

Adjoint Lifts and Modular Endomorphism Algebras

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Theme of today's talk

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Arithmetic information about an object attached to a modular form is sometimes contained in the Fourier coefficients of a suitable lift of the original form.

Shimura-Shintani-Waldspurger Correspondence

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of integral weight

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Shimura-Shintani-Waldspurger correspondence:

$$\begin{aligned} \mathrm{PGL}_2 &\mapsto \widetilde{\mathrm{SL}}_2 \\ f &\mapsto g = \mathrm{HI}(f). \end{aligned}$$

We show that the ramification of the

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under the

Gelbart-Jacquet map:

$$\begin{aligned} \mathrm{GL}_2 &\mapsto \mathrm{GL}_4 \\ f &\mapsto g = \mathrm{Ad}(f). \end{aligned}$$

Modular forms and motives

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a primitive classical cusp form of

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- level $N \geq 1$, and,
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M_f is a rank 2 motive over \mathbb{Q} with coefficients in $E = \mathbb{Q}(a_n)$.

Modular endomorphism algebras

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In particular, we study its:

- 1 Crossed product structure
- 2 Brauer class.

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Let

$\Gamma := \{ \gamma \in \text{Aut}(E) \mid (\gamma, \chi_\gamma) \text{ is an extra twist for } f \} = \text{abelian group.}$

Crossed product algebra

We define a 2-cocycle on Γ as follows. For $\gamma, \delta \in \Gamma$, let

$$c(\gamma, \delta) = \frac{G(\chi_\gamma^{-1})G(\chi_\delta^{-\gamma})}{G(\chi_{\gamma\delta}^{-1})} \in E,$$

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Aside: What is the structure of X_f when f has CM?

Brauer class of X_f

Let $F = E^\Gamma$. Then X is a central simple algebra over F .

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Remark: It suffices to specify X locally! Recall:

$${}_2\text{Br}(F) \hookrightarrow \bigoplus_{v|F} ({}_2\text{Br}(F_v) = \mathbb{Z}/2)$$

$$X \mapsto \left(X_v = X \otimes_F F_v = \begin{cases} 0 & \text{if } X_v \text{ splits,} \\ 1 & \text{if } X_v \text{ ramifies.} \end{cases} \right)_v$$

NB: $F \subset E$ is a totally real field since

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Aside: So X fits into Albert's classification of algebras with involution. What is the involution on X ?

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- $k \geq 2$, $p \geq 2$, χ_γ arbitrary: Banerjee-Ghate (recent work).

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We now make the following simple observation:

Lemma

Let ν be the cyclotomic character. Then

$$\text{trace}((\rho_\Pi \otimes \nu^{k-1})(\text{Frob}_p)) = a_p^2 \epsilon(p)^{-1}.$$

Adjoint lifts continued

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Theorem (Reinterpretation of previous result)

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Thus, we should try and understand the roots of ϕ in the cases of bad reduction as well!

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The corresponding filtered module was written down explicitly in Ghatté-Mézard \sim 2009.

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$$\rho_{\pi}|_{G_p} \sim \begin{pmatrix} \lambda(p \cdot a_p) & * \\ 0 & \lambda(a_p) \end{pmatrix}.$$

So the roots of crystalline Frobenius are

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Steinberg case

Say now $N_p \neq C_p$, with $N_p = 1$ and $C_p = 0$.
So π_p is an unramified twist of Steinberg.

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- The formula above completely specifies the ramification at the Steinberg places, in all weights.

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Question (Infinite slope)

Say p is a prime of either good or bad reduction, with $a_p = 0$. Is it possible to give a purely local criterion which specifies the ramification of X_v , for $v|p$?

Other cases continued

In the good reduction case, one can prove:

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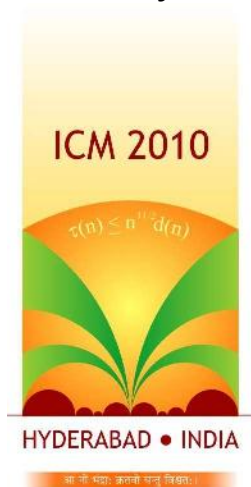
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Q: Is this useful in determining X_v when $p \nmid N$ and $a_p = 0$??

Thank you



See you in Goa in August, 2010
(Two ICM satellite conferences in Number Theory)