NOTES ON K-THEORY OF 3-MANIFOLD GROUPS

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ABSTRACT. These are the notes of 3 lectures on K-theory of 3-manifold groups given in the Instructional Workshop on Low Dimensional Topology and Geometric Group Theory, Bangalore ISI. These notes are expository in nature and for detailed accounts a list of references are given.

CONTENTS

1. INTRODUCTION

There are two fundamental questions in Topology.

- When is a space homotopy equivalent to a finite simplicial (or CW) complex?
- Are two homotopy equivalent spaces homeomorphic?

The second question has several exotic variations. For example one asks when are two homeomorphic smooth manifolds diffeomorphic?

In fact there is another question in between the first and the second. We will talk about it in Section 4.

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One easily constructs examples to show that the answer to the second question is NO. Therefore one asks when is this answer YES? Together with the first question this led to the breakthrough in high dimensional topology and the formidable industrial development in the subject. On the algebraic side this is connected to two new disciplines: Algebraic K-theory and L-theory. In recent times Geometry came into picture and made another breakthrough in computing the invariant already defined. Some geometric assumption gave answers to several questions which was not possible to answer by algebraic methods.

In these lectures we will try to present the Algebraic K-theory part and describe the major results and techniques. Finally we will see, assuming the Geometrization conjecture of Thurston, how all these techniques are applied to a complete calculation of the Algebraic K-theory of 3-manifold groups. Before the application of Geometry in the subject one could compute K-theory of only a class of 3-manifold groups. Throughout by 'K-theory' we mean 'lower K-theory'.

2. 3-manifolds: preliminaries

Throughout the lectures we only considered orientable manifolds.

In this section we recall some preliminaries on 3-manifold topology. By a 3-*manifold group* we mean the fundamental group of a 3-manifold. A 3-manifold is called *irreducible* if every embedding of the 2-sphere in the 3-manifold extends to an embedding of the 3-disc. And a 3-manifold is called *prime* if whenever it is a connected sum of two 3-manifolds then one of the summand is homeomorphic to the 3-spheres. The only 3-manifolds which are prime but not irreducible are the $\mathbb{S}^2$ bundles over the circle ([[5], lemma 3.13]). And by Kneser's prime decomposition theorem any compact 3-manifold can be written as a connected sum of finitely many prime 3-manifolds ([[5], theorem 3.15]). Thus we have the following lemma.

Lemma 2.1. *A compact* 3*-manifold group is a free product of finitely many irreducible* 3*-manifold groups and a free group of finite rank.*

An embedded closed 2-manifold F in a 3-manifold is called 2-*sided* if the normal bundle of F in M is trivial, that is homeomorphic to $F \times \mathbb{R}$. For example an orientable surface embedded in an orientable 3-manifold is always 2-sided. By a *Haken* 3-manifold we mean it is irreducible and contains an embedded 2-sided π_1-injective closed 2-manifold of infinite fundamental group. Such an embedded 2-manifold is called an *incompressible* surface. Also we recall that there is a unique decomposition of any compact Haken 3-manifold by cutting the 3-manifold along finitely many incompressible tori so that each piece is either Seifert fibered or hyperbolic. This is known as JSJT (Jaco-Shalen, Johannson and Thurston) (see [8], [9] and [21]) decomposition. Recall that a *graph manifold* is either a Seifert fibered space or is a Haken 3-manifold which has only Seifert fibered pieces in its JSJT decomposition or equivalently which is obtained by gluing along tori boundary components of finitely many Seifert fibered 3-manifolds.

2.1. **Geometrization Conjecture.** Above we have learnt about JSJT decomposition of Haken 3-manifolds. It is known that any of the pieces of the decomposition of a Haken 3-manifold admits a geometric structure and there are only eight such geometries possible in dimension 3. By a *geometric structure* in a manifold M one means a complete locally homogeneous Riemannian metric in the interior of M. The universal cover $\tilde{M}$ of M with the lifted metric is a geometric model space. In dimension 3 there are eight model spaces: namely $\mathbb{E}^n$ (Euclidean), $\mathbb{S}^n$ (spherical), $\mathbb{H}^n$ (hyperbolic), $\mathbb{H}^2 \times \mathbb{E}$, $\mathbb{S}^2 \times \mathbb{E}$, $\widetilde{\mathbb{SL}}(2, \mathbb{R})$, $\mathbb{N}il$ and $\mathbb{S}ol$. This means that if a 3-manifold M admits a geometric structure then the universal cover $\tilde{M}$ with the induced metric is isometric to one of the eight geometries above. It may happen that M admits more than one geometric structures, but if M is closed (that is compact with empty boundary) and admits a geometric structure then the geometric structure is unique in the sense that in this case $\tilde{M}$ is isometric to one and only one of the above geometries. For more details about geometric structure in dimension 3 see [15].

The Geometrization Conjecture of Thurston states that any compact 3-manifold can be decomposed into pieces so that each such piece admits geometric structure in the interior. Let us now talk about the kind of decomposition one can think of. Firstly we have already seen the prime decomposition of a 3-manifold. This was a decomposition along embedded two sided spheres. Thus we can concentrate on prime 3-manifolds only. Also recall that a prime 3-manifold is either irreducible or an $\mathbb{S}^2$-bundle over $\mathbb{S}^1$. On the other hand an $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ has $\mathbb{S}^2 \times \mathbb{E}$ structure. Thus, we can consider only irreducible

3-manifolds. Thus, next is to split an irreducible 3-manifold along surfaces of genus ≥ 1. Of course there always exists embedded surfaces of any genus in a 3-manifold. But splitting along such surfaces does not produce anything interesting. Therefore one looks at embedded incompressible surfaces as we have already seen in the JSJT decomposition. On the other hand there may not exists any incompressible surfaces of genus ≥ 1. In fact Thurston proved that most 3-manifolds do not have incompressible surfaces (that is these are non-Haken). Now if the Geometrization Conjecture is true then any non-Haken 3-manifold has to support a geometric structure. One proves from the properties of the eight geometric model spaces that any 3-manifold having geometry other then $\mathbb{H}^n$ is Seifert fibered. Consequently conjecturally a 3-manifold with finite fundamental group has spherical geometry and a non-Haken irreducible 3-manifold with infinite fundamental group either has a hyperbolic geometry or is Seifert fibered. Thus we have the following.

Geometrization Conjecture. ([22]) *Let M be a compact irreducible 3-manifold. Then one of the following holds.*

- *M is Seifert fibered.*
- *M is Haken.*
- *M is hyperbolic.*

3. $\tilde{K}_0(.)$

Let us recall the following definition.

Definition 3.1. *Two spaces X and Y are said to be homotopy equivalent if there are continuous maps $f : X \to Y$ and $g : Y \to X$ so that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Here $\simeq$ means 'homotopic to'.*

Historically the first question (in the Introduction) was dealt in two steps. To describe this we need another definition.

Definition 3.2. *A space X is said to be dominated by another space Y if there are maps $f : X \to Y$ and $g : Y \to X$ so that $g \circ f \simeq id_X$. And X is called finitely dominated if in addition Y is a finite CW-complex.*

It is not difficult to show that (see theorem 3.9 of [23] or exercise G6 in chapter 7 of [16]) a finitely dominated space is homotopy equivalent to a CW-complex. But to show whether this complex is also finite needs a much more difficult methods and techniques. This is answered in a very important and celebrated series of papers by C.T.C. Wall (see [25] and [26]). This leads to an obstruction which lies in an abelian group called the *reduced projective class* group and denoted by $\tilde{K}_0(\mathbb{Z}[\pi_1(X)])$. Interestingly the answer to this question depends only on the fundamental group of X.

Before going further let us define reduced projective class group of a ring. Let R be a ring with unity. Let $\mathcal{P}$ denotes the free abelian group generated by the set of all isomorphism classes of finitely generated projective R-modules. Let $K_0(R)$ be the quotient of $\mathcal{P}$ by the relations $[P] + [Q] = [P \oplus Q]$ where $[P], [Q] \in \mathcal{P}$. Note that $K_0(R)$ becomes a covariant functor from the category of rings with unity to the category of abelian groups. Thus the homomorphism $\mathbb{Z} \to R$ sending 1 to the unity of R induces a map $i : K_0(\mathbb{Z}) \to K_0(R)$. The *reduced projective class* group $\tilde{K}_0(R)$ is by definition the quotient $K_0(R)/i(K_0(\mathbb{Z}))$. By abusing notation we will denote elements of $\tilde{K}_0(R)$ by the same notation $[P]$. For details on the projective class groups see [14] or any standard book on Algebraic K-theory.

For technical reason we need to assume that the space X is locally connected and semilocally simply connected. (Since we need to consider the universal cover of X). Let us call such a space a *nice space*. Let $S_*(X)$ be the singular chain complex of X. When X is a simplicial (or CW-) complex then we denote the simplicial (or cellular) chain complex of X by $C_*(X)$. Recall that when X is a simplicial (or CW-) complex then $S_*(X)$ and $C_*(X)$ are chain equivalent.

Let X be a nice space and $\tilde{X}$ be its universal cover. Then the fundamental group $\pi_1(X) = G$ (say) acts on the singular chain complex $S_*(\tilde{X})$ via Deck transformations. This makes $S_*(\tilde{X})$ a chain complex of free $R(= \mathbb{Z}[G])$-module. Here $\mathbb{Z}[G]$ is the integral group ring of G. In general $S_i(\tilde{X})$ is a very large R-module (not even countably generated sometime). On the other hand if X were a finite complex then it follows that $S_*(\tilde{X})$ is a finite chain complex of finitely generated free R-module. For an arbitrary nice space we can make the situation better by assuming that X is finitely dominated. C.T.C. Wall proves that in this case $S_*(\tilde{X})$ is chain equivalent to a finite chain complex of finitely generated projective R-module, say $P_*(X)$. Consider the object $\chi(P_*(X)) = \Sigma_i[P_i] \in \tilde{K}_0(R)$. $\chi(P_*(X))$ is called the *Wall obstruction* of the space X.

Theorem 3.1. *(C.T.C. Wall) Let X be a finitely dominated nice space and $R = \mathbb{Z}[\pi_1(X)]$. Then X is homotopy equivalent to a finite CW-complex if and only if $\chi(P_*(X)) = 0$ in $\tilde{K}_0(R)$.*

We now talk about another very natural question in topology whose answer also lies in the reduced projective class group.

- When is a non-compact manifold the interior of a compact manifold with boundary?

Well, obviously the first problem will be if the fundamental group is not finitely presented and hence the manifold should have finitely many ends to start with. Secondly the manifold should have finitely generated homology groups. Thirdly, assuming the manifold is orientable, near the ends the manifold should behave like a compact manifold product with an interval. As a consequence a necessary condition is that outside a big enough compact subset the manifold should have homology of a compact manifold of one lesser dimension. For simplicity assume that the manifold has only one end. Another consequence is that if Y_i are increasing sequence of compact subset then the fundamental groups of the complements of Y_i should stabilize to a finitely presented group $\pi_1(U) = G$ at certain stage, where U is the complement of Y_i for some large i. Under these necessary conditions one shows that $C_*(\tilde{U})$ is chain equivalent to a finite chain complex $P_*(U)$ of finitely generated projective $R = \mathbb{Z}[\pi_1(U)]$-module.

Theorem 3.2. *(L. Siebenmann) Under the above necessary conditions if $\chi(P_*(U)) = \Sigma_i P_i(U) = 0$ in $\tilde{K}_0(R)$ and if $dimU \neq 3, 4, 5$ then the manifold is the interior of a compact manifold with boundary.*

For detailed account on Siebenmann's work see [27].

For some computations of the reduced projective class groups for many group rings see next section.

4. $K_1(.)$ and $Wh(.)$

On the passage from homotopy equivalence to topological equivalence (that is homeomorphism) there lies an important notion of equivalence in topology; the *simple homotopy equivalence*. The celebrated s-cobordism theorem and the proof of generalized Poincare Conjecture are the most famous examples which circled around the notion of simple homotopy equivalence. Even in such a classical vast area of topology the most basic question is not yet answered; namely, if any homotopy equivalence between two finite aspherical complexes is homotopic to a simple homotopy equivalence, which is known as *Whitehead's conjecture*. There is an even stronger conjecture which asks; if any homotopy equivalence between two aspherical manifolds is homotopic to a homeomorphism. This is also known as *Borel's conjecture*. A result of Chapman says that any homeomorphism is a simple homotopy equivalence. So the first step to prove Borel's conjecture is to verify Whitehead's conjecture.

Now we come to the interesting complication in topology imposed by dimension. For example the techniques and tricks used in manifold topology of dimension ≤ 4 and in dimension ≥ 5 are completely different. For this reason there is a (clear!) division in manifold topology; one is called *low dimensional topology* and the other *high dimensional topology*.

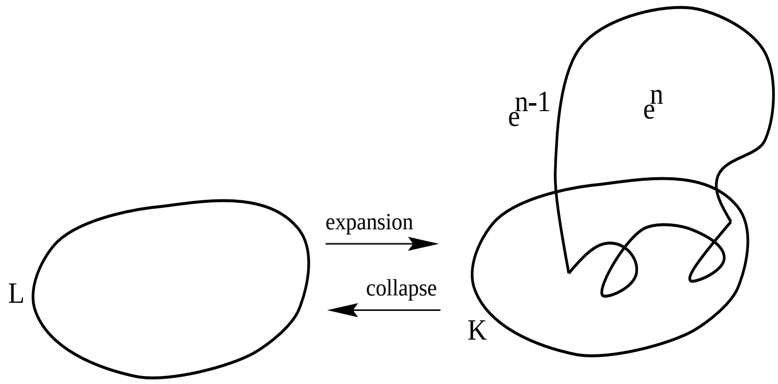
Dimension 2 is understood completely in the topological, piecewise linear or smooth category. Thurston's Geometrization programme gives a conjectural picture in dimension 3. Also in dimension ≤ 3 any manifold supports a unique topological, piecewise linear or smooth structure. In dimension greater or equal to 5 an enormous literature exists; where there are enough machinery, language to attack a problem. The critical dimension is 4. In this dimension even the s-cobordism theorem (see below) is not yet known. So far this is proved for 4-manifolds with some restriction on the fundamental group; namely, groups with subexponential growth. Also another interesting fact is that a 4-manifold can support infinitely many smooth structure. Even our familiar Euclidean 4-space $\mathbb{R}^4$ has infinitely many smooth structure.

4.1. **Simple homotopy equivalence and Whitehead group.** The reference for this subsection is [2]. Let K and K' be two finite CW-complexes. A homotopy equivalence $f : K \to K'$ is called a *simple homotopy equivalence* provided f is homotopic to a composition of maps of the following kind: $K = K_0 \to K_1 \to \cdots \to K_s = K'$ where the arrows are either an elementary expansion or collapse. A pair of CW-complexes (K, L) is called an *elementary collapse* (and we say K *collapses to* L or L *expands to* K) if the followings are satisfied:

- $K = L \cup e^{n-1} \cup e^n$ where e^{n-1} and e^n are not in L (e^i denotes an i-cell),

- there exist a ball pair $(\mathbb{D}^n, \mathbb{D}^{n-1})$ and a map $\phi : \mathbb{D}^n \to K$ such that
  (a) ϕ is a characteristic map for e^n
  (b) $\phi|_{\mathbb{D}^{n-1}}$ is a characteristic map for $\mathbb{D}^{n-1}$
  (c) $\phi(P^{n-1}) \subset L^{n-1}$, where $P^{n-1} = \overline{\mathbb{D}^n - \mathbb{D}^{n-1}}$.

From the following figure the definition of an expansion (or collapse) will be clear.



Given any homotopy equivalence $f : K \to K'$, there is an obstruction $\tau(f)$ which lies in an abelian group $Wh(\pi_1(K'))$ (defined below) detecting if f is a simple homotopy equivalence.

Let R be a ring with unity. Let $GL_n(R)$ be the invertible $n \times n$ matrices with entries in R and $E_n(R)$ be the subgroup of elementary matrices. By definition an *elementary matrix*, denoted $E_{ij}(a)$, has 1 on the diagonal entry, $a \in R - \{0\}$ at the (i, j)-th position and the remaining entries are 0. Define $GL(R) = \lim_{n\to\infty} GL_n(R)$, $E(R) = \lim_{n\to\infty} E_n(R)$. Here the limit is taken over the following maps:

$$GL_n(R) \to GL_{n+1}(R)$$

$$\begin{pmatrix} A \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

The following lemma shows that $E(R)$ is also the commutator subgroup of $GL(R)$.

Lemma 4.1. *(Whitehead's lemma) Let R be a ring. Then the commutator subgroup of $GL(R)$ and of $E(R)$ is $E(R)$.*

Proof. At first note the following easy to prove identities.

$E_{ij}(a)E_{ij}(b) = E_{ij}(a+b);$ (1)
$E_{ij}(a)E_{kl}(b) = E_{kl}(b)E_{ij}(a)$, $j \neq k$ and $i \neq l$; (2)
$E_{ij}(a)E_{jk}(b)E_{ij}(a)^{-1}E_{jk}(b)^{-1} = E_{ik}(ab)$, i, j, k distinct; (3)
$E_{ij}(a)E_{ki}(b)E_{ij}(a)^{-1}E_{ki}(b)^{-1} = E_{kj}(-ba)$, i, j, k distinct. (4)

Also note that any upper triangular or lower triangular matrix with 1 on the diagonal belongs to $E(R)$.

Since $E(R) \subset GL(R)$ we have $[E(R), E(R)] \subset [GL(R), GL(R)]$. Using (3) we find that $E_{ij}(a) = [E_{ik}(a), E_{kj}(1)]$ provided i, j and k are distinct. Hence any generator of $E(R)$ is a commutator of two other generators of $E(R)$. Hence $E(R) = [E(R), E(R)]$. We only need to check that $[GL(R), GL(R)] \subset E(R)$. So let $A, B \in GL_n(R)$. Then note the following identity.

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

Now we check that all the factors on the right hand side belong to $E_{2n}(R)$. This follows from the following. Let $A \in GL_n(R)$. Then the following equality is easy to check.

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Again note the following.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Now recall our earlier remark that any upper triangular or lower triangular matrix with 1 on the diagonal belong to $E(R)$.

This completes the proof of the Whitehead Lemma. $\square$

Define $K_1(\mathbb{Z}[\pi]) = \frac{GL(\mathbb{Z}[\pi])}{E(\mathbb{Z}[\pi])}$. The *Whitehead group* $Wh(\pi)$ of π is by definition $\frac{K_1(\mathbb{Z}[\pi])}{\sigma(<\pi,-\pi>)}$. Here $\sigma :< \pi, -\pi >\to K_1(\mathbb{Z}[\pi])$ is defined by sending $\pm g$ to the 1×1 matrix $(\pm g)$ for all $g \in \pi$.

Note that multiplying a matrix A by an elementary matrix from left (or right) makes an elementary row (or column) operation in A. Now applying elementary row and column operations one can transform an invertible integral matrix to I or $-I$. This shows that $Wh((1)) =$

0. (Though the matrix multiplication induces the group operation in $Wh(.)$ we write it additively since $Wh(.)$ is abelian.)

We recall below a transparent geometric definition of Whitehead group which is naturally isomorphic to the above one. The proof of this isomorphism is very long and can be found in [2]. But we will give the map from this geometric definition to the algebraic definition above. For computational purpose the algebraic definition is more useful as we will see later.

Let K be a fixed finite CW-complex. Let $\mathcal{W}(K)$ be the collection of all pairs of finite complexes (L, K) so that K is a strong deformation retract of L. For any two objects $(L_1, K), (L_2, K) \in \mathcal{W}$ define $(L_1, K) \equiv (L_2, K)$ if and only if L_1 and L_2 are simple homotopical equivalent relative to the subcomplex K. Here by a relative simple homotopy equivalence we mean that in the definition of simple homotopy equivalence we do not collapse (or expand) any cells contained in K. Let $Wh(K) = \mathcal{W}/\equiv$. Let $[L_1, K]$ and $[L_2, K]$ be two classes in $Wh(K)$. Define $[L_1, K] \oplus [L_2, K] = [L_1 \cup_K L_2, K]$. Here $L_1 \cup_K L_2$ is the disjoint union of L_1 and L_2 identified along the common subcomplex K. This defines an abelian group structure on $Wh(K)$ called the Whitehead group of K.

Now let (L, K) be an element in $\mathcal{W}$. Consider the universal cover $(\tilde{L}, \tilde{K})$. It can be checked that the inclusion $\tilde{K} \subset \tilde{L}$ is a homotopy equivalence. Equip $(\tilde{L}, \tilde{K})$ with the CW-complex structure lifted from the CW-structure of (L, K). Let $C_*(\tilde{L}, \tilde{K})$ be the cellular chain complex of $(\tilde{L}, \tilde{K})$. The covering action of $\pi_1(L)$ on $(\tilde{L}, \tilde{K})$ induces an action on $C_*(\tilde{L}, \tilde{K})$ and makes it a chain complex of $\mathbb{Z}[\pi_1(L)]-$module. In fact it is a finitely generated free acyclic chain complex of $\mathbb{Z}[\pi_1(L)]$ module. Let d be the boundary map. One can find a contraction map δ of degree $+1$ of $C_*(\tilde{L}, \tilde{K})$ so that $d\delta + \delta d = id$ and $\delta^2 = 0$. Consider the module homomorphism $d + \delta : \bigoplus_{i=0}^{\infty} C_{2i+1}(\tilde{L}, \tilde{K}) \to \bigoplus_{i=0}^{\infty} C_{2i}(\tilde{L}, \tilde{K})$. It turns out that this homomorphism is an isomorphism of $\mathbb{Z}[\pi_1(L)]$-modules. The image and range of this homomorphism are finitely generated free module with a preferred basis coming from the CW-structure on $(\tilde{L}, \tilde{K})$. We consider the matrix of this homomorphism $d + \delta$ which is an invertible matrix with entries in $\mathbb{Z}[\pi_1(L)]$ and hence lies in $GL_n(\mathbb{Z}[\pi_1(L)])$ for some n. We take the image of this matrix in $Wh(\pi_1(L))$. The proof that this map (say τ) sending (L, K) to this image in $Wh(\pi_1(L))$ is an isomorphism is given in [2].

Now consider a homotopy equivalence $f : K \to K'$ between two finite CW-complexes. Let M_f be the mapping cylinder of the map f. Consider the pair (M_f, K). As f is a homotopy equivalence it is easy to check that $(M_f, K) \in \mathcal{W}$. Now we recall that f is homotopic to a simple homotopy equivalence if and only if $\tau([M_f, K])$ is the trivial element in $Wh(\pi_1(K))$.

We are in a position to state the s-cobordism theorem now.

Theorem 4.1. *(**S-cobordism theorem**) (Barden, Mazur, Stallings) Let M_1 and M_2 be two compact manifolds of dimension ≥ 5 if they have empty boundary and of dimension ≥ 6 otherwise. Let W be another compact manifold with two boundary components M_1 and M_2. Assume that both the inclusions $M_i \to W$, $i = 1, 2$ are homotopy equivalences. If $\tau([W, M_1]) = 1$ in $Wh(\pi_1(M_1))$ then $W \simeq M_1 \times I$. In particular $M_1 \simeq M_2$.*

Remark 4.1. When $\partial W = M_1 \cup M_2$ and the inclusions $M_i \subset W$ are homotopy equivalence so that $\tau([W, M_1]) = 1$, then we say W is an s-cobordism between M_1 and M_2.

Here $\simeq$ is a homeomorphism, a piecewise linear homeomorphism or a diffeomorphism according as the manifolds are topological, piecewise linear or smooth respectively.

5. Computing $\tilde{K}_0(.)$ and $Wh(.)$

Now we come to results about computing the two invariants we defined so far, namely the reduced projective class group and the Whitehead group.

The Whitehead conjecture asks if $Wh(\pi) = 1$ for any finitely presented torsion free group. So far this has been checked for several classes of groups: for free abelian groups by Bass-Heller-Swan ([1]) ; for free nonabelian groups by Stallings ([17]); for the fundamental group of any complete nonpositively curved Riemannian manifold by Farrell and Jones ([3]; for the fundamental group of finite $CAT(0)$ complexes by B. Hu. ([7]). Waldhausen proved that the Whitehead group of the fundamental group of any Haken 3-manifold vanishes ([24]). Some results like Whitehead group of finite groups are also known: $Wh(F) = 0$, when F is a finite cyclic group of order $1, 2, 3, 4$ and 6 and $Wh(F)$ is infinite when F is any other finite cyclic group. Also $Wh(S_n) = 0$, here S_n is the symmetric group on n letters. Below we show by a little calculation that there is an element of infinite order in $Wh(\mathbb{Z}_5)$. In fact this group is infinite cyclic, but that needs a difficult proof and we do not talk about it here.

Lemma 5.1. *There exist an element of infinite order in* $Wh(\mathbb{Z}_5)$.

Proof. Let t be the generator of $\mathbb{Z}_5$. Let $a = 1 - t - t^{-1}$. Then

$$(1 - t - t^{-1})(1 - t^2 - t^3) = 1 - t - t^{-1} - t^2 + t^3 + t - t^3 + t^{-1} + t^2 = 1.$$

Hence a is a unit in $\mathbb{Z}[\mathbb{Z}_5]$. Define $\alpha : \mathbb{Z}[\mathbb{Z}_5] \to \mathbb{C}$ by sending t to $e^{2\pi i/5}$. Then α sends $\{\pm g : g \in G\}$ to the roots of unity in $\mathbb{C}$. Hence $x \mapsto |\alpha(x)|$ defines a homomorphism from $Wh(\mathbb{Z}_5)$ into R^*_+, the nonzero positive real numbers. Next note the following.

$$|\alpha(a)| = |1 - e^{2\pi i/5} - e^{-2\pi i/5}| = |1 - 2cos\frac{2\pi}{5}| \approx 0.4.$$

This proves that α defines an element of infinite order in $Wh(\mathbb{Z}_5)$. $\square$

For Whitehead groups of general finite groups see [11].

We have already seen that the Whitehead group of the trivial group is trivial. Next simplest question is what is $Wh(C)$? Where C is the infinite cyclic group. Well, this has already been computed by G. Higman in 1940 even before Whitehead group was defined.

Theorem 5.1. *([6]) The Whitehead group of the infinite cyclic group is trivial.*

One can ask generalization of this result in two possible directions; that is what is $Wh(F_r)$ or $Wh(C^n)$? Where F_r is the non-abelian free group on r generators and C^n is the abelian free group on n generators. This question has been answered completely.

Theorem 5.2. *([17]) Let G_1 and G_2 be two groups and let $G_1 * G_2$ denotes their free product. Then $Wh(G_1 * G_2) \simeq Wh(G_1) \oplus Wh(G_2)$.*

Using Higman's result we get the following important corollary.

Corollary 5.1. $Wh(F_r) = 0$.

Theorem 5.3. *([1])* $Wh(C^n) = 0$.

We have already mentioned before the following result.

Theorem 5.4. *([24]) Let M be a compact Haken 3-manifold. Then $Wh(\pi_1(M)) = \tilde{K}_0(\mathbb{Z}[\pi_1(M)]) = 0$.*

An important result relating Whitehead group and the reduced projective class group is the Bass-Heller-Swan formula.

Theorem 5.5. *([1]) Let G be a finitely generated group. Then*

$$Wh(G \times C) \simeq Wh(G) \oplus \tilde{K}_0(\mathbb{Z}[G]) \oplus N \oplus N.$$

Where N is some Nil-group.

Theorem 5.3 is in fact a corollary of the above formula.

As we are mostly interested in showing the vanishing of the Whitehead group we do not recall the definition of the group N. For details see [14].

Corollary 5.2. *$\tilde{K}_0(\mathbb{Z}[C^n]) = 0$, where C^n is the free abelian group on n generators.*

The most recent fundamental breakthrough in calculating the Whitehead group is the following theorem by Farrell and Jones.

Let us first recall that a *nonpositively curved manifold* is a smooth complete Riemannian manifold with all the sectional curvature ≤ 0.

Theorem 5.6. *([3]) Let M be a closed nonpositively curved manifold. Then $Wh(\pi_1(M)) = \tilde{K}_0(\mathbb{Z}[\pi_1(M)]) = 0$.*

6. K-theory of 3-manifold groups

Finally we show how a complete calculation of the K-theory of a 3-manifold group can be done using the Geometrization conjecture and the results from the last section.

Using Lemma 2.1 and the Geometrization conjecture we get the following.

Fundamental group of a 3-manifold under Geometrization Conjecture. *Let M be a compact 3-manifold. Assume that the Geometrization conjecture is true. Then $\pi_1(M)$ is isomorphic to the following free product*

$$F_r * \pi_1(M_1) * \cdots * \pi_1(M_s) * \pi_1(N_1) * \cdots * \pi_1(N_t) * \pi_1(P_1) * \cdots * \pi_1(P_u)$$

where F_r is a free group of rank r, M_i's are Seifert fibered spaces, N_i's are Haken and P_i's are hyperbolic 3-manifolds.

For K-theory of finite groups we directly refer the reader to [11].

Now we are ready to state and prove the theorem calculating the K-theory of 3-manifold groups.

Theorem 6.1. *Let M be a compact 3-manifold with torsion free fundamental group and assume that the Geometrization Conjecture is true. Then $Wh(\pi_1(M)) = \tilde{K}_0(\mathbb{Z}[\pi_1(M)]) = 0$.*

Proof. We only describe the proof for the vanishing of the Whitehead group. From above and by Theorem 5.2 we only need to show that the Whitehead group vanishes when M is either a Seifert fibered, Haken or hyperbolic.

When M is Haken then it follows from Theorem 5.4. The hyperbolic case is a consequence of Theorem 5.6. So we need to consider only the Seifert fibered space case. There are Seifert fibered spaces which contains incompressible surface and hence they are Haken. The following Proposition describes the non-Haken Seifert fibered spaces.

Proposition 6.1. [12] *Let M be a Seifert fibered space with infinite fundamental group and which is not Haken. Then the base surface of M is the two sphere and with three cone points of indices p, q and r satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$.*

Remark 6.1. The orbifold fundamental group of the above base surface is also known as *triangle group* and is denoted by $Q(p, q, r)$. In the case $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ the group $Q(p, q, r)$ is a discrete group of isometries of the Euclidean plane. In the other case it is a discrete group of isometries of the hyperbolic plane and it is called a *hyperbolic triangle group*.

When the above inequality is an equality the fundamental group of M is virtually poly-cyclic. In this case the theorem follows from a result of Farrell and Hsiang (see [4]). The remaining case is completed by Plotnick in [13]. As the proof of Plotnick uses another very important technique in K-theory we sketch the proof below.

Recall that a finite group G is called *hyperelementary* if it is a semidirect product of a cyclic group and a p-group for some prime p.

Proposition 6.2. *(Hyperelementary Induction Technique) (*[19], [20]*) Let π and H be two groups with H finite and a surjective homomorphism $f : \pi \to H$. Assume $Wh(f^{-1}(G)) = 0$ for all hyperelementary subgroup G of H. Then $Wh(\pi) = 0$.*

The main idea of Plotnick is that in the case when $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ there is a finite quotient H of $Q(p,q,r)$ which is not hyperelementary (see also lemma 1.2 of [18]). And then he shows that the inverse image of any hyperelementary subgroup of H under the composite homomorphism $\pi_1(M)\to Q(p,q,r)\to H$ is either the fundamental group of a Haken manifold or of a non-Haken Seifert fibered space (see section 3 of [13]). In the second possibility the orbifold fundamental group of the base surface is again a hyperbolic triangle group and is a subgroup of $Q(p,q,r)$. At this point the following two important facts about hyperbolic triangle groups are used.

- $Q(p,q,r)$ is not a proper subgroup of itself. which means that any injective homomorphism $Q(p,q,r)\to Q(p,q,r)$ is also surjective.
- a hyperbolic triangle group contains only finitely many hyperbolic triangle groups.

Now using induction on the number of hyperbolic triangle groups contained in $Q(p,q,r)$ and by the above Proposition and by Theorem 5.4 we complete the proof of the Theorem. $\square$

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