GS 2025, Mathematics: Stage II

Instructions

1. Every claim needs a justification.
2. If a question consists of two parts, (a) and (b), you may use part (a) to solve part (b), even if you have not worked (a) out.
3. Do NOT write your name or affiliation or any personal detail other than application number (of the form GS2025MTHPHDxxxxxx or GS2025MTHIPHxxxxxx) on the paper.
4. There are 8 problems in this paper. Each of these carries 10 points.
5. Use both sides of each sheet for writing your answers.
6. Extra/rough sheets: Three extra pages have been provided. If these do not suffice, you can ask the invigilator for more sheets.
7. Extra sheet etiquette:
   - On the top of each extra sheet, write clearly which problem is being attempted on that sheet. **Do not do more than one problem on one extra sheet.**
   - Write your application number clearly at the top of each extra sheet.
   - All extra sheet(s) should be stapled onto this answer booklet, whether or not you consider them rough work.
8. If a given sheet contains part of your work on a particular problem, and that work is continued on some other page, indicate this clearly.
9. No books, notes, electronic devices etc. are allowed.
10. $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, \dots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
11. $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
12. All rings are assumed to be associative and containing a multiplicative identity denoted by 1.

1. Let $\{P_n\}_{n\geq 1}$ be a sequence of real polynomials in $\mathbb{R}[x]$ converging uniformly to a continuous function $P : \mathbb{R} \to \mathbb{R}$. Show that P is a polynomial.

2. Suppose that there are 5 red balls and 10 black balls in a box at time $t = 0$. At any time $t = 0, 1, 2, \ldots$, we pick a ball at random from the box and then return the ball to the box with 1 extra ball of the same color. Show that the probability that one picks a red ball at any time t remains constant at $\frac{1}{3}$.

3. Let $f : [0, \infty) \to \mathbb{R}$ be a function with continuous derivative that satisfies the differential equation $f'(x) = -f(x) + \sin(f(x)/2)$ and $f(0) = 1$. Show that $\lim_{x\to\infty} e^{x/4} f(x) = 0$.

4. (a) Let (X, d) be a metric space equipped with an action of a finite group G by isometries. Consider the function $d' : X \times X \to \mathbb{R}$ defined by $d'(x, y) = \min\{d(x, gy) | g \in G\}$. Prove that d' factorizes as $d' = \overline{d} \circ \pi$ where $\pi : X \times X \to X/G \times X/G$ is the natural projection and where $\overline{d}$ is a well-defined metric on the quotient set X/G.

   (b) Prove that the topology on X/G induced by the above metric $\bar{d}$ is the quotient topology.

   (c) Let $\mathbb{RP}^2$ denote the quotient space $(\mathbb{R}^3 \setminus 0)/\sim$ equipped with the quotient topology, where for $x, y \in (\mathbb{R}^3 \setminus 0)$ we say $x \sim y$ if and only if $x = cy$ for some $c \in \mathbb{R}^\times$. Prove that $\mathbb{RP}^2$ is a compact metric space.

5. (a) Show that a finite subgroup of the multiplicative group of a field is cyclic.

   (b) Let G be a finite commutative subgroup of $D^\times$, the group of units of D, where D is a division algebra. Show that G is a cyclic group.

6. Let $n \geq 2$ be a positive integer.

   (a) Let A be an $n \times (n-1)$ real matrix such that the sum of the entries in each column is 0. For each $1 \leq i \leq n$, let A_i be the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row of A. Prove that the number $(-1)^i \det(A_i)$ is independent of i.

   (b) Let A be an $n \times n$ real matrix such that the sum of the entries in each row of A, as well as the sum of the entries in each column of A, is 0. For any $1 \leq i, j \leq n$ let A_{ij} be the matrix obtained by deleting the i-th row and the j-th column of A. Prove that the number $(-1)^{i+j} \det(A_{ij})$ is independent of i and j.

7. Let V be the $\mathbb{R}$-vector space $\mathbb{R}^m$, for a positive integer m, and let $T \in \mathrm{End}_{\mathbb{R}}(V)$. Suppose there exists $v_0 \in V$ such that $\{T^n v_0 \mid n \in \mathbb{N}\}$ spans V. Show that there exists an open dense subset $U \subset \mathbb{R}^m = V$ such that for all $v \in U$, $\{T^n v \mid n \in \mathbb{N}\}$ spans V.

8. Show that there is no positive integer $n > 1$ such that n divides $2^n - 1$.