

ON THE ISOMORPHISM CONJECTURE FOR GROUPS ACTING ON TREES

S.K. ROUSHON

ABSTRACT. We study the Fibered Isomorphism conjecture of Farrell and Jones for groups acting on trees. We show that under certain conditions the conjecture is true for groups acting on trees when the stabilizers satisfy the conjecture. These conditions are satisfied in several cases of the conjecture. We prove some general results on the conjecture for the pseudoisotopy theory for groups acting on trees with residually finite vertex stabilizers. In particular, we study situations when the stabilizers belong to the following classes of groups: polycyclic groups, finitely generated nilpotent groups, closed surface groups, finitely generated abelian groups and virtually cyclic groups.

Finally, we provide explicit examples of groups on which the results of this article can be applied and show that these groups were not considered before. Furthermore, we deduce that these groups are neither hyperbolic nor $CAT(0)$.

CONTENTS

1. Introduction and statements of Theorems	2
2. Statements of the conjecture and some propositions	6
3. Graphs of groups	10
4. Residually finite groups	14
5. Basic results on the Isomorphism conjecture	16
6. Proofs of the Propositions	20
7. Proofs of the Theorems	27
8. Examples	27
References	31

2000 *Mathematics Subject Classification*. Primary: 19D35, 57N37. Secondary: 55N91.

Key words and phrases. pseudoisotopy theory, K -theory, fibered isomorphism conjecture, equivariant homology theory, group action on trees, $CAT(0)$ -groups.

November 16, 2011.

1. INTRODUCTION AND STATEMENTS OF THEOREMS

In this article we study the Fibered Isomorphism conjecture of Farrell and Jones for groups acting on trees. Originally the conjecture was stated for the pseudoisotopy theory, algebraic K -theory and for L -theory. Here we prove some results for the pseudoisotopy theory case of the conjecture. The methods of proofs of our results hold for the other theories under certain conditions. We make these conditions explicit here. It is well known that the pseudoisotopy version of the conjecture yields computations of the Whitehead groups and lower K -groups of the associated groups and also implies the conjecture in lower K -theory (that is in dimension ≤ 1). The main problem we are concerned with is the following.

Problem. *Assume the Fibered Isomorphism conjecture for the stabilizers of the action of a group on a tree. Show that the group also satisfies the conjecture.*

A particular case of the problem is to show that the conjecture is true for a generalized free product $G_1 *_H G_2$ (HNN -extension $G *_H H$) of groups assuming it is true for the groups G_1 , G_2 and H (G and H).

In [[19], Reduction Theorem] we solved the problem when the edge stabilizers are trivial. In this article we show it when the edge stabilizers are finite and in addition either the vertex stabilizers are residually finite or the group surjects onto another group with certain properties. We also solve the problem, under a certain condition, assuming that the vertex stabilizers are polycyclic. This condition is both necessary and sufficient for the fundamental group to be subgroup separable when the vertex stabilizers are finitely generated nilpotent. We further consider cases when the stabilizers are virtually cyclic or surface groups. Next, we prove a result when the stabilizers are finitely generated abelian. Graph of groups of the last type were studied before in [24] where the L -theory was computed when the stabilizers are finitely generated free abelian groups. We note that in this case the result in [24] implies the L -theory version of the Isomorphism conjecture.

Before we come to the statements of the main results let us recall that a *graph of groups* consists of a graph \mathcal{G} (that is an one-dimensional CW -complex) and to each vertex v or edge e of \mathcal{G} there is associated a group \mathcal{G}_v (called the *vertex group* of the vertex v) or \mathcal{G}_e (called the *edge group* of the edge e) respectively with the assumption that for each edge e and for its two end vertices v and w (say) there are injective group homomorphisms $\mathcal{G}_e \rightarrow \mathcal{G}_v$ and $\mathcal{G}_e \rightarrow \mathcal{G}_w$. (If $v = w$ then also we demand two injective homomorphisms as above). The *fundamental group* $\pi_1(\mathcal{G})$ of the graph \mathcal{G} can be defined so that in the simple cases of

graphs of groups where the graph has two vertices and one edge or one vertex and one edge the fundamental group is the amalgamated free product or the HNN -extension respectively. See [8] for some more on this subject.

Throughout the article we make the following conventions:

1. A graph is assumed to be locally finite and connected.
2. We use the same notation for a graph of groups and its underlying graph. $V_{\mathcal{G}}$ and $E_{\mathcal{G}}$ denotes respectively the set of all vertices and edges of \mathcal{G} .
3. And a group is assumed to be countable.

Recall that given two groups G and H the *wreath product* $G \wr H$ is by definition the semidirect product $G^H \rtimes H$ where the action of H on G^H is the regular action and G^H denotes the direct sum of copies of G indexed by the elements of H .

Now we state our main results.

In the statements of the results we say that the FIC^P ($FICwF^P$) is true for a group G if the Farrell-Jones Fibered Isomorphism conjecture for the pseudoisotopy theory is true for the group G (for $G \wr H$ for any finite group H). We mention here that the advantage of taking the $FICwF^P$ is that it passes to finite index over-groups (see Proposition 5.2).

Theorem 1.1. *Let \mathcal{G} be a graph of groups so that the edge groups are finite. Then $\pi_1(\mathcal{G})$ satisfies the $FICwF^P$ if one of the following conditions is satisfied.*

- (1). *The vertex groups are residually finite and satisfy the $FICwF^P$.*
- (2). *There is a homomorphism $f : \pi_1(\mathcal{G}) \rightarrow Q$ onto another group Q so that the restriction of f to any vertex group has finite kernel and Q satisfies the $FICwF^P$.*

This Theorem is a special case of Proposition 2.2 (1) respectively Proposition 2.1 (b). See Section 7.

Remark 1.1. There is a large class of residually finite groups for which the $FICwF^P$ is true. For example, any group which contains a group from the following examples as a subgroup of finite index is a residually finite group satisfying the $FICwF^P$.

1. Polycyclic groups ([9]).
2. Artin full braid groups ([12], [21]).
3. Compact 3-manifold groups ([19], [20]).
4. Compact surface groups ([9]).
5. Fundamental groups of hyperbolic Riemannian manifolds ([9]).
6. Crystallographic groups ([9]).

Here we remark that by an email, dated July 11, 2008, Wolfgang Lück informed the author that the proof of the FIC^P for a certain class of polycyclic groups as stated in [[9], Theorem 4.8] is not yet complete. However, this result and the corresponding result in the L -theory case of the conjecture were already used by several authors in published papers. The proof in the L -theory case is completed recently in [1] and the pseudoisotopy case is in progress. We note here that if a polycyclic group G is also virtually nilpotent then the FIC^P for G is already proved in [[9], Lemma 4.1]

Definition 1.1. A graph of groups \mathcal{G} is said to satisfy the *intersection property* if for each connected subgraph of groups \mathcal{G}' of \mathcal{G} , $\cap_{e \in E_{\mathcal{G}'}} \mathcal{G}'_e$ contains a subgroup which is normal in $\pi_1(\mathcal{G}')$ and is of finite index in some edge group. We say \mathcal{G} is of *finite-type* if the graph is finite and all the vertex groups are finite.

Theorem 1.2. *Let \mathcal{G} be a graph of groups. Let \mathcal{D} be a collection of finitely generated groups satisfying the following.*

- *Any element $C \in \mathcal{D}$ has the following properties. Quotients and subgroups of C belong to \mathcal{D} . C is residually finite and the $FICwF^P$ is true for the mapping torus $C \rtimes \langle t \rangle$ for any action of the infinite cyclic group $\langle t \rangle$ on C .*

Assume that the vertex groups of \mathcal{G} belong to \mathcal{D} . Then the $FICwF^P$ is true for $\pi_1(\mathcal{G})$ if \mathcal{G} satisfies the intersection property.

Theorem 1.2 is a special case of Proposition 2.2 (2) and is proved in Section 7.

As a consequence of Theorem 1.2 we prove the following.

Let us first recall that a group G is called *subgroup separable* if the following is satisfied. For any finitely generated subgroup H of G and $g \in G - H$ there is a finite index normal subgroup K of G so that $H \subset K$ and $g \notin K$. Equivalently, a group is subgroup separable if the finitely generated subgroups of G are closed in the profinite topology of G . A subgroup separable group is therefore residually finite. The classes of subgroup separable and residually finite groups are extensively studied in group theory and Manifold Topology.

Definition 1.2. Let \mathcal{G} be a graph of groups. An edge e of \mathcal{G} is called a *finite edge* if the edge group \mathcal{G}_e is finite. \mathcal{G} is called *almost a tree of groups* if there are finite edges e_1, e_2, \dots so that the components of $\mathcal{G} - \{e_1, e_2, \dots\}$ are trees. If we remove all the finite edges from a graph of groups then the components of the resulting graph are called *component subgraphs*.

Theorem 1.3. *Let \mathcal{G} be a graph of finitely generated groups. Then the FICwFP is true for $\pi_1(\mathcal{G})$ if one of the following five conditions is satisfied.*

(1). *The vertex groups are virtually polycyclic and \mathcal{G} satisfies the intersection property.*

(2). *The vertex groups of \mathcal{G} are finitely generated nilpotent and $\pi_1(\mathcal{G})$ is subgroup separable.*

(3). *The vertex and the edge groups of any component subgraph (see Definition 1.2) are fundamental groups of closed surfaces of genus ≥ 2 . Given a component subgraph \mathcal{H} which has at least one edge there is a subgroup $C < \cap_{e \in E_{\mathcal{H}}} \mathcal{H}_e$ which is finite index in some edge group and is normal in $\pi_1(\mathcal{H})$.*

(4). *The vertex groups are virtually cyclic and the graph is almost a tree. Further assume that either the edge groups are finite or the infinite vertex groups are abelian.*

(5). *\mathcal{G} is almost a tree of groups and the vertex and the edge groups of any component subgraph of \mathcal{G} are finitely generated abelian and of the same rank.*

For examples of graphs of groups satisfying the hypothesis in (3) see Example 3.1.

In the next section we start with stating the general Fibered Isomorphism conjecture for equivariant homology theory ([2]). We show that under certain conditions a group acting on a tree satisfies this general conjecture provided the stabilizers also satisfy the conjecture. Then we restrict to the pseudoisotopy case of the conjecture and prove the Theorems.

We work in this general setting as we will apply these methods in the L -theory version of the conjecture in later works ([22], [23]). We will see that using these methods we can prove some stronger results in the L -theory case.

Finally, in Examples 8.1, 8.2 and 8.3 we provide explicit examples of groups for which the results of this paper can be applied to prove the Fibered Isomorphism conjecture in the pseudoisotopy case and show that these groups were not considered before. We further show that the groups in these examples are neither $CAT(0)$ nor hyperbolic. Here we note that in a recent paper ([3]) Bartels and Lück have proved the Fibered Isomorphism conjectures in L - and lower K -theory case for hyperbolic groups and for $CAT(0)$ -groups which act on finite dimensional $CAT(0)$ -spaces.

Acknowledgment. This work started during my visit to the Mathematisches Institut, Universität Münster, Münster, Germany under a

Fellowship of the Alexander von Humboldt-Stiftung. I would like to thank my host Wolfgang Lück for making this visit possible and Arthur Bartels for some useful comments during the initial stage of this work. I am also grateful to F. T. Farrell for some helpful email communications. Finally thanks to Henrik Rueping for pointing out an error in an earlier version of the paper.

2. STATEMENTS OF THE CONJECTURE AND SOME PROPOSITIONS

In this section we recall the statement of the Isomorphism conjecture for equivariant homology theories (see [[2], Section 1]) and state some propositions.

Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in R -modules for R a commutative associative ring with unit. An equivariant homology theory assigns to a group G a G -homology theory \mathcal{H}_*^G which, for a pair of G -CW complex (X, A) , produces a \mathbb{Z} -graded R -module $\mathcal{H}_n^G(X, A)$. For details see [[15], Section 1].

A *family* of subgroups of a group G is defined as a set of subgroups of G which is closed under taking subgroups and conjugations. If \mathcal{C} is a class of groups which is closed under isomorphisms and taking subgroups then we denote by $\mathcal{C}(G)$ the set of all subgroups of G which belong to \mathcal{C} . Then $\mathcal{C}(G)$ is a family of subgroups of G . For example \mathcal{VC} , the class of virtually cyclic groups, is closed under isomorphisms and taking subgroups. By definition a virtually cyclic group has a cyclic subgroup of finite index. Also \mathcal{FIN} , the class of finite groups is closed under isomorphisms and taking subgroups.

Given a group homomorphism $\phi : G \rightarrow H$ and a family \mathcal{C} of subgroups of H define $\phi^*\mathcal{C}$ to be the family of subgroups $\{K < G \mid \phi(K) \in \mathcal{C}\}$ of G . Given a family \mathcal{C} of subgroups of a group G there is a G -CW complex $E_{\mathcal{C}}(G)$ which is unique up to G -equivalence satisfying the property that for $H \in \mathcal{C}$ the fixpoint set $E_{\mathcal{C}}(G)^H$ is contractible and $E_{\mathcal{C}}(G)^H = \emptyset$ for H not in \mathcal{C} .

(Fibered) Isomorphism conjecture. ([2], Definition 1.1]) *Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in R -modules. Let G be a group and \mathcal{C} be a family of subgroups of G . Then the Isomorphism conjecture for the pair (G, \mathcal{C}) states that the projection $p : E_{\mathcal{C}}(G) \rightarrow pt$ induces an isomorphism*

$$\mathcal{H}_n^G(p) : \mathcal{H}_n^G(E_{\mathcal{C}}(G)) \simeq \mathcal{H}_n^G(pt)$$

for $n \in \mathbb{Z}$.

And the *Fibered Isomorphism conjecture* for the pair (G, \mathcal{C}) states that for any group homomorphism $\phi : K \rightarrow G$ the *Isomorphism conjecture* is true for the pair $(K, \phi^*\mathcal{C})$.

Let \mathcal{C} be a class of groups which is closed under isomorphisms and taking subgroups.

Definition 2.1. If the (Fibered) Isomorphism conjecture is true for the pair $(G, \mathcal{C}(G))$ we say that the $(F)IC_{\mathcal{C}}$ is true for G or simply say $(F)IC_{\mathcal{C}}(G)$ is satisfied. Also we say that the $(F)ICwF_{\mathcal{C}}(G)$ is satisfied if the $(F)IC_{\mathcal{C}}$ is true for $G \wr H$ for any finite group H . Finally a group homomorphism $p : G \rightarrow K$ is said to *satisfy the $FIC_{\mathcal{C}}$ or the $FICwF_{\mathcal{C}}$* if for $H \in p^*\mathcal{C}(K)$ the $FIC_{\mathcal{C}}$ or the $FICwF_{\mathcal{C}}$ is true for H respectively.

Let us denote by P, K, L , and KH the equivariant homology theories arise in the stable topological pseudoisotopy theory, algebraic K -theory, L -theory and in the homotopy K -theory ([2]) respectively. The conjectures corresponding to these equivariant homology theories are denoted by $(F)IC_{\mathcal{C}}^X$ (or $FICwF_{\mathcal{C}}^X$) where $X = P, K, L$ or KH respectively. For the first three theories we shorten the notation $(F)IC_{\mathcal{V}\mathcal{C}}^X$ (or $FICwF_{\mathcal{V}\mathcal{C}}^X$) to $(F)IC^X$ (or $FICwF^X$). And we shorten $(F)IC_{\mathcal{F}\mathcal{L}\mathcal{N}}^{KH}$ (or $FICwF_{\mathcal{F}\mathcal{L}\mathcal{N}}^{KH}$) to $(F)IC^{KH}$ (or $FICwF^{KH}$). The Isomorphism conjectures $(F)IC^P$, $(F)IC^K$ and $(F)IC^L$ are equivalent to the Farrell-Jones conjectures stated in (§1.7) §1.6 in [9]. (For details see [[2], Sections 5 and 6] for the K and L theories and see [[16], 4.2.1 and 4.2.2] for the pseudoisotopy theory.)

Definition 2.2. We say that $\mathcal{T}_{\mathcal{C}}$ (${}_w\mathcal{T}_{\mathcal{C}}$) is satisfied if for a graph of groups \mathcal{G} with vertex groups from the class \mathcal{C} the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$) for $\pi_1(\mathcal{G})$ is true.

Let us now assume that \mathcal{C} contains all the finite groups. We say that ${}_t\mathcal{T}_{\mathcal{C}}$ (${}_f\mathcal{T}_{\mathcal{C}}$) is satisfied if for a graph of groups \mathcal{G} with trivial (finite) edge groups and vertex groups belonging to the class \mathcal{C} , the $FIC_{\mathcal{C}}$ for $\pi_1(\mathcal{G})$ is true. If we replace the $FIC_{\mathcal{C}}$ by the $FICwF_{\mathcal{C}}$ then we denote the corresponding properties by ${}_{wt}\mathcal{T}_{\mathcal{C}}$ (${}_{wf}\mathcal{T}_{\mathcal{C}}$). Clearly ${}_w\mathcal{T}_{\mathcal{C}}$ implies $\mathcal{T}_{\mathcal{C}}$ and ${}_{w*}\mathcal{T}_{\mathcal{C}}$ implies ${}_*\mathcal{T}_{\mathcal{C}}$ where $*$ = t or f .

And we say that $\mathcal{P}_{\mathcal{C}}$ is satisfied if for $G_1, G_2 \in \mathcal{C}$ the product $G_1 \times G_2$ satisfies the $FIC_{\mathcal{C}}$.

$\mathcal{F}\mathcal{P}_{\mathcal{C}}$ is satisfied if whenever the $FIC_{\mathcal{C}}$ is true for two groups G_1 and G_2 then the $FIC_{\mathcal{C}}$ is true for the free product $G_1 * G_2$.

We denote the above properties for the equivariant homology theories P, K, L or KH with only a super-script by P, K, L or KH respectively. For example, $\mathcal{T}_{\mathcal{C}}$ for P is denoted by \mathcal{T}^P since in all the first three cases we set $\mathcal{C} = \mathcal{V}\mathcal{C}$ and for KH we set $\mathcal{C} = \mathcal{F}\mathcal{L}\mathcal{N}$.

We will show in this article and in [22] that the above properties are satisfied in several instances of the conjecture.

For the rest of this section we assume that \mathcal{C} is also closed under quotients and contains all the finite groups.

Proposition 2.1. (Graphs of groups). *Let $\mathcal{C} = \mathcal{FIN}$ or \mathcal{VC} . Let \mathcal{G} be a graph of groups and there is a homomorphism $f : \pi_1(\mathcal{G}) \rightarrow Q$ onto another group Q so that the restriction of f to any vertex group has finite kernel. If the $FIC_{\mathcal{C}}(Q)$ ($FICwF_{\mathcal{C}}(Q)$) is satisfied then the $FIC_{\mathcal{C}}(\pi_1(\mathcal{G}))$ ($FICwF_{\mathcal{C}}(\pi_1(\mathcal{G}))$) is also satisfied provided one of the following holds. (In the $FICwF_{\mathcal{C}}$ -case in addition assume that $\mathcal{P}_{\mathcal{C}}$ is satisfied.)*

- (a). $\mathcal{T}_{\mathcal{C}}$ (${}_w\mathcal{T}_{\mathcal{C}}$) is satisfied.
- (b). The edge groups of \mathcal{G} are finite and ${}_f\mathcal{T}_{\mathcal{C}}$ (${}_w{}_f\mathcal{T}_{\mathcal{C}}$) is satisfied.
- (c). The edge groups of \mathcal{G} are trivial and ${}_t\mathcal{T}_{\mathcal{C}}$ (${}_w{}_t\mathcal{T}_{\mathcal{C}}$) is satisfied.

Proposition 2.1 will be proved in Section 5.

For definition of continuous $\mathcal{H}_*^?$ in the following statement see [[2], Definition 3.1]. Also see Proposition 5.1.

Proposition 2.2. (Graphs of residually finite groups) *Assume that $\mathcal{P}_{\mathcal{C}}$ and ${}_w{}_t\mathcal{T}_{\mathcal{C}}$ are satisfied. Let \mathcal{G} be a graph of groups. If \mathcal{G} is infinite then assume that $\mathcal{H}_*^?$ is continuous.*

(1). *Assume that the edge groups of \mathcal{G} are finite and the vertex groups are residually finite. If the $FICwF_{\mathcal{C}}$ is true for the vertex groups of \mathcal{G} , then it is true for $\pi_1(\mathcal{G})$.*

For the next three items assume that $\mathcal{C} = \mathcal{VC}$.

(2). *Let \mathcal{D} be the collection of groups as defined in Theorem 1.2 replacing ‘ $FICwF^P$ ’ by ‘ $FICwF_{\mathcal{VC}}$ ’. Assume that the vertex groups of \mathcal{G} belong to \mathcal{D} . Then the $FICwF_{\mathcal{VC}}$ is true for $\pi_1(\mathcal{G})$ if \mathcal{G} satisfies the intersection property.*

(3). *Assume that the vertex groups of \mathcal{G} are virtually polycyclic and the $FICwF_{\mathcal{VC}}$ is true for virtually polycyclic groups. Then the $FICwF_{\mathcal{VC}}$ is true for $\pi_1(\mathcal{G})$ provided either \mathcal{G} satisfies the intersection property or the vertex groups are finitely generated nilpotent and $\pi_1(\mathcal{G})$ is subgroup separable.*

(4). *Assume that the vertex and the edge groups of any component subgraph (see Definition 1.2) are fundamental groups of closed surfaces of genus ≥ 2 and for every component subgraph \mathcal{H} which has at least one edge there is a subgroup $C < \cap_{e \in \mathcal{H}} \mathcal{H}_e$ which is of finite index in some edge group and is normal in $\pi_1(\mathcal{H})$. Then the $FICwF_{\mathcal{VC}}$ is true for $\pi_1(\mathcal{G})$ provided the $FICwF_{\mathcal{VC}}$ is true for the fundamental groups of closed 3-manifolds which fibers over the circle.*

Let F^∞ denote a countable infinitely generated free group and \mathbb{Z}^∞ denotes a countable infinitely generated abelian group. Also $G \rtimes H$ denotes a semidirect product with respect to an arbitrary action of H on G . When H is infinite cyclic and generated by the symbol t , we denote it by $\langle t \rangle$

Proposition 2.3. (Graphs of abelian groups). *Let \mathcal{G} be a graph of groups whose vertex groups are finitely generated abelian and let \mathcal{H}_*^2 be continuous.*

(1). *Assume that the $FIC_{\mathcal{V}\mathcal{C}}$ is true for $F^\infty \rtimes \langle t \rangle$ and that $\mathcal{P}_{\mathcal{V}\mathcal{C}}$ is satisfied. Then the $FIC_{\mathcal{V}\mathcal{C}}$ is true for $\pi_1(\mathcal{G})$ provided one of the following holds.*

(a). *\mathcal{G} is a tree and the vertex groups of \mathcal{G} are torsion free.*

For the next two items assume that the $FICwF_{\mathcal{V}\mathcal{C}}$ is true for $F^\infty \rtimes \langle t \rangle$.

(b). *\mathcal{G} is a tree.*

(c). *\mathcal{G} is not a tree and the $FIC_{\mathcal{V}\mathcal{C}}$ is true for $\mathbb{Z}^\infty \rtimes \langle t \rangle$ for any countable infinitely generated abelian group \mathbb{Z}^∞ .*

(2). *Assume that $\mathcal{P}_{\mathcal{V}\mathcal{C}}$ and ${}_{wt}\mathcal{T}_{\mathcal{V}\mathcal{C}}$ are satisfied. Further assume that \mathcal{G} is almost a tree of groups and the vertex and the edge groups of any component subgraph of \mathcal{G} have the same rank. Then the $FICwF_{\mathcal{V}\mathcal{C}}$ is true for $\pi_1(\mathcal{G})$ provided one of the followings is satisfied.*

(i). *The $FICwF_{\mathcal{V}\mathcal{C}}$ is true for $\mathbb{Z}^\infty \rtimes \langle t \rangle$ for any countable infinitely generated abelian group \mathbb{Z}^∞ .*

(ii). *The vertex and the edge groups of any component subgraph of \mathcal{G} have rank equal to 1.*

(3). *The $IC_{\mathcal{F}LN}^L$ is true for $\pi_1(\mathcal{G})$ provided the vertex groups of \mathcal{G} are torsion free.*

In (1) of Proposition 2.3 if we assume that the $FICwF_{\mathcal{V}\mathcal{C}}$ is true for $\mathbb{Z}^\infty \rtimes \langle t \rangle$ and for $F^\infty \rtimes \langle t \rangle$ then one can deduce from the same proof, using (3) of Proposition 5.2 instead of Lemma 5.2, that the $FICwF_{\mathcal{V}\mathcal{C}}$ is true for $\pi_1(\mathcal{G})$ irrespective of whether \mathcal{G} is a tree or not.

Proposition 2.4. (${}_w\mathcal{T}^P$). *Let \mathcal{G} be a graph of virtually cyclic groups so that the graph is almost a tree. Further assume that either the edge groups are finite or the infinite vertex groups are abelian. Then the $FICwF^P$ is true for $\pi_1(\mathcal{G})$.*

An immediate corollary of the Proposition is the following.

Corollary 2.1. (${}_{wf}\mathcal{T}^P$). *${}_{wf}\mathcal{T}^P$ (and hence ${}_{wt}\mathcal{T}^P$) is satisfied.*

Remark 2.1. We remark here that it is not yet known if the FIC^P is true for the HNN-extension $G = C *_C$, with respect to the maps $id : C \rightarrow C$ and $f : C \rightarrow C$ where C is an infinite cyclic group, id is

the identity map and $f(u) = u^2$ for $u \in C$. This was already mentioned in the introduction in [11]. Note that G is isomorphic to the semidirect product $\mathbb{Z}[\frac{1}{2}] \rtimes \langle t \rangle$, where t acts on $\mathbb{Z}[\frac{1}{2}]$ by multiplication by 2. The main problem with this example is that $\mathbb{Z}[\frac{1}{2}] \rtimes \langle t \rangle$ is not subgroup separable.

Now we recall that a property (*tree property*) similar to \mathcal{T}_C was defined in [[2], Definition 4.1]. The tree property of [2] is stronger than \mathcal{T}_C . Corollary 4.4 in [2] was proved under the assumption that the tree property is satisfied. In the following proposition we state that this is true with the weaker assumption that \mathcal{T}_{FLN} is satisfied. The proof of the Proposition goes exactly in the same way as the proof of [[2], Corollary 4.4]. It can also be deduced from Proposition 2.1.

Proposition 2.5. *Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be an exact sequence of groups. Assume that \mathcal{T}_{FLN} is satisfied and K acts on a tree with finite stabilizers and that the $FIC_{FLN}(Q)$ is satisfied. Then the $FIC_{FLN}(G)$ is also satisfied.*

3. GRAPHS OF GROUPS

In this section we prove some results on graphs of groups needed for the proofs of the Theorems and Propositions.

We start by recalling that by Bass-Serre theory a group acts on a tree without inversion if and only if the group is isomorphic to the fundamental group of a graph of groups (the Structure Theorem I.4.1 in [8]). Therefore, throughout the paper by ‘action of a group on a tree’ we will mean an action without inversion.

Lemma 3.1. *Let \mathcal{G} be a finite and almost a tree of groups (see Definition 1.2). Then there is another graph of groups \mathcal{H} with the following properties.*

- (1). $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{H})$.
- (2). *Either \mathcal{H} has no edge or the edge groups of \mathcal{H} are finite.*
- (3). *The vertex groups of \mathcal{H} are of the form $\pi_1(\mathcal{K})$ where \mathcal{K} varies over subgraphs of groups of \mathcal{G} which are maximal with respect to the property that the underlying graph of \mathcal{K} is a tree and the edge (if there is any) groups of \mathcal{K} are all infinite.*

Proof. The proof is by induction on the number of finite edges of the graph. Recall that an edge is called a finite edge if the corresponding edge group is finite (Definition 1.2). If the graph has no finite edge then by definition of almost a graph of groups \mathcal{G} is a tree. In this case we can take \mathcal{H} to be a graph consisting of a single vertex and associate the group $\pi_1(\mathcal{G})$ to the vertex. So assume \mathcal{G} has n finite edges and that

the Lemma is true for graphs with $\leq n - 1$ finite edges. Let e be an edge of \mathcal{G} with \mathcal{G}_e finite. If $\mathcal{G} - \{e\}$ is connected then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) *_{\mathcal{G}_e}$ where $\mathcal{G}_1 = \mathcal{G} - \{e\}$ is a graph with $n - 1$ finite edges. By the induction hypothesis there is a graph of groups \mathcal{H}_1 satisfying (1), (2) and (3) for \mathcal{G}_1 . Let v_1 and v_2 be the vertices of \mathcal{G} to which the ends of e are attached and let v'_1 and v'_2 be the vertices of \mathcal{H}_1 so that \mathcal{G}_{v_i} is a subgroup of $\mathcal{H}_{v'_i}$ for $i = 1, 2$. (Note that v_1 and v_2 could be the same vertex). Define \mathcal{H} by attaching an edge e' to \mathcal{H}_1 so that the ends of e' are attached to v'_1 and v'_2 and associate the group \mathcal{G}_e to e' . The injective homomorphisms $\mathcal{H}_{e'} \rightarrow \mathcal{H}_{v'_i}$ for $i = 1, 2$ are defined by the homomorphisms $\mathcal{G}_e \rightarrow \mathcal{G}_{v_i}$. It is now easy to check that \mathcal{H} satisfies (1), (2) and (3) for \mathcal{G} . On the other hand if $\mathcal{G} - \{e\}$ has two components say \mathcal{G}_1 and \mathcal{G}_2 then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) *_{\mathcal{G}_e} \pi_1(\mathcal{G}_2)$ where \mathcal{G}_1 and \mathcal{G}_2 has $\leq n - 1$ finite edges. Using the induction hypothesis and a similar argument as above we complete the proof of the Lemma. \square

Lemma 3.2. *A finitely generated group contains a free subgroup of finite index if and only if the group acts on a tree with finite stabilizers. And a group acts on a tree with trivial stabilizers if and only if the group is free.*

Proof. By [[8], Theorem IV.1.6] a group contains a free subgroup of finite index if and only if the group acts on a tree with finite stabilizers and the stabilizers have bounded order. The proof of the Lemma now follows easily. \square

We will also need the following two Lemmas.

Lemma 3.3. *Let \mathcal{G} be a graph of finitely generated abelian groups so that the underlying graph of \mathcal{G} is a tree. Then the restriction of the abelianization homomorphism $\pi_1(\mathcal{G}) \rightarrow H_1(\pi_1(\mathcal{G}), \mathbb{Z})$ to each vertex group of the tree of groups \mathcal{G} is injective.*

Proof. If the tree \mathcal{G} is finite and the vertex groups are finitely generated free abelian then it was proved in [[24], Lemma 3.1] that there is a homomorphism $\pi_1(\mathcal{G}) \rightarrow A$ onto a free abelian group so that the restriction of this homomorphism to each vertex group is injective. In fact it was shown there that the abelianization homomorphism $\pi_1(\mathcal{G}) \rightarrow H_1(\pi_1(\mathcal{G}), \mathbb{Z})$ is injective when restricted to the vertex groups and then since the vertex groups are torsion free $\pi_1(\mathcal{G}) \rightarrow A = H_1(\pi_1(\mathcal{G}), \mathbb{Z}) / \{torsion\}$ is also injective on the vertex groups. The same proof goes through, without the torsion free vertex group assumption, to prove the Lemma when \mathcal{G} is finite. Therefore we mention the additional arguments needed in the infinite case. At first write \mathcal{G}

as an increasing union of finite trees \mathcal{G}_i of finitely generated abelian groups. Consider the following commutative diagram.

$$\begin{array}{ccc} \pi_1(\mathcal{G}_i) & \longrightarrow & H_1(\pi_1(\mathcal{G}_i), \mathbb{Z}) \\ \downarrow & & \downarrow \\ \pi_1(\mathcal{G}_{i+1}) & \longrightarrow & H_1(\pi_1(\mathcal{G}_{i+1}), \mathbb{Z}) \end{array}$$

Note that the left hand side vertical map is injective. And by the finite tree case the restriction of the two horizontal maps to each vertex group of the respective trees are injective. Now since group homology and fundamental group commute with direct limit, taking limit completes the proof of the Lemma. \square

Lemma 3.4. *Let \mathcal{G} be a finite graph of finitely generated groups satisfying the following.*

P. *Each edge group is of finite index in the end vertex groups of the edge. Also assume that the intersection of the edge groups contains a subgroup C (say) which is normal in $\pi_1(\mathcal{G})$ and is of finite index in some edge (e say) group.*

Then $\pi_1(\mathcal{G})/C$ is isomorphic to the fundamental group of a finite-type (see Definition 1.1) graph of groups. Consequently \mathcal{G} has the intersection property.

Proof. The proof is by induction on the number of edges of the graph.

Induction hypothesis. (IH_n) For any finite graph of groups \mathcal{G} with $\leq n$ edges which satisfies **P**, $\pi_1(\mathcal{G})/C$ is isomorphic to the fundamental group of a graph of groups whose underlying graph is the same as that of \mathcal{G} and the vertex groups are finite and isomorphic to \mathcal{G}_v/C where $v \in V_{\mathcal{G}}$.

If \mathcal{G} has one edge e then $C < \mathcal{G}_e$ and C is normal in $\pi_1(\mathcal{G})$. First assume e disconnects \mathcal{G} . Since \mathcal{G}_e is of finite index in the end vertex groups and C is also of finite index in \mathcal{G}_e , C is of finite index in the end vertex groups of e . Therefore $\pi_1(\mathcal{G})/C$ has the desired property. The argument is same when e does not disconnect \mathcal{G} .

Now assume \mathcal{G} has n edges and satisfies **P**.

Let us first consider the case that there is an edge e' other than e so that $\mathcal{G} - \{e'\} = \mathcal{D}$ (say) is a connected graph. Note that \mathcal{D} has $n - 1$ edges and satisfies **P**.

Hence by IH_{n-1} , $\pi_1(\mathcal{D})/C$ is isomorphic to the fundamental group of a finite-type graph of groups with \mathcal{D} as the underlying graph and the vertex groups are of the form \mathcal{D}_v/C .

Let v be an end vertex of e' . Then \mathcal{D}_v/C is finite. Also by hypothesis $\mathcal{G}_{e'}$ is of finite index in $\mathcal{G}_v = \mathcal{D}_v$. Therefore $\mathcal{G}_{e'}/C$ is also finite. This completes the proof in this case.

Now if every edge $e' \neq e$ disconnects \mathcal{G} then $\mathcal{G} - \{e\}$ is a tree. Let e' be an edge other than e so that one end vertex v' (say) of e' has valency 1. Such an edge exists because $\mathcal{G} - \{e\}$ is a tree. Let $\mathcal{D} = \mathcal{G} - (\{e'\} \cup \{v'\})$. Then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{D}) *_{\mathcal{G}_{e'}} \mathcal{G}_{v'}$. Let v'' be the other end vertex of e' . Then by the induction hypothesis C is of finite index in $\mathcal{G}_{v''} = \mathcal{D}_{v''}$. Hence C is of finite index in $\mathcal{G}_{e'}$. Also $\mathcal{G}_{e'}$ is of finite index in $\mathcal{G}_{v'}$. Therefore C is also of finite index in $\mathcal{G}_{v'}$.

This completes the proof. \square

The following Lemma and Example give some concrete examples of graphs of groups with intersection property.

Lemma 3.5. *Let \mathcal{G} be a finite graph of groups so that all the vertex and the edge groups are finitely generated abelian and of the same rank r (say) and the underlying graph of \mathcal{G} is a tree. Then $\cap_{e \in E_{\mathcal{G}}} \mathcal{G}_e$ contains a rank r free abelian subgroup C which is normal in $\pi_1(\mathcal{G})$ so that $\pi_1(\mathcal{G})/C$ is isomorphic to the fundamental group of a graph of groups whose underlying graph is \mathcal{G} and vertex groups are finite and isomorphic to \mathcal{G}_v/C where $v \in V_{\mathcal{G}}$.*

Proof. The proof is by induction on the number of edges. If the graph has one edge e then clearly \mathcal{G}_e is normal in $\pi_1(\mathcal{G})$, for \mathcal{G}_e is normal in the two end vertex groups. This follows from Lemma 3.7. So by induction assume that the Lemma is true for graphs with $\leq n-1$ edges. Let \mathcal{G} be a finite graph with n edges and satisfies the hypothesis of the Lemma. Consider an edge e which has one end vertex v (say) with valency 1. Such an edge exists because the graph is a tree. Let v_1 be the other end vertex of e . Then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}') *_{\mathcal{G}_e} \mathcal{G}_v$. Here $\mathcal{G}' = \mathcal{G} - (\{e\} \cup \{v\})$. Clearly by induction hypothesis there is a finitely generated free abelian normal subgroup $C_1 < \cap_{e' \in E_{\mathcal{G}'}} \mathcal{G}'_{e'}$ of rank r of $\pi_1(\mathcal{G}_1)$ satisfying all the required properties. Now note that $C_1 \cap \mathcal{G}_e = C'$ (say) is of finite index in \mathcal{G}'_{v_1} and also in C_1 and C' has rank r . This follows from the following easy to verify Lemma. Now since C_1 is finitely generated and C' is of finite index in C_1 we can find a characteristic subgroup $C (< C')$ of C_1 of finite index. Therefore C has rank r and is normal in $\pi_1(\mathcal{G}')$, since C_1 is normal in $\pi_1(\mathcal{G}')$. Obviously C is normal in \mathcal{G}_v . Therefore, we again use Lemma 3.7 to conclude that C is normal in $\pi_1(\mathcal{G})$. The other properties are clearly satisfied. This completes the proof of the Lemma.

Lemma 3.6. *Let G be a finitely generated abelian group of rank r . Let G_1, \dots, G_k be rank r subgroups of G . Then $\cap_{i=1}^{i=k} G_i$ is of rank r and of finite index in G .*

□

Example 3.1. Let G and H be finitely generated groups and let H be a finite index normal subgroup of G . Let $f : H \rightarrow G$ be the inclusion. Consider a finite tree of groups \mathcal{G} whose vertex groups are copies of G and the edge groups are copies of H . Also assume that the maps from the edge groups to the vertex groups (defining the tree of group structure) are f . Then \mathcal{G} has the intersection property.

Lemma 3.7. *Let $G = G_1 *_H G_2$ be a generalized free product. If H is normal in both G_1 and G_2 then H is normal in G .*

Proof. The proof follows by using the normal form of elements in a generalized free product. See [[17], p. 72]. □

4. RESIDUALLY FINITE GROUPS

In this section we recall and also prove some basic results we need on residually finite groups. For this section we abbreviate ‘residually finite’ by \mathcal{RF} .

Lemma 4.1. *The fundamental group of a finite graph of \mathcal{RF} groups with finite edge groups is \mathcal{RF} .*

Proof. The proof is by induction on the number of edges. If there is no edge then there is nothing to prove. So assume the Lemma for graphs with $\leq n - 1$ edges. Let \mathcal{G} be a graph of groups with n edges and satisfies the hypothesis. It follows that $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) *_F \pi_1(\mathcal{G}_2)$ or $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) *_F$ where F is a finite group and \mathcal{G}_i satisfy the hypothesis of the Lemma and has $\leq n - 1$ edges. Also note that $\pi_1(\mathcal{G}_1) *_F \pi_1(\mathcal{G}_2)$ can be embedded as a subgroup in $(\pi_1(\mathcal{G}_1) * \pi_1(\mathcal{G}_2)) *_F$. Therefore by the induction hypothesis and since free product of \mathcal{RF} groups is again \mathcal{RF} (see [13]) we only need to prove that, for finite H , $G *_H$ is \mathcal{RF} if so is G . But, this follows from [6] or [[7], Theorem 2]. This completes the proof of the Lemma. □

Lemma 4.2. *Let $1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$ be an extension of groups so that K and H are \mathcal{RF} . Assume that any finite index subgroup of K contains a subgroup K' so that K' is normal in G and G/K' is \mathcal{RF} . Then G is \mathcal{RF} .*

Proof. Let $g \in G - K$ and $g' \in H$ be the image of g in H . Since H is \mathcal{RF} there is a finite index subgroup H' of H not containing g' . The inverse image of H' in G is a finite index subgroup not containing g .

Next let $g \in K - \{1\}$. Choose a finite index subgroup of K not containing g . By hypothesis there is a finite index subgroup K' of K which is normal in G and does not contain g . Also G/K' is \mathcal{RF} . Now applying the previous case we complete the proof. \square

Lemma 4.3. *Let \mathcal{G} be a finite graph of finitely generated \mathcal{RF} groups satisfying the intersection property. Assume the following. ‘Given an edge e and an end vertex v of e , for every subgroup E of \mathcal{G}_e which is normal in \mathcal{G}_v , the quotient \mathcal{G}_v/E is again \mathcal{RF} .’ Then $\pi_1(\mathcal{G})$ is \mathcal{RF} .*

Proof. Using Lemma 4.1 we can assume that all the edge groups of \mathcal{G} are infinite. Now the proof is by induction on the number of edges of the graph \mathcal{G} . Clearly the induction starts because if there is no edge then the Lemma is true. Assume the result for all graphs satisfying the hypothesis with number of edges $\leq n - 1$ and let \mathcal{G} be a graph of groups with n number of edges and satisfying the hypothesis of the Lemma. By the intersection property there is a normal subgroup K , contained in all the edge groups, of $\pi_1(\mathcal{G})$ which is of finite index in some edge group. Hence we have the following.

K. $\pi_1(\mathcal{G})/K$ is isomorphic to the fundamental group of a finite graph of \mathcal{RF} groups (by hypothesis the quotient of a vertex group by K is \mathcal{RF}) with n edges and some edge group is finite. Also it is easily seen that this quotient graph of groups has the intersection property.

By the induction hypothesis and by Lemma 4.1 $\pi_1(\mathcal{G})/K$ is \mathcal{RF} .

Now we would like to apply Lemma 4.2 to the exact sequence.

$$1 \rightarrow K \rightarrow \pi_1(\mathcal{G}) \rightarrow \pi_1(\mathcal{G})/K \rightarrow 1.$$

Let H be a finite index subgroup of K . Since K is finitely generated (being of finite index in a finitely generated group) we can find a finite index characteristic subgroup H' of K contained in H . Hence H' is normal in $\pi_1(\mathcal{G})$. It is now easy to see that **K** is satisfied if we replace K by H' . Hence $\pi_1(\mathcal{G})/H'$ is \mathcal{RF} .

Therefore, by Lemma 4.2 $\pi_1(\mathcal{G})$ is \mathcal{RF} . \square

Lemma 4.4. *Let \mathcal{G} be a finite graph of virtually cyclic groups so that either the edge groups are finite or the infinite vertex groups are abelian and the associated graph is almost a tree. Then $\pi_1(\mathcal{G})$ is \mathcal{RF} .*

Proof. Applying Lemma 4.1 and using the definition of almost a graph of groups we can assume that the graph is a tree and all the edge

groups are infinite. Now using Lemma 3.5 we see that the hypothesis of Lemma 4.3 is satisfied. This proves the Lemma. \square

Lemma 4.5. *Let \mathcal{G} be a finite graph of groups whose vertex and edge groups are fundamental groups of closed surfaces of genus ≥ 2 . Also assume that the intersection of the edge groups contains a subgroup C (say) which is normal in $\pi_1(\mathcal{G})$ and is of finite index in some edge group. Then $\pi_1(\mathcal{G})$ contains a normal subgroup isomorphic to the fundamental group of a closed surface so that the quotient is isomorphic to the fundamental group of a finite-type graph of groups. Also $\pi_1(\mathcal{G})$ is \mathcal{RF} .*

Proof. We need the following Lemma.

Lemma 4.6. *Let S be a closed surface. Let G be a subgroup of $\pi_1(S)$. Then G is isomorphic to the fundamental group of a closed surface if and only if G is of finite index in $\pi_1(S)$.*

Proof. The proof follows from covering space theory. \square

Therefore, using the above Lemma we get that the edge groups of \mathcal{G} are of finite index in the end vertex groups of the corresponding edges. Hence by Lemma 3.4 $\pi_1(\mathcal{G})/C$ is the fundamental group of a finite-type graph of groups. This proves the first statement.

Now by Lemma 4.1 $\pi_1(\mathcal{G})/C$ is \mathcal{RF} . Next, by [5] closed surface groups are \mathcal{RF} . Then using the Lemma above, it is easy to check that the hypothesis of Lemma 4.2 is satisfied for the following exact sequence.

$$1 \rightarrow C \rightarrow \pi_1(\mathcal{G}) \rightarrow \pi_1(\mathcal{G})/C \rightarrow 1.$$

Hence $\pi_1(\mathcal{G})$ is \mathcal{RF} . \square

5. BASIC RESULTS ON THE ISOMORPHISM CONJECTURE

In this section we recall some known facts as well as deduce some basic results on the Isomorphism conjecture.

\mathcal{C} always denotes a class of groups closed under isomorphisms and taking subgroups unless otherwise mentioned.

We start by noting that if the $\text{FIC}_{\mathcal{C}}$ is true for a group G then the $\text{FIC}_{\mathcal{C}}$ is also true for any subgroup H of G . We will refer to this fact as the *hereditary property* in this paper.

By the Algebraic Lemma in [12] if G is a normal subgroup of K then K can be embedded in the wreath product $G \wr (K/G)$. We will be using this fact throughout the paper without explicitly mentioning it.

Lemma 5.1. *If the $\text{FICwF}_{\mathcal{C}}(G)$ is satisfied then the $\text{FICwF}_{\mathcal{C}}(L)$ is also satisfied for any subgroup L of G .*

Proof. Note that given a group H , $L \wr H$ is a subgroup of $G \wr H$. Now use the hereditary property of the $\text{FIC}_{\mathcal{C}}$. \square

Proposition 5.1. *Assume that $\mathcal{H}_*^?$ is continuous. Let G be a group and $G = \cup_{i \in I} G_i$ where G_i 's are increasing sequence of subgroups of G so that the $\text{FIC}_{\mathcal{C}}(G_i)$ is satisfied for $i \in I$. Then the $\text{FIC}_{\mathcal{C}}(G)$ is also satisfied. And if the $\text{FICwF}_{\mathcal{C}}(G_i)$ is satisfied for $i \in I$ then the $\text{FICwF}_{\mathcal{C}}(G)$ is also satisfied.*

Proof. The first assertion is the same as the conclusion of [[2], Proposition 3.4] and the second one is easily deducible from it, since given a group H , $G \wr H = \cup_{i \in I} (G_i \wr H)$. \square

Remark 5.1. Since the fundamental group of an infinite graph of groups can be written as an increasing union of fundamental groups of finite subgraphs, throughout rest of the paper we consider only finite graphs. The infinite case will then follow if the corresponding equivariant homology theory satisfies the assumption of Proposition 5.1. Examples of such equivariant homology theories are P , K , L , and KH . See Section 8.

The following Lemma from [2] is crucial for proofs of the results in this paper. This result in the context of the original Fibered Isomorphism conjecture ([9]) was proved in [[9], Proposition 2.2].

Lemma 5.2. (*[[2], Lemma 2.5]*) *Let \mathcal{C} be also closed under taking quotients. Let $p : G \rightarrow Q$ be a surjective group homomorphism and assume that the $\text{FIC}_{\mathcal{C}}$ is true for Q and for p . Then G satisfies the $\text{FIC}_{\mathcal{C}}$.*

Proposition 5.2. *Let \mathcal{C} be as in the statement of the above Lemma. Assume that $\mathcal{P}_{\mathcal{C}}$ is satisfied.*

(1). *If the $\text{FIC}_{\mathcal{C}}$ ($\text{FICwF}_{\mathcal{C}}$) is true for G_1 and G_2 then the $\text{FIC}_{\mathcal{C}}$ ($\text{FICwF}_{\mathcal{C}}$) is true for $G_1 \times G_2$.*

(2). *Let G be a finite index normal subgroup of a group K . If the $\text{FICwF}_{\mathcal{C}}(G)$ is satisfied then the $\text{FICwF}_{\mathcal{C}}(K)$ is also satisfied.*

(3). *Let $p : G \rightarrow Q$ be a group homomorphism. If the $\text{FICwF}_{\mathcal{C}}$ is true for Q and for p then the $\text{FICwF}_{\mathcal{C}}$ is true for G .*

Proof. The proof of (1) is essentially two applications of Lemma 5.2. First apply it to the projection $G_1 \times G_2 \rightarrow G_2$. Hence to prove the Lemma we need to check that the $\text{FIC}_{\mathcal{C}}$ is true for $G_1 \times H$ for any $H \in \mathcal{C}(G_2)$. Now fix $H \in \mathcal{C}(G_2)$ and apply Lemma 5.2 to the projection $G_1 \times H \rightarrow G_1$. Thus we need to show that the $\text{FIC}_{\mathcal{C}}$ is true for $K \times H$ where $K \in \mathcal{C}(G_1)$. But this is exactly $\mathcal{P}_{\mathcal{C}}$ which is true by hypothesis.

Next note that given a group H , $(G_1 \times G_2) \wr H$ is a subgroup of $(G_1 \wr H) \times (G_2 \wr H)$. Therefore the $\text{FICwF}_{\mathcal{C}}$ is true for $G_1 \times G_2$ if it is true for G_1 and G_2 .

For (2) let $H = K/G$. Then K is a subgroup of $G \wr H$. Let L be a finite group then it is easy to check that

$$\begin{aligned} K \wr L &< (G \wr H) \wr L \simeq G^{H \times L} \rtimes (H \wr L) \\ &< G^{H \times L} \wr (H \wr L) < \Pi_{|H \times L| \text{-times}}(G \wr (H \wr L)). \end{aligned}$$

The isomorphism in the above display follows from [[11], Lemma 2.5] with respect to some action of $H \wr L$ on $G^{H \times L}$ (replace X by A and Y by B in [[11], Lemma 2.5]). The second inclusion follows from the Algebraic Lemma in [12].

Now using (1) and by hypothesis we complete the proof.

For (3) we need to prove that the $\text{FIC}_{\mathcal{C}}$ is true for $G \wr H$ for any finite group H . We now apply Lemma 5.2 to the homomorphism $G \wr H \rightarrow Q \wr H$. By hypothesis the $\text{FIC}_{\mathcal{C}}$ is true for $Q \wr H$. So let $S \in \mathcal{C}(Q \wr H)$. We have to prove that the $\text{FIC}_{\mathcal{C}}$ is true for $p^{-1}(S)$. Note that $p^{-1}(S)$ contains $p^{-1}(S \cap Q^H)$ as a normal subgroup of finite index. Therefore using (2) it is enough to prove the $\text{FICwF}_{\mathcal{C}}$ for $p^{-1}(S \cap Q^H)$. Next, note that $S \cap Q^H$ is a subgroup of $\prod_{h \in H} L_h$, where L_h is the image of $S \cap Q^H$ under the projection to the h -th coordinate of Q^H , and since \mathcal{C} is closed under taking quotient $L_h \in \mathcal{C}(Q)$. Hence $p^{-1}(S \cap Q^H)$ is a subgroup of $\prod_{h \in H} p^{-1}(L_h)$. Since by hypothesis the $\text{FICwF}_{\mathcal{C}}$ is true for $p^{-1}(L_h)$ for $h \in H$, using (1), (2) and Lemma 5.1 we see that the $\text{FICwF}_{\mathcal{C}}$ is true for $p^{-1}(S \cap Q^H)$. This completes the proof. \square

Corollary 5.1. *$\mathcal{P}_{\mathcal{V}\mathcal{C}}$ implies that the $\text{FIC}_{\mathcal{V}\mathcal{C}}$ is true for finitely generated abelian group.*

Proof. The proof is immediate from (1) of Proposition 5.2 since the $\text{FIC}_{\mathcal{V}\mathcal{C}}$ is true for virtually cyclic groups. \square

Remark 5.2. In (2) of Proposition 5.2 if we assume that the $\text{FIC}_{\mathcal{C}}(G)$ is satisfied instead of the $\text{FICwF}_{\mathcal{C}}(G)$ then it is not known how to deduce the $\text{FIC}_{\mathcal{C}}(K)$. Even in the case of the $\text{FIC}_{\mathcal{V}\mathcal{C}}$ and when G is a free group it is open. However if G is free then the $\text{FIC}^P(K)$ is satisfied by results of Farrell-Jones. See the proof of Proposition 5.5 for details. Also using a recent result of Bartels and Lück ([3]) it can be shown by the same method that the $\text{FIC}^L(K)$ is satisfied.

Proposition 5.3. *${}_t\mathcal{T}_{\mathcal{C}}$ (${}_{wt}\mathcal{T}_{\mathcal{C}}$) implies that the $\text{FIC}_{\mathcal{C}}$ ($\text{FICwF}_{\mathcal{C}}$) is true for any free group. And if \mathcal{C} contains the finite groups then ${}_f\mathcal{T}_{\mathcal{C}}$ (${}_{wf}\mathcal{T}_{\mathcal{C}}$) implies that the $\text{FIC}_{\mathcal{C}}$ ($\text{FICwF}_{\mathcal{C}}$) is true for a finitely generated group which contains a free subgroup of finite index.*

Proof. The proof follows from Lemma 3.2. \square

Corollary 5.2. ${}_t\mathcal{T}_C$ (${}_{wt}\mathcal{T}_C$) and \mathcal{P}_C imply that the FIC_C ($FICwF_C$) is true for finitely generated free abelian groups.

Proof. The proof is a combination of Proposition 5.3 and (1) of Proposition 5.2. \square

Let F^n denote a finitely generated free group of rank n .

Lemma 5.3. Assume that the FIC_C is true for $\mathbb{Z}^\infty \rtimes \langle t \rangle$ ($F^\infty \rtimes \langle t \rangle$) for any countable infinitely generated abelian group \mathbb{Z}^∞ , then the FIC_C is true for $\mathbb{Z}^n \rtimes \langle t \rangle$ ($F^n \rtimes \langle t \rangle$) for all $n \in \mathbb{N}$. Here all actions of t on the corresponding groups are arbitrary.

And the same holds if we replace the FIC_C by the $FICwF_C$.

Proof. The proof is an easy consequence of the hereditary property and Lemma 5.1. \square

Lemma 5.4. If the $FICwF_{\mathcal{V}C}$ is true for $F^\infty \rtimes \langle t \rangle$ for any action of t on F^∞ , then $\mathcal{P}_{\mathcal{V}C}$ is satisfied.

Proof. Note that $\mathbb{Z} \times \mathbb{Z}$ is a subgroup of $F^\infty \rtimes_t \langle t \rangle$, where the suffix ‘ t ’ denotes the trivial action of $\langle t \rangle$ on F^∞ . Hence the $FICwF_{\mathcal{V}C}$ is true for $\mathbb{Z} \times \mathbb{Z}$, that is the $FIC_{\mathcal{V}C}$ is true for $(\mathbb{Z} \times \mathbb{Z}) \wr F$ for any finite group F . On the other hand for two virtually cyclic groups C_1 and C_2 the product $C_1 \times C_2$ contains a finite index free abelian normal subgroup (say H) of rank ≤ 2 (see Lemma 6.1), and therefore $C_1 \times C_2$ is a subgroup of $H \wr F$ for a finite group F . Using the hereditary property we conclude that $\mathcal{P}_{\mathcal{V}C}$ is satisfied. \square

Proposition 5.4. The $FICwF^P$ is true for any virtually polycyclic group.

Proof. By [[9], Proposition 2.4] the FIC^P is true for any virtually poly-infinite cyclic group. Also a polycyclic group is virtually poly-infinite cyclic. Now it is easy to check that the wreath product of a virtually polycyclic group with a finite group is virtually poly-infinite cyclic. This completes the proof. \square

Since the product of two virtually cyclic groups is virtually polycyclic an immediate corollary is the following. But we give a proof of the Corollary independent of Proposition 5.4.

Corollary 5.3. \mathcal{P}^P is satisfied. Also $FICwF^P$ is true for any virtually cyclic group.

Proof of Corollary 5.3. At first note that the FIC^P is true for virtually cyclic groups. Hence for the first part we only have to prove that the FIC^P is true for $V_1 \times V_2$ where V_1 and V_2 are two infinite virtually cyclic groups. Note that $V_1 \times V_2$ contains a finite index free abelian normal subgroup, say A , on two generators. Therefore $V_1 \times V_2$ embeds in $A \wr H$ for some finite group H . Since A is isomorphic to the fundamental group of a flat 2-torus, FIC^P is true for $A \wr H$. See Fact 3.1 and Theorem A in [12]. Therefore FIC^P is true for $V_1 \times V_2$ by the hereditary property. This proves that \mathcal{P}^P is satisfied.

The proof of the second part is similar since for any virtually cyclic group V and for any finite group H , $V \wr H$ is either finite or embeds in a group of the type $A \wr H'$ for some finite group H' and where A is isomorphic to a free abelian group on $|H|$ number of generators and therefore A is isomorphic to the fundamental group of a flat $|H|$ -torus. Then we can again apply Fact 3.1 and Theorem A from [12]. \square

We will also need the following proposition.

Proposition 5.5. *Let \mathcal{G} be a graph of finite groups. Then $\pi_1(\mathcal{G})$ satisfies the FICwF^P .*

Proof. By Remark 5.1 we can assume that the graph is finite. Lemma 3.2 implies that we need to show that the FICwF^P is true for finitely generated groups which contains a free subgroup of finite index. Now it is a formal consequence of results of Farrell-Jones that the FICwF^P is true for a free group. For details see [[11], Lemma 2.4]. Also compare [[12], Fact 3.1]. Next since \mathcal{P}^P is satisfied (Corollary 5.3) using (2) of Proposition 5.2 we complete the proof. \square

A version of Proposition 5.5 can also be proven for FICwF^L using the result that the FIC^L is true for a group which acts on a finite dimensional $\text{CAT}(0)$ -space from [3].

6. PROOFS OF THE PROPOSITIONS

Recall that \mathcal{C} always denotes a class of groups which is closed under isomorphism, taking subgroups and quotients and contains all the finite groups.

The proofs of the Propositions appear in the following sequence: 2.3-2.4-2.1-2.2.

Proof of Proposition 2.3. Proof of (1). Since $\mathcal{H}_*^?$ is continuous by Remark 5.1 we can assume that the graph \mathcal{G} is finite.

1(a). Since the graph \mathcal{G} is a tree and the vertex groups are torsion free by Lemma 3.3 the restriction of the homomorphism $f : \pi_1(\mathcal{G}) \rightarrow$

$H_1(\pi_1(\mathcal{G}), \mathbb{Z})/\{\text{torsion}\} = A(\text{say})$ to each vertex group is injective. Let K be the kernel of f . Let T be the tree on which $\pi_1(\mathcal{G})$ acts for the graph of group structure \mathcal{G} . Then K also acts on T with vertex stabilizers $K \cap g\mathcal{G}_v g^{-1} = (1)$ where $g \in \pi_1(\mathcal{G})$ and $v \in V_{\mathcal{G}}$. Hence by Lemma 3.2 K is a free group (not necessarily finitely generated). Next note that A is a finitely generated free abelian group and hence the $\text{FIC}_{\mathcal{V}\mathcal{C}}$ is true for A by Corollary 5.1 and by hypothesis. Now applying Lemma 5.2 to the homomorphism $f : \pi_1(\mathcal{G}) \rightarrow A$ and noting that a torsion free virtually cyclic group is either trivial or infinite cyclic (see [[10], Lemma 2.5]) we complete the proof. We also need to use Lemma 5.3 when K is finitely generated.

1(b). The proof of this case is almost the same as that of the previous case.

Let $A = H_1(\pi_1(\mathcal{G}), \mathbb{Z})$. Now A is a finitely generated abelian group and hence the $\text{FIC}_{\mathcal{V}\mathcal{C}}$ is true for A by Corollary 5.1. Next we apply Lemma 5.2 to $p : \pi_1(\mathcal{G}) \rightarrow A$. Again by Lemma 3.3 the kernel K of this homomorphism acts on a tree with trivial stabilizers and hence K is free. Let V be a virtually cyclic subgroup of A with $C < V$ be infinite cyclic subgroup of finite index in V . Let C be generated by t . Then the inverse image $p^{-1}(V)$ contains $K \rtimes \langle t \rangle$ as a normal finite index subgroup. By hypothesis the $\text{FICwF}_{\mathcal{V}\mathcal{C}}$ is true for $K \rtimes \langle t \rangle$. Now using (2) of Proposition 5.2 we see that the $\text{FICwF}_{\mathcal{V}\mathcal{C}}$ is true for $p^{-1}(V)$ and in particular the $\text{FIC}_{\mathcal{V}\mathcal{C}}$ is true for $p^{-1}(V)$. This completes the proof of 1(b).

1(c). Since the graph \mathcal{G} is not a tree it is homotopically equivalent to a wedge of r circles for $r \geq 1$. Then there is a surjective homomorphism $p : \pi_1(\mathcal{G}) \rightarrow F^r$ where F^r is a free group on r generators. And the kernel K of this homomorphism is the fundamental group of the universal covering $\tilde{\mathcal{G}}$ of the graph of groups \mathcal{G} . Hence K is the fundamental group of an infinite tree of finitely generated abelian groups. Now we would like to apply Lemma 5.2 to the homomorphism $p : \pi_1(\mathcal{G}) \rightarrow F^r$. By hypothesis and by Lemma 5.3 the $\text{FIC}_{\mathcal{V}\mathcal{C}}$ is true for any semidirect product $F^n \rtimes \langle t \rangle$, hence the $\text{FIC}_{\mathcal{V}\mathcal{C}}$ is true for F^r by the hereditary property. Since F^r is torsion free, by Lemma 5.2, we only have to check that the $\text{FIC}_{\mathcal{V}\mathcal{C}}$ is true for the semidirect product $K \rtimes \langle t \rangle$ for any action of $\langle t \rangle$ on K .

By Lemma 3.3 the restriction of the following map to each vertex group of $\tilde{\mathcal{G}}$ is injective.

$$\pi_1(\tilde{\mathcal{G}}) \rightarrow H_1(\pi_1(\tilde{\mathcal{G}}), \mathbb{Z}).$$

Let us denote the range group and the kernel of the homomorphism in the above display by A and B be respectively. Since the commutator subgroup of a group is characteristic we have the following exact sequence of groups induced by the above homomorphism.

$$1 \rightarrow B \rightarrow K \rtimes \langle t \rangle \rightarrow A \rtimes \langle t \rangle \rightarrow 1$$

for any action of $\langle t \rangle$ on K . Recall that $K = \pi_1(\tilde{\mathcal{G}})$. Now let K acts on a tree T which induces the tree of groups structure $\tilde{\mathcal{G}}$ on K . Hence B also acts on T with vertex stabilizers equal to $B \cap g\tilde{\mathcal{G}}_v g^{-1} = (1)$ where $v \in V_{\tilde{\mathcal{G}}}$ and $g \in K$. This follows from the fact that the restriction to any vertex group of $\tilde{\mathcal{G}}$ of the homomorphism $K \rightarrow A$ is injective. Thus B acts on a tree with trivial stabilizers and hence B is a free group by Lemma 3.2.

Next note that the group A is a countable infinitely generated abelian group. Now we can apply Lemma 5.2 to the homomorphism $K \rtimes \langle t \rangle \rightarrow A \rtimes \langle t \rangle$ (and use Lemma 5.3 if B is finitely generated) and (2) of Proposition 5.2 in exactly the same way as we did in the proof of 1(b). This completes the proof of 1(c).

Proofs of 2(i) and 2(ii). Let e_1, e_2, \dots, e_k be the finite edges of \mathcal{G} so that each of the connected components $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ of $\mathcal{G} - \{e_1, e_2, \dots, e_k\}$ is a tree of finitely generated abelian groups of the same rank. By Lemma 3.5 a finite tree of finitely generated abelian groups of the same rank has the intersection property. Therefore using Lemma 4.3 we see that such a tree of groups has residually finite fundamental group.

Now we check that the $\text{FICwF}_{\mathcal{V}\mathcal{C}}$ is true for $\pi_1(\mathcal{G}_i)$ for $i = 1, 2, \dots, n$.

Assume that \mathcal{G}_i is a graph of finitely generated abelian groups of the same rank (say r). Then by Lemma 3.5 $\pi_1(\mathcal{G}_i)$ contains a finitely generated free abelian normal subgroup A of rank r so that the quotient $\pi_1(\mathcal{G}_i)/A$ is isomorphic to the fundamental group of a graph of finite-type. Hence by the assumption ${}_{wt}\mathcal{T}_{\mathcal{V}\mathcal{C}}$ the $\text{FICwF}_{\mathcal{V}\mathcal{C}}$ is true for $\pi_1(\mathcal{G}_i)/A$. Now we would like to apply (3) of Proposition 5.2 to the following exact sequence.

$$1 \rightarrow A \rightarrow \pi_1(\mathcal{G}_i) \rightarrow \pi_1(\mathcal{G}_i)/A \rightarrow 1.$$

Note that using (2) of Proposition 5.2 it is enough to prove that the $\text{FICwF}_{\mathcal{V}\mathcal{C}}$ is true for $A \rtimes \langle t \rangle$. In case 2(i) this follows from the hypothesis and Lemma 5.3. And in case 2(ii) note that $A \rtimes \langle t \rangle$ contains a rank 2 free abelian subgroup of finite index. (See Lemma 6.1.) Therefore we can again apply (2) of Proposition 5.2 and Lemma 5.1 to see that the $\text{FICwF}_{\mathcal{V}\mathcal{C}}$ is true for $A \rtimes \langle t \rangle$.

Now observe that there is a finite graph of groups \mathcal{H} so that the edge groups of \mathcal{H} are finite and the vertex groups are isomorphic to $\pi_1(\mathcal{G}_i)$ where i varies over $1, 2, \dots, n$ and $\pi_1(\mathcal{H}) \simeq \pi_1(\mathcal{G})$. This follows from Lemma 3.1 since \mathcal{G} is almost a tree of groups.

Next we apply (1) of Proposition 2.2 (since $\pi_1(\mathcal{G}_i)$ is residually finite for $i = 1, 2, \dots, n$) to \mathcal{H} to complete the proof of (2).

Proof of (3). For the proof of (3) of Proposition 2.3 recall that in [[24], Corollary 3.5] it is proved that the UNil-groups with respect to free product with amalgamation decomposition of $\pi_1(\mathcal{G})$ along any edge group of \mathcal{G} vanish and hence using the discussion before Theorem 0.13 in [2] we complete the proof of (3). Since there is no UNil-groups we do not need to tensor the assembly map by $\mathbb{Z}[\frac{1}{2}]$ in [[2], Theorem 0.13]. \square

Proof of Proposition 2.4. Lemma 3.1 implies that there is a graph of groups \mathcal{H} with the same fundamental group as of \mathcal{G} so that the edge groups of \mathcal{H} are finite and the vertex groups are either finite or fundamental groups of finite trees of groups with rank 1 finitely generated abelian vertex and edge groups. Now note that by Lemma 3.5 the tree of group (say \mathcal{K}) corresponding to a vertex group of \mathcal{H} satisfies the hypothesis of Lemma 3.4.

Therefore \mathcal{K} has the intersection property. Also by Lemma 4.4 $\pi_1(\mathcal{K})$ is residually finite. Since by Corollary 5.3 the FICwF^P is true for virtually cyclic groups we can apply (1) and (3) of Proposition 2.2 to complete the proof of the Proposition provided we check that ${}_{wt}\mathcal{T}^P$ and \mathcal{P}^P are satisfied. By Corollary 5.3 the last condition is satisfied.

We now check the first condition. So let \mathcal{G} be a graph of virtually cyclic groups with trivial edge groups. Using Remark 5.1 we can assume that the graph is finite. Hence by Lemma 6.2 below $\pi_1(\mathcal{G})$ is a free product of a finitely generated free group and the vertex groups of \mathcal{G} . By Corollary 5.3 the FICwF^P is true for any virtually cyclic group and by Proposition 5.5 and Lemma 3.2 it is true for finitely generated free groups. Therefore we can apply [[19], Theorem 3.1] to conclude that the FICwF^P is true for $\pi_1(\mathcal{G})$. \square

Here we remark that when we applied (3) of Proposition 2.2 in the above proof of Proposition 2.4 we only needed the fact that the FICwF^P be true for the mapping torus of a virtually cyclic group (see the proof of (3) of Proposition 2.2). We note that for this we do not need Proposition 5.4, instead it can easily be deduced from the proof of Corollary 5.3. We just need to mention that the mapping torus of a virtually cyclic group is either virtually cyclic or it contains a two generators free abelian normal subgroup of finite index (see the following Lemma).

Lemma 6.1. *Suppose a group G has a virtually cyclic normal subgroup V_1 with virtually cyclic quotient V_2 . Then G is virtually cyclic when either V_1 or V_2 is finite and G contains a two-generator free abelian normal subgroup of finite index otherwise.*

Proof. We have the following exact sequence.

$$1 \rightarrow V_1 \rightarrow G \rightarrow V_2 \rightarrow 1.$$

For the first assertion there is nothing to prove when V_2 is finite. So let V_1 is finite and V_2 is infinite and C is an infinite cyclic subgroup of V_2 of finite index. Let p denote the homomorphism $G \rightarrow V_2$. Then $p^{-1}(C)$ has a finite normal subgroup with quotient C . Therefore $p^{-1}(C)$ is virtually infinite cyclic. Let C' be an infinite cyclic subgroup of $p^{-1}(C)$ of finite index. Hence C' is also an infinite cyclic subgroup of G of finite index. This proves the first assertion. For the second assertion let V_1 is also infinite and D is an infinite cyclic subgroup of V_1 of finite index. Then $p^{-1}(C) \simeq D \rtimes C$. Since C and D are both infinite cyclic the only possibilities for $p^{-1}(C)$ are that it contains a free abelian subgroup on two generators of index either 1 or 2. This proves the Lemma. \square

Proof of Proposition 2.1. Let T be the tree on which the group $\pi_1(\mathcal{G})$ acts so that the associated graph of groups is \mathcal{G} . For the proof, by Lemma 5.2 ((3) of Proposition 5.2) we need to show that the $\text{FIC}_{\mathcal{C}}$ ($\text{FICwF}_{\mathcal{C}}$) is true for the homomorphism f . We prove (a) assuming $\mathcal{T}_{\mathcal{C}}$ (${}_w\mathcal{T}_{\mathcal{C}}$).

Let $H \in \mathcal{C}(Q)$. Note that $f^{-1}(H)$ also acts on the tree T with stabilizers $f^{-1}(H) \cap \{\text{stabilizers of the action of } \pi_1(\mathcal{G})\}$. Since the restriction of f to the vertex groups of \mathcal{G} have finite kernels we get that $f^{-1}(H) \cap \{\text{stabilizers of the action of } \pi_1(\mathcal{G})\}$ is an extension of a finite group by a subgroup of H . If $\mathcal{C} = \mathcal{FLN}$ then these stabilizers also belong to \mathcal{C} . If $\mathcal{C} = \mathcal{VC}$ then using Lemma 6.1 we see that the stabilizers again belong to \mathcal{C} .

Therefore the associated graph of groups of the action of $f^{-1}(H)$ on the tree T has vertex groups belonging to \mathcal{C} .

Now using $\mathcal{T}_{\mathcal{C}}$ (${}_w\mathcal{T}_{\mathcal{C}}$) we conclude that the $\text{FIC}_{\mathcal{C}}$ ($\text{FICwF}_{\mathcal{C}}$) is true for $f^{-1}(H)$. This completes the proof of (a).

For the proofs of (b) and (c) just replace $\mathcal{T}_{\mathcal{C}}$ (${}_w\mathcal{T}_{\mathcal{C}}$) by ${}_f\mathcal{T}_{\mathcal{C}}$ (${}_{wf}\mathcal{T}_{\mathcal{C}}$) and ${}_t\mathcal{T}_{\mathcal{C}}$ (${}_{wt}\mathcal{T}_{\mathcal{C}}$) respectively in the above proof. Also use the corresponding assumption on the edge groups of the graph of groups \mathcal{G} . \square

Corollary 6.1. (Free products). *Let \mathcal{G} be a finite graph of groups with trivial edge groups so that the vertex groups satisfy the $\text{FIC}_{\mathcal{C}}$*

(FICwF $_C$). If \mathcal{P}_C and ${}_t\mathcal{T}_C$ (${}_{wt}\mathcal{T}_C$) are satisfied then the FIC $_C$ (FICwF $_C$) is true for $\pi_1(\mathcal{G})$.

Proof. The proof combines the following two Lemmas. \square

Lemma 6.2. *Let \mathcal{G} be a finite graph of groups with trivial edge groups. Then there is an isomorphism $\pi_1(\mathcal{G}) \simeq G_1 * \cdots * G_n * F$ where G_i 's are vertex groups of \mathcal{G} and F is a free group.*

Proof. The proof is by induction on the number of edges of the graph. If the graph has no edge then there is nothing to prove. So assume \mathcal{G} has n edges and that the Lemma is true for graphs with $\leq n - 1$ edges. Let e be an edge of \mathcal{G} . If $\mathcal{G} - \{e\}$ is connected then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) * \mathbb{Z}$ where $\mathcal{G}_1 = \mathcal{G} - \{e\}$ is a graph with $n - 1$ edges. On the other hand if $\mathcal{G} - \{e\}$ has two components say \mathcal{G}_1 and \mathcal{G}_2 then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) * \pi_1(\mathcal{G}_2)$ where \mathcal{G}_1 and \mathcal{G}_2 has $\leq n - 1$ edges. Therefore by induction we complete the proof of the Lemma. \square

Lemma 6.3. *Assume \mathcal{P}_C and ${}_t\mathcal{T}_C$ (${}_{wt}\mathcal{T}_C$) are satisfied. If the FIC $_C$ (FICwF $_C$) is true for G_1 and G_2 then the FIC $_C$ (FICwF $_C$) is true for $G_1 * G_2$.*

Proof. Consider the surjective homomorphism $p : G_1 * G_2 \rightarrow G_1 \times G_2$. By (1) of Proposition 5.2 the FIC $_C$ (FICwF $_C$) is true for $G_1 \times G_2$.

Now note that the group $G_1 * G_2$ acts on a tree with trivial edge stabilizers and vertex stabilizers conjugates of G_1 or G_2 . Therefore the restrictions of p to the stabilizers of this action of $G_1 * G_2$ on the tree are injective. Hence we are in the situation of (c) of Proposition 2.1,

This completes the proof of the Lemma. \square

Remark 6.1. We recall here that in the case of the FICwF P the Lemma 6.3 coincides with the Reduction Theorem (Theorem 3.1) in [19].

Proof of Proposition 2.2. Since the equivariant homology theory is assumed to be continuous when the graph is infinite, we can assume that \mathcal{G} is a finite graph of groups.

(1). By hypothesis the edge groups of \mathcal{G} are finite and the vertex groups are residually finite and satisfy the FICwF $_C$. By Lemma 4.1 $\pi_1(\mathcal{G})$ is residually finite. Let F_1, F_2, \dots, F_n be the edge groups. Let $1 \neq g \in \cup_{i=1}^n F_i$. Since $\pi_1(\mathcal{G})$ is residually finite there is a finite index normal subgroup N_g of $\pi_1(\mathcal{G})$ so that $g \in \pi_1(\mathcal{G}) - N_g$. Let $N = \cap_{g \in \cup_{i=1}^n F_i} N_g$. Then N is a finite index normal subgroup of $\pi_1(\mathcal{G})$ so that $N \cap (\cup_{i=1}^n F_i) = \{1\}$.

Let T be a tree on which $\pi_1(\mathcal{G})$ acts so that the associated graph of group structure on $\pi_1(\mathcal{G})$ is \mathcal{G} . Hence N also acts on T . Since N is

normal in $\pi_1(\mathcal{G})$ and $N \cap (\cup_{i=1}^n F_i) = \{1\}$, the edge stabilizers of this action are trivial and the vertex stabilizers are subgroups of conjugates of vertex groups of \mathcal{G} . Therefore by Corollary 6.1 the FICwF $_{\mathcal{C}}$ is true for N . Now (2) of Proposition 5.2 completes the proof of (1).

(2). The proof is by induction on the number of edges. If there is no edge then there is one vertex and hence by hypothesis the induction starts. Since by hypothesis the FICwF $_{\mathcal{V}\mathcal{C}}$ is true for $C \rtimes \langle t \rangle$ for $C \in \mathcal{D}$ it is true for C also by Lemma 5.1. Therefore assume that the result is true for graphs with $\leq n - 1$ edges which satisfy the hypothesis. So let \mathcal{G} be a finite graph of groups which satisfies the hypothesis and has n edges. Since \mathcal{G} has the intersection property there is a normal subgroup N of $\pi_1(\mathcal{G})$ contained in all the edge groups and of finite index in some edge group, say \mathcal{G}_e of the edge e . Let \mathcal{G}_1 be the graph of groups with \mathcal{G} as the underlying graph and the vertex and the edge groups are \mathcal{G}_x/N where x is a vertex and an edge respectively. Then $\pi_1(\mathcal{G}_1) \simeq \pi_1(\mathcal{G})/N$. Let $\mathcal{G}_2 = \mathcal{G}_1 - \{e\}$. It is now easy to check that the connected components of \mathcal{G}_2 satisfy the hypotheses and also has the intersection property. Also, since \mathcal{G}_2 has $n - 1$ edges, by the induction hypothesis $\pi_1(\mathcal{H})$ satisfies the FICwF $_{\mathcal{V}\mathcal{C}}$ where \mathcal{H} is a connected component of \mathcal{G}_2 . We now use Lemma 4.3 to conclude that $\pi_1(\mathcal{H})$ is also residually finite where \mathcal{H} is as above. Therefore by (1) $\pi_1(\mathcal{G}_1)$ satisfies the FICwF $_{\mathcal{V}\mathcal{C}}$. Next we apply (3) of Proposition 5.2 to the homomorphism $\pi_1(\mathcal{G}) \rightarrow \pi_1(\mathcal{G}_1)$. Note that using (2) of Proposition 5.2 it is enough to show the FICwF $_{\mathcal{V}\mathcal{C}}$ is true for the inverse image of any infinite cyclic subgroup of $\pi_1(\mathcal{G}_1)$, but such a group is of the form $N \rtimes \langle t \rangle$. Now since N is a subgroup of the edge groups of \mathcal{G} and \mathcal{D} is closed under taking subgroups we get $N \in \mathcal{D}$, as by hypothesis all the vertex groups of \mathcal{G} belong to \mathcal{D} . Again by definition of \mathcal{D} the FICwF $_{\mathcal{V}\mathcal{C}}$ is true for $N \rtimes \langle t \rangle$. This completes the proof of (2).

(3). The proof of (3) follows from (2). For the polycyclic case we only need to note that virtually polycyclic groups are residually finite and quotients and subgroups of virtually polycyclic groups are virtually polycyclic. Also mapping torus of a virtually polycyclic group is again virtually polycyclic.

And for the nilpotent case recall from [18] that the fundamental group of a finite graph of finitely generated nilpotent groups is subgroup separable if and only if the graph of groups satisfies the intersection property. Also note that finitely generated nilpotent groups are virtually polycyclic.

(4). Note that by hypothesis the FICwF $_{\mathcal{V}\mathcal{C}}$ is true for closed surface groups and by [5] closed surface groups are residually finite. Therefore by Lemma 4.5 and by (1) it is enough to consider a finite graph of

groups whose vertex and edge groups are infinite closed surface groups and the graph of groups satisfies the hypothesis. Again by Lemma 4.5 we have the exact sequence: $1 \rightarrow H \rightarrow \pi_1(\mathcal{G}) \rightarrow \pi_1(\mathcal{G})/H \rightarrow 1$. Where H is a closed surface group and $\pi_1(\mathcal{G})/H$ is isomorphic to the fundamental group of a finite-type graph of groups and hence contains a finitely generated free subgroup of finite index by Lemma 3.2. Hence by $wt\mathcal{T}_{\mathcal{V}\mathcal{C}}$ and by (2) of Proposition 5.2 the $\text{FICwF}_{\mathcal{V}\mathcal{C}}$ is true for $\pi_1(\mathcal{G})/H$. Now apply (3) of Proposition 5.2 to the homomorphism $\pi_1(\mathcal{G}) \rightarrow \pi_1(\mathcal{G})/H$. Using (2) of Proposition 5.2 it is enough to prove the $\text{FICwF}_{\mathcal{V}\mathcal{C}}$ for $H \rtimes \langle t \rangle$. Since H is a closed surface group the action of $\langle t \rangle$ on H is induced by a diffeomorphism of a surface S so that $\pi_1(S) \simeq H$. Hence $H \rtimes \langle t \rangle$ is isomorphic to the fundamental group of a closed 3-manifold which fibers over the circle. Therefore, using the hypothesis we complete the proof. \square

7. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. By Corollary 2.1 $wf\mathcal{T}^P$ is satisfied and hence $wt\mathcal{T}^P$ is also satisfied. Next by Corollary 5.3 \mathcal{P}^P is satisfied. Therefore the proof of (1) is completed using (1) of Proposition 2.2.

For the proof of (2) apply (b) of Proposition 2.1 and Corollary 2.1. \square

Proof of Theorem 1.2. Corollaries 2.1, 5.3 and (2) of Proposition 2.2 prove the Theorem. \square

Proof of Theorem 1.3. Corollaries 2.1, 5.3, Proposition 5.4 and (3) of Proposition 2.2 prove (1) and (2) of the Theorem.

To prove (3) we need only to use (4) of Proposition 2.2, Corollary 2.1 and that the FICwF^P is true for the fundamental groups of 3-manifolds fibering over the circle from [19] and [20].

(4) is same as Proposition 2.4.

The proof of (5) follows from 2(i) of Proposition 2.3, Corollaries 2.1 and 5.3. Since for the proof of 2(i) of Proposition 2.3 the weaker hypothesis that ‘the $\text{FICwF}_{\mathcal{V}\mathcal{C}}$ is true for $\mathbb{Z}^n \rtimes \langle t \rangle$ for all n ’ is enough. Further note that this weaker hypothesis is true for the FICwF^P by Proposition 5.4. \square

8. EXAMPLES

We recall some of the known and unknown results about the FIC^P , FIC^L , FIC^K and the FIC^{KH} which are related to this paper.

Known results:

- ${}_{wf}\mathcal{T}^P$ (Corollary 2.1), ${}_{wf}\mathcal{T}^L$ ([22]), ${}_{wf}\mathcal{T}^K$ ([2], Theorem 11.4) (when the underlying ring is regular), \mathcal{T}^{KH} ([2], Theorem 11.1) are known.
- \mathcal{P}^P is satisfied (Corollary 5.3). It is known that the FICwF^P is true for any virtually poly-infinite cyclic group (Proposition 5.4). Consequently, the FICwF^P is true for $\mathbb{Z}^n \rtimes \langle t \rangle$ for any $n \in \mathbb{N}$. Also see the footnote in Section 1.
- It is known that if a group G is the limit of a directed system of subgroups $\{G_i\}_{i \in I}$ directed under inclusions and the $\text{FIC}^X(G_i)$ is satisfied for $i \in I$ then the $\text{FIC}^X(G)$ is also satisfied for $X = P, K$ or L . See [[11], Theorem 7.1]. Compare Proposition 5.1. And the same is true for the FIC^{KH} . See proposition 3.4 and [[2], Theorem 11.1].
- \mathcal{FP}^X is already known for $X = P$ (Lemma 6.3) and for $X = K$. \mathcal{FP}^{KH} is also known ([2]). And in the case of the FIC^L , it is proved recently in [22].

Unknown results:

- \mathcal{T}^K and \mathcal{P}^K are not known. See [[2], Remark 8.5]. Also \mathcal{T}^P is not known for a general situation. Except for some cases which we checked in Proposition 2.4.
- It is not yet proved that the FIC^P is true for $F^n \rtimes \langle t \rangle$ for $n \in \mathbb{N} \cup \{\infty\}$. Though some important special cases are known. For example if we assume that the action of $\langle t \rangle$ on F^n (here n could be ∞) is induced by a diffeomorphism of a surface then it is proved that the FIC^P is true for $F^n \rtimes \langle t \rangle$. See [19] and [20]. Now for a countable infinitely generated torsion free abelian group \mathbb{Z}^∞ it was shown in [[11], Corollary 4.4] that the $\text{FICwF}^P(\mathbb{Z}^\infty \rtimes \langle t \rangle)$ is satisfied if every nearly crystallographic group ([[11], Definition]) satisfies the FIC^P .

Below we describe some examples of groups for which the results in this article are applicable. Furthermore, we show that these groups are new and also they are neither $CAT(0)$ nor hyperbolic.

Example 8.1. Graphs of virtually polycyclic groups: We consider an amalgamated free product H of two nontrivial infinite virtually polycyclic groups over a finite group. Next we recall that in [6] the Fibered Isomorphism conjecture in the pseudoisotopy case was proved for the class of groups which act cocompactly and properly discontinuously on symmetric simply connected nonpositively curved Riemannian manifolds. In the proof of [[4], Theorem A] it was noted that the condition ‘symmetric’ can be replaced by ‘complete’ if we consider torsion free groups. See [[12], Theorem A] also. We choose polycyclic groups so

that H does not belong to this class. One example of such a polycyclic group S is of the type $1 \rightarrow Z \rightarrow S \rightarrow Z^2 \rightarrow 1$ as described below. Here Z is an infinite cyclic group.

Let G be the Lie group consisting of those 3×3 matrices with real number entries whose diagonal entries are all equal to 1, entries below the diagonal are all equal to 0, and entries above the diagonal are arbitrary. Note that G is diffeomorphic to Euclidean 3-space. Let S be the subgroup of G whose entries above the diagonal are restricted to be integers. Then S is discrete and cocompact. Let E be the coset space of G by S . It clearly fibers over the 2-torus with fiber the circle. And the fundamental group of E is S which is nilpotent but not abelian. On the other hand in [25] it was shown that the fundamental group of a closed nonpositively curved manifold which is nilpotent must be abelian. This shows that E can not support a nonpositively curved Riemannian metric. Now by [[14], Corollary 2.6] it follows that S can not even embed in a group (called *Hadamard groups*) which acts discretely and cocompactly on a complete simply connected nonpositively curved space (that is a $CAT(0)$ -space).

Now consider $H_i = S \times F_i$ or $H_i = S \wr F_i$ where F_i is a finite group for $i = 1, 2$. Next we take amalgamated free product of H_1 and H_2 , $H = H_1 *_F H_2$ along some finite group F . Then H does not embed in a Hadamard group as before by [[14], Corollary 2.6] and H is not virtually polycyclic. But H satisfies the $FICwF^P$ by (1) of Theorem 1.1 and Proposition 5.4.

Example 8.2. Graphs of residually finite groups with finite edge groups: Let S be the fundamental group of a compact Haken 3-manifold which does not support any nonpositively curved Riemannian metric. Such 3-manifolds can easily be constructed by cutting along an incompressible torus in a compact Haken 3-manifold and then gluing differently. See [14] for this kind of construction. Next let H_1 and H_2 be two residually finite groups for which the $FICwF^P$ is true and such that S is embedded in H_1 . It is easy to construct such H_1 , for instance take $H_1 = S * G * F_1$ or $H_1 = (S \times G) \wr F_1$ or any such combination where G is a finitely generated free group and F_1 is a finite group. By the same argument as in Example 8.1 (and using [19] and [20]) it follows that $H = H_1 *_F H_2$ (along some finite group F) satisfies the $FICwF^P$ but is neither virtually polycyclic nor embeds in a Hadamard group.

Example 8.3. (Almost) a tree of finitely generated abelian groups where the vertex and the edge groups of any component subgraph have the same rank: Fundamental group of such a graph of groups can get very complicated, for example in the simplest

case of amalgamated free product of two infinite cyclic groups over an infinite cyclic group, that is $H = \mathbb{Z} *_Z \mathbb{Z}$ produces the (p, q) -torus knot group where the two inclusions $\mathbb{Z} \rightarrow \mathbb{Z}$ defining the amalgamation are multiplications by p and q , $(p, q) = 1$. Though the FICwF^P is known for knot groups ([19]), most of the other groups in this class, as far as we know, are new for which we prove the FICwF^P .

Let us now note that the group H considered in the above examples is not hyperbolic as it contains a free abelian subgroup on more than one generator.

Remark 8.1. We conclude by remarking that in this paper [[9], Theorem 4.8] (Proposition 5.4) is used in the proofs of (1), (2) (when the polycyclic or the nilpotent groups are not virtually cyclic), (3) and (5) (when the ranks of the abelian groups are ≥ 2) of Theorem 1.3. See the proof of Corollary 5.3 and the discussion after the proof of Proposition 2.4. In this connection we note here that using the recent work of Bartels and Lück in [3] all the results in the Introduction of this article can be deduced in the L -theory case of the Fibered Isomorphism conjecture. The same proofs will go through. But for this we need to use the L -theory version of [[9], Theorem 4.8] in the proofs of the particular cases of the items of Theorem 1.3 as mentioned above. See [1] for the proof of [[9], Theorem 4.8] in the L -theory case.

REFERENCES

- [1] A. Bartels, F.T. Farrell and W. Lück, The Farrell-Jones conjecture for cocompact lattices in virtually connected Lie groups, arXiv:1101.0469.
- [2] A. Bartels and W. Lück, Isomorphism conjecture for homotopy K -theory and groups acting on trees, *J. Pure Appl. Algebra*, 205 (2006), 660-696.
- [3] ———, The Borel conjecture for hyperbolic and CAT(0)-groups, arXiv:0901.0442 [math.GT].
- [4] E. Berkove, F. T. Farrell, D. Juan-Pineda and K. Pearson, The Farrell-Jones isomorphism conjecture for finite co-volume hyperbolic actions and the algebraic K -theory of Bianchi groups, *Trans. Amer. Math. Soc.* 352 (2000), no. 12, 5689-5702.
- [5] G. Baumslag, On generalised free products, *Math. Z.*, 78 (1962), 423-438.
- [6] B. Baumslag and M. Tretkoff, Residually finite HNN extensions, *Comm. Algebra*, 6 (1978), no. 2, 179-194.
- [7] D. Cohen, Residual finiteness and Britton's lemma, *J. London Math. Soc.*, 16 (1977), no. 2, 232-234.
- [8] W. Dicks and M.J. Dunwoody, Groups acting on graphs, Cambridge Studies in Advanced Mathematics 17, Cambridge University Press, 1989.
- [9] F.T. Farrell and L.E. Jones, Isomorphism conjectures in algebraic K -theory, *J. Amer. Math. Soc.*, 6 (1993), 249-297.
- [10] ———, The lower algebraic K -theory for virtually infinite cyclic groups, *K-theory*, 9 (1995), 13-30.
- [11] F.T. Farrell and P.A. Linnell, K -Theory of solvable groups, *Proc. London Math. Soc.* (3), 87 (2003), 309-336.
- [12] F.T. Farrell and Sayed K. Roushon, The Whitehead groups of braid groups vanish, *Internat. Math. Res. Notices*, no. 10 (2000), 515-526.
- [13] K.W. Gruenberg, Residual properties of infinite soluble groups, *Proc. London Math. Soc.*, 7 (1957) 29-62.
- [14] M. Kapovich and B. Leeb, Actions of discrete groups on nonpositively curved spaces, *Math. Ann.*, 306 (1996) 341-352.
- [15] W. Lück, Chern characters for equivariant homology theories and applications to K - and L -theory, *J. Reine Angew. Math.*, 543 (2002), 193-234.
- [16] W. Lück and H. Reich, The Baum-Connes and the Farrell-Jones conjectures in K - and L -theory, Handbook of K -theory Volume 2, editors: E.M. Friedlander, D.R. Grayson, (2005) 703-842, Springer.
- [17] R.C. Lyndon and P.E. Schupp, Combinatorial group theory, Classics in Mathematics, Springer, 1977.
- [18] V. Metaftsis and E. Raptis, Subgroup separability of graphs of nilpotent groups, *J. Group Theory*, 7 (2004), no. 2, 265-273.
- [19] S.K. Roushon, The Farrell-Jones isomorphism conjecture for 3-manifold groups, *J. K-Theory* 1 (2008) 49-82.
- [20] ———, The isomorphism conjecture for 3-manifold groups and K -theory of virtually poly-surface groups, *J. K-Theory* 1 (2008) 83-93.
- [21] ———, Algebraic K -theory of groups wreath product with finite groups, *Topology Appl.*, 154 (2007), 1921-1930.
- [22] ———, The fibered isomorphism conjecture in L -theory, *Topology Appl.*, 157 (2010), 508-515. doi:10.1016/j.topol.2009.10.012.

- [23] ———, The isomorphism conjecture in L -theory: graphs of groups, math.KT/0709.4592v2.
- [24] C.W. Stark, L -theory and graphs of free abelian groups, *J. Pure Appl. Algebra* 47 (1987), 299-309.
- [25] S.T. Yau, On the fundamental group of compact manifolds of non-positive curvature, *Ann. of Math. (2)* 93 (1971), 579-585.

SCHOOL OF MATHEMATICS, TATA INSTITUTE, HOMI BHABHA ROAD, MUMBAI
400005, INDIA

E-mail address: `roushon@math.tifr.res.in`

URL: `http://www.math.tifr.res.in/~roushon/`