NOTES ON FIXED POINT AND DUALITY THEOREMS IN ALGEBRAIC TOPOLOGY

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Abstract. These are the notes of lectures for the **Advanced Instructional School in Algebraic Topology**, June 13 to July 02, 2011, North-Eastern Hill University, Shillong, India.

1. Introduction

In these lectures we plan to study some fundamental duality theorems and a fixed point theorem in Algebraic Topology and their applications. Mainly we will follow the book of Hatcher ([1]).

We will study some Algebraic Topological properties which are determined by the local properties of a certain class of spaces, called *manifolds*. We will need to consider mainly compact spaces, but proofs of results will force us to consider noncompact spaces as well. The *duality* theorems will show a strong symmetry and relations in homology and cohomology groups under certain (orientabilty) hypothesis. This will help us answer some questions in Topology which are otherwise difficult to see or prove. Furthermore, a fixed point theorem will be proved which is more general than the Brouwer fixed point theorem.

2. Manifolds and Orientation: Definitions, (Local) Homological Properties

A *manifold* is a Hausdorff topological space which locally looks like the Euclidean space. That is, every point of the topological space has a neighbourhood homeomorphic to some Euclidean space $\mathbb{R}^n$. Obviously any open set of $\mathbb{R}^n$ is a manifold. The standard example of an n-dimensional manifold is the n-sphere:

$$\mathbb{S}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

This can be proved using stereographic projection. Also the product of two manifolds is again a manifold with respect to the product topology. Furthermore, using the definition of a covering space it follows that for a covering space $\tilde{X} \to X$, $\tilde{X}$ is a manifold if and only if

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X is a manifold. More generally two locally homeomorphic spaces are manifolds if one of them is.

The very first homological property of a manifold is local. Let X be an n-dimensional manifold and $x \in X$. Then

$$H_i(X, X - \{x\}; \mathbb{Z}) \simeq H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z})$$
$$\simeq \tilde{H}_{i-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}) \simeq \tilde{H}_{i-1}(\mathbb{S}^{n-1}; \mathbb{Z}).$$

The first isomorphism is by excision, the second one follows from the long homology exact sequence for the pair $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ and noting that $\mathbb{R}^n$ is contractible. Therefore, we get the following. For $n \geq 1$ $\tilde{H}_i(X, X - \{x\}; \mathbb{Z}) \simeq \mathbb{Z}$ for $i = n$ and zero otherwise. Hence the local homology of a manifold determines the dimension of the manifold if it is connected.

The most crucial hypothesis we will need is called *orientability* of a manifold. The definition is in terms of local homology which satisfy some consistency condition. Intuitively in dimension 1 an orientation is a process to fix a direction locally which continuously follow along the space. In dimension 2 it is to decide an anticlockwise or clockwise direction which amounts to the choice of a generator of $H_2(X, X - \{x\}; \mathbb{Z}) \simeq H_1(\mathbb{S}^1; \mathbb{Z}) \simeq \mathbb{Z})$. Here observe that this $\mathbb{S}^1$ is nothing but a small circle around the point x and the orientation at x is just a choice of a direction on this circle. In case of $\mathbb{R}^n$ we see that this property is invariant under rotation and gets reversed under reflection along a line. Since a rotation of $\mathbb{S}^n$ is of degree one and hence preserves the choice of the generator of the local homology and on the other hand a reflection sends a generator of the local homology to its negative. Furthermore, a choice of a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ determines a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$ at any other point $y \in \mathbb{R}^n$. This can be deduced by the following. Let B be a ball containing both the points x and y. Then there is the following canonical isomorphisms.

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z}) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n - B)$$
$$\simeq H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\}).$$

This idea will help us to define an orientable manifold in general.

Definition 2.1. Let M be an n-dimensional manifold. Then an *orientation* at a point $x \in M$ is a choice of a generator of $H_n(M, M - \{x\}; \mathbb{Z}) \simeq \mathbb{Z}$. An *orientation* for M is a function $x \mapsto \mu_x$ for all $x \in M$ which satisfies the following consistency condition. Each $x \in M$ has a neighbourhood $\mathbb{R}^n \subset M$ containing a ball B centered at x such that μ_y for each $y \in B$ is the image of one generator μ_B

of $H_n(M, M - B) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n - B) \simeq \mathbb{Z}$ under the natural map $H_n(M, M - B) \simeq H_n(M, M - \{y\})$. If a manifold has an orientation then it is called *orientable.*

Let us now do a general construction of orientable manifold. We show that every manifold has a two-sheeted orientable covering.

Let $\tilde{M} = \{\mu_x \mid x \in M$ and μ_x is a generator of $H_n(M, M - \{x\})\}$.

The map $\mu_x \mapsto x$ is a two-to-one surjective map. We now topologize $\tilde{M}$ to make this map a covering space projection. Let $B \subset \mathbb{R}^n \subset M$ be an open ball in M and μ_B is a generator of $H_n(M, M - B)$ (which is isomorphic to $\mathbb{Z}$). Then define $U(\mu_B) = \{\mu_x \in \tilde{M} \mid x \in B$ and μ_x is the image of μ_B under $H_n(M, M-B) \to H_n(M, M-\{x\})\}$. The sets $U(\mu_B)$ form a base for a topology on $\tilde{M}$ which makes the projection a covering map.

We leave it as an exercise that $\tilde{M}$ is an orientable manifold.

Before we give some concrete examples let us observe the following.

Proposition 2.1. *Let M be connected. Then M is orientable if and only if $\tilde{M}$ has two components.*

Proof. Since M is connected $\tilde{M}$ either has one or two components. If $\tilde{M}$ has two components then each component projects homeomorphically onto M. Since $\tilde{M}$ is orientable, each of its components are also orientable and therefore M is orientable being homeomorphic to an orientable manifold.

Conversely if M is orientable, it has exactly two orientations since M is connected. Each if these orientations defines a component of $\tilde{M}$. □

$\tilde{M}$ is called the *orientation cover* of M.

For example the cylinder is the orientation cover of the Möbius band and the 2-torus is the orientation cover of the Klein Bottle. Since both the cylinder and the 2-torus are connected, the Möbius band and the Klein Bottle are non-orientable.

An interesting consequence of the above proposition is that if M is connected and $\pi_1(M)$ does not contain any index 2 subgroup (for example if M is simply connected) then M is orientable. Thus any manifold with finite odd order fundamental group is orientable.

Now we proceed to define another useful cover of M.

$$M_{\mathbb{Z}} = \{\mu_x \mid x \in M, \mu_x \in H_n(M, M - \{x\})\}.$$

We topologize $M_{\mathbb{Z}}$ in the same way as $\tilde{M}$ so that $M_{\mathbb{Z}} \to M$ becomes a $\mathbb{Z}$-sheeted covering map. Let $B \subset \mathbb{R}^n \subset M$ be an open ball in M and $\mu_B \in H_n(M, M - B)$. Then define $U(\mu_B) = \{\mu_x \in M_{\mathbb{Z}} \mid x \in B$ and μ_x is the image of μ_B under $H_n(M, M-B) \to H_n(M, M-\{x\})\}$.

The sets $U(\mu_B)$ form a base for a topology on $M_{\mathbb{Z}}$ which makes the projection a covering map.

Note that the topology of $M_{\mathbb{Z}}$ is so defined that a continuous section $\alpha : M \to M_{\mathbb{Z}}$ such that $\alpha(x)$ is a generator of $H_n(M, M - \{x\})$ for all $x \in M$ defines an orientation of M.

We can also define orientability of manifolds when we take the homology groups with coefficients in a more general commutative ring with unity R. An element $u \in R$ is called a generator of R if $Ru = R$. That is if u is an unit in R.

Now we say a manifold M is R-orientable if the same conditions are satisfied as in $\mathbb{Z}$-orientabilty but with coefficients in R and demanding that each μ_x is a generator of R. Similarly we can also define the R-sheeted covering M_R of M. Finally we say M is *R-orientable* if there is a section of $M_R \to R$ taking values in units in the ring R.

We are now in a position to deduce some important homological properties of manifolds related to orientability.

A manifold is called *closed* if it is compact.

Theorem 2.1. *Let R be a commutative ring with unity. Let M be a closed connected manifold.*

(a) If M is R-orientable then the map $H_n(M; R) \to H_n(M, M - \{x\}; R)$ is an isomorphism for all $x \in M$.

(b) If M is not R-orientable the map $H_n(M; R) \to H_n(M, M - \{x\}; R)$ is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.

(c) $H_i(M; R) = 0$ for all $i \geq n + 1$.

Remark 2.1. Note that (a) implies that there exists a class $[M] \in H_n(M; R)$ whose images $[M]_x$ in $H_n(M, M - \{x\}; R)$ for $x \in M$ define an orientation in M. $[M]$ is called the *fundamental class* for this orientation on M.

Conversely, if there exists such a class $[M] \in H_n(M; R)$ so that $[M]_x$ is a generator of $H_n(M, M - \{x\}; R)$ for all $x \in M$ then this defines an orientation on M. Since if $B \subset \mathbb{R}^n \subset M$ is a ball then for all $x \in B$ the map $H_n(M; R) \mapsto H_n(M; M - \{x\}; R)$ factors through $H_n(M, M - B; R)$ which implies that the map $x \mapsto [M]_x$ is a continuous section of $M_R \to M$ with images generators of R. Furthermore, this implies that M is compact. Since $[M]_x$ will be zero for x not in the image of a cycle representing $[M]$ but on the other hand this image is compact. $[M]$ is also called an *orientation class* of M.

The Theorem follows from the following Lemma. This Lemma will be used throughout our course of lectures.

Lemma 2.1. *Let M be a connected manifold of dimension n and $A \subset M$ be a compact subset. Then the following hold.*

(a) $H_i(M, M-A; R) = 0$ for $i \geq n+1$ and a class in $H_n(M, M-A; R)$ is zero if and only if its image in $H_n(M, M-\{x\}; R)$ is zero for all $x \in A$.

(b) If $x \mapsto \alpha_x$ is a section of the covering map $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M, M-A; R)$ whose image in $H_n(M, M-\{x\}; R)$ is α_x for all $x \in A$.

Proof of Theorem 2.1. Since M is compact we apply Lemma 2.1 for the case $A = M$. Then (c) of Theorem is immediate from (a) of the Lemma.

Let $\Gamma_R(M)$ be the set of sections of $M_R \to M$. Then $\Gamma_R(M)$ is an abelian group (by pointwise addition) and a module over R. Define $H_n(M; R) \to \Gamma_R(M)$ by $\alpha \mapsto \{\alpha_x\}$ where $\{\alpha_x\}$ is the section which sends x to α_x. (a) and $b)$ of the Lemma imply that this homomorphism is an isomorphism.

Next we note the following observation.

Any element of $\Gamma_R(M)$ is uniquely determined by its value at one point in M. This can be checked by using the continuity of the section.

Proof of (a) is immediate using (a) and (b) of the Lemma and the above observation.

Proof of (b): the injectivity parts follows from (a) of the Lemma. The remaining part follows from the structure of the covering projection $M_R \to M$ as we describe below.

Note that there is a canonical isomorphism $H_n(M, M-\{x\}; R) \simeq H_n(M, M-\{x\}; \mathbb{Z}) \otimes R$. Each $r \in R$ determines a subcovering space M_r of M_R consisting of the points $\pm\mu_x \otimes x \in H_n(M, M-\{x\}; R)$ for μ_x a generator of $H_n(M, M-\{x\}; \mathbb{Z})$. If r has order 2 in R then $r = -r$ so M_r is just a copy of M, and otherwise M_r is isomorphic to the two-sheeted cover $\tilde{M}$. M_R is the union of these M_r's which are disjoint except for the equality $M_r = M_{-r}$. □

Proof of Lemma 2.1. The proof is in several steps.

(1) If the Lemma is true for compact subsets A, B and $A \cap B$ then it is true for their union $A \cup B$.

Let us first write down a Mayer-Vietoris long exact sequence:

$$\cdots \longrightarrow H_{n+1}(M, M-A\cap B) \longrightarrow H_n(M, M-A\cup B) \xrightarrow{\phi}$$

$$H_n(M, M-A) \oplus H_n(M, M-B) \xrightarrow{\psi} H_n(M, M-A\cap B)$$

Where $\phi(\alpha) = (\alpha, -\alpha)$ and $\psi(\alpha, \beta) = \alpha + \beta$.

Plugging the hypothesis in the above exact sequence we get that $H_i(M, M - A \cup B) = 0$ for $i \geq n+1$. Furthermore, the above exact sequence reduces to the following.

$$0 \longrightarrow H_n(M, M - A \cup B) \stackrel{\phi}{\longrightarrow} H_n(M, M - A) \oplus H_n(M, M - B)$$
$$\stackrel{\psi}{\longrightarrow} H_n(M, M - A \cap B)$$

Let $\alpha \in H_n(M, M - A \cup B)$ such that $\alpha_x = 0$ in $H_n(M, M - \{x\})$ for all $x \in A \cup B$. Since the map $H_n(M, M - A \cup B) \to H_n(M, M - \{x\})$ factors through $H_n(M, M - A)$ we get $(\alpha_A)_x = 0$ for all $x \in A$. So by hypothesis $\alpha_A = 0$ and similarly $\alpha_B = 0$. Since ϕ is injective $\alpha = 0$. This proves (a).

Proof of (b) for $A \cup B$. Let $x \mapsto \alpha_x$ be a section of $M_R \to M$. By hypothesis there exists $\alpha_A \in H_n(M, M - A)$ and $\alpha_B \in H_n(M, M - B)$ such that $(\alpha_A)_x = \alpha_x$ for all $x \in A$ and $(\alpha_B)_x = \alpha_x$ for all $x \in B$. Now note that $\psi(\alpha_A)$ and $\psi(\alpha_B)$ have the defining properties of $\alpha_{A \cap B}$ and hence by uniqueness $\psi(\alpha_A) = \psi(\alpha_B) = \alpha_{A \cap B}$. Therefore $\psi(\alpha_A, -\alpha_B) = 0$. By exactness of the sequence there is $\alpha_{A \cup B} \in H_n(M, M - A \cup B)$ such that $\phi(\alpha_{A \cup B}) = (\alpha_A, -\alpha_B)$. This $\alpha_{A \cup B}$ maps to α_A and α_B, so $\alpha_{A \cup B}$ has images α_x for all $x \in A \cup B$ since α_A and α_B have also the same property. This proves (b).

(2) Reduction to $M = \mathbb{R}^n$. Let $A \subset M$ be compact. Then there exist open sets $U_1, U_2, \ldots, U_m$ and compact sets $A_i \subset U_i$ for $i = 1, 2, \ldots, m$ such that U_i is homeomorphic to $\mathbb{R}^n$ and $A = \cup_i A_i$. We apply (1) to $A_1 \cup A_2 \cup \ldots \cup A_{m-1}$ and A_m and by induction we see that we need to prove the Lemma for compact set $A \subset \mathbb{R}^n \subset M$. By excision we are reduced to the case $M = \mathbb{R}^n$.

(3) $M = \mathbb{R}^n$ and A is a finite subcomplex of some triangulation of $\mathbb{R}^n$.

If A has only one simplex then the Lemma is obvious. Thus we can use an induction on the number of simplices and apply (1) to complete the proof in this case.

(4) Let A be an arbitrary compact subset of $\mathbb{R}^n$. Let $\alpha \in H_i(\mathbb{R}^n, \mathbb{R}^n - A)$ be represented by the cycle z and C be the image of the singular simplices in ∂z. Then C is compact. Let $\delta = \text{dist}(A, C) > 0$. Let Δ^n be a large simplex in $\mathbb{R}^n$ such that $A \subset \Delta^n$. Subdivide Δ^n sufficient number of times so that diameter of each simplex is $< \delta$. Let $K = \{\sigma \mid \sigma \cap A \neq \emptyset \text{ where } \sigma \subset \Delta^n \text{ is a simplex}\}$.

Then K is a simplicial complex disjoint from C. Also K is finite. Clearly z defines a cycle and hence an element $\alpha_K \in H_i(\mathbb{R}^n, \mathbb{R}^n - K)$ which goes to α in $H_i(\mathbb{R}^n, \mathbb{R}^n - A)$. If $i > n$ then by (3) $H_i(\mathbb{R}^n, \mathbb{R}^n - K) = 0$ and hence $\alpha = 0$. Therefore $H_i(\mathbb{R}^n, \mathbb{R}^n - A) = 0$.

If $i = n$ and α_x is zero in $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ for all $x \in A$ then $\alpha_x = 0$ for all $x \in K$. Therefore by (3) $\alpha_K = 0$ and hence $\alpha = 0$. This proves (a).

To check (b) choose a large ball B containing A. Then the isomorphisms $H_n(\mathbb{R}^n, \mathbb{R}^n - B) \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ factors through $H_n(\mathbb{R}^n, \mathbb{R}^n - A)$. (b) now easily follows using continuity of a section.

Finally, the uniqueness in (b) follows from (a). □

We have the following easy consequence.

Corollary 2.1. *Let M be a closed manifold. Then $H_n(M; \mathbb{Z}_2) = \mathbb{Z}_2$. And $H_n(M; \mathbb{Z}) = 0$ or $\mathbb{Z}$ if and only if M is non-orientable or orientable respectively.*

We can deduce some more as can be seen in the following.

Corollary 2.2. *Let M be a closed connected n-manifold. Then the torsion subgroup of $H_{n-1}(M; \mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}_2$ if M is non-orientable.*

Proof. This is an application of the universal coefficient theorem for homology.

$$H_n(M; G) \simeq (H_n(M; \mathbb{Z}) \otimes G) \oplus Tor(H_{n-1}(M; \mathbb{Z}), G).$$

Let M be orientable. If possible assume that $H_{n-1}(M, \mathbb{Z})$ contains some torsion. Then clearly for some prime p, $H_n(M; \mathbb{Z}_p)$ would be larger than the $\mathbb{Z}_p$ coming from $H_n(M; \mathbb{Z})$. In the non-orientable case, $H_n(M; \mathbb{Z}_m)$ is either $\mathbb{Z}_2$ or 0 depending on whether m is odd or even. Therefore the torsion part of $H_{n-1}(M; \mathbb{Z})$ must be $\mathbb{Z}_2$. □

Another application of the lemma is the following homological result for noncompact manifolds.

Proposition 2.2. *If M is a connected noncompact manifold of dimension n, then $H_i(M; R) = 0$ for $i \geq n$.*

Proof. Let $[z] \in H_i(M; R)$ be an element represented by a cycle z. Note that the image of z in M is compact. Hence we can find an open set U in M containing this image so that $\bar{U}$ is compact. Let $V = M - \bar{U}$. Consider the following part of the long exact homology sequence of the triple $(M, U \cup V, V)$.

$$\begin{array}{ccccc}
H_{i+1}(M, U \cup V; R) & \longrightarrow & H_i(U \cup V, V; R) & \longrightarrow & H_i(M, V; R) \\
 & & \approx \uparrow & & \uparrow \\
 & & H_i(U; R) & \longrightarrow & H_i(M; R)
\end{array}$$

By Lemma 2.1 for $i > n$ $H_{i+1}(M, U \cup V; R) = H_i(M, V; R) = 0$ since both V and $U \cup V$ are complement of compact sets.

Therefore $H_i(U; R) = 0$. Thus z is a boundary in U and hence in M. Hence $H_i(M; R) = 0$ for $i > n$.

$i = n$ case. Since M is connected the image $[z]_x$ of the class $[z]$ in $H_n(M, M - \{x\}; R)$ must either be zero for all $x \in M$ or nonzero for all x. Since M is noncompact and $[z]_x$ is always zero for all x not in the image of z and the image of z is compact. Thus by Lemma 2.1 z represents zero in $H_n(M, V; R)$ and hence in $H_i(U; R)$ also since $H_{n+1}(M, U \cup V; R) = 0$ by Lemma 2.1 again. So $[z] = 0$ in $H_n(M; R)$ and since z was an arbitrary choice we get $H_n(M; R) = 0$. □

3. Cap Product, Statements of Poincaré duality for Compact and Non-compact Manifolds

If M is a closed R-orientable manifold of dimension n the Poincaré duality theorem says that there is an isomorphism

$$H^i(M; R) \to H_{n-i}(M; R)$$

for each i. In this section we set up the methods to define the above homomorphism and to prove a crucial lemma. The proof will require the consideration of noncompact manifolds in some inductive steps and define a similar map for noncompact manifold. But in the noncompact case the ordinary cohomology will not work. Instead we need to define something called *cohomology with compact support.*

3.1. **Cap product.** Let X be an arbitrary space and R be a commutative ring with unity. Let $k \geq l$ be nonnegative integers. Then we define the *cap product*

$$\frown : C_k(X; R) \times C^l(X; R) \to C_{k-l}(X; R)$$

in the following way.

For $\sigma : \Delta^k \to X$ a singular k-simplex and a cochain $\phi : C^l(X; R) = hom(C_l(X; R), R)$

$$\sigma \frown \phi = \phi(\sigma|_{[v_0,\ldots,v_l]})\sigma|_{[v_l,\ldots,v_k]}.$$

We now check the following formula.

$$\partial(\sigma \frown \phi) = (-1)^l(\partial\sigma \frown \phi - \sigma \frown \delta\phi).$$

$$\partial\sigma \frown \phi = \Sigma_{i=0}^l (-1)^i \phi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{l+1}]})\sigma|_{[v_{l+1},\ldots,v_k]}$$
$$+\Sigma_{i=l+1}^k (-1)^i \phi(\sigma|_{[v_0,\ldots,v_l]})\sigma|_{[v_l,\ldots,\hat{v_i},\ldots,v_k]}.$$

$$\sigma \frown \delta\phi = \Sigma_{i=0}^{l}(-1)^i \phi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{l+1}]})\sigma|_{[v_{l+1},\ldots,v_k]}.$$

On the other hand we have

$$\partial(\sigma \frown \phi) = \Sigma_{i=l}^{k}(-1)^{i-l} \phi(\sigma|_{[v_0,\ldots,v_l]})\sigma|_{[v_l,\ldots,\hat{v_i},\ldots,v_k]}.$$

This proves the assertion

$$\partial(\sigma \frown \phi) = (-1)^l(\partial\sigma \frown \phi - \sigma \frown \delta\phi).$$

Thus if σ is a cycle and ϕ is a cocycle then clearly $\sigma \frown \phi$ is a cycle. Furthermore the following are satisfied.

(1) $\partial\sigma = 0$ implies $\sigma \frown \phi$ is a boundary if ϕ is a coboundary.

(2) $\delta\phi = 0$ implies $\sigma \frown \phi$ is a boundary if σ is a boundary.

These above facts imply that there is an induced cap product map

$$\frown: H_k(X;R) \times H^l(X;R) \to H_{k-l}(X;R)$$

which is bilinear.

There are similar cap product map in relative situation as follows.

$$\frown: H_k(X,A;R) \times H^l(X;R) \to H_{k-l}(X,A;R)$$
$$\frown: H_k(X,A;R) \times H^l(X,A;R) \to H_{k-l}(X;R)$$
$$\frown: H_k(X,A \cup B;R) \times H^l(X,A;R) \to H_{k-l}(X,B;R)$$

There is also a naturality property satisfied by the cap product. Let $f : X \to Y$ be a continuous function, $\alpha \in H_k(X)$ and $\phi \in H^l(Y)$. Consider the following diagram.

$$\begin{array}{ccccc} H_k(X) & \times & H^l(X) & \xrightarrow{\frown} & H_{k-l}(X) \\ \downarrow f_* & & \uparrow f^* & & \downarrow f_* \\ H_k(Y) & \times & H^l(Y) & \xrightarrow{\frown} & H_{k-l}(Y) \end{array}$$

Then the naturality of the cap product says the following.

$$f_*(\alpha) \frown \phi = f_*(\alpha \frown f^*(\phi))$$

The above can be checked by replacing $f\sigma$ for σ in the definition of cap product.

Now we are in a position to define the homomorphism in the Poincaré duality theorem.

Let M be a closed R-oriented n-dimensional manifold and $[M] \in H_n(M;R)$ be the fundamental class of the orientation. Then the cap product with $[M]$ defines a map

$$D_M : H^k(M;R) \to H_{n-k}(M;R).$$

That is $D_M(\phi) = [M] \frown \phi$ for $\phi \in H^k(M;R)$.

Then the theorem says that D_M is an isomorphism for all k.

Theorem 3.1 (Poincaré duality theorem)**.** *Let M be a closed connected R-orientable n-dimensional manifold with fundamental class $[M]$. Then the map $D_M : H^k(M;R) \to H_{n-k}(M;R)$ is an isomorphism for all k.*

Example For a closed orientable manifold with a triangulation the fundamental class will look like $\Sigma \pm \sigma_i$ where σ_i's are top dimensional simplices in the triangulation of the manifold. This can be checked using Lemma 2.1. That is the two facts that the map $H_n(M : R) \to H_n(M, M - \{x\}; R)$ is an isomorphism and the fundamental class goes to a generator of the local homology at x.

The proof of the duality theorem is again in the same spirit of the proof of Lemma 2.1. The argument is by induction and we need to consider non-compact manifold in the process. We have to state a duality theorem for non-compact manifolds. For which we need a different cohomology group.

3.2. **Cohomology with compact support.** To motivate the situation we first consider the simplicial case. Let X be a locally finite simplicial complex and G be an abelian group. Consider the simplicial cochain group $C^i(X;G) = \mathit{hom}(C_i(X;G), G)$ and define a subgroup $C^i_c(X;G)$ consisting of cochains which take nonzero value only on finitely many simplices. It is then easy to see that the coboundary map restricts to these cochain groups. Since by the locally finiteness condition an n-simplex is intersected by only finitely many simplices of $(n+1)$-dimension. Therefore we get a subcomplex $C^*_c(X;G)$ of $C^*(X;G)$ called the *compactly supported cochain complex* and cohomology of this complex is called the *compactly supported cohomology groups* of X.

Example. Equip $\mathbb{R}$ with the simplicial structure where the integer points are zero simplices. Then show that $H^i_c(\mathbb{R};G) = 0$ for $i \neq 1$ and G for $i = 1$.

Let us now see the singular situation. Let X be an arbitrary topological space. Define

$$C^i_c(X;G) = \{\phi \in C^i(X;G) \mid \text{ there exists a compact set } K_\phi \text{ such that } \phi \text{ takes zero value on any chain in } X - K_\phi\}$$

Then again $C^*_c(X;G)$ forms a subcomplex of $C^*(X;G)$ and its' cohomology groups are called the *compactly supported cohomology groups* of X.

Remark 3.1. Here we remark that in the above definition if we demand that ϕ should vanish only on simplices in $X - K_\phi$ then the resulting groups will not form a subcomplex because the coboundary map will not be defined.

For our purpose we need an alternate way to look at these groups. Let $K \subset X$ be a compact subset. Consider the relative cochain subcomplex $C^*(X, X-K; G) \subset C^*(X; G)$. For compact subsets $K \subset L$ we get homomorphism $C^i(X, X-K; G) \to C^i(X, X-L; G)$ which induces homomorphism in cohomology $H^i(X, X-K; G) \to H^i(X, X-L; G)$. Also the set of all compact subsets of X forms a directed set under the partial ordering induced by inclusion of sets. Therefore we can take the direct limit of the collection $\{H^i(X, X-K; G)\}_K$ and then it is easy to show that it is isomorphic to the compactly supported cohomology we defined above.

Therefore we can think of compactly supported cohomology as the direct limit of local cohomology supported at compact sets.

Example. It is easy to show that $H^i_c(\mathbb{R}^n; G) = 0$ for $i \neq n$ and $= G$ for $i = n$. Just consider the closed balls as the compact sets. Since the closed balls form a cofinal subset of the set of all compact subsets.

Now we are in a position to define duality for noncompact manifolds.

Let M be an R-oriented n-dimensional manifold possibly noncompact. We define a duality map $D_M : H^k_c(M; R) \to H_{n-k}(M; R)$ in the following way.

Let $K \subset L \subset M$ be compact subsets of M. Then we have the following diagram.

$$\begin{array}{ccccc}
H_n(M, M-K) & \times & H^k(M, M-K) & \xrightarrow{\frown} & H_{n-k}(M) \\
\uparrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle =} \\
H_n(M, M-L) & \times & H^k(M, M-L) & \xrightarrow{\frown} & H_{n-k}(M)
\end{array}$$

By Lemma 2.1 there exist unique classes $\mu_K \in H_n(M, M-K)$ and $\mu_L \in H_n(M, M-L)$ restricting to the given orientation at each point of K and L respectively.

By uniqueness we get $i_*(\mu_L) = \mu_K$. By naturality of cap product we get

$$\mu_K \frown \alpha = i_*(\mu_L) \frown \alpha = \mu_L \frown i^*(\alpha)$$

for $\alpha \in H^k(M, M-K)$.

Thus we get $\mu_K \frown \alpha = \mu_L \frown i^*(\alpha)$. This shows that the duality map $H^k(M, M-K) \to H_{n-k}(M)$ factors through $H^k(M, M-L)$. Hence we get a homomorphism in the limit.

$$D_M : H^k_c(M; R) \to H_{n-k}(M; R)$$

Theorem 3.2 (Poincaré duality theorem for noncompact manifold)**.** *Let M be an R-orientable manifold (possibly non-compact) of dimension n. Then D_M is an isomorphism for all k.*

The proof of this general theorem depends on the following lemma.

Lemma 3.1. *Let U and V be two open subsets of M such that $M = U\cup V$, then there is a diagram of Mayer-Vietoris sequences commutative up to sign.*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^k_c(U\cap V) & \longrightarrow & H^k_c(U)\oplus H^k_c(V) & \longrightarrow & H^k_c(M) & \longrightarrow \\
& \big\downarrow{\scriptstyle D_{U\cap V}} & & \big\downarrow{\scriptstyle D_U\oplus - D_V} & & \big\downarrow{\scriptstyle D_M} & \\
\cdots \longrightarrow & H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U)\oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow & H^{k+1}_c(U\cap V) & \longrightarrow \cdots \\
& \big\downarrow{\scriptstyle D_{U\cap V}} & \\
\longrightarrow & H_{n-k-1}(U\cap V) & \longrightarrow \cdots
\end{array}$$

Proof. Let $K\subset U$ and $L\subset V$ be compact subsets. Then we have Mayer-Vietoris sequences fitting in a diagram as below. Now we need to introduce the following notation: $(X, X-A) := (X|A)$.

$$\begin{array}{ccccc}
\cdots \longrightarrow & H^k(M|K\cap L) & \longrightarrow & H^k(M|K)\oplus H^k(M|L) & \longrightarrow \\
& \big\downarrow{\scriptstyle \approx} & & \big\downarrow{\scriptstyle \approx} & \\
& H^k(U\cap V|K\cap L) & & H^k(U|K)\oplus H^k(V|L) & \\
& \big\downarrow{\scriptstyle \mu_{K\cap L}\frown} & & \big\downarrow{\scriptstyle \mu_K\frown\oplus -\mu_L\frown} & \\
\cdots \longrightarrow & H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U)\oplus H_{n-k}(V) & \longrightarrow
\end{array}$$

$$\begin{array}{cccc}
\longrightarrow & H^k(M|K\cup L) & \xrightarrow{\delta} & H^{k+1}(M|K\cap L) \\
& \big\downarrow{\scriptstyle \mu_{K\cup L}\frown} & & \big\downarrow{\scriptstyle \mu_{K\cap L}\frown} \\
\longrightarrow & H_{n-k}(M) & \xrightarrow{\partial} & H_{n-k-1}(U\cap V)
\end{array}$$

Above the isomorphisms $\approx$ are induced by excision. Taking limit varying the compact sets $K\subset U$ and $L\subset V$ and assuming that the above diagram is commutative completes the proof of the lemma. Since any compact subset of $U\cap V$ is contained in a compact set of the form $K\cap L$ where $K\subset U$ and $L\subset V$ are compact. Similar for $U\cup V$. Also since direct limit of exact sequences is exact.

We need to show that the diagram involving K and L above is commutative up to sign. The commutativity of the first and the second square is trivial. We have to check it for the third square only.

Let $A = M - K$ and $B = M - L$. Let us first compute the connecting homomorphisms δ and ∂.

Note that δ is obtained from the following short exact sequence of cochain complexes.

$$0 \to C^*(M, A + B) \to C^*(M, A) \oplus C^*(M, B) \to C^*(M, A \cap B) \to 0$$

where $C^*(M, A + B)$ is the group of cochains on M which vanish on chains in A and chains in B. Let $\phi \in C^*(M, A \cap B)$ be a cocycle. Then $\phi = \phi_A - \phi_B$ for $\phi_A \in C^*(M, A)$ and $\phi_B \in C^*(M, B)$. So $\delta\phi = \delta\phi_A - \delta\phi_B = 0$ which implies $\delta\phi_A = \delta\phi_B$. Therefore, $\delta[\phi] = [\delta\phi_A] = [\delta\phi_B]$. Similarly for $[z] \in H_*(M)$ $\partial[z] = [\partial z_U] = [\partial z_V]$ where $z_U \in C^*(U)$ and $z_V \in C^*(V)$.

Remark 3.2. Here is a correction to the above paragraph. Note that $C * (M, A + B)$ is not the same as $C^*(M, A \cup B)$ but the inclusion $C^*(M, A \cup B) \subset C^*(M, A + B)$ induces isomorphism on cohomology. So the correct interpretation of the connecting homomorphism will be $\delta[\phi] = [\delta\phi_A + \delta\psi]$ for some $\psi \in C^*(M, A + B)$.

Remark 3.3. Also notice that there are two different types of δ (∂), one is the coboundary for the cochain (chain) complex and another is the connecting homomorphism in the Mayer-Vietoris sequence for cohomology (homology).

By barycentric subdivision the class $\mu_{K\cup L}$ can be represented by a chain α of the form $\alpha = \alpha_{U-L} + \alpha_{U\cap V} + \alpha_{V-K}$ of sum of chains in $U - L$, $U \cap V$ and in $V - K$ respectively.

Note that under the map $H_n(M|K \cup L) \xrightarrow{i_*} H_n(M|K \cap L)$ the class $\mu_{K\cup L}$ goes to $i_*([\alpha_{U\cap V}])$ since α_{U-L} and α_{V-K} both lies outside $K \cap L$. Hence by uniqueness of $\mu_{K\cap L}$ it is represented by $\alpha_{U\cap V}$. Similarly $\alpha_{U-L} + \alpha_{U\cap V}$ represents μ_K.

Let us now write the third square in the following form where we identified $H^{k+1}(M|K \cap L)$ with $H^{k+1}(U \cap V|K \cap L)$ using excision.

$$\begin{array}{ccc} H^k(M|K \cup L) & \xrightarrow{\delta} & H^{k+1}(U \cap V|K \cap L) \\ \downarrow{\scriptstyle \mu_{K\cup L}\frown} & & \downarrow{\scriptstyle \mu_{K\cap L}\frown} \\ H_{n-k}(M) & \xrightarrow{\partial} & H_{n-k-1}(U \cap V) \end{array}$$

Let ϕ be a cocycle representing an element in $H^k(M|K \cup L)$. Then $\delta\phi = \delta\phi_A$ and hence ϕ goes to $\alpha \frown \delta\phi_A = \alpha_{U\cap V} \frown \delta\phi_A = \partial\alpha_{U\cap V} \frown \phi_A$. Since $0 = \partial(\alpha_{U\cap V} \frown \phi_A) = (-1)^k(\partial\alpha_{U\cap V} \frown \phi_A - \alpha_{U\cap V} \frown \delta\phi_A)$.

Now going the other way in the above square we get ϕ goes to $\alpha \frown \phi = (\alpha_{U-L} \frown \phi) + (\alpha_{U\cap V} \frown \phi + \alpha_{V-K} \frown \phi)$. Here we have written the above in chains in U and V in separate bracket. This will ease to compute the connecting homomorphism. Therefore we need to calculate the value of $\partial(\phi_{U-L} \frown \phi)$ only.

Note that $\partial(\phi_{U-L} \frown \phi) = (-1)^k \partial\phi_{U-L} \frown \phi$ since $\delta\phi = 0$. Which is equal to $(-1)^k \partial\alpha_{U-L} \frown \phi_A$ since $\partial\phi_{U-L} \frown \phi_B = 0$ for ϕ_B being zero on chains in $B = M - L$. Finally we get $(-1)^{k+1}\partial\alpha_{U\cap V} \frown \phi_A$ since $\partial(\alpha_{U-L} + \alpha_{U\cap V}) \frown \phi_A = 0$ for $\partial(\alpha_{U-L} + \alpha_{U\cap V})$ being a chain in $U - K$ and $\alpha_{U-L} + \alpha_{U\cap V}$ represents μ_K and ϕ_A vanishes in $A = M - K$.

Thus the diagram commutes up to sign. □

4. Proof of the Poincaré duality Theorem

Proof of the Poincaré duality theorem. We need to first deduce two inductive steps: finite and infinite.

(A) Let $M = U \cup V$ where U and V are open sets of M. Assume that D_U, D_V and $D_{U\cap V}$ are isomorphism. Then by Lemma 3.1 D_M is an isomorphism.

(B) Let M be the union of open sets $U_1 \subset U_2 \subset \cdots$ such that the duality maps D_{U_i} are isomorphisms, then so is D_M since the following point are satisfied.

• $\lim_{K\subset U_i} H^k(M, M - K) \simeq^{ex} \lim_{K\subset U_i} H^k(U_i, U_i - K) \simeq H^k_c(U_i)$.

• Since there are more compact sets in U_{i+1} than in U_i we have a natural map $\lim_{K\subset U_i} H^k(M, M - K) \to \lim_{K\subset U_{i+1}} H^k(M, M - K)$. Thus we have maps $H^k_c(U_i) \to H^k_c(U_{i+1})$.

• Notice that any compact set in M is contained in some U_i. Hence $\lim_i H^k_c(U_i) = H^k_c(M)$.

• Furthermore $H_{n-k}(M) = \lim_i H_{n-k}(U_i)$.

The map D_M is then the limit of the isomorphisms D_{U_i} and hence is an isomorphism.

Now we complete the proof of the theorem in three steps.

(1) Let $M = \mathbb{R}^n$. Regard M as the interior of the n-simplex Δ^n. Then the duality map can be identified with the map $H^k(\Delta^n, \partial\Delta^n)(\simeq H^k_c(\Delta^n)) \to H_{n-k}(\Delta^n)$. Where (fundamental class) $[\Delta^n] \in H_n(\Delta^n, \partial\Delta^n)$ is the homology class of the identity map $\Delta^n \to \Delta^n$ and the above map is the cap product with $[\Delta^n]$.

The only nontrivial map is when $k = n$. A generator of $H^n(\Delta^n, \partial\Delta^n)$ is represented by a cocycle ϕ which takes the value 1 on $[\Delta^n]$.

It is easy to check that $[\Delta^n] \frown [\phi]$ is a generator of $H_0(\Delta^n)$.

(2) Let M be an arbitrary open set of $\mathbb{R}^n$. Let us write M as a union of convex open subsets U_i (for example open balls). Define $V_i = \cup_{j<i} U_j$. Note that D_{U_i} are isomorphism for all i by (1). Also $U_i \cap V_i$ and V_i both are intersections of $(n-1)$ convex subsets. By induction on the convex open sets in the cover, D_{U_i} and $D_{U_i\cap V_i}$ are isomorphisms. Now

since $V_{i+1} = U_i \cup V_i$ we see that $D_{V_{i+1}}$ is an isomorphism. By (B) D_M is an isomorphism since M is an increasing union of the open sets V_i.

(3) If M is a finite or countable infinite union of open sets U_i homeomorphic to $\mathbb{R}^n$ the two applications of (2) and on the second application replace 'convex open sets' by 'open sets' proves that D_M is an isomorphism.

Thus we have completed the proof of the Poincaré duality theorem for closed manifolds and manifolds which are countable union of open sets homeomorphic to $\mathbb{R}^n$.

For manifolds which cannot be written in the above form we can use Zorn's Lemma to complete the proof.

Let M be an arbitrary non-compact connected manifold of dimension n. Consider the collection

$$\mathcal{A} = \{U \mid U \subset M \text{ open and } D_U \text{ is an isomorphism}\}.$$

Then $\mathcal{A}$ is partially ordered and if $\mathcal{B} \subset \mathcal{A}$ is a totally ordered subset then for $V = \cup\{B \mid B \in \mathcal{B}\}$ D_V is an isomorphism by (B). Hence by Zorn's Lemma there is a maximal element $C \in \mathcal{A}$ for which D_C is an isomorphism. We claim that $M = C$. If not then there is a point $x \in M - C$. Let $B \subset \mathbb{R}^n$ be a ball neighbourhood of x in M. Then D_U is an isomorphism by (1) and $D_{U\cap V}$ is an isomorphism by (2). Therefore using (A) we see that $D_{C\cup U}$ is an isomorphism which is a contradiction to the maximality of $\mathcal{A}$. $\square$

5. Lefschetz and Alexander duality Theorem: Statements and Proofs

Let $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. A Hausdorff topological space M is called a *manifold with boundary* if each $x \in M$ has a neighbourhood homeomorphic either to $\mathbb{R}^n$ or to $\mathbb{R}^n_+$. Let $x \in M$ then $H_n(M, M - \{x\}; R) = 0$ if x corresponds to a point on $x_n = 0$ and it is R if x corresponds to a point on $x_n > 0$. The subspace on M consisting of the points of the former type is called the *boundary* of M and is denoted by ∂M and is characterized by the above homological observation. Also it is easy to show that ∂M is a manifold of dimension $n-1$ without boundary. For example $\mathbb{R}^n_+$ itself is a manifold with boundary. Also $\mathbb{D}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \Sigma_{i=1}^n x_i^2 \leq 1\}$ is a manifold with $\mathbb{S}^{n-1}$ as boundary. More generally a closed manifold minus an open ball is an example of such a manifold.

M is called R-orientable if $M - \partial M$ is R-orientable.

A *collar* neighbourhood of the boundary ∂M of a manifold with boundary M is the image of an embedding $\partial M \times [0, 1) \to M$ such that $\phi(x, 0) = x$ for all $x \in \partial M$.

Proposition 5.1. *The boundary of any compact manifold with boundary has a collar neighbourhood.*

Proof. See the proof in the book of Hatcher. It basically uses the existence of partition of unity for compact manifolds. □

The Lefschetz duality theorem is the relative version of the Poincaré duality for compact manifolds with nonempty boundary. The proof is also immediate from the Poincaré duality theorem for noncompact manifolds.

Theorem 5.1 (Lefschetz duality theorem)**.** *Let M be an R-oriented compact connected manifold with boundary. Then there is an isomorphism*

$$D_M : H^k(M, \partial M; R) \to H_{n-k}(M; R)$$

for all k.

Proof. Let us first define the map D_M. Using the existence of a collar neighbourhood it is easy to deduce that $H_i(M, \partial M; R)$ and $H_i(M - \partial M, \partial M \times (0, \epsilon))$ are isomorphic for all i. Hence by Lemma 2.1 there is a fundamental class $[M] \in H_n(M, \partial M; R)$ which restricts to the orientation in each point of $M - \partial M$. Therefore taking cap product with $[M]$ gives a map $D_M : H^k(M, \partial M; R) \to H_{n-k}(M; R)$. Next, using the existence of collar neighbourhood we see that $H^k(M, \partial M)$ and $H_c^K(M - \partial M; R)$ are isomorphic and similarly $H_{n-k}(M; R)$ and $H_{n-k}(M - \partial M; R)$ are isomorphic. This completes the proof of the theorem. □

Theorem 5.2 (Alexander duality theorem)**.** *Let $K \subset \mathbb{S}^n$ be a compact locally contractible nonempty proper subspace. Then $\tilde{H}_k(\mathbb{S}^n - K; \mathbb{Z}) \simeq \tilde{H}^{n-k-1}(K; \mathbb{Z})$ for all i.*

It is important to note here that the theorem implies that the homology of the complement of K depends only on K as long as it is not too wild a space. However for an arbitrary compact subspace there is Alexander duality but we need to consider Čech cohomology theory.

Proof. Let $k \neq 0$. Then we have the following deductions.

$$\begin{aligned}
\tilde{H}_k(\mathbb{S}^n - K) &= H_k(\mathbb{S}^n - K) \\
&\simeq H_c^{n-k}(\mathbb{S}^n - K)(\text{ Poincaré duality }) \\
&\simeq \lim_{K \subset U} H^{n-k}(\mathbb{S}^n - K, U - K) \\
&\simeq \lim_{K \subset U} H^{n-k}(\mathbb{S}^n, U)(\text{ Excision })
\end{aligned}$$

$$\simeq \lim_{K\subset U} \tilde{H}^{n-k-1}(U) \text{ if } i \neq 0$$
$$\simeq \tilde{H}^{n-k-1}(K)$$

The second isomorphism is by definition of compactly supported cohomology. The fourth isomorphism is obtained from the long exact sequence of cohomology for pairs. The final isomorphism is described below.

Since K is locally contractible it has a neighbourhood U_0 in $\mathbb{S}^n$ of which K is a retract. Therefore in the limit above we can take only those neighbourhood of K which are contained in U_0. These open sets will retract to K by restricting the retraction of U_0. This information can be used to show that $\lim_{K\subset U} H^*(U) \to H^*(K)$ is surjective.

To check injectivity we need to find an open set $V \subset U \subset U_0$ containing K such that the inclusion $V \to U$ is homotopic to the retraction $V \to K$. Let us first prove injectivity of $\lim_{K\subset U} H^*(U) \to H^*(K)$ using this fact. Existence of the above V implies that the inclusion induced map $H^*(U) \to H^*(V)$ factors through $H^*(K)$. Hence if some element of $H^*(U)$ goes to zero in $H^*(K)$ then it goes to zero in $H^*(V)$ also. Thus proving the required injectivity.

Let us now prove the existence of V. Let $U \subset U_0$ and $r : U \to K$ be the retraction. Define $U \times I \to \mathbb{R}^n \subset \mathbb{S}^n$ by the following formula $(u, t) \mapsto u(1 - t) + tr(u)$. This is a homotopy of the identity to the retraction $U \to K$. Also it clearly takes $K \times I$ to K. Using compactness of I we can therefore find an open set $V \subset U$ containing K which has all the required property.

We still have to consider the case of $i = 0$. In this case the fourth isomorphism above does not hold. To get around this problem we use the naturality of the first three isomorphism and map them to the case when $K = U = \emptyset$ and then take the kernel of these maps to get the isomorphism $\tilde{H}_0(\mathbb{S}^n) \simeq \lim_{K\subset U} \tilde{H}^{n-1}(U)$ which is then identified with $\tilde{H}^{n-1}(K)$ as we did above.

This completes the proof of the Alexander duality theorem. □

6. Examples and Applications

In this section we derive some consequences of the duality theorems we proved.

Let M be a closed manifold of dimension n. The very first implication of the Poincaré duality theorem is that $H_i(M; \mathbb{Z}_2) = H^i(M; \mathbb{Z}_2) = 0$ for $i > n$. Now if M is also orientable then further we have got isomorphism $H_k(M; \mathbb{Z}) \to H^{n-k}(M; \mathbb{Z})$ for all k. This implies the above vanishing results with integer coefficients as well. Let us now recall that a closed manifold has finitely generated homology. (see Corollaries A.8 and A.9 in the Appendix in [1]). The universal coefficient theorem says the following.

$$H^k(M;\mathbb{Z}) \simeq hom(H_k(M;\mathbb{Z}),\mathbb{Z}) \oplus Ext(H_{k-1}(M,\mathbb{Z}),\mathbb{Z}).$$

Since homology groups of M are finitely generated using properties of Ext we get the following.

$$Ext(H_{n-1}(M;\mathbb{Z}),\mathbb{Z}) = \text{ torsion part of } H_{n-1}(M;\mathbb{Z}) = T_{n-1}(M;\mathbb{Z})(\text{say}).$$

Furthermore, note that $hom(H_n(M;\mathbb{Z}),\mathbb{Z}) \simeq H_n(M;\mathbb{Z})/T_n(M;\mathbb{Z})$. Hence $H^n(M;\mathbb{Z}) \simeq H_n(M;\mathbb{Z})/T_n(M;\mathbb{Z}) \oplus T_{n-1}(M;\mathbb{Z})$. Now let us assume that M is oriented and apply Poincaré duality and see that

$$H_k(M;\mathbb{Z}) \simeq H^{n-k}(M;\mathbb{Z}) \simeq H_{n-k}(M;\mathbb{Z})/T_{n-k}(M;\mathbb{Z}) \oplus T_{n-k-1}(M;\mathbb{Z}).$$

Therefore we get the following two equalities for closed oriented manifolds:

(1). $H_k(M;\mathbb{Z})/T_k(M;\mathbb{Z}) \simeq H_{n-k}(M;\mathbb{Z})/T_{n-k}(M;\mathbb{Z})$.

(2). $T_k(M;\mathbb{Z}) \simeq T_{n-k-1}(M;\mathbb{Z})$.

Thus the free parts of $H_k(M;\mathbb{Z})$ and $H_{n-k}(M;\mathbb{Z})$ are isomorphic and the torsion parts of $H_k(M;\mathbb{Z})$ and $H_{n-k-1}(M;\mathbb{Z})$ are isomorphic.

Exercise. Check the above two equalities (1) and (2) for the manifold $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n-times) directly using Künneth formula.

We now deduce an interesting application of the Alexander duality theorem. Let K be the Klein's bottle. We show that K cannot be embedded in $\mathbb{R}^3$. Because, if it is so then K can also be embedded in $\mathbb{S}^3$ and then by the Alexander duality theorem we get $\tilde{H}_k(\mathbb{S}^3 - K) \simeq \tilde{H}^{2-k}(K)$. Now, by the universal coefficients theorem we have $H^{2-k}(K) \simeq hom(H_{2-k}(K),\mathbb{Z}) \oplus Ext(H_{1-k}(K),\mathbb{Z})$. Put $k = 0$. Then $\tilde{H}_0(\mathbb{S}^3 - K)$ is free. Hence $\tilde{H}^2(K) = H^2(K)$ is free. Therefore $Ext(H_1(K),\mathbb{Z})$ is trivial. Which implies $H_1(K)$ is free. That is a contradiction as $H_1(K)$ contains $\mathbb{Z}_2$.

For more exercises on these lectures we refer the reader to the corresponding chapters in the book [1]. Some exercises have been accumulated in the section Exercises in this notes.

7. Lefschetz Fixed Point Theorem and its Application

In this lecture we will prove a general version of the Brouwer fixed point theorem. The question is that given a finite simplicial complex X and a continuous map $f : X \to X$ find a condition which will ensure that there is a point $x \in X$ such that $f(x) = x$. A more general question is that given two such spaces X and Y and two continuous maps $f, g : X \to Y$ find a condition which will guarantee that there is a point $x \in X$ such that $f(x) = g(x)$. This last question of which the first one is a consequence was answered by Lefschetz. For a detailed information on fixed point theorems of Lefschetz see the book [2].

To state the Lefschetz fixed point theorem we need to define a homological invariant called *Lefschetz number*. Recall that given an $n \times n$ matrix $A = \{a_{ij}\}$ (say over the integers) the trace of A, denoted, $tr(A)$ is $\Sigma_i a_{ii}$. Given two matrices A and B, $tr(AB) = tr(BA)$. Thus under conjugation the trace of a matrix does not change and hence if we represent the linear map $A : \mathbb{Z}^n \to \mathbb{Z}^n$ under a change of coordinates by a different matrix then the trace does not change. Therefore, without any confusion we can talk about trace of a linear map. Furthermore, given a finitely generated abelian group G and a homomorphism $f : G \to G$ we define the trace of f, $tr(f)$ as the trace of the induced linear map $f : G/\{torsion\} \to G/\{torsion\}$.

Definition 7.1. Let X be a finite simplicial complex and $f : X \to X$ be a continuous map. Then the *Lefschetz number* $\tau(f)$ is defined as $\Sigma_i (-1)^i tr(f_* : H_i(X, \mathbb{Z}) \to H_i(X, \mathbb{Z}))$.

Theorem 7.1. *Let X and f be as in the Definition. If $\tau(f) \neq 0$ then there is a point $x \in X$ such that $f(x) = x$.*

An immediate corollary is the Brouwer fixed point theorem. More generally, if X has homology of a point or has finite homology groups then any continuous map has a fixed point. For a concrete example let $X = \mathbb{RP}^{2k}$ then we know that the homology groups of X are finite and therefore any continuous map $f : \mathbb{RP}^{2k} \to \mathbb{RP}^{2k}$ has a fixed point. It is evident that the compactness hypothesis is necessary. As for example take $\mathbb{R}^1$ and f be a translation by some non-zero number. Then $\tau(f) = 1$ but f has no fixed point.

Proof of Theorem 7.1. At first equip X with a metric d. Which can easily be done by giving Euclidean metric in every simplex so that common edges from different simplices gets the same metric from the bigger simplices. Or it can be done by embedding the simplicial complex in a Δ^n in some Euclidean space and then take the restricted Euclidean metric.

Since X is compact there is an $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$. Choose a subdivision of L of X so that the star of any simplex in L has diameter $< \epsilon/2$. Applying the simplicial approximation theorem there is a further subdivision K of L and a simplicial map $g : K \to L$ homotopic to f. By construction this g has the property that for each simplex σ of K, $f(\sigma)$ is contained in the star of the simplex $g(\sigma)$.

Then $g(\sigma)\cap\sigma=\emptyset$ for each simplex σ of K since for $x\in\sigma$, σ lies within distance $\epsilon/2$ of x and $g(\sigma)$ lies within distance $\epsilon/2$ of $f(x)$, while $d(x,f(x))>\epsilon$.

Since f and g are homotopic, $\tau(f)=\tau(g)$. Since g is simplicial it takes the n-skeleton K^n of K to the n-skeleton L^n of L, but since K is a subdivision of L $L^n\subset K^n$. Thus g induces chain map $H_n(K^n,K^{n-1})\to H_n(K^n,K^{n-1})$ on the cellular chain complex to itself of K. We claim the following.

$$\tau(g)=\Sigma_n(-1)^n tr(g_*:H_n(K^n,K^{n-1})\to H_n(K^n,K^{n-1})).$$

Recall that $H_n(K^n,K^{n-1})$ is a free abelian group on the n-simplices of K, now since $g(\sigma)\cap\sigma=\emptyset$ for all simplex σ in K we see that $tr(g_*)=0$ and hence $\tau(g)=0$ which is a contradiction.

Now we only need to prove the above displayed equality. Let $C_n:=H_n(K^n,K^{n-1})$. Then we have the following two commutative diagram where the vertical maps are induced by g.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ker(\partial_n) & \longrightarrow & C_n & \longrightarrow & img(\partial_n) & \longrightarrow & 0\\
 & & \downarrow g_{K_n} & & \downarrow g_{C_n} & & \downarrow g_{I_n} & & \\
0 & \longrightarrow & ker(\partial_n) & \longrightarrow & C_n & \longrightarrow & img(\partial_n) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & img(\partial_{n+1}) & \longrightarrow & ker(\partial_n) & \longrightarrow & H_n(K) & \longrightarrow & 0\\
 & & \downarrow g_{I_{n+1}} & & \downarrow g_{K_n} & & \downarrow g_{H_n} & & \\
0 & \longrightarrow & img(\partial_{n+1}) & \longrightarrow & ker(\partial_n) & \longrightarrow & H_n(K) & \longrightarrow & 0
\end{array}$$

It is easy to prove the following two equalities. $tr(g_{C_n})=tr(g_{K_n})+tr(g_{I_n})$ and $tr(g_{K_n})=tr(g_{H_n})+tr(g_{I_{n+1}})$. Substitute the second into the first and then multiply by $(-1)^n$ and take sum over n. The $tr(g_{I_n})$ terms will cancel out and that will prove the claim. □

8. Exercises

1. Show that the following manifolds are non-orientable: $\mathbb{RP}^{2k}$ for k a positive integer, Möbius band and Klein bottle.
2. Let M be a closed manifold of dimension n. Assume that $\pi_1(M)$ is finite of odd order. Prove that $H_{n-1}(M;\mathbb{Z})$ is torsion free.
3. Show that no non-orientable surface can be embedded in $\mathbb{R}^3$.
4. Show that removing finitely many points from the interior (that is $M-\partial M$) of a manifold does not affect the orientability of the manifold.
5. Let M be a closed manifold and $A\subset M$ is a compact subset. Prove that there is an isomorphism $H_n(M,M-A)\to\Gamma_A(M)$ where $\Gamma_A(M)$ is the R-module of sections of the covering projection $p:M_R\to M$

on the subset A, that is the continuous maps $q : A \to M_R$ such that $p \circ q = id_A$.

6. Let A be a compact locally contractible space and $f, g : A \to \mathbb{S}^n$ be two embeddings. Show that $H_n(M - f(A))$ is isomorphic to $H_n(M - g(A))$ for all n.

7. Show that the Poincaré duality theorem for R-orientable manifolds is equivalent to the isomorphism of the duality map

$$D_{\Delta^n} : H^n((\Delta^n, \partial\Delta^n); R) \to H_0(\Delta^n; R).$$

8. Show that any continuous map $f : M(G, 2k+1) \to M(G, 2k+1)$ has a fixed point.

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