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The Isomorphism Conjecture for Groups with Generalized Free Product Structure

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Abstract

In this article we study the K- and L-theory of groups acting on trees. We consider the problem in the context of the fibered isomorphism conjecture of Farrell and Jones. We show that in the class of residually finite groups it is enough to prove the conjecture for finitely presented groups with one end. Also, we deduce that the conjecture is true for the fundamental groups of graphs of finite groups and of trees of virtually cyclic groups. To motivate the reader we include a survey on some classical works on this subject.

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Contents

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1 Introduction

We study the computation of K- and L-theory, in terms of generalized homology theories, of groups acting on trees from the computation of the respective theories for the vertex and the edge stabilizers. We also give a survey on some classical works on this subject.

To motivate the reader we begin with some of the fundamental questions in Topology, which are answered by the K- and L-theory of groups.

- When is a space homotopy equivalent to a finite CW-complex?
- When is a CW-complex homotopy equivalent to a manifold?
- Are two homotopy equivalent closed manifolds homeomorphic? When the manifolds are closed and aspherical a positive answer is expected and it is also known as the *Borel Conjecture*.
- When can we place a boundary to a noncompact manifold?
- When is a map $f : M \to \mathbb{S}^1$ from a manifold M homotopic to a fiber bundle projection?

These questions were well studied over the last several decades. Lower K-theory (that is, the Whitehead group $Wh(G)$, reduced projective class group $\tilde{K}_0(\mathbb{Z}[G])$, negative K-groups $K_{-i}(\mathbb{Z}[G])$, $i \geqslant 1$) and the surgery L-groups $L_n(\mathbb{Z}[G])$ of the fundamental group G contain the answers to the above and many more questions. The subject is enormous and, therefore, we describe some of these obstruction groups and the contexts.

1.1 Lower K-groups and surgery groups

One can easily construct examples to answer the above questions in the negative. Therefore, we need several necessary conditions.

A connected space X is called *finitely dominated* if there is a connected finite complex Y and maps $f : X \to Y$ and $g : Y \to X$ so that $g \circ f$ is homotopic to the identity map of X. Note that this is half of saying that X is homotopy equivalent to a finite complex. It is also known that if X is finitely dominated then it is homotopy equivalent to a countable complex, which can be checked using some facts from [46]. Therefore, to investigate whether a space is homotopy equivalent to a finite complex we need to make the assumption that X is finitely dominated. The *Wall finiteness obstruction* ([66, 67]) is produced from a finitely dominated connected space X, which is an element $\omega(X) \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ so that X is homotopy equivalent to a finite complex if and only if $\omega(X) = 0$ in $\tilde{K}_0(\mathbb{Z}[\pi_1(X)])$. Furthermore, given an element ω in $\tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ there is a finitely dominated space whose obstruction is ω. Let us recall that the *reduced projective class group* $\tilde{K}_0(\mathbb{Z}[G])$ is by definition the free abelian group generated by the isomorphism classes $[P]$ of finitely generated projective $\mathbb{Z}[G]$-modules P modulo the following two relations

$$[P] + [Q] - [P \oplus Q], [F]$$

where F is a finitely generated free $\mathbb{Z}[G]$-module. See [48] for some more basic information on this group.

The fourth problem also can be solved using the reduced projective class group. So let M be a noncompact connected smooth orientable manifold. We would like to know whether there exist a compact manifold N so that $N-\partial N$ is diffeomorphic to M. Obviously the first obstruction will be if the fundamental group of M is not finitely presented and hence M should have finitely many ends to start with. Secondly the manifold should have finitely generated homology. Thirdly, near the ends M should behave like a compact manifold product with an interval. As a consequence a necessary condition is that outside a big enough compact subset the manifold should have the homology of a compact manifold of one less dimension. For simplicity assume that the manifold has only one end. Another necessity is that if Y_i is an increasing sequence of compact subsets of N, then the fundamental groups of the complements of Y_i should stabilize to a finitely presented group $\pi_1(U) = G$ at a certain stage, where U is the complement of Y_i for some large i. Such an end is also called a *tame* end. Under these necessary conditions one shows that $C_*(\tilde{U})$ is chain equivalent to a finite chain complex $P_*(U)$ of finitely generated projective $R = \mathbb{Z}[\pi_1(U)]$-modules.

In [62] it was shown that if the Euler characteristic $\chi(P_*(U)) = \sum_i (-1)^i [P_i(U)] = 0$ in $\tilde{K}_0(R)$ and if $\dim U \neq 3, 4, 5$ then M is the interior of a compact manifold with boundary. For a detailed account on this work see [69].

If we want a finite complex to be homotopy equivalent to a closed orientable manifold there are several more necessary conditions needed. The first condition is that the homology of the complex must satisfy Poincaré duality. Such a complex is called a *Poincaré complex.* Let X be a connected finite complex and for some n, $H_n(X, \mathbb{Z}) \simeq \mathbb{Z}$. Let $[X] \in H_n(X, \mathbb{Z})$ be a generator such that the cap product with $[X]$ gives an isomorphism $H^q(X, \mathbb{Z}) \to H_{n-q}(X, \mathbb{Z})$ for all q. Then X is called a *Poincaré complex* with *orientation* $[X]$ and dimension n.

The second condition needed is the existence of a bundle over the complex, which has properties similar to the normal bundle of a manifold embedded in some Euclidean space. We describe this below.

Let M be a closed connected oriented smooth manifold of dimension n. By the Whitney Embedding Theorem we can embed this manifold in $\mathbb{R}^{n+k} \subset \mathbb{S}^{n+k}$ for $k \geqslant n$. Let ν_M be the normal bundle of M in $\mathbb{S}^{n+k}$. Let τ_M be the tangent bundle of M. Then $\tau_M \oplus \nu_M$ is the product bundle as $M \subset \mathbb{S}^{n+k} - \{\infty\} = \mathbb{R}^{n+k}$. Let N be the subset of ν_M consisting of vectors of length $< \epsilon$, with respect to some Riemannian metric, for some $\epsilon > 0$. Then N is an open neighborhood of M in $\mathbb{S}^{n+k}$ and in fact diffeomorphic to the total space $E(\nu_M)$ of ν_M. The one point compactification of $T(\nu_M)$ is called the *Thom space* of ν_M. On the other hand the one point compactification N^* of N is homeomorphic to $\mathbb{S}^{n+k}/\mathbb{S}^{n+k} - N$. Hence, we get a map $\alpha : \mathbb{S}^{n+k} \to N^* \simeq T(\nu_M)$. One can check that α induces an isomorphism $H_{n+k}(\mathbb{S}^{n+k}, \mathbb{Z}) \to H_{n+k}(T(\nu_M), \mathbb{Z})$ and sends the canonical generator of $H_{n+k}(\mathbb{S}^{n+k}, \mathbb{Z})$ to the generator of $H_{n+k}(T(\nu_M), \mathbb{Z})$, which comes from the orientation class $[M]$ via the Thom isomorphism. In this sense this map is of degree 1.

Therefore, for a Poincaré complex X to be homotopy equivalent to a closed oriented smooth manifold it is necessary that there should be a bundle ξ on X and a degree 1 map $\alpha : \mathbb{S}^{n+k} \to T(\xi)$. Once this information is given we can apply

Thom Transversality Theorem to homotope α to β so that $\beta^{-1}(X)$ ($:= K$) is a (oriented) submanifold of $\mathbb{S}^{n+k}$. Furthermore, the map β restricted to the normal bundle ν_K of K gives a linear bundle map onto ξ and $\beta|_K$ is of degree 1. A data of this type as in the following commutative diagram is called a *normal map* and is denoted by (f, b).

$$\begin{array}{ccc} \nu_M & \xrightarrow{f} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{b} & X \end{array}$$

Where, M is a closed connected smooth oriented manifold, X is a Poincaré complex, ν_M is the normal bundle in some embedding of M in $\mathbb{S}^{n+k}$ and b is a degree 1 map. As we saw this map b is obtained from the Transversality Theorem but is nowhere close to be a homotopy equivalence. In the simply connected and odd high dimension case in fact b can be homotoped to a homotopy equivalence ([12]). So the next general step is to apply Surgery theory to b to get another normal map, which is normally cobordant to the previous one, and to try to get closer to a homotopy equivalence. To achieve this we need that b induces isomorphisms on the homotopy groups level. This can be done up to dimension $< [\frac{n}{2}]$ and, therefore, by Poincaré duality the main problem lies in dimension $[\frac{n}{2}]$. The complication in this middle dimension gives rise to Wall's *Surgery obstruction groups* $L^h_*(\mathbb{Z}[\pi_1(X)])$. That is, given a normal map (f, b), there is an obstruction $\sigma(f, b)$, which lies in the group $L^h_*(\mathbb{Z}[\pi_1(X)])$, whose vanishing will ensure that the normal map can be normally cobordant to another normal map (f', b'), where b' is a homotopy equivalence. See [68].

Remark 1.1.1. These surgery groups depend only on the fundamental group and on its orientation character $\omega : \pi_1(X) \to \mathbb{Z}_2$. Here as we are dealing with the oriented case this homomorphism is trivial. In the general situation we need to add the ω in the notation of the surgery groups. But in this article we avoid it for simplicity.

Remark 1.1.2. The upper script h to the notation of the surgery groups is due to "homotopy equivalence". There are problems, when one asks for "simple homotopy equivalence" in the normal map. Then a different surgery problem appears and gives rise to the surgery groups $L^s_n(\mathbb{Z}[-])$. There are many other decorated surgery groups for different surgery problems, like $L_n^{\langle -\infty \rangle}(\mathbb{Z}[-])$ mentioned before. But all of them coincide once we have the lower K-theory vanishing result of the group. This is checked using Rothenberg's exact sequence. For example if the Whitehead group of a group G vanishes, then, $L^h_n(\mathbb{Z}[G]) = L^s_n(\mathbb{Z}[G])$ for all n.

One further remark is that all surgery groups are 4-periodic. That is, for example, $L^h_n(\mathbb{Z}[-]) = L^h_{n+4}(\mathbb{Z}[-])$ for all n.

Once we have established that a complex is homotopy equivalent to a manifold, the next question is about the uniqueness of such a manifold. In surgery theory one gets two such manifolds M_1 and M_2 (when they exist) as h-cobordant, that is, there is a manifold W with two boundary components homeomorphic to M_1 and M_2 and the inclusions $M_i \subset W$ are homotopy equivalences. Given such an

h-cobordism W there is an obstruction $\tau(W, M_1)$, which lies in a quotient of the K_1 of the integral group ring of the fundamental group of M_1 defined below.

Let Γ be a group and $\mathbb{Z}[\Gamma]$ be its integral group ring. $K_1(\mathbb{Z}[\Gamma])$ is by definition $\frac{GL(\mathbb{Z}\Gamma)}{[GL(\mathbb{Z}\Gamma),GL(\mathbb{Z}\Gamma)]}$ where $GL(\mathbb{Z}\Gamma) = \lim_{n\to\infty} GL_n(\mathbb{Z}\Gamma)$. There is a map from Γ to $K_1(\mathbb{Z}[\Gamma])$ sending an element $g \in \Gamma$ to the 1×1 matrix $< g >$. Define the *Whitehead group* $Wh(\Gamma)$ of Γ as the quotient $\frac{K_1(\mathbb{Z}[\Gamma])}{\{\pm<g> \mid g\in\Gamma\}}$.

Now, given an h-cobordism W between two connected manifolds M_1 and M_2 with $\dim W \geqslant 6$ there is an element $\tau(W, M_1) \in Wh(\pi_1(M_1))$ with the property: $\tau(W, M_1) = 0$ implies that W is homeomorphic to $M_1 \times I$ where $I = [0, 1]$. This is called the *s-cobordism Theorem*. See [40] and [41]. One consequence of this theorem is the Poincaré conjecture in high dimensions. Furthermore, given an element $\tau \in Wh(\pi_1(M_1))$ there is an h-cobordism W over M_1 realizing τ.

There is another interpretation of the Whitehead group, which says that given a homotopy equivalence $f : K \to L$ between two connected finite complexes K and L there is an element $\tau(f) \in Wh(\pi_1(K))$ whose vanishing ensures that the map f is homotopic to a *simple homotopy equivalence*. See [19].

There are similar interpretation of the reduced projective class groups and negative K-groups in terms of some "special" kind of h-cobordism called *bounded h-cobordism*. See [47] for some more on this matter.

For the last problem in the list we started with one needs $Wh(-)$, $\tilde{K}_0(\mathbb{Z}[-])$ and one more obstruction group called the Farrell Nil group. See [23]. We describe the steps briefly. Let $f : M \to \mathbb{S}^1$ be a map where M is a closed connected smooth manifold. We need to find whether f can be homotoped to a fiber bundle projection. At first, obviously we need $f_* : \pi_1(M) \to \pi_1(\mathbb{S}^1)$ to be surjective. Secondly, the kernel of f_* must be finitely presented as it should be the fundamental group of a fiber of a possible fibration. This can be achieved by looking at the infinite cyclic covering $\tilde{M}$ of M corresponding to the kernel of f_*. Hence, $\tilde{M}$ must be finitely dominated and, then, we use the Wall finiteness obstruction, which lies in $\tilde{K}_0(\mathbb{Z}[\pi_1(\tilde{M})])$. Next, a Nil group is defined in [23], which contains the obstruction to the existence of a framed submanifold N of M representing f so that we get an h-cobordism $(\overline{M - N}, N_1, N_2)$. Here N_1 and N_2 are the two boundary components of $\overline{M - N}$. Finally, we need to make this h-cobordism a product to achieve a fiber bundle projection. There comes the dimension restriction and a Whitehead torsion type obstruction for the application of the s-cobordism theorem.

1.2 Methods of computations

For computation of the lower K-groups and surgery groups there are generally two methods.

Method A. Using the following two classical assembly maps

$$A_K : H_*(BG, \mathbb{K}) \to K_*(\mathbb{Z}[G])$$

and

$$A_L : H_*(BG, \mathbb{L}) \to L_*(\mathbb{Z}[G]).$$

It is still an open problem that A_K and A_L are isomorphisms for torsion free groups, which also imply the Borel conjecture. Recall that the Novikov conjecture asks for the rational injectivity of A_K and A_L. In the case where G has torsion, A_K and A_L need not be surjective or injective.

There are now two issues. First, we need to study the maps A_K and A_L, and then compute the domain homology theories using the Atiyah-Hirzebruch spectral sequence.

In this method, the breakthrough came in the works of Farrell, Hsiang and Jones. See, for example, [27–29, 31] and [32]. They introduced geometric methods to study the assembly maps, and were profoundly successful in proving the Borel and the Novikov conjectures for flat, hyperbolic, and more generally for non-positively curved Riemannian manifolds. There are results based on these works for several other classes of groups. See, for example, [2, 9, 35, 36, 49, 50] and [52–59]. Their methods were extended to prove the Borel conjecture in the hyperbolic and CAT(0) groups cases ([6]). The Frobenius induction technique was also fruitfully used in [26, 30, 63] and in [4].

Method B. Groups built from the following two classical constructions inductively (also known as *generalized free product* ([65])):

- Amalgamated free product $G = G_1 *_H G_2$.
- HNN-extension $G = K*_L$.

For example, surface groups and fundamental groups of Haken 3-manifolds ([64]) can be built from the trivial group using the above two constructions. The combination of the two constructions inductively (also known as *generalized free product structure* on the group ([65])) is also equivalent to the situation when the group acts on a tree. Consequently, given a group G acting on a tree one would like to compute the K- and L-theory of G in terms of the computation for the vertex and edge stabilizers of the action.

In the next section we follow the second method to get the computation of the K- and the L-theory of new classes of groups from known cases.

There is a generalization of the assembly map we discussed above to have a suitable set up for groups with torsion. Farrell and Jones replaced the domain of the assembly map by some other homology theory and conjectured ((*fibered*) *isomorphism conjecture* [33]) the new assembly maps to be isomorphisms for all groups. This new homology theory incorporates the information of the obstruction groups of the virtually cyclic subgroups of the group. In other words, the conjecture says that the K- and L-theory of any group are concentrated at the K- and L-theories of the virtually cyclic subgroups of the group. When restricted to torsion free groups these new assembly maps give the classical ones above. This conjecture has profound applications in Geometry, Topology and Algebra and imply many of the standard conjectures in high dimensional topology, for example the Borel conjecture, the Novikov conjecture, vanishing of the lower K-theory for torsion free groups, etc.

We will discuss this aspect of the subject in Section 3 and will study the isomorphism conjecture in the context of groups acting on trees. One basic result we prove is that to prove the fibered isomorphism conjecture for the class of residually finite groups it is enough to prove the fibered isomorphism conjecture for finitely presented residually finite groups with one end. We also investigate cases where the vertex stabilizers belong to some well-known classes of groups; for example virtually cyclic groups, finitely generated abelian groups, polycyclic groups and nilpotent groups.

To end this Introduction we mention that the lower K-theory information is already embedded in the surgery theory. So it helps in deduction of many topological question, which needs surgery theory if we have an ad hoc information of the lower K-theory. Sometimes to avoid this problem one defines surgery theory (denoted by $L_*^{\langle -\infty \rangle}$) from where the embedded lower K-theory information is removed. For example, in the isomorphism conjecture in L-theory the assembly map is expected to be an isomorphism for the $L_*^{\langle -\infty \rangle}$-theory only.

2 Lower K-theory and surgery L-theory of groups acting on trees

In this section we survey some classical works on the K- and L-theory of groups acting on trees and also mention some recent developments.

2.1 Lower K-theory of groups acting on trees

We recall the work from [65] on the K-theory of groups with generalized free product structure. Under certain (regularity) conditions a Mayer-Vietoris type exact sequence is produced in [65] for K-theory and as a consequence the author computed lower K-theory for several classes of groups including knot groups, and more generally fundamental groups of submanifolds of the 3-sphere and compact Haken 3-manifolds. In the general situation the existence of a Mayer-Vietoris type exact sequence is obstructed by certain Nil groups. These Nil groups vanish if some algebraic assumptions are made on the group ring of the group.

Recall that by the Bass-Serre theory, given a group acting on a tree (without inversion) there is an associated graph of groups whose fundamental group is isomorphic to the group. See [21].

Let us recall that a *graph of groups* consists of a graph $\mathcal{G}$ (that is a one-dimensional CW-complex) and to each vertex v or edge e of $\mathcal{G}$ there is associated a group $\mathcal{G}_v$ (called the *vertex group* of the vertex v) or $\mathcal{G}_e$ (called the *edge group* of the edge e) respectively with the assumption that for each edge e and for its two end vertices v and w (say) there are injective group homomorphisms $\mathcal{G}_e \to \mathcal{G}_v$ and $\mathcal{G}_e \to \mathcal{G}_w$. (If $v = w$ then we also demand two injective homomorphisms as above.) The *fundamental group* $\pi_1(\mathcal{G})$ of a finite graph $\mathcal{G}$ is defined inductively so that in the simple cases of graphs of groups where the graph has two vertices and one edge or one vertex and one edge the fundamental group is the amalgamated free product or the HNN-extension respectively. For an infinite graph it is then defined taking limit on finite subgraphs. See [21] for some more on this subject.

Throughout the article we make the following conventions:

We use the same notation for a graph of groups and its underlying graph. $V_\mathcal{G}$ and $E_\mathcal{G}$ are respectively the set of all vertices and edges of a graph $\mathcal{G}$.

Theorem 2.1.1 ([65]). *Let $\mathcal{G}$ be a graph of groups with $\pi_1(\mathcal{G}) \simeq G$ and $\{G_v\}_{v\in V_{\mathcal{G}}}$ and $\{G_e\}_{e\in E_{\mathcal{G}}}$ are the vertex and edge groups of $\mathcal{G}$ respectively. Then there is an exact sequence:*

$$\bigoplus_{e\in E_{\mathcal{G}}} Wh(G_e) \to \bigoplus_{v\in V_{\mathcal{G}}} Wh(G_v) \to Wh(G) \to$$

$$Nil \oplus (\bigoplus_{e\in E_{\mathcal{G}}} \tilde{K}_0(\mathbb{Z}[G_e])) \to \bigoplus_{v\in V_{\mathcal{G}}} \tilde{K}_0(\mathbb{Z}[G_v])$$

where Nil is the Nil group.

The lower K-theory is also computed as the low degree homotopy groups of a certain space called the *Whitehead space* defined in [65]. The homotopy groups, denoted by $Wh_i(-)$, of the Whitehead space then become obstructions for the K-theory to form a generalized homology theory. In other words we have the following:

$$\cdots \to H_i(BG, \mathbb{K}) \to K_i(G) \to Wh_i(G) \to H_{i-1}(BG, \mathbb{K}) \to \cdots .$$

Now, there is a trick which helps in computation using the above theorem. This says that if G acts on a tree and A is another group then $G \times A$ also acts on the same tree with stabilizers product of the stabilizers of G with A. In the particular case where A is free abelian, it is known that $\tilde{K}_0(\mathbb{Z}[G\times A])$ is a subgroup of $Wh(G \times A \times \mathbb{Z})$. Therefore, from an a priori information that the Nil group Nil vanishes one gets many interesting vanishing results using the Five Lemma argument.

Let us now talk about under what conditions the Nil group vanishes.

Proposition 2.1.1 ([65]). *Let G be as above. Then $Nil = 0$ if for any edge group G_e the group ring $\mathbb{Z}[G_e]$ is regular coherent.*

Here it is interesting to note that the vanishing of the Nil group depends only on the edge groups. It does not depend on the vertex groups or on the way the edge groups are embedded in the group G.

Let us recall that a ring R is called *coherent* if its finitely presented modules form an abelian category and it is called *regular coherent* if, in addition, each finitely presented R-module has a finite-dimensional projective resolution. If a group G acts on a tree there is a way to check if $\mathbb{Z}[G]$ is regular coherent.

Remark 2.1.1. Here we remark that if $\mathbb{Z}[G]$ is regular coherent then G is torsion free. See [5, Lemma 4.3].

Proposition 2.1.2 ([65]). *Let G be as above. Then $\mathbb{Z}[G]$ is regular coherent if $\mathbb{Z}[G_v]$ is regular coherent for each vertex v and $\mathbb{Z}[G_e]$ is regular noetherian for each edge e.*

Therefore, if a group G can be constructed inductively by amalgamated free product and HNN-extension from the trivial group then $\mathbb{Z}[G]$ is regular coherent. For example if G is one of the following groups: a free abelian group, a free group, the fundamental group of a 2-manifold and the fundamental group of a Haken 3-manifold ([64]), then $\mathbb{Z}[G]$ is regular coherent.

The exact sequence in the theorem is obtained from the low degree part of the long homotopy exact sequence of a certain fibration. At first, the problem is reformulated in the un-reduced K-theory case. We describe this below briefly.

First, the concept of generalized free product of groups is transferred to the category of rings and free product with amalgamation and generalized Laurent extensions of rings are defined as the ring theoretic setup of the problem. Next, passing to Quillen's K-theory spaces from rings, the problem of the relation between various constituent spaces is analyzed. For example in the case of amalgamated free product of rings one has rings A, B and C, where C is embedded as a ring in A and B in a *pure* ([65]) way, the resulting ring R is the colimit of the associated diagram. Then the existence of a Nil space $\tilde{K}\mathcal{NIL}(C, A', B')$ is proved and it is shown that the loop space $\Omega K(R)$ of the K-theory space of R is the direct product, up to homotopy, of $\tilde{K}\mathcal{NIL}(C, A', B')$ and the homotopy fiber of a map $K(C) \to K(A) \times K(B)$. And a similar situation occurs in the generalized Laurent extension case. The homotopy groups of the Nil space computes the Nil groups. The vanishing of the Nil groups is guaranteed by the following theorem.

Theorem 2.1.2 ([65]). *If the ring C is regular coherent then the corresponding Nil space is contractible.*

The following Mayer-Vietoris type exact sequence is then deduced from the long homotopy exact sequence of a certain fibration.

$$\cdots \to K_i(C) \to K_i(A) \oplus K_i(B) \to K_i(R) \to K_{i-1}(C) \to \cdots \to K_0(R).$$

When C is not regular coherent the exotic Nil terms appear with $K_*(C)$. Finally, to complete the proof of the theorem one needs the Mayer-Vietoris sequence of generalized homology theory and the two long exact sequences above.

Now, we state a corollary.

Corollary 2.1.1 ([65]). *There is a class of groups $\mathcal{CL}$ containing the following groups such that for any $G \in \mathcal{CL}$ the Whitehead space of G is contractible and hence their lower K-theory vanish.*

- *free groups and free abelian groups.*
- *poly-infinite cyclic groups.*
- *torsion free one-relator groups.*
- *fundamental groups of 2-manifolds different from the projective plane.*
- *fundamental groups of compact orientable 3-manifolds any irreducible summand of which either has non-empty boundary, or is simply connected, or is Haken.*
- *fundamental groups of submanifolds of the 3-sphere.*
- *subgroups of the groups listed above.*

2.2 Surgery L-theory of groups acting on trees

The problem in the surgery L-theory case was studied in [17, 14, 16, 13] and [61]. Many variations of the problem were studied by mathematicians before. Here we recall the above works, which give a strong generalization of previous results. As in the K-theory case here also similar UNil obstruction groups were defined, which together with some lower K-theory non-triviality obstruct Mayer-Vietoris type exact sequence for surgery groups. Again under some group theoretic (*square root closed*) condition it was shown that the UNil groups vanish. Here it is to be noted that in contrast to the situation of the vanishing of the Nil groups where the sufficient condition was that only the amalgamated group be regular coherent, for the vanishing of the UNil group the square root closed nature of the embedding of the amalgamated group in the whole group is relevant.

Let M and N be closed manifolds, K a codimension 1 submanifold of N and $f : M \to N$ a homotopy equivalence.

Splitting problem. When can f be homotoped to a map $g : M \to N$ so that $g^{-1}(K)$ is a submanifold of M and $g|_{g^{-1}(K)}$ and $g|_{M-g^{-1}(K)}$ are homotopy equivalences?

If such a g exists then we say f is *splittable.*

The answer to this problem is related to the computation of the surgery obstruction groups of a generalized free product in terms of its constituent groups.

Let M be a manifold and N be a codimension 1 submanifold of M. Recall that N is called *two sided* in M if N has a neighborhood V in M such that $V - N$ has two components or equivalently the normal bundle of N in M is trivial. If both M and N are orientable then N is always two sided in M. A subgroup H of a group G is called *square root closed* if for any $g \in G$, $g^2 \in H$ implies $g \in H$.

We would also need the existence of the exact sequence involving Whitehead groups and reduced projective class groups of the constituent groups of a generalized free product as in Theorem 2.1.1.

Theorem 2.2.1 ([17]). *Let M and N be two closed connected manifolds of dimension $n \geqslant 5$ and $f : M \to N$ a homotopy equivalence. Let K be a codimension 1 submanifold of N. For simplicity of the presentation assume that $N - K$ has two components N_1 and N_2. Assume the following*:

- *K is two sided in N.*
- *$\pi_1(K) \to \pi_1(N)$ is injective.*
- *$\pi_1(K)$ is square root closed in $\pi_1(N)$.*

Then f is splittable if and only if the Whitehead torsion $\tau(f) \in Wh(\pi_1(N))$ of f is in the image of the map

$$Wh(\pi_1(N_1)) \oplus Wh(\pi_1(N_2)) \to Wh(\pi_1(N)).$$

Now, recall that there is a map defined in [65] (Theorem 2.1.1)

$$\phi : Wh(\pi_1(N)) \to \tilde{K}_0(\mathbb{Z}[\pi_1(K)]) \oplus Nil \to \tilde{K}_0(\mathbb{Z}[\pi_1(K)]).$$

Theorem 2.2.2 ([17]). *Under the same hypothesis as in the above theorem if $\phi(\tau(f)) = 0$ then there exists an h-cobordism W between M and another manifold M' with a map $F : W \to N$ extending f so that $F|_{M'}$ is splittable.*

Next, we relate the above results with the Mayer-Vietoris exact sequence for surgery groups. The reduced surgery groups $\tilde{L}^h_n(\mathbb{Z}[G])$ are defined as the kernel of the homomorphism $L^h_n(\mathbb{Z}[G]) \to L^h_n(\mathbb{Z}[\langle 1\rangle])$ induced from the trivial map $G \to \langle 1\rangle$.

First, we deduce a consequence of Theorem 2.2.1.

Theorem 2.2.3 ([17]). *Assume M, N and f are as above and N is the connected sum of two manifolds N_1 and N_2. Assume the dimension of M is $\geqslant 5$ and the fundamental groups of N_1 and N_2 have no elements of order 2. Then M is also a connected sum of two manifolds M_1 and M_2 and $f|_{M_i} : M_i \to N_i$ is a homotopy equivalence for $i = 1, 2$.*

The above theorem is not true if the fundamental group has an element of order 2. In [15] it was shown that there is a smooth $4k+1$ dimensional manifold M ($k \geqslant 1$) and a simple homotopy equivalence $M \to \mathbb{RP}^{4k+1}\#\mathbb{RP}^{4k+1}$ so that M is not a nontrivial connected sum in any of the categories: smooth, piecewise linear or smooth.

Now, we come to the more general existence of a Mayer-Vietoris exact sequence for surgery groups. Below we state it for the situation of amalgamated free product. A similar statement is also there for the HNN-extension case.

Theorem 2.2.4 ([14, 16]). *Let H be a square root closed subgroup of two groups G_1 and G_2. Let $G = G_1 *_H G_2$ and A be the kernel of the map $\phi : Wh(G) \to \tilde{K}_0(\mathbb{Z}[H])$ defined above.*

(1) *Then there is a long exact sequence of surgery groups*

$$\cdots \to L^h_n(\mathbb{Z}[H]) \to L^h_n(\mathbb{Z}[G_1]) \oplus L^h_n(\mathbb{Z}[G_2]) \to L^A_n(\mathbb{Z}[G]) \to^{\partial} L^h_{n-1}(\mathbb{Z}[H]) \to \cdots .$$

Here $L^A_n(\mathbb{Z}[-])$ consists of $\sigma(f, b)$ of those normal maps (f, b) for which the Whitehead torsions of b lie in the subgroup A.

(2) *Let B be the kernel of the map*

$$\tilde{K}_0(\mathbb{Z}[H]) \to \tilde{K}_0(\mathbb{Z}[G_1]) \oplus \tilde{K}_0(\mathbb{Z}[G_2]).$$

Therefore, B is isomorphic to $Wh(G)/A$. Then there is an exact sequence

$$\cdots \to H^{n+1}(\mathbb{Z}_2; B) \to L^A_n(\mathbb{Z}[G]) \to L^{Wh(G)}_n(\mathbb{Z}[G]) \to H^n(\mathbb{Z}_2; B) \to \cdots$$

where the $\mathbb{Z}_2$-action comes from the orientation character $\omega : G \to \mathbb{Z}_2$ of the group G, which we suppressed in the notation of the surgery groups.

Here note that $L^{Wh(G)}_n(\mathbb{Z}[-]) = L^h_n(\mathbb{Z}[-])$.

If we do not assume the square root closed condition then a new term appears called the UNil-groups whose vanishing is needed for the existence of the above exact sequence. More explicitly, assume that $A = Wh(G)$ (which implies $L^A_n(\mathbb{Z}[G]) = L^h_n(\mathbb{Z}[G])$) then there is a subgroup $\mathrm{UNil}^h_n(\mathbb{Z}[H]; \mathbb{Z}[\hat{G}_1], \mathbb{Z}[\hat{G}_2])$ of $L^h_n(\mathbb{Z}[G])$ so that the above Mayer-Vietoris exact sequence exists replacing $L^h_n(\mathbb{Z}[G])$ by the quotient

$$L^h_n(\mathbb{Z}[G])/\mathrm{UNil}^h_n(\mathbb{Z}[H]; \mathbb{Z}[\hat{G}_1], \mathbb{Z}[\hat{G}_2]).$$

Where $\mathbb{Z}[\hat{G_i}]$ denotes the H-subbimodule of $\mathbb{Z}[G_i]$ additively generated by $G_i - H$, for $i = 1, 2$. The groups $\text{UNil}_n^h(\mathbb{Z}[H]; \mathbb{Z}[\hat{G_1}], \mathbb{Z}[\hat{G_2}])$ are 2-primary and vanish if G is square root closed. See [16]. The group $\text{UNil}_n^h(\mathbb{Z}[\langle 1\rangle]; \mathbb{Z}[\hat{\mathbb{Z}}_2], \mathbb{Z}[\hat{\mathbb{Z}}_2])$ for the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ is related to the question of homotopy invariance of connected sum we discussed before in the context of projective spaces. The calculation of these UNil groups was only done recently in [20] and [3]. In [24] it was shown that the UNil groups have exponent at the most 4.

Remark 2.2.1. We get from the above exact sequences that the Mayer-Vietoris type exact sequence for surgery groups always exists modulo 2-torsions. Therefore, if we change the ring from $\mathbb{Z}$ to $\mathbb{Z}[\frac{1}{2}]$ then the Mayer-Vietoris exact sequence always exists. Hence, using the Five Lemma, the Mayer-Vietoris exact sequence above with coefficients in $\mathbb{Z}[\frac{1}{2}]$ and the Mayer-Vietoris exact sequence for generalized homology theory we get that if a group G acts on a tree so that for the vertex and edge stabilizers the assembly map in surgery theory (see Introduction) is an isomorphism then so it is for G. This was also stated in [5, Theorem 0.13].

We know that there is a large class of (torsion free) groups for which the lower K-theory vanishes. See Section 3. Therefore, the square root closed condition is the only obstruction in any of these situations. We used this square root closed condition together with the result of [65] for Haken 3-manifold groups to deduce a vanishing theorem for structure sets (which is equivalent to saying that the assembly map is an isomorphism) for a large class of 3-manifolds in [50] and [49]. Recently, we could deduce this vanishing theorem for all aspherical 3-manifolds in [58] due to some stronger developments in this area.

The above theorem gives important calculations of surgery groups. We state the cases of free groups and surface groups.

If G_1 and G_2 are two groups with no elements of order 2 (for example if they are torsion free) then it follows from Theorem 2.2.4 that

$$\tilde{L}_n^h(\mathbb{Z}[G_1 * G_2]) = \tilde{L}_n^h(\mathbb{Z}[G_1]) \oplus \tilde{L}_n^h(\mathbb{Z}[G_2])$$

and

$$\tilde{L}_n^s(\mathbb{Z}[G_1 * G_2]) = \tilde{L}_n^s(\mathbb{Z}[G_1]) \oplus \tilde{L}_n^s(\mathbb{Z}[G_2]).$$

This gives the following calculation.

Corollary 2.2.1 ([14]). *Let F_m be the free group on m generators. Then*

$$L_n^h(\mathbb{Z}[F_m]) = \begin{cases} \mathbb{Z} & \text{if } n = 4k, \\ \mathbb{Z}^m & \text{if } n = 4k+1, \\ \mathbb{Z}_2 & \text{if } n = 4k+2, \\ \mathbb{Z}_2^m & \text{if } n = 4k+3. \end{cases}$$

Note that since the Whitehead group of the free groups vanishes (Corollary 2.1.1) the same calculation holds for the groups $L_n^s(\mathbb{Z}[F_m])$. See Remark 1.1.2.

Since for the trivial group we already know the calculation of the surgery groups (restrict to $m = 0$ in the free group case), let S be a non-simply connected closed surface.

Corollary 2.2.2 ([14]). *Let $G = \pi_1(S)$. If S is orientable of genus $g \geqslant 1$ then*

$$L_n^h(\mathbb{Z}[G]) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n = 4k, \\ \mathbb{Z}^{2g} & \text{if } n = 4k+1, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n = 4k+2, \\ \mathbb{Z}_2^{2g} & \text{if } n = 4k+3. \end{cases}$$

If $G = \mathbb{Z}_2$ then the respective calculation is the following

$$L_n^h(\mathbb{Z}[G]) = \begin{cases} \mathbb{Z}_2 & \text{if } n = 4k, \\ 0 & \text{if } n = 4k+1, \\ \mathbb{Z}_2 & \text{if } n = 4k+2, \\ 0 & \text{if } n = 4k+3. \end{cases}$$

Finally, if S is the connected sum of the Klein bottle and the orientable surface of genus g then the corresponding calculation takes the following form:

$$L_n^h(\mathbb{Z}[G]) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } n = 4k, \\ \mathbb{Z}^{2g+1} & \text{if } n = 4k+1, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n = 4k+2, \\ \mathbb{Z}_2^{2g+2} & \text{if } n = 4k+3. \end{cases}$$

Again since the Whitehead group of surface groups vanishes (Corollary 2.1.1) we have $L_n^h(\mathbb{Z}[G]) = L_n^s(\mathbb{Z}[G])$ where G is a surface group as above.

Remark 2.2.2. In this connection, Cappell conjectured that for any knot group $G = \pi_1(\mathbb{S}^3 - k)$, where k is a knot in the 3-sphere $\mathbb{S}^3$, the abelianization homomorphism $G \to \mathbb{Z}$ induces an isomorphism on the surgery groups. That is, $L_n(\mathbb{Z}[G]) \to L_n(\mathbb{Z}[\mathbb{Z}])$ is an isomorphism for all n. This was known as *Cappell's conjecture* and he proved it in the case where the commutator subgroup of the knot group is finitely generated. We proved this conjecture for all knot groups in [1]. A similar computation was give in [51] for link groups. Recently, an explicit computation of the surgery groups of the classical Artin pure braid groups (denoted by PB_m) was given in [59].

Theorem 2.2.5 ([59]). *If PB_m is the classical Artin pure braid group on m strands, then we have the following*

$$L_n^h(PB_m) = \begin{cases} \mathbb{Z} & \text{if } n = 4k, \\ \mathbb{Z}^{\frac{m(m+1)}{2}} & \text{if } n = 4k+1, \\ \mathbb{Z}_2 & \text{if } n = 4k+2, \\ \mathbb{Z}_2^{\frac{m(m+1)}{2}} & \text{if } n = 4k+3. \end{cases}$$

The vanishing of the lower K-theory of PB_m was proved in [2]. Therefore, the same calculation holds for the surgery groups for simple homotopy equivalence.

Theorem 2.2.4 is proved by a more general (relative) version of Theorem 2.2.1. The crucial point is to describe the boundary map ∂. The other maps are defined by the inclusion maps with proper sign. First, note that by Theorem 2.1.1 the image of $Wh(G_1) \oplus Wh(G_2) \to Wh(G)$ is the same as the kernel A of $Wh(G) \to \tilde{K}_0(\mathbb{Z}[H])$. Now, given $\alpha \in L_n^A(\mathbb{Z}[G])$ one represents this by a normal map (f, b), which satisfies the statement of the splitting theorem and one gets f to be transverse to the splitting submanifold (whose fundamental group is H) and then one restricts the normal map to the inverse of this splitting submanifold to get a normal map giving an element in $L_{n-1}^h(\mathbb{Z}[H])$. The exactness of the sequence is again proved by some geometric methods. For details we refer the reader to [13] and [61].

3 The isomorphism conjecture for groups acting on trees

In this section we study the fibered isomorphism conjecture of Farrell and Jones for groups acting on trees. Originally, the conjecture was stated for pseudo-isotopy theory, algebraic K-theory and for L-theory ([33]). One can deduce a lower algebraic K-theory version of the conjecture (that is in dimension $\leqslant 1$) from the pseudo-isotopy version of the conjecture. Here we prove some results for the pseudo-isotopy theory, L-theory and in the lower K-theory. The methods of proofs of our results hold for all equivariant homology theories under certain conditions. We make these conditions explicit here.

It is well known that the pseudo-isotopy version of the conjecture yields the following:

• Computation of the Whitehead group and the lower K-groups of the associated groups.

• It implies the conjecture in lower K-theory (that is in dimension $\leqslant 1$). In particular, it proves that the Whitehead group, the reduced projective class group and the lower K-groups in dimension $\leqslant -2$ vanish for torsion free groups.

• Computation of the algebraic K-groups of the groups in all dimensions after tensoring with the rationals in terms of the ordinary rational homology groups. See [33].

$$K_n(\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq H_n(BG, \mathbb{Q}) \oplus \bigoplus_{k=1}^{\infty} H_{n-4k-1}(BG, \mathbb{Q}).$$

• Together with the conjecture in L-theory this also gives the following computations of the homotopy groups of the space of homeomorphisms (diffeomorphisms) for closed aspherical manifolds M of dimension $m \geqslant 11$ and for $1 \leqslant i \leqslant \frac{m-7}{3}$. See [25].

$$\pi_i(Top(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \text{center}(\pi_1(M)) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{for } i = 1 \\ 0 & \text{for } i \geqslant 2. \end{cases}$$

Also under the same hypothesis one has the following

$$\pi_i(Diff(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \text{center}(\pi_1(M)) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{for } i = 1, \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M; \mathbb{Q}) & \text{for } i \geqslant 2 \text{ and } m \text{ odd}, \\ 0 & \text{for } i \geqslant 2 \text{ and } m \text{ even}. \end{cases}$$

The main problem we are concerned with is the following:

Problem. Assume the fibered isomorphism conjecture for the stabilizers of the action of a group on a tree. Show that the group also satisfies the conjecture.

In [52, Reduction Theorem] we solved the problem when the edge stabilizers are trivial. In this article we show it when the edge stabilizers are finite, and, in addition, either the vertex stabilizers are residually finite or the group surjects onto another group with certain properties. We also solve the problem, under a certain condition, assuming that the vertex stabilizers are polycyclic. This condition is both necessary and sufficient for the fundamental group to be subgroup separable when the vertex stabilizers are finitely generated nilpotent. Next, we prove some results when the stabilizers are abelian groups. Graphs of groups of the last type were studied before in [63] where the L-theory was computed when the stabilizers are free abelian groups.

A positive answer to the problem implies that the fibered isomorphism conjecture is true for one-relator groups and for solvable groups. See Subsection 3.8 for details.

3.1 Statements of theorems

Now, we state our main results. Before that we recall some definitions. Given two groups G and H the *wreath product* $G \wr H$ is by definition the semidirect product $G^H \rtimes H$ where the action of H on G^H is the regular action and G^H is the direct sum of copies of G indexed by the elements of H.

In the statements of the results we say that the FIC^P (FICwF^P) is true for a group G if the Farrell-Jones fibered isomorphism conjecture for pseudo-isotopy theory is true for the group G (for $G \wr H$ for any finite group H).

The importance of the version FICwF^P (*fibered isomorphism conjecture wreath product with finite groups*) was first realized in the proof of the conjecture for braid groups in [36]. Later it was defined in [52] to facilitate proving the conjecture for a 3-manifold group from the knowledge of the conjecture for some finite sheeted covering. The main advantage in this version of the conjecture is that it passes to finite index overgroups (Proposition 3.5.2).

Theorem 3.1.1. *Let $\mathcal{G}$ be a graph of groups such that the edge groups are finite. Then $\pi_1(\mathcal{G})$ satisfies the* FICwF^P *if one of the following conditions is satisfied.*

(1) *The vertex groups are residually finite and satisfy the* FICwF^P.

(2) *There is a homomorphism $f : \pi_1(\mathcal{G}) \to Q$ onto another group Q such that the restriction of f to any vertex group has finite kernel and Q satisfies the* FICwF^P.

This theorem is a special case of Proposition 3.2.2 (1) respectively Proposition 3.2.1 (b). See Section 3.7.

We state now the following immediate corollary.

Corollary 3.1.1. *Assume that the* FICwF^P *is true for all one-ended, finitely presented residually finite groups. Then the* FICwF^P *is true for all residually finite groups.*

Let G be a residually finite group. Then by [35, Theorem 7.1] it is enough to consider finitely presented residually finite group. Since finitely presented groups are accessible ([22]), by the above Corollary we need to show that the conjecture is true for one-ended finitely presented residually finite groups to prove the conjecture for all residually finite groups.

Remark 3.1.1. There is a large class of residually finite groups for which the FICwF^P is true. For example, any group which contains a group from the following examples as a subgroup of finite index is a residually finite group satisfying the FICwF^P.

1. Polycyclic groups ([33]).
2. Artin full braid groups ([36, 54]).
3. Compact 3-manifold groups ([52, 53]).
4. Compact surface groups ([33]).
5. Fundamental groups of hyperbolic Riemannian manifolds ([33]).
6. Crystallographic groups ([33]).

Definition 3.1.1. A graph of groups $\mathcal{G}$ is said to satisfy the *intersection property* if for each connected subgraph of groups $\mathcal{G}'$ of $\mathcal{G}$, $\cap_{e\in E_{\mathcal{G}'}}\mathcal{G}'_e$ contains a subgroup which is normal in $\pi_1(\mathcal{G}')$ and is of finite index in some edge group. We say that $\mathcal{G}$ is of *finite-type* if the graph is finite and all the vertex groups are finite.

Theorem 3.1.2. *Let* $\mathcal{G}$ *be a graph of groups. Let* $\mathcal{D}$ *be a collection of finitely generated groups satisfying the following.*

- *Any element* $C \in \mathcal{D}$ *has the following properties. Quotients and subgroups of* C *belong to* $\mathcal{D}$. C *is residually finite and the* FICwF^P *is true for the mapping torus* $C \rtimes \langle t\rangle$ *for any action of the infinite cyclic group* $\langle t\rangle$ *on* C.

  Assume that the vertex groups of $\mathcal{G}$ *belong to* $\mathcal{D}$. *Then the* FICwF^P *is true for* $\pi_1(\mathcal{G})$ *if* $\mathcal{G}$ *satisfies the intersection property.*

Theorem 3.1.2 is a special case of Proposition 3.2.2 (2) and is proved in Section 3.7.

As a consequence of Theorem 3.1.2 we prove the following.

Let us first recall that a group G is called *subgroup separable* if the following is satisfied. For any finitely generated subgroup H of G and $g \in G - H$ there is a finite index normal subgroup K of G so that $H \subset K$ and $g \in G - K$. Equivalently, a group is subgroup separable if the finitely generated subgroups of G are closed in the profinite topology of G. A subgroup separable group is therefore residually finite.

Definition 3.1.2. Let $\mathcal{G}$ be a graph of groups. An edge e of $\mathcal{G}$ is called a *finite edge* if the edge group $\mathcal{G}_e$ is finite. $\mathcal{G}$ is called *almost a tree of groups* if there are finite edges $e_1, e_2, \ldots$ so that the components of $\mathcal{G} - \{e_1, e_2, \ldots\}$ are trees. If we remove all the finite edges from a graph of groups then the components of the resulting graph are called *component subgraphs.*

Theorem 3.1.3. *Let $\mathcal{G}$ be a graph of groups. Then the* FICwF^P *is true for $\pi_1(\mathcal{G})$ if one of the following five conditions is satisfied.*

(1) *The vertex groups are virtually polycyclic and $\mathcal{G}$ satisfies the intersection property.*

(2) *The vertex groups of $\mathcal{G}$ are finitely generated nilpotent and $\pi_1(\mathcal{G})$ is subgroup separable.*

(3) *The vertex and the edge groups of any component subgraph* (*Definition* 3.1.2) *are fundamental groups of closed surfaces of genus $\geqslant 2$. Given a component subgraph $\mathcal{H}$ which has at least one edge there is a subgroup $C < \cap_{e\in E_{\mathcal{H}}}\mathcal{H}_e$, which is of finite index in some edge group and is normal in $\pi_1(\mathcal{H})$.*

(4) *$\mathcal{G}$ is almost a tree of groups and the vertex groups of any component subgraph of $\mathcal{G}$ are finitely generated abelian and of the same rank.*

(5) *$\mathcal{G}$ is a tree of finitely generated abelian groups and either it has only one edge or* FICwF^P *is true for any graph of infinite cyclic groups.*

For examples of graphs of groups satisfying the hypothesis in (3) see Example 3.3.1.

Using some recent results we deduce the following proposition.

Proposition 3.1.1. *Let $\mathcal{G}$ be a tree of virtually cyclic groups. Then the fibered isomorphism conjectures in L-theory and lower K-theory are true for $\pi_1(\mathcal{G})\wr H$ for any finite group H.*

Remark 3.1.2. Although we stated the above theorems for the pseudo-isotopy version of the conjecture, the same statements are also true in the L-theory version of the conjecture. See [56] and Remark 3.10.1.

We start by stating the general fibered isomorphism conjecture for equivariant homology theory ([5]). We show that under certain conditions a group acting on a tree satisfies this general conjecture provided the stabilizers also satisfy the conjecture. Then we restrict to the pseudo-isotopy case of the conjecture and prove the theorems.

Finally, in Examples 3.9.1, 3.9.2 and 3.9.3 we provide explicit examples of groups for which the results of this paper can be applied to prove the fibered isomorphism conjecture in the pseudo-isotopy and L-theory case. We further show that the groups in these examples are neither CAT(0) nor hyperbolic.

3.2 Statements of the conjecture and some propositions

We now recall the statement of the isomorphism conjecture for equivariant homology theories ([5, Section 1]) and we state some propositions.

Let $\mathcal{H}^-_*$ be an equivariant homology theory with values in R-modules for R a commutative associative ring with unit. An equivariant homology theory assigns to a group G a G-homology theory $\mathcal{H}^G_*$, which for a pair of G-CW complex (X, A), produces a $\mathbb{Z}$-graded R-module $\mathcal{H}^G_n(X, A)$. For details see [42, Section 1].

A *family* of subgroups of a group G is defined as a set of subgroups of G, which is closed under taking subgroups and conjugations. If $\mathcal{C}$ is a class of groups, which is closed under isomorphisms and taking subgroups then we denote by $\mathcal{C}(G)$ the set of all subgroups of G, which belong to $\mathcal{C}$. Then $\mathcal{C}(G)$ is a family of subgroups of G. For example $\mathcal{VC}$, the class of virtually cyclic groups, is closed under isomorphisms and taking subgroups. By definition a virtually cyclic group has a cyclic subgroup of finite index. Also $\mathcal{FIN}$, the class of finite groups is closed under isomorphisms and taking subgroups.

Given a group homomorphism $\phi : G \to H$ and a family $\mathcal{C}$ of subgroups of H define $\phi^*\mathcal{C}$ to be the family of subgroups $\{K < G \ : \ \phi(K) \in \mathcal{C}\}$ of G. Given a family $\mathcal{C}$ of subgroups of a group G there is a G-CW complex $E_\mathcal{C}(G)$, which is unique up to G-equivalence, satisfying the property that for $H \in \mathcal{C}$ the fixpoint set $E_\mathcal{C}(G)^H$ is contractible and $E_\mathcal{C}(G)^H = \varnothing$ for H not in $\mathcal{C}$.

(Fibered) Isomorphism conjecture ([5, Definition 1.1]). *Let $\mathcal{H}^-_*$ be an equivariant homology theory with values in R-modules. Let G be a group and $\mathcal{C}$ be a family of subgroups of G. Then the isomorphism conjecture for the pair $(G, \mathcal{C})$ states that the projection $p : E_\mathcal{C}(G) \to pt$ to the point pt induces an isomorphism*

$$\mathcal{H}^G_n(p) : \mathcal{H}^G_n(E_\mathcal{C}(G)) \simeq \mathcal{H}^G_n(pt)$$

for $n \in \mathbb{Z}$.

The fibered isomorphism conjecture for the pair $(G, \mathcal{C})$ states that for any group homomorphism $\phi : K \to G$ the isomorphism conjecture is true for the pair $(K, \phi^*\mathcal{C})$.

Let $\mathcal{C}$ be a class of groups which is closed under isomorphisms and taking subgroups.

Definition 3.2.1. If the (fibered) isomorphism conjecture is true for the pair $(G, \mathcal{C}(G))$ we say that the (F)IC$_\mathcal{C}$ *is true for* G or simply say (F)IC$_\mathcal{C}(G)$ *is satisfied.* Also we say that the (F)ICwF$_\mathcal{C}(G)$ *is satisfied* if the (F)IC$_\mathcal{C}$ is true for $G \wr H$ for any finite group H. Finally, a group homomorphism $p : G \to K$ is said to *satisfy* the FIC$_\mathcal{C}$ or *the* FICwF$_\mathcal{C}$ if for $H \in p^*\mathcal{C}(K)$ the FIC$_\mathcal{C}$ or the FICwF$_\mathcal{C}$ is true for H respectively.

The (fibered) isomorphism conjectures in the pseudo-isotopy theory, K-theory and in the L-theory are equivalent to the Farrell-Jones conjectures stated in (Section 1.7) Section 1.6 in [33]. (For details see [5, Sections 5 and 6] for the K and L theories and see [43, Sections 4.2.1 and 4.2.2] for the pseudo-isotopy theory.)

Definition 3.2.2. We say that $\mathcal{T}_\mathcal{C}$ (${}_w\mathcal{T}_\mathcal{C}$) is satisfied if for a graph of groups $\mathcal{G}$ with vertex groups from the class $\mathcal{C}$ the FIC$_\mathcal{C}$ (FICwF$_\mathcal{C}$) for $\pi_1(\mathcal{G})$ is true.

Let us now assume that $\mathcal{C}$ contains all the finite groups. We say that ${}_t\mathcal{T}_\mathcal{C}$ (${}_f\mathcal{T}_\mathcal{C}$) is satisfied if for a graph of groups $\mathcal{G}$ with trivial (finite) edge groups and vertex groups belonging to the class $\mathcal{C}$, the FIC$_\mathcal{C}$ for $\pi_1(\mathcal{G})$ is true. If we replace the $\mathrm{FIC}_{\mathcal{C}}$ by the $\mathrm{FICwF}_{\mathcal{C}}$ then we denote the corresponding properties by ${}_{wt}\mathcal{T}_{\mathcal{C}}$ (${}_{wf}\mathcal{T}_{\mathcal{C}}$). Clearly ${}_{w}\mathcal{T}_{\mathcal{C}}$ implies $\mathcal{T}_{\mathcal{C}}$ and ${}_{w*}\mathcal{T}_{\mathcal{C}}$ implies ${}_{*}\mathcal{T}_{\mathcal{C}}$ where $* = t$ or f.

And we say that $\mathcal{P}_{\mathcal{C}}$ is satisfied if for $G_1, G_2 \in \mathcal{C}$ the product $G_1 \times G_2$ satisfies the $\mathrm{FIC}_{\mathcal{C}}$.

$\mathcal{FP}_{\mathcal{C}}$ is satisfied if whenever the $\mathrm{FIC}_{\mathcal{C}}$ is true for two groups G_1 and G_2 then the $\mathrm{FIC}_{\mathcal{C}}$ is true for the free product $G_1 * G_2$.

We denote the above properties for the equivariant homology theories P, K, L or KH with only a super-script by P, K, L or KH respectively. For example, $\mathcal{T}_{\mathcal{C}}$ for P is denoted by $\mathcal{T}^P$ since in all the first three cases we set $\mathcal{C} = \mathcal{VC}$ and for KH we set $\mathcal{C} = \mathcal{FIN}$.

We show in this article and in [55] that the above properties are satisfied in several instances of the conjecture.

For the rest of this section we assume that $\mathcal{C}$ is also closed under quotients and contains all the finite groups.

Proposition 3.2.1 (Graphs of groups)**.** *Let $\mathcal{C} = \mathcal{FIN}$ or $\mathcal{VC}$. Let $\mathcal{G}$ be a graph of groups and assume there is a homomorphism $f : \pi_1(\mathcal{G}) \to Q$ onto another group Q so that the restriction of f to any vertex group has finite kernel. If the* $\mathrm{FIC}_{\mathcal{C}}(Q)$ ($\mathrm{FICwF}_{\mathcal{C}}(Q)$) *is satisfied then the* $\mathrm{FIC}_{\mathcal{C}}(\pi_1(\mathcal{G}))$ ($\mathrm{FICwF}_{\mathcal{C}}(\pi_1(\mathcal{G}))$) *is also satisfied provided one of the following holds.* (*In the* $\mathrm{FICwF}_{\mathcal{C}}$*-case assume in addition that $\mathcal{P}_{\mathcal{C}}$ is satisfied.*)

(*a*) *$\mathcal{T}_{\mathcal{C}}$ (${}_{w}\mathcal{T}_{\mathcal{C}}$) is satisfied.*

(*b*) *The edge groups of $\mathcal{G}$ are finite and ${}_{f}\mathcal{T}_{\mathcal{C}}$ (${}_{wf}\mathcal{T}_{\mathcal{C}}$) is satisfied.*

(*c*) *The edge groups of $\mathcal{G}$ are trivial and ${}_{t}\mathcal{T}_{\mathcal{C}}$ (${}_{wt}\mathcal{T}_{\mathcal{C}}$) is satisfied.*

For the definition of continuous $\mathcal{H}^-_*$ in the following statement see [5, Definition 3.1]. See also Proposition 3.5.1.

Proposition 3.2.2 (Graphs of residually finite groups)**.** *Assume that $\mathcal{P}_{\mathcal{C}}$ and ${}_{wt}\mathcal{T}_{\mathcal{C}}$ are satisfied. Let $\mathcal{G}$ be a graph of groups. If $\mathcal{G}$ is infinite then assume that $\mathcal{H}^-_*$ is continuous.*

(1) *Assume that the edge groups of $\mathcal{G}$ are finite and the vertex groups are residually finite. If the* $\mathrm{FICwF}_{\mathcal{C}}$ *is true for the vertex groups of $\mathcal{G}$, then it is true for $\pi_1(\mathcal{G})$.*

For the next three items assume that $\mathcal{C} = \mathcal{VC}$.

(2) *Let $\mathcal{D}$ be the collection of groups defined in Theorem* 3.1.2 *replacing "FICwF^P" by "$\mathrm{FICwF}_{\mathcal{VC}}$". Assume that the vertex groups of $\mathcal{G}$ belong to $\mathcal{D}$. Then the* $\mathrm{FICwF}_{\mathcal{VC}}$ *is true for $\pi_1(\mathcal{G})$ if $\mathcal{G}$ satisfies the intersection property.*

(3) *Assume that the vertex groups of $\mathcal{G}$ are virtually polycyclic and that the* $\mathrm{FICwF}_{\mathcal{VC}}$ *is true for virtually polycyclic groups. Then the* $\mathrm{FICwF}_{\mathcal{VC}}$ *is true for $\pi_1(\mathcal{G})$ provided either $\mathcal{G}$ satisfies the intersection property or the vertex groups are finitely generated nilpotent and $\pi_1(\mathcal{G})$ is subgroup separable.*

(4) *Assume that the vertex and the edge groups of any component subgraph* (*Definition* 3.1.2) *are fundamental groups of closed surfaces of genus $\geqslant 2$ and for every component subgraph $\mathcal{H}$, which has at least one edge there is a subgroup $C < \cap_{e\in\mathcal{H}}\mathcal{H}_e$ which is of finite index in some edge group and is normal in $\pi_1(\mathcal{H})$. Then*

the $\mathrm{FICwF}_{\mathcal{VC}}$ *is true for* $\pi_1(\mathcal{G})$ *provided the* $\mathrm{FICwF}_{\mathcal{VC}}$ *is true for the fundamental groups of closed* 3*-manifolds which fiber over the circle.*

We denote a countable infinitely generated free group by F^∞ and a countable infinitely generated abelian group by $\mathbb{Z}^\infty$. Also $G \rtimes H$ is a semidirect product with respect to an arbitrary action of H on G. When H is infinite cyclic and generated by the symbol t; we denote it by $\langle t \rangle$.

Proposition 3.2.3 (Graphs of abelian groups)**.** *Let* $\mathcal{G}$ *be a graph of groups whose vertex groups are abelian and let* $\mathcal{H}^-_*$ *be continuous.*

(1) *Assume that the* $\mathrm{FIC}_{\mathcal{VC}}$ *is true for* $F^\infty \rtimes \langle t \rangle$ *and that* $\mathcal{P}_{\mathcal{VC}}$ *is satisfied. Then the* $\mathrm{FIC}_{\mathcal{VC}}$ *is true for* $\pi_1(\mathcal{G})$ *provided one of the following holds.*

(a) $\mathcal{G}$ *is a tree and the vertex groups of* $\mathcal{G}$ *are finitely generated and torsion free.*

For the next two items assume that the $\mathrm{FICwF}_{\mathcal{VC}}$ *is true for* $F^\infty \rtimes \langle t \rangle$*.*

(b) $\mathcal{G}$ *is a tree.*

(c) $\mathcal{G}$ *is not a tree and the* $\mathrm{FIC}_{\mathcal{VC}}$ *is true for* $\mathbb{Z}^\infty \rtimes \langle t \rangle$ *for any countable infinitely generated abelian group* $\mathbb{Z}^\infty$*.*

(2) *Assume that* $\mathcal{P}_{\mathcal{VC}}$ *and* ${}_{wt}\mathcal{T}_{\mathcal{VC}}$ *are satisfied. Furthermore, assume that* $\mathcal{G}$ *is almost a tree of groups and the vertex and the edge groups of any component subgraph of* $\mathcal{G}$ *are finitely generated and have the same rank. Then the* $\mathrm{FICwF}_{\mathcal{VC}}$ *is true for* $\pi_1(\mathcal{G})$ *provided one of the followings is satisfied.*

(i) *The* $\mathrm{FICwF}_{\mathcal{VC}}$ *is true for* $\mathbb{Z}^n \rtimes \langle t \rangle$ *for all positive integers* n*.*

(ii) *The vertex and the edge groups of any component subgraph of* $\mathcal{G}$ *have rank equal to* 1*.*

(3) *Assume that* $\mathcal{P}_{\mathcal{VC}}$ *and* ${}_{w}\mathcal{T}_{\mathcal{VC}}$ ($\mathcal{T}_{\mathcal{VC}}$) *are satisfied. Furthermore, assume that* $\mathcal{G}$ *is a tree of groups. Then the* $\mathrm{FICwF}_{\mathcal{VC}}$ ($\mathrm{FIC}_{\mathcal{VC}}$) *is true for* $\pi_1(\mathcal{G})$*.*

In (1) of Proposition 3.2.3 if we assume that the $\mathrm{FICwF}_{\mathcal{VC}}$ is true for $\mathbb{Z}^\infty \rtimes \langle t \rangle$ and for $F^\infty \rtimes \langle t \rangle$ then one can deduce from the same proof, using (3) of Proposition 3.5.2 instead of Lemma 3.5.2, that the $\mathrm{FICwF}_{\mathcal{VC}}$ is true for $\pi_1(\mathcal{G})$ irrespective of whether $\mathcal{G}$ is a tree or not.

Proposition 3.2.4 (${}_{w}\mathcal{T}^P$)**.** *Let* $\mathcal{G}$ *be a graph of virtually cyclic groups so that the graph is almost a tree. Furthermore, assume that either the edge groups are finite or the infinite vertex groups are abelian. Then the* FICwF^P *is true for* $\pi_1(\mathcal{G})$*.*

An immediate corollary of the proposition is the following.

Corollary 3.2.1 (${}_{wf}\mathcal{T}^P$)**.** ${}_{wf}\mathcal{T}^P$ *(and hence* ${}_{wt}\mathcal{T}^P$*) is satisfied.*

Remark 3.2.1. We remark here that it is not yet known whether the FIC^P is true for the HNN-extension $G = C*_C$, with respect to the maps $id : C \to C$ and $f : C \to C$ where C is an infinite cyclic group, id is the identity map and $f(u) = u^2$ for $u \in C$. This was mentioned in the introduction of [35]. Note that G is isomorphic to the semidirect product $\mathbb{Z}[\frac{1}{2}] \rtimes \langle t \rangle$, where t acts on $\mathbb{Z}[\frac{1}{2}]$ by multiplication by 2. The main problem with this example is that $\mathbb{Z}[\frac{1}{2}] \rtimes \langle t \rangle$ is not subgroup separable.

Now, we recall that a property (*tree property*) similar to $\mathcal{T}_{\mathcal{C}}$ was defined in [5, Definition 4.1]. The tree property of [5] is stronger than $\mathcal{T}_{\mathcal{C}}$. Corollary 4.4 in [5] was proved under the assumption that the tree property is satisfied. In the following proposition we state that this is true with the weaker assumption that $\mathcal{T}_{\mathcal{FIN}}$ is satisfied. The proof of the proposition goes exactly in the same way as the proof of [5, Corollary 4.4]. It can also be deduced from Proposition 3.2.1.

Proposition 3.2.5. *Let* $1 \to K \to G \to Q \to 1$ *be an exact sequence of groups. Assume that* $\mathcal{T}_{\mathcal{FIN}}$ *is satisfied and* K *acts on a tree with finite stabilizers and that the* $\mathrm{FIC}_{\mathcal{FIN}}(Q)$ *is satisfied. Then the* $\mathrm{FIC}_{\mathcal{FIN}}(G)$ *is also satisfied.*

3.3 Graphs of groups

In this subsection, we prove some results on graphs of groups needed for the proofs of the theorems and propositions.

We start by recalling that by Bass-Serre theory, a group acts on a tree without inversion if and only if the group is isomorphic to the fundamental group of a graph of groups (the Structure Theorem I.4.1 in [21]). Therefore, throughout the paper, by "action of a group on a tree" we will mean an action without inversion.

Lemma 3.3.1. *Let* $\mathcal{G}$ *be a finite and almost a tree of groups* (*Definition* 3.1.2). *Then there is another graph of groups* $\mathcal{H}$ *with the following properties.*

(1) $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{H})$.

(2) *Either* $\mathcal{H}$ *has no edge or the edge groups of* $\mathcal{H}$ *are finite.*

(3) *The vertex groups of* $\mathcal{H}$ *are of the form* $\pi_1(\mathcal{K})$ *where* $\mathcal{K}$ *varies over subgraphs of groups of* $\mathcal{G}$ *which are maximal with respect to the property that the underlying graph of* $\mathcal{K}$ *is a tree and the edge* (*if there is any*) *groups of* $\mathcal{K}$ *are all infinite.*

Proof. The proof is by induction on the number of edges of the graph. Recall that an edge is called a finite edge if the corresponding edge group is finite (Definition 3.1.2). If the graph has no finite edge then by definition of almost a graph of groups $\mathcal{G}$ is a tree. In this case we can take $\mathcal{H}$ to be a graph consisting of a single vertex and associate the group $\pi_1(\mathcal{G})$ to the vertex. So assume $\mathcal{G}$ has n finite edges and that the lemma is true for graphs with $\leqslant n-1$ finite edges. Let e be an edge of $\mathcal{G}$ with $\mathcal{G}_e$ finite. If $\mathcal{G}-\{e\}$ is connected then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1)*_{\mathcal{G}_e}$ where $\mathcal{G}_1 = \mathcal{G}-\{e\}$ is a graph with $n-1$ finite edges. By the induction hypothesis there is a graph of groups $\mathcal{H}_1$ satisfying (1), (2) and (3) for $\mathcal{G}_1$. Let v_1 and v_2 be the vertices of $\mathcal{G}$ to which the ends of e are attached and let v_1' and v_2' be the vertices of $\mathcal{H}_1$ so that $\mathcal{G}_{v_i}$ is a subgroup of $\mathcal{H}_{v_i'}$ for $i=1,2$. (Note that v_1 and v_2 could be the same vertex.) Define $\mathcal{H}$ by attaching an edge e' to $\mathcal{H}_1$ so that the ends of e' are attached to v_1' and v_2' and associate the group $\mathcal{G}_e$ to e'. The injective homomorphisms $\mathcal{H}_{e'} \to \mathcal{H}_{v_i'}$ for $i=1,2$ are defined by the homomorphisms $\mathcal{G}_e \to \mathcal{G}_{v_i}$. It is now easy to check that $\mathcal{H}$ satisfies (1), (2) and (3) for $\mathcal{G}$. On the other hand if $\mathcal{G}-\{e\}$ has two components say $\mathcal{G}_1$ and $\mathcal{G}_2$ then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) *_{\mathcal{G}_e} \pi_1(\mathcal{G}_2)$ where $\mathcal{G}_1$ and $\mathcal{G}_2$ have $\leqslant n-1$ finite edges. Using the induction hypothesis and a similar argument as above we complete the proof of the lemma. □

Lemma 3.3.2. *A finitely generated group contains a free subgroup of finite index if and only if the group acts on a tree with finite stabilizers. And a group acts on a tree with trivial stabilizers if and only if the group is free.*

Proof. By [21, Theorem IV.1.6] a group contains a free subgroup of finite index if and only if the group acts on a tree with finite stabilizers and the stabilizers have bounded order. The proof of the lemma now follows easily. □

We will also need the following two Lemmas.

Lemma 3.3.3. *Let $\mathcal{G}$ be a graph of finitely generated abelian groups so that the underlying graph of $\mathcal{G}$ is a tree. Then the restriction of the abelianization homomorphism $\pi_1(\mathcal{G}) \to H_1(\pi_1(\mathcal{G}), \mathbb{Z})$ to each vertex group of the tree of groups $\mathcal{G}$ is injective.*

Proof. If the tree $\mathcal{G}$ is finite and the vertex groups are finitely generated free abelian then it was proved in [63, Lemma 3.1] that there is a homomorphism $\pi_1(\mathcal{G}) \to A$ onto a free abelian group so that the restriction of this homomorphism to each vertex group is injective. In fact it was shown there that the abelianization homomorphism $\pi_1(\mathcal{G}) \to H_1(\pi_1(\mathcal{G}), \mathbb{Z})$ is injective when restricted to the vertex groups, and, then since the vertex groups are torsion free $\pi_1(\mathcal{G}) \to A = H_1(\pi_1(\mathcal{G}), \mathbb{Z})/\{torsion\}$ is also injective on the vertex groups. The same proof goes through without the torsion free vertex group assumption to prove the lemma when $\mathcal{G}$ is finite. Therefore, we mention the additional arguments needed in the infinite case. First, write $\mathcal{G}$ as an increasing union of finite trees $\mathcal{G}_i$ of finitely generated abelian groups. Consider the following commutative diagram

$$\begin{array}{ccc} \pi_1(\mathcal{G}_i) & \longrightarrow & H_1(\pi_1(\mathcal{G}_i), \mathbb{Z}) \\ \downarrow & & \downarrow \\ \pi_1(\mathcal{G}_{i+1}) & \longrightarrow & H_1(\pi_1(\mathcal{G}_{i+1}), \mathbb{Z}) \end{array}$$

Note that the left hand side vertical map is injective. By the finite tree case the restriction of the two horizontal maps to each vertex group of the respective trees are injective. Now, since group homology and fundamental group commute with direct limit, taking limit completes the proof of the Lemma. □

Lemma 3.3.4. *Let $\mathcal{G}$ be a finite graph of finitely generated groups satisfying the following.*

P. *Each edge group is of finite index in the end vertex groups of the edge. Also assume that the intersection of the edge groups contains a subgroup C (say), which is normal in $\pi_1(\mathcal{G})$ and is of finite index in some edge e (say) group.*

Then $\pi_1(\mathcal{G})/C$ is isomorphic to the fundamental group of a finite-type (*Definition* 3.1.1) *graph of groups. Consequently $\mathcal{G}$ has the intersection property.*

Proof. The proof is by induction on the number of edges of the graph.

Induction hypothesis (IH_n)**.** For any finite graph of groups $\mathcal{G}$ with $\leqslant n$ edges, which satisfies **P**, $\pi_1(\mathcal{G})/C$ is isomorphic to the fundamental group of a graph of groups whose underlying graph is the same as that of $\mathcal{G}$ and the vertex groups are finite and isomorphic to $\mathcal{G}_v/C$ where $v \in V_{\mathcal{G}}$.

If $\mathcal{G}$ has one edge e then $C < \mathcal{G}_e$ and C is normal in $\pi_1(\mathcal{G})$. First, assume e disconnects $\mathcal{G}$. Since $\mathcal{G}_e$ is of finite index in the end vertex groups and C is also of finite index in $\mathcal{G}_e$, C is of finite index in the end vertex groups of e. Therefore, $\pi_1(\mathcal{G})/C$ has the desired property. The argument is the same when e does not disconnect $\mathcal{G}$.

Now, assume $\mathcal{G}$ has n edges and satisfies **P**.

Let us first consider the case where there is an edge e' other than e so that $\mathcal{G} - \{e'\} = \mathcal{D}$ (say) is a connected graph. Note that $\mathcal{D}$ has $n-1$ edges and satisfies **P**.

Hence, by IH_{n-1}, $\pi_1(\mathcal{D})/C$ is isomorphic to the fundamental group of a finite-type graph of groups with $\mathcal{D}$ as the underlying graph and the vertex groups are of the form $\mathcal{D}_v/C$.

Let v be an end vertex of e'. Then $\mathcal{D}_v/C$ is finite. Also by hypothesis $\mathcal{G}_{e'}$ is of finite index in $\mathcal{G}_v = \mathcal{D}_v$. Therefore, $\mathcal{G}_{e'}/C$ is also finite. This completes the proof in this case.

Now, if every edge $e' \neq e$ disconnects $\mathcal{G}$ then $\mathcal{G} - \{e\}$ is a tree. Let e' be an edge other than e so that one end vertex v' (say) of e' has valency 1. Such an edge exists because $\mathcal{G} - \{e\}$ is a tree. Let $\mathcal{D} = \mathcal{G} - (\{e'\} \cup \{v'\})$. Then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{D}) *_{\mathcal{G}_{e'}} \mathcal{G}_{v'}$. Let v'' be the other end vertex of e'. Then by the induction hypothesis C is of finite index in $\mathcal{G}_{v''} = \mathcal{D}_{v''}$. Hence, C is of finite index in $\mathcal{G}_{e'}$. Also $\mathcal{G}_{e'}$ is of finite index in $\mathcal{G}_{v'}$. Therefore, C is also of finite index in $\mathcal{G}_{v'}$.

This completes the proof. □

The following lemma and example give some concrete examples of graphs of groups with intersection property.

Lemma 3.3.5. *Let $\mathcal{G}$ be a finite graph of groups so that all the vertex and the edge groups are finitely generated abelian and of the same rank r (say) and the underlying graph of $\mathcal{G}$ is a tree. Then $\cap_{e\in E_{\mathcal{G}}}\mathcal{G}_e$ contains a rank r free abelian subgroup C which is normal in $\pi_1(\mathcal{G})$ so that $\pi_1(\mathcal{G})/C$ is isomorphic to the fundamental group of a graph of groups whose underlying graph is $\mathcal{G}$ and the vertex groups are finite and isomorphic to $\mathcal{G}_v/C$ where $v \in V_{\mathcal{G}}$.*

Proof. The proof is by induction on the number of edges. If the graph has one edge e then clearly $\mathcal{G}_e$ is normal in $\pi_1(\mathcal{G})$, for $\mathcal{G}_e$ is normal in the two end vertex groups. This follows from Lemma 3.3.7. So by induction assume that the lemma is true for graphs with $\leqslant n-1$ edges. Let $\mathcal{G}$ be a finite graph with n edges and satisfies the hypothesis of the Lemma. Consider an edge e which has one end vertex v (say) with valency 1. Such an edge exists because the graph is a tree. Let v_1 be the other end vertex of e. Then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}') *_{\mathcal{G}_e} \mathcal{G}_v$. Here $\mathcal{G}' = \mathcal{G} - (\{e\} \cup \{v\})$. Clearly by the induction hypothesis there is a finitely generated free abelian normal subgroup $C_1 < \cap_{e'\in E_{\mathcal{G}'}} {\mathcal{G}'}_{e'}$ of rank r of $\pi_1(\mathcal{G}_1)$ satisfying all the required properties. Now, note that $C_1 \cap \mathcal{G}_e = C'$ (say) is of finite index in ${\mathcal{G}'}_{v_1}$ and also in C_1 and C' has rank r. This follows from the following easy to verify Lemma. Now, since C_1 is finitely generated and C' is of finite index in C_1 we can find a characteristic subgroup

$C(< C')$ of C_1 of finite index. Therefore, C has rank r and is normal in $\pi_1(\mathcal{G}')$, since C_1 is normal in $\pi_1(\mathcal{G}')$. Obviously C is normal in $\mathcal{G}_v$. Therefore, we again use Lemma 3.3.7 to conclude that C is normal in $\pi_1(\mathcal{G})$. The other properties are clearly satisfied. This completes the proof of the Lemma. □

Lemma 3.3.6. *Let G be a finitely generated abelian group of rank r and let $G_1, \ldots, G_k$ be rank r subgroups of G. Then $\cap_{i=1}^{i=k} G_i$ is of rank r and of finite index in G.*

Example 3.3.1. Let G and H be finitely generated groups and let H be a finite index normal subgroup of G. Let $f : H \to G$ be the inclusion. Consider a finite tree of groups $\mathcal{G}$ whose vertex groups are copies of G and the edge groups are copies of H. Also assume that the maps from the edge groups to the vertex groups (defining the tree of group structure) are f. Then $\mathcal{G}$ has the intersection property.

Lemma 3.3.7. *Let $G = G_1 *_H G_2$ be a generalized free product. If H is normal in both G_1 and G_2 then H is normal in G.*

Proof. The proof follows by using the normal form of elements in a generalized free product. See [44, p. 72]. □

3.4 Residually finite groups

We now recall and also prove some basic results we need on residually finite groups. For this section we abbreviate “residually finite” by $\mathcal{RF}$.

Lemma 3.4.1. *The fundamental group of a finite graph of $\mathcal{RF}$ groups with finite edge groups is $\mathcal{RF}$.*

Proof. The proof is by induction on the number of edges. If there is no edge then there is nothing to prove. So assume the lemma for graphs with $\leqslant n-1$ edges. Let $\mathcal{G}$ be a graph of groups with n edges and satisfying the hypothesis. It follows that $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) *_F \pi_1(\mathcal{G}_2)$ or $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) *_F$ where F is a finite group and $\mathcal{G}_i$ satisfy the hypothesis of the lemma and have $\leqslant n-1$ edges. Also note that $\pi_1(\mathcal{G}_1) *_F \pi_1(\mathcal{G}_2)$ can be embedded as a subgroup in $(\pi_1(\mathcal{G}_1) * \pi_1(\mathcal{G}_2)) *_F$. Therefore, by the induction hypothesis and since a free product of $\mathcal{RF}$ groups is again $\mathcal{RF}$ ([37]) we only need to prove that, for finite H, $G *_H$ is $\mathcal{RF}$ if so is G. But, this follows from [8] or [18, Theorem 2]. This completes the proof of the Lemma. □

Lemma 3.4.2. *Let $1 \to K \to G \to H \to 1$ be an extension of groups so that K and H are $\mathcal{RF}$. Assume that any finite index subgroup of K contains a subgroup K' so that K' is normal in G and G/K' is $\mathcal{RF}$. Then G is $\mathcal{RF}$.*

Proof. Let $g \in G - K$ and $g' \in H$ be the image of g in H. Since H is $\mathcal{RF}$ there is a finite index subgroup H' of H not containing g'. The inverse image of H' in G is a finite index subgroup not containing g.

Next, let $g \in K - \{1\}$. Choose a finite index subgroup of K not containing g. By hypothesis there is a finite index subgroup K' of K, which is normal in G

and does not contain g. Also G/K' is $\mathcal{RF}$. Now, applying the previous case we complete the proof. □

Lemma 3.4.3. *Let $\mathcal{G}$ be a finite graph of finitely generated $\mathcal{RF}$ groups satisfying the intersection property. Assume the following. "Given an edge e and an end vertex v of e, for every subgroup E of $\mathcal{G}_e$, which is normal in $\mathcal{G}_v$, the quotient $\mathcal{G}_v/E$ is again $\mathcal{RF}$." Then $\pi_1(\mathcal{G})$ is $\mathcal{RF}$.*

Proof. Using Lemma 3.4.1 we can assume that all the edge groups of $\mathcal{G}$ are infinite. Now, the proof is by induction on the number of edges of the graph $\mathcal{G}$. Clearly the induction starts because if there is no edge then the lemma is true. Assume the result for all graphs satisfying the hypothesis with number of edges $\leqslant n-1$ and let $\mathcal{G}$ be a graph of groups with n number of edges and satisfying the hypothesis of the lemma. By the intersection property there is a normal subgroup K, contained in all the edge groups, of $\pi_1(\mathcal{G})$, which is of finite index in some edge group. Hence, we have the following:

K. $\pi_1(\mathcal{G})/K$ is isomorphic to the fundamental group of a finite graph of $\mathcal{RF}$ groups (by hypothesis the quotient of a vertex group by K is $\mathcal{RF}$) with n edges and some edge group is finite. Also it is easily seen that this quotient graph of groups has the intersection property.

By the induction hypothesis and by Lemma 3.4.1 $\pi_1(\mathcal{G})/K$ is $\mathcal{RF}$.

Now, we would like to apply Lemma 3.4.2 to the exact sequence

$$1 \to K \to \pi_1(\mathcal{G}) \to \pi_1(\mathcal{G})/K \to 1.$$

Let H be a finite index subgroup of K. Since K is finitely generated (being of finite index in a finitely generated group) we can find a finite index characteristic subgroup H' of K contained in H. Hence, H' is normal in $\pi_1(\mathcal{G})$. It is now easy to see that **K** is satisfied if we replace K by H'. Hence, $\pi_1(\mathcal{G})/H'$ is $\mathcal{RF}$.

Therefore, by Lemma 3.4.2 $\pi_1(\mathcal{G})$ is $\mathcal{RF}$. □

Lemma 3.4.4. *Let $\mathcal{G}$ be a finite graph of virtually cyclic groups so that either the edge groups are finite or the infinite vertex groups are abelian and the associated graph is almost a tree. Then $\pi_1(\mathcal{G})$ is $\mathcal{RF}$.*

Proof. Applying Lemma 3.4.1 and using the definition of almost a graph of groups we can assume that the graph is a tree and all the edge groups are infinite. Now, using Lemma 3.3.5 we see that the hypothesis of Lemma 3.4.3 is satisfied. This proves the Lemma. □

Lemma 3.4.5. *Let $\mathcal{G}$ be a finite graph of groups whose vertex and edge groups are fundamental groups of closed surfaces of genus $\geqslant 2$. Also assume that the intersection of the edge groups contains a subgroup C (say), which is normal in $\pi_1(\mathcal{G})$ and is of finite index in some edge group. Then $\pi_1(\mathcal{G})$ contains a normal subgroup isomorphic to the fundamental group of a closed surface so that the quotient is isomorphic to the fundamental group of a finite-type graph of groups. Also $\pi_1(\mathcal{G})$ is $\mathcal{RF}$.*

Proof. We need the following Lemma.

Lemma 3.4.6. *Let S be a closed surface. Let G be a subgroup of $\pi_1(S)$. Then G is isomorphic to the fundamental group of a closed surface if and only if G is of finite index in $\pi_1(S)$.*

Proof. The proof follows from covering space theory. □

Therefore, using the above lemma we get that the edge groups of $\mathcal{G}$ are of finite index in the end vertex groups of the corresponding edges. Hence, by Lemma 3.3.4 $\pi_1(\mathcal{G})/C$ is the fundamental group of a finite-type graph of groups. This proves the first statement.

Now, by Lemma 3.4.1 $\pi_1(\mathcal{G})/C$ is $\mathcal{RF}$. Next, by [7], closed surface groups are $\mathcal{RF}$. Then using the above lemma, it is easy to check that the hypothesis of Lemma 3.4.2 is satisfied for the following exact sequence:

$$1 \to C \to \pi_1(\mathcal{G}) \to \pi_1(\mathcal{G})/C \to 1.$$

Hence, $\pi_1(\mathcal{G})$ is $\mathcal{RF}$. □

3.5 Basic results on the isomorphism conjecture

$\mathcal{C}$ is always a class of groups closed under isomorphisms and taking subgroups unless otherwise mentioned.

We start by noting that if the $\mathrm{FIC}_{\mathcal{C}}$ is true for a group G then the $\mathrm{FIC}_{\mathcal{C}}$ is also true for any subgroup H of G. We will refer to this fact as the *hereditary property* in this paper.

By the Algebraic lemma in [36] if G is a normal subgroup of K then K can be embedded in the wreath product $G \wr (K/G)$. We will be using this fact throughout the paper without explicitly mentioning it.

Lemma 3.5.1. *If the* $\mathrm{FICwF}_{\mathcal{C}}(G)$ *is satisfied then the* $\mathrm{FICwF}_{\mathcal{C}}(L)$ *is also satisfied for any subgroup L of G.*

Proof. Note that given a group H, $L \wr H$ is a subgroup of $G \wr H$. Now, use the hereditary property of the $\mathrm{FIC}_{\mathcal{C}}$. □

Proposition 3.5.1. *Assume that $\mathcal{H}^-_*$ is continuous. Let G be a group and $G = \cup_{i\in I} G_i$ where the G_i's are an increasing sequence of subgroups of G so that the* $\mathrm{FIC}_{\mathcal{C}}(G_i)$ *is satisfied for $i \in I$. Then the* $\mathrm{FIC}_{\mathcal{C}}(G)$ *is also satisfied. And if the* $\mathrm{FICwF}_{\mathcal{C}}(G_i)$ *is satisfied for $i \in I$ then the* $\mathrm{FICwF}_{\mathcal{C}}(G)$ *is also satisfied.*

Proof. The first assertion is the same as the conclusion of [5, Proposition 3.4] and the second one is easily deducible from it, since given a group H, $G \wr H = \cup_{i\in I}(G_i \wr H)$. □

Remark 3.5.1. Since the fundamental group of an infinite graph of groups can be written as an increasing union of fundamental groups of finite subgraphs, throughout rest of the paper we consider only finite graphs. The infinite case will then follow if the corresponding equivariant homology theory satisfies the assumption of Proposition 3.5.1. Examples of such equivariant homology theories appear in the isomorphism conjecture for pseudo-isotopy theory, L-theory and for K-theory. See [35, Theorem 7.1].

The following lemma from [5] is crucial for the proofs of the results in this paper. This result in the context of the original fibered isomorphism conjecture ([33]) was proved in [33, Proposition 2.2].

Lemma 3.5.2 ([5, Lemma 2.5]). *Let $\mathcal{C}$ be also closed under taking quotients. Let $p : G \to Q$ be a surjective group homomorphism and assume that the* $\text{FIC}_{\mathcal{C}}$ *is true for Q and for p. Then G satisfies the* $\text{FIC}_{\mathcal{C}}$.

Proposition 3.5.2. *Let $\mathcal{C}$ be as in the statement of the above lemma. Assume that $\mathcal{P}_{\mathcal{C}}$ is satisfied.*

(1) *If the* $\text{FIC}_{\mathcal{C}}$ *is true for G_1 and G_2 then the* $\text{FIC}_{\mathcal{C}}$ *is true for $G_1 \times G_2$. And, the same statement also holds for the* $\text{FICwF}_{\mathcal{C}}$.

(2) *Let G be a finite index normal subgroup of a group K. If the* $\text{FICwF}_{\mathcal{C}}(G)$ *is satisfied then the* $\text{FICwF}_{\mathcal{C}}(K)$ *is also satisfied.*

(3) *Let $p : G \to Q$ be a group homomorphism. If the* $\text{FICwF}_{\mathcal{C}}$ *is true for Q and for p then the* $\text{FICwF}_{\mathcal{C}}$ *is true for G.*

Proof. The proof of (1) is essentially two applications of Lemma 3.5.2. First, apply it to the projection $G_1 \times G_2 \to G_2$. Hence, to prove the lemma we need to check that the $\text{FIC}_{\mathcal{C}}$ is true for $G_1 \times H$ for any $H \in \mathcal{C}(G_2)$. Now, fix $H \in \mathcal{C}(G_2)$ and apply Lemma 3.5.2 to the projection $G_1 \times H \to G_1$. Thus we need to show that the $\text{FIC}_{\mathcal{C}}$ is true for $K \times H$ where $K \in \mathcal{C}(G_1)$. But this is exactly $\mathcal{P}_{\mathcal{C}}$, which is true by hypothesis.

Next, note that given a group H, $(G_1 \times G_2) \wr H$ is a subgroup of $(G_1 \wr H) \times (G_2 \wr H)$. Therefore, the $\text{FICwF}_{\mathcal{C}}$ is true for $G_1 \times G_2$ if it is true for G_1 and G_2.

For (2) let $H = K/G$. Then K is a subgroup of $G \wr H$. Let L be a finite group; then it is easy to check that

$$K \wr L < (G \wr H) \wr L \simeq G^{H \times L} \rtimes (H \wr L)$$

$$< G^{H \times L} \wr (H \wr L) < \Pi_{|H \times L|-times}(G \wr (H \wr L)).$$

The isomorphism in the above display follows from [35, Lemma 2.5] with respect to some action of $H \wr L$ on $G^{H \times L}$ (replace X by A and Y by B in [35, Lemma 2.5]). The second inclusion follows from the Algebraic Lemma in [36].

Now, using (1) and by hypothesis we complete the proof.

For (3) we need to prove that the $\text{FIC}_{\mathcal{C}}$ is true for $G \wr H$ for any finite group H. We now apply Lemma 3.5.2 to the homomorphism $G \wr H \to Q \wr H$. By hypothesis the $\text{FIC}_{\mathcal{C}}$ is true for $Q \wr H$. So let $S \in \mathcal{C}(Q \wr H)$. We have to prove that the $\text{FIC}_{\mathcal{C}}$ is true for $p^{-1}(S)$. Note that $p^{-1}(S)$ contains $p^{-1}(S \cap Q^H)$ as a normal subgroup of finite index. Therefore, using (2) it is enough to prove the $\text{FICwF}_{\mathcal{C}}$ for $p^{-1}(S \cap Q^H)$. Next, note that $S \cap Q^H$ is a subgroup of $\prod_{h \in H} L_h$, where L_h is the image of $S \cap Q^H$ under the projection to the h-th coordinate of Q^H, and since $\mathcal{C}$ is closed under taking quotient, $L_h \in \mathcal{C}(Q)$. Hence, $p^{-1}(S \cap Q^H)$ is a subgroup of $\prod_{h \in H} p^{-1}(L_h)$. Since by hypothesis the $\text{FICwF}_{\mathcal{C}}$ is true for $p^{-1}(L_h)$ for $h \in H$, using (1), (2) and Lemma 3.5.1 we see that the $\text{FICwF}_{\mathcal{C}}$ is true for $p^{-1}(S \cap Q^H)$. This completes the proof. □

Corollary 3.5.1. *$\mathcal{P}_{\mathcal{VC}}$ implies that the* $\mathrm{FIC}_{\mathcal{VC}}$ *is true for finitely generated abelian groups.*

Proof. The proof is immediate from (1) of Proposition 3.5.2 since the $\mathrm{FIC}_{\mathcal{VC}}$ is true for virtually cyclic groups. □

Remark 3.5.2. In (2) of Proposition 3.5.2, if we assume that the $\mathrm{FIC}_{\mathcal{C}}(G)$ is satisfied instead of the $\mathrm{FICwF}_{\mathcal{C}}(G)$, then it is not known how to deduce the $\mathrm{FIC}_{\mathcal{C}}(K)$. Even in the case of the $\mathrm{FIC}_{\mathcal{VC}}$ and when G is a free group it is open. However if G is free then the $\mathrm{FIC}^P(K)$ is satisfied by results of Farrell-Jones. See the proof of Proposition 3.5.5 for details. Also using a recent result ([6]) it can be shown by the same method that the $\mathrm{FIC}^L(K)$ is satisfied.

Proposition 3.5.3. *${}_t\mathcal{T}_{\mathcal{C}}$ (${}_{wt}\mathcal{T}_{\mathcal{C}}$) implies that the* $\mathrm{FIC}_{\mathcal{C}}$ ($\mathrm{FICwF}_{\mathcal{C}}$) *is true for any free group. And if $\mathcal{C}$ contains the class of all finite groups then ${}_f\mathcal{T}_{\mathcal{C}}$ (${}_{wf}\mathcal{T}_{\mathcal{C}}$) implies that the* $\mathrm{FIC}_{\mathcal{C}}$ ($\mathrm{FICwF}_{\mathcal{C}}$) *is true for a finitely generated group, which contains a free subgroup of finite index.*

Proof. The proof follows from Lemma 3.3.2. □

Corollary 3.5.2. *${}_t\mathcal{T}_{\mathcal{C}}$ (${}_{wt}\mathcal{T}_{\mathcal{C}}$) and $\mathcal{P}_{\mathcal{C}}$ imply that the* $\mathrm{FIC}_{\mathcal{C}}$ ($\mathrm{FICwF}_{\mathcal{C}}$) *is true for finitely generated free abelian groups.*

Proof. The proof is a combination of Proposition 3.5.3 and (1) of Proposition 3.5.2. □

Let F^n be a finitely generated free group of rank n.

Lemma 3.5.3. *Assume that the* $\mathrm{FIC}_{\mathcal{C}}$ *is true for $\mathbb{Z}^\infty \rtimes \langle t\rangle$ ($F^\infty \rtimes \langle t\rangle$) for any countable infinitely generated abelian group $\mathbb{Z}^\infty$, then the* $\mathrm{FIC}_{\mathcal{C}}$ *is true for $\mathbb{Z}^n \rtimes \langle t\rangle$ ($F^n \rtimes \langle t\rangle$) for all $n \in \mathbb{N}$. Here all actions of t on the corresponding groups are arbitrary.*

And the same holds if we replace the $\mathrm{FIC}_{\mathcal{C}}$ *by the* $\mathrm{FICwF}_{\mathcal{C}}$.

Proof. The proof is an easy consequence of the hereditary property and Lemma 3.5.1. □

Lemma 3.5.4. *If the* $\mathrm{FICwF}_{\mathcal{VC}}$ *is true for $F^\infty \rtimes \langle t\rangle$ for any action of t on F^∞, then $\mathcal{P}_{\mathcal{VC}}$ is satisfied.*

Proof. Note that $\mathbb{Z}\times\mathbb{Z}$ is a subgroup of $F^\infty \rtimes_t \langle t\rangle$, where the suffix "t" denotes the trivial action of $\langle t\rangle$ on F^∞. Hence, the $\mathrm{FICwF}_{\mathcal{VC}}$ is true for $\mathbb{Z}\times\mathbb{Z}$, that is, the $\mathrm{FIC}_{\mathcal{VC}}$ is true for $(\mathbb{Z}\times\mathbb{Z})\wr F$ for any finite group F. On the other hand, for two virtually cyclic groups C_1 and C_2, the product $C_1\times C_2$ contains a finite index free abelian normal subgroup (say H) of rank $\leqslant 2$ (Lemma 3.6.1), and, therefore, $C_1\times C_2$ is a subgroup of $H\wr F$ for a finite group F. Using the hereditary property we conclude that $\mathcal{P}_{\mathcal{VC}}$ is satisfied. □

Proposition 3.5.4. *The* FICwF^P *is true for any virtually polycyclic group.*

Proof. By [33, Proposition 2.4] the FIC^P is true for any virtually poly-infinite cyclic group. Also a polycyclic group is virtually poly-infinite cyclic. Now, it is easy to check that the wreath product of a virtually polycyclic group with a finite group is virtually poly-infinite cyclic. This completes the proof. □

Since the product of two virtually cyclic groups is virtually polycyclic, an immediate corollary is the following. We give a proof of the corollary which is independent of Proposition 3.5.4.

Corollary 3.5.3. *$\mathcal{P}^P$ is satisfied. Also* FICwF^P *is true for any virtually cyclic group.*

Proof of Corollary 3.5.3. First, note that the FIC^P is true for virtually cyclic groups. Hence, for the first part we only have to prove that the FIC^P is true for $V_1 \times V_2$ where V_1 and V_2 are two infinite virtually cyclic groups. Note that $V_1 \times V_2$ contains a finite index free abelian normal subgroup, say A, on two generators. Therefore, $V_1 \times V_2$ embeds in $A \wr H$ for some finite group H. Since A is isomorphic to the fundamental group of a flat 2-torus, FIC^P is true for $A \wr H$. See Fact 3.1 and Theorem A in [36]. Therefore, FIC^P is true for $V_1 \times V_2$ by the hereditary property. This proves that $\mathcal{P}^P$ is satisfied.

The proof of the second part is similar since for any virtually cyclic group V and for any finite group H, $V \wr H$ is either finite or embeds in a group of the type $A \wr H'$ for some finite group H' and where A is isomorphic to a free abelian group on $|H|$ number of generators and, therefore, A is isomorphic to the fundamental group of a flat $|H|$-torus. Then we can again apply Fact 3.1 and Theorem A from [36]. □

We will also need the following proposition.

Proposition 3.5.5. *Let $\mathcal{G}$ be a graph of finite groups. Then $\pi_1(\mathcal{G})$ satisfies the* FICwF^P.

Proof. By Remark 3.5.1 we can assume that the graph is finite. Lemma 3.3.2 implies that we need to show that the FICwF^P is true for finitely generated groups, which contains a free subgroup of finite index. Now, it is a formal consequence of results of Farrell-Jones that the FICwF^P is true for a free group. For details see [35, Lemma 2.4]. Also compare [36, Fact 3.1]. Next, since $\mathcal{P}^P$ is satisfied (Corollary 3.5.3) using (2) of Proposition 3.5.2 we complete the proof. □

Proposition 3.5.5 was used in [38] to calculate the lower K-groups of virtually free groups.

3.6 Proofs of the propositions

Recall that $\mathcal{C}$ is always a class of groups which is closed under isomorphism, taking subgroups and quotients and contains all the finite groups.

Proof of Proposition 3.2.3. Proof of (1). Since $\mathcal{H}^-_*$ is continuous by Remark 3.5.1 we can assume that the graph $\mathcal{G}$ is finite.

1(a). Since the graph $\mathcal{G}$ is a tree and the vertex groups are torsion free, we can apply Lemma 3.3.3 to see that the restriction of the homomorphism $f : \pi_1(\mathcal{G}) \to H_1(\pi_1(\mathcal{G}), \mathbb{Z})/\{torsion\} = A(say)$ to each vertex group is injective. Let K be the kernel of f. Let T be the tree on which $\pi_1(\mathcal{G})$ acts for the graph of group structure $\mathcal{G}$. Then K also acts on T with vertex stabilizers $K \cap g\mathcal{G}_v g^{-1} = (1)$ where $g \in \pi_1(\mathcal{G})$ and $v \in V_{\mathcal{G}}$. Hence, by Lemma 3.3.2 K is a free group (not necessarily finitely generated). Next, note that A is a finitely generated free abelian group and hence the $\text{FIC}_{\mathcal{VC}}$ is true for A by Corollary 3.5.1 and by hypothesis. Now, applying Lemma 3.5.2 to the homomorphism $f : \pi_1(\mathcal{G}) \to A$ and noting that a torsion free virtually cyclic group is either trivial or infinite cyclic ([34, Lemma 2.5]) we complete the proof. We also need to use Lemma 3.5.3 when K is finitely generated.

1(b). The proof of this case is almost the same as that of the previous case.

Let $A = H_1(\pi_1(\mathcal{G}), \mathbb{Z})$. Now, A is a finitely generated abelian group and hence the $\text{FIC}_{\mathcal{VC}}$ is true for A by Corollary 3.5.1. Next, we apply Lemma 3.5.2 to $p : \pi_1(\mathcal{G}) \to A$. Again by Lemma 3.3.3 the kernel K of this homomorphism acts on a tree with trivial stabilizers and hence K is free. Let V be a virtually cyclic subgroup of A with $C < V$ an infinite cyclic subgroup of finite index in V. Let C be generated by t. Then the inverse image $p^{-1}(V)$ contains $K \rtimes \langle t \rangle$ as a normal finite index subgroup. By hypothesis the $\text{FICwF}_{\mathcal{VC}}$ is true for $K \rtimes \langle t \rangle$. Now, using (2) of Proposition 3.5.2 we see that the $\text{FICwF}_{\mathcal{VC}}$ is true for $p^{-1}(V)$ and in particular, the $\text{FIC}_{\mathcal{VC}}$ is true for $p^{-1}(V)$. This completes the proof of 1(b).

1(c). Since the graph $\mathcal{G}$ is not a tree it is homotopically equivalent to a wedge of r circles for $r \geqslant 1$. Then there is a surjective homomorphism $p : \pi_1(\mathcal{G}) \to F^r$ where F^r is a free group on r generators. And the kernel K of this homomorphism is the fundamental group of the universal covering $\widetilde{\mathcal{G}}$ of the graph of groups $\mathcal{G}$. Hence, K is the fundamental group of an infinite tree of finitely generated abelian groups. Now, we would like to apply Lemma 3.5.2 to the homomorphism $p : \pi_1(\mathcal{G}) \to F^r$. By hypothesis and by Lemma 3.5.3 the $\text{FIC}_{\mathcal{VC}}$ is true for any semidirect product $F^n \rtimes \langle t \rangle$, hence the $\text{FIC}_{\mathcal{VC}}$ is true for F^r by the hereditary property. Since F^r is torsion free, by Lemma 3.5.2, we only have to check that the $\text{FIC}_{\mathcal{VC}}$ is true for the semidirect product $K \rtimes \langle t \rangle$ for any action of $\langle t \rangle$ on K.

By Lemma 3.3.3, the restriction of the following map to each vertex group of $\widetilde{\mathcal{G}}$ is injective:

$$\pi_1(\widetilde{\mathcal{G}}) \to H_1(\pi_1(\widetilde{\mathcal{G}}), \mathbb{Z}).$$

Let A and B be the range group and the kernel of the homomorphism respectively in the above display. Since the commutator subgroup of a group is characteristic we have the following exact sequence of groups induced by the above homomorphism

$$1 \to B \to K \rtimes \langle t \rangle \to A \rtimes \langle t \rangle \to 1$$

for any action of $\langle t \rangle$ on K. Recall that $K = \pi_1(\widetilde{\mathcal{G}})$. Now, let K acts on a tree T, which induces the tree of groups structure $\widetilde{\mathcal{G}}$ on K. Hence, B also acts on T with vertex stabilizers equal to $B \cap g\widetilde{\mathcal{G}}_v g^{-1} = (1)$ where $v \in V_{\widetilde{\mathcal{G}}}$ and $g \in K$. This follows from the fact that the restriction to any vertex group of $\widetilde{\mathcal{G}}$ of the homomorphism

$K \to A$ is injective. Thus B acts on a tree with trivial stabilizers and hence B is a free group by Lemma 3.3.2.

Next, note that the group A is a countable infinitely generated abelian group. Now, we can apply Lemma 3.5.2 to the homomorphism $K \rtimes \langle t \rangle \to A \rtimes \langle t \rangle$ (and use Lemma 3.5.3 if B is finitely generated) and (2) of Proposition 3.5.2 in exactly the same way as we did in the proof of 1(b). This completes the proof of 1(c).

Proofs of 2(i) *and* 2(ii). Let $e_1, e_2, \ldots, e_k$ be the finite edges of $\mathcal{G}$ so that each of the connected components $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ of $\mathcal{G} - \{e_1, e_2, \ldots, e_k\}$ is a tree of finitely generated abelian groups of the same rank. By Lemma 3.3.5 a finite tree of finitely generated abelian groups of the same rank has the intersection property. Therefore, using Lemma 3.4.3 we see that such a tree of groups has residually finite fundamental group.

Now, we check that the FICwF$_{\mathcal{VC}}$ is true for $\pi_1(\mathcal{G}_i)$ for $i = 1, 2, \ldots, n$.

Assume that $\mathcal{G}_i$ is a graph of finitely generated abelian groups of the same rank (say r). Then by Lemma 3.3.5 $\pi_1(\mathcal{G}_i)$ contains a finitely generated free abelian normal subgroup A of rank r so that the quotient $\pi_1(\mathcal{G}_i)/A$ is isomorphic to the fundamental group of a graph of finite type. Hence, by the assumption ${}_{wt}\mathcal{T}_{\mathcal{VC}}$ the FICwF$_{\mathcal{VC}}$ is true for $\pi_1(\mathcal{G}_i)/A$. Now, we would like to apply (3) of Proposition 3.5.2 to the following exact sequence:

$$1 \to A \to \pi_1(\mathcal{G}_i) \to \pi_1(\mathcal{G}_i)/A \to 1.$$

Note that using (2) of Proposition 3.5.2 it is enough to prove that the FICwF$_{\mathcal{VC}}$ is true for $A \rtimes \langle t \rangle$. In case 2(i) this follows from the hypothesis and Lemma 3.5.3. And in case 2(ii) note that $A \rtimes \langle t \rangle$ contains a rank 2 free abelian subgroup of finite index (Lemma 3.6.1). Therefore, we can again apply (2) of Proposition 3.5.2 and Lemma 3.5.1 to see that the FICwF$_{\mathcal{VC}}$ is true for $A \rtimes \langle t \rangle$.

Now, observe that there is a finite graph of groups $\mathcal{H}$ so that the edge groups of $\mathcal{H}$ are finite and the vertex groups are isomorphic to $\pi_1(\mathcal{G}_i)$ where i varies over $1, 2, \ldots, n$ and $\pi_1(\mathcal{H}) \simeq \pi_1(\mathcal{G})$. This follows from Lemma 3.3.1 since $\mathcal{G}$ is almost a tree of groups.

Next, we apply (1) of Proposition 3.2.2 (since $\pi_1(\mathcal{G}_i)$ is residually finite for $i = 1, 2, \ldots, n$) to $\mathcal{H}$ to complete the proof of (2).

Proof of (3). First, assume that the tree is finite. By Lemma 3.3.3, there is a homomorphism $\pi_1(\mathcal{G}) \to H_1(\pi_1(\mathcal{G}), \mathbb{Z})$ whose restriction to any vertex group is injective. A limit argument proves the FICwF$_{\mathcal{VC}}$ for $H_1(\pi_1(\mathcal{G}), \mathbb{Z})$ in case some vertex group is infinitely generated. Now, (2) can be applied to complete the proof of (3). □

Proof of Proposition 3.2.4. Lemma 3.3.1 implies that there is a graph of groups $\mathcal{H}$ with the same fundamental group as $\mathcal{G}$ so that the edge groups of $\mathcal{H}$ are finite and the vertex groups are either finite or fundamental groups of finite trees of groups with rank 1 finitely generated abelian vertex and edge groups. Now, note that by Lemma 3.3.5 the tree of group (say $\mathcal{K}$) corresponding to a vertex group of $\mathcal{H}$ satisfies the hypothesis of Lemma 3.3.4.

Therefore, $\mathcal{K}$ has the intersection property. Also by Lemma 3.4.4 $\pi_1(\mathcal{K})$ is residually finite. Since by Corollary 3.5.3 the FICwF^P is true for virtually cyclic groups, we can apply (1) and (3) of Proposition 3.2.2 to complete the proof of the Proposition provided we check that ${}_{wt}\mathcal{T}^P$ and $\mathcal{P}^P$ are satisfied. By Corollary 3.5.3 the last condition is satisfied.

We now check the first condition. So let $\mathcal{G}$ be a graph of virtually cyclic groups with trivial edge groups. Using Remark 3.5.1 we can assume that the graph is finite. Hence, by Lemma 3.6.2 below, $\pi_1(\mathcal{G})$ is a free product of a finitely generated free group and the vertex groups of $\mathcal{G}$. By Corollary 3.5.3, the FICwF^P is true for any virtually cyclic group and by Proposition 3.5.5 and Lemma 3.3.2 it is true for finitely generated free groups. Therefore, we can apply [52, Theorem 3.1] to conclude that the FICwF^P is true for $\pi_1(\mathcal{G})$. □

We remark that, when we applied (3) of Proposition 3.2.2 in the above proof of Proposition 3.2.4, we only needed the fact that the FICwF^P be true for the mapping torus of a virtually cyclic group (see the proof of (3) of Proposition 3.2.2). We note that for this we do not need Proposition 3.5.4, instead it can easily be deduced from the proof of Corollary 3.5.3. We just need to mention that the mapping torus of a virtually cyclic group is either virtually cyclic or it contains a two-generator free abelian normal subgroup of finite index (see the following lemma).

Lemma 3.6.1. *Suppose a group G has a virtually cyclic normal subgroup V_1 with virtually cyclic quotient V_2. Then G is virtually cyclic, when either V_1 or V_2 is finite, and otherwise G contains a two-generator free abelian normal subgroup of finite index.*

Proof. We have the following exact sequence:

$$1 \to V_1 \to G \to V_2 \to 1.$$

For the first assertion there is nothing to prove when V_2 is finite. So let V_1 be finite and V_2 infinite and let C be an infinite cyclic subgroup of V_2 of finite index. Let p be the homomorphism $G \to V_2$. Then $p^{-1}(C)$ has a finite normal subgroup with quotient C. Therefore, $p^{-1}(C)$ is virtually infinite cyclic. Let C' be an infinite cyclic subgroup of $p^{-1}(C)$ of finite index. Hence, C' is also an infinite cyclic subgroup of G of finite index. This proves the first assertion. For the second assertion let V_1 be also infinite and D an infinite cyclic subgroup of V_1 of finite index. Then $p^{-1}(C) \simeq D \rtimes C$. Since C and D are both infinite cyclic the only possibilities for $p^{-1}(C)$ are that it contains a free abelian subgroup on two generators of index either 1 or 2. This proves the Lemma. □

Proof of Proposition 3.2.1. Let T be the tree on which the group $\pi_1(\mathcal{G})$ acts so that the associated graph of groups is $\mathcal{G}$. For the proof, by Lemma 3.5.2 ((3) of Proposition 3.5.2) we need to show that the FIC$_\mathcal{C}$ (FICwF$_\mathcal{C}$) is true for the homomorphism f. We prove (a) assuming $\mathcal{T}_\mathcal{C}$ (${}_w\mathcal{T}_\mathcal{C}$).

Let $H \in \mathcal{C}(Q)$. Note that $f^{-1}(H)$ also acts on the tree T with stabilizers $f^{-1}(H) \cap \{$stabilizers of the action of $\pi_1(\mathcal{G})\}$. Since the restrictions of f to the vertex groups of $\mathcal{G}$ have finite kernels we get that $f^{-1}(H) \cap \{$stabilizers of the action of $\pi_1(\mathcal{G})\}$ is an extension of a finite group by a subgroup of H. If $\mathcal{C} = \mathcal{FIN}$

then these stabilizers also belong to $\mathcal{C}$. If $\mathcal{C} = \mathcal{VC}$ then using Lemma 3.6.1 we see that the stabilizers again belong to $\mathcal{C}$.

Therefore, the associated graph of groups of the action of $f^{-1}(H)$ on the tree T has vertex groups belonging to $\mathcal{C}$.

Now, using $\mathcal{T}_{\mathcal{C}}$ ($_w\mathcal{T}_{\mathcal{C}}$) we conclude that the FIC$_{\mathcal{C}}$ (FICwF$_{\mathcal{C}}$) is true for $f^{-1}(H)$. This completes the proof of (a).

For the proofs of (b) and (c) just replace $\mathcal{T}_{\mathcal{C}}$ ($_w\mathcal{T}_{\mathcal{C}}$) by $_f\mathcal{T}_{\mathcal{C}}$ ($_{wf}\mathcal{T}_{\mathcal{C}}$) and $_t\mathcal{T}_{\mathcal{C}}$ ($_{wt}\mathcal{T}_{\mathcal{C}}$) respectively in the above proof. Also use the corresponding assumption on the edge groups of the graph of groups $\mathcal{G}$.

□

Corollary 3.6.1 (Free products). *Let $\mathcal{G}$ be a finite graph of groups with trivial edge groups so that the vertex groups satisfy the* FIC$_{\mathcal{C}}$ (FICwF$_{\mathcal{C}}$). *If $\mathcal{P}_{\mathcal{C}}$ and $_t\mathcal{T}_{\mathcal{C}}$ ($_{wt}\mathcal{T}_{\mathcal{C}}$) are satisfied then the* FIC$_{\mathcal{C}}$ (FICwF$_{\mathcal{C}}$) *is true for $\pi_1(\mathcal{G})$.*

Proof. The proof combines the following two Lemmas. □

Lemma 3.6.2. *Let $\mathcal{G}$ be a finite graph of groups with trivial edge groups. Then there is an isomorphism $\pi_1(\mathcal{G}) \simeq G_1 * \cdots * G_n * F$ where G_i's are vertex groups of $\mathcal{G}$ and F is a free group.*

Proof. The proof is by induction on the number of edges of the graph. If the graph has no edge then there is nothing to prove. So assume $\mathcal{G}$ has n edges and that the lemma is true for graphs with $\leqslant n-1$ edges. Let e be an edge of $\mathcal{G}$. If $\mathcal{G} - \{e\}$ is connected then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) * \mathbb{Z}$ where $\mathcal{G}_1 = \mathcal{G} - \{e\}$ is a graph with $n-1$ edges. On the other hand if $\mathcal{G} - \{e\}$ has two components say $\mathcal{G}_1$ and $\mathcal{G}_2$ then $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{G}_1) * \pi_1(\mathcal{G}_2)$ where $\mathcal{G}_1$ and $\mathcal{G}_2$ has $\leqslant n-1$ edges. Therefore, by induction we complete the proof of the lemma. □

Lemma 3.6.3. *Assume that the properties $\mathcal{P}_{\mathcal{C}}$ and $_t\mathcal{T}_{\mathcal{C}}$ ($_{wt}\mathcal{T}_{\mathcal{C}}$) are satisfied. If the* FIC$_{\mathcal{C}}$ (FICwF$_{\mathcal{C}}$) *is true for G_1 and G_2 then the* FIC$_{\mathcal{C}}$ (FICwF$_{\mathcal{C}}$) *is true for $G_1 * G_2$.*

Proof. Consider the surjective homomorphism $p : G_1 * G_2 \to G_1 \times G_2$. By (1) of Proposition 3.5.2 the FIC$_{\mathcal{C}}$ (FICwF$_{\mathcal{C}}$) is true for $G_1 \times G_2$.

Now, note that the group $G_1 * G_2$ acts on a tree with trivial edge stabilizers and vertex stabilizers conjugates of G_1 or G_2. Therefore, the restrictions of p to the stabilizers of this action of $G_1 * G_2$ on the tree are injective. Hence, we are in the situation of (c) of Proposition 3.2.1.

This completes the proof of the Lemma. □

Remark 3.6.1. We recall that in the case of the FICwF^P, Lemma 3.6.3 coincides with the Reduction Theorem (Theorem 3.1) in [52].

Proof of Proposition 3.2.2. Since the equivariant homology theory is assumed to be continuous when the graph is infinite, we can assume that $\mathcal{G}$ is a finite graph of groups.

(1) By hypothesis the edge groups of $\mathcal{G}$ are finite and the vertex groups are residually finite and satisfy the FICwF$_{\mathcal{C}}$. By Lemma 3.4.1 $\pi_1(\mathcal{G})$ is residually

finite. Let $F_1, F_2, \ldots, F_n$ be the edge groups. Let $1 \neq g \in \cup_{i=1}^n F_i$. Since $\pi_1(\mathcal{G})$ is residually finite there is a finite-index normal subgroup N_g of $\pi_1(\mathcal{G})$ so that $g \in \pi_1(\mathcal{G}) - N_g$. Let $N = \cap_{g \in \cup_{i=1}^n F_i} N_g$. Then N is a finite-index normal subgroup of $\pi_1(\mathcal{G})$ so that $N \cap (\cup_{i=1}^n F_i) = \{1\}$.

Let T be a tree on which $\pi_1(\mathcal{G})$ acts so that the associated graph of group structure on $\pi_1(\mathcal{G})$ is $\mathcal{G}$. Hence, N also acts on T. Since N is normal in $\pi_1(\mathcal{G})$ and $N \cap (\cup_{i=1}^n F_i) = \{1\}$, the edge stabilizers of this action are trivial and the vertex stabilizers are subgroups of conjugates of vertex groups of $\mathcal{G}$. Therefore, by Corollary 3.6.1 the FICwF$_{\mathcal{C}}$ is true for N. Now, (2) of Proposition 3.5.2 completes the proof of (1).

(2) The proof is by induction on the number of edges. If there is no edge then there is one vertex and hence by hypothesis the induction starts. Since by hypothesis the FICwF$_{\mathcal{VC}}$ is true for $C \rtimes \langle t \rangle$ for $C \in \mathcal{D}$ it is true for C also by Lemma 3.5.1. Therefore, assume that the result is true for graphs with $\leqslant n-1$ edges, which satisfy the hypothesis. So let $\mathcal{G}$ be a finite graph of groups which satisfies the hypothesis and has n edges. Since $\mathcal{G}$ has the intersection property there is a normal subgroup N of $\pi_1(\mathcal{G})$ contained in all the edge groups and of finite index in some edge group, say $\mathcal{G}_e$ of the edge e. Let $\mathcal{G}_1$ be the graph of groups with $\mathcal{G}$ as the underlying graph and the vertex and the edge groups are $\mathcal{G}_x/N$ where x is a vertex and an edge respectively. Then $\pi_1(\mathcal{G}_1) \simeq \pi_1(\mathcal{G})/N$. Let $\mathcal{G}_2 = \mathcal{G}_1 - \{e\}$. It is now easy to check that the connected components of $\mathcal{G}_2$ satisfy the hypotheses and also have the intersection property. Also, since $\mathcal{G}_2$ has $n-1$ edges, by the induction hypothesis $\pi_1(\mathcal{H})$ satisfies the FICwF$_{\mathcal{VC}}$ where $\mathcal{H}$ is a connected component of $\mathcal{G}_2$. We now use Lemma 3.4.3 to conclude that $\pi_1(\mathcal{H})$ is also residually finite where $\mathcal{H}$ is as above. Therefore, by (1) $\pi_1(\mathcal{G}_1)$ satisfies the FICwF$_{\mathcal{VC}}$. Next, we apply (3) of Proposition 3.5.2 to the homomorphism $\pi_1(\mathcal{G}) \to \pi_1(\mathcal{G}_1)$. Note that using (2) of Proposition 3.5.2 it is enough to show the FICwF$_{\mathcal{VC}}$ is true for the inverse image of any infinite cyclic subgroup of $\pi_1(\mathcal{G}_1)$, but such a group is of the form $N \rtimes \langle t \rangle$. Now, since N is a subgroup of the edge groups of $\mathcal{G}$ and $\mathcal{D}$ is closed under taking subgroups we get $N \in \mathcal{D}$, as by hypothesis all the vertex groups of $\mathcal{G}$ belong to $\mathcal{D}$. Again by definition of $\mathcal{D}$ the FICwF$_{\mathcal{VC}}$ is true for $N \rtimes \langle t \rangle$. This completes the proof of (2).

(3) The proof of (3) follows from (2). For the polycyclic case we only need to note that virtually polycyclic groups are residually finite and quotients and subgroups of virtually polycyclic groups are virtually polycyclic. Also the mapping torus of a virtually polycyclic group is again virtually polycyclic.

For the nilpotent case, recall from [45] that the fundamental group of a finite graph of finitely generated nilpotent groups is subgroup separable if and only if the graph of groups satisfies the intersection property. Also note that finitely generated nilpotent groups are virtually polycyclic.

(4) Note that by hypothesis the FICwF$_{\mathcal{VC}}$ is true for closed surface groups and by [7] closed surface groups are residually finite. Therefore, by Lemma 3.4.5 and by (1) it is enough to consider a finite graph of groups whose vertex and edge groups are infinite closed surface groups and the graph of groups satisfies the hypothesis. Again by Lemma 3.4.5 we have the exact sequence: $1 \to H \to \pi_1(\mathcal{G}) \to \pi_1(\mathcal{G})/H \to 1$ where H is a closed surface group and $\pi_1(\mathcal{G})/H$ is isomorphic

to the fundamental group of a finite-type graph of groups and hence contains a finitely generated free subgroup of finite index by Lemma 3.3.2. Hence, by ${}_{wt}\mathcal{T}_{\mathcal{VC}}$ and by (2) of Proposition 3.5.2 the FICwF$_{\mathcal{VC}}$ is true for $\pi_1(\mathcal{G})/H$. Now, apply (3) of Proposition 3.5.2 to the homomorphism $\pi_1(\mathcal{G}) \to \pi_1(\mathcal{G})/H$. Using (2) of Proposition 3.5.2 it is enough to prove the FICwF$_{\mathcal{VC}}$ for $H \rtimes \langle t\rangle$. Since H is a closed surface group the action of $\langle t\rangle$ on H is induced by a diffeomorphism of a surface S so that $\pi_1(S) \simeq H$. Hence, $H \rtimes \langle t\rangle$ is isomorphic to the fundamental group of a closed 3-manifold which fibers over the circle. Therefore, using the hypothesis we complete the proof. □

3.7 Proofs of the theorems

Proof of Theorem 3.1.1. By Corollary 3.2.1 ${}_{wf}\mathcal{T}^P$ is satisfied and hence ${}_{wt}\mathcal{T}^P$ is also satisfied. Next, by Corollary 3.5.3 $\mathcal{P}^P$ is satisfied. Therefore, the proof of (1) is completed using (1) of Proposition 3.2.2.

For the proof of (2) apply (b) of Proposition 3.2.1 and Corollary 3.2.1. □

Remark 3.7.1. Using Proposition 3.5.4, the same proof works to prove the more general statement in case (2), when the kernel of f restricted to any vertex group is virtually polycyclic.

Proof of Theorem 3.1.2. Corollaries 3.2.1, 3.5.3 and (2) of Proposition 3.2.2 prove the theorem. □

Proof of Theorem 3.1.3. Corollaries 3.2.1, 3.5.3, Proposition 3.5.4 and (3) of Proposition 3.2.2 prove (1) and (2) of the theorem.

To prove (3) we need only to use (4) of Proposition 3.2.2, Corollary 3.2.1 and that the FICwF^P is true for the fundamental groups of 3-manifolds fibering over the circle from [52] and [53].

The proof of (4) follows from 2(i) of Proposition 3.2.3, Corollaries 3.2.1 and 3.5.3. Since the FICwF$_{\mathcal{VC}}$ is true for $\mathbb{Z}^n \rtimes \langle t\rangle$ for all n by Proposition 3.5.4.

For the proof of (5) first assume that the graph is finite and has more than one edge. Then get a homomorphism $f : \pi_1(\mathcal{G}) \to A := H_1(\pi_1(\mathcal{G}), \mathbb{Z})$ using Lemma 3.3.3 where the restriction of f to any vertex group is injective. Now, we apply Corollaries 3.2.1 and 3.5.3 and Proposition 3.5.2 to the above homomorphism. Therefore, one only needs that $f^{-1}(C)$ for any infinite cyclic subgroup C of A satisfies the FICwF^P. But the restricted action of $f^{-1}(C)$ on the tree (on which $\pi_1(\mathcal{G})$ acts) has infinite cyclic or trivial vertex stabilizers. Therefore, we conclude the proof using the Reduction Theorem in [52] and the hypothesis.

Next, if the graph has one edge then $\pi_1(\mathcal{G}) \simeq G_1 *_H G_2$ where G_1, G_2 and H are finitely generated abelian. Then H is a normal subgroup of $\pi_1(\mathcal{G})$ with quotient $G_1/H * G_2/H$. The proof now follows the similar steps as in the previous case using Proposition 3.5.2. □

Proof of Proposition 3.1.1. Since the conjecture in L-theory and lower K-theory commutes with direct limits (Remark 3.5.1) we can assume that the tree of virtually cyclic groups is finite. Also since the conjecture is true for virtually cyclic groups we can start proving the proposition by induction on the number of edges. We need the following lemma.

Lemma 3.7.1. *Let $G *_V G'$ be an amalgamated free product where G and G' act on finite-dimensional* CAT(0) *spaces properly, cocompactly and isometrically and V is a virtually cyclic group. Then $G *_V G'$ also acts on a finite-dimensional* CAT(0) *space properly, cocompactly and isometrically.*

Proof. The lemma follows easily from the proof of [11, Corollary 11.19, p.357]. □

The proof of the proposition now follows from [6] and noting that any virtually cyclic group acts on a finite-dimensional CAT(0) space properly, cocompactly and isometrically. □

3.8 Some deductions

We now show that a positive solution to the Problem in the beginning of this section will imply that the conjecture is true for solvable and for one-relator groups. We will have to show that both classes of groups can be obtained, in some suitable sense, from groups which act on trees so that the vertex stabilizers satisfy the conjecture.

Solvable groups. Recall that in [56] we had shown that the FICwF in both pseudo-isotopy and L-theory is true for a virtually solvable group if and only if the same conjecture is true for closely crystallographic groups. A closely crystallographic group is by definition of the form $A \rtimes C$ where A is a torsion free abelian group and C is an infinite cyclic group so that A is an irreducible $\mathbb{Q}[C]$-module. Now, note that the semidirect product structure gives an action of the closely crystallographic group on a tree with stabilizers isomorphic to A. Since the conjecture is true for A, a positive solution to the Problem will imply the conjecture for any virtually solvable group.

One-relator groups. Let G be a one-relator group. Then $G \simeq H * F$ where H is a finitely generated one-relator group and F is a free group. Therefore, by [56, Reduction Theorem], Corollary 3.2.1 and Proposition 3.5.3, we can assume that G is finitely generated. Let l be the length of the relator in the group G. Then by a result of Bieri ([10]) G acts on a tree with stabilizers subgroups of one-relator groups with the relator lengths $\leqslant l-1$. Since a one-relator group with relator length 1 is a free product of a free group and a finite cyclic group, by induction on l, using again [56, Reduction Theorem], Corollary 3.2.1 and Proposition 3.5.3 and using a positive solution to the problem, we deduce the conjecture for all one-relator groups.

3.9 Examples

Below we describe some examples of groups for which the results in this article are applicable. Furthermore, we show that these groups are new and that they are neither CAT(0) nor hyperbolic.

Example 3.9.1. (Almost) a tree of finitely generated abelian groups where the vertex and the edge groups of any component subgraph have the same rank: Fundamental group of such a graph of groups can get very complicated, for example in the simplest case of amalgamated free product of two infinite cyclic groups over an infinite cyclic group, that is $H = Z *_Z Z$ where Z is infinite cyclic, produces the (p, q)-torus knot group where the two inclusions $Z \to Z$ defining the amalgamation are power raised by p and q, $(p, q) = 1$. Though the FICwF^P is known for knot groups ([52]), most of the other groups in this class for which we prove the FICwF^P are new. We further recall that in general these examples of groups where the edge groups have rank $\geqslant 2$ are not CAT(0) (see the discussion after the proof of Proposition 6.8 on page 500 in [11]).

Example 3.9.2. Graphs of virtually polycyclic groups: We consider an amalgamated free product H of two nontrivial infinite virtually polycyclic groups over a finite group. Next, we recall that in [6] the fibered isomorphism conjecture in the pseudo-isotopy case was proved for the class of groups which act cocompactly and properly discontinuously on symmetric simply connected nonpositively curved Riemannian manifolds. In the proof of [9, Theorem A] it was noted that the condition "symmetric" can be replaced by "complete" if we consider torsion free groups. See also [36, Theorem A]. We choose polycyclic groups so that H does not belong to this class. One example of such a polycyclic group S is of the type $1 \to Z \to S \to Z^2 \to 1$ as described below, where Z is infinite cyclic.

Let G be the Lie group consisting of those 3×3 matrices with real entries whose diagonal entries are all equal to 1, entries below the diagonal are all equal to 0, and entries above the diagonal are arbitrary. Note that G is diffeomorphic to Euclidean 3-space. Let S be the subgroup of G whose entries above the diagonal are restricted to be integers. Then S is discrete and cocompact. Let E be the coset space of G by S. It clearly fibers over the 2-torus with fiber the circle. And the fundamental group of E is S, which is nilpotent but not abelian. On the other hand, in [70] it was shown that the fundamental group of a closed nonpositively curved manifold, which is nilpotent must be abelian. This shows that E cannot support a nonpositively curved Riemannian metric. Now, by [39, Corollary 2.6] it follows that S cannot even embed in a group (called *Hadamard group*), which acts discretely and cocompactly on a complete simply connected nonpositively curved space (that is a CAT(0)-space).

Now, consider $H_i = S \times F_i$ or $H_i = S \wr F_i$ where F_i is a finite group for $i = 1, 2$. Next, we take the amalgamated free product of H_1 and H_2, $H = H_1 *_F H_2$ along some finite group F. Then H does not embed in a Hadamard group as before by [39, Corollary 2.6] and H is not virtually polycyclic. But H satisfies the FICwF^P by (1) of Theorem 1.1 and Proposition 5.4.

Example 3.9.3. Graphs of residually finite groups with finite edge groups: Let S be the fundamental group of a compact Haken 3-manifold, which does not support any nonpositively curved Riemannian metric. Such 3-manifolds can easily be constructed by cutting along an incompressible torus in a compact Haken 3-manifold and then gluing differently. See [39] for this kind of construction. Next, let H_1 and H_2 be two residually finite groups for which the FICwF^P is true and such that S is embedded in H_1. It is easy to construct such H_1, for instance take $H_1 = S * G * F_1$ or $H_1 = (S \times G) \wr F_1$ or any such combination where G is a finitely generated free group and F_1 is a finite group. By the same argument as in Example 3.9.2 (and using [52] and [53]) it follows that $H = H_1 *_F H_2$ (along some finite group F) satisfies the FICwF^P but is neither virtually polycyclic nor embeds in a Hadamard group.

Let us now note that the group H considered in the above examples is also not hyperbolic as it contains a free abelian subgroup on more than one generator.

Finally, we remark that in a recent paper ([6]) the fibered isomorphism conjectures in L- and lower K-theory are proved for hyperbolic groups and for CAT(0)-groups, which act on finite-dimensional CAT(0)-spaces.

3.10 Open problems

We state some open problems related to this article on the (fibered) isomorphism conjecture.

A. Show that the (fibered) isomorphism conjecture is true for $A \rtimes \mathbb{Z}$ for a torsion-free abelian group A and for an arbitrary action of $\mathbb{Z}$ on A. Note that a positive answer to this problem will imply the conjecture for all solvable groups. See [56].

B. This is a more general situation compared to **A**. Show that the conjecture is true for $G \rtimes \mathbb{Z}$ assuming the conjecture for G. This is a very important open problem and will imply the conjecture for poly-free groups. It is open even when G is finitely generated and free. For certain situations the answers are known, for example, when G is a surface group and the action is realizable by a diffeomorphism of the surface. See [56].

C. Prove (1) of Theorem 3.1.1 without the assumption "residually finite". This will imply that one only needs to prove the fibered isomorphism conjecture for finitely presented groups with one end.

D. Prove the (fibered) isomorphism conjecture for the fundamental group of a graph of virtually cyclic groups. Even for the graph of infinite cyclic groups this is an open problem. If the underlying graph is a tree, then in Proposition 3.1.1 we proved it for the L-theory and lower K-theory case.

Remark 3.10.1. Finally, we remark that in this section, [33, Theorem 4.8] is used (see Proposition 3.5.4) in the proofs of (1), (2) (when the polycyclic or the nilpotent groups are not virtually cyclic), (3) and (5) (when the ranks of the abelian groups are $\geqslant 2$) of Theorem 3.1.3. See the proof of Corollary 3.5.3 and the discussion after the proof of Proposition 3.2.4. In this connection we note here that using the recent work in [6] all the results in this section can be deduced in the L-theory case of the fibered isomorphism conjecture. The same proofs will go through. But for this we need to use the L-theory version of [33, Theorem 4.8] in the proofs of the particular cases of the items of Theorem 3.1.3 as mentioned above. See [4] for the proof of [33, Theorem 4.8] in the L-theory case.

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