Adjoint Lifts and Modular Endomorphism Algebras

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Theme of today's talk

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Arithmetic information about an object attached to a modular form is sometimes contained in the Fourier coefficients of a suitable lift of the original form.

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under the

Shimura-Shintani-Waldspurger correspondence:

$$\begin{aligned} \mathrm{PGL}_2 &\mapsto \widetilde{\mathrm{SL}}_2 \\ f &\mapsto g = \mathsf{HI}(f). \end{aligned}$$

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under the

Gelbart-Jacquet map:

$$\begin{aligned} \mathrm{GL}_2 &\mapsto \mathrm{GL}_4 \\ f &\mapsto g = \mathrm{Ad}(f). \end{aligned}$$

Modular forms and motives

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a primitive classical cusp form of

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M_f is a rank 2 motive over $\mathbb{Q}$ with coefficients in $E = \mathbb{Q}(a_n)$.

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In particular, we study its:

1. Crossed product structure
2. Brauer class.

Extra twists

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A pair (γ, χ_γ) *where*

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Let

$$\Gamma \quad := \quad \{\gamma \in \mathrm{Aut}(E) \,|\, (\gamma, \chi_\gamma) \text{ is an extra twist for } f\} \;=\; \text{abelian group.}$$

Crossed product algebra

We define a 2-cocycle on Γ as follows. For γ, $\delta \in \Gamma$, let

$$c(\gamma, \delta) = \frac{G(\chi_\gamma^{-1})G(\chi_\delta^{-\gamma})}{G(\chi_{\gamma\delta}^{-1})} \in E,$$

where $G(\chi)$ is the Gauss sum of χ.

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Aside: What is the structure of X_f when f has CM?

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Remark: It suffices to specify X locally! Recall:

$$\begin{aligned} {}_2\mathrm{Br}(F) &\hookrightarrow \bigoplus_{v|F} \left({}_2\mathrm{Br}(F_v) = \mathbb{Z}/2 \right) \\ X &\mapsto \left(X_v = X \otimes_F F_v = \begin{cases} 0 & \text{if } X_v \text{ splits,} \\ 1 & \text{if } X_v \text{ ramifies.} \end{cases} \right)_v \end{aligned}$$

Infinite places

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Aside: So X fits into Albert's classification of algebras with involution. What is the involution on X?

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- $k \geq 2$, $p \geq 2$, χ_γ arbitrary: Banerjee-Ghate (recent work).

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Lemma

Let ν be the cyclotomic character. Then

$$trace((\rho_\Pi \otimes \nu^{k-1})(Frob_p)) = a_p^2 \epsilon(p)^{-1}.$$

Adjoint lifts continued

Proof: Say $\rho_\pi(\mathrm{Frob}_p) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$.

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Theorem (Reinterpretation of previous result)

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p-adic Hodge theory

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Thus, we should try and understand the roots of ϕ in the cases of bad reduction as well!

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The corresponding filtered module was written down explicitly in Ghate-Mézard $\sim$ 2009.

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Note: $\alpha\beta = pa_p^2 = \epsilon'(p)p^{k-1}$. We compute:

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Say now $N_p \neq C_p$, with $N_p = 1$ and $C_p = 0$.
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- The formula above completely specifies the ramification at the Steinberg places, in all weights.

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Question (Infinite slope)

Say p is a prime of either good or bad reduction, with $a_p = 0$. Is it possible to give a purely local criterion which specifies the ramification of X_v, for $v|p$?

Other cases continued

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Q: Is this useful in determining X_v when $p \not| N$ and $a_p = 0$??

Thank you

See you in Goa in August, 2010
(Two ICM satellite conferences in Number Theory)