - 2m, \mathbb{C})).$$

We will call the projection of $T_\beta^*\widetilde{(\delta(f))}|_{\mathbb{H}}$ into the m-th component by $T_\beta^*\widetilde{(\delta_{2n-2m}(f))}$, slightly abusing notation since the subscript $2n - 2m$ should technically be outside the parentheses. For each m, define

$$(8) \qquad g^\alpha(\underline{z}) = \begin{cases} \frac{f^\alpha(\underline{z}) + (-1)^{n+1-\alpha+m} f^{2n+2-\alpha}(\underline{z})}{\binom{2n+2}{\alpha}} & \text{if } \alpha = 0, 1, \ldots, n, \\ \frac{f^{n+1}(\underline{z})}{\binom{2n+2}{n+1}} & \text{if } \alpha = n + 1. \end{cases}$$

Then, we have

$$(9) \qquad T_\beta^*\widetilde{(\delta_{2n-2m}(f))}(x, t) = \sum_{l=0}^{2n-2m} (A_l dx + 2B_l dt) t^{2n-m-l} X^l Y^{2n-2m-l},$$

where

$$A_l = \sum_{\alpha=0}^{n+1} (-1)^\alpha g^\alpha(T_\beta(x, t)) a(m, l, \alpha),$$

$$B_l = \sum_{\alpha=0}^{n+1} (-1)^\alpha g^\alpha(T_\beta(x, t)) b(m, l, \alpha),$$

with $a(m, l, \alpha)$ and $b(m, l, \alpha)$ the integers defined at the end of [Gha99, §5].

For any $n \geq 0$ and any $\mathbb{Z}[1/n!]$-algebra A, there is an $\mathrm{SL}_2(\mathbb{Z})$-equivariant pairing [Gha99, Lemma 4]

$$\langle\,,\,\rangle : L(n, A) \otimes L(n, A) \to A,$$

which induces by Poincare duality a pairing

$$\langle\,,\,\rangle : H^1_c(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(n, A)) \otimes H^1(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(n, A)) \to H^2_c(\Gamma_0^\beta(N)\backslash\mathbb{H}, A) \to A,$$

where the last map $H^2_c(\Gamma_0^\beta(N)\backslash\mathbb{H}, A) \to A$ is given by integrating a compactly supported 2-form on a fundamental domain $[\Gamma_0^\beta(N)\backslash\mathbb{H}]$ of $\Gamma_0^\beta(N)\backslash\mathbb{H}$. We will use this pairing when $A = \mathbb{C}$, $A = E$ is a p-adic number field with $p > n$, and with $A = \mathcal{O}_E$, its ring of integers. When $A = \mathbb{C}$, the pairing can be extended to

$$\langle\,,\,\rangle : H^1_{cusp}(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(n, \mathbb{C})) \otimes H^1(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(n, \mathbb{C})) \to H^2_{cusp}(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathbb{C}) \to \mathbb{C} \cup \{\infty\}.$$

For each m, there is an Eisenstein differential form $E^\beta_{2n-2m+2}$ for $\Gamma^\beta_0(N)$ given by

$$(10) \qquad E^\beta_{2n-2m+2}(s,z) = \sum_{\gamma \in \Gamma^\beta_0(N)_\infty \backslash \Gamma^\beta_0(N)} \gamma^{-1} \cdot \gamma^*(\omega\, y^s),$$

where $\omega = (X - zY)^{2n-2m}dz$. One may check that

$$(11) \quad E^\beta_{2n-2m+2}(s,z) = \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^\beta_0(N)_\infty \backslash \Gamma^\beta_0(N)} \frac{1}{(cz+d)^{2n-2m+2}|cz+d|^{2s}} \cdot y^s\,\omega.$$

We view $E^\beta_{2n-2m+2}$ as an element of $H^1(\Gamma^\beta_0(N)\backslash\mathbb{H}, \mathcal{L}(n,\mathbb{C}))$. We now wish to evaluate

$$\langle T^*_\beta(\delta_{2n-2m}(f)), E^\beta_{2n-2m+2}\rangle,$$

following [Gha99, §6.3]. We have

$$\langle T^*_\beta(\delta_{2n-2m}(f)), E^\beta_{2n-2m+2}\rangle = \int_{[\Gamma^\beta_0(N)\backslash\mathbb{H}]} \langle T^*_\beta(\delta_{2n-2m}(f))(x,t), E^\beta_{2n-2m+2}(x,t)\rangle$$

$$= \int_{[\Gamma^\beta_0(N)\backslash\mathbb{H}]} \langle T^*_\beta(\widetilde{\delta_{2n-2m}}(f)), \widetilde{E^\beta_{2n-2m+2}}\rangle,$$

where the $\sim$ indicates that we have twisted the differential forms by the action of the matrix $\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}$. Using a standard unwinding argument, the last integral becomes

$$\int_0^\infty \int_0^1 \langle T^*_\beta(\widetilde{\delta_{2n-2m}}(f)), \widetilde{\omega} t^s\rangle,$$

where $\widetilde{\omega} = (X - itY)^{2n-2m}dz$. Using the expression (9) for $T^*_\beta(\widetilde{\delta_{2n-2m}}(f))(x,t)$ and the definition of the pairing, we have

$$\int_0^\infty \int_0^1 \langle T^*_\beta(\widetilde{\delta_{2n-2m}}(f)), \widetilde{\omega} t^s\rangle = \int_0^\infty \int_0^1 \sum_{l=0}^{2n-2m} i^{l+1} A_l t^{2n-m+s}\, dxdt$$

$$- 2\int_0^\infty \int_0^1 \sum_{l=0}^{2n-2m} i^l B_l t^{2n-m+s}\, dxdt.$$

We denote the first integral by I_1 and the second integral by I_2. We now compute I_1 using the definition of A_l as

$$I_1 = \sum_{l=0}^{2n-2m} i^{l+1} \sum_{\alpha=0}^{n+1} (-1)^\alpha a(m,l,\alpha) \int_0^\infty \int_0^1 g^\alpha(T_\beta(x,t)) t^{2n-m+s}\, dxdt.$$

Using the Fourier expansion for the α-th component of f, see [Gha99, (7)] with $a_1 = 1$, we get

$$f^\alpha(T_\beta(x,t)) = t\binom{2n+2}{\alpha} \left[\sum_{\xi \in F^\times} c(\xi d) \left(\frac{\xi}{i|\xi|}\right)^{n+1-\alpha} K_{\alpha-n-1}(4\pi t|\xi|) e_F(\xi(x+\beta)) \right],$$

where $e_F(w) = e^{2\pi i \mathrm{Tr}_{F/\mathbb{Q}}(w)}$. Using (8) and plugging this into the expression for I_1, we get

$$I_1 = \sum_{l=0}^{2n-2m} i^{l+1} \sum_{\alpha=0}^{n} (-1)^\alpha a(m, l, \alpha) \int_0^\infty \sum_{\xi \in F^\times} c(\xi d) t^{2n-m+1+s}$$

$$\left(\left(\frac{\xi}{i|\xi|} \right)^{n+1-\alpha} K_{\alpha-n-1}(4\pi t|\xi|) + (-1)^{n+m+1-\alpha} \left(\frac{\xi}{i|\xi|} \right)^{\alpha-n-1} K_{n+1-\alpha}(4\pi t|\xi|) \right) dt$$

$$\int_0^1 e_F(\xi(x+\beta))dx + \sum_{l=0}^{2n-2m} i^{l+1}(-1)^{n+1} a(m, l, n+1) \int_0^\infty \sum_{\xi \in F^\times} c(\xi d) t^{2n+1-m+s}$$

$$K_0(4\pi t|\xi|)dt \int_0^1 e_F(\xi(x+\beta))dx.$$

The only terms $c(\xi d)$ that survive are when $\xi = \frac{r}{\sqrt{-D}}$, for some $0 \neq r \in \mathbb{Z}$, and in this case $\int_0^1 e_F(\xi x)dx = 1$.

$$I_1 = \sum_{l=0}^{2n-2m} i^{l+1} \sum_{\alpha=0}^{n} (-1)^\alpha a(m, l, \alpha) \sum_{r \neq 0} e_F(r\beta/\sqrt{-D})c(r) \left(\frac{-r}{|r|} \right)^{n+1-\alpha}$$

$$\int_0^\infty t^{2n+1-m+s} \left[K_{\alpha-n-1} \left(\frac{4\pi t|r|}{\sqrt{D}} \right) + (-1)^{n+m+1-\alpha} K_{n+1-\alpha} \left(\frac{4\pi t|r|}{\sqrt{D}} \right) \right] dt$$

$$+ \sum_{l=0}^{2n-2m} i^{l+1}(-1)^{n+1} a(m, l, n+1) \sum_{r \neq 0} e_F(r\beta/\sqrt{-D})c(r) \int_0^\infty t^{2n+1-m+s} K_0 \left(\frac{4\pi t|r|}{\sqrt{D}} \right) dt.$$

The Bessel functions have the property [Gha99, Lemma 7]

$$\int_0^\infty K_\nu(at)t^{\mu-1}dt = 2^{\mu-2}a^{-\mu}\Gamma\left(\frac{\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right).$$

This implies that the two Bessel functions in the sum above will cancel each other unless $\alpha \equiv n+1+m \mod (2)$. Setting $s' = 2n+2-m+s$, we have

$$I_1 = \frac{(-1)^{n+1}\sqrt{D}^{s'}}{2(2\pi)^{2n+2-m+s}} \sum_{l=0}^{2n-2m} i^{l+1} \sum_{\substack{\alpha=0 \\ \alpha \equiv n+1+m\ (2)}}^{n} (-1)^m a(m, l, \alpha) \sum_{0 \neq r \in \mathbb{Z}} e_F(r\beta/\sqrt{-D})c(r)$$

$$\left(\frac{-r}{|r|} \right)^{n+1-\alpha} \frac{1}{|r|^{s'}} \Gamma\left(\frac{n+1-m+\alpha+s}{2} \right) \Gamma\left(\frac{3n+3-m-\alpha+s}{2} \right)$$

$$+ \frac{(-1)^{n+1}\sqrt{D}^{s'}}{4(2\pi)^{2n+2-m+s}} \sum_{l=0}^{2n-2m} i^{l+1} a(m, l, n+1) \sum_{0 \neq r \in \mathbb{Z}} e_F(r\beta/\sqrt{-D})c(r) \frac{1}{|r|^{s'}}$$

$$\Gamma\left(\frac{2n+2-m+s}{2} \right)^2.$$

We will take $\beta = \frac{b\sqrt{-D}}{2}$ for some rational number b. Then the term $e_F(r\beta/\sqrt{-D}) = e^{2\pi irb}$.

Now we break the sum over r into a sum over positive integers and a sum over negative integers. The term $\left(\frac{-r}{|r|} \right)^{n+1-\alpha}$ equals $(-1)^m$ when r is positive and is 1 when r is negative. The second sum over r does not have such a term, so we assume that m is even in order to be able to put these terms together into a single term. The terms $c(r)$ and $|r|^{s'}$ are

obviously independent of the sign of r. So finally, we have

$$
(12) \quad
\begin{aligned}
I_1 = {}& \frac{(-1)^{n+1}\sqrt{D}^{s'}}{2(2\pi)^{2n+2-m+s}} \sum_{r=1}^{\infty} \left(e^{2\pi irb} + e^{-2\pi irb}\right) \frac{c(r)}{r^{s'}} \sum_{l=0}^{2n-2m} i^{l+1} \\
& \sum_{\substack{\alpha=0 \\ \alpha \equiv n+1+m \ (2)}}^{n+1} a(m,l,\alpha)\Gamma\left(\frac{n+1-m+\alpha+s}{2}\right)\Gamma\left(\frac{3n+3-m-\alpha+s}{2}\right),
\end{aligned}
$$

where there is an extra factor of $\frac{1}{2}$ in the $\alpha = n+1$ term, which we will adjust for. By a similar computation, I_2 will also have an expression in terms of $b(m,l,\alpha)$. Putting together these two expressions, we get that

$$
\langle T_\beta^*(\delta_{2n-2m}(f)), E_{2n-2m+2}^\beta(s)\rangle = \frac{\sqrt{D}^{s'}}{(2\pi)^{2n+2-m+s}} \sum_{r=1}^{\infty} \left(e^{2\pi irb} + e^{-2\pi irb}\right) \frac{c(r)}{r^{s'}} \cdot G_\infty'(s,f),
$$

where we collect all the combinations of Gamma factors appearing in both I_1 and I_2 and denote it by $G_\infty'(s,f)$.

Now let $\chi : (\mathbb{Z}/p^j\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive character (so $j = j_\chi$) and recall that

$$
G(s, \bar{\chi}, f) = L_N(2s - 2k + 2, \bar{\chi}^2) \sum_{r=1}^{\infty} \frac{c(r)\bar{\chi}(r)}{r^s}.
$$

Substituting the formula

$$
\bar{\chi}(r) = \frac{1}{G(\chi)} \sum_{a \mod p^j} \chi(a)e^{2\pi ira/p^j}
$$

in the above equation, we get

$$
G(s', \bar{\chi}, f) = L_N(2n - 2m + 2 + 2s, \bar{\chi}^2)\frac{1}{G(\chi)} \sum_{a \mod p^j} \chi(a) \sum_{r=1}^{\infty} \frac{c(r)}{r^{s'}}e^{2\pi ira/p^j}.
$$

Now assume that χ is an even character, i.e., $\chi(-1) = 1$. Then grouping together the terms coming from a and $-a$, we get

$$
G(s', \bar{\chi}, f) = L_N(2n - 2m + 2 + 2s, \bar{\chi}^2)\frac{1}{G(\chi)} \sum_{a \in R} \chi(a) \sum_{r=1}^{\infty} \frac{c(r)}{r^{s'}}(e^{2\pi ira/p^j} + e^{-2\pi ira/p^j}),
$$

where R is half of the representatives modulo p^j such that if $a \in R$, then $-a \notin R$. We now write $G(s', \chi, f)$ in terms of the inner product considered earlier

$$
(13) \quad
\begin{aligned}
G(\chi)G(s', \bar{\chi}, f) = {}& \frac{(2\pi)^{2n+2-m+s}}{G_\infty'(s,f)\sqrt{D}^{s'}} L_N(2n - 2m + 2 + 2s, \bar{\chi}^2) \\
& \sum_{a \in R} \chi(a)\langle T_\beta^*(\delta_{2n-2m}(f)), E_{2n-2m+2}^\beta(s)\rangle,
\end{aligned}
$$

with $\beta = a\sqrt{-D}/2p^j$.

Let $G_\infty(s,f) = G_\infty'(s,f)\Gamma(s+2n-2m+2)$. Dividing both sides of the above equation by the period $G(\bar{\chi}^2)(2\pi)^{2n-2m+2}$, we obtain

$$
\begin{aligned}
\frac{G(\chi)G(s', \bar{\chi}, f)}{G(\bar{\chi}^2)(2\pi)^{2n-2m+2}} = {}& \frac{(2\pi)^{2n+2-m+s}}{G_\infty(s,f)\sqrt{D}^{s'}} \cdot \frac{L_N(2n-2m+2+2s, \bar{\chi}^2)}{(2\pi)^{2n-2m+2}G(\bar{\chi}^2)} \cdot \Gamma(s+2n-2m+2) \\
& \cdot \sum_{a \in R} \chi(a)\langle T_\beta^*(\delta_{2n-2m}(f)), E_{2n-2m+2}^\beta(s)\rangle.
\end{aligned}
$$

We evaluate this expression at $s = 0$. Note that $G_\infty(0, f) \neq 0$, by [Gha99, §6.4] using some special arguments, and by Lanphier-Skogman and Ochiai [LSO14] as a consequence of their proof of [Gha99, Conjecture 1]. The special value $L_N(2n - 2m + 2, \bar{\chi}^2)$ becomes rational after dividing by the period $G(\bar{\chi}^2)(2\pi)^{2n-2m+2}$. We denote this ratio by $L^\circ(2n - 2m + 2, \bar{\chi}^2)$. We get

(14)
$$\frac{G(\chi)G(2n - m + 2, \bar{\chi}, f)}{G(\bar{\chi}^2)\Omega_\infty} = L^\circ(2n-2m+2, \bar{\chi}^2) \sum_{a \in R} \chi(a) \langle T_\beta^*(\delta_{2n-2m}(f)), E_{2n-2m+2}^\beta(0) \rangle.$$

Here Ω_∞ is defined as

$$\Omega_\infty = \frac{(2\pi)^{4n-3m+4}\Gamma(2n - 2m + 2)}{G_\infty(0, f)\sqrt{D}^{2n-m+2}}.$$

We now conclude rationality properties of the special values $G(2n - 2m + 2, \bar{\chi}, f)$ from equation (14). Choose a period $\Omega(f)$ such that after dividing by this period, the differential form

$$\delta^\circ(f) := \frac{\delta(f)}{\Omega(f)} \in H^1_{cusp}(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(\underline{n}, E))$$

takes rational values. Here E is a sufficiently large p-adic field, containing the field of rationality of the form f, which we also view as a subfield of $\mathbb{C}$ after fixing an isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. Then

$$T_\beta^*\delta(f)|_\mathbb{H} = \Omega(f) \cdot T_\beta^*\delta^\circ(f)|_\mathbb{H},$$

noting that if $\sqrt{-D} \in E$, which we assume, then the image $T_\beta^*\delta^\circ(f)|_\mathbb{H}$ of $\delta^\circ(f)$ under the map

$$T_\beta^*|_\mathbb{H} : H^1_{cusp}(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(\underline{n}, E)) \to H^1_{cusp}(\Gamma_0^\beta(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(\underline{n}, E)) \to H^1_{cusp}(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(\underline{n}, E)),$$

is also rational. Since Clebsch-Gordan preserves rationality, for $0 \leq m \leq n$, we obtain that

$$T_\beta^*(\delta_{2n-2m}(f)) = \Omega(f) \cdot T_\beta^*(\delta_{2n-2m}^\circ(f)),$$

where $T_\beta^*(\delta_{2n-2m}^\circ(f)) \in H^1_{cusp}(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(2n - 2m, E))$ is also rational.

The rational cuspidal class $T_\beta^*(\delta_{2n-2m}^\circ(f))$ is cohomologous to a compactly supported rational class which has the same value when paired with $E_{2n-2m+2}^\beta(0)$ (see the proof of [Gha99, Theorem 1]). Since the differential form $E_{2n-2m+2}^\beta(0)$ coming from the Eisenstein series is E-rational, at least when $m \neq n$ (see Proposition 1 in Section 4 below), and the pairing between compactly supported rational classes and such classes preserves E-rationality, the following theorem follows from (14), if E contains the field of rationality of χ, which we again assume.

Theorem 1 (Rationality result for twisted Asai L-values)**.** *Let E be a sufficiently large p-adic number field with $p \nmid 2ND$. Let $0 \leq m < n$ be even and χ be even. Then*

$$\frac{G(\chi)G(2n - m + 2, \bar{\chi}, f)}{G(\bar{\chi}^2)\Omega(f)\Omega_\infty} \in E.$$

This result matches with [Gha99, Theorem 1] when χ is trivial. In that theorem it was assumed that the finite part of the central character of f is non-trivial primarily to deal with the rationality of the Eisenstein series when $m = n$. In this paper, we have assumed (for simplicity) that the finite part of the central character of f is trivial. We could still probably include the case $m = n$ in the theorem above, by using the rationality of the Eisenstein series $E_2^\beta(0, z) - pE_2^\beta(0, pz)$ instead (see [Gha99, Remark 2]).

4. Rationality of Eisenstein cohomology classes

We start by recalling the following result that goes back to Harder [Har81], [Har87]. See also [Hid94, §10].

Lemma 1. *Eisenstein cohomology classes corresponding to Eisenstein series whose constant terms at every cusp are rational are rational cohomology classes.*

Proof. We use notation in this proof that is independent of the rest of the paper. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and $\mathcal{L}(n, \mathbb{C})$ denote the sheaf of locally constant sections of $\pi : \Gamma \backslash (\mathbb{H} \times L(n, \mathbb{C})) \to \Gamma \backslash \mathbb{H}$. Consider the restriction map to boundary cohomology given by

$$R : H^1(\Gamma \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C})) \to H^1_{\partial}(\Gamma \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C})) := \bigoplus_{\xi} H^1(\Gamma_{\xi} \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C})),$$

where ξ varies through the cusps of Γ. We know that

$$H^1(\Gamma \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C})) = H^1_{cusp}(\Gamma \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C})) \oplus H^1_{Eis}(\Gamma \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C})),$$

where H^1_{cusp} and H^1_{Eis} are the cuspidal and Eisenstein part of cohomology respectively. The restriction of R to H^1_{Eis} is an isomorphism

$$R : H^1_{Eis}(\Gamma \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C})) \to \bigoplus_{\xi} H^1(\Gamma_{\xi} \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C}))$$

$$\omega \mapsto c_{\xi}(0, \omega),$$

where $c_{\xi}(0, \omega)$ is the differential form corresponding to the "constant term" in the Fourier expansion at the cusp ξ of the differential form ω corresponding to the underlying Eisenstein series. Clearly R preserves the rational structures on both sides. The following fact is due to Harder.

Fact: There exists a section

$$M : \oplus_{\xi} H^1(\Gamma_{\xi} \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C})) \to H^1(\Gamma \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C}))$$

of R preserving rational structures on both sides.

Now let $\omega \in H^1_{Eis}(\Gamma \backslash \mathbb{H}, \mathcal{L}(n, \mathbb{C}))$ be such that $R(\omega)$ is rational. Then $M(R(\omega))$ is rational and $R(M(R(\omega))) = R(\omega)$. Since R is an isomorphism we have $M(R(\omega)) = \omega$. Hence, ω is rational. Thus the Eisenstein class ω is rational if and only if the constant term in the Fourier expansion at every cusp is rational. This proves the lemma. $\square$

Proposition 1. *If $m \neq n$, then the Eisenstein differential form*

$$E^{\beta}_{2n-2m+2}(0) \in H^1(\Gamma^{\beta}_0(N) \backslash \mathbb{H}, \mathcal{L}(2n - 2m, E))$$

is rational, for a sufficiently large p-adic number field E.

Proof. Recall that $\beta = \frac{a\sqrt{-D}}{2p^j}$ if $j \geq 1$ (and $\beta = 0$ if $j = 0$). We claim that $\Gamma^{\beta}_0(N) = \gamma^{-1}_{\beta} \Gamma_0(\mathfrak{N}) \gamma_{\beta} \cap \mathrm{SL}_2(\mathbb{Z})$ is independent of a. We do this by showing that

$$(15) \qquad \Gamma^{\beta}_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \ : \ a \equiv d \mod p^j, \ c \equiv 0 \mod Np^{2j} \right\}.$$

Indeed, if $j = 0$, (15) holds trivially, since in this case $\gamma_{\beta} = 1$, so both sides of (15) are equal to $\Gamma_0(N)$. So assume that $j \geq 1$. Since p is odd and we are considering representatives $a \in R = (\mathbb{Z}/p^j\mathbb{Z})^{\times}/\{\pm 1\}$, by replacing a by $p^j - a$ if necessary, we may

assume that all $a \in R$ are even, so that $p^j \sqrt{-D}^{-1} \beta \in \mathbb{Z}$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{N})$ and $\gamma_\beta = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$(16) \qquad \gamma_\beta^{-1} \gamma \gamma_\beta = \begin{pmatrix} a - c\beta & b - d\beta + (a - c\beta)\beta \\ c & d + c\beta \end{pmatrix}.$$

Assume that the matrix in (16) is in $\mathrm{SL}_2(\mathbb{Z})$, so is in $\Gamma_0^\beta(N)$. Then $a - c\beta$, $b - d\beta + (a - c\beta)\beta$, c, $d + c\beta \in \mathbb{Z}$. Note that $c \in \mathbb{Z} \Leftrightarrow c \in N\mathbb{Z}$. Since $a - c\beta$ and $d + c\beta \in \mathbb{Z}$, we have

$$\begin{aligned}
b - d\beta + (a - c\beta)\beta \in \mathbb{Z} &\Leftrightarrow (b + c\beta^2) + (a - c\beta - d - c\beta)\beta \in \mathbb{Z} \\
&\Leftrightarrow \Re(b) + c\beta^2 \in \mathbb{Z} \text{ and } \Im(b) = i(a - c\beta - d - c\beta)\beta \\
&\Leftrightarrow p^{2j} | c \text{ and } a - c\beta \equiv d + c\beta \mod p^j,
\end{aligned}$$

since $p \nmid 2D$ and both $\Re(b)$, $\sqrt{D}^{-1} \Im(b) \in \frac{1}{2}\mathbb{Z}$. Therefore $\Gamma_0^\beta(N)$ is contained in the right hand side of (15). On the other hand, if γ is any matrix on the right hand side of (15), then by replacing β by $-\beta$ in (16), one checks that $\gamma_\beta \gamma \gamma_\beta^{-1} \in \Gamma_0(\mathfrak{N})$. It follows that equality holds in (15).

From (15), we have

$$\mathrm{SL}_2(\mathbb{Z})_\infty = \left\{ \left(\begin{smallmatrix} \pm 1 & n \\ 0 & \pm 1 \end{smallmatrix}\right) : n \in \mathbb{Z} \right\} \subset \Gamma_0^\beta(N).$$

Thus $\Gamma_0^\beta(N)_\infty = \mathrm{SL}_2(\mathbb{Z})_\infty$. Also, note that the coset $\Gamma_0^\beta(N)_\infty \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\Gamma_0^\beta(N)_\infty \backslash \Gamma_0^\beta(N)$ contains all the matrices of $\Gamma_0^\beta(N)$ whose bottom row equals $\pm(c, d)$. Hence $\Gamma_0^\beta(N)_\infty \backslash \Gamma_0^\beta(N)$ is in bijection with the set

$$\Lambda := \{(c, d) \in \mathbb{Z}^2 \smallsetminus \{(0, 0)\} : (c, d) = 1, c \equiv 0 \mod Np^{2j}, d \equiv \pm 1 \mod p^j\} / \{\pm 1\}.$$

For each integer $k \geq 3$ and $(u, v) \in (\mathbb{Z}/Np^{2j})^2$, consider the Eisenstein series

$$E_k^{(u,v)}(z) := \sum_{\substack{(c,d) \equiv (u,v) \mod Np^{2j} \\ (c,d) = 1}} \frac{1}{(cz + d)^k}.$$

This Eisenstein series differs from the Eisenstein series in [DS05, (4.4)] by a factor of $\epsilon_{Np^{2j}} = \frac{1}{2}$ or 1. By (11), we have

$$\begin{aligned}
(17) \qquad E_{2n-2m+2}^\beta(0, z) &= \sum_{(c,d) \in \Lambda} \frac{1}{(cz + d)^{2n-2m+2}} \cdot \omega \\
&= \frac{1}{2} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \mod Np^{2j} \\ v \equiv \pm 1 \mod p^j}} E_{2n-2m+2}^{(u,v)}(z) \cdot \omega,
\end{aligned}$$

noting that $2n - 2m + 2 \geq 4$, since $m \neq n$. By [DS05, (4.6)], for $k \geq 3$, we have

$$(18) \qquad E_k^{(u,v)}(z) = \sum_{l \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^\times} \zeta_+^l(k, \mu) G_k^{l^{-1}(u,v)}(z),$$

where

$$\zeta_+^l(k, \mu) := \sum_{\substack{m=1 \\ m \equiv l \mod Np^{2j}}}^{\infty} \frac{\mu(m)}{m^k},$$

$\mu(\cdot)$ is the Möbius function, and

$$G_k^{(u,v)}(z) := \sideset{}{'}\sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\equiv(u,v)\ \mathrm{mod}\ Np^{2j}}} \frac{1}{(cz+d)^{-k}}.$$

We will obtain the q-expansion of the Eisenstein series $E_{2n-2m+2}^{\beta}(0,z) = \sum_{n=0}^{\infty} a_n q^n$, using (17), (18) and the q-expansion of the Eisenstein series above using facts from [DS05].

Let $k \geq 2$ an integer and φ be the Euler totient function. For $v \in (\mathbb{Z}/Np^{2j})^{\times}$, we have

$$
\begin{aligned}
\zeta^v(k) &:= \sideset{}{'}\sum_{d\equiv v\ \mathrm{mod}\ Np^{2j}} d^{-k} \\
&= \sum_{\substack{d=1 \\ d\equiv v\ \mathrm{mod}\ Np^{2j}}}^{\infty} d^{-k} + (-1)^k \sum_{\substack{d=1 \\ d\equiv -v\ \mathrm{mod}\ Np^{2j}}}^{\infty} d^{-k} \\
&= \frac{1}{\varphi(Np^{2j})}\left(\sum_{\psi\ \mathrm{mod}\ Np^{2j}} \psi(v)^{-1}L(k,\psi) + (-1)^k \sum_{\psi\ \mathrm{mod}\ Np^{2j}} \psi(-v)^{-1}L(k,\psi) \right) \\
&= \frac{1}{\varphi(Np^{2j})}\left(\sum_{\psi\ \mathrm{mod}\ Np^{2j}} (1 + (-1)^k\psi(-1))\psi(v)^{-1}L(k,\psi) \right),
\end{aligned}
$$
(19)

where the penultimate step follows from [DS05, Page 122]. If k is even, then $(1 + (-1)^k\psi(-1))$ is equal to 2 (resp. 0) if ψ is even (resp. odd). A similar expression for $\zeta_+^l(k,\mu)$ in terms of Dirichlet L-functions can also be derived. For $l \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^{\times}$, by the orthogonality relations, we have

$$
\begin{aligned}
\zeta_+^l(k,\mu) &= \sum_{m=1}^{\infty} \frac{1}{\varphi(Np^{2j})} \sum_{\psi\ \mathrm{mod}\ Np^{2j}} \psi(l)^{-1}\psi(m)\ \mu(m)m^{-k} \\
&= \frac{1}{\varphi(Np^{2j})} \sum_{\psi\ \mathrm{mod}\ Np^{2j}} \psi(l)^{-1} \sum_{m=1}^{\infty} \psi(m)\ \mu(m)m^{-k} \\
&= \frac{1}{\varphi(Np^{2j})} \sum_{\psi\ \mathrm{mod}\ Np^{2j}} \psi(l)^{-1}L(k,\psi)^{-1},
\end{aligned}
$$
(20)

where the last step follows from by multiplying the corresponding L-functions. Therefore

$$
\begin{aligned}
\zeta_+^l(k,\mu) + \zeta_+^{-l}(k,\mu) &= \frac{1}{\varphi(Np^{2j})} \sum_{\psi\ \mathrm{mod}\ Np^{2j}} (\psi(l)^{-1} + \psi(-l)^{-1})L(k,\psi)^{-1} \\
&= \frac{2}{\varphi(Np^{2j})} \sum_{\substack{\psi\ \mathrm{even} \\ \psi\ \mathrm{mod}\ Np^{2j}}} \psi(l)^{-1}L(k,\psi)^{-1}.
\end{aligned}
$$
(21)

By (17) and (18), we have

$$
(22)\quad E_{2n-2m+2}^{\beta}(0,z) = \frac{1}{2} \sum_{\substack{(u,v)\in(\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u\equiv 0\ \mathrm{mod}\ Np^{2j} \\ v\equiv \pm 1\ \mathrm{mod}\ p^j}} \sum_{l\in(\mathbb{Z}/Np^{2j}\mathbb{Z})^{\times}} \zeta_+^l(2n-2m+2,\mu)G_{2n-2m+2}^{l^{-1}(u,v)}(z)\cdot\omega.
$$

For simplicity, let $k = 2n - 2m + 2$. By the description of the set Λ, we have $(c,d) \in \Lambda$ implies that $Np^{2j} \mid c$ and $(c,d) = 1$, so the congruence class v of d mod Np^{2j} has order

Np^{2j}. Therefore, for $m < n$, by [DS05, Theorem 4.2.3], we have

$$(23) \qquad G_k^{(u',v')}(z) = \zeta^{v'}(k) + \frac{(-2\pi i)^k}{(k-1)!(Np^{2j})^k} \sum_{l=1}^{\infty} \sigma_{k-1}^{(u',v')}(l) e^{2\pi i l z/Np^{2j}},$$

for tuples (u', v') occurring in (22), where

$$\sigma_{k-1}^{(u',v')}(l) = \sum_{\substack{l'|l \\ l/l' \equiv 0 \bmod Np^{2j}}} \text{sgn}(l') l'^{k-1} e^{2\pi i v' l'/Np^{2j}}.$$

Constant term:

By (22) and (23), the constant term a_0 in the q-expansion of $E_k^{\beta}(0, z)$ equals

$$a_0 = \frac{1}{2} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \bmod Np^{2j} \\ v \equiv \pm 1 \bmod p^j}} \sum_{l \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^{\times}} \zeta_+^l(k, \mu) \zeta^{l^{-1}v}(k)$$

$$\overset{(*)}{=} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \bmod Np^{2j} \\ v \equiv \pm 1 \bmod p^j}} \sum_{l \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^{\times}} \frac{1}{\varphi(Np^{2j})^2} \sum_{\psi \bmod Np^{2j}} \psi(l)^{-1} L(k, \psi)^{-1} \sum_{\substack{\psi_1 \bmod Np^{2j} \\ \psi_1 \text{ even}}} \psi_1(l^{-1}v)^{-1} L(k, \psi_1)$$

$$= \frac{1}{\varphi(Np^{2j})^2} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \bmod Np^{2j} \\ v \equiv \pm 1 \bmod p^j}} \sum_{\substack{\psi, \psi_1 \bmod Np^{2j} \\ \psi_1 \text{ even}}} \psi_1(v)^{-1} \frac{L(k, \psi_1)}{L(k, \psi)} \sum_{l \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^{\times}} \psi_1 \psi^{-1}(l)$$

$$\overset{(**)}{=} \frac{1}{\varphi(Np^{2j})} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \bmod Np^{2j} \\ v \equiv \pm 1 \bmod p^j}} \sum_{\substack{\psi_1 \bmod Np^{2j} \\ \psi_1 \text{ even}}} \psi_1(v)^{-1}$$

$$= \frac{1}{2\varphi(Np^{2j})} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \bmod Np^{2j} \\ v \equiv \pm 1 \bmod p^j}} \sum_{\psi_1 \bmod Np^{2j}} (\psi_1(v)^{-1} + \psi_1(-v)^{-1})$$

$$\overset{(***)}{=} 1,$$

where $(*)$ follows from (19) and (20), and $(**)$ and $(***)$ follow from the orthogonality relations.

Higher Fourier coefficients:

Clearly $\sigma_{k-1}^{(0,v')}(l) = 0$ if $Np^{2j} \nmid l$. So assume that l is a multiple of Np^{2j}. Say $l = Np^{2j} l''$. Then

$$\sigma_{k-1}^{(0,v')}(l) = \sum_{l'|l''} \text{sgn}(l') l'^{k-1} e^{2\pi i v' l'/Np^{2j}},$$

which is clearly E-rational if E contains a sufficiently large cyclotomic number field depending on j. From (22) and (23), we see that the coefficient $a_{l''}$ of $q^{l''}$ in the Fourier expansion of $E_k^{\beta}(0, z)$ equals

$$a_{l''} = \frac{1}{2} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \bmod Np^{2j} \\ v \equiv \pm 1 \bmod p^j}} \sum_{n \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^{\times}} \zeta_+^n(k, \mu) \frac{(-2\pi i)^k}{(k-1)!(Np^{2j})^k} \sigma_{k-1}^{n^{-1}(0,v)}(Np^{2j} l'').$$

If $j = 0$, one checks that the formula for $a_{l''}$ above reduces to a well-known expression (see [Miy89, Theorem 7.1.3 and (7.1.30)]), and in particular $a_{l''} \in \mathbb{Q}$ is rational. So assume that $j > 0$. Then

$$a_{l''} = \frac{1}{2} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \bmod Np^{2j} \\ v \equiv 1 \bmod p^j}} \sum_{n \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^\times} (\zeta_+^n(k,\mu) + \zeta_+^{-n}(k,\mu)) \frac{(-2\pi i)^k}{(k-1)!(Np^{2j})^k} \sigma_{k-1}^{n^{-1}(0,v)}(Np^{2j}l'')$$

$$\overset{(21)}{=} \frac{1}{\varphi(Np^{2j})} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \bmod Np^{2j} \\ v \equiv 1 \bmod p^j}} \sum_{n \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^\times} \sum_{\substack{\psi \bmod Np^{2j} \\ \psi \text{ even}}} \psi(n)^{-1} L(k,\psi)^{-1} \frac{(-2\pi i)^k}{(k-1)!(Np^{2j})^k}$$

$$\cdot \sigma_{k-1}^{n^{-1}(0,v)}(Np^{2j}l'')$$

$$\overset{(24)}{=} \frac{-2k}{\varphi(Np^{2j})} \sum_{\substack{(u,v) \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^2 \\ u \equiv 0 \bmod Np^{2j} \\ v \equiv 1 \bmod p^j}} \sum_{n \in (\mathbb{Z}/Np^{2j}\mathbb{Z})^\times} \sum_{\substack{\psi \bmod Np^{2j} \\ \psi \text{ even}}} \psi(n)^{-1} \left(\frac{C_\psi}{Np^{2j}}\right)^k \frac{1}{G(\psi)B_{k,\bar{\psi}}}$$

$$\cdot \sigma_{k-1}^{n^{-1}(0,v)}(Np^{2j}l''),$$

where in the last step we have used the following special value result for the Dirichlet L-function:

$$(24) \qquad L(k,\psi) = -\frac{(-2\pi i)^k G(\psi) B_{k,\bar{\psi}}}{2k! C_\psi^k} \text{ if } \psi \text{ is even and } k > 0 \text{ is even,}$$

where C_ψ denotes the conductor of ψ. Thus $a_{l''}$ is again E-rational for a sufficiently large p-adic number field E containing an appropriate cyclotomic number field.

Summarizing, the computations above show that $E_{2n-2m+2}^\beta(0,z)$ has an E-rational q-expansion $\sum_{n=0}^\infty a_n q^n$ (at the cusp ∞) if E contains a sufficiently large cyclotomic number field (which depends on j). By [DS05, Proposition 4.2.1], since $E_{2n-2m+2}^{(u,v)}|_\gamma = E_{2n-2m+2}^{(u,v)\gamma}$, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the Eisenstein series $E_{2n-2m+2}^\beta(0,z)$ has an E-rational q-expansion at each cusp ξ of $\Gamma_0^\beta(N)$. The proposition now follows from Lemma 1. $\qquad\square$

5. TOWARDS INTEGRALITY

Note that the map $T_\beta^*|_\mathbb{H}$ can also be described as the pull-back of a differential form via the map

$$S_\beta : \Gamma_0^\beta(N) \backslash \mathbb{H} \to \Gamma_0(\mathfrak{n}) \backslash \mathcal{H}$$

given by sending

$$x + it \mapsto \gamma_\beta \begin{bmatrix} x & -t \\ t & x \end{bmatrix}.$$

We now choose $\delta^\circ(f)$ such that it generates $\bar{H}_{cusp}^1(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(\underline{n}, \mathcal{O}_E))[f]$, which is a rank one $\mathcal{O}_E$-submodule of $\bar{H}_{cusp}^1(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(\underline{n}, \mathcal{O}_E))$, where $\mathcal{O}_E$ is the valuation ring of E and $\bar{H}^1$ denotes the image of the integral cohomology in the rational cohomology under the natural map. We correspondingly refine the period $\Omega(f)$ so that $\Omega(f) \in \mathbb{C}^\times/\mathcal{O}_E^\times$.

Since $\beta = \frac{a\sqrt{-D}}{2p^j}$, we have $\gamma_\beta^{-1} \cdot P \in L(\underline{n}, \frac{1}{p^{2nj}}\mathcal{O}_E)$, for $P \in L(\underline{n}, \mathcal{O}_E)$. Thus, the map S_β does not preserve cohomology with integral coefficients, but instead induces a map

$$(25) \qquad S_\beta^* : \bar{H}_{cusp}^1(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(\underline{n}, \mathcal{O}_E)) \to \bar{H}_{cusp}^1(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(\underline{n}, \frac{1}{p^{2nj}}\mathcal{O}_E)),$$

on cohomology.

Lemma 2. *Assume $p > n$. Then under the Clebsch-Gordan decomposition* (7), *we have*

$$S_\beta^*(\bar{H}_{cusp}^1(\Gamma_0(\mathfrak{n})\backslash\mathcal{H}, \mathcal{L}(\underline{n}, \mathcal{O}_E))) \longmapsto \bigoplus_{m=0}^n \bar{H}_{cusp}^1(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(2n - 2m, \frac{1}{p^{j(2n-m)}}\mathcal{O}_E)).$$

Proof. Let $\nabla = \left(\frac{\partial^2}{\partial X \partial \overline{Y}} - \frac{\partial^2}{\partial \overline{X} \partial Y}\right)$. By [Gha99, Lemma 2], the projection to the m-th component in (7) is induced by $P(X, Y, \overline{X}, \overline{Y}) \mapsto \frac{1}{m!^2}\nabla^m P(X, Y, \overline{X}, \overline{Y})\mid_{\substack{\overline{X}=X \\ \overline{Y}=Y}}$. Clearly the projection continues to be defined with $\mathcal{O}_E$ coefficients if $p > n$. As remarked in (25), S_β^* does not preserve integrality. However, since

$$\nabla(\gamma_\beta^{-1} \cdot X^{n-k}Y^k\overline{X}^{n-l}\overline{Y}^l) =$$
$$l\frac{\partial}{\partial X}X^{n-k}(Y - \beta X)^k\overline{X}^{n-l}(\overline{Y} + \beta\overline{X})^{l-1} - k\frac{\partial}{\partial \overline{X}}X^{n-k}(Y - \beta X)^{k-1}\overline{X}^{n-l}(\overline{Y} + \beta\overline{X})^l,$$

we see that if $P \in L(\underline{n}, \mathcal{O}_E)$, the total power of p^j in the denominator goes down by one after applying ∇ to $\gamma_\beta^{-1} \cdot P$. Iterating this, we see $\nabla^m(\gamma_\beta^{-1} \cdot P) \in L(2n - 2m, \frac{1}{p^{j(2n-m)}}\mathcal{O}_E)$, for $m = 0, \ldots, n$, proving the lemma. $\square$

We now assume that the prime p is greater than n, so that we may apply the lemma above. Let

$$(26) \qquad S_\beta^*(\delta_{2n-2m}^\circ(f)) \in \bar{H}_{cusp}^1(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(2n - 2m, \frac{1}{p^{j(2n-m)}}\mathcal{O}_E))$$

be the image of $\delta^\circ(f)$ under the map (25) followed by the projection to the m-th component in the Clebsch-Gordan decomposition in Lemma 2. Again note the slight abuse of notation, since the subscript $2n - 2m$ should be outside the brackets.

By Proposition 1, we know that

$$(27) \qquad E_{2n-2m+2}^\beta(0) \in \frac{1}{p^{c_j}}\bar{H}^1(\Gamma_0^\beta(N)\backslash\mathbb{H}, \mathcal{L}(2n - 2m, \mathcal{O}_E)),$$

for some integer $c_j \geq 0$, depending on j.

Let S denote the finite set of excluded primes above (i.e., $p \mid 2ND$ and $p \leq n$), which we extend to include the primes $p < 2n + 4$. We remark that if $p \notin S$, then $p > 2n$ which ensures that the duality pairing $\langle , \rangle$ is a well-defined pairing on cohomology with integral coefficients $L(2n - 2m, \mathcal{O}_E)$.

For the refined period $\Omega(f)$ defined above, we get the following partial integrality result.

Proposition 2. *Suppose p is not in the finite set of primes S, and that E is a sufficiently large p-adic number field as above. Let $0 \leq m < n$ be even and χ be an even character of conductor p^{j_χ}. Then*

$$(28) \qquad \frac{G(\chi)G(2n - m + 2, \bar{\chi}, f)}{\Omega(f)G(\bar{\chi}^2)\Omega_\infty} \in \frac{\mathcal{O}_E}{p^{j_\chi(4n-3m+3)+c_{j_\chi}}}.$$

Proof. Indeed, this follows from the fact that by (14) the special value in the statement of the proposition has a cohomological description in terms of integrals of the form

$$\int_{[\Gamma_0^\beta(N)\backslash\mathbb{H}]} S_\beta^*\delta_{2n-2m}^\circ(f) \wedge E_{2n-m+2}^\beta.$$

These are integrals of cohomology classes with specifiable denominator over an integral cycle, hence belong to $\mathcal{O}_E$ with specifiable denominator. The size of the denominator can be computed from (26) and (27), taking $j = j_\chi$, and the fact that the Dirichlet L-value in (14) satisfies $L^\circ(2n - 2m + 2, \bar{\chi}^2) \in \frac{1}{p^{j_\chi(2n-2m+3)}}\mathcal{O}_E$ (which in turn follows easily from a

special values result like (24), noting that the corresponding twisted Bernoulli numbers lie in $\frac{1}{p^{j_\chi}}\mathcal{O}_E$ by a standard formula for these numbers involving the usual Bernoulli numbers up to B_{2n+2}, and by the well-known result of von Staudt-Clausen which says that p does not divide the denominators of these Bernoulli numbers since $p - 1 > 2n + 2$, by the definition of the set S). $\square$

6. Constructing bounded distributions

Finally, we now define our p-adic distribution by the formula

$$\mu^\circ_{2n-m+2} = \frac{1}{\Omega(f)\Omega_\infty} \cdot \mu_{2n-m+2}.$$

These distributions are certainly defined whenever $2n - m + 2 \geq k + 2$ which is the same as $m \leq n - 2$, but may possibly be defined for all $0 \leq m \leq n$, by analytic continuation.

We wish to show that μ°_{2n-m+2} is a bounded distribution and hence a measure. To this end we recall the notion of abstract Kummer congruences.

Theorem 2 (Abstract Kummer congruences). *Let $Y = \mathbb{Z}_p^\times$, let $\mathcal{O}_p$ be the ring of integers of $\mathbb{C}_p$ and let $\{f_i\}$ be a collection of continuous functions in $C(Y, \mathcal{O}_p)$ such that the $\mathbb{C}_p$-linear span of $\{f_i\}$ is dense in $C(Y, \mathbb{C}_p)$. Let $\{a_i\}$ be a system of elements with $a_i \in \mathcal{O}_p$. Then the existence of an $\mathcal{O}_p$-valued measure μ on Y with the property*

$$\int_Y f_i \, d\mu = a_i$$

is equivalent to the following congruences: for an arbitrary choice of elements $b_i \in \mathbb{C}_p$ almost all zero, and for $n \geq 0$, we have

$$\sum_i b_i f_i(y) \in p^n \mathcal{O}_p, \text{ for all } y \in Y \implies \sum_i b_i a_i \in p^n \mathcal{O}_p.$$

We apply this theorem with f_i the collection of Dirichlet characters χ of $(\mathbb{Z}/p^j\mathbb{Z})^\times$, for all $j \geq 1$, thought of as functions of $Y = \mathbb{Z}_p^\times$, and with $a_\chi \in \mathcal{O}_p$ the values of $\mu(\chi)$, for a given $\mathbb{C}_p$-valued distribution μ on Y. To prove that μ is an $\mathcal{O}_p$-valued measure on Y, it suffices to prove Kummer congruences of the more specialized form

$$(29) \qquad \sum_\chi \chi^{-1}(a)\chi(y) \in p^{j-1}\mathcal{O}_p, \text{ for all } y \in Y \implies \sum_\chi \chi^{-1}(a)\mu(\chi) \in p^{j-1}\mathcal{O}_p,$$

where χ varies over all characters mod p^j, for a fixed $j \geq 1$, and where the first congruence in (29) follows from the identity $\sum_\chi \chi^{-1}(a)\chi = \phi(p^j)\mathbb{1}_{a+p^j\mathbb{Z}_p}$, for $\mathbb{1}_{a+p^j\mathbb{Z}_p}$ the characteristic function of the coset $a + p^j\mathbb{Z}_p \subset \mathbb{Z}_p^\times$. Indeed, then the second congruence in (29) shows that μ is $\mathcal{O}_p$-valued on $\mathbb{1}_{a+p^j\mathbb{Z}_p}$, whence on all $\mathcal{O}_p$-valued step functions on $\mathbb{Z}_p$, whence on all $\mathcal{O}_p$-valued continuous functions on Y.

Claim. *The Kummer congruences* (29) *hold for* $\mu = \mu^\circ_{2n-m+2}$, *for $m \leq n - 2$ even.*

In order to prove this claim we must show that the second sum $\sum_\chi \chi^{-1}(a)\mu^\circ_{2n-m+2}(\chi)$ in (29) should firstly a) be *integral* and secondly b) be in $p^{j-1}\mathcal{O}_p$. Now (28) shows that for any even character χ and m even:

(30)
$$\int \chi \, d\mu^\circ_{2n-m+2} = \frac{2p^{j_\chi(2n-m+1)}G(\bar\chi^2)}{\kappa^{j_\chi}} \frac{G(\chi)G(2n-m+2, \bar\chi, f)}{\Omega(f)G(\bar\chi^2)\Omega_\infty} \in \frac{1}{p^{j_\chi(2n-2m+2)+c_{j_\chi}}}\mathcal{O}_E,$$

at least if κ is a unit, which we have assumed. For odd characters χ, the integral above vanishes. Thus (30) shows that the second sum above is in $\frac{1}{p^{j(2n-2m+2)+c_j}}\mathcal{O}_p$, with $c_j =$

$\max_\chi c_{j_\chi}$. This is still quite far from the integrality claimed in part a). Assuming that part a) holds, one must then further prove the congruence in part b).

In any case, assuming the Claim, we have that μ°_{2n-m+2} is a measure, for $0 \le m \le n-2$ even.

Let $x_p : \mathbb{Z}_p^\times \to \mathcal{O}_p$ be the usual embedding. We now wish to glue the measures μ°_{2n-m+2}, for $0 \le m$ even, into one measure μ° satisfying (see [CP89, Lemma 4.4], noting $q(V)$ there equals 1)

$$(31) \qquad \int_{\mathbb{Z}_p^\times} \chi \, d\mu^\circ = (-1)^{m/2} \int_{\mathbb{Z}_p^\times} x_p^m \chi \, d\mu^\circ_{2n-m+2}.$$

To do this, we again appeal to the abstract Kummer congruences in the theorem above. For the f_i, we consider a slightly larger class of functions than the Dirichlet characters χ above, namely those of the form $x_p^{-m} \cdot \chi$, for $0 \le m \le n$, with m even. We set $a_{m,\chi} = (-1)^{m/2} \mu^\circ_{2n-m+2}(\chi) \in \mathcal{O}_p$, which should be equal to $\mu^\circ(x_p^{-m}\chi)$, by (31) above. We now assume that

Claim. *The $a_{m,\chi}$ satisfy the abstract Kummer congruences:*

$$\sum_{m,\chi} b_{m,\chi}(x_p^{-m}\chi)(y) \in p^{j-1}\mathcal{O}_p, \text{ for all } y \in Y \implies \sum_{m,\chi} b_{m,\chi}a_{m,\chi} \in p^{j-1}\mathcal{O}_p.$$

It would then follow from Theorem 2 that there is a measure μ° such that (31) holds. Note that the Kummer congruences in the latter claim actually imply the ones in the former claim for μ°_{2n-m+2}, by choosing the $b_{m',\chi} = \chi^{-1}(a)$ if $m' = m$, and $b_{m',\chi} = 0$ if $m' \ne m$. We expect that the proof of these Kummer congruences should be similar to the Kummer congruences proved by Panchishkin in his construction of the p-adic Rankin product L-function attached to two cusp forms f and g, described in detail in [Pan88] (see also [CP04], and [GV19] where a sign similar to the one occurring in (31) is corrected).

Since μ° and μ°_{2n+2} agree on a dense set of functions, namely all χ, the measure μ° is just the measure μ°_{2n+2}. We now define the p-adic Asai L-function as the Mellin transform of the measure $\mu^\circ = \mu^\circ_{2n+2}$:

$$L_p(\chi) = \int_{\mathbb{Z}_p^\times} \chi(a) \, d\mu^\circ_{2n+2}, \quad \text{ for all } \chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times.$$

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