Homology, Homotopy and Applications, vol. 14(1), 2012, pp.1–17

THE ISOMORPHISM CONJECTURE IN L-THEORY: GRAPHS OF GROUPS

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(*communicated by Jonathan M. Rosenberg*)

Abstract

We study the fibered isomorphism conjecture of Farrell and Jones in L-theory for groups acting on trees. In several cases we prove the conjecture. This includes wreath products of abelian groups and free metabelian groups. We also deduce the conjecture in pseudoisotopy theory for these groups. Finally in 2. of Theorem 1.2, we prove the L-theory version of Theorem 1.2 in the 2003 paper by Farrell and Linnell.

1. Introduction and statements of results

The classification problem for high-dimensional manifolds needs the study of two classes of obstruction groups. One is the lower K-groups (that is, K-theory in dimension $\leqslant 1$) (pseudoisotopy-theory), and the other is the surgery L-groups (surgery theory) of the group ring of the fundamental group. The Farrell-Jones isomorphism conjecture gives a unified approach for computations and understanding of both these classes of groups. If this conjecture is true for the pseudoisotopy theory as well as for the surgery theory, then other results for example, the Borel conjecture, the Novikov conjecture and the Hsiang conjecture will be immediate consequences (see [**9**]). The Farrell-Jones conjecture predicts that one needs to consider only virtually cyclic subgroups of a group for computations of the above obstruction groups of the group.

In this second article we are concerned about the fibered isomorphism conjecture in surgery theory for groups acting (without inversion) on trees or equivalently for the fundamental groups of graphs of groups. Also, we prove the conjecture for a certain class of virtually solvable groups in both the pseudoisotopy and surgery theory. The fibered isomorphism conjecture in surgery theory for various classes of groups were proved in [**15**]. Also, some machinery was set up in [**15**] which are crucial in this paper.

The fibered isomorphism conjecture is stronger and has the hereditary property. Also, it allows one to consider groups with torsion in induction steps, although the final aim is to prove results for torsion free groups. This technique was first used in [**8**] to prove the conjecture in the pseudoisotopy case for Artin full braid groups. The general methods in [**14**] and [**15**] extend this feature further by considering the

Received June 3, 2010, revised October 1, 2010; published on April 25, 2012.
2000 Mathematics Subject Classification: Primary: 19G24, 19J25; Secondary: 55N91.
Key words and phrases: group action on trees, graph of groups, fibered isomorphism conjecture, L-theory, surgery group.
Article available at `http://intlpress.com/HHA/v14/n1/a1` and `doi:10.4310/HHA.2012.v14.n1.a1`

conjecture always for groups wreath products with finite groups. This simplifies proofs and proves stronger results.

In most of our results of the fibered isomorphism conjecture in the equivariant homology theory ([**2**]), we need the assumption that ${}_{wt}\mathcal{T}_{\mathcal{VC}}$, $\mathcal{P}_{\mathcal{VC}}$ and $\mathcal{L}_{\mathcal{VC}}$ (see Definition 2.2) are satisfied. We checked before that these conditions are satisfied for the $L^{\langle -\infty\rangle}$ and for the pseudoisotopy version of the conjecture. (See [**14**] and [**15**].) In [**4**, Theorem 0.1] it is included that $\mathcal{L}_{\mathcal{VC}}$ and $\mathcal{P}_{\mathcal{VC}}$ are satisfied for the K-theory case of the conjecture.

Formally, the conjecture in surgery theory says that a certain assembly map in $L^{\langle -\infty\rangle}$-theory is an isomorphism. A weaker version of the conjecture is that the assembly map is an isomorphism after tensoring with $\mathbb{Z}[\frac{1}{2}]$. This eliminates the UNiL groups of Cappell, and the tensored assembly map can be proven to be an isomorphism for a larger class of groups. In addition to some general results, we also prove the isomorphism of this tensored assembly map for a large class of groups acting on trees.

For two groups G and H, $G\wr H$ denotes the (restricted) wreath product with respect to the regular action of H on G^H. By definition, G^H denotes the group under coordinatewise multiplication of the tuples $\{\{g_h\}_{h\in H}\ |\ g_h\in G_h$ for $h\in H$ and $g_h=1$ for all but finitely many $h\in H\}$, where G_h's are the copies of G indexed by H. And the action of H on G^H is such that $h'\in H$ sends an element of G_h to the corresponding element of $G_{h(h')^{-1}}$.

If the fibered isomorphism conjecture is true for $G\wr F$ for all finite groups F for the $L^{\langle -\infty\rangle}$, $\underline{L}^{\langle -\infty\rangle}=L^{\langle -\infty\rangle}\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{2}]$ or for the pseudoisotopy theory, then we say respectively that the FICwF^L, FICwF$^{\underline{L}}$ or FICwF^P is true for G.

Throughout the article a 'graph' is assumed to be connected and locally finite, and groups are assumed to be discrete and countable.

Definition 1.1. A finitely generated group G is called *closely crystallographic* if it is of the form $A\rtimes C$, where A is torsion free abelian, C is infinite cyclic and A is irreducible as a $\mathbb{Q}[C]$-module.

When C is virtually cyclic then G was defined as *nearly crystallographic* in [**7**, Definition].

Our first theorem is the following:

Theorem 1.2.

1. *Let G be a group which contains a subgroup H of finite index so that H belongs to one of the following classes:*
   (i) $A\wr B$ where A and B are both abelian.
   (ii) Free metabelian groups. That is, it is the quotient of a free group by its second derived subgroup.
   (iii) $A\rtimes\mathbb{Z}$, where A is torsion abelian.

   Then the $FICwF^L$ and the $FICwF^P$ are satisfied for G.
2. *If the $FICwF^L$ ($FICwF^P$) is true for all closely crystallographic groups, then the $FICwF^L$ ($FICwF^P$) is true for all virtually solvable groups.*

Remark 1.3. It is not yet known if the fibered isomorphism conjecture is true for all metabelian groups. The simplest case for which it is unknown is $\mathbb{Z}[\frac{1}{2}]\rtimes\mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{Z}[\frac{1}{2}]$ is multiplication by 2. One can show that $\mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z}$ can not be embedded in $A \wr B$ where A and B are both abelian. I thank Chuck Miller for explaining this fact to me. On the other hand, by a result of Magnus, free metabelian groups can be embedded in such a wreath product.

Although our method does not work to deduce the fibered isomorphism conjecture in the closely crystallographic case, the isomorphism conjecture can be proved for these groups in surgery theory for all the decorations.

Theorem 1.4. *The isomorphism conjecture in L^i-theory is true for closely crystallographic groups where $i = \langle -\infty \rangle, h$ or s.*

Remark 1.5. Here we recall that in [**7**, Theorem 1.2] it was proved that the fibered isomorphism conjecture in the pseudoisotopy theory is true for any virtually solvable groups if the same is true for any nearly crystallographic groups. Thus B of Theorem 1.2 is the L-theory version of [**7**, Theorem 1.2].

The following is an Important Assertion in the fibered isomorphism conjecture. In general it is not yet known.

IA(K). *K is a normal subgroup of a group G with infinite cyclic quotient. If the $FICwF_{\mathcal{VC}}(K)$ is satisfied, then the $FICwF_{\mathcal{VC}}(G)$ is also satisfied.*

Theorem 1.6. *Let $\mathcal{G}$ be a graph of groups with finite edge groups.*

1. *If the vertex groups are residually finite and the $FICwF^L$ is true for the vertex groups of $\mathcal{G}$, then the $FICwF^L$ is true for $\pi_1(\mathcal{G})$.*
2. *Assume that there is a homomorphism $f\colon \pi_1(\mathcal{G}) \to Q$. Then the following statements hold:*
   - *(i) If the kernels of the restriction of f to the vertex groups of $\mathcal{G}$ are finitely generated, residually finite and satisfy the $FICwF^L$, then the $FICwF^L$ is true for $\pi_1(\mathcal{G})$ provided the same is true for Q and the IA(V) is satisfied for all vertex groups V of $\mathcal{G}$ in the L-theory case.*
   - *(ii) If the kernels of the restriction of f to the vertex groups of $\mathcal{G}$ are virtually polycyclic and the $FICwF^L$ is true for Q, then the $FICwF^L$ is true for $\pi_1(\mathcal{G})$.*

Let us now recall from [**14**] the following definitions: $V_{\mathcal{G}}$ and $E_{\mathcal{G}}$ denote respectively the set of all vertices and edges of a graph of groups $\mathcal{G}$. $\mathcal{G}_x$ denotes a vertex or an edge group for x a vertex or an edge respectively. An edge e of $\mathcal{G}$ is called a *finite edge* if the edge group $\mathcal{G}_e$ is finite. $\mathcal{G}$ is called *almost a tree of groups* if there are finite edges $e_1, e_2, \ldots$ so that the components of $\mathcal{G} - \{e_1, e_2, \ldots\}$ are trees. If we remove all the finite edges from a graph of groups, then we call the components of the resulting graph *component subgraphs.* A graph of groups $\mathcal{G}$ is said to satisfy the *intersection property* if for each connected subgraph of groups $\mathcal{G}'$ of $\mathcal{G}$, $\cap_{e\in E_{\mathcal{G}'}}\mathcal{G}'_e$ contains a subgroup which is normal in $\pi_1(\mathcal{G}')$ and is of finite index in some edge group. A group G is called *subgroup separable* if for any finitely generated subgroup H of G and for any $g \in G - H$ there is a finite index normal subgroup N of G so that $H \subset N$ and $g \in G - N$.

Theorem 1.7. *The $FICwF^L$ is true for $\pi_1(\mathcal{G})$ where $\mathcal{G}$ satisfies one of the following:*

1. *$\mathcal{G}$ is a graph of poly-cyclic groups with intersection property.*
2. *$\mathcal{G}$ is a graph of finitely generated nilpotent groups with $\pi_1(\mathcal{G})$ subgroup separable.*
3. *The vertex groups are virtually cyclic, and any component subgraph is either a single vertex or a tree of abelian groups.*
4. *The vertex and edge groups of any component subgraph are finitely generated abelian and of the same rank, and any component subgraph is a tree.*

Finally we state our results in the $\underline{L}^{\langle -\infty \rangle}$-theory case.

Let $\mathcal{D}$ be a class of groups which is closed under isomorphism. For a graph $\mathcal{G}$ we denote by $\mathcal{D}_{\mathcal{G}}$ the class of graphs of groups whose vertex and edge groups belong to $\mathcal{D}$ and the underlying graph is $\mathcal{G}$.

Theorem 1.8.

1. *If the $FICwF^{\underline{L}}(\pi_1(\mathcal{T}))$ is satisfied for all tree of groups $\mathcal{T}$, then the*
$$FICwF^{\underline{L}}(\pi_1(\mathcal{G}))$$
is satisfied for all graph of groups $\mathcal{G}$.
2. *If the $FICwF^{\underline{L}}(\pi_1(\mathcal{H}))$ is satisfied for all $\mathcal{H} \in \mathcal{D}_{\mathcal{T}}$ and for all tree $\mathcal{T}$, then the $FICwF^{\underline{L}}(\pi_1(\mathcal{H}))$ is satisfied for all $\mathcal{H} \in \mathcal{D}_{\mathcal{G}}$ and for all graphs $\mathcal{G}$.*

Theorem 1.9. *If $\mathcal{G}$ is a graph of finitely generated abelian groups, then the $FICwF^{\underline{L}}$ is true for $\pi_1(\mathcal{G})$.*

Theorem 1.10. *Let $\mathcal{G}$ be a graph of groups with finite edge groups.*

1. *If the vertex groups are residually finite and the $FICwF^{\underline{L}}$ is true for the vertex groups of $\mathcal{G}$, then the $FICwF^{\underline{L}}$ is true for $\pi_1(\mathcal{G})$.*
2. *Assume that there is a homomorphism $f\colon \pi_1(\mathcal{G}) \to Q$ and the $FICwF^{\underline{L}}$ is true for Q. If the kernels of the restriction of f to the vertex groups of $\mathcal{G}$ are residually finite and satisfy the $FICwF^{\underline{L}}$, then the $FICwF^{\underline{L}}$ is true for $\pi_1(\mathcal{G})$.*

Acknowledgements

I would like to thank F.T. Farrell for some helpful e-mail communications. Also, I am grateful to Wolfgang Lück for pointing out an error in a preprint which initiated some of the results in this paper.

2. Statement of the isomorphism conjecture and some basic results

Now we proceed to describe the formal statement of the conjecture (see [**2**]) and introduce some notations.

Let $\mathcal{H}^?_*$ be an equivariant homology theory with values in R-modules for R a commutative associative ring with unit. In this article we are considering the special case $R = \mathbb{Z}$.

In this section we always assume that a class of groups $\mathcal{C}$ is closed under isomorphisms, taking subgroups and taking quotients. We denote by $\mathcal{C}(G)$ the set of all subgroups of a group G which belong to $\mathcal{C}$. In this case $\mathcal{C}(G)$ is said to be a *family of subgroups* of G. It follows that $\mathcal{C}(G)$ is closed under taking subgroups and conjugation.

Given a group homomorphism $\phi\colon G \to H$ and $\mathcal{C}$ a family of subgroups of H define $\phi^*\mathcal{C}$ by the family of subgroups $\{K < G \mid \phi(K) \in \mathcal{C}\}$ of G. For a family $\mathcal{C}$ of subgroups of a group G there is a G-CW complex $E_{\mathcal{C}}(G)$, which is unique up to G-equivalence, satisfying the property that for each $H \in \mathcal{C}$ the fixpoint set $E_{\mathcal{C}}(G)^H$ is contractible and $E_{\mathcal{C}}(G)^H = \emptyset$ for H not in $\mathcal{C}$.

The *isomorphism conjecture* for the pair $(G, \mathcal{C})$ states that the projection $p\colon E_{\mathcal{C}}(G) \to pt$ to the point pt induces an isomorphism

$$\mathcal{H}^G_n(p)\colon \mathcal{H}^G_n(E_{\mathcal{C}}(G)) \simeq \mathcal{H}^G_n(pt)$$

for $n \in \mathbb{Z}$.

And the *fibered isomorphism conjecture* for the pair $(G, \mathcal{C})$ states that for any group homomorphism $\phi\colon K \to G$ the isomorphism conjecture is true for the pair $(K, \phi^*\mathcal{C})$.

Definition 2.1 ([**14**, Definition 2.1])**.** Let $\mathcal{C}$ be a class of groups. If the (fibered) isomorphism conjecture is true for the pair $(G, \mathcal{C}(G))$, then we say that the (F)IC$_{\mathcal{C}}$ is true for G or simply say (F)IC$_{\mathcal{C}}(G)$ *is satisfied*. We also say that the (F)ICwF$_{\mathcal{C}}(G)$ is satisfied if the (F)IC$_{\mathcal{C}}$ is true for $G \wr H$ for any finite group H.

Clearly, if $H \in \mathcal{C}$ then the (F)IC$_{\mathcal{C}}(H)$ is satisfied.

Let us denote by P, $\underline{L}$ and L, the equivariant homology theories that arise for the pseudoisotopy theory, $\underline{L}^{\langle -\infty \rangle}$-theory and for the $L^{\langle -\infty \rangle}$-theory respectively. We also denote the corresponding conjectures with respect to the class of groups $\mathcal{VC}$ by (F)IC^X ((F)ICwF^X) where $X = P$, $\underline{L}$ or L.

Definition 2.2 ([**14**, Definition 2.2])**.** We say that ${}_{wt}\mathcal{T}_{\mathcal{C}}$ (${}_{wf}\mathcal{T}_{\mathcal{C}}$) is satisfied if, for a graph of groups $\mathcal{G}$ with trivial (finite) edge groups and the vertex groups belonging to the class $\mathcal{C}$, the FICwF$_{\mathcal{C}}$ for $\pi_1(\mathcal{G})$ is true. And ${}_{w}\mathcal{T}_{\mathcal{C}}$ stands for the same property without any restriction on the edge groups.

Also, we say that $\mathcal{P}_{\mathcal{C}}$ is satisfied if for $G_1, G_2 \in \mathcal{C}$ the product $G_1 \times G_2$ satisfies the FIC$_{\mathcal{C}}$.

We further say that $\mathcal{L}_{\mathcal{C}}$ is satisfied if for any directed sequence of groups $\{G\}_{i\in I}$ for which the FIC$_{\mathcal{C}}(G_i)$ is satisfied for $i \in I$, then the FIC$_{\mathcal{C}}(\lim_{i\in I} G_i)$ is satisfied.

We denote the above properties for the equivariant homology theories P, $\underline{L}$ and L with a superscript by the corresponding theory. For example, $\mathcal{P}_{\mathcal{C}}$ for L is denoted by $\mathcal{P}^L_{\mathcal{C}}$.

We now recall some results we need to prove the theorems.

Lemma 2.3. *Assume that $\mathcal{L}_{\mathcal{C}}$ is satisfied. If for a directed sequence of groups $\{G\}_{i\in I}$ the FICwF$_{\mathcal{C}}(G_i)$ is satisfied for $i \in I$, then the FICwF$_{\mathcal{C}}$ is true for* $\lim_{i\in I} G_i$.

Proof. Given a finite group F note the following equality:

$$\left(\lim_{i\in I}(G_i)\right) \wr F = \lim_{i\in I}(G_i \wr F).$$

The proof now follows. □

The following is easy to prove and is known as the hereditary property of the fibered isomorphism conjecture:

Lemma 2.4. *If the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$) is true for a group G, then the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$) is true for any subgroup H of G.*

Lemma 2.5 ([**15**, Lemma 2.2])**.** *Assume that $\mathcal{P}_{\mathcal{C}}$ is satisfied.*

1. *If G_1 and G_2 satisfy the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$), then $G_1 \times G_2$ satisfies the $FIC_{\mathcal{C}}$ ($FICwF_{\mathcal{C}}$).*
2. *Let G be a finite index subgroup of a group K. If the group G satisfies the $FICwF_{\mathcal{C}}$, then K also satisfies the $FICwF_{\mathcal{C}}$.*
3. *Let $p\colon G \to Q$ be a group homomorphism. If the $FICwF_{\mathcal{C}}$ is true for Q and for $p^{-1}(H)$ for all $H \in \mathcal{C}(Q)$, then the $FICwF_{\mathcal{C}}$ is true for G. If $\mathcal{C} = \mathcal{VC}$, then using 2. it is enough to consider $H \in \mathcal{C}(Q)$ to be infinite cyclic.*

Lemma 2.6 ([**14**, Corollary 5.3] and [**15**, Lemma 2.11])**.** *The properties $\mathcal{P}^P_{\mathcal{VC}}$, $\mathcal{P}^L_{\mathcal{VC}}$, $\mathcal{P}^{\underline{L}}_{\mathcal{VC}}$ and $\mathcal{P}^{\underline{L}}_{\mathcal{FIN}}$ are satisfied.*

Lemma 2.7 ([**14**, Corollary 2.1] and [**15**, Lemma 2.14])**.** *The properties ${}_{wt}\mathcal{T}^P_{\mathcal{VC}}$, ${}_{w}\mathcal{T}^{\underline{L}}_{\mathcal{FIN}}$, ${}_{wt}\mathcal{T}^L_{\mathcal{VC}}$ and ${}_{wt}\mathcal{T}^{\underline{L}}_{\mathcal{VC}}$ are satisfied.*

The proofs of the properties $\mathcal{P}$ and $\mathcal{T}$ in the L-theory case were given in [**15**], [**6**, Theorem 2.1 and Remark 2.1.3]. See Remark 5.4 regarding the present status of the proof of [**6**, Theorem 2.1 and Remark 2.1.3]. Here we sketch alternate proofs of the above properties using some recent results of Bartels and Lück in [**3**]. In fact we can even prove ${}_{wf}\mathcal{T}^L_{\mathcal{VC}}$.

Alternate proofs of $\mathcal{P}^L_{\mathcal{VC}}$, ${}_{w}\mathcal{T}^{\underline{L}}_{\mathcal{FIN}}$ and ${}_{wt}\mathcal{T}^L_{\mathcal{VC}}$. The proofs of these facts in the pseudoisotopy case of the fibered isomorphism conjecture were given in [**14**]. The same proofs also apply in the L-theory case if we use [**3**]. We describe below the changes required.

For $\mathcal{P}^L_{\mathcal{VC}}$ replace P by L and use [**3**, Theorem B] in the proof of [**14**, Corollary 5.3]. Also see [**16**, Section 3] for more on this matter.

For a proof of ${}_{wt}\mathcal{T}^L_{\mathcal{VC}}$ use the last paragraph of the proof of [**14**, Proposition 2.4] after replacing P by L and use [**3**, Theorem B]. We also need to use some basic deductions from [**16**, Section 3]. The proof of the second property is immediate. $\square$

For the proof of ${}_{wf}\mathcal{T}^L_{\mathcal{VC}}$ we again use the proof of [**14**, Proposition 2.4].

The following lemma is another ingredient for the proofs of the theorems.

Lemma 2.8 ([**7**, Theorem 7.1])**.** *The properties $\mathcal{L}^P_{\mathcal{VC}}$, $\mathcal{L}^L_{\mathcal{VC}}$, $\mathcal{L}^{\underline{L}}_{\mathcal{VC}}$ and $\mathcal{L}^{\underline{L}}_{\mathcal{FIN}}$ are satisfied.*

Finally we recall the following two lemmas:

Lemma 2.9 ([**14**, Lemma 6.3])**.** *Assume that $\mathcal{P}_{\mathcal{C}}$ and ${}_{wt}\mathcal{T}_{\mathcal{C}}$ are satisfied. If the $FICwF_{\mathcal{C}}$ is true for G_1 and G_2, then the $FICwF_{\mathcal{C}}$ is true for $G_1 * G_2$.*

Lemma 2.10. *Let $\mathcal{FIN} \subset \mathcal{C}$ and assume that $\mathcal{P}_{\mathcal{C}}$, $\mathcal{L}_{\mathcal{C}}$ and ${}_{wt}\mathcal{T}_{\mathcal{C}}$ are satisfied. Then the $FICwF_{\mathcal{C}}$ is true for G where G is either a virtually abelian group or a virtually free group.*

Proof. By Lemma 2.3 we can assume that the group G is finitely generated. At first assume that G is virtually abelian. Using 2. of Lemma 2.5 we reduce to the case of finitely generated abelian groups. Since $\mathcal{FIN} \subset \mathcal{C}$ and since the conjecture is true for members of $\mathcal{C}$, it is enough to prove the lemma for finitely generated free abelian groups. Now 1. of Lemma 2.5 implies that we need to consider only the infinite cyclic group. Since the fundamental group of a graph of groups with trivial stabilizers is a free group, we are done using ${}_{wt}\mathcal{T}_{\mathcal{C}}$ and Lemma 2.4.

Next assume that G is finitely generated and virtually free. Again using 2. of Lemma 2.5 it is enough to assume that G is a finitely generated free group. Now note that a free group is isomorphic to the fundamental group of a graph of groups whose vertex groups are trivial. Hence using ${}_{wt}\mathcal{T}_{\mathcal{C}}$ we complete the proof. □

3. Proofs of the theorems

For the proof of Theorem 1.2 we prove the following theorem for the conjecture in equivariant homology theory. The advantage of this general statement is that it works for the conjecture in any equivariant homology theory, and to prove Theorem 1.2 we just have to show that the hypotheses are satisfied both for the pseudoisotopy case and for the $L^{\langle -\infty \rangle}$-theory case.

Theorem 3.1. *Assume that ${}_{wt}\mathcal{T}_{\mathcal{VC}}$, $\mathcal{P}_{\mathcal{VC}}$ and $\mathcal{L}_{\mathcal{VC}}$ are satisfied. Then the following hold:*

1. *The $FICwF_{\mathcal{VC}}$ is true for G if G contains $A \wr B$ as a subgroup of finite index, where A and B are abelian groups.*
2. *The $FICwF_{\mathcal{VC}}$ is true for any virtually free metabelian group.*
3. *The $FICwF_{\mathcal{VC}}$ is true for $A \rtimes \mathbb{Z}$ where A is torsion abelian.*
4. *The $FICwF_{\mathcal{VC}}$ is true for any virtually solvable group provided it is true for any closely crystallographic group.*

Remark 3.2. In the algebraic K-theory version of the conjecture, $\mathcal{P}_{\mathcal{VC}}$ and $\mathcal{L}_{\mathcal{VC}}$ are known. (See [**4**, Theorem 0.1].) But it is not yet known if ${}_{wt}\mathcal{T}_{\mathcal{VC}}$ is also satisfied. If this is the case, then, together with the results in this paper, most of the results from [**14**] and [**15**] will be true for the fibered isomorphism conjecture in algebraic K-theory.

Here we should recall that it was proved in [**7**, Lemma 4.3] that the pseudoisotopy version of the fibered isomorphism conjecture is true for $(\mathbb{Z}^n \wr \mathbb{Z}) \wr F$, where F is a finite group. That is, the FICwF^P is true for $\mathbb{Z}^n \wr \mathbb{Z}$.

We now begin the proof of Theorem 3.1.

Using 2. of Lemma 2.5 it is enough to prove the FICwF$_{\mathcal{VC}}$ for $A \wr B$, where B is infinite for free metabelian groups and for solvable groups under the respective hypotheses as in 1., 2. and 3. Also, we will use the fact that the FICwF$_{\mathcal{VC}}$ is true for virtually abelian groups during the proof. See Lemma 2.10.

Proof of 1. At first we reduce the situation to the case $A \wr \mathbb{Z}$. Let $B = \lim_{j \in J} B_j$, where $\{B_j\}$ is an increasing sequence of finitely generated subgroups of B. Then we get the following equality:

$$A \wr B = \lim_{j \in J}(A \wr B_j).$$

Therefore, from now on we can assume that B is finitely generated. If B has rank equal to k, then $B = \mathbb{Z}^k \times F_1$ where F_1 is finite. Hence $A \wr B$ contains $A^{\mathbb{Z}^k \times F_1} \rtimes \mathbb{Z}^k = A^{F_1} \wr \mathbb{Z}^k$ as a subgroup of finite index. Therefore we can use 2. of Lemma 2.5 to reduce the situation to the case $A \wr \mathbb{Z}^k$. If $k \geqslant 2$ then note the following equality: Let $\mathbb{Z}^k = B_1 \times B_2$ where B_1 and B_2 are both nontrivial. Then

$$\begin{aligned} A \wr (B_1 \times B_2) = A^{B_1 \times B_2} \rtimes (B_1 \times B_2) &< (A^{B_1 \times B_2} \rtimes B_1) \times (A^{B_1 \times B_2} \rtimes B_2) \\ &\simeq (A^{B_2} \wr B_1) \times (A^{B_1} \wr B_2). \end{aligned}$$

In the above display, for $i = 1, 2$, the action of B_i on $A^{B_1 \times B_2}$ is the restriction of the regular action of $B_1 \times B_2$ on $A^{B_1 \times B_2}$. Note that the restricted action of B_1 (B_2) is again regular on $(A^{B_2})^{B_1}$ ($(A^{B_1})^{B_2}$). And the second inequality is easily checked by showing that the map

$$A^{B_1 \times B_2} \rtimes (B_1 \times B_2) \to (A^{B_1 \times B_2} \rtimes B_1) \times (A^{B_1 \times B_2} \rtimes B_2)$$

defined by

$$(x, (b_1, b_2)) \mapsto ((x, b_1), (x, b_2))$$

for $x \in A^{B_1 \times B_2}$, and $(b_1, b_2) \in B_1 \times B_2$ is an injective homomorphism.

Therefore, using 1. of Lemma 2.5 and by the hereditary property, it is enough to prove the FICwF$_{\mathcal{VC}}$ for groups of the form $A \wr \mathbb{Z}$ where A is abelian.

Since A is countable abelian we can write it as a limit of finitely generated abelian subgroups A_i. Now note that $A \wr \mathbb{Z} = (\lim_{i \in I} A_i) \wr \mathbb{Z} = \lim_{i \in I}(A_i \wr \mathbb{Z})$. Hence by Lemma 2.3 it is enough to prove the FICwF$_{\mathcal{VC}}$ for $A_i \wr \mathbb{Z}$.

Therefore, from now on we can assume that A is finitely generated.

Next note the equality in the following lemma. This was obtained in the proof of [**7**, Lemma 4.3].

Lemma 3.3. *Let A be an abelian group. Then the following equality holds:*

$$A \wr \mathbb{Z} = \lim_{n \to \infty} (A^{n+1} *_{A^n}),$$

where the HNN *extension $A^{n+1} *_{A^n} = H_n$ (say) is obtained using the following inclusions:*

$$\begin{aligned} i_j \colon A^n &\to A^{n+1}, \\ i_1(a_1, \ldots, a_n) &\mapsto (a_1, \ldots, a_n, 0), \\ i_2(a_1, \ldots, a_n) &\mapsto (0, a_1, \ldots, a_n). \end{aligned}$$

Again by Lemma 2.3 we need to prove the FICwF$_{\mathcal{VC}}$ for H_n.

We have a surjective homomorphism $p \colon H_n \to A^{n+1} \rtimes_\alpha \mathbb{Z} = \underline{H}_n$ (say), where

$$\alpha(a_1, \ldots, a_{n+1}) = (a_{n+1}, a_1, \ldots, a_n).$$

Recall that A is a finitely generated abelian group. Let B be a finitely generated free abelian subgroup of A of finite index. Clearly, α leaves B^{n+1} invariant. Therefore, $B^{n+1} \rtimes_\alpha \mathbb{Z} = \underline{G}_n$ (say) is a finite index subgroup of $\underline{H}_n$. Hence $p^{-1}(\underline{G}_n)$ is a finite index subgroup of H_n. Obviously $p^{-1}(\underline{G}_n) = B^{n+1}*_{B^n} = G_n$ (say), where the HNN extension G_n is obtained by the same maps i_j as we defined above.

We now use 2. of Lemma 2.5 to reduce the situation to G_n. That is, we need to prove the FICwF$_{\mathcal{VC}}$ for G_n. We would like to apply 3. of Lemma 2.5 to $p\colon G_n \to \underline{G}_n$. From now on we follow the proof of Lemma 4.3 in [**7**, page 314].

Let C be a virtually cyclic subgroup of $\underline{G}_n$. Since $\underline{G}_n$ is torsion free, C is either trivial or infinite cyclic.

Since G_n is an HNN-extension it acts on a tree with vertex stabilizer conjugates of B^{n+1} and edge stabilizers conjugates of B^n. Therefore $\ker(p)$ also acts on this tree, and it follows that the stabilizers of this restricted action are trivial. Hence $\ker(p)$ is a free group by [**14**, Lemma 3.2].

When C is infinite cyclic then in the proof of [**7**, Lemma 4.3] (see paragraphs 2 and 3 in [**7**, page 316]), it was deduced that $p^{-1}(C)$ is a direct limit of finitely generated subgroups C_i (say) so that each C_i is a subgroup of a finite free product $K * \cdots * K$, where K is isomorphic to a direct product of a finitely generated free group and an infinite cyclic group.

Now since ${}_{wt}\mathcal{T}_{\mathcal{VC}}$ is satisfied the FICwF$_{\mathcal{VC}}$ is true for free groups. See Lemma 2.10. Therefore the FICwF$_{\mathcal{VC}}$ is true for $\ker(p)$. Also, by Lemma 2.9, the FICwF$_{\mathcal{VC}}$ is true for the free product of two groups if the FICwF$_{\mathcal{VC}}$ is true for each free summand and ${}_{wt}\mathcal{T}_{\mathcal{VC}}$ and $\mathcal{P}_{\mathcal{VC}}$ are satisfied. Therefore, in addition, using 1. of Lemma 2.5 we deduce that the FICwF$_{\mathcal{VC}}$ is true for $K * \cdots * K$ and hence for C_i also by Lemma 2.4. Finally, by Lemma 2.3, we conclude that the FICwF$_{\mathcal{VC}}$ is true for $p^{-1}(C)$.

Therefore G_n satisfies the FICwF$_{\mathcal{VC}}$ for each n. This completes the proof of 1. □

Proof of 2. Let G be a free metabelian group. Then the Magnus Embedding Theorem ([**11**]) says that G can be embedded as a subgroup of a group of the form $A \wr B$ where A and B are abelian. The proof of 2. now follows from 1. using Lemma 2.4. □

Proof of 3. The proof follows the steps of the proof of [**7**, Corollary 4.2].

Using Lemma 2.3 we assume that $G = A \wr \mathbb{Z}$ is finitely generated. This makes A a finitely generated $\mathbb{Z}[\mathbb{Z}]$-module via the conjugation action of G on A. Hence A has finite exponent.

Let us first assume that we have proved the result when this exponent is a prime. To complete the proof we now use induction on the exponent, say τ. If $\tau = 1$ then there is nothing to prove. So assume $\tau = pq \geqslant 2$ and p is a prime. Note that pA is a normal subgroup of G, and hence we have the following two exact sequences:

$$1 \to pA \to G \to G_1 \to 1,$$

$$1 \to A/pA \to G_1 \to \mathbb{Z} \to 1.$$

Note that the exponent of A/pA is p, and hence the FICwF$_{\mathcal{VC}}$ is true for G_1 by assumption. Next, the exponent of pA is $q < \tau$, and hence by the induction hypothesis and applying 3. of Lemma 2.5 to the homomorphism $G \to G_1$ we are done.

Let us now assume that the exponent of A is a prime p and complete the proof. This makes A a finitely generated $\mathbb{Z}_p[\mathbb{Z}]$-module. Since $\mathbb{Z}_p[\mathbb{Z}]$ is a PID, A has a decomposition in free part and torsion part as a $\mathbb{Z}_p[\mathbb{Z}]$-module. Let A_0 be the free part. Then A_0 is a normal subgroup of G. Let C be an infinite cyclic subgroup of G which goes onto $\mathbb{Z}$ under the map $G \to \mathbb{Z}$. Then A_0C is a finite index subgroup of

G, and also $A_0C \simeq \mathbb{Z}_p^n \wr \mathbb{Z}$, where n is the rank of A_0 as a free $\mathbb{Z}_p[\mathbb{Z}]$-module. Hence using 2. of Lemma 2.5 we are done once we show that the FICwF$_{\mathcal{VC}}$ is true for $\mathbb{Z}_p^n \wr \mathbb{Z}$.

Let $B = \mathbb{Z}_p^n$. Then by Lemma 3.3 we have the following equality:

$$B \wr \mathbb{Z} \simeq \lim_{k\to\infty} B^{k+1} *_{B^k} .$$

Next note that $B^{k+1}*_{B^k}$ is finitely generated and isomorphic to the fundamental group of a graph of finite groups and hence contains a free subgroup of finite index (see [**14**, Lemma 3.2]). Finally, using Theorem 1.9, we complete the proof. □

Proof of 4. The proof uses the method of the proof of [**7**, Corollary 4.4].

For a solvable group G we say that it is *n-step* solvable if $G^{(n)} = (1)$ and $G^{(n-1)} \neq (1)$, where $G^{(i)}$ denotes the i-th derived subgroup of G.

Let G be an n-step solvable group. The proof of 4. is by induction on n. So assume that if the FICwF$_{\mathcal{VC}}$ is true for all closely crystallographic groups, then it is true for all k-step solvable groups for $k \leqslant n-1$.

We have an exact sequence $1 \to G^{(2)} \to G \to G/G^{(2)} \to 1$. By 3. of Lemma 2.5 and by the induction hypothesis it is enough to prove the FICwF$_{\mathcal{VC}}$ for 2-step solvable groups, since for any infinite cyclic subgroup of $G/G^{(2)}$ the inverse image under the quotient map $G \to G/G^{(2)}$ is an $(n-1)$-step solvable group.

Therefore we have reduced the proof to the following situation:

$$1 \to G^{(1)} \to G \to G/G^{(1)} \to 1.$$

Here $G^{(1)}$ and $G/G^{(1)}$ are both abelian.

By Lemma 2.10 we can assume that $G/G^{(1)}$ is infinite. Again applying 3. of Lemma 2.5 to the map $G \to G/G^{(1)}$, we see that it is enough to prove the FICwF$_{\mathcal{VC}}$ for the group $G = A \rtimes \mathbb{Z}$ where A is an abelian group.

Let A_T be the subgroup of A consisting of all elements of finite order. Then A_T is a characteristic subgroup of A, and hence we have an exact sequence

$$1 \to A_T \to G \to G/A_T \to 1.$$

Note that $G/A_T \simeq (A/A_T) \rtimes \mathbb{Z}$.

Therefore, by 2. and 3. of Lemma 2.5, it is enough to prove the FICwF$_{\mathcal{VC}}$ for G for the following two individual cases:

Case (a). *A is torsion abelian.*

Case (b). *A is torsion free abelian.*

Proof of Case (a). This case is the same as 2. of Theorem 3.1. □

Proof of Case (b). Note that by Lemma 2.3 we may assume that A is finitely generated as a $\mathbb{Q}[\mathbb{Z}]$-module (see the proof of [**7**, Corollary 4.4]). As $\mathbb{Q}[\mathbb{Z}]$ is a principal ideal domain, $A \simeq X \oplus Y$, where X is the sum of free and Y is the sum of finite $\mathbb{Q}$-dimensional $\mathbb{Q}[\mathbb{Z}]$-submodules of A. Let $m = \dim Y$ and n be the number of free parts in X. Note that Y is a normal subgroup of G. The proof is now by induction first on n and then on m. If $m = n = 0$ then G is infinite cyclic so there is nothing to prove. So assume that $n = 0$ and $m > 0$. Let Y_0 be an irreducible $\mathbb{Q}[\mathbb{Z}]$-submodule of Y. Then we have an exact sequence

$$1 \to Y_0 \to G \to G/Y_0 \to 1.$$

By induction and by 2. and 3. of Lemma 2.5 it is enough to prove the FICwF$_{\mathcal{VC}}$ for $Y_0 \rtimes \mathbb{Z}$, which is true by hypothesis since $Y_0 \rtimes \mathbb{Z}$ is a closely crystallographic group.

Next assume that $n > 0$. Then it follows that G/Y is isomorphic to $\mathbb{Q}^n \wr \mathbb{Z}$ for which 1. shows that the FICwF$_{\mathcal{VC}}$ is true. Now again we apply 2. and 3. of Lemma 2.5 to the homomorphism $G \to G/Y$, and hence we need only to show the FICwF$_{\mathcal{VC}}$ when G/Y is infinite cyclic. But this is again the case $n = 0$ treated above. □

This completes the proof of 4. of Theorem 3.1. □

Proof of Theorem 1.2. The proof is immediate from Theorem 3.1, Lemmas 2.6, 2.7 and 2.8. □

Proof of Theorem 1.4. Let G be a closely crystallographic group. Recall that then G is nearly crystallographic. Since nearly crystallographic groups are linear (see the paragraph after [**7**, Definition]), the following hold by [**7**, Theorem 1.1] and the discussion following it:

$$\mathrm{Wh}(G) = \tilde{K}_0(\mathbb{Z}[G]) = K_i(\mathbb{Z}[G]) = 0$$

for all negative integers i.

Let $G = A \rtimes \mathbb{Z}$ where A is torsion free abelian. Since A is a direct limit of its finitely generated subgroups and since the functors in the above display commute with direct limit, the display also holds if we replace G by A.

As the Whitehead groups of the groups (G and A) we are considering vanish, the surgery L-groups of these groups with different decorations coincide. Therefore, we denote the surgery groups by the simple notation $L_n(-)$.

Let us first show that the non-connective assembly map in L-theory is an isomorphism for G. That is,

$$H_n(K(G, 1), \underline{\mathbb{L}}_0) \to L_n(\mathbb{Z}[G])$$

is an isomorphism.

Since the isomorphism of the above assembly map is invariant under taking direct limit of groups and since the map is an isomorphism for finitely generated free abelian groups ([**5**]), it follows that $H_n(K(A, 1), \underline{\mathbb{L}}_0) \to L_n(\mathbb{Z}[A])$ is also an isomorphism.

Let us now recall the following Ranicki's Mayer-Vietoris type exact sequence of surgery groups ([**12**]) for G:

$$\cdots \to L_{n+1}(G) \to L_n(A) \to L_n(A) \to L_n(G) \to \cdots .$$

There is a similar exact sequence for the homology theory $H_n(-, \underline{\mathbb{L}}_0)$. Now, since the assembly map is natural, a five lemma argument implies that $H_n(K(G, 1), \underline{\mathbb{L}}_0) \to L_n(\mathbb{Z}[G])$ is an isomorphism.

Next, the above K-theoretic vanishing result and an application of the Rothenberg's exact sequence (see the proof of Corollary 5.3) imply that the IC$^{L^i}(G)$ is satisfied for $i = \langle -\infty \rangle, h$ or s. □

We now begin the proof of Theorem 1.6.

Proof of 1. 1. is an immediate consequence of [**14**, (1) of Proposition 2.2] and Lemmas 2.6, 2.7 and 2.8. Recall that in [**14**, (1) of Proposition 2.2] we assumed that the equivariant homology theory should be continuous when the graph of groups is infinite. But there this continuity assumption was used to get Lemma 2.8 for the corresponding homology theory. Since we have noted that for L-theory Lemma 2.8 is true, we do not need this assumption here. □

Proof of 2.(i). 2.(i) follows using Lemmas 2.6, 2.7 and 2.8 and the following Proposition 3.4. □

Proof of 2.(ii). At first recall that virtually polycyclic groups are residually finite, and the FICwF^L is true for virtually polycyclic groups by [**15**, Theorem 1.1 and (iv) of Theorem 1.3]. Also, note that by the same result, $\mathrm{IA}(K)$ is true in the L-theory case for any virtually polycyclic group K. This completes the proof using 2.(ii). □

Proposition 3.4. *Assume the same hypotheses as in 2.(i) of Theorem 1.6 replacing L by an arbitrary equivariant homology theory and, in addition, assume that ${}_{wt}\mathcal{T}_{\mathcal{VC}}$, $\mathcal{P}_{\mathcal{VC}}$ and $\mathcal{L}_{\mathcal{VC}}$ are satisfied. Then the $FICwF_{\mathcal{VC}}$ is true for $\pi_1(\mathcal{G})$.*

Proof. We need to apply 3. of Lemma 2.5 to the homomorphism $f\colon \pi_1(\mathcal{G}) \to Q$. Let C be an infinite cyclic subgroup of Q. Then $f^{-1}(C)$ is isomorphic to the fundamental group of a graph of groups whose edge groups are finite, and vertex groups are subgroups of groups of the form $K \rtimes \mathbb{Z}$, where K is finitely generated and residually finite and by $\mathrm{IA}(K)$ $K \rtimes \mathbb{Z}$ satisfies the $\mathrm{FICwF}_{\mathcal{VC}}$. The proof will be completed by [**14**, (1) of Proposition 2.2] once we show that $K \rtimes \mathbb{Z}$ is residually finite. We apply [**14**, Lemma 4.2]. That is, we have to show that given any finite index subgroup K' of K there is a (finite index) subgroup K'' of K' which is normal in $K \rtimes \mathbb{Z}$, and the quotient $(K \rtimes \mathbb{Z})/K''$ is residually finite. Since K is finitely generated we can find a finite index characteristic subgroup K'' (and hence normal in $K \rtimes \mathbb{Z}$) of K contained in K'. Then $(K \rtimes \mathbb{Z})/K''$ is residually finite since it is virtually cyclic by [**14**, Lemma 6.1].

This completes the proof of the proposition. □

Proof of Theorem 1.7. The proofs of 1. and 2. follow using the following:

(i) Finitely generated nilpotent groups are virtually polycyclic.
(ii) Lemmas 2.6, 2.7 and 2.8.
(iii) [**14**, (3) of Proposition 2.2], which says that the statements 1. and 2. are true for general equivariant homology theories $\mathcal{H}^?_*$ if $\mathcal{H}^?_*$ is continuous and $\mathcal{P}_{\mathcal{VC}}$ and ${}_{wt}\mathcal{T}_{\mathcal{VC}}$ are satisfied. In the proof of [**14**, (3) of Proposition 2.2] we needed the fact that $\mathcal{L}_{\mathcal{VC}}$ is satisfied which is implied by the hypothesis that $\mathcal{H}^?_*$ is continuous (see [**14**, Proposition 5.1]). This completes the argument using the previous item.

The proof of 3. follows from the following: At first assume that the graph of groups is finite which we can by Lemma 2.8.

(i) $\pi_1(\mathcal{G}) \simeq \pi_1(\mathcal{H})$, where $\mathcal{H}$ is a graph of groups whose edge groups are finite and each vertex group is either virtually cyclic or fundamental group of a tree of infinite virtually cyclic abelian groups. See [**14**, Lemma 3.1].

(ii) The vertex groups of $\mathcal{H}$ are residually finite. See [**14**, Lemma 4.4].

(iii) The vertex groups satisfy FICwF^L. Use 1. and [**14**, Lemma 3.5] which implies that the graph of groups $\mathcal{G}$ has the intersection property.

For the proof of 4. we need the following:

(i) Lemmas 2.6, 2.7 and 2.8.

(ii) ([**14**, 2(i) of Proposition 2.3]). Here note that for the proof of [**14**, 2(i) of Proposition 2.3] we needed that the FICwF$_{\mathcal{VC}}$ is true for $\mathbb{Z}^n \rtimes \mathbb{Z}$ for all n, which is the case for the FICwF^L [**15**, Theorem 1.1 and (iv) of Theorem 1.3].

This completes the proof of Theorem 1.7. □

Proof of Theorem 1.8. Let $\mathcal{G}$ be a graph of groups. If $\mathcal{G}$ is a tree then there is nothing to prove. So assume that it is not a tree. Then there is a surjective homomorphism $f\colon \pi_1(\mathcal{G}) \to F$ where F is a countable free group, and the kernel of f is a tree of groups (the universal covering graph of groups of $\mathcal{G}$). Now using the hypothesis, Lemma 2.10 and 2. and 3. of Lemma 2.5 we complete the proof of 1.

For the proof of 2. we just need to note that the universal covering graph of groups of $\mathcal{G}$ is a tree of groups whose class of vertex and edge groups is same as that of $\mathcal{G}$. □

Proof of Theorem 1.9. By 2. of Theorem 1.8 we can assume that the graph of groups is a tree of finitely generated abelian groups. Next, by [**14**, Lemma 3.3] there is a surjective homomorphism $p\colon \pi_1(\mathcal{G}) \to H_1(\pi_1(\mathcal{G}), \mathbb{Z})$ so that the restriction of p to any vertex group has trivial kernel. This implies that the kernel of p acts on a tree with trivial stabilizers, and hence it is a free group. Now using 2. and 3. of Lemma 2.5 and Lemma 2.10 we complete the proof. □

Proof of Theorem 1.10. The proof of 1. follows from Lemmas 2.6, 2.7, 2.8 and [**14**, (1) of Proposition 2.2]. The proof of 2. is routine using 1. of Theorem 1.10 and 2. and 3. of Lemma 2.5. The only fact we need to mention is that a virtually residually finite group is residually finite. □

4. Some special cases

In this section we deduce some results for the following simple cases of graphs of groups. This is contrary to the situation of ascending HNN extension for which the fibered isomorphism conjecture is still not proved. The simplest case is the groups $\mathbb{Z}*_{\mathbb{Z}}$ where the two inclusions $\mathbb{Z} \to \mathbb{Z}$ are identity and multiplication by 2. Note here that $\mathbb{Z}*_{\mathbb{Z}} = \mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z}$. See Remark 1.3.

Proposition 4.1. *Let G and A be two groups. Let $i_j\colon A \to G$ be two injective homomorphisms for $j = 1, 2$. Assume that there exists an automorphism $\alpha\colon G \to G$ with the property that $\alpha(i_1(a)) = i_2(a)$ for all $a \in A$. Then the $FICwF^{\underline{L}}$ is satisfied for the* HNN*-extension $G*_A$ (defined by the two homomorphisms i_1 and i_2) provided G also satisfies the $FICwF^{\underline{L}}$.*

Proposition 4.2. *Let G_1 and G_2 be two groups. Let A be a group with two injective homomorphisms $i_j\colon A \to G_j$ for $j = 1, 2$. Assume that there is an isomorphism $\alpha\colon G_1 \to G_2$ with the property that $\alpha(i_1(a)) = i_2(a)$ for each $a \in A$. Then the*

$FICwF^{\underline{L}}$ is satisfied for the generalized free product $G_1 *_A G_2$ (defined by the two homomorphisms i_1 and i_2) provided G_1 (or G_2) also satisfies the $FICwF^{\underline{L}}$.

The following is an immediate corollary of Proposition 4.2:

Corollary 4.3. *Let M and P be two compact manifolds with nonempty connected π_1-injective boundaries and let $f\colon M \to P$ be a homotopy equivalence so that $f|_{\partial M}\colon \partial M \to \partial P$ is a homeomorphism. Then the $FICwF^{\underline{L}}$ is true for $\pi_1(M \cup_\partial P)$ if the $FICwF^{\underline{L}}$ is true for $\pi_1(M)$. Here $M \cup_\partial P$ is the union of M and P glued along the boundary via the map f.*

Proof of Proposition 4.1. At first note that there is an obvious surjective homomorphism $f\colon G*_A \to G \rtimes_\alpha \langle t\rangle$. Using 2. of Lemma 2.5 it follows that the FICwF$^{\underline{L}}$ is true for $G \rtimes \langle t\rangle$ for any action of $\langle t\rangle$ over G. Now note that the group $G*_A$ acts on a tree with vertex groups conjugates of G and edge groups conjugates of A, and also that the restrictions of f to the vertex groups are injective. Therefore 2. of Theorem 1.10 completes the proof. □

Proof of Proposition 4.2. Let us consider the free product $G = G_1 * G_2$. Then there are two inclusions j_1 and j_2 from A to G defined by i_1 and i_2, and there is an isomorphism $\tilde{\alpha}\colon G \to G$ defined by α so that $\tilde{\alpha}(j_1(a)) = j_2(a)$. Next note that there is an embedding $G_1 *_A G_2 \to G*_A$, where $G_1 *_A G_2$ is defined with respect to i_1 and i_2 and $G*_A$ is defined with respect to j_1 and j_2. Hence by Lemma 2.4 it is enough to prove the FICwF$^{\underline{L}}$ for $G*_A$. Since by Lemmas 2.6, 2.7 and 2.9 the FICwF$^{\underline{L}}$ is true for G, we are done using Proposition 4.1. □

Remark 4.4. Propositions 4.1 and 4.2 can be proven for arbitrary homology theories and with respect to the class $\mathcal{FIN}$ of finite groups if we add the extra assumptions that ${}_{wt}\mathcal{T}_{\mathcal{FIN}}$ and $\mathcal{L}_{\mathcal{FIN}}$ are satisfied. We have already mentioned in the introduction that ${}_{wt}\mathcal{T}_{\mathcal{FIN}}$ in the K-theory case is still not known.

5. Some consequences

The following are some of the well-known consequences of the isomorphism conjecture:

Corollary 5.1. *If Γ is a torsion free group for which the fibered isomorphism conjecture in pseudoisotopy theory is true, then the following holds:*

The Whitehead group $\mathrm{Wh}(\Gamma)$, the lower K-groups $K_{-i}(\mathbb{Z}\Gamma)$ for $i \geqslant 1$ and the reduced projective class group $\tilde{K}_0(\mathbb{Z}\Gamma)$ vanish.

Corollary 5.2. *In addition to the hypothesis of the previous corollary, if the isomorphism conjecture in $L^{\langle -\infty\rangle}$-theory is also true for the group Γ, then the following holds:*

The following assembly map is an isomorphism for all n and for $j = \langle -\infty\rangle, h$ and s:

$$H_n(B\Gamma; \mathbf{L}^j(\mathbb{Z})) \to L_n^j(\mathbb{Z}\Gamma).$$

Note that the above two corollaries give further evidence to the Whitehead conjecture and the integral Novikov conjecture respectively. The Whitehead conjecture says that the Whitehead group of any torsion free group vanishes, and the integral Novikov conjecture says that the above assembly map is split injective for torsion free groups.

Corollaries 5.1 and 5.2 together imply the following:

Corollary 5.3 (Generalized Borel conjecture)**.** *Let M be a closed aspherical manifold with $\pi_1(M)$ isomorphic to G, where G satisfies the fibered isomorphism conjecture for the pseudoisotopy and the L-theory cases. Then $M \times \mathbb{D}^k$ satisfies the Borel conjecture for $\dim(M) + k \geqslant 5$. That is, if $f\colon N \to M \times \mathbb{D}^k$ is a homotopy equivalence from another compact manifold so that $f|_{\partial N}\colon \partial N \to M \times \mathbb{S}^{k-1}$ is a homeomorphism, then f is homotopic, relative to boundary, to a homeomorphism.*

Finally we recall that, in our earlier works together with the present article, we have proved the fibered isomorphism conjecture both for the pseudoisotopy theory and for the $L^{-\langle\infty\rangle}$-theory for a large class of groups.

Below we sketch the proofs of the above corollaries.

The arguments for the proofs of Corollaries 5.1 and 5.2 and 5.3 are already in the literature. We briefly recall the proofs and then refer to the original sources.

Proof of Corollary 5.1. See [**6**, 1.6.5]. Also see [**8**, Theorem D]. □

Proof of Corollary 5.2. The isomorphism conjecture in $L^{\langle-\infty\rangle}$-theory for torsion free groups implies the isomorphism of the assembly map

$$H_n(B\Gamma; \mathbf{L}^{\langle j\rangle}(\mathbb{Z})) \to L_n^{\langle j\rangle}(\mathbb{Z}\Gamma)$$

for $j = -\infty$. See [**6**, 1.6.1] for details.

Now recall the following Rothenberg exact sequence:

$$\cdots \to L_n^{\langle i+1\rangle}(R) \to L_n^{\langle i\rangle}(R) \to \hat{H}^n(\mathbb{Z}/2; \tilde{K}_i(R))$$

$$\to L_{n-1}^{\langle i+1\rangle}(R) \to L_{n-1}^{\langle i\rangle}(R) \to \cdots,$$

where $R = \mathbb{Z}\Gamma$ and $i \leqslant 1$. Recall that $L_n^{\langle 1\rangle} = L_n^h$ and $L_n^{\langle-\infty\rangle}$ is the limit of $L_n^{\langle i\rangle}$. Now using Corollary 5.1 and by a five lemma argument we get the isomorphism of the assembly map

$$H_n(B\Gamma; \mathbf{L}^h(\mathbb{Z})) \to L_n^h(\mathbb{Z}\Gamma).$$

Using a similar Rothenberg exact sequence which connects the surgery groups with h and s decorations and the Tate cohomology which appears is with coefficient in the Whitehead group, one can show the following isomorphism:

$$H_n(B\Gamma; \mathbf{L}^s(\mathbb{Z})) \to L_n^s(\mathbb{Z}\Gamma).$$

See [**10**, Section 1.5] for details and for other related features. □

Proof of Corollary 5.3. Let us first recall the surgery exact sequence. This sequence is for simple homotopy types and for the surgery groups with the decoration 's'. Since the Whitehead group of the group G in the present situation vanishes, there is no difference between 's' and 'h', and therefore we do not use any decoration.

$$\cdots \to H_n(X;\underline{\mathbb{L}_0}) \to L_n(\pi_1(X)) \to \mathcal{S}_n(X) \to H_{n-1}(X;\underline{\mathbb{L}_0}) \to \cdots,$$

where $\mathcal{S}_*(-)$ is the total surgery obstruction group of Ranicki and $\underline{\mathbb{L}_0}$ is a 1-connective Ω-spectrum with 0-space homotopically equivalent to G/TOP. If X is a compact n-dimensional manifold ($n \geqslant 5$), then the following part of the above surgery exact sequence

$$\cdots \to \mathcal{S}_{n+2}(X) \to H_{n+1}(X;\underline{\mathbb{L}_0}) \to L_{n+1}(\pi_1(X))$$

$$\to \mathcal{S}_{n+1}(X) \to H_n(X;\underline{\mathbb{L}_0}) \to L_n(\pi_1(X))$$

is identified with the original surgery exact sequence

$$\cdots \to \mathcal{S}^{\mathrm{Top}}(X \times \mathbb{D}^1, \partial(X \times \mathbb{D}^1)) \to [X \times \mathbb{D}^1, \partial(X \times \mathbb{D}^1); G/\mathrm{TOP}, *]$$

$$\to L_{n+1}(\pi_1(X)) \to \mathcal{S}^{\mathrm{Top}}(X) \to [X; G/\mathrm{TOP}] \to L_n(\pi_1(X)).$$

In particular, for an n-dimensional closed manifold X there is the following identification:

$$\mathcal{S}_{n+k+1}(X) = \mathcal{S}^{\mathrm{Top}}(X \times \mathbb{D}^k, \partial(X \times \mathbb{D}^k)).$$

Here $\mathcal{S}^{\mathrm{Top}}(P, \partial P)$ denotes the structure set of a compact manifold P. When the Whitehead group $\mathrm{Wh}(\pi_1(P)) = (1)$ and $\dim(P) \geqslant 5$ (which is the case in the present situation), the structure set can be defined in the following simpler way: $\mathcal{S}^{\mathrm{Top}}(P, \partial P)$ is the set of all equivalence classes of homotopy equivalences $f\colon (N, \partial N) \to (P, \partial P)$ from compact manifolds $(N, \partial N)$ so that $f|_{\partial N}\colon \partial N \to \partial P$ is a homeomorphism. Here two such maps $f_i\colon (N_i, \partial N_i) \to (P, \partial P)$ for $i = 1, 2$ are said to be equivalent if there is a homeomorphism $h\colon (N_1, \partial N_1) \to (N_2, \partial N_2)$ so that $f_2 \circ h$ is homotopic to f_1 relative to boundary, that is, during the homotopy the map on the boundary is constant.

Next, there is a homomorphism $H_k(X;\underline{\mathbb{L}_0}) \to H_k(X;\mathbf{L}(\mathbb{Z}))$ which is an isomorphism for $k > n$ and is injective for $k = n$.

Now using the fact that M is aspherical and applying Corollaries 5.1 and 5.2, we see that $\mathcal{S}^{\mathrm{Top}}(M \times \mathbb{D}^k)$ contains only one element for $n + k \geqslant 5$. This completes the proof of Corollary 5.3.

For some more details with related references, see [**10**, Theorem 1.28] or [**6**, 1.6.3]. □

Remark 5.4. In view of the footnote in [**14**, Introduction], we finally remark that [**6**, Remark 2.1.3] is used in this paper in the following statements: 2.(ii) of Theorem 1.6, 1. and 2. of Theorem 1.7 and 4. of Theorem 1.7 when the vertex groups of any component subgraph has rank > 1. The complete proof of [**6**, Remark 2.1.3] is given in [**1**].

References

[**1**] A. Bartels, F.T. Farrell and W. Lück, The Farrell-Jones Conjecture for cocompact lattices in virtually connected Lie groups, `arXiv:1101.0469`.

[2] A. Bartels and W. Lück, Isomorphism Conjecture for homotopy K-theory and groups acting on trees, *J. Pure Appl. Algebra* **205** (2006), no. 3, 660–696.

[3] A. Bartels and W. Lück, The Borel Conjecture for hyperbolic and CAT(0)-groups, *Ann. of Math. (2)* **175** (2012), no. 2, 631–689.

[4] A. Bartels, W. Lück and H. Reich, On the Farrell-Jones Conjecture and its applications, *J. Topol.* **1** (2008), no. 1, 57–86.

[5] F.T. Farrell and W.-C. Hsiang, Topological characterization of flat and almost flat Riemannian manifolds $M^n (n \neq 3, 4)$, *Amer. J. Math.* **105** (1983), no. 3, 641–672.

[6] F.T. Farrell and L.E. Jones Isomorphism conjectures in algebraic K-theory, *J. Amer. Math. Soc.* **6** (1993), no. 2, 249–297.

[7] F.T. Farrell and P.A. Linnell, K-Theory of solvable groups, *Proc. London Math. Soc. (3)* **87** (2003), no. 2, 309–336.

[8] F.T. Farrell and S.K. Roushon, The Whitehead groups of braid groups vanish, *Internat. Math. Res. Notices* **10** (2000), no. 10, 515–526.

[9] W. Lück, The Farrell-Jones conjecture in algebraic K- and L-theory, *Enseign. Math. (2)* **54** (2008), 140–141, in *Guido's book of conjectures*, a gift to Guido Mislin on the occasion of his retirement from ETHZ, June 2006, Collected by Indira Chatterji, *Enseign. Math. (2)* **54** (2008), no. 1–2, 3–189.

[10] W. Lück and H. Reich, The Baum–Connes and the Farrell–Jones Conjectures in K- and L-theory, in *Handbook of K-theory* Volume 2 (E.M. Friedlander and D.R. Grayson, eds.), 703–842, Springer-Verlag, New York, 2005.

[11] W. Magnus, On a theorem of Marshall Hall, *Ann. of Math. (2)* **40** (1939), no. 4, 764–768.

[12] A. Ranicki, Algebraic L-theory. III. Twisted Laurent extensions in *Algebraic K-Theory. III: Hermitian K-theory and geometric application* (Proc. Conf., Seattle, Washington, 1972), Lecture Notes in Math. **343**, Springer-Verlag, New York, 1973, 412–463.

[13] S.K. Roushon, The Farrell-Jones isomorphism conjecture for 3-manifold groups, *J. K-theory* **1** (2008), no. 1, 49–82.

[14] S.K. Roushon, On the isomorphism conjecture for groups acting on trees, `math.KT/0510297v5`.

[15] S.K. Roushon, The fibered isomorphism conjecture in L-theory, *Topology Appl.* **157** (2010), no. 2, 508–515.

[16] S.K. Roushon, Surgery groups of fundamental groups of hyperplane arrangement complements, *Arch. Math. (Basel)* **96** (2011), no. 5, 491–500.

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