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A SUFFICIENT CONDITION FOR A 2-DIMENSIONAL ORBIFOLD TO BE GOOD

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Abstract. We prove that a connected 2-dimensional orbifold with finitely generated and infinite orbifold fundamental group is good. We also describe all the good 2-dimensional orbifolds with finite orbifold fundamental groups.

1. Introduction

We start with a short introduction to orbifolds.

The concept of orbifold was first introduced in [2], and was called 'V-manifold'. Later, it was revived in [4], with the new name *orbifold*, and *orbifold fundamental group* of a connected orbifold was defined.

Definition 1.1. An *orbifold* is a second countable and Hausdorff topological space M, which at every point looks like the quotient space of $\mathbb{R}^n$, for some n, by some finite group action. More precisely, there is an open covering $\{U_i\}_{i\in I}$ of M together with a collection $\mathcal{O}_M = \{(\tilde{U}_i, p_i, G_i)\}_{i\in I}$ (called *orbifold charts*), where for each $i \in I$, $\tilde{U}_i$ is an open set in $\mathbb{R}^n$, for some n, G_i is a finite group acting on $\tilde{U}_i$ and $p_i : \tilde{U}_i \to U_i$ is the quotient map, via an identification of $\tilde{U}_i/G_i$ with U_i by some homeomorphism.

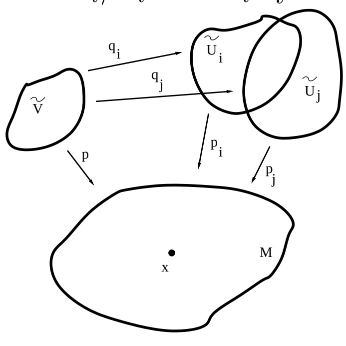

Figure 1: Orbifold charts

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Furthermore, the following compatibility condition is satisfied. Given $x \in U_i \cap U_j$, for $i, j \in I$ there is a neighborhood $V \subset U_i \cap U_j$ of x and a chart $(\tilde{V}, p, G) \in \mathcal{O}_M$ together with embeddings $q_i : \tilde{V} \to \tilde{U}_i$ and $q_j : \tilde{V} \to \tilde{U}_j$ such that, $p_i \circ q_i = p$ and $p_j \circ q_j = p$ are satisfied.

The pair $(M, \mathcal{O}_M)$ is called an *orbifold* and M is called its *underlying space*. The finite groups $\{G_i\}_{i \in I}$ are called the *local groups*. The union of the images (under p_i) of the fixed point sets of the action of G_i on $\tilde{U}_i$, when G_i acts non-trivially, is called the *singular set*. Points outside the singular set are called *regular points*. In case the local group is cyclic (of order k) acting by rotation about the origin on the Euclidean space, the image of the origin is called a *cone point* of *order* k. If the local group at some point acts trivially then it is a regular point. A regular point is also called a *manifold point*.

Clearly, if all the local groups are either trivial or acts trivially then the orbifold is a manifold. The easiest example of an orbifold is the quotient of a manifold by a finite group. Also see Example 1.1 below.

In the rest of the paper we will not use the full notation of an orbifold as defined above, unless it is explicitly needed. In general, we will use the same letter M both for the orbifold and its underlying space, which will be clear from the context.

As in the case of a manifold, *dimension* of a connected orbifold is defined. One can also define an *orbifold with boundary* in the same way we define a manifold with boundary. In the definition we have to replace $\mathbb{R}^n$ by the upper half space $\mathbb{R}^n_+$.

In this paper we consider orbifolds with boundary (may be empty).

A boundary component of the underlying space of an orbifold has two types, one which we call *manifold boundary* and the other *orbifold boundary*. These are respectively defined as points on the boundary (of the underlying space) with local group acting trivially or non-trivially.

The notion of *orbifold covering space* was defined in [4]. We refer the reader to this source for the basic materials and examples. But, we recall below the definition and properties we need.

Definition 1.2. A connected orbifold $(\hat{M}, \mathcal{O}_{\hat{M}})$ is called an *orbifold covering space* of a connected orbifold $(M, \mathcal{O}_M)$ if there is a surjective map $f : \hat{M} \to M$, called an *orbifold covering map*, such that given $x \in M$ there is an orbifold chart $(\tilde{U}, p, G)$ with $x \in U$ and the following is satisfied.

• $f^{-1}(U) = \cup_{i\in I} V_i$, where for each $i \in I$, $(\tilde{V}_i, q_i, H_i) \in \mathcal{O}_{\hat{M}}$, V_i is a component of $f^{-1}(U)$, there is an injective homomorphism $\rho_i : H_i \to G$ and V_i is homeomorphic to $\tilde{U}/(\rho_i(H_i))$ making the two squares in the following diagram commutative.

$$\begin{array}{ccccc} \tilde{U}/(\rho_i(H_i)) & \longrightarrow & V_i & \xleftarrow[q_i]{} & \tilde{V}_i \\ \downarrow & & \downarrow{\scriptstyle f|_{V_i}} & & \downarrow{\scriptstyle \tilde{f}} \\ \tilde{U}/G & \longrightarrow & U & \xleftarrow[p]{} & \tilde{U}, \end{array}$$

where $\tilde{f}$ is ρ_i-equivariant.

Here note that the map on the underlying spaces of an orbifold covering map need not be a covering map in the ordinary sense.

Example 1.1. Given a group Γ and a properly discontinuous action of Γ on a manifold M, the quotient space M/Γ has an orbifold structure and the quotient map $M \to M/\Gamma$ is an orbifold covering map. See [4, Proposition 5.2.6]. Furthermore, if H is a subgroup of Γ, then the map $M/H \to M/\Gamma$ is an orbifold covering map. Therefore, the quotient map of a finite group action on an orbifold is always an orbifold covering map.

In general, an orbifold need not have a manifold as an orbifold covering space.

Definition 1.3 ([4])**.** If an orbifold has an orbifold covering space which is a manifold, then the orbifold is called *good* or *developable*.

In the above example M/Γ is a good orbifold.

Remark 1.1. One can show that a good compact 2-dimensional orbifold has a finite sheeted orbifold covering space, which is a manifold [3, Theorem 2.5]. In the case of closed (that is, compact and the underlying space has empty boundary) 2-dimensional orbifolds, only the sphere with one cone point and the sphere with two cone points of different orders are not good orbifolds. See the figure below. Also, see [4, Theorem 5.5.3].

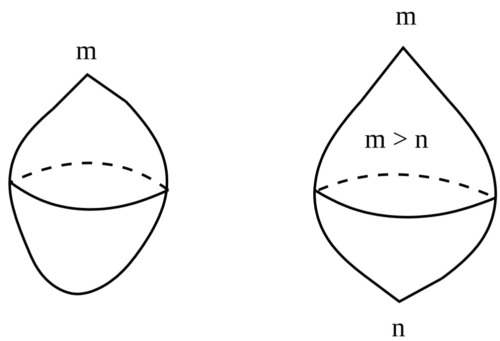

Figure 2: Bad orbifolds

Definition 1.4. Let $f : \hat{M} \to M$ be an orbifold covering map of connected orbifolds. Let $x \in M$ be a regular point and $\hat{x} \in \hat{M}$ with $f(\hat{x}) = x$. Then $\hat{M}$ is called the *universal orbifold cover* of M if given any other connected orbifold covering $g : N \to M$ with $g(y) = x$, for $y \in N$, there is a unique $h : \hat{M} \to N$ so that $h(\hat{x}) = y$, h is an orbifold covering map and $h \circ g = f$.

Definition 1.5. The universal orbifold cover always exists for a connected orbifold M [4, Proposition 5.3.3] and its group of Deck transformations is defined as the *orbifold fundamental group* of M. It is denoted by $\pi_1^{orb}(M)$.

In Remark 1.1 we have already seen a classification of closed good 2-dimensional orbifolds. But in the literature I did not find any condition which will say when an orbifold (compact or not) is good.

In this article we use the above classification of good closed orbifolds, to prove the following theorem which gives a sufficient condition for an arbitrary 2-dimensional orbifold to be good.

Theorem 1.1. *Let S be a connected 2-dimensional orbifold with finitely generated and infinite orbifold fundamental group. Then, S is good.*

Remark 1.2. There are 2-dimensional good orbifolds with finite orbifold fundamental groups. The simplest example is the 2-dimensional disc quotiented out by the action of a finite cyclic group acting by rotation around the center. See Remark 2.1 for a classification of such orbifolds.

2. Proof of Theorem 1.1

For the proof of Theorem 1.1 we need the following two lemmas, which follow from standard techniques and covering space theory of orbifolds.

Lemma 2.1. *Let M be a connected orbifold and $f : \tilde{M} \to M$ be a connected covering space of the underlying space of M. Then, $\tilde{M}$ has an orbifold structure induced from M by f, so that f is an orbifold covering map.*

Proof. Choose an open covering of $\tilde{M}$ which consists of inverse images under f of evenly covered open subsets of M which are associated to orbifold charts of M, and then pull back the finite group actions using f. That gives the required orbifold structure on $\tilde{M}$. □

Lemma 2.2. *Let $q : \tilde{M} \to M$ be a finite sheeted orbifold covering map between two connected orbifolds. Then, the induced map $q_* : \pi_1^{orb}(\tilde{M}) \to \pi_1^{orb}(M)$ is an injection, and the image $q_*(\pi_1^{orb}(\tilde{M}))$ is a finite index subgroup of $\pi_1^{orb}(M)$.*

Proof. See [1, Corollary 2.4.5]. □

Proof of Theorem 1.1. Let S be a 2-dimensional orbifold as in the statement. Note that, the underlying space of a 2-dimensional orbifold is a 2-dimensional (smooth) manifold. (This follows easily from the discussion below on the possible types of singular points on a 2-dimensional orbifold and the fact that manifolds of dimension ≤ 3 has unique smooth structure. Also see [3, p. 422, last paragraph]). Therefore, after going to a two sheeted cover of the underlying space of S and using Lemma 2.1, we can assume that the underlying space of S is an orientable 2-dimensional manifold. (In such a situation, it is standard to call the orbifold *orientable*).

For the remaining part of the proof we refer the reader to the discussion on pages 422-424 of [3].

It is known that, S has three types of singularities: cone points, reflector lines and corner reflectors [3, p. 422].

Note that, other than the cone points the remaining singular set contributes to the orbifold boundary components of the orbifold. We take double of S along the orbifold boundary, then we get an orientable orbifold $\tilde{S}$ which is a 2-sheeted orbifold covering of S and $\tilde{S}$ has only cone singularities [3, p. 423]. Furthermore, since $\pi_1^{orb}(\tilde{S})$ is a finite index subgroup of $\pi_1^{orb}(S)$ (Lemma 2.2), $\pi_1^{orb}(\tilde{S})$ is again infinite and finitely generated.

So, to prove the Theorem we only have to show that $\tilde{S}$ is a good orbifold.

Note that, $\tilde{S}$ has no orbifold boundary component. If there is any manifold boundary of $\tilde{S}$, then we take the double of $\tilde{S}$ along these boundary components and denote it by $D\tilde{S}$. Therefore, $D\tilde{S}$ is an orientable orbifold with no boundary in its underlying space and only has cone singularities. Also, if $D\tilde{S}$ is good then so is $\tilde{S}$.

We now prove that $D\tilde{S}$ is good.

First, we consider the case when $D\tilde{S}$ is noncompact. The cone points form a discrete subset of $D\tilde{S}$ and hence we can write the underlying space of $D\tilde{S}$ as an increasing union of compact orientable sub-manifolds S_i, $i \in \mathbb{N}$ with no singular points on the boundary of S_i. (This can be done by taking a proper smooth map from the underlying space of $D\tilde{S}$ to $\mathbb{R}$). Let DS_i be the double of S_i. DS_i is an orbifold with twice as many cone points as S_i. Suppose DS_i is of genus g_i and has k_i cone points with orders $p_1, p_2, \ldots, p_{k_i}$. Then, DS_i is a closed orientable orbifold with only cone singularities, and by [3, p. 424] the orbifold fundamental group of DS_i is given by the following presentation.

$$\pi_1^{orb}(DS_i) \simeq \langle a_1, b_1, \ldots, a_{g_i}, b_{g_i}, x_1, \ldots, x_{k_i} \mid x_j^{p_j} = 1,\ j = 1, 2, \ldots, k_i,$$
$$\Pi_{j=1}^{g_i}[a_j, b_j]x_1x_2 \cdots x_{k_i} = 1\rangle.$$

Hence, the abelianization of $\pi_1^{orb}(DS_i)$ is isomorphic to $\mathbb{Z}^{2g_i} \oplus K_{k_i-1}$. Where, K_{k_i-1} is a finite abelian group with $k_i - 1$ number of generators. Furthermore, $g_1 \leq g_2 \leq \cdots$ and $k_1 \leq k_2 \leq \cdots$.

This shows that, since $\pi_1^{orb}(D\tilde{S})$ is finitely generated (as $\pi_1^{orb}(\tilde{S})$ is finitely generated), there is an i_0 such that $g_j = g_l$ and $k_j = k_l$ for all $j, l \geq i_0$.

Therefore, $D\tilde{S}$ has finitely many cone points, all contained in S_{i_0}, and outside S_{i_0}, $D\tilde{S}$ is a finite union of components, each homeomorphic to the infinite cylinder $\mathbb{S}^1 \times (0, \infty)$. Next, we cut the infinite ends of $D\tilde{S}$ at some finite stage and denote the resulting compact orbifold again by S_{i_0}.

Note that, here $\pi_1^{orb}(S_{i_0})$ is a subgroup of $\pi_1^{orb}(DS_{i_0})$ and hence $\pi_1^{orb}(DS_{i_0})$ is infinite. Hence, from the classification of closed 2-dimensional orbifolds using geometry (see [4, Theorem 5.5.3]), it follows that DS_{i_0} is a good orbifold. Since, DS_{i_0} has even number of cone points, and in the case of two cone points they have the same orders. See Figure 2. Clearly, then S_{i_0}, and hence $D\tilde{S}$ (which is homeomorphic to the interior of S_{i_0}) has an orbifold covering space, which is a manifold.

Next, if $D\tilde{S}$ is compact then the same argument as in the above paragraph shows that it is good. This completes the proof of the Theorem. □

Remark 2.1. We end the paper with a remark on 2-dimensional orbifolds with finite orbifold fundamental group. As in the proof of the Theorem, after going to a finite sheeted covering and then doubling along manifold boundary components, we can assume that the orbifold (say, S) is orientable and has finitely many cone singularities. Ignoring the cone points we have a surjective homomorphism $\pi_1^{orb}(S) \to \pi_1(S)$. Hence, the fundamental group of the underlying space of S is finite and therefore, the underlying space is $\mathbb{S}^2$ or $\mathbb{R}^2$. It is easy to see that in the last case S is good and in the $\mathbb{S}^2$ case we already have a classification (Remark 1.1).

References

[1] Chen, W., On a notion of maps between orbifolds. II. Homotopy and CW-complex, *Commun. Contemp. Math.*, **8** (2006), no. 6, 763–821.

[2] Satake, I., The Gauss-Bonnet theorem for V-manifolds, *J. Math. Soc. Japan*, **9** (1957), 464–492.

[3] Scott, P., The geometries of 3-manifolds, *Bull. London Math. Soc.*, **15** no. 5 (1983), 401–487.

[4] Thurston, W. P., *Three-dimensional Geometry & Topology*, December 1991 version, Mathematical Sciences Research Institute Notes, Berkeley, California.

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