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MANIFOLD TOPOLOGY: A PRELUDE

S. K. ROUSHON
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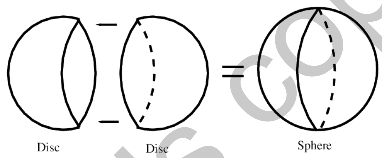
Abstract. In this expository article we describe some of the fundamental questions in Manifold Topology and the obstruction groups which give answers to the questions.

1. Introduction and some basic concepts

In Topology we study spaces and properties of spaces which are invariant under 'deformation'. For example, we do not distinguish between a solid rubber ball and a solid rubber cube because one can be deformed into another without tearing. Next, we can build new spaces by gluing 'simple' pieces. For example, take two 2-discs and glue them along their boundaries, we get a sphere. See the following picture.



This way one constructs very complicated useful spaces and then, we ask when are any two such objects same under deformation. To answer such questions we need to define invariants of spaces which remain same under deformations. Many a time this method is useful to give answers in negative and also sometime one finds a complete set of invariants to give a positive answer.

Note that, there are mainly two kinds of deformations (or equivalences) we deal with in this subject; one is 'homeomorphism' and the other is 'homotopy equivalence'. The first one only 'stretches' the underlying space without tearing and the second one 'squeezes' or 'thickens' the underlying space continuously.

There are various other kinds of deformations we come across, e.g., *weak homotopy equivalence*, *simple homotopy equivalence*, *diffeomorphism* and *PL-homeomorphism*.

We urge the reader to look at the reference [16], or any other text book on Algebraic Topology for the remainder of this section.

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1.1. **Homotopy Theory.**

Definition 1.1.1. Two topological spaces X and Y are called *homeomorphic* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$, here id_* denotes the identity map. When this happens we call f (or g) a *homeomorphism.*

Definition 1.1.2. Two continuous maps $f, g : X \to Y$ are called *homotopic* if there is a continuous map $F : X \times I \to Y$, such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$. If $x_0 \in X$ and $y_0 \in Y$ are given such that $f(x_0) = y_0$ and $g(x_0) = y_0$, then f and g are called *homotopic relative to base point* if, in addition, $F(x_0, t) = y_0$ for all $t \in I$.

Definition 1.1.3. Two topological spaces X and Y are called *homotopy equivalent* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. When this happens we call f (or g) a *homotopy equivalence.*

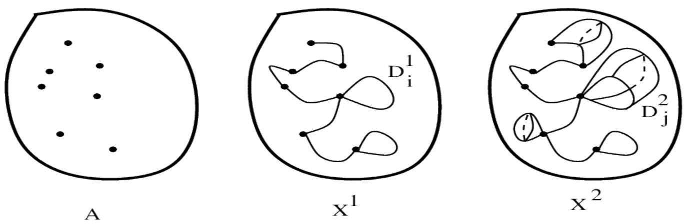
Obviously, homeomorphism is stronger than homotopy equivalence, but there are times when homotopy equivalence implies homeomorphism. These kinds of instances are major breakthroughs in Topology.

In this article we will be considering spaces which are 'CW-complexes' and 'manifolds'. CW-complexes are spaces which are built from the following subspaces of $\mathbb{R}^n$.

$$\mathbb{D}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{i=n} x_i^2 \leq 1\}.$$

Here, $\mathbb{D}^n$ is called an *n-disc* and n its *dimension.* The boundary $\partial\mathbb{D}^n$ is defined as $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{i=n} x_i^2 = 1\}$, it is also called the $(n-1)$-sphere $\mathbb{S}^{n-1}$. These subspaces are the simple pieces we referred above.

To avoid unnecessary hypothesis, we define a *finite CW-complex.* We begin with a finite set A with discrete topology. Then, consider finitely many maps $\phi_i^1 : \mathbb{S}^0 \to A$, for $i = 1, 2, \ldots, j_1$. Construct the quotient space X^1 of the disjoint union $A \cup_{i=1}^{i=j_1} \mathbb{D}_i^1$, where each $\mathbb{D}_i^1$ is a copy of $\mathbb{D}^1$, under the relation: $\phi_i^1(x)$ is identified with x where $x \in \mathbb{S}^0$. Suppose we have constructed X^{n-1}. Next, consider continuous maps $\phi_i^n : \mathbb{S}^{n-1} \to X^{n-1}$, for $i = 1, 2, \ldots, j_n$, and construct the quotient space



CW–complex construction

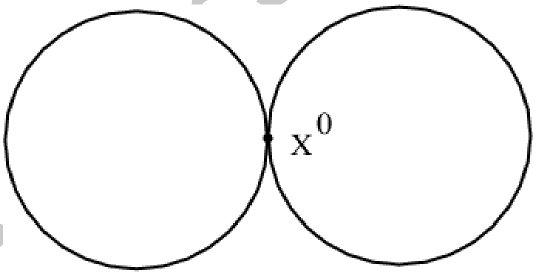
$X := X^n$ of X^{n-1} disjoint union with j_n copies of $\mathbb{D}^n$, in a similar way as defined in the case $n = 1$. X is called a *finite CW-complex* and X^k is called its *k-skeleton.* Above we show the construction of a CW-complex pictorially. Also, we say X^k is obtained from X^{k-1} by attaching j_k k-discs by the *characteristic maps* ϕ_i^k, for $i = 1, 2, \ldots, j_k$. Note that each $\mathbb{D}_i^k - \mathbb{S}^{k-1}$ is embedded in X and is called a *k-cell.* The maximum value of k, for which X has a k-cell but no cells of higher dimensions, is called the *dimension* of the CW-complex. We call X finite as there are only finitely many cells in X. We can similarly define an infinite CW-complex which has finitely many cells in each dimension. To define CW-complex with arbitrary number of cells in each dimension is tricky and we avoid it here.

A subspace Y of a CW-complex X is called a *subcomplex* of X if Y is closed in X and is a union of cells of X. We call the pair (X, Y) a *CW-pair.* The dimension of the pair (X, Y) is defined considering the cells in X which are not in Y, and the k-skeleton of the pair (X, Y) is defined as $Y \cup X^k$. Therefore, the dimension of the pair (X, Y) could be less than the dimension of Y.

There is yet another class of spaces called *polyhedra* which is a special class of CW-complexes, in this case the attaching maps are injective.

In this article by a 'complex' we will always mean a CW-complex.

Example 1.1.1. Let $X^0 = \{*\}$ be a singleton and let $\phi_1^1, \phi_2^1 : \mathbb{S}^0 \to X^0$ be the obvious maps. Then X^1 is the one point union (or *wedge*) of two circles, called the *figure eight.* It is a finite complex. More generally, a wedge of finitely many spheres of (different dimensions) is also a finite complex.

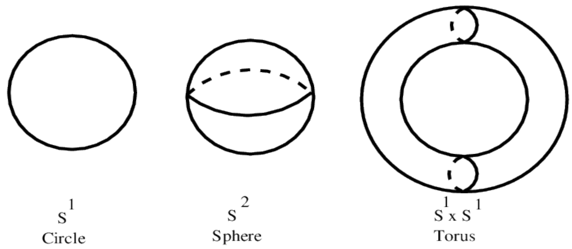


X^1 = Figure eight

Next, we define manifolds and give examples.

Definition 1.1.4. A Hausdorff second countable space M is called a *topological manifold with boundary* if any point $x \in M$ has a neighborhood V homeomorphic to an open set in $\mathbb{R}_+^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$, for some n. V is called an *Euclidean neighborhood* of x. n is called the *dimension* of M if the same n works for all points of M. The points on M which corresponds to the points in $\mathbb{R}_+^n$ with $x_n = 0$ are called the *boundary* points. A compact manifold with empty boundary is called *closed.* Similarly, one defines *smooth* or *differentiable* manifolds. There is yet another class of manifolds called *piecewise linear* or PL-manifolds.

By 'invariance of domain' (see Theorem 1.2.2), if M is connected then the dimension of M is well-defined. Below we give examples of closed manifolds of dimensions 1 and 2. An example of a manifold with boundary is $\mathbb{D}^n$. One can construct many other examples of manifolds with boundary by removing Euclidean neighborhoods of points from a manifold. The figure eight is a space which is not a manifold, the 0-cell is the troubling point.



Examples of manifolds

For some basics on manifold theory see the book [6] or any other book on manifolds. But, we recall the concept of tangent space which is intuitive. In the case of differentiable manifolds, by Whitney embedding theorem, a compact manifold M of dimension n can be embedded in the Euclidean space $\mathbb{R}^N$, for $N \geq 2n$. Now, we consider the set TM of disjoint union of all lines in $\mathbb{R}^N$ which are tangent to the manifold at some point. The collection of these lines form a manifold of dimension $2n$. And the set of all lines tangent to M at a given point is the translate of some n-dimensional vector subspace of $\mathbb{R}^N$. Therefore, there is a map $TM \to M$ which becomes a vector bundle projection of rank n (see Definition 1.2.1), called the *tangent bundle* of M. For example, one can show that for $\mathbb{S}^1$ the tangent bundle is, in fact, the cylinder $\mathbb{S}^1 \times \mathbb{R}$. Also, the tangent bundle of $\mathbb{R}^n$ is the trivial bundle $\mathbb{R}^n \times \mathbb{R}^n$.

Next, we recall a very crucial and important theorem in Homotopy Theory, called the *Whitehead theorem* which is used to study many fundamental questions in Manifold Topology, we will see in the next section. There are algebraic invariants called *homotopy groups* associated to topological spaces. It is defined as the homotopy classes of base point preserving maps from the n-sphere $\mathbb{S}^n$ with a base point to the topological space X with a base point, say x_0. It is denoted by $\pi_n(X, x_0)$. For $n \geq 1$ this set has a group structure and for $n = 0$ it is the set of all path components of X. For $n = 1$ it is called the *fundamental group* of the space. The Whitehead theorem says the following.

Theorem 1.1.1. *Let X and Y be complexes and $f : (X, x_0) \to (Y, f(x_0))$ be a map. If the induced homomorphisms $f_*^q : \pi_q(X, x_0) \to \pi_q(Y, f(x_0))$ are isomorphisms for all $x_0 \in X$ and $q \geq 1$ and a bijection for $q = 0$, then, f is a homotopy equivalence.*

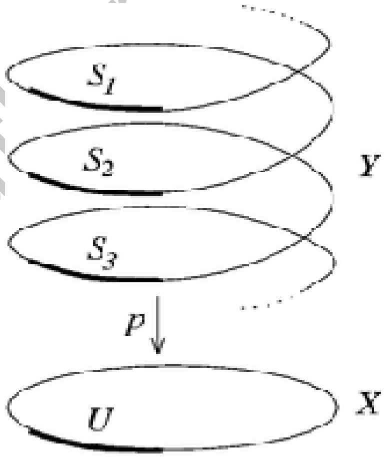
When the hypothesis of the theorem is satisfied, f is called a *weak homotopy equivalence*. See Theorem 7.5.9 and Corollary 7.6.24 in [16] for a proof and related materials.

For finite dimensional complexes, even a stronger statement is true. If X and Y are k-dimensional and $f : X \to Y$ induces isomorphisms on homotopy groups for all $q < n$, for some $n > k$ and induces a surjective homomorphism for $q = n$, then f is a homotopy equivalence. This implies that although a homotopy equivalence induces isomorphisms on homotopy groups in all dimensions; for finite complexes, the converse is true with a weaker assumption. That is, we only need to assume that f induces isomorphism in homotopy groups in low dimensions, although for most finite complexes there are nonzero homotopy groups in arbitrary high dimensions. This fact is very crucial in the proofs of many theorems in Manifold Topology as we highlighted above.

At this point we inform the reader about the significance of the theory of covering spaces, which is intimately related to fundamental group.

Definition 1.1.5. Given a topological space X, a *covering space* is defined to be a topological space Y together with a continuous map $p : Y \to X$, called *covering map*, which has the following property: given any point $x \in X$, there is a neighborhood U of x in X, such that $p^{-1}(U)$ is a disjoint union of open sets S_i, $i \in I$, of Y and $p|_{S_i} : S_i \to U$ is a homeomorphism for all $i \in I$. Here I is an indexing set.

For example, any homeomorphism is a covering map. An example of a covering



Non-trivial covering projection

map which is not a homeomorphism is the exponential map $\mathbb{R} \to \mathbb{S}^1$. A pictorial view of this covering map is given above.

One then defines, given a fixed space X, its universal covering space $\tilde{X}$ to be the maximal element in the category of all covering spaces of X. For technical reason, we need to assume that the space X is connected, locally path connected and semilocally simply connected, to ensure existence of the universal cover of X. Let us call such a space a *nice space*. Complexes are nice spaces.

The study of covering spaces helps us to understand the fundamental group well, through group action on spaces. (We recall here that, we say a group G *acts* on a space if there is a group homomorphism from G to the group of homeomorphisms of the space.) In fact, if X has a universal covering space, then the fundamental group G of X acts on $\tilde{X}$, so that the quotient $\tilde{X}/G = X$. This is the first instance one sees a connection between topological spaces and groups. This also transfers a topological problem into a problem in group theory and solves topological problems with the help of group theory and vice versa. For example, using covering space theory it becomes easy to prove that subgroup of a free group is free. Also, one checks using the above mentioned example that the fundamental group of the circle is infinite cyclic, which in turn proves the following Brouwer fixed point theorem in dimension 2.

Theorem 1.1.2. *For any map $f : \mathbb{D}^2 \to \mathbb{D}^2$, there is a point $x \in \mathbb{D}^2$ such that $f(x) = x$.*

In fact, this is true for $\mathbb{D}^n$ also, which requires the concept of homology theory.

1.2. **Homology Theory.** Homology theory is basically 'formal sum' of maps or of subspaces of a space, and then introducing 'natural' relations among elements of these sums to define invariants of the space. Initially, the subject is difficult to appreciate, but after certain amount of work one sees beautiful applications, which otherwise are impossible to see. One indication we have already given in the Brouwer fixed point theorem. Another advantage of this subject, in contrast to homotopy theory, is that the invariants, although defined in a difficult way, are not very difficult to compute.

The very first example is $H_0(X)$, of a topological space X. It is by definition, the free abelian group generated by the path components of X. Note that, if two spaces are homotopy equivalent then they have isomorphic $H_0(-)$.

Let us define this group differently, which will give the motivation for the definition of higher dimensional invariants.

Let $C_0(X)$ be the free abelian group generated by the points of X, and $C_1(X)$ be the free abelian group generated by all continuous maps $\sigma_1 : [0,1] \to X$. Define $\partial_1 : C_1(X) \to C_0(X)$ by $partial_1(\sigma_1) = \sigma_1(1) - \sigma_1(0)$, for all $\sigma_1 \in C_1(X)$. Then, ∂_1 is a homomorphism of abelian groups. It can be shown that $H_0(X)$ is isomorphic to $C_0(X)/\partial_1(C_1(X))$. Let ∂_q be the convex hull of the vectors $e_i \in \mathbb{R}^n$, for $i = 0, 1, 2, \ldots, n$, where $e_0 = (0, 0, \ldots, 0)$ and $e_i = (0, 0, \ldots, 1, \ldots, 0)$, where 1 is at the i-th place. Note that $\partial_1 = [0,1]$. Now, more generally, define $C_q(X)$ to be the free abelian group generated by all continuous maps $\sigma_q : \partial_q \to X$. Next, define the map (called *boundary map*) $\partial_q : C_q(X) \to C_{q-1}(X)$ by

$$\partial_q(\sigma_q) = \sigma_q^0 - \sigma_q^1 \ldots + (-1)^q \sigma_q^q.$$

Here, σ_q^i evaluated at $(x_1, x_2, \ldots, x_{q-1}) \in \partial_{q-1}$ is equal to σ_q evaluated at $(x_1, x_2, \ldots, 0(i-th), \ldots, x_{q-1}) \in \partial_q$. σ_q^i is said to be the *i-th face* of σ_q. One now checks that $\partial_{q-1} \circ \partial_q = 0$ for all q. Define $C_i(X) = 0$ if $i \leq -1$. We call $\{C_*(X), \partial_*\}$ the *singular chain complex* of X. It is easily checked that the kernel Z_q of ∂_q contains the image B_q of ∂_{q+1}. Define the q-the *singular homology group* of X as the quotient group Z_q/B_q and it is denoted by $H_q(X, \mathbb{Z})$. For a complex X, there is a similar object called *cellular chain complex*, denoted by $\{S_*(X), \partial_*\}$. The group $S_q(X)$ is defined as the free abelian group generated by the q-cells of X. In this particular case the boundary map is defined using singular homology theory. But, one can show that the singular and cellular homology (that is the homology of the chain complex $S_*(X)$) are isomorphic for complexes.

One can define the homology $H_*(X, A; \mathbb{Z})$ of pairs (X, A) by defining the associated chain complex as the quotient chain complex $\{C_*(X, A), \partial_*\}$, where $C_i(X, A) = C_i(X)/C_i(A)$.

Also, one constructs a dual class of groups $\{C^*(X)\}$ whose elements $C^i(X)$ are defined as $hom(C_i(X), \mathbb{Z})$ and define *coboundary* maps $\delta^i : C^i(X) \to C^{i+1}(X)$. Now, renaming $C^i(X)$ as $C_{-i}(X)$ one gets a chain complex and the homology groups of this chain complex are called the *(integral) cohomology groups* of X, and is denoted by $H^i(X, \mathbb{Z})$.

Below we give couple of well known examples of homology computation.

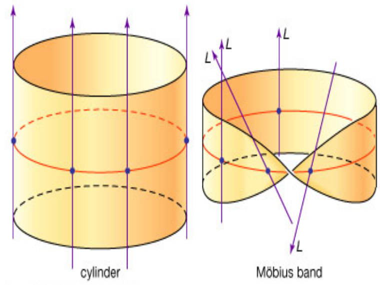
Example 1.2.1. $H_n(\mathbb{R}^m, \mathbb{Z}) = 0$ for $n \neq 0$ and $H_0(\mathbb{R}^m, \mathbb{Z}) = \mathbb{Z}$. This is also true for any contractible space. A space is *contractible* if the identity map is homotopic to the constant map.

Example 1.2.2. $H_n(\mathbb{S}^m, \mathbb{Z}) = 0$ if $n \neq m$ and $n, m \neq 0$. $H_0(\mathbb{S}^m, \mathbb{Z}) = H_m(\mathbb{S}^m, \mathbb{Z}) = \mathbb{Z}$ if $m \neq 0$. And $H_0(\mathbb{S}^0, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$, $H_n(\mathbb{S}^0, \mathbb{Z}) = 0$ for $n \neq 0$.

Now, we motivate the reader to the notion of orientability and few more concepts we need.

In Topology sometimes it is helpful to embed a space X into another space Y and look at its 'surroundings' or the 'complement' to understand X and Y. Probably, the first such situation one observes is in General Topology, where, one can show that a space X is Hausdorff if and only if the image of the embedding $X \to X \times X$, sending $x \mapsto (x, x)$ is closed with respect to the product topology on $X \times X$.

We want to study such surroundings of a manifold and more generally, for complexes later. We begin with one example. Consider the following picture of a cylinder and a Möbius band. Both of these spaces are obtained as quotients of the square $[0, 1] \times [0, 1]$ under certain identifications of its boundary.



In the cylinder case $(0, t)$ is identified with $(1, t)$ and in the Möbius band case $(0, t)$ is identified with $(1, 1 - t)$ for all $t \in [0, 1]$. The middle line $\{(\frac{1}{2}, t) \| \ t \in [0, 1]\}$ goes to the mid circles in the cylinder and in the Möbius band. Therefore we have got embedding of the circle in the two different spaces. We leave it an exercise to check that the cylinder and the Möbius bands are not homeomorphic.

There is yet another aspect to these two examples. Replace $[0, 1] \times [0, 1]$ by $[0, 1] \times \mathbb{R}$ and do the same identifications, one gets the open cylinder and the open Möbus band. Now, over each point on the mid circle in each of the example, there lies the real line $\mathbb{R}$. This gives the impression of a bundle of real lines on the circle. But we get two different spaces although they look the same locally, that is around any point of the mid circle the two spaces look like $(\epsilon, 1 - \epsilon) \times \mathbb{R}$. One can show these are the only two different $\mathbb{R}$-bundle spaces over the circle. This motivates the following definition.

Definition 1.2.1. A surjective continuous map $p : E \to B$ is called a *fiber bundle projection*, E is called the *total space* and B is called the *base space* if the following are satisfied. Each point $b \in B$ has a neighborhood U_b, such that $p^{-1}(U_b)$ is homeomorphic to $p^{-1}(b) \times U_b$ by a homeomorphism f with $p|_{p^{-1}(U_b)} = \pi_2 \circ f$. Here π_2 denotes the second projection. It is called a *vector bundle* of rank n if the fibers $p^{-1}(x) = \mathbb{R}^n$ for all $x \in B$ and the restriction of f to each fiber is a linear isomorphism. It is called a *trivial bundle* if U_b can be taken to be the whole space B.

We now see an important aspect associated to the Möbius band and cylinder, which is crucial in Manifold Topology. Consider the mid circles in the two examples. Draw a line L at some point, say c, on the mid circle and perpendicular to the circle and give a direction to the line. Now start moving the line, keeping it perpendicular to the circle, along the circle in a fixed direction. When we reach the point c, the direction of the line gets reversed in the case of Möbius band and remain the same in the cylinder case. This gives the fundamental concept of *orientability* of manifolds. This is also intimately related to the concept of homology of manifolds. Let us try to understand this phenomenon differently. Consider another line T at the point c which is tangent to the mid circle of the Möbius band. Then, L and T form a basis of the tangent space of the manifold at c. Now, as we move the frame $\{L, T\}$ along the circle maintaining the position and direction of the lines and come back to c, we see the basis of the tangent space changed to another one with the change matrix having determinant negative in the case of Möbius band. But there is no change in the cylinder case.

We state a theorem, whose proof can be found in any Differential Topology book.

Theorem 1.2.1. *Let M be a closed manifold of dimension n. Then the following are equivalent.*

1. $H_n(M, \mathbb{Z}) = \mathbb{Z}$.

2. The following statement holds for the image C of any embedding of the circle $\mathbb{S}^1$ in the manifold. At any point on C, take any framing F_1 (that is, a basis of the tangent space of M at the point), and then move along the circle with the framing, when we come back to the point again, we get another framing F_2. Then, the determinant of the matrix which sends the framing F_1 to F_2 is positive.

If one of the conditions in the statement of the theorem is satisfied then, we call the manifold *orientable*. In such a situation one of the generators of $H_n(M, \mathbb{Z})$, denoted $[M]$, is called the *fundamental class* or *the orientation class* of M. Once an orientation class is fixed, the manifold is called *oriented*. One can show that the top homology is either trivial or infinite cyclic for any closed manifold.

In this context we make a definition which we will require later.

Definition 1.2.2. A map $f : M \to N$ between two n-dimensional closed oriented manifolds is said to be of degree k if $f_*([M]) = k[N]$.

For example, the map $\mathbb{S}^1 \to \mathbb{S}^1$ defined by $z \mapsto z^k$ has degree k.

Now, take two Möbius bands and attach their boundaries together to get the Klein bottle or attach a disc to the boundary of the Möbius band to get the real projective plane. By the above theorem and the remark following it, both these manifolds are not orientable and their top homology is trivial. Furthermore, note that the open Möbius band is embedded in both the examples, this embedding is called the *normal bundle* of the mid circle in the embedding. Similarly the open cylinder is embedded in the torus, and in this case also this embedded object is called the normal bundle of the mid circle. Therefore, we see examples where the normal bundles can be trivial or non-trivial depending on the ambient manifold. In fact, one can show that if the submanifold (in our case the mid circle) and the ambient manifold are both orientable, the normal bundle of the submanifold is always trivial.

We recall Poincaré duality theorem which gives an important relationship between homology and cohomology of an oriented manifold.

Theorem 1.2.2. *Let M be a closed oriented manifold of dimension n. There is an isomorphism, called the duality isomorphism, obtained by taking cap product with the fundamental class*

$$\cap[M] : H^k(M, \mathbb{Z}) \to H_{n-k}(M, \mathbb{Z})$$

for all k.

Using homology theory one can also prove the following very important *invariance of domain* theorem.

Theorem 1.2.3. *If an open set in* $\mathbb{R}^n$ *is homeomorphic to an open set in* $\mathbb{R}^m$*, then* $m = n$*.*

This theorem is needed to ensure the well-definedness of dimension of a connected manifold.

1.3. **Algebra.** Here we recall some basic Algebra used in this article.

Let R be a ring with unity. An (left) *R-module* M is an abelian group together with an action of R, that is a map $R \times M \to M$, sending (r, m) to an element, denoted by rm, of M satisfying the following properties. For all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$, $(r_1 + r_2)m = r_1m + r_2m$, $r(m_1 + m_2) = rm_1 + rm_2$, $(r_1r_2)m = r_1(r_2m)$ and $1m = m$. Equivalently, an R-module structure on an abelian group M is nothing but a ring homomorphism from R to the endomorphism ring $End(M)$ of M. For example, any abelian group is a $\mathbb{Z}$-module.

In this article we mostly consider the (integral) group ring $\mathbb{Z}[G]$ of a group G. The group ring, by definition, consists of the formal finite sums $\Sigma_{i=1}^k r_{g_i} g_i$, where $g_i \in G$ and $r_{g_i} \in \mathbb{Z}$, for $i = 1, 2, \ldots, k$. Equivalently, the group ring can be defined as the set of maps $f : G \to \mathbb{Z}$, such that f takes zero value on all but finitely many elements in G, and the operations are defined by the following. Given f and g in $\mathbb{Z}[G]$, $f + g : G \to \mathbb{Z}$ denotes the map $(f + g)(\alpha) = f(\alpha) + g(\alpha)$ and fg denotes the map $(fg)(\alpha) = \Sigma_{\alpha=uv} f(u)g(v)$. The summation is well defined since there are only finitely many elements of G on which f or g takes nonzero values.

We now give a special example which we require in this article. Let X be a nice space and $\tilde{X}$ be its universal cover. Then, one knows that the fundamental group G of X acts on $\tilde{X}$ as a group of covering transformations. That is, we have a map $G \times \tilde{X} \to \tilde{X}$, so that the induced map from G to the group of homeomorphisms of $\tilde{X}$ is a homomorphism. It is easy to check that this induces a homomorphism from G to the group of isomorphisms of $C_i(\tilde{X})$ for each i. It now follows that each $C_i(\tilde{X})$ becomes a free $\mathbb{Z}[G]$-module. When X is a complex then also one can get such a module structure, by using the induced complex structure on the universal cover. For example, one gives a complex structure on $\mathbb{R}$ lifting a complex structure on $\mathbb{S}^1$ using the exponential map and then make $S_i(\mathbb{R})$, a free $\mathbb{Z}[\mathbb{Z}]$-module.

We now recall the definition and some examples of a particular class of R-modules, which are useful in Topology.

Definition 1.3.1. A *projective R-module* is by definition a direct summand of a free R-module. That is, an R-module P is called projective, if there exists another R-module Q such that $P \oplus_R Q$ is isomorphic to a free R-module.

Of course, free R-modules are projective. We now give an example of an R-module which is projective, but not free.

Example 1.3.1. Let $R = \mathbb{Z}_6$ and $M = \mathbb{Z}_3 \oplus_R \mathbb{Z}_2$, then $M = R$, and $\mathbb{Z}_3$ and $\mathbb{Z}_2$ are R-modules and hence they are projective. Obviously, they are not free.

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2. Some questions

To motivate the reader we begin with some of the fundamental questions in this subject.

• When is a topological space homotopy equivalent to a finite complex?

• When is a finite complex homotopy equivalent to a compact manifold?

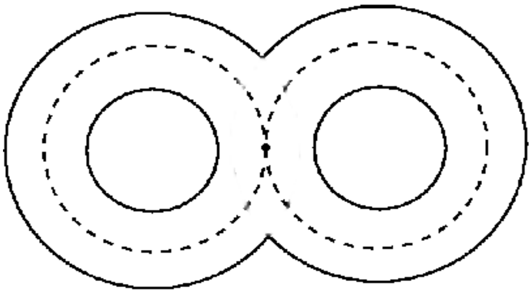
• Are two homotopy equivalent closed manifolds homeomorphic? That is, we are asking about the uniqueness of the manifold in the previous question.

The examples given below, which show that the answer to the above questions are in general no, follow the same order as the questions.

• Note that, the necessary conditions the space should satisfy are those known properties of a finite complex which are invariant under homotopy equivalence. We may not need to assume all these properties at a time. At some point we will see that only some of the properties are enough or we will hit on an obstruction.

We begin with the number of path components. We know this number is homotopy invariant, that is, if two spaces are homotopy equivalent then they have the same number of path components. Therefore, if we want our finite complex to be path connected, then we have to assume that the space we started with should also be path connected. Compactness need not be preserved under homotopy equivalence, so we do not consider this. The most basic homotopy invariant we study in Algebraic Topology is the fundamental group. A finite complex has finitely generated fundamental group. Therefore, the wedge of infinitely many circles or the Euclidean plane with all points whose both coordinates are integers deleted, can never be homotopy equivalent to a finite complex, as they have infinitely generated fundamental groups. The further conditions we need to put on the space are discussed in the next section.

• First, note that any finite complex is homotopy equivalent to a compact manifold with nonempty boundary. This can be obtained by first taking a compact polyhedron homotopy equivalent to the finite complex, and then taking the regular neighborhood of the polyhedron after embedding it in some Euclidean space. We see this with an example. Consider the figure eight embedded in $\mathbb{R}^2$. The regular neighborhood is shown in the picture below, which is a 2-dimensional manifold with three boundary components.



Regular neighborhood of Figure eight.

Therefore, the question is to ask for a closed manifold. Next, recall that a manifold has vanishing homology groups in higher dimensions, therefore, the complex should not have cells of arbitrary high dimensions. For example, the wedge of spheres $\mathbb{S}^n$ for $n = 1, 2, \dots$ can never be homotopy equivalent to any closed manifold. Hence the complex should be assumed to be finite dimensional. Furthermore, homological properties are homotopy invariants, for example, the homology groups must satisfy Poincaré duality (Theorem 1.2.2). There are further conditions required which are described in the next section.

We now give examples of finite complexes which are not homotopy equivalent to closed manifold as they do not satisfy Poincaré duality.

Example 2.0.1. Let X be the figure eight. Then $H_i(X, \mathbb{Z}) = 0$ for $i \geq 2$, since X is an one-dimensional complex. Therefore, if there is any manifold homotopy equivalent to X, then it must be one-dimensional. Note that, $H_1(X, \mathbb{Z})$ has rank 2, but an orientable closed one-dimensional manifold must have first homology of rank 1 by Poincaré duality. For a similar reason (taking homology with coefficient in $\mathbb{Z}_2$) it follows that X is not homotopy equivalent to any closed non-orientable manifold. Similarly, one shows that the wedge of two spheres of the same dimension also gives an example of a finite complex not homotopy equivalent to any closed manifold.

• There are homotopy equivalent manifolds which are not homeomorphic. For example, the lens spaces $L(7, 1)$ and $L(7, 2)$ are homotopy equivalent but not homeomorphic. There is another kind of equivalence called *simple homotopy equivalence* which lies in between homotopy equivalence and homeomorphism. We will study this later in this article. It can be shown that the above two spaces are not even simple homotopy equivalent.

These questions were well studied over the last several decades. The Whitehead group $Wh(G)$, reduced projective class group $\tilde{K}_0(\mathbb{Z}[G])$, and the surgery L-groups $L_n(\mathbb{Z}[G])$ of the fundamental group G contain the answers to the above and many more questions. Mainly, the breakthrough on classifying manifolds started during 1960 to 1970. Then, there were results proved about computing the above obstruction groups for small classes of groups. Several conjectures were formulated which still remain open. The second breakthrough happened with works which use geometry and controlled topology. Apart from geometry and controlled topology, Frobenius induction technique was also found to be very useful. Finally, in 1993 Farrell and Jones formulated an important conjecture, now popularly known as *Farrell-Jones Isomorphism conjecture* ([5]), which captures the subject in a single statement and implies all the previous conjectures. Since then, an enormous amount of work has been done by many authors and it is still an ongoing front line area of research to prove the Isomorphism conjecture for different classes of groups.

In this article we describe the above obstruction groups and see how they answer the questions. Also, we recall some of the classical results. In a future article we plan to review the development that took place during the last two decades.

The subject we are exposing in this article is enormous and, therefore, we urge the reader to look at the sources in the reference list to know more about it.

3. In more detail

In this section we see exactly the conditions needed and how the obstruction groups appear in solving the questions in the previous section. We have already recalled many of the basics required in this section. There are times when we will use some new concepts, which we do not recall due to technical reason. We refer the reader to look at the corresponding sources. But this lacking will not prevent the reader from understanding the core ideas behind the proofs.

3.1. **Reduced projective class group $\tilde{K}_0(-)$.** Recall the definition of homotopy equivalence. There are two conditions which need to be satisfied. We start with the following definition.

Definition 3.1.1. A space X is said to be *dominated* by another space Y if there are maps $f : X \to Y$ and $g : Y \to X$ so that $g \circ f \simeq id_X$. And X is called *finitely dominated* if in addition Y is a finite complex.

Note that this is half of saying that X is homotopy equivalent to Y. It is also known that if X is finitely dominated then it is homotopy equivalent to a countable complex, which can be checked using some facts from [12] (or see Theorem 3.9 of [19] or Exercise G6 in Chapter 7 of [16]). In fact, the *mapping telescope* of $f \circ g$ becomes homotopy equivalent to X. Therefore, to investigate whether a space X is homotopy equivalent to a finite complex, we can assume that X is a complex. But, to show whether a complex is homotopy equivalent to a finite complex, one needs more difficult work. This is answered in a very important and celebrated series of papers by C.T.C. Wall (see [22] and [23]). This leads to an obstruction which lies in an abelian group called the *reduced projective class* group and denoted by $\tilde{K}_0(\mathbb{Z}[\pi_1(X)])$. Interestingly, the answer to this question depends only on the fundamental group of X.

Before going further let us define reduced projective class group of a ring. Let R be a ring with unity. Let $\mathcal{P}$ denote the free abelian group generated by the set of all isomorphism classes of finitely generated projective R-modules. Let $K_0(R)$ be the quotient of $\mathcal{P}$ by the relations $[P] + [Q] = [P \oplus Q]$ where $[P], [Q] \in \mathcal{P}$. Note that, $K_0(R)$ is a covariant functor from the category of rings with unity to the category of abelian groups. Thus the homomorphism $\mathbb{Z} \to R$ sending 1 to the unity of R induces a map $i : K_0(\mathbb{Z}) \to K_0(R)$. The *reduced projective class* group $\tilde{K}_0(R)$ is by definition the quotient $K_0(R)/i(K_0(\mathbb{Z}))$. By abusing notation we will denote elements of $\tilde{K}_0(R)$ by the same notation $[P]$. For details on the projective class groups see [15] or any standard book on Algebraic K-theory. Below we give some examples.

Example 3.1.1. Let $R = \mathbb{Z}$, the ring of integers. Then one knows that, since $\mathbb{Z}$ is a principal ideal domain, any finitely generated projective module over $\mathbb{Z}$ is free. Therefore, we have $\tilde{K}_0(\mathbb{Z}) = 0$. Same conclusion holds if R is a field.

In this subject one encounters only integral group ring $\mathbb{Z}[G]$ of a group G. See Section 4 for more on this. Although, more abstract coefficients are considered for the purpose of a general framework and applications in other areas of Mathematics.

Recall that, any finitely dominated space is homotopically equivalent to a (countable) complex, and hence up to homotopy such a space is nice.

Let X be a nice space and $\tilde{X}$ be its universal cover. Then, recall that $C_*(\tilde{X})$ is a chain complex of free $R(= \mathbb{Z}[G])$-module. In general $C_i(\tilde{X})$ is a very large R-module. On the other hand if X were a finite complex then it follows that $S_*(\tilde{X})$ is a finite chain complex of finitely generated free R-module. For an arbitrary nice space we can make the situation better by assuming that X is finitely dominated. C.T.C. Wall proves that in this case $S_*(\tilde{X})$ is chain equivalent to a finite chain complex of finitely generated projective R-modules, say $P_*(X)$. Consider the object $\chi(P_*(X)) = \Sigma_i(-1)^i[P_i(X)] \in \tilde{K}_0(R)$. $\chi(P_*(X))$ is called the *Wall finiteness obstruction* of the space X.

Theorem 3.1.1 (C.T.C. Wall). *Let X be a finitely dominated space and $R = \mathbb{Z}[\pi_1(X)]$. Then X is homotopy equivalent to a finite complex if and only if $\chi(P_*(X)) = 0$ in $\tilde{K}_0(R)$. Furthermore, given an element ω in $\tilde{K}_0(R)$ there is a finitely dominated space X' with fundamental group isomorphic to $\pi_1(X)$, such that $\omega = \chi(P_*(X'))$.*

This gives a complete solution to the first problem. But computing the reduced projective class group of a group is another story. We will talk about it in the next section.

3.2. **Surgery groups $L_*(-)$.** If we want a finite complex to be homotopy equivalent to a closed (orientable) manifold there are several more necessary conditions needed. One such condition is that the homology of the complex must satisfy

Poincaré duality as mentioned in Section 1. Such a complex is called a *Poincaré complex*. We give the precise definition below.

Definition 3.2.1. Let X be a connected finite complex and for some n, $H_n(X, \mathbb{Z}) \simeq \mathbb{Z}$. Let $[X] \in H_n(X, \mathbb{Z})$ be a generator such that the cap product with $[X]$ gives an isomorphism $H^q(X, \mathbb{Z}) \to H_{n-q}(X, \mathbb{Z})$ for all q. Then X is called a *Poincaré complex* with *fundamental class* $[X]$ and dimension n.

The second condition needed is the existence of a bundle over the complex, which has properties similar to the normal bundle of a manifold embedded in some Euclidean space. We describe this below.

Let M be a closed connected oriented smooth manifold of dimension n. By the Whitney Embedding Theorem we can embed this manifold in $\mathbb{R}^{n+k} \subset \mathbb{S}^{n+k}$ for $k \geq n$. Let ν_M be the normal bundle of M in $\mathbb{S}^{n+k}$. Let τ_M be the tangent bundle of M. Then $\tau_M \oplus \nu_M$ is the product bundle as $M \subset \mathbb{S}^{n+k} - \{\infty\} = \mathbb{R}^{n+k}$. Let N be the subset of ν_M consisting of vectors of length $< \epsilon$, with respect to some Riemannian metric, for some $\epsilon > 0$. Then N is an open neighborhood of M in $\mathbb{S}^{n+k}$ and in fact diffeomorphic to the total space $E(\nu_M)$ of ν_M. The one point compactification of $E(\nu_M)$ is called the *Thom space* of ν_M. On the other hand the one point compactification N^* of N is homeomorphic to $\mathbb{S}^{n+k}/\mathbb{S}^{n+k} - N$. Hence, we get a map $\alpha : \mathbb{S}^{n+k} \to N^* \simeq T(\nu_M)$. One can check that α induces an isomorphism $H_{n+k}(\mathbb{S}^{n+k}, \mathbb{Z}) \to H_{n+k}(T(\nu_M), \mathbb{Z})$ and sends the canonical generator of $H_{n+k}(\mathbb{S}^{n+k}, \mathbb{Z})$ to the generator of $H_{n+k}(T(\nu_M), \mathbb{Z})$, which comes from the fundamental class $[M]$ via the Thom isomorphism. In this sense this map is of degree 1.

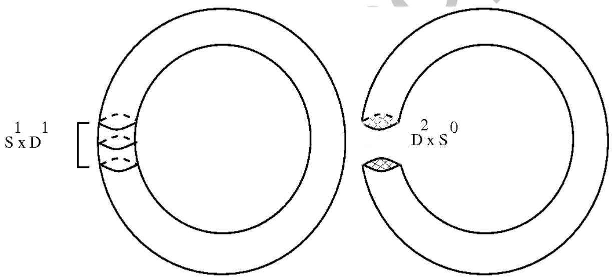
Therefore, for a Poincaré complex X to be homotopy equivalent to a closed oriented smooth manifold it is necessary that there should be a real vector bundle ξ on X and a degree 1 map $\alpha : \mathbb{S}^{n+k} \to T(\xi)$. The Thom space $T(\xi)$ in this generality is defined as follows. Consider the fiber bundle ξ^* over X obtained by taking the one point compactification of the fibers of the bundle ξ. The bundle ξ^* admits a section $s : X \to \xi^*$, sending a point of X to the point at infinity of the fiber over the point. Then $T(\xi)$ is defined as the quotient of the total space of ξ^* by $s(X)$.

Once this information is given we can apply Thom Transversality Theorem to homotope α to β so that $\beta^{-1}(X)$ $(:= K)$ is a (oriented) submanifold of $\mathbb{S}^{n+k}$. Furthermore, the map β restricted to the normal bundle ν_K of K gives a linear bundle map onto ξ and $\beta|_K$ is of degree 1. A data of this type as in the following commutative diagram is called a *normal map* and is denoted by (f, b),

$$\begin{array}{ccc} \nu_M & \xrightarrow{f} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{b} & X \end{array}$$

where, M is a closed connected smooth oriented manifold, X is a Poincaré complex, ν_M is the normal bundle in some embedding of M in $\mathbb{S}^{n+k}$, and b is a degree 1 map. Here we remark that not all Poincaré complexes admit a normal map. See the example on p.32-33 in [11]. We saw that this map b is obtained from the Transversality Theorem but it is nowhere close to being a homotopy equivalence. In the simply connected and odd high dimension case, in fact, b can be homotoped to a homotopy equivalence ([2]). In the general case, the next general step is to apply Surgery theory to b to get another normal map, which is normally cobordant to the previous one, and to try to get closer to a homotopy equivalence. To achieve this we need that b induces isomorphisms on the homotopy groups level and then apply Whitehead theorem (Theorem 1.1.1).

We digress here a bit to show how surgery works to get rid of some homotopy group element from the kernel of b^q_*. Let $M = \mathbb{S}^1 \times \mathbb{S}^1$ as in the picture. And, suppose we want to get rid of the fundamental group element generated by the first circle, which lies in the kernel of $b^1_* : \pi_1(M, m) \to \pi_1(X, b(m))$. In the picture we show a tubular neighborhood of the circle, this is an embedded $\mathbb{S}^1 \times \mathbb{D}^1$ (also called an 1-handle). Note that $\partial(\mathbb{S}^1 \times \mathbb{D}^1) = \partial(\mathbb{D}^2 \times \mathbb{S}^0)$. Now remove the interior of this handle and replace it by $\mathbb{D}^2 \times \mathbb{S}^0$. This is called



Surgery on an 1−handle

the *surgery on the* 1*-handle*. The resulting manifold is a 2-sphere and b can be extended to this new manifold as the restrictions of b to the circles $\mathbb{S}^1 \times \mathbb{S}^0$ are homotopic to the constant maps. This new b^1_* does not have any kernel.

More generally, suppose an element α in the kernel of b^q_* is represented by the embedding of $\mathbb{S}^q$ in M. Since both these manifolds are orientable, the normal bundle of the image of $\mathbb{S}^q$ in M is trivial and hence this image gives rise to an embedding of $\mathbb{S}^q \times \mathbb{D}^{n-q}$ (take the disc bundle of the normal bundle), called a *q-handle*, in M, where M is n-dimensional. A *surgery* along this q-handle is removing the interior of $\mathbb{S}^q \times \mathbb{D}^{n-q}$ and attaching $\mathbb{D}^{q+1} \times \mathbb{S}^{n-q-1}$ to the resulting manifold. This operation kills the kernel element α.

These surgery operations can be done up to dimension $< [\frac{n}{2}]$ and, therefore, by Poincaré duality the main problem lies in dimension $[\frac{n}{2}]$. The complication in this middle dimension gives rise to the Wall's *Surgery obstruction group* $L^h_n(\mathbb{Z}[\pi_1(X)])$.

That is, given a normal map (f, b), there is an obstruction $\sigma(f, b)$, which lies in the group $L_n^h(\mathbb{Z}[\pi_1(X)])$, whose vanishing will ensure that the normal map can be normally cobordant to another normal map (f', b'), where b' is a homotopy equivalence. See [24].

Remark 3.2.1. These surgery groups depend only on the fundamental group and on its orientation character $\omega : \pi_1(X) \to \mathbb{Z}_2$. Here, as we are dealing with the oriented case, this homomorphism is trivial. In the general situation we need to incorporate ω in the the surgery groups. But in this article we avoid it for simplicity.

Remark 3.2.2. The upper script 'h' to the notation of the surgery groups is due to 'homotopy equivalence'. There are problems, when one asks for 'simple homotopy equivalence' in the normal map. Then a different surgery problem appears and gives rise to the surgery groups $L_*^s(\mathbb{Z}[-])$. There are many other decorated surgery groups for different surgery problems, like $L_n^{\langle -\infty \rangle}(\mathbb{Z}[-])$. But all of them coincide, once we have the lower K-theory vanishing result of the group. This is checked using Rothenberg's exact sequence. For example, if the Whitehead group of a group G vanishes, then, $L_n^h(\mathbb{Z}[G]) = L_n^s(\mathbb{Z}[G])$ for all n.

Remark 3.2.3. One further remark is that the surgery groups are 4-periodic. That is, $L_n^h(\mathbb{Z}[-]) = L_{n+4}^h(\mathbb{Z}[-])$ for all n and, in fact, is true for all decorations. This is obtained by showing that the surgery obstruction of a normal map on a complex X of dimension n is same as the surgery obstruction of the corresponding normal map on $X \times \mathbb{CP}^2$.

To end this subsection we give a nice application of surgery theory.

Theorem 3.2.1. *There are infinitely many distinct closed manifolds homotopy equivalent to the projective space* $\mathbb{CP}^k$ *for* $k \geq 3$*. And, this is true in any of the categories; topological, smooth or piecewise linear (PL).*

3.3. **Whitehead group** $Wh(-)$**.** Once we have established that a complex is homotopy equivalent to a manifold, the next question is about the uniqueness of the manifold. In surgery theory one gets two such manifolds M_1 and M_2 (when they exist) as h-cobordant, which we define below.

Definition 3.3.1. Let M_1 and M_2 be two connected closed n-dimensional manifolds. A compact manifold W of dimension $n + 1$ with two boundary components M_1 and M_2 is called a *cobordism* between M_1 and M_2. In such a situation M_1 and M_2 are called *cobordant.* If the inclusions $M_i \subset W$ are homotopy equivalences then W is called an *h-cobordism* between M_1 and M_2. *And* M_1 *and* M_2 *are called h-cobordant.*

Using Morse Theory, one can show that two closed manifolds are cobordant if and only if one of the manifolds can be obtained from the other by finitely many surgery operations.

Given such an h-cobordism W there is an obstruction $\tau(W, M_1)$, which lies in a quotient (called the *Whitehead group* and is denoted by $Wh(\pi_1(M_1))$) of the K_1 of the integral group ring of the fundamental group of M_1. When $\dim W \geq 6$ the element $\tau(W, M_1) \in Wh(\pi_1(M_1))$ has the following property: $\tau(W, M_1) = 0$ implies that W is homeomorphic to $M_1 \times I$ where $I = [0, 1]$. This is called the *s-cobordism theorem* stated below as Theorem 3.4.1. See [9] and [10]. One consequence of this theorem is the Poincaré conjecture in high dimensions.

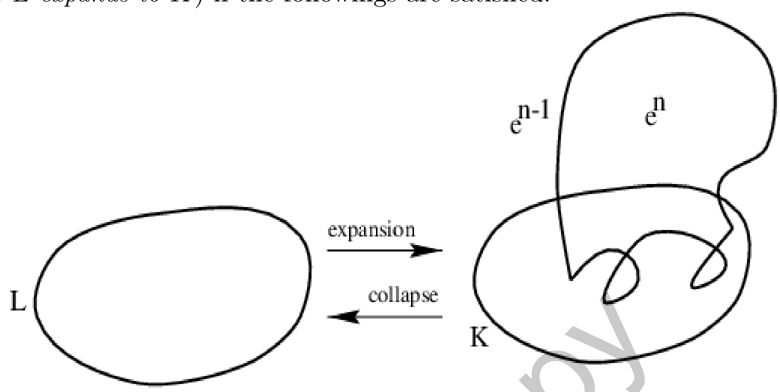
There are similar interpretation of the reduced projective class groups and negative K-groups in terms of some 'special' kind of h-cobordism called *bounded h-cobordism*. See [14] for some more on this matter.

We now recall the original interpretation of the Whitehead group, which says that given a homotopy equivalence $f : K \to L$ between two connected finite complexes K and L there is an element $\tau(f) \in Wh(\pi_1(K))$ whose vanishing ensures that the map f is homotopic to a *simple homotopy equivalence*. See [4]. We already gave a hint to this in Section 2. We describe this important subject below to a certain extent.

The simple homotopy equivalences lie in between homotopy equivalences and homeomorphisms. Even in such a classical vast area of topology the most basic question is not yet answered; namely, if any homotopy equivalence between two finite aspherical complexes is homotopic to a simple homotopy equivalence, which is known as *Whitehead's conjecture* in K-theory. There is an even stronger conjecture which asks; if any homotopy equivalence between two aspherical manifolds is homotopic to a homeomorphism. This is known as *Borel's conjecture*. Recall that, a connected complex X is called *aspherical* if $\pi_i(X) = 0$ for all $i \geq 2$, or equivalently, the universal cover of X is contractible. A result of Chapman says that any homeomorphism of complexes is a simple homotopy equivalence. So, the first step to prove the Borel's conjecture is to verify Whitehead's conjecture in K-theory.

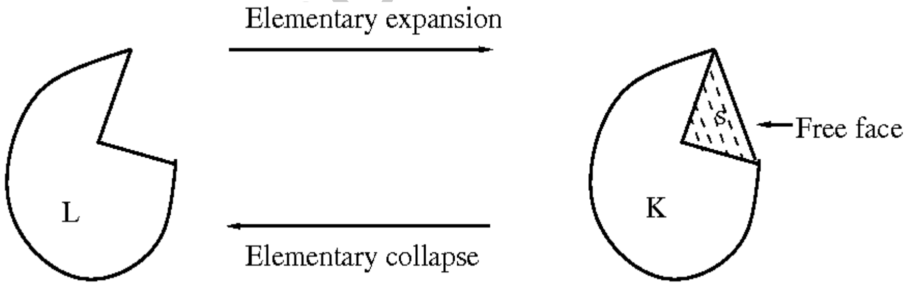
As far as the Borel's conjecture is concerned, dimension 2 is understood completely in the topological, piecewise linear or smooth category. Perelman's proof of the Thurston's Geometrization conjecture completes the picture in dimension 3. Recall that, in dimension ≤ 3 any manifold supports a unique topological, piecewise linear or smooth structure. In dimension greater or equal to 5 an enormous literature exists; where there are enough machinery, language to attack a problem. The critical dimension is 4. In this dimension even the s-cobordism theorem is not yet known. So far this is proved for 4-manifolds with some restriction on the fundamental group; namely, groups with subexponential growth. Also another interesting fact is that a 4-manifold can support infinitely many smooth structures. Even our familiar Euclidean 4-space $\mathbb{R}^4$ has infinitely many smooth structures.

3.4. **Simple homotopy equivalence and Whitehead group.** The reference for this subsection is [4]. Let K and K' be two finite complexes. A homotopy equivalence $f : K \to K'$ is called a *simple homotopy equivalence* provided f is homotopic to a composition of maps of the following kind: $K = K_0 \to K_1 \to \cdots \to K_s = K'$ where the arrows are either an elementary expansion or collapse. A pair of complexes (K, L) is called an *elementary collapse* (and we say K *collapses to* L or L *expands to* K) if the followings are satisfied:



- $K = L \cup e^{n-1} \cup e^n$ where e^{n-1} and e^n are not in L (e^i denotes an i-cell),
- there exist a ball pair $(\mathbb{D}^n, \mathbb{D}^{n-1})$ and a map $\phi : \mathbb{D}^n \to K$ such that
  (a) $\phi|_{\partial\mathbb{D}^n}$ is a characteristic map for e^n
  (b) $\phi|_{\partial\mathbb{D}^{n-1}}$ is a characteristic map for e^{n-1}
  (c) $\phi(P^{n-1}) \subset L^{n-1}$, where $P^{n-1} = \overline{\mathbb{D}^n - \mathbb{D}^{n-1}}$.

From the above figure the definition of an expansion (or collapse) will be clear.

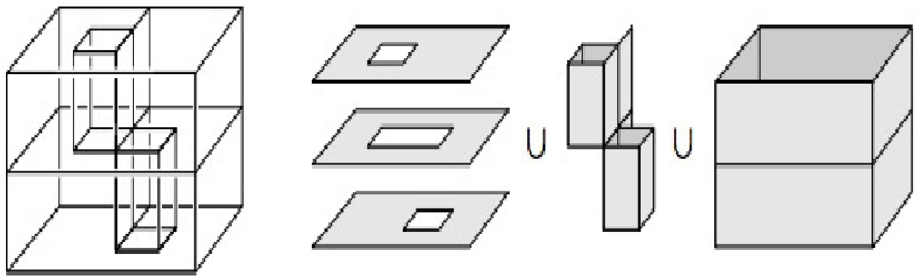


Collapsing a simplex

Above is yet another example in a more concrete situation of a polyhedron. In the example, L is a polyhedron to which we introduce the new simplex s, but the new simplex has a face which is not the face of any other simplex, such a face is called a *free face*. A free face gives the freedom to collapse without changing the homotopy type of the complex.

We now come to a popular example of a contractible topological space which has a 2-dimensional polyhedron structure but has no free face. This is called

Bing's house with two rooms. In this example one has to first expand and then collapse. The picture explains the space. There are two rooms with two different entrances. It is easy to see if we start filling the rooms with square blocks (which are elementary expansions) then at the end we get a three dimensional cube, which can be collapsed to a point as it has free faces.



Bing's house with two rooms

There is an algebraic picture of the above topological construction, which we describe now.

Given any homotopy equivalence $f : K \to K'$, there is an obstruction $\tau(f)$ which lies in an abelian group $Wh(\pi_1(K'))$ (defined below) detecting if f is a simple homotopy equivalence.

Let R be a ring with unity. Let $GL_n(R)$ be the multiplicative group of invertible $n \times n$ matrices with entries in R and $E_n(R)$ be the subgroup of elementary matrices. By definition an *elementary matrix*, denoted by $E_{ij}(a)$, for $i \neq j$, has 1 on the diagonal entries, $a \in R - \{0\}$ at the (i, j)-th position and the remaining entries are 0. Define $GL(R) = \lim_{n\to\infty} GL_n(R)$, $E(R) = \lim_{n\to\infty} E_n(R)$. Here the limit is taken over the following maps:

$$GL_n(R) \to GL_{n+1}(R)$$

$$\begin{pmatrix} A \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

The following lemma shows that $E(R)$ is also the commutator subgroup of $GL(R)$.

Lemma 3.4.1 (Whitehead's lemma)**.** *Let R be a ring with unity. Then the commutator subgroup of $GL(R)$ and of $E(R)$ is $E(R)$.*

Proof. At first note the following easy to prove identities.

$E_{ij}(a)E_{ij}(b) = E_{ij}(a + b)$; (1)

$E_{ij}(a)E_{kl}(b) = E_{kl}(b)E_{ij}(a)$, $j \neq k$ and $i \neq l$; (2)

$E_{ij}(a)E_{jk}(b)E_{ij}(a)^{-1}E_{jk}(b)^{-1} = E_{ik}(ab)$, i, j, k distinct; (3)

$E_{ij}(a)E_{ki}(b)E_{ij}(a)^{-1}E_{ki}(b)^{-1} = E_{kj}(-ba)$, i, j, k distinct. (4)

Also note that any upper triangular or lower triangular matrix with 1 on the diagonal belongs to $E(R)$.

Since $E(R) \subset GL(R)$ we have $[E(R), E(R)] \subset [GL(R), GL(R)]$. Using (3) we find that $E_{ij}(a) = [E_{ik}(a), E_{kj}(1)]$ provided i, j and k are distinct. Hence any generator of $E(R)$ is a commutator of two other generators of $E(R)$. Hence $E(R) = [E(R), E(R)]$. We only need to check that $[GL(R), GL(R)] \subset E(R)$. So let $A, B \in GL_n(R)$. Then note the following identity.

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

Now we check that all the factors on the right hand side belong to $E_{2n}(R)$. This follows from the following. Let $A \in GL_n(R)$. Then the following equality is easy to check.

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Again note the following.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Now recall our earlier remark that any upper triangular or lower triangular matrix with 1 on the diagonal belong to $E(R)$. This completes the proof of the Whitehead Lemma. □

Definition 3.4.1. Define $K_1(\mathbb{Z}[\pi]) = GL(\mathbb{Z}[\pi])/E(\mathbb{Z}[\pi])$. The *Whitehead group* $Wh(\pi)$ of π is by definition $K_1(\mathbb{Z}[\pi])/N$. Here N is the subgroup of $K_1(\mathbb{Z}[\pi])$ generated by the 1×1 matrices (g) and $(-g)$, for $g \in \pi$.

Note that multiplying a matrix A by an elementary matrix from left (or right) makes an elementary row (or column) operation in A. Now by applying elementary row and column operations one can transform an invertible integral matrix to I or $-I$. This shows that $Wh((1)) = 0$. (Though the matrix multiplication induces the group operation in $Wh(-)$ we write it additively, since $Wh(-)$ is abelian.)

We recall below a transparent topological definition of Whitehead group which is naturally isomorphic to the above one. The proof of this isomorphism is very long and can be found in [4]. But we will give the map from this topological definition to the algebraic definition above. For computational purposes the algebraic definition is more useful as we will see later.

Let K be a fixed finite complex. Let $\mathcal{W}(K)$ be the collection of all pairs of finite complexes (L, K) so that K is a strong deformation retract of L. For any two objects $(L_1, K), (L_2, K) \in \mathcal{W}$ define $(L_1, K) \equiv (L_2, K)$ if and only if L_1 and L_2 are simple homotopically equivalent relative to the subcomplex K. Here by a relative simple homotopy equivalence we mean that in the definition of simple homotopy equivalence we do not collapse (or expand) any cells contained in K. Let $Wh(K) = \mathcal{W}/\equiv$. Let $[L_1, K]$ and $[L_2, K]$ be two classes in $Wh(K)$. Define

$[L_1, K] \oplus [L_2, K] = [L_1 \cup_K L_2, K]$. Here $L_1 \cup_K L_2$ is the disjoint union of L_1 and L_2 identified along the common subcomplex K. This defines an abelian group structure on $Wh(K)$ called the *topological Whitehead group* of K.

Now let (L, K) be an element in $\mathcal{W}$. Consider the universal cover $(\tilde{L}, \tilde{K})$. It can be checked that the inclusion $\tilde{K} \subset \tilde{L}$ is a homotopy equivalence. Equip $(\tilde{L}, \tilde{K})$ with the complex structure lifted from the complex structure of (L, K). Let $S_*(\tilde{L}, \tilde{K})$ be the cellular chain complex of $(\tilde{L}, \tilde{K})$. The covering action of $\pi_1(L)$ on $(\tilde{L}, \tilde{K})$ induces an action on $S_*(\tilde{L}, \tilde{K})$ and makes it a chain complex of $\mathbb{Z}[\pi_1(L)]$-modules. In fact, it is a finitely generated free acyclic chain complex of $\mathbb{Z}[\pi_1(L)]$-modules. Let d be the boundary map. One can find a contraction map δ of degree $+1$ of $S_*(\tilde{L}, \tilde{K})$ so that $d\delta + \delta d = id$ and $\delta^2 = 0$. Consider the module homomorphism $d + \delta : \bigoplus_{i=0}^{\infty} S_{2i+1}(\tilde{L}, \tilde{K}) \to \bigoplus_{i=0}^{\infty} S_{2i}(\tilde{L}, \tilde{K})$. It turns out that this homomorphism is an isomorphism of $\mathbb{Z}[\pi_1(L)]$- modules. The image and range of this homomorphism are finitely generated free modules with a preferred basis coming from the complex structure on $(\tilde{L}, \tilde{K})$. We consider the matrix of this homomorphism $d + \delta$ which is an invertible matrix with entries in $\mathbb{Z}[\pi_1(L)]$ and hence lies in $GL_n(\mathbb{Z}[\pi_1(L)])$ for some n. We take the image of this matrix in $Wh(\pi_1(L))$. The proof that this map (say τ) sending (L, K) to this image in $Wh(\pi_1(L))$ is an isomorphism is given in [4].

Now, consider a homotopy equivalence $f : K \to K'$ between two finite complexes. Let M_f be the mapping cylinder of the map f. Recall that, the *mapping cylinder* M_f of f is by definition, the quotient space of the disjoint union $(K \times I) \cup K'$, under the identifications $(k, 1) = f(k)$, for all $k \in K$.

Consider the pair (M_f, K). Here K is identified with $K \times \{0\}$ in M_f. As f is a homotopy equivalence it is easy to check that $(M_f, K) \in \mathcal{W}$. Now, we recall that f is a simple homotopy equivalence if and only if $\tau([M_f, K])$ is the trivial element in $Wh(\pi_1(K))$.

Finally, we state the s-cobordism theorem. Smale proved the theorem in the simply connected case and it is known as the h-cobordism theorem. He received the Fields medal for this proof, which also implies the high dimensional Poincaré conjecture.

Theorem 3.4.1 (**S-cobordism theorem**). *(Barden, Mazur, Stallings) Let M_1 and M_2 be two compact connected manifolds of dimension ≥ 5 if they have empty boundary, and of dimension ≥ 6 otherwise. Let W be an h-cobordism between M_1 and M_2. If $\tau([W, M_1]) = 0$ in $Wh(\pi_1(M_1))$ then $W \simeq M_1 \times I$. In particular $M_1 \simeq M_2$. Furthermore, any element of $Wh(\pi_1(M_1))$ is realized by an h-cobordism.*

Remark 3.4.1. An h-cobordism between M_1 and M_2 with $\tau([W, M_1]) = 0$, is called an s-cobordism between M_1 and M_2.

Here $\simeq$ denotes a homeomorphism, a piecewise linear homeomorphism or a diffeomorphism according as the manifolds are topological, piecewise linear or smooth respectively.

4. Known results on $\tilde{K}_0(-)$, $Wh(-)$ and $L_*(-)$

Now we come to results about computing the invariants we encountered so far, namely the reduced projective class group, the Whitehead group and the surgery obstruction groups.

The Whitehead conjecture in K-theory asks if $Wh(\pi) = 0$, for any finitely presented torsion free group. This has been checked for several classes of groups: for free abelian groups by Bass-Heller-Swan ([1]) ; for free nonabelian groups by Stallings ([17]); for the fundamental group of any complete nonpositively curved Riemannian manifold by Farrell and Jones ([5]); for the fundamental group of finite $CAT(0)$ complexes by B. Hu. ([8]). Waldhausen proved that the Whitehead group of the fundamental group of any Haken 3-manifold vanishes ([21]). Some results like Whitehead group of finite groups are also known: $Wh(F) = 0$, when F is a finite cyclic group of order $1, 2, 3, 4$ and 6 and $Wh(F)$ is infinite when F is any other finite cyclic group. Also $Wh(S_n) = 0$, here S_n is the symmetric group on n letters. Below, we show by a little calculation that there is an element of infinite order in $Wh(\mathbb{Z}_5)$. In fact this group is infinite cyclic, but that needs a difficult proof and we do not talk about it here.

Lemma 4.0.1. *There exists an element of infinite order in* $Wh(\mathbb{Z}_5)$.

Proof. Let t be the generator of $\mathbb{Z}_5$. Let $a = 1 - t - t^{-1}$. Then

$$(1 - t - t^{-1})(1 - t^2 - t^3) = 1 - t - t^{-1} - t^2 + t^3 + t - t^3 + t^{-1} + t^2 = 1.$$

Hence a is a unit in $\mathbb{Z}[\mathbb{Z}_5]$. Define $\alpha : \mathbb{Z}[\mathbb{Z}_5] \to \mathbb{C}$ by sending t to $e^{2\pi i/5}$. Then α sends $\{\pm g \mid g \in \mathbb{Z}_5\}$ to the roots of unity in $\mathbb{C}$. Hence $x \mapsto |\alpha(x)|$ defines a homomorphism from $Wh(\mathbb{Z}_5)$ into R^*_+, the nonzero positive real numbers. Next note the following.

$$|\alpha(a)| = |1 - e^{2\pi i/5} - e^{-2\pi i/5}| = |1 - 2cos\frac{2\pi}{5}| \approx 0.4.$$

This proves that α defines an element of infinite order in $Wh(\mathbb{Z}_5)$. □

For Whitehead groups of general finite groups see [13]. We have already seen that the Whitehead group of the trivial group is trivial. Next simplest question is what is $Wh(C)$, where C is the infinite cyclic group? Well, this has already been computed by G. Higman in 1940 even before Whitehead group was defined.

Theorem 4.0.1 ([7]). *The Whitehead group of the infinite cyclic group is trivial.*

The proof is also given in [4]. One can ask for generalization of this result in two possible directions; namely, what is $Wh(F_r)$ and $Wh(C^n)$? Here F_r is the non-abelian free group on r generators and C^n is the abelian free group on n generators. This question has been answered completely.

Theorem 4.0.2 ([17]). *Let G_1 and G_2 be two groups and let $G_1 * G_2$ denotes their free product. Then $Wh(G_1 * G_2)$ is isomorphic to $Wh(G_1) \oplus Wh(G_2)$ and $\tilde{K}_0(G_1 * G_2)$ is isomorphic to $\tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)$.*

Using Higman's result we get the following important corollary.

Corollary 4.0.1. $Wh(F_r) = 0$.

Theorem 4.0.3 ([1]). $Wh(C^n) = 0$.

We have already mentioned before the following result.

Theorem 4.0.4 ([21], [5], [8]). *Let M be one of the following spaces: a compact Haken 3-manifold, a complete nonpositively curved Riemannian manifold or a finite $CAT(0)$-complex. Then $Wh(\pi_1(M)) = \tilde{K}_0(\mathbb{Z}[\pi_1(M)]) = 0$.*

An important result relating Whitehead group and the reduced projective class group is the Bass-Heller-Swan formula.

Theorem 4.0.5 ([1]). *Let G be a finitely generated group. Then*

$$Wh(G \times C) \simeq Wh(G) \oplus \tilde{K}_0(\mathbb{Z}[G]) \oplus N \oplus N,$$

where N is some Nil-group.

Theorem 4.0.3 is, in fact, a corollary of the above formula.

As we are mostly interested in showing the vanishing of the Whitehead group we do not recall the definition of the group N. For details see [15].

The following two corollaries are now easily deduced.

Corollary 4.0.2. $\tilde{K}_0(\mathbb{Z}[C^n]) = 0$.

Corollary 4.0.3. $\tilde{K}_0(\mathbb{Z}[F_r]) = 0$.

It is also known that $\tilde{K}_0(\mathbb{Z}[G])$ of any finite group G is finite. See [18].

The surgery L-groups involves extremely complicated algebra. We recall here only the computation for the non-abelian free groups, which also gives the computations for the trivial group and the infinite cyclic group.

Theorem 4.0.6 ([3]). *Let F_m be a free group on m generators. Then $L_n(F_m) = \mathbb{Z}, \mathbb{Z}^m, \mathbb{Z}_2, \mathbb{Z}_2^m$ for $n = 4k, 4k+1, 4k+2, 4k+3$ respectively.*

5. Open problems

Finally, to end this article we state the well known conjectures we already mentioned in this article. These conjectures are still open.

Conjecture 1. (W. C. Hsiang) *$\tilde{K}_0(\mathbb{Z}[G])$ of any torsion free group G vanishes.*

Conjecture 2. (J. H. C. Whitehead) *The Whitehead group of any torsion free group vanishes.*

Note that, using Bass-Heller-Swan formula it follows that Conjecture 2 implies Conjecture 1.

Conjecture 3. (A. Borel) *Two aspherical closed manifolds are homeomorphic if their fundamental groups are isomorphic.*

We should mention here that the above conjectures are already known for a large class of groups. We will discuss this in a future article. To explain these groups we need to describe many other concepts which have not been covered here.

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S. K. Roushon
School of Mathematics, Tata Institute of Fundamental Research
Homi Bhabha Road, Mumbai-400005, Maharshtra, India
E-mail *roushon@math.tifr.res.in* ;
URL: `http://www.math.tifr.res.in/~ roushon/`