GS 2026, Mathematics: Stage II
Instructions

1. Every claim needs a justification.
2. If a question consists of two parts, (i) and (ii), you may use part (i) to solve part (ii), even if you have not worked (i) out.
3. Do NOT write your name or affiliation or any personal detail other than application number (of the form GS2026MTHPHDxxxxxx or GS2026MTHIPHxxxxxx) on the paper.
4. There are 10 problems in this paper. Each of these carries 10 points. **You need to solve 8 problems ONLY.**
5. Use both sides of each sheet for writing your answers.
6. Extra/rough sheets: Three extra pages have been provided. If these do not suffice, you can ask the invigilator for more sheets.
7. Extra sheet etiquette:
   - On the top of each extra sheet, write clearly which problem is being attempted on that sheet. **Do not do more than one problem on one extra sheet.**
   - Write your application number clearly at the top of each extra sheet.
   - All extra sheet(s) should be stapled onto this answer booklet, whether or not you consider them rough work.
8. If a given sheet contains part of your work on a particular problem, and that work is continued on some other page, indicate this clearly.
9. No books, notes, electronic devices etc. are allowed.
10. $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, \dots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
11. $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
12. All rings are assumed to be associative and containing a multiplicative identity denoted by 1.

1. Let $S \subset \mathbb{Z}$ be such that $\#S = n$. Define the $m-$fold sum of S as

$$\underbrace{S + \cdots + S}_{m \text{ copies}} := \{s_1 + \cdots + s_m : s_1, \cdots, s_m \in S\}.$$

Prove that the maximum possible value of $\#(\underbrace{S + \cdots + S}_{m \text{ copies}})$ is $\binom{n+m-1}{m}$.

2. Let $p(x), q(x) \in \mathbb{Z}[x]$. Prove that the following are equivalent:

(i) The polynomial $q(x)$ is a multiple of $p(x)$ in $\mathbb{Z}[x]$.

(ii) For any commutative ring R and $r \in R$ such that $p(r) = 0$, we have that $q(r) = 0$.

3. Let a, b, c be three distinct integers. Then prove that the polynomial

$$(x-a)(x-b)(x-c) - 1$$

is irreducible in $\mathbb{Z}[x]$.

4. Prove that at least half of the permutations of the set $\{1, 2, \ldots, 2026\}$ fix at least one point.

5. If $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function that vanishes outside a bounded interval, then prove that

$$\sup\{|f(x)| : x \in \mathbb{R}\} \leq \int_{\mathbb{R}} |f'(x)| dx.$$

6. Prove that the topological subspace

$$\{(a, b, c) \in \mathbb{R}^3 : a + b + c + abc = 0\}$$

of $\mathbb{R}^3$ is connected.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that vanishes outside a bounded interval such that:

$$f(x) \geq 0 \ \forall \ x \in \mathbb{R}, \quad \text{and} \quad \int_{\mathbb{R}} f(x)\, dx = 1.$$

Define a sequence of functions $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$ by

$$f_n(x) = f(x - n).$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function that vanishes at infinity; i.e., $\lim\limits_{|x| \to \infty} g(x) = 0$. Prove that:

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) g(x)\, dx = 0.$$

8. Let k, n be positive integers such that $k \leq n$. Let $v_1, \ldots, v_k \in \mathbb{R}^n$ be a collection of linearly independent vectors. Define an equivalence relation $\sim$ on $\mathbb{R}^n$ by declaring that $x \sim y$ if and only if there exists a tuple $(a_1, \ldots, a_k) \in \mathbb{Z}^k$ such that

$$x - y = \sum_{i=1}^{k} a_i v_i.$$

Show that the quotient space $\mathbb{R}^n/\sim$ is homeomorphic to the product space $(\mathbb{S}^1)^k \times \mathbb{R}^{n-k}$.

Here, $\mathbb{R}^m$ carries the standard topology for all m, and $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ carries the subspace topology inherited from $\mathbb{R}^2$.

9. Let $H \leq S_n$ be a subgroup whose order is a power of 2. Let K be the group

$$(H \times H) \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $H \times H$ by permuting its factors.

(i) Prove that there exists an injective homomorphism $K \hookrightarrow S_{2n}$.

(ii) Assume that $H \leq S_n$ is a Sylow 2-subgroup, and that n is a power of 2. Use any injective homomorphism $K \hookrightarrow S_{2n}$ to view K as a subgroup of S_{2n}. Show that $K \leq S_{2n}$ is a Sylow 2-subgroup.

10. Fix an integer $m \geq 1$. Let $\mathbb{C}[x]_{\leq m}$ be the complex vector space of all polynomials of degree at most m with complex coefficients equipped with the usual polynomial addition and scalar multiplication. For $(a, b) \in \mathbb{C}^2$, define the linear map $T_{a,b} : \mathbb{C}[x]_{\leq m} \to \mathbb{C}[x]_{\leq m}$ by

$$(T_{a,b} f)(x) = f(ax + b).$$

(i) Prove that if $a \neq 1$, then $T_{a,b}$ is diagonalizable.
Hint: It might help to consider the cases where a is zero or a is a root of unity separately.

(ii) Prove that $T_{1,b}$ is diagonalizable if and only if $b = 0$.