Lectures on
The Theory of Functions of
Several Complex Variables

By

B. Malgrange

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Raghavan Narasimhan

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Author

B. Malgrange
Institut Fourier, Université de Grenoble I
Laboratoire de Mathématiques Pures
38402 Saint-Martin-d'Hères Cedex
FRANCE

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Part I

Domains of Holomorphy

Chapter 1

Cauchy's formula and elementary consequences

1. Let C^n be the space of n complex variables $(z_1, \ldots, z_n)$. We write simply z for $(z_1, \ldots, z_n)$. Let $z_j = x_j + iy_j$, $j = 1, \ldots, n$, and let Ω be an opne set in C^n. Suppose that $f(z) = f(z_1, \ldots, z_n)$ is a complex-valued function defined in Ω which is (once) continuously differentiable as a function of the $2n$ real variables $x_1, y_1; \ldots; x_n, y_n$. 1

Set, by definition,

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right)$$

and

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right).$$

Definition. $f(z)$ is said to be *holomorphic* in Ω is $\dfrac{\partial f}{\partial \bar{z}_j} = 0$, $j = 1, \ldots, n$, at every point of Ω. (These equations generalize the Cauchy-Riemann equations to the case of functions of several variables).

A definition which is equivalent to the above, is t the following: $f(z)$ is said to have a complex derivative at $a \in \Omega$ if, for any $b \in C^n$, λ

complex,

$$\lim_{\lambda \to 0} \frac{f(a + \lambda b) - f(a)}{\lambda}$$

exists. $f(z)$ is said to be holomorphic in Ω if it has a complex derivative at every $a \in \Omega$.

Definition. A polydisc, with centre at the origin is the set K of points such that

$$|z_1| \leq \rho_1, |z_2| \leq \rho_2, \ldots, |z_n| \leq \rho_n, \rho_1, \rho_2, \ldots, \rho_n > 0.$$

These inequalities are, for brevity, written

$$|z| \leq \rho.$$

2 Let Γ denote the set of points $z \in C^n$ for which

$$|z_1| = \rho_1, \ldots, |z_n| = \rho_n.$$

Consider a function $f(z)$ holomorhpic in a neighbourhood of K and denote by c_j the curve $|z_j| = \rho_j$ in the z_j-plane. Then the following theorem holds.

Cauchy's formula. If z is a point with $|z| < \rho$ (i.e., $|z_j| < \rho_j$, $j = 1, \ldots, n$), then

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \underset{\Gamma}{\int \cdots \int} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n.$$

(The integral $\underset{\Gamma}{\int \ldots \int} g(\zeta_1, \ldots, \zeta_n) d\zeta_1 \ldots d\zeta_n$, is defined, for continuous g, to be $\int_{c_1} d\zeta_1 \ldots \int_{c_n} g(\zeta_1, \ldots, \zeta_n) d\zeta_n$ and is independent of the order in which the repeated integration is performed.)

Proof. Repeated application of the Cauchy formula for holomorphic functions of one complex variable gives

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{c_1} \frac{f(\zeta_1, z_2, \ldots, z_n)}{\zeta_1 - z_1} d\zeta_1$$

$$
\begin{aligned}
&= \frac{1}{(2\pi i)^2}\int_{c_1}\frac{d\zeta_1}{\zeta_1 - z_1}\int_{c_2}\frac{f(\zeta_1,\zeta_2,z_3,\dots,z_n)}{\zeta_2 - z_2}d\zeta_2\\
&= \dots = \frac{1}{(2\pi i)^n}\int_{c_1}\frac{d\zeta_1}{\zeta_1 - z_1}\int_{c_2}\frac{d\zeta_2}{\zeta_2 - z_2}\dots\\
&\qquad\int_{c_n}\frac{f(\zeta_1,\dots,\zeta_n)}{\zeta_n - z_n}d\zeta_n\\
&= \frac{1}{(2\pi i)^n}\underset{\Gamma}{\int\cdots\int}\frac{f(\zeta_1,\dots,\zeta_n)}{(\zeta_1 - z_1)\dots(\zeta_n - z_n)}d\zeta_1\dots d\zeta_n.
\end{aligned}
$$

□

2. If f is a complex valued continuous function on a compact space K, we define

$$\|f\|_K = \sum_{x\in K}|f(x)|.$$

Definition. A series $\sum\limits_{m\in N^n} a_m(z)$ of complex valued continuous functions 3
on a compact space K is said to converge normally if

$$\sum_{m\in N}\|a_m\|_K < +\infty.$$

(N is the set of non-negative integers).

If $a_m(z)$ are functions defined in an open set Ω, we say the series converges normally in Ω if it converges normally on every compact subset of Ω.

The following are simple consequences of Cauchy's formula.

Proposition 1. *If $f(z)$ is holomorphic in a neighbourhood U of a polydisc K, then*

$$f(z_1,\dots,z_n) = \sum_{(j_1,\dots,j_n)\in N^n} a_{j_1,\dots,j_n} z_1^{j_1}\dots z_n^{j_n} \text{ for } z\in K. \tag{1}$$

The series converges normally on K.

The proposition is proved simply by applying Cauchy's formula to a polydisc K' wiht $K\subset \overset{o}{K'}\subset K'\subset U$ ($\overset{o}{K'}$ is the interior of K').

The coefficients $a_j = a_{j_1,\ldots,j_n}$ in (1) are given by

$$a_{j_1,\ldots,j_n} = \frac{1}{(2\pi i)^n}\int\limits_{\Gamma}\ldots\int \frac{f(\zeta_1,\ldots,\zeta_n)}{\zeta_1^{j_1+1}\ldots\zeta_n^{j_n+1}}d\zeta_1\ldots d\zeta_n$$
$$= \frac{1}{(2\pi)^n}\int_0^{2\pi}\ldots\int_0^{2\pi}\frac{f(\rho_1 e^{i\theta_1},\ldots,\rho_n e^{i\theta_n})}{\rho_1^{j_1}\ldots\rho_n^{j_n}}$$
$$e^{-i(j_1\theta_1+\cdots+j_n\theta_n)}d\theta_1\ldots d\theta_n \tag{2}$$

From the expansion of $f(z)$ as a power series (1), it is seen that $f(z)$ is indefinitely differentiable and that

$$\frac{\partial^{j_1+\ldots+j_n}f(0)}{\partial z_1^{j_1}\ldots\partial z_n^{j_n}} = j_1!\ldots j_n! a_{j_1,\ldots j_n}$$

4 so that the expansion is unique. By differentiation in Cauchy's formula, we have also

$$\frac{\partial^{j_1+\ldots+j_n}f(z)}{\partial z_1^{j_1}\ldots\partial z_{z_n}^{j_n}} = \frac{j_1!\ldots j_n!}{(2\pi i)^n}\int\limits_{\Gamma}\ldots\int\frac{f(\zeta_1,\ldots,\zeta_n)}{(\zeta_1-z_1)^{j_1+1}\ldots(\zeta_n-z_n)^{j_n+1}}d\zeta_1\ldots d\zeta_n \tag{3}$$

If $f(z)$ is assumed to be holomorphic only in the interior of K, we can write

$$f(z) = \sum_{J\in N^n} a_J z^J$$

where

$$a_J = \frac{1}{(2\pi i)^n}\int\limits_{\Gamma'}\ldots\int\frac{f(\zeta_1\ldots,\zeta_n)}{\zeta_1^{j_1+1}\ldots\zeta_n^{j_n+1}}d\zeta_1\ldots d\zeta_n,$$

Γ' being a set of points of the form

$$|z_1| = r_1,\ldots,|z_n| = r_n,\quad 0 < r_j < \rho_j,\quad j = 1,\ldots,n.$$

The series converges normally in the interior of K.

If $f(z)$ is holomorphic in a neighbourhood of K and $M = \max\limits_{z\in\Gamma} |f(z)| = \max\limits_{z\in K}|f(z)|$ (the latter equality is easily established using Cauchy's formula), then

$$|a_J| \le \frac{M}{\rho J}$$

i.e.

$$|a_{j_1,\ldots,j_n}| \leq M/(\rho_1^{j_1} \ldots \rho_n^{j_n}).$$

This follows at once from the expression (2) for a_J as an integral in terms of $f(z)$. The inequalities are called the *Cauchy inequalities*.

Proposition 2. *Let $\{f_k(z)\}$ be a sequence of functions holomorphic in Ω and suppose that $\{f_k(z)\}$ converges, uniformly on every compact subset of Ω, to a function $f(z)$. Then $f(z)$ is holomorphic in Ω.*

Proof. Since $f_k(z) \to f(z)$ uniformly in a neighbourhood of any $z \in \Omega$, 5
$f(z)$ is continuous in Ω. Also, in a polydisc K about any point of Ω (lying wholly in Ω), $f_k(z)$ verifies Cauchy's formula, and $\{f_k(z)\}$ being uniformly convergent on K, so does $f(z)$, from which it follows easily that $f(z)$ is holomorphic in Ω.

Also, using the integral (3) for the derivatives of a holomorphic function, one proves that the derivatives (opf all orders) of $f_k(z)$ converge to the corresponding derivatives of $f(z)$, uniformly on every compact subset of Ω.

Another interpretation of the Proposition 2 is as follows. If $\mathscr{C}_\Omega, \mathscr{H}_\Omega$ denote the sets of continuous and holomorphic functions in Ω respectively, $\mathscr{C}_\Omega$ and $\mathscr{H}_\Omega$ are vector spaces over the field of complex numbers. We recall that one may topologize $\mathscr{C}_\Omega, \mathscr{H}_\Omega$ by putting on them the topology of uniform convergence on compact sets, namely if $f_n \in \mathscr{C}_\Omega$ (or $\mathscr{H}_\Omega$), $f_n \to 0$ if $\|f_n\|_K \to 0$ as $n \to \infty$ for every compact $K \subset \Omega$. A fundamental system of neighbourhoods of the origin is given by the sets $\mathscr{U}(K_m, 1/m)$, where $\mathscr{U}(K, a)(a > 0)$ is the set of $f \in \mathscr{C}_\Omega$ (resp. $\mathscr{H}_\Omega$) for which $\|f\|_K < a$, and $\{K_m\}$ is a sequence of compact sets with

$$K_m \subset \overset{o}{K}_{m+1}, \quad \bigcup_{m=1}^{\infty} \overset{o}{K}_m = \Omega.$$

$\mathscr{C}_\Omega$ is an $(\mathscr{F})$-space, i.e., it has a countable fundamental system of neighbourhood of 0 and is complete.

Proposition 2 may be expressed by saying that $\mathscr{H}_\Omega$ *is a closed subspace of* $\mathscr{C}_\Omega$. □

Proposition 3. *Every bounded closed set Φ in $\mathscr{H}_\Omega$ is compact.* 6

(A bounded set Φ in any topological vector space is a set such that to any neighbourhood $\mathscr{U}$ of the origin, there exists a $\lambda > 0$ such that $\Phi \subset \lambda\mathscr{U}$. $\mathscr{C}_\Omega$ or $\mathscr{H}_\Omega$, we may say equivalently that a set Φ is bounded if $\sum\limits_{f\in\Phi} \|f\|_K < +\infty$ for every compact $K \subset \Omega$.).

Proof. Let K be any compact subset of Ω. Let Φ_K be the set of functions f_K, where f_K is the restriction of $f \in \Phi$ to K. We prove that Φ_K is equicontinuous from which it follows by means of Ascoli's theorem (see e.g. Bourbaki: Topologie Générale, Chap.X, p.48) that Φ_K is relatively compact in $\mathscr{C}_K$.

Choose a compact set K' so that $K \subset \overset{o}{K'} \subset K' \subset \Omega$. K has positive distance from the boundary of K'. Since $\sup\limits_{f\in\Phi} \|f\|_{K'} < +\infty$ Cauchy's inequlity applied to the derivatives of the f in a suitable polydisc about an arbitrary point of K, contained in K' shows that

$$\sup_{f\in\Phi} \max_j \|\frac{\partial f}{\partial z_j}\|_K = M_K < \infty.$$

Since $\dfrac{\partial f}{\partial \bar{z}_j} = 0$, it follows from the definitions of $\dfrac{\partial f}{\partial z_j}$, $\dfrac{\partial f}{\partial \bar{z}_j}$ that $\dfrac{\partial f}{\partial x_j}$, $\dfrac{\partial f}{\partial y_j}$, are bounded uniformly on K, for $f \in \Phi$ (actually, $\|\dfrac{\partial f}{\partial x_j}\|_K \leq M_K$, $\|\dfrac{\partial f}{\partial y_j}\|_K \leq M_K$). The mean value theorem now shows that Φ_K is equicontinuous.

To prove now that Φ is compact, it suffices to prove that any sequence $\{f_m\}$, $f_m \in \Phi$ has a limit point in $\mathscr{C}_\Omega$. We choose a sequence $\{K_m\}$ of compact sets K_m with $K_m \subset \overset{\circ}{K}_{m+1}$, $\bigcup_{m=1}^\infty K_m = \Omega$. Since Φ_K is relatively compact for every compact $K \subset \Omega$, we choose, inductively, subsequences $\{f_{m,y}\}$ of $\{f_m\}$, such that each is a subsequence of the pre-
7 ceding, while if $f_{m,\nu+1} = f_{m',\nu_0}$ then $m' > m$, and $\{f_{m,\nu}\}$ converges uniformly on K_ν. Since $\bigcup \overset{\circ}{K}_m = \Omega$, the subsequence $\{f_{\nu\nu}\}$ of $\{f_m\}$ converges to a limit in $\mathscr{C}_\Omega$ and Proposition 3 is proved.

Suppose given a subset Φ of $\mathscr{H}_\Omega$ such that $\sup\limits_{f\in\Phi}|f(z)| < +\infty$ for every $z \in \Omega$. The question arises as to what one can assert about the boundedness of the set Φ. The following result holds. $\square$

Proposition 4. *There exists an open set $\Omega' \subset \Omega$, which is dense in Ω such that $\Phi_{\Omega'}$ (the set of the restrictions to Ω' of functions of Φ is a bounded set in $\mathscr{H}_{\Omega'}$.)*

The proposition and its proof remain valid if $\mathscr{H}_\Omega$ is replaced by $\mathscr{C}_\Omega$. We need the following

Theorem of Baire. Let U be an open set $\subset \mathbb{C}^n$ and $\{\mathscr{O}\}$, $k = 1, 2, \ldots$ a sequence of open sets $\subset U$, such that each $\mathscr{O}_k$ is dense in U. The $\bigcap\limits_{k=1}^{\infty} \mathscr{O}_k$ is dense in U.

Proof. Since $\mathscr{O}_1$ is dense in U, given any open ball $B \subset U$, $\mathscr{O}_1 \cap B$ contains an open ball B_1. In the same way, $\mathscr{O}_2 \cap \frac{1}{2}B_1$ ($\frac{1}{2}B_1$ is the ball with the same centre as B_1 and half its radius) contains an open ball B_2 and this process can be continued. Clearly $\bigcap\limits_{k=1}^{\infty} \bar{B}_k \neq 0$ and since $\bar{B}_k \subset B_{k-1}$, $\bigcap\limits_{k=1}^{\infty} B_k \neq 0$. Also $B_k \subset \mathscr{O}_k \cap B$, so that $\bigcap\limits_{k=1}^{\infty} \mathscr{O}_k \cap B \neq 0$ and $\bigcap\limits_{k=1}^{\infty} \mathscr{O}_k$ is dense in U. $\square$

Proof of the proposition.

Let U be an arbitrary open set $\subset \Omega$. Let $\mathscr{O}_k \subset U$ be the set of $z \in U$ for which there exists at least one $f \in \Phi$ for which $|f(z)| > k$. Clearly $\mathscr{O}_k$ is open. Also since for any $z \in \Omega$, $\sup\limits_{f\in\Phi}|f(z)| < +\infty$, $\bigcap\limits_{k=1}^{\infty} \mathscr{O}_k = 0$. By Baire's theorem, at least one $\mathscr{O}_k$ is not dense in U. Hence U **8**
contains an open set $\mathscr{O}_U$ contained in the complement of $\mathscr{O}_k$ for some k and the functions of Φ are uniformly bounded for $z \in \mathscr{O}_U$(by k). If we set $\Omega' = \bigcup\limits_{U} \mathscr{O}_U$, Ω' is clearly open and dense in Ω. If $z \in \Omega'$, z possesses a neighbourhood on which Φ is uniformly bounded and so Φ is bounded uniformly on every compact subset of Ω' (by the Borel-Lebesgue lemma).

Chapter 2

Reinhardt domains and circular domains

1 Reinhardt domains

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Definition. A Reinhardt domain is an open set $\Omega \subset C^n$ such that

$$(z_1, \ldots, z_n) \epsilon \Omega \text{ implies } (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \epsilon \Omega$$

for all real $\theta_1, \ldots, \theta_n$.

Theorem 1. *Let $\Omega \subset C^n$ be a connected Reinhardt domain containing 0 and suppose that $f(z)$ is holomorphic in Ω, Then $f(z)$ can be written*

$$f(z) = \sum_{m \epsilon N^n} a_m z^m$$

in Ω, the series converging normally in Ω. Such an expansion is unique.

Proof. Let Ω_ϵ be the set of $z \epsilon \Omega$ which have distance $> \epsilon |z| (|z^2 = \sum |z_i|^2)$ from the boundary of Ω. Let $\Omega'_\epsilon \subset \Omega_\epsilon$ be the connected component of 0. Then

$$\bigcup_{\epsilon > 0} \Omega'_\epsilon = \Omega.$$

For, if $z\epsilon\Omega$, we can join z to the origin by a path in Ω. This path has a distance > 0 from the boundary of Ω. If ϵ is small enough the path lies in Ω_ϵ and so in Ω'_ϵ. In particular, $z\epsilon\Omega'_\epsilon$.

Define, for $z\epsilon\Omega'_\epsilon$

$$g(z_1,\ldots,z_n) = \frac{1}{(2\pi i)^n} \int\limits_{|t_1|=1+\epsilon} \cdots \int\limits_{|t_n|=1+\epsilon} \frac{f(t_1z_1,\ldots,t_nz_n)}{(t_1-1)\ldots(t_n-1)} dt_1\ldots dt_n.$$

The integral is defined, for if $(z_1,\ldots,z_n)\epsilon\Omega'_\epsilon$, then $((1+\epsilon)z_1,\ldots,(1+\epsilon)z_n)\epsilon\Omega$ for the distance between these two points is $\epsilon|z|$. Hence, since Ω is a Reinhardt domain, $(t_1z_1,\ldots,t_nz_n)\epsilon\Omega$ for all $(t_1,\ldots,t_n)$ with $|t_j| = 1+\varepsilon$. By differentiation under the integral, it is seen that $g(z)$ is holomorphic
10 in Ω'_ε. Moreover, if we choose a polydisc $K\subset\Omega'_\varepsilon$ with centre 0 such that $(1+\epsilon)K\subset\Omega$, then for $z\epsilon\overset{\circ}{K}$, $(t_1z_1,\ldots,t_nz_n)\epsilon\Omega$ for all $|t_j|\le 1+\epsilon$ so that, by Cauchy's formula,

$$g(z_1,\ldots,z_n) = f(z_1,\ldots,z_n), z\epsilon\overset{\circ}{K}.$$

Now, if two holomorphic functions in an open, connected set U coincide in some open set in U, then they coincide in the whole of U. (See the principle of analytic continuation in III). Hence

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{|t_1|=1+\epsilon} \cdots \int\limits_{|t_n|=1+\epsilon} \frac{f(t_1z_1,\ldots,t_nz_n)}{(t_1-1)\ldots(t_n-1)} dt_1\ldots dt_n$$

in Ω'_ϵ. Moreover,

$$\frac{1}{(t_1-t)\ldots(t_n-1)} = \frac{1}{t_1\ldots t_n} \sum_{(m_1,\ldots,m_n)\epsilon N^n} \frac{1}{t_1^{m_1}}\cdots\frac{1}{t_n^{m_n}}$$

the series being normally convergent on $|t_j| = 1+\epsilon$, $j = 1,2,\ldots,n$. Hence

$$f(z_1,\ldots,z_n) = \sum_{(m_1,\ldots,m_n)\epsilon N^n} \phi_{m_1\ldots m_n}(z)$$

where

$$\varphi_{m_1\ldots m_n}(z) = \frac{1}{(2\pi i)^n} \int\limits_{|t_1|=1+\epsilon} \cdots \int\limits_{|t_n|=1+\epsilon} \frac{f(t_1z_1,\ldots,t_nz_n)}{t_1^{m_1+1}\ldots t_n^{m_n+1}} dt_1\ldots dt_n.$$

Exactly as we proved above that $g(z) = f(z)$ in Ω'_ϵ, we prove, using formula (3) for the derivatives of a holomorphic function, that

$$\phi_{m_1\ldots m_n}(z) = \frac{1}{m_1!\ldots m_n!}\ \frac{\partial^{m_1+\cdots m_n} f(t_1 z_1,\ldots,t_n z_n)}{\partial t_1^{m_1}\ldots\partial t_n^{m_n}}\bigg|\, t_j = 0$$
$$= \frac{z_1^{m_1}\ldots z_n^{m_n}}{m_1!\ldots m_n!}\ \frac{\partial^{m_1+\cdots+m_n} f(0)}{\partial z_1^{m_1}\ldots\partial z_n^{m_n}}.$$

From the integral representation of ϕ, it is seen that if z lies in a compact subset $K \subset \Omega$ and ϵ is small enough 11

$$|\phi_{m_1\ldots m_n}(z)| \leq \frac{M_K}{(1+\epsilon)^{m_1}\ldots(1+\epsilon)^{m_n}}$$

where M_K depends only on K, so that the series converges normally on K. The expansion in the whole of Ω follows easily by letting $\epsilon \to 0$.

The uniqueness is obvious. □

2 Domains of convergence of powe series

Let

$$\sum_{m\epsilon N^n} a_m z^m$$

be a given power series. We define its domain of convergence, D, as follows: $z\epsilon D$ if the series converges absolutely in a neighbourhood of z. D is clearly open. We define the set B as the set of z for which there exists $C > 0$ such that $|a_m z^m| \leq C$ for all $m\epsilon N^n$. Clearly $D \subset B^\circ$ (interior of B). But we can prove that $D = B^o$. This follows from

Abel's lemma: If $z = (z_1,\ldots,z_n)\epsilon B$, then $(\alpha_1 z_1,\ldots,\alpha_n z_n)\epsilon D$ if $|\alpha_1| < 1,\ldots,|\alpha_n| < 1$ and the series converges normally in D.

Proof. If $|a_m z^m| \leq C$, the general term of the series $\sum\limits_{m\epsilon N^n} a_{m_1\ldots m_n}\times (\alpha_1 z_1)^{m_1}\ldots(\alpha_n z_n)^{m_n}$ is majorized by $C|\alpha_1|^{m_1}\ldots|\alpha_n|^{m_n}$ and the result follows. □

Consequences of Abel's lemma

(a) $D = B^\circ$. This is obvious.

(b) The series represents a holomorphic function in D (by I, Prop 2)

12 (c) D is a Reinhardt domain. More, if $(z_1, \ldots, z_n)$ then $(\alpha_1 z_1, \ldots, \alpha_n z_n) \epsilon D$ if $|\alpha_1| \leq 1, \ldots |\alpha_n| \leq 1$,

and so D is a union of polydiscs (all with centre).

We give an example, due to Hartogs, of a domain in C^2 that any holomorphic function in the domain can be extended to a larger domain.

Let G be the domain consisting of the points z with

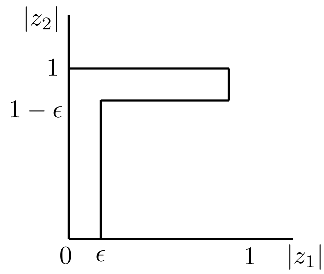
$$|z_1| < \epsilon, \quad |z_2| < 1$$

and the points with

$$|z_1| < 1, \quad 1 - \epsilon < |z_2| < 1.$$

In the figure we have indicated the variation in $|z_1|$, $|z_2|$. By Theorem 1, any function holomorphic in G can be expanded in a power series in G and, by Abel's lemma, the power series represents a holomorphic function in the open polydisc

$$|z_1| < 1, \quad |z_2| < 1.$$



A special case is the following:

Let Ω be an open set in $C^n (n > 1)$ and a a point of Ω. Let $f(z)$ be holomorphic in $\Omega - a$. Then $f(z)$ can be continued holomorphically throughour Ω.

Consequence (c) of Abel's lemma shows that the domain conver-
13 gence of a power series is a Reinhardt domain, but this is not all; not every Reinhardt domain, which is the union of open polydiscs, is the domain of convergence of a power series. We can, however, characterize the domains of convergence of power series.

Let $D^* \subset R^n$ be the set consisting of the points $(\log|z_1|, \ldots, \log|z_n|)$ where $(z_1, \ldots, z_n) \epsilon D$ i.e. D^* is the image of D under the mapping $\phi : C^n \to R^n$ defined by $\phi(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|)$. Let B^* be the image of B under ϕ (D and B are the sets of points defined on p.11 associated with the power series). If $(\rho_1, \ldots, \rho_n) \epsilon D^*$ then $(\rho_1 - t_1, \ldots, \rho_n - t_n) \epsilon D^*$ if $t_1 \geq 0, \ldots, t_n \geq 0$.

The following fundamental result holds.

Theorem 2. *D^* is a convex set in R^n.*

Proof. Since $D = B^0$, $D^* = \overset{\circ}{B^*}$ so that it suffices to prove that B^* is convex. Now $B^* = \bigcup\limits_{C>0} B^*_C$, where B^*_C is the image, under ϕ, of the set of $z \epsilon C^n$ where $|a_m z^m| \leq C$ for all $m \epsilon N^n$. Since $B^*_C \subset B^*_{C'}$, if $C < C'$ it is enough to prove that B^*_C is convex. Also $B^*_C = \bigcap\limits_{m \epsilon N^n} B^*_{C,m}$, where $B^*_{C,m}$ is the image of the set of $z \epsilon C^n$, $|a_m z^m| \leq C$ for a fixed m, under ϕ. Thus, we have only to prove that $B^*_{C,m}$ is convex. Now $B^*_{C,m}$ is the image of the set of $z \epsilon C^n$ for which $\left|a_{m_1 \ldots m_n} z_1^{m_1} \ldots z_n^{m_n}\right| \leq C$ and so is the set of points $(\rho_1, \ldots, \rho_n) \epsilon R^n (\rho_j = \log|z_j|)$ at which

$$\log|a_{m_1 \ldots m_n}| + m_1 \rho_1 + \cdots + m_n \rho_n \leq \log C,$$

which, being a half space in R^n, is convex. This proves Theorem 2.

For example, consider a series convergent in the domain in C^2 con- 14
sisting of the points $|z_1| < 1$, $|z_2| < e$, and the points $|z_1| < e$, $|z_2| < 1$. It has the image shown in Fig. (a) (the set α) in the $(|z_1|, |z_2|)$-plane and D^* contains the set α^* in Fig. (b). Since D^* is convex, it contains also the set β^* in Fig. (b) and so the series converges at the points in C^2 which are mapped by

$$\phi : z \to (\log|z_1|, \log|z_2|)$$

into the set β' in Fig. (a).

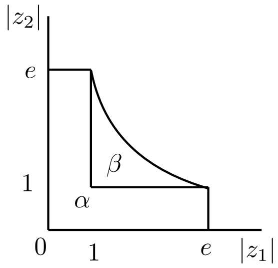


Fig. (a)

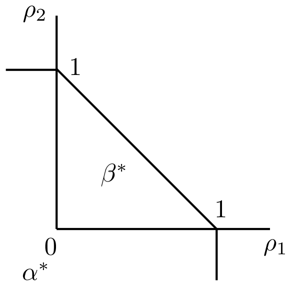


Fig. (b)

The converse of Theorem 2 is true; that is if a Reinhardt domain, which is the union of polydiscs, is such that its image under ϕ is convex, then it is (precisely) the domain of convergence of a power series. We shall prove this later (in VII, p.44) □

3 Circular domains

We consider next the expansion of holomorphic functions in series of homogeneous polynomials.

Definition. An open set $\Omega \subset C^n$ is said to be a *circular domain* if $z \epsilon \Omega$
15 implies $e^{i\theta} z \epsilon \Omega$ i.e. $(e^{i\theta} z_1, \ldots, e^{i\theta} z_n) \epsilon \Omega$ for all real θ.

Theorem 3. *Let Ω be a connected circular domain and let $0 \epsilon \Omega$. Suppose $f(z)$ to be holomorphic in Ω. Then $f(z)$ can be expanded in a series of homogeneous polynomials,*

$$f(z) = \sum_{k=0}^{\infty} P_k(z) \text{ in } \Omega$$

($P_k(z)$ is homogeneous, of degree k in $z_1, \ldots, z_n$) and the series converges normally in Ω. An expansion of this form is unique.

Proof. Define Ω'_ϵ as in Theorem 1 and consider the integral

$$\frac{1}{2\pi i} \int\limits_{|t|=1+\epsilon} \frac{f(tz_1, \ldots, tz_n)}{t-1}\, dt.$$

Exactly as in the proof of Theorem 1, we choose a polydisc K about the origin such that $(1+\epsilon)K \subset \Omega'_\epsilon$ and then, if $z\epsilon K$, $(tz_1, \ldots, tz_n)\epsilon\Omega$ for $|t| \leq 1$ so that, the integral being holomorphic in Ω'_ϵ we have, by Cauchy's formula,

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int\limits_{|t|=1+\epsilon} \frac{f(tz_1, \ldots, tz_n)}{t-1} dt.$$

in $\overset{\circ}{K}$ and so, since Ω'_ϵ is connected, in Ω'_ϵ. Since

$$\frac{1}{t-1} = \frac{1}{t} \sum_{k=0}^{\infty} \frac{1}{t^k},$$

the series converging normally on $|t| = 1+\epsilon$, we have

$$f(z_1, \ldots, z_n) = \sum_{k=0}^{\infty} P_k(z)$$

where

$$P_k(z) = \frac{1}{2\pi i} \int\limits_{|t|=1+\epsilon} \frac{f(tz_1, \ldots, tz_n)}{t^{k+1}}\, dt.$$

As in the proof of Theorem 1, the series converges normally in Ω'_ϵ and, 16
repeating the above argument, it follows that

$$P_k(z) = \frac{1}{k!} \left| \frac{d^k f(tz_1, \ldots, tz_n)}{} \right|_{t=0}$$
$$= \frac{1}{k!} \sum_{m_1+\cdots+m_n=k} \frac{\partial^k f(0)}{\partial^{m_1} z_1 \ldots z_n^{m_n}} z_1^{m_1} \ldots z_n^{m_n}$$

The theorem follows by letting $\epsilon \to 0$.

It is easily seen that if $f(z)$ has an expansion $\sum\limits_{k=0}^{\infty} P_k(z)$ which converges uniformly in a neighbourhood of 0, then $P_k(z)$ has the above form. This proves Theorem 3.

The above theorem shows that if a function $f(z)$ is holomorphic is a circular domain Ω (which is connected and contains 0) then it can be holomorphically continued to $\bigcup\limits_{0\le t\le 1}(t\Omega)$. This is proved in the same way as Abel's lemma. $\square$

Chapter 3

Complex analytic manifolds

Definition. A Hausdorff, topological space V^n is called a *topological* 17
manifold of dimension n ($n \geq 0$ an integer) if it has followsing property: every point $a \in V^n$ has a neighbourhood homeomorphic to an open set $\Omega \subset R^n$.

(Note that a space having this property is not automatically Hausdorff.)

V^n is said to be countable at infinity, if V^n is a countable union of compact sets.

We next recall, without proof, the following two propositions.

Proposition 1. *If V^n is connected the following three conditions are equivalent:*

(1) V^n is countable at infinity.

(2) V^n is paracompact (i.e. any open covering $(U_i)_{i \in I}$ admits a locally finite refinement, namelyl there is another open convering $(W_j)_{j \in J}$ each set of which is contained in at least one U_i and such that any point has a neighbourhood intersecting only a finite number of the W_j)

(3) V^n has a countable open base.

Proposition 2 (Poincaré - Volterra theorem)**.** *If V^n, W^n are two n dimensional manifolds and if (1) V^n is connected, (2) W^n is countable at infinity, (3) there exists a continuous mapping $\phi : V^n \to W^n$ which is a local homeomorphism (i.e. every $a \in V^n$ has an open neighbourhood*
18 *which is mapped homeomorphically on an open set of W^n). Then V^n is countable at infinity.*

Differentiable manifolds. Let V^n be a topological manifold of dimension n. by an (indefinitely) differentiable or C^∞-structure on V^n is meant a family $\{\mathscr{O}_i\}_{i\in I}$ of open sets $\subset V^n$ which cover V^n, and mappings $\{f_i\}_{i\in I}$ such that f_i maps $\mathscr{O}_i$ homeomorphically onto an open subset $\tilde{\mathscr{O}}_i \subset R^n$ and such that the mappings $f_i \circ f_j^{-1}$ and $f_j \circ f_i^{-1}$ are C^∞-mappings of $f_j(\mathscr{O}_i \cap \mathscr{O}_j)$, $f_i(\mathscr{O}_i \cap \mathscr{O}_j)$ respectively, onto $f_i(\mathscr{O}_i \cap \mathscr{O}_j)$, $f_j(\mathscr{O}_i \cap \mathscr{O}_j)$, i.e. if the correspondence

$$f_i(\mathscr{O}_i \cap \mathscr{O}_j) \longleftrightarrow f_j(\mathscr{O}_j \cap \mathscr{O}_j)$$

is C^∞.

(Note that it may happen that one can define more than one differentiable structure on a manifold V^n, so that a C^∞-manifold is not a special topological manifold, but is a topological manifold with an additional structure).

If $\{\mathscr{O}'_k, f'_k\}$ defines a C^∞-structure on V^n, we say that it defines the same structure as $\{\mathscr{O}_i, f_i\}$ if (and only if) the correspondence $f_i(\mathscr{O}_i \cap \mathscr{O}'_k) \longleftrightarrow f'_k(\mathscr{O}_i \cap \mathscr{O}'_k)$ is C^∞. If the intersections $\mathscr{O}_i \cap \mathscr{O}_j$ or $\mathscr{O}_i \cap \mathscr{O}'_k$ are empty we take it that the condition is satisfied. We make a similar convention whenever we speak of properties of mappings of $\mathscr{O}_i \cap \mathscr{O}_j$ or $\mathscr{O}_i \cap \mathscr{O}'_k$, without stating this explicitly.

If ϕ is a mapping of V to W, where V, W are C^∞-manifolds (not necessarily of the same dimension) with C^∞-structures $\{\mathscr{O}_i, f_i\}$, $\{\mathscr{O}'_j, f'_j\}$ we say that ϕ is a *differentiable or a C^∞-mapping* if $f'_j \circ \phi \circ f_i^{-1}$ is a
19 C^∞-mapping, of $f_i(\mathscr{O}_i \cap \varphi^{-1}(\mathscr{O}'))$. If V and W have the same dimension, we say that ϕ is a *diffeomorphism* if ϕ is a homeomorphism of V onto W and if ϕ and ϕ^{-1} are differentiable.

System of local coordinates

Definition. Let V be a C^∞-manifold and let $a \in V$. If $(X_1, \ldots, X_n)$ is a system of **real valued** functions in an open neighbourhood W of a such that the mapping $\phi : W \to R^n$ defined by $b \to (X_1(b), \ldots, X_n(b))$ for every $b \in W$, is a diffeomorphism, then $(X_1, \ldots, X_n)$ are said to form a *system of local coordinates at a*.

If $Y_1, \ldots, Y_n$ are C^∞-functions in a neighbourhood of a, they form a system of local coordinates at a if and only if the Jacobian

$$\frac{D(Y_1, \ldots, Y_n)}{D(X_1, \ldots, X_n)}$$

does not vanish at a. The proof follows easily from the implicit function theorem.

Complex analytic manifolds.

Definition. Let Ω, Ω' be two open sets in C^n, $f_1, \ldots, f_n$ complex valued functions in Ω. We say that $f = (f_1, \ldots, f_n)$ is an **analytic isomorphism** of Ω onto Ω' if the mapping $f = (f_1, \ldots, f_n)$ of $\Omega \to C^n$ is a diffeomorphism of Ω onto Ω' and the functions $f_1, \ldots, f_n$ are holomorphic in Ω.

The composite of analytic isomorphisms is an analytic isomorphicm. Moreover, the inverse of an analytic isomorphism is also one. This is seen as follows:

Let $(f_1, \ldots, f_n)$ be n holomorphic functions of $z_1, \ldots, z_n$, and let $J =$ **20**
$\det\left(\frac{\partial f_k}{\partial z_l}\right)$ and $f_k = f_k^{(1)} + i f_k^{(2)}$, $z_k = x_k + i y_k$. Then the determinant $\begin{vmatrix} A & B \\ C & D \end{vmatrix}$, where

$$A = \left(\frac{\partial f_k^{(1)}}{\partial x_l}\right), \quad B = \left(\frac{\partial f_k^{(2)}}{\partial x_l}\right), \quad C = \left(\frac{\partial f_k^{(1)}}{\partial y_l}\right), \quad D = \left(\frac{\partial f_k^{(2)}}{\partial y_l}\right)$$

is equal to $\begin{vmatrix} A_1 & B_1 \\ C_1 & D_1 \end{vmatrix}$, where

$$A_1 = \left(\frac{\partial f_k}{\partial z_l}\right), \quad B_1 = \left(\frac{\partial \overline{f_k}}{\partial z_l}\right), \quad C_1 = \left(\frac{\partial f_k}{\partial \overline{z_l}}\right), \quad D_1 = \left(\frac{\partial \overline{f_k}}{\partial \overline{z_l}}\right)$$

and, by the Cauchy-Riemann equations, this equals $|J|^2$ so that, since $(f_1, \ldots, f_n) = f$ is a diffeomorphism, $J \neq 0$. We can now easily compute $\dfrac{\partial z}{\partial \bar{f}_k}$ and show that it is zero which proves the statement.

Definition. Let V be a topological manifold of dimension $2n$. We shall identify R^{2n} with C^n. A system $\{\mathscr{O}_i, f_i\}_{i\in I}$, where $\{\mathscr{O}_i\}_{i\in I}$ is an open covering of V and f_i are mappings, $f_i : \mathscr{O}_i \to \widetilde{\mathscr{O}}_i \subset C^n$, is said to define a *complex analytic structure* if the mapping $f_i \circ f_k^{-1}$ defines an analytic isomorphism of $f_j(\mathscr{O}_i \cap \mathscr{O}_j)$ onto $f_i(\mathscr{O}_i \cap \mathscr{O}_j)$, i.e., if the correspondence

$$f_i(\mathscr{O}_i \cap \mathscr{O}_j) \longleftrightarrow f_j(\mathscr{O}_i \cap \mathscr{O}_j)$$

is an analytic isomorphism for every i, $j \in I$. Two systems $\{\mathscr{O}_i, f_i\}$,
21 $\{\mathscr{O}'_k, f'_k\}$ define the same complex analytic structure if the correspondence $f_i(\mathscr{O}_i \cap \mathscr{O}'_k) \longleftrightarrow f'_k(\mathscr{O}_i \cap \mathscr{O}'_k)$ is an analytic isomorphism for every i, k.

We call V a complex analytic manifold of (complex) dimension n.

Definition. Let V^n, W^m be two complex analytic manifolds, with structures $\{\mathscr{O}_i, f_i\}$, $\{\mathscr{O}'_j, f'_j\}$ and ϕ a map $V^n \to W^m$. ϕ is said to be an *analytic mapping* if the mappings $f'_j \circ \phi \circ f_i^{-1}$ are analytic mappings of $\tilde{\mathscr{O}}_j$ to C^m, i.e., if the component functions are holomorphic in $\tilde{\mathscr{O}}_i$. ϕ is called an *analytic isomorphism* of V^n onto W^m if it is an analytic map and is, moreover, a diffeomorphism.

Local coordinates.

Let V be a complex, analytic manifold. A system of n complex valued functions $(z_1, \ldots, z_n)$ in a neighbourhood of a point $a \in V$ is said to form a system of local coordinates at a if there exists an open set W, $a \in W$ such that the mapping $\phi : W \to \tilde{W} \subset C^n$ defined by $b \to (z_1(b), \ldots, z_n(b))$ is an analytic isomorphism n holomophic functions $t_1, \ldots, t_n$ (i.e., analytic mappings into C^1) in a neighbourhood of a form system of local coordinates if and only if the determinant

$$J = \det \left[\frac{\partial t_k}{\partial z_l}\right]_a \neq 0.$$

This follows from the fact that Jacobian of the $\mathscr{R}t_i$, $\Im t_j$ in terms of the $\mathscr{R}z_i$, $\Im z_j$ equals $|J|^2 \neq 0$ and the remark on the inverse of an analytic isomorphism.

The question arises as to which theorems on holomorphic functions in an open set $\Omega \subset C^n$ generalize to holomorphic functions on a complex analytic manifold V. The following theorems, and their proofs, do. **22**

(1) $\mathscr{H}_V$ is closed in $\mathscr{C}_V$ (the notation is the obvious one).

(2) A closed bounded set Φ in $\mathscr{H}_V$ is compact. The diagonal process fails if V is not countable at infinity, but it is easy to prove that any ultrafilter in Φ converges, by considering, for any compact $K \subset V$, the restrictions to K of the functions in the elements of the ultrafilter.

(3) If $\Phi \subset \mathscr{H}_V$ (or $\mathscr{C}_V$) and $\sum\limits_{f \in \Phi} |f(a)| < +\infty$ for every $a \in V$, then there exists an open set $V' \subset V$ dense in V such that $\Phi_{V'}$ (the set of restrictions to V' of the functions of Φ) is a bounded set in $\mathscr{H}_{V'}$ (respectively $\mathscr{C}_{V'}$).

This is because Baire's theorem (used in I, Prop.4) is true for open subsets of locally compact spaces or complete metric spaces (and any manifold is locally compact).

It is also sometimes of interest to apply Baire's theorem in $\mathscr{H}_V$, but this necessitates the assumption that V is countable at infinity when $\mathscr{H}_V$ is a Fréchet space and so a complete metric space.

The principle of analytic continuation.

Let V be a connected, W an arbitrary, complex analytic manifold, and let f_1, f_2 be two analytic mappings of V into W. Let $\{\mathscr{O}_i, \phi_i\}$, $\{\mathscr{O}'_j, \phi'_j\}$ define the structures on V, W respectively.

Let $a \in \mathscr{O}_i$, $f_1(a) = f_2(a) \in \mathscr{O}'_j$. We say that f_1, f_2 have the same derivatives at a if the components of the mappings $\phi'_j \circ f_1 \circ \phi_i^{-1}$ and $\phi'_j \circ f_2 \circ \phi_j^{-1}$ have the same derivatives (of all orders) at $\phi_i(a)$. This definition is independent of the $\mathscr{O}_i, \mathscr{O}'_j$ containing a, $f_1(a)$ respectively, and of the systems chosen to define the analytic structures.

The following theorem then holds: **23**

Theorem. (1) *(The strong principle of analytic continuation).*

If f_1 and f_2 and all their derivatives coincide at a point $a \in V$, then $f_1 = f_2$ everywhere on V.

(2) *(The weak principle of analytic continuation).*

If $f_1 = f_2$ in an open set on V, $f_1 = f_2$ everywhere on V.

Proof. The weak principle follows at once from the strong principle so that we have only to prove the latter. Let E be the set of points where f_1 and f_2 and all their derivatives coincide. It is clear that E is closed. Suppose now that $b \in E$. Then all the components of the functions $\phi'_k \circ f_1 \circ \phi_l^{-1}$ and $\phi'_k \circ f_2 \circ \phi_l^{-1}(b \in \mathscr{O}_l, f_1(b) = f_2(b) \in \mathscr{O}'_k)$ and all their derivatives coincide at $\phi_l(b)$. By expanding in power series about $\phi_l(b)$, it follows that these components and their derivatives coincide in a neighbourhood of $\phi_l(b)$ so that E is open. Since $a \in E$ and V is connected, $E = V$ and the theorem follows. $\square$

Chapter 4

Analytic Continuation

All the manifolds considered in this and the next lecture are assumed to be connected. 24

Definition. Let V, W be two complex analytic manifolds of complex dimension n, ϕ a mapping of $V \to W$. We say that ϕ is a *local analytic isomorphism* if every point $a \in V$ has an open neighbourhood $\mathscr{O}$ such that ϕ restricted to $\mathscr{O}$ is an analytic isomorphism.

We say that V is spread in W and that ϕ spreads V in W if ϕ is a local analytic isomorphism of V in W.

One may define the continuation of a holomorphic function f on V, to the manifold W in which V is spread by ϕ by saying that g is the continuation of f to W if $f = g \circ \phi$. If such a g exists, it is unique, for if $g' \circ \phi = f = g \circ \phi$, then since ϕ is a local homemorphism, $g = g'$ in an open set on W, and W being connected, $g = g'$ on W.

Example. (i) V is an open set $\subset W$, ϕ is the inclusion map $\phi(a) = a$ for every $a \in V$. The functions in V which can be continued to W are precisely the restrictions to V of holomorphic functions on W.

(ii) V is a convering space of W and ϕ the natural projection. The functions on V which can be continued to W are those functions which have the same value at all points which lie over one point of W.

However, this definition of continuation turns out to be too general
25 to be of use. To have interesting theorems, it is necessary to restrict the definition, and we introduce therefore the following

Definition . Let V be a complex analytic manifold, ϕ a map which spreads V in C^n. (The necessary and sufficient condition that such a ϕ exist is that there are n global functions on V which form a system of local coordinates at each point of V). Let (V', ϕ') be another such pair, ϕ' spreading V' in C^n. Suppose also that there exists a map $\psi : V \to V'$ which spreads V in V', such that $\phi = \phi' \circ \psi$ (ϕ, ϕ', ψ are assumed to be given and fixed). Let f be a holomorphic function on V. We say that f' is the continuation of f from V to V' if f' is a holomorphic function on V' such that $f = f' \circ \psi$.

Maximal continuation.

Let (V, ϕ) be a pair consisting of the (n dimensional) complex analytic manifold V and a spread ϕ of V in C^n. Let f be a holomorphic function on V. Suppose that there exists another such pair $(\tilde{V}, \tilde{\phi})$ with the following properties:

(i) f can be continued to $(\tilde{V}, \tilde{\phi})$, i.e., there exists a holomorphic function $\tilde{f}$ on $\tilde{V}$ and a spread $\tilde{\psi}$ of V in $\tilde{V}$ such that $f = \tilde{f} \circ \tilde{\psi}$, $\phi = \tilde{\phi} \circ \tilde{\psi}$.

(ii) If f' is a continuation of f to the pair (V', ϕ') and ψ is the spread of V in V' such that $f = f' \circ \psi$, $\phi = \phi' \circ \psi$, then there exists a spread χ of V' in $\tilde{V}$ such that $f' = \tilde{f} \circ \chi$ and such that the mapping $\tilde{\psi} : V \to \tilde{V}$ factorises into $\tilde{\psi} = \chi \circ \psi$. Then we can show that we have also $\phi' = \tilde{\phi} \circ \psi$ and we call $(\tilde{V}, \tilde{\phi}, \tilde{\psi}, \tilde{f})$ a *maximal continuation* of (V, ϕ, f).

26 To consider the problem of the existence and uniquencess of a maximal continuation, we shall have to introduce the so-called *sheaf of germs of holomorphic functions*.

Let W be a complex analytic manifold (of complex dimension n). Let $a \in W$. Consider the set of all holomorphic functions in open sets containing the point a. We introduce an equivalence relation in this set of functions by identifying two functions f, g, if, in a neighbourhood of a, $f = g$. The equivalence classes are called germs of *holomorphic functions* at a. f_a will stand for a germ at a. It is clear that $f_a(a)$ has an unambiguous meaing.

Denote by $\mathscr{O}_a$ the set of germs at a. *The sheaf of germs of holomorphic functions on W* is defined to be $\underline{\mathscr{O}}_W = \underline{\mathscr{O}} = \bigcup\limits_{a \in W} \mathscr{O}_a$. A complex analytic structure can be put on $\underline{\mathscr{O}}$ in the following way.

Let $a \in W$ and let $(a, f_a) \in \mathscr{O}_a \cdot f_a$ is defined by a holomorphic function f in a neighbourhood U of a. Also, for every $b \in U$, f defines a germ f_b at b. We define $\bigcup\limits_{b \in U} (b, f_b)$ to be a neighbourhood of a. It is easy to verify that this defines a topology on $\underline{\mathscr{O}}$.

Proposition 1. *$\underline{\mathscr{O}}$ is a Hausdorff space.*

Proof. Let $(a, f_a) \neq (b, g_b)$ be two points of $\mathscr{O}$.

If $a \neq b$ we choose neighbourhoods U_a, V_b of a, b in W such that f_a is determined by f in U_a, g_b by g in V_b and $U_a \cap V_b = 0$. Then $\bigcup\limits_{c \in U_a} (c, f_c)$, $\bigcup\limits_{d \in V_b} (d, g_d)$ are disjoint neighbourhoods of (a, f_a), (b, g_b) in $\underline{\mathscr{O}}$. $\square$

If $a = b$, then $f_a \neq g_a$. Let U be a connected neighbourhood of a on **27**
W such that f_a, g_a are defined by holomorphic functions f, g in U. Then the neighbourhoods $\bigcup\limits_{c \in U} (c, f_c)$, $\bigcup\limits_{c \in U} (c, g_c)$ of (a, f_a) (a, g_a) are disjoint, for if $(c, f_c) = (c, g_c)$, then f, g coincide in a neighbourhood of c, and since U is connected, $f = g$ in U so that $f_a = g_a$ which is not the case.

Let p be the projection $\underline{\mathscr{O}} \to W$ defined by $p(a, f_a) = a$. This is a mapping of $\underline{\mathscr{O}}$ onto W. It is a local homeormorphism as follows at once from the definition of the topology on $\underline{\mathscr{O}}$. It is clear now, how p can be used to carry over the complex analytic structure from W to $\underline{\mathscr{O}}$.

Let now V be an n dimensional complex analytic manifold, ϕ a spread of V in C^n. Let $\underline{\mathscr{O}}$ be the sheaf of germs of holomorphic functions on C^n. Let f be a holomorphic function on V. Let $a \in V$ and U a neighbourhood of a such that ϕ, restricted to U is an analytic isomorphism. Then the holomorphic function $f \circ \phi^{-1}$ in $\phi(U)$ defines a germ in $\underline{\mathscr{O}}$, viz., $(f \circ \phi^{-1})_{\phi(a)}$. We define a mapping $\tilde{\psi} : V \to \underline{\mathscr{O}}$ by setting $\tilde{\psi}(a) = (\phi(a), (f \circ \phi^{-1})_{\phi(a)}) \in \underline{\mathscr{O}}$. $\tilde{\psi}$ is a local analytic isomorphism which spreads (V, ϕ) in $\underline{\mathscr{O}}$.

Let $\tilde{V}$ be the connected component in $\underline{\mathscr{O}}$ of $\phi(V)$, and define $\tilde{\phi}$ to be the restriction to $\tilde{V}$ of the projection $p : \underline{\mathscr{O}} \to C^n$ and define $\tilde{f}(b, g_b) = g_b(b)$ for $(b, g_b) \in V$. Clearly $\phi = \tilde{\phi} \circ \psi$ and $f = \tilde{f} \circ \psi$. Hence $(\tilde{V}, \tilde{\phi}, \tilde{\psi}, \tilde{f})$ is a continuation of (V, ϕ, f). We assert that it is a maximal continuation. Suppose that (V', ϕ', f') is any continuation of (V, ϕ, f). Let ψ be a local analytic isomorphism $V \to V'$ such that $f = f' \circ \psi$, $\phi = \phi' \circ \psi$. We
28 can apply the above reasoning to (V', ϕ', f') to continue it to $(\tilde{V}', \tilde{\phi}', \tilde{f}')$. The point $\psi(a) \in V'(a \in V)$ is mapped onto $(\phi'(\psi(a)), (f' \circ \phi'^{-1})_{\phi'(\psi(a))})$ and this $= (\phi(a), (f \circ \phi^{-1})_{\phi(a)})$. Hence the images of V and V' have a common point in $\underline{\mathscr{O}}$ and by the definition of $\tilde{V}$, $\tilde{V}'$ we have $\tilde{V} = \tilde{V}'$. It follows easily that $(\tilde{V}, \tilde{\phi}, \tilde{\psi}, \tilde{f})$ is maximal.

If now $(\tilde{V}, \tilde{\phi}, \tilde{f})$ and $(\tilde{V}', \tilde{\phi}', \tilde{f}')$ are both maximal continuations of (V, ϕ, f), $\tilde{\psi} : V \to \tilde{V}$, $\tilde{\psi}' : V \to \tilde{V}'$ the corresponding spreads of V in $\tilde{V}$, $\tilde{V}'$ respectively, then there exists a spread χ of $\tilde{V}$ in $\tilde{V}'$ such that $\tilde{f} = \tilde{f}' \circ \chi$, $\tilde{\psi}' = \chi \circ \tilde{\psi}$, $\tilde{\phi} = \tilde{\phi}' \circ \chi$. Using the similar spread χ_1 of $\tilde{V}'$ in $\tilde{V}$ it is easily shown that χ is an analytic isomorphism. Thus we have the following

Theorem. *Let V be a complex analytic manifold, ϕ a spread of V in C^n. Let f be a holomorphic function on V. Then there exists a maximal continuation $(\tilde{V}, \tilde{\phi}, \tilde{\psi}, \tilde{f})$ of (V, ϕ, f). If $(\tilde{V}, \tilde{\phi}, \tilde{\psi}, \tilde{f})$ and $(\tilde{V}', \tilde{\phi}', \tilde{\psi}', \tilde{f})$ are two maximal continuations, then there exists an analytic isomorphism $\chi : V \to V'$ such that $\tilde{f} = \tilde{f}' \circ \chi$, $\tilde{\psi} = \chi \circ \tilde{\psi}$, $\tilde{\phi} = \tilde{\phi}' \circ \chi$.*

Chapter 5

Envelopes of Holomorphy

Let V be a complex analytic manifold of (complex) dimension n and let ϕ spread V in C^n. Let $F = (f_i)_{i\in I}$ be a subset of the set of all holomorphic functions on V. We say that $(V, \phi, (f_i)_{i\in I})$ is continuted to $(V', \phi', \psi, (f_i')_{i\in I})$ if there exists a complex analytic manifold V', a spread ϕ' of V' in C^n, a system $(f_i')_{i\in I}$ of holomorphic functions f_i' on V' and a local isomorphism ψ of V into V' such that $\phi = \phi' \circ \psi$ and $f_i = f_i' \circ \psi$ for $i \in I$. This process is called *simultaneous continuation* of $(f_i)_{i\in I}$ to (V', ϕ') from (V, ϕ). A *maximal continuation* of $(f_i)_{i\in I}$ is a continuation $(\tilde{V}, \tilde{\phi}, \tilde{\psi}, (\tilde{f}_i)_{i\in I})$ such that if $(V', \phi', \psi', (f_i')_{i\in I})$ is any continuation of $(V, \phi, (f_i)_{i\in I})$, then there is a local isomorphism χ of $V' \to \tilde{V}$ such that for all $i \in I$, $f_i' = \tilde{f}_i \circ \chi$ and $\tilde{\psi} = \chi \circ \psi'$. It follows that $\phi' = \tilde{\psi} \circ \chi$. 29

We can prove the existence and uniquencess of a maximal (simultaneous) continuation of a given system $(V, \phi, (f_i)_{i\in I})$ in a manner similar to the proof of the theorem in IV.

Consider an open neighbourhood U of a point $a \in C^n$. Let $(g_i)_{i\in I}$ be a family of holomorphic functions in U (indexed by I). Let $(g_i')_{i\in I}$ be another such family defined in a neighbourhood U' of a. Identify $(g_i)_{i\in I}$ and $(g_i')_{i\in I}$ if there existsaneighbourhood W of a, $W \subset U \cap U'$ such that $g_i = g_i'$ in W for every $i \in I$. Denote by $(g_i)_a$ an equivalence class of the set of all families $(g_i)_{i\in I}$ of holomorphic functions, such that all functions of one family are defined in a fixed neighbourhood of a by the equivalence relation defined by this identification. The set of all

$(g_i)_a$ is denoted by $\mathscr{O}_{I,a}$. Let $\mathscr{O}_I = \bigcup\limits_{a \in C^n} \mathscr{O}_{I,a}$. Then $\underline{\mathscr{O}_I}$ is a sheaf. The
30 topology on $\underline{\mathscr{O}_I}$ is defined exactly as before: if $(a, (g_i)_a) \in \underline{\mathscr{O}_I}$ and U is a neighbourhood of a, $(g_i)_{i \in I}$ a family of holomorphic functions in U defining $(g_i)_a$, then $\mathscr{U} = \bigcup\limits_{b \in U} (b, (g_i)_b)$ [$(g_i)_b$ is the equivalence class defined by $(g_i)_{i \in I}$ at b] is an open neighbourhood of $(a, (g_i)_a)$. Exactly as in IV, we put on $\underline{\mathscr{O}_I}$ a complex analytic structure and define a mapping ψ from (V, ϕ) to $\underline{\mathscr{O}_I}$: $\psi(a) = (\psi(a), (g_i \circ \phi^{-1})_{\phi(a)})$ and show that this indeed gives us a maximal continuation. The uniqueness is proved in the same way as in IV.

The most important case is that in which F consists of all holomorphic functions on V. In this case the maximal continuation $(\tilde{V}, \tilde{\phi}, \tilde{\psi})$ is called the *envelope of holomorphy* of (V, ϕ).

Let V be a complex analytic manifold, ϕ_1, ϕ_2 two local analytic isomorphisms of V into C^n. Let $F = (f_i)_{i \in I}$ be a family of holomorphic functions on V. Let $(\tilde{V}_1, \tilde{\phi}_1, \tilde{\psi}_1, \tilde{f}_{1i})$, $(\tilde{V}_2, \tilde{\phi}_2, \tilde{\psi}_2, \tilde{f}_{2i})$ be the maximal continuations of (V, ϕ_1, f_i), (V, ϕ_2, f_i) respectively. The two continuations are said to be *isomorphic* if there exists an analytic isomorphism χ of $\tilde{V}_1$ onto $\tilde{V}_2$ such that $\tilde{\psi}_2 = \chi \circ \tilde{\psi}_1$ and $\tilde{f}_{1i} = \tilde{f}_{2i} \circ \chi$ for all $i \in I$.

We have the following

Theorem 1. *Let $F = (f_i)_{i \in I}$ consist of all the holomorphic functions on the complex analytic manifold V. Let ϕ_1, ϕ_2 be two maps which spread V in C^n. Let $(\tilde{V}_1, \tilde{\phi}_1, \tilde{\psi}_1)$, $(\tilde{V}_2, \tilde{\phi}_2, \tilde{\psi}_2)$ be the envelopes of holomorphy of (V, ϕ_1), (V, ϕ_2) respectively. Then $(\tilde{V}_1, \tilde{\phi}_1, \tilde{\psi}_1)$ and $(\tilde{V}_2, \tilde{\phi}_2, \tilde{\phi}_2)$ are isomorphic.*

31 *Proof.* Consider the components of the mapping $\phi_2 : V \to C^n$. They are holomorphic functions on V and they can be continued to $\tilde{V}_1$ and this gives us a mapping ϕ_2' of $\tilde{V}_1$ to C^n, such that $\phi_2 = \phi_2' \circ \tilde{\psi}_1$. Let J be the Jacobian of ϕ_2 with respect to the local coordinates defined by ϕ_1. Then J is a holomorphic function on V, and since ϕ_2 is a local isomorphism, $J \neq 0$ so that $1/J$ is holomorphic on V. Hence $1/J$ (resp. J) has a continuation $(\widetilde{1/J})_1$ (resp. $\tilde{J}_1$) to $\tilde{V}_1$. Clearly we have $(\widetilde{1/J})_1 X \tilde{J}_1 = 1$ on the image of V by $\tilde{\psi}_1$ in $\tilde{V}_1$ and since $\tilde{V}_1$ is connected, $(\widetilde{1/J})_1 X \tilde{J}_1 = 1$ everywhere on V_1 so that $\tilde{J}_1 \neq 0$ throughout V_1. Moreover $\tilde{J}_1$ is the Jacobian of ϕ_2' with respect to the local coordinates defined by $\tilde{\phi}_1$ so that

ψ_2' *is a local analytic isomorphism,* and spreads $\tilde{V}_1$ in C^n. The situation is explained by the following diagram:

$$\begin{tikzcd}
 & & V \arrow[dl, "\tilde{\psi}_1"'] \arrow[dr, "\tilde{\psi}_2"] \arrow[ddll, "\phi_1"'] \arrow[ddrr, "\phi_2"] & & \\
 & \widetilde{V}_1 \arrow[rr, shift left, "\chi_1"] \arrow[dl, "\tilde{\phi}_1"'] \arrow[drrr, "\phi_2'"] & & \widetilde{V}_2 \arrow[ll, shift left, "\chi_2"] \arrow[dr, "\tilde{\phi}_2"] & \\
C_1^n & & & & C_2^n
\end{tikzcd}$$

Since $(\tilde{V}_2, \tilde{\phi}_2, \tilde{\psi}_2)$ is maximal, this implies that there exists a local analytic isomorphism $\chi_1 : \tilde{V}_1 \to \tilde{V}_2$ such that $\tilde{\psi}_2 = \chi_2 \circ \tilde{\psi}_1$ and $\tilde{f}_{1i} = \tilde{f}_{2i} \circ \chi_1$. In the same way, we prove that there exists a local analytic isomorphism $\chi_2 : \tilde{V}_2 \to \tilde{V}_1$ such that $\tilde{\psi}_1 = \chi_2 \circ \tilde{\psi}_2$.

It follows that $\tilde{\psi}_2 = \chi_1 \circ \chi_2 \circ \tilde{\psi}_2$ so that $\chi_1 \circ \chi_2 = I_{\tilde{V}_2}$ (the identity **32**
mapping of $\tilde{V}_2$) on the image of V under $\tilde{\psi}_2$ in $\tilde{V}_2$ and hence, by the principle of analytic continuation, on all $\tilde{V}_2$. Similarly $\chi_2 \circ \chi_1 = I_{\tilde{V}_1}$ on $\tilde{V}_1$, so that χ_1 is an analytic isomorphism of $\tilde{V}_1$ onto $\tilde{V}_2$. Since $\tilde{\psi}_2 = \chi_1 \circ \tilde{\psi}_1$ and $\tilde{f}_{1i} = \tilde{f}_{2i} \circ \chi_1$, this proves the theorem.

We can prove also the following □

Theorem 2. *Let V, V' be complex analytic manifolds, and let ϕ, ϕ' spread V, V' in C^n. Let g spread V in V'. (It is not required that $\phi = \phi' \circ g$. Let $(\tilde{V}, \tilde{\phi}, \tilde{\psi})$, $(\tilde{V}', \tilde{\phi}', \tilde{\psi}')$ be the envelopes of holomorphy of (V, ϕ), (V', ϕ'). Then there exists a local analytic isomorphism $\tilde{g}$ of $\tilde{V}'$ such that the following diagram is commutative:*

$$\begin{tikzcd}
V \arrow[r, "g"] \arrow[d, "\tilde{\psi}"'] & V' \arrow[d, "\tilde{\psi}'"] \\
\tilde{V} \arrow[r, "\tilde{g}"'] & \tilde{V}'
\end{tikzcd}$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & V \arrow[rrr, "g"] \arrow[dr, "\tilde{\psi}"] \arrow[ddl, "\phi"'] & & & V' \arrow[dl, "\tilde{\psi}'"'] \arrow[ddr, "\phi'"] & \\
 & & \tilde{V} \arrow[r, "\tilde{g}"] \arrow[dll, "\tilde{\phi}"'] \arrow[drrr, "\bar{\phi}"] & \tilde{V}' \arrow[drr, "\tilde{\phi}'"] & & \\
C_1^n & & & & & C_2^n
\end{tikzcd}$$

V is spread in C^n by $\phi' \circ g$ and by Theorem 1, since $\tilde{V}$ is maximal, there is a local analytic isomorphism $\bar{\phi} : \tilde{V} \to C^n$ such that $\phi' \circ g = \bar{\phi} \circ \tilde{\psi}$.
33 Also, $\tilde{V}'$ is a maximal continuation for the functions on V induced by g from those on V', so that, since $\tilde{V}$ is maximal for *all* functions on V, $\tilde{V}$ is "intermediate" between V and $\tilde{V}'$ and there is a spread $\tilde{g}$ of $\tilde{V}$ in $\tilde{V}'$ with the properties we require.

This has the following □

Corollary . *Let V be a complex analytic manifold and σ an analytic automorphism of V. Let V be spread in C^n by ϕ, and $(\tilde{V}, \tilde{\phi}, \tilde{\psi})$ be the envelope of holomorphy of (V, ϕ). Then, there exists an analytic automorphism $\tilde{\sigma}$ of $\tilde{V}$ such that the following diagram is commutative:*

$$\begin{tikzcd}
V \arrow[r, "\sigma"] \arrow[d, "\tilde{\psi}"'] & V \arrow[d, "\tilde{\psi}"] \\
\tilde{V} \arrow[r, "\tilde{\sigma}"'] & \tilde{V}
\end{tikzcd}$$

Definition 1. *Let V be a complex analytic manifold, ϕ a spread of V in C^n. Let $(\tilde{V}, \tilde{\phi}, \tilde{\psi})$ be the envelope of holomorphy of (V, ϕ).*

(V, ϕ) is called a domain of holomorphy of $\tilde{\psi}$ is an analytic isomorphism of V onto $\tilde{V}$.

Note that, in the definition above, (V, ϕ) is not called a domain of holomorphy if V and $\tilde{V}$ are isomorphic, but only if $\tilde{\psi}$ is an isomorphism. Nevertheless, Theorem 1 shows that if V is a complex analytic manifold,

ϕ_1, ϕ_2, two mappings which spread V in C^n, (V, ϕ_1) is a domain of holomorphy if and only if (V, ϕ_2) is. For, if $(\tilde{V}_1, \tilde{\phi}_1, \tilde{\psi}_1)$, $(\tilde{V}_2, \tilde{\phi}_2, \tilde{\psi}_2)$ are the envelopes of (V, ϕ_1), (V, ϕ_2), then there is an analytic isomorphism χ of V_1 onto $\tilde{V}_2$ such that $\tilde{\psi}_2 = \tilde{\psi}_1 \circ \chi$ and $\tilde{\psi}_1$ is an isomorphism if and only if $\tilde{\psi}_2$ is. This justifies the following

Definition 2. *Let V be a complex analytic manifold which can be spread in C^n. Then V is called a domain of holomorphy, if, for some spread ϕ of V in C^n, (V, ϕ) is a domain of holomorphy.* **34**

Theorem 1 is, in general, false, if, instead of considering the family of all holomorphic functions on V, we take only a subfamily. The following is a counter-example.

Let Γ be the complex ζ-plane, C the complex z-plane. Spread Γ in C by the two mappings $\phi_1(\zeta) = \zeta$ and $\phi_2(\zeta) = e^{\zeta}$ respectively. Let $f(\zeta) = e^{\zeta}$. The maximal continuation of (Γ, ϕ_1, f) is $(C, \tilde{f}_1)$ where $\tilde{f}_1(z) = e^z$. Since Γ is the universal convering surface of $C^* = C - (0)$ under the projection $\phi_2(\zeta)$ (which naturally has the same value at all points lying over one point on C^*) and f as a mapping $\Gamma \to C$ coincides with ϕ_2, the maximal continuation of (Γ, ϕ_2, f) is $(C, \tilde{f}_2)$ where $\tilde{f}_2(z) = z$. But (since e.g. z has zeros and e^z has not) there is no isomorphism of C into itself taking z to e^z.

Chapter 6

Domains of Holomorphy: Convexity Theory

An important problem is to find necessary and sufficient conditions under which a manifold which can be spread in C^n is a domain of holomorphy. The results that are presented below are due to CartanThullen [2] (see also [1]). 35

$S(z, r)$ is the open polydisc with centre z and radius r in C^n, i.e., the set of points $z' \in C^n$ with

$$|z'_j - z_j| < r, \quad j = 1, \ldots, n.$$

Definition 1. *Let V be a complex manifold spread in C^n by ϕ. Let $z \in V$. By the polydisc $S(z, r) \subset V$ with centre z and radius r is meant the open set $\mathscr{O}$ containing z (if it exists) such that the restriction of ϕ to $\mathscr{O}$ is an analytic isomorphism of θ onto $S(\phi(z), r) \subset C^n$.*

By the distance of $z \in V$ to the boundary of V, $d(z)$, is meant the radius of the maximal polydisc $S(z, r) \subset V$; the distance $d(K)$ of a compact set K to the boundary of V is $\inf\limits_{z \in K} d(z)$. $\{d(z), d(K)$ depend of course, on $\phi\}$.

Definition. Let K be a compact subset of the complex manifold V, and let $\mathscr{C}$ be a family of holomorphic functions on V. The $\mathscr{C}$-envelope of K,

$\hat{K}_{\mathscr{C}}$ is the set of $z \in V$ for which there exists a $c_z > 0$ such that for all $f \in \mathscr{C}$, $|f(z)| \leq c_z \|f\|_K$ ($\|f\|_K = \sup\limits_{x \in K} |f(x)|$, see p.2).

Examples. (a) Let $V = C$ be the complex plane, $\mathscr{C}$ the family of polynomials. If K is a compact set $\subset C$ and L is the union of the relatively
36 compact components of the complement of K (relatively compact in C), then it can be shown that $\hat{K}_{\mathscr{C}} = K \cup L$. If, on the other hand, we consider $C^* = C - (0)$ and take K to be an annulus enclosing 0, $\mathscr{C}$ as the set of all holomorphic functions in C^*, then $\hat{K}_{\mathscr{C}} = K$.

(b) In $V = C^2$ if we take K to be the closure of the domain in the example on p.12, viz., $|z_1| \leq 1$, $|z_2| \leq \epsilon$; $1 - \epsilon \leq |z_1| \leq 1$, $|z_2| \leq 1$, $\mathscr{C}$ to be the family of all polynomials (or all holomorphic functions) in C^2, then $\hat{K}_{\mathscr{C}}$ is the polydisc $|z_1| \leq 1$, $|z_2| \leq 1$.

(c) If on the manifold V, $\mathscr{C}$ is such that $f \in \mathscr{C}$ implies $f^p \in \mathscr{C}$ for every integer $p > 0$, then

$$\hat{K}_{\mathscr{C}} = \left\{ z \in V \middle| |f(z)| \leq \|f\|_K \text{ for every } f \in \mathscr{C} \right\}.$$

It the set of points defined above is denoted K_1, clearly $K_1 \subset \hat{K}_{\mathscr{C}}$. If $z \in \hat{K}_{\mathscr{C}}$, $|f(z)|^p \leq c_z \|f\|_K^p$, $|f(z)| \leq c_z^{1/p} \|f\|_K$ and letting $p \to \infty$, it follows that $z \in K_1$.

Let V be a complex manifold and let ϕ spread V in C^n. Let f be a holomorphic function on V and $z \in V$. Let U be an open neighbourhood of z such that $\phi|U$ is an isomorphism. Then $\dfrac{\partial f(z)}{\partial z_i}$ is defined to be $\dfrac{\partial (f \circ \phi^{-1})(\phi(z))}{\partial z_i}$, ϕ^{-1} being the inverse of $\phi|U$.

Theorem 1. *Let V be a complex manifold and suppose that ϕ spreads V in C^n. Let $\mathscr{C}$ be a family of holomorphic functions on V, stable for derivation, i.e., $f \in \mathscr{C}$ implies $\dfrac{\partial f}{\partial z_i} \in \mathscr{C}$. Suppose also that the canonial mapping of V to the maximal continuation of V is an isomorphism. If K is an arbitrary compact subset of V, then the distances of K and $\hat{K}_{\mathscr{C}}$ to the boundary of V are the same.*

37 *Proof.* Since V is itself maximal for $\mathscr{C}$, it is clearly sufficient to prove

the following:

If $z \in \hat{K}_{\mathscr{C}}$ and $0 < \rho < d(K)$, then any $f \in \mathscr{C}$ can be continued to $S(z,\rho)$, i.e., that the functions $f \circ \phi^{-1}$ in a neighbourhood of $\phi(z)$, can be continued to the polydisc $S(\phi(z),\rho) \subset C^n$.

Let $L = \bigcup_{x\in K} \overline{S(x,\rho)}$. Then L is the continuous image of the compact space $K \times \overline{S(0,\rho)}\{S(0,\rho) \subset C^n\}$ and so L is compact. Let, for $f \in \mathscr{C}$, $M(f) = \sup_{z\in L} |f(z)|$. It follows from the Cauchy inequalities that $|D^J f(z)| \leq \dfrac{J!M(f)}{\rho^{|J|}}$ (where $J = (j_1, \ldots, j_n)$, D^J is the operator $\dfrac{\partial^{|J|}}{\partial z_1^{j_1} \ldots \partial z_n^{j_n}}$, $|J| = j_1 + \cdots + j_n$ and $J! = j_1! \ldots j_n!$). From the definition of $\hat{K}_{\mathscr{C}}$ and the fact that $\mathscr{C}$ is stable for derivation, it follows that if $z \in \hat{K}_{\mathscr{C}}$, $|D^J f(z)| \leq \dfrac{C_z J! M(f)}{\rho^{|J|}}$. If, therefore,

$$g(z') = \sum_{J\in N^n} \frac{D^J f(z)}{J!} (\phi(z') - \phi(z))^J$$

for any $z \in \hat{K}_{\mathscr{C}}$, the series converges normally in the polydisc $S(\phi(z), \rho) \subset C^n$ and it is clear that it continues f to $S(\phi(z),\rho)$ for every $f \in \mathscr{C}$. The theorem follows. $\square$

Theorem 2. *Let V be a complex analytic manifold, ϕ a spread of V in C^n. Let $\mathscr{C}$ be a family of holomorphic functions on V having the following properties:*

(1°) *$\mathscr{C}$ is a closed subalgebra of $\mathscr{H}_V$, $1 \in \mathscr{C}$.*

(2°) *If $\phi(z) = \phi(z')(z \neq z')$, then there exists a function $f \in \mathscr{C}$ such that $f_z \neq f_{z'}$ (f_a is the germ $f_a = (f \circ \phi^{-1})_{\phi(a)}$; see IV and V).*

(3°) *If K is a compact subset of V and $z \in \hat{K}_{\mathscr{C}}$, $S(z,r)$, the maximal (open) polydisc about z in V contains points not in $\hat{K}_{\mathscr{C}}$.* **38**

Then, there exists a function $g \in \mathscr{C}$ such that g cannot be continued outside (V,ϕ).

Proof. The main step in the proof is to construct a function $g \in \mathscr{C}$ such that

(a) if $\phi(z) = \phi(z')$, then $g_z \neq g_{z'}$ and it is clearly enough that for a countable dense set $\{z_m\}$ on V, $g_{z_m} \neq g_{z'_m}$ if z'_m is any point such that $\phi(z_m) = \phi(z'_m)$. (The existence of a countable dense set follows from the Poincaré-Volterra theorem);

(b) for a countable dense set $\{z_m\}$ of points on V, let $S(z_m, r_m)$ denote the maximal open polydisc about z_m. Then $g(z)$ has zeros of arbitrarily large multiplicity in every $S(z_m, r_m)$.

□

The proof then divides into three steps.

Step 1. The existence of the function g implies that (V, ϕ) is the maximal domain for g.

Proof. Let $(\tilde{V}, \tilde{\phi}, \tilde{\psi})$ be the maximal domain of g. V consists of pairs $(\phi(z), (g \circ \phi^{-1})_{\phi(z)}) = (\phi(z), g_z)$. We show that $\tilde{\psi}$ is one-one. If $(\phi(z), g_z) = (\phi(z'), g_{z'})$ then $\phi(z) = \phi(z')$, $g_z = g_{z'}$ and since, if $z \neq z'$, $g_z \neq g_{z'}$ this implies that $z = z'$. $\tilde{\psi}$ therefore identifies V with an open subset of $\tilde{V}$. In what follows, we assume this identification made, and show that $V = \tilde{V}$. In the first place, $S(z_m, r_m)$ is the maximal polydisc about z_m in $\tilde{V}$ for if it were not, $S(z_m, r_m)$ is relatively compact in $\tilde{V}$ and $\tilde{g}$, the continuation of g to $\tilde{V}$ cannot have zeros of unbounded multiplicity in $S(z_m, r_m)$. Now
39 suppose that $V \neq \tilde{V}$. Since $\tilde{V}$ is connected, there is a point $b \in \tilde{V}$ so that $b \notin V$, $b \in \bar{V}$. If c is near enough to b it is clear that there is a polydisc $S(c, \rho)$ containing b. But if c is a z_m, then there is a polydisc $S(z_m, \rho) \subset \tilde{V}$, $\not\subset V$ which is not the case. This completes Step 1. □

Step 2. Construction of a funtion $f \in \mathscr{C}$ having property (b). Let $S_m = S(z_m, r_m)$ and consider the following sequence of polydiscs:

$$S_1, S_2, S_1, S_2, S_3, S_1, S_2, S_3, S_4, \ldots$$

Denote its p-th term by $\sum_p$. Let K_p be a sequence of compact sets $\subset V$ such that $K_p \subset \overset{o}{K}_{p+1}$ and $\bigcup\limits_{p=1}^{\infty} K_p = V$. Now, by property 3° in the hypotheses of Theorem 2, $\sum_p \not\subset (\hat{K}_p)_{\mathscr{C}}$. Hence there is a function $h \in \mathscr{C}$ and a point $z(p) \in \sum_p$ so that $|h(z^{(p)})| > ||h||_{K_p}$, by example ($c$) on p.36, since $\mathscr{C}$ is an algebra. If $f_p(z) = (h(z)/h(z^{(p)}))^\nu$ and ν is large enough, f_p satisfies $f_p \in \mathscr{C}$, $|f_p(z)| \leq 2^{-p}$ on K_p, $f_p(z^{(p)}) = 1$ with $z^{(p)} \in \sum_p$. Let $f = \prod\limits_{p=1}^{\infty}(1 - f_p)^p$. It is easily verified that this product converges in $\mathscr{H}_V$ and that $f \not\equiv 0$. Since $\mathscr{C}$ is a closed subalgebra of $\mathscr{H}_V$, $f \in \mathscr{C}$. Also f has a zero of order at least p in $\sum_p$ and since each $S_m = S(z_m, r_m)$ occurs infinitely often in the sequence $\{\sum_p\}$, this concludes Step 2.

Step 3. Modification of the function f, such that the resulting function has properties (a) and (b).

Let $\mathscr{C}_f$ be the closure of the set of all function fh, $h \in \mathscr{C}$, where f is the function constructed in Step 2. Since $\mathscr{C}$ is closed, $\mathscr{C}_f \subset \mathscr{C}$ and trivially, each $g \in \mathscr{C}_f$ has property (b).

Let (X_m) be a countable dense set on V and (Y_m) the set of all points having $\phi(Y_m) = \phi(X_m)$ (this set is countable by the Poincaré-Volterra theorem). Let $\mathscr{O}(m, Y_m)$ be the set of functions $h \in \mathscr{C}_f$ such that $h_{X_m} \neq h_{Y_m}$ and $\phi(X_m) = \phi(Y_m)$.Clearly $\mathscr{O}(m, Y_m)$ is open in $\mathscr{C}_f$. We prove below that each $\mathscr{O}(m, Y_m)$ is dense in $\mathscr{C}_f$. It then follows from Baire's theorem applied to $\mathscr{C}_f$ ($\mathscr{C}_f$ is a complete metrizable space since V is countable at infinity) that $\mathscr{O} = \cap\mathscr{O}(m, Y_m)$ is dense in $\mathscr{C}_f$ and if $g \in \mathscr{O}$, $g \not\equiv 0$, g has properties (a) and (b). Thus to complete Step 3, it remains only to prove that $\mathscr{O}(m, Y_m)$ is dense in $\mathscr{C}_f$. 40

Let $k \in \mathscr{C}_f$, $\phi(Y_m) = \phi(X_m)$. If $k_{Y_m} \neq k_{X_m}$, $k \in \mathscr{O}(m, Y_m)$. If $k \notin \mathscr{O}(m, Y_m)$, let $h \in \mathscr{C}_f$ be so that $h_{X_m} \neq h_{Y_m}$ (h exists: for if f has this property one may take $h = f$ while if $fX_m = f_{Y_m}$ and l is such that $l_{X_m} \neq l_{Y_m}$ (hypothesis (2°)) one may take $h = fl$). Then, if $|\lambda|$ is small enough, $k + \lambda h$ defines different germs at X_m and Y_m, and is in the closure of the functions $k + \lambda h$, $|\lambda|$ small, which are in $\mathscr{O}(m, Y_m)$ and $k \in \overline{\mathscr{O}(m, Y_m)}$. This proves that $\mathscr{O}(m, Y_m)$ is dense in $\mathscr{C}_f$ and thus completes Step 3 and with Steps 1, 2 and 3, the proof of Theorem 2 is complete.

Corollary. *Under the hypotheses of Theorem 2*

(i) (V, ϕ) *is a domain of holomorphy.*

(ii) (V, ϕ) *is the maximal continuation of* $(V, \phi, \mathscr{C})$.

Theorems 1 and 2 show that the following are equivalent:

(a) (V, ϕ) *is a domain of holomophy.*

(b) If $\phi(z) = \phi(z')$, $z \neq z'$ *there exists* $f \in \mathscr{H}_V$ *so that* $f_z \neq f_{z'}$, *and, if* K *is a compact subset of* V, $d(K) = d(\hat{K}_{\mathscr{H}_V})$.

41 *The last condition may be replaced by the apparently weaker condition that for any* $z \in \hat{K}_{\mathscr{H}_V}$ *the maximal polydisc about* z *contains points not in* $\hat{K}_{\mathscr{H}_V}$. *Also, it follows from Theorems 1 and 2 that if* V *is a domain of holomorphy, then there exists a function* $g \in \mathscr{H}_V$ *which separates points in the sense that if* $z \neq z'$, *and* $\phi(z) = \phi(z')$, *then* $g_z \neq g_{z'}$, *such that* g *cannot be continued outside* V, *i.e., if the family of all functions on* V *cannot be continued simultaneously outside* V, *there is one function which itself cannot be continued.*

Chapter 7

Convexity Theory (continued)

1. The maximal continuation of a family of holomorphic functions on a manifold V (spread in C^n) was defined in V. Certain analogous concepts will now be defined. 42

Let V be a complex manifold spread in C^n by ϕ and let $\mathscr{C}$ be a family of holomorphic functions on V.

An *N-continuation* of $(V, \phi, \mathscr{C})$ is a continuation $(V', \phi', \psi', \mathscr{C}')$ such that $\{f_i\}_{i \in I}$ is any subfamily of $\mathscr{C}$, normally convergent in V and f'_i is the continuation of f_i, then the family $\{f'_i\}_{i \in I}$ converges normally in V'. A maximal N-continuation is now defined in the same way as was maximal continuation.

The following two concepts are defined similarly.

A *maximal U-continuation*: the property considered is that of convergence of sequence of functions of $\mathscr{C}$, in $\mathscr{H}_V$.

Maximal B-continuation: The property considered is that of boundedness of subfamilies in $\mathscr{H}_V$.

In the same way as before, one can prove the existence and uniqueness of maximal $N-$, $U-$ and $B-$ continuations.

The proof of 6, Theorem 1 on p.36 gives us the following result.

Theorem 1′. *Let V be a complex manifold, ϕ a spread of V in C^n. Let $\mathscr{C} \subset \mathscr{H}_V$ and suppose that $\mathscr{C}$ is stable derivation. If (V, ϕ) is itself the* *maximal $N-$ continuation (or $B-$ or $U-$ continuation) of $(V, \phi, \mathscr{C})$, then the distances of K and $\hat{K}_{\mathscr{C}}$ to be boundary of V are the same.*

43 6 Theorem 2 (p.37) also has an analogue:

Theorem 2′. *Let V be a complex manifold, ϕ a spread of V in C^n. Suppose $\mathscr{C}$ has the following properties:*

(1°) *$f \in \mathscr{C}$ implies $\lambda f \in \mathscr{C}$ for every complex λ;*

(2°) *If $\phi(x) = \phi(y)$, there exists $f \in \mathscr{C}$ such that $f_z \neq f_y$;*

(3°) *If $K \subset V$ is compact, and $z \in \hat{K}_{\mathscr{C}}$, the maximal polydisc about z contains points not in $\hat{K}_{\mathscr{C}}$.*

Finally, let $(\tilde{V}, \tilde{\phi}, \tilde{\psi}, \tilde{\mathscr{C}})$ be the maximal $N-$, $B-$, or U-continuation of $(V, \phi, \mathscr{C})$.

Then, $\tilde{\psi}$ is an isomorphism of V onto $\tilde{V}$.

Proof. As in Step 1, in the proof of 6 Theorem 2, $\tilde{\psi}$ is one-one (into). We now construct a sequence of functions (f_p) as follows.

Choose a countable dense set $\{z_m\}$ in V and let S_m be the maximal polydisc about z_m in V. Consider the sequence.

$$S_1, S_2, S_1, S_2, S_3, \ldots$$

and denote its p-th term by $\sum_p$. Let $\{K_p\}$ be a sequence of compact sets so that $K_p \subset \overset{o}{K}_{p+1}$, $\bigcup\limits_{p=1}^{\infty} K_p = V$. Then, by hypothesis (3°), there is a point $z^{(p)} \in \sum_p$, $(z(p) \notin (\hat{K}_p)_{\mathscr{C}})$ and (by the definition of $(\hat{K}_p)_{\mathscr{C}}$) a function $f \in \mathscr{C}$ so that $|f(z^{(p)})| > 2^{2p} \|f\|_{K_p}$. This gives rise to a sequence of functions $\{f_p\}$ such that $\|f_p\|_{K_p} \leq 2^{-p}$, $|f_p(z^{(p)})| > 2^p$. The fact that ψ is onto is proved by reasoning analogous to Step 1 of 6, Theorem 2. □

Examples. (a) Let $\theta \subset C^n$ be a (univalent) Reinhardt domain and let $\mathscr{C}$ be the family of monomials λz^J (λ complex). It is possible to find $\hat{K}_{\mathscr{C}}$
44 very simply. Let K' be the set of $(t_1 z_1, \ldots, t_n z_n)$, where $(z_1, \ldots, z_n) \in K$ and $|t_1| \leq 1, \ldots, |t_n| \leq 1$. Clearly $K' \subset \hat{K}_{\mathscr{C}}$. Also it is clear that

$\hat{K}_{\mathscr{C}}$ is completely characterised by its image in the $(|z_1|, \ldots, |z_n|)$-space, and so in the $(\rho_1, \ldots, \rho_n)$-space $(\rho_j = \log |z_j|)$. As in II, let $K^{'*}$ be the image of K', $\hat{K}^*_{\mathscr{C}}$ the image of $\hat{K}_{\mathscr{C}}$ in the $(\rho_1, \ldots, \rho_n)$ space. K'^* has the following propety: if $(\rho_1, \ldots, \rho_n) \in K'^*$, then $(\rho_1 - a_1, \ldots, \rho_n - a_n) \in K'^*$ if $a_j \geq 0$, $j = 1, \ldots, n$. Also $\hat{K}^*_{\mathscr{C}}$ is defined by inequalitics $j_1\rho_1 + \cdots + j_n\rho_n \leq \log ||z^J||_K$, so that $\hat{K}^*_{\mathscr{C}}$ is clearly convex and so contains the convex closure of K'^*. Since the convex closure of K'^* is the intersection of all closed half spaces containing K'^*, it follows easily that $\hat{K}^*_{\mathscr{C}}$ is the convex closure of K'^* and this gives us the $\mathscr{C}$-envelope of K.

This leads to a necessary and sufficient condition for a Reinhardt domain $\mathscr{O}$, containing 0, to be a domain of holomorphy. If $\mathscr{O}$ is a domain of holomorphy, then by 2 Theorem 1, $\mathscr{O}$ is the maximal N-continuation domain of the family of all monomials. If $\mathscr{O}$ is the maximal N-continuation domain of the monomials, then by Theorem 1′ and 6, Theorem 2, $\mathscr{O}$ is a domain of holomorphy (since the envelope of a compact set with respect to the monomials is trivially larger than that with respect to all holomorphic functions). By the above results, this is so if and only if the image $\mathscr{O}^*$ of $\mathscr{O}$ in the $(\rho_1, \ldots, \rho_n)$-space is convex and such that if $(\rho_1, \ldots, \rho_n) \in \mathscr{O}^*$ then $(\rho_1 - a_1, \ldots, \rho_n - a_n) \in \mathscr{O}^*$ when $a_j \geq 0$. By 6, Theorem 2, it follows that if $\mathscr{O}$ is a Reinhardt domain which is the union of polydiscs, with centre 0, and $\mathscr{O}^*$ is convex, then there is a power series such that $\mathscr{O}$ is precisely the domain of convergence of this power series, a result which was stated on p.14 (Converse of 2, Theorem 2).

(b) Let $\mathscr{O}$ be an open set in C, and let $\mathscr{C} = \mathscr{H}_{\mathscr{O}}$. If K is a compact **45**
subst of $\mathscr{O}$ and L is the union of the relatively compact components of the complement of $K(in\mathscr{O})$, then $\hat{K}_{\mathscr{H}_{\mathscr{O}}} = K \cup L$. It is easy to see from 6, Theorem 2, that $\mathscr{O}$ is a domain of holomophy. In fact, it can be proved that $\hat{K}_{\mathscr{H}_{\mathscr{O}}}$ is compact.

3. Some remarks on domains of holomorphy.

Proposition 1. *Let $\mathscr{O}$ be a (univalent) domain in C^n. The following three conditions are equivalent:*

1) $\mathscr{O}$ is a domain of holomorphy;

2) *If K is a compact subset of $\mathscr{O}$ and $z \in \hat{K}_{\mathscr{H}}$ (where $\mathscr{H} = \mathscr{H}_{\mathscr{O}}$), the maximal polydisc about z in $\mathscr{O}$ contains points not in $\hat{K}_{\mathscr{H}}$;*

3) *If K is a compact subset of $\mathscr{O}$, then $\hat{K}_{\mathscr{H}}$ is compact.*

Proof. After 6, Theorems 1 and 2, it is enough to prove that 1) implies 3). Clearly $\hat{K}_{\mathscr{H}}$ is closed in $\mathscr{O}$, since $\mathscr{H}$ is an algebra and $d(K) = d(\hat{K}_{\mathscr{H}})$. Also $\hat{K}_{\mathscr{H}}$ is bounded in C^n: the function z_i is holomorphic in $\mathscr{O}$ and, by definition of $\hat{K}_{\mathscr{H}}$, $|z'_i| \leq ||z_i||_K$ for every $z' \in \hat{H}_{\mathscr{H}}$.

Suppose $\hat{K}_{\mathscr{H}}$ were not compact. There is then a sequence $\{X_p\}$ of $X_p \in \hat{K}_{\mathscr{H}}$ having no limit point in $\hat{K}_{\mathscr{H}}$. However $\{X_p\}$ has a limit point $X \in C^n$ since $\hat{K}_{\mathscr{H}}$ is bounded. Now X belongs to the boundary of $\mathscr{O}$ since $\hat{K}_{\mathscr{H}}$ is closed in $\mathscr{O}$, so that X has a distance > 0 from K. But if $\mathscr{O}$ is a domain of holomorphy, this implies that X has a positive distance from $\hat{K}_{\mathscr{H}}$ which is not the case, and $\hat{K}_{\mathscr{H}}$ is compact. □

Definition. Let V be a complex manifold. V is called *holomorph-convex* if the $\mathscr{H}_V$-envelope of every compact set is compact.

46 **Proposition 2.** *Let V be a manifold spread in C^n and suppose that V is holomorph-convex. Then V is a domain of holomorphy.*

Let $(\tilde{V}, \tilde{\phi}, \tilde{\psi})$ be the envelope of (V, ϕ).

The proof of Step 1 in 6 Theorem 2 shows that

(1°) *V is a covering spaces of $\tilde{V}$ under projection $\tilde{\psi}$.*

(2°) *Over any point of V lie only finitely many points of V: by definition of $\tilde{V}, \tilde{\psi}$, all holomorphic functions on V have the same value at all points over one point of $\tilde{V}$. If X is any point of $\tilde{V}$, it follows that all points lying over X belong to the $\mathscr{H}_V$-envelope of X and since $\tilde{\psi}$ is a local homeomorphism, this set cannot have a limit point; since V is holomorph-convex, this set must be finite.*

(3°) *$\tilde{V}$ is holomorph-convex: this follows from* (2°).

(4°) *If $x, y \in \tilde{V}$, $x \neq y$, there is a holomorphic function f on $\tilde{V}$ such that $f(x) \neq f(y)$: if $\tilde{\phi}(x) \neq \tilde{\phi}(y)$, this is obvious; if $\tilde{\phi}(x) = \tilde{\phi}(y)$, then, since $\tilde{V}$ is a domain of holomorphy, there exists a function g*

such that $g_x \neq g_y$ and, by going to a derivative of g of sufficiently large order, the existence of f follows.

(1°), (2°), (3°), and (4°) imply that V is a domain of holomorphy by a theorem of J-P. Serre [1, Chap. XX] which is a consequence of Theorem A and B of Oka-Cartan-Serre on Stein manifolds and these will be proved later.

Exercises

1. Let V be a connected complex manifold spread in C^n by ϕ. Suppose that there exists a n-parameter family T^n of analytic automorphisms of V, $\sigma(\alpha_1, \ldots, \alpha_n)$ where the α are real numbers $\mod 2\pi$, such that, if $\phi(z) = (\phi_1(z), \ldots, \phi_n(z))$, then $\phi \circ \sigma(\alpha_1, \ldots, \alpha_n)(z) = (e^{i\alpha_1}\phi_1(z), \ldots e^{i\alpha_n}\phi_n(z))$ for all $\alpha \cdot (V, \phi)$ is then called *a Reinhardt domain*. 47

Prove the following two results:

(a) If $(\tilde{V}, \tilde{\phi})$ is the envelope of holomorphy of (V, ϕ), then $(\tilde{V}, \tilde{\phi})$ is a Reinhardt domain.

(b) If there exists a point $z \in V$ so that $\phi(z) = 0$, then every holomorphic function on V can be expanded in a series $\sum a_J \phi(z)^J$ on V, which converges normally in V. Deduce that $\tilde{\phi}$ is one-one, i.e., $\tilde{V}$ is univalent.

2. Let $\mathscr{O}$ be an open set in C^{n+1} and $(z_1, \ldots, z_n, w)$ be a generic point of $\mathscr{O}$. $\mathscr{O}$ is called a *Hartogs domain* if $(z, w) \in \mathscr{O}$ implies $(z, e^{i\alpha}w) \in \mathscr{O}$ for every real α. Let now $\mathscr{O}$ be a connected Hartogs domain such that there is a point $(z, 0) \in \mathscr{O}$. Then, if $f(z, w)$ is holomorphic in $\mathscr{O}$, prove that $f(z, w)$ can be expanded in a series of the form

$$f(z, w) = \sum_{p=0}^{\infty} a_p(z, w) w^p,$$

where the $a_p(z, w)$ are holomorphic functions in $\mathscr{O}$ which are locallyl independent of w, i.e., $\dfrac{\partial a_p}{\partial w} = 0$ in $\mathscr{O}$, and such that the series converges normally in $\mathscr{O}$.

3. Let $\mathscr{O}$ be the following open set in C^2: **48**

$$\begin{cases} -3 < \mathscr{R}z < 0, \quad |w| < e^{\mathscr{R}z}, \Im z \quad \text{arbitrary} \\ 0 \le \mathscr{R}z < 3, \quad e^{-1/\mathscr{R}z} < |w| < 1, \Im z \quad \text{arbitrary}. \end{cases}$$

($\mathscr{O}$ is a Hartogs domain). Prove that the $a_p(z, w)$ of Exercise 2 are independent of w and deduce that every holomorphic function in $\mathscr{O}$ can be continued to $\mathscr{O}'$, the union of $\mathscr{O}$ and the set of points (z, w) with $0 < \mathscr{R}z < 3$, $|2| \le e^{-1/\mathscr{R}z}$.

Prove also that the mapping $(z, w) \to (e^{i\pi/2}z, w)$ spreads $\mathscr{O}$, but cannot be continued univalently to $\mathscr{O}'$.

Remark. Exercise 1 gives an example of a non-univalent domain whose envelope is univalent, and Exercise 3 gives an example of a univalent domain whose envelope is not univalent.

In the following two exercises, D will denote the closed unit disc $|z| \le 1$ in the C-plane.

4. If $f(z)$ is holomorphic in a neighbourhood of D,

$$\log|f(z)| \le \frac{1}{2\pi} \int_0^{2\pi} \log|f(e^{i\theta})| \mathscr{R} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta$$

5. Let $f_p(z)$, $(p = 1, 2, \ldots)$ be holomorphic in neighbourhoods of D and suppose that (i) $\overline{\lim\limits_{p\to\infty}} \dfrac{1}{p} \log \|f_p\|D < +\infty$, (ii) $\overline{\lim\limits_{p\in\infty}} \dfrac{1}{p} \log|f_p(z)| \le M$, $z \in D$. Prove that $\overline{\lim\limits_{p\in\infty}} \dfrac{1}{p} \log \|f_p\|_D \le M$.

6. (Hartog's main theorem"). Let θ be an open set in C and let $\{f_p(z)\}$ be a sequence of holomorphic functions in θ. The domain of absolute (normal) convergence of the series

$$\sum_{p=0}^{\infty} f_p(z) w^p$$

49 is defined to be the set of points $(z_0, w_0) \in C^2$ such that $\sum\limits_{p=0}^{\infty} f_p(z) w^p$

converges absolutely (normally) in a neighbourhood of (z_0, w_0). Let $R(z)(\overline{R}(z))$ be the greatest number (≥ 0) such that the set $\{z \in \mathscr{O}, |w| < R(z)(\overline{R}(z))$ is contained in the domain of absolute (normal) convergence.

Prove that if $\overline{R}(z) > 0$, the $R(z) = \overline{R}(z)$ (use Exercise 5).

7. Let $\{P_n(z_1, z_2)\}$ be a sequence of homogeneous polynomials (P_n is of degree n) in z_1, z_2. Consider the series

$$\sum_{n=0}^{\infty} P_n(z_1, z_2).$$

The domain of absolute convergence, δ, is defined as the set of $(z_1^{(0)}, z_2^{(0)}) \in C^2$ such tht in a neighbourhood of $(z_1^{(0)}, z_2^{(0)})$ the series $\overset{\infty}{\sum} P_n(z_1, z_2)$ converges absolutely.

Prove the following statements.

(a) If $(z_1, z_2) \in \Delta$ then $(tz_1, tz_2) \in \Delta$ if $0 < |t| \leq 1$.

(b) If Δ is not empty, $(0, 0) \in \Delta$ and the series converges normally near $(0, 0)$.

(c) The series converges normally in Δ.

(Also due to Hartogs; for (b), use Baire's theorem and the maximum principle; for (c) use Exercise 6).

Bibliography

[1] *H. Cartan*: *Séminaire E.N.S.*, 1951/52 (especially lectures VII-X). **50**

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Part II

Differential Properties of the Cube

Chapter 8

d''-cohomology on the cube

1 Differential forms

Let $\mathscr{O} \subset R^n$ be an open set. The concept of (C^∞)-differential form is assumed known. A differential form $\overset{r}{\omega}$ of degree r in $\mathscr{O}$ has a representation 51

$$\overset{r}{\omega} = \sum_{i_1<\ldots<i_r} a_{i_1\ldots i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r} \tag{1}$$

where $\wedge$ is the sign of exterior multiplication. The $a_{i_1\ldots i_r}$ are C^∞ - functions. Also we define the partial derivatives of a form (1) by

$$\frac{\partial \overset{r}{\omega}}{\partial x_i} = \sum_{i_1<\ldots<i_r} \frac{\partial a_{i_1\ldots i_r}}{\partial x_i} dx_{i_1} \wedge \ldots \wedge dx_{i_r}.$$

The differential $d\overset{r}{\omega}$ of the form (1) is defined by

$$d\overset{r}{\omega} = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \overset{r}{\omega}}{\partial x_i}.$$

The operator d has the following properties:

(a) d is a local operator: if $\omega = \omega'$ in an open set U, $d\omega = d\omega'$ in U.

(b) d is linear on the forms considered as a vector space over the complex numbers (but not as a module over C^∞-functions).

(c) $d(\overset{p}{\omega} \wedge \overset{q}{\omega}) = d\overset{p}{\omega} \wedge \overset{q}{\omega} + (-1)^p \overset{p}{\omega} \wedge d\overset{q}{\omega}$.

(d) $dd = 0$.

(e) d is invariant under diffeomorphisms.

52 Of course, d has the property that for functions f (forms of degree 0), $df = \sum \dfrac{\partial f}{\partial x_i} dx_i$ is the ordinary differential of f.

This property with (b), (c), and (d) characterize d completely. The following result, called *Poincaré's theorem* holds:

Let $\mathscr{O}$ be an open ball in R^n (or $\mathscr{O} = R^n$). Let $\overset{p}{\omega}$ be a form of degree p in $\mathscr{O}$, such that $d\overset{p}{\omega} = 0$. Then, there exists a form $\overset{p-1}{\pi}$ of degree $p-1$, such that

$$d\overset{p-1}{\pi} = \overset{p}{\omega}.$$

This will not a proved here. See for instance [6].

2 The operators d' and d''

We now identify C^n and R^{2n}, and set $z_j = x_j + ix_{n+j}$, $j = 1, \ldots, n$, where $(x_1, \ldots, x_{2n})$ are the coordinates in R^{2n}. Then an r-form $\overset{r}{\omega}$ (form of degree r) in $\mathscr{O}$ can be written uniquely in the form

$$\overset{r}{\omega} = \sum_{\substack{p+q=r \\ p,q \geq 0}} \overset{(p,q)}{\omega}$$

where

$$\overset{(p,q)}{\omega} = \sum_{\substack{0<i_1<\ldots<i_p \leq n \\ 0<j_1<\ldots<j_q \leq n}} a_{i_1 \ldots i_p\, j_1 \ldots j_q} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$$

$\overset{(p,q)}{\omega}$ is said to be of *type* (p, q). Its degree is, of course, $p + q = r$. It is easy to verify that one has, for every form ω,

$$d\omega = \sum_{j=1}^{n} dz_j \wedge \frac{\partial \omega}{\partial z_j} + \sum_{j=1}^{n} d\overline{z}_j \wedge \frac{\partial \omega}{\partial \overline{z}_j}.$$

53 The first sum is denoted by $d'\omega$, the second by $d''\omega \cdot d'$, d'' are operators of degree $+1$, i.e., an r-form goes into an $(r+1)$-form under d', d''; more precisely d' *is of type (1,0)*, i.e., a form of type (p, q) goes into one of type $(p+1, q)$, while d'' *is of type (0,1)*, taking forms of type (p, q) into forms of type $(p, q+1)$.

The operators d', d'' have properties similar to those of d. They are the following:

(a) d', d'' are the local operators.

(b) d', d'' are linear.

(c) $d'(\overset{p}{\omega} \wedge \overset{q}{\omega}) = d'\overset{p}{\omega} \wedge \overset{q}{\omega} + (-1)^p \overset{p}{\omega} \wedge d'\overset{q}{\omega}$ and similarly for d''.

(d) $d'd' = 0$, $d'd'' + d''d' = 0$, $d''d'' = 0$.

(e) d', d'' are invariant under analytic isomorphisms (but not under diffeomorphisms). A form ω is said to be d', (d'') *closed* if $d'\omega = 0$ $(d''\omega = 0)$.

3 Triviality of d''-cohomology on a cube

A form ω is said to be *holomorphic* if it is of type $(p, 0)$ and hte coefficient of $dz_{i_1} \ldots dz_{i_p}$ are holomorphic for all $i_1 < \ldots < i_p$. A $(p, 0)$ form ω is holomorphic if and only if $d''\omega = 0$. In particular, a function f is holomorphic if and only if $d'' f = 0$.

For forms of type (p, q) with $q \geq 1$, we prove an analogue of Poincaré's theorem, due to A. Grothendieck.

A *clsoed cube* in C^n is a set in C^n defined by inequalities

$$|\mathscr{R} z_j| \leq a_j, \quad |\mathscr{I} z_j| \leq b_j, \quad a_j, \quad b_j > 0.$$

Theorem 1. *Triviality of d''-cohomology on a cube. Let K be a closed cube $\subset C^n$. Let ω be a form of type (p, q), $q \geq 1$ defined in a neighbourhood of K and suppose that $d''\omega = 0$. Then there exists a neighbourhood U of K and a form π of type $(p, q-1)$ in U such that $d''\pi = \omega$ in U.* 54

We need a lemma.

Lemma. *Let $\alpha(z,\lambda,\mu)$ be a complex function defined for $z \in U$ (U a neighbourhood of the closed unit square Δ with centre 0 in the z-plane), $\lambda \in \mathscr{O} \subset C^l$, $\mu \in \Omega \subset R^m$. Suppose that λ is differentiable in all its variables, and is holomorhic in $\lambda_1,\ldots,\lambda_l$ ($\lambda = (\lambda_1,\ldots,\lambda_l)$). Then*

$$f(z) = f(z,\lambda,\mu) = \frac{1}{2\pi i}\iint_\Delta \frac{\alpha(\zeta,\lambda,\mu)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

is differentiable in all its variables in $\overset{o}{\Delta} x \mathscr{O} x \Omega$ and is a holomorphic function in λ, such that

$$\frac{\partial f}{\partial \bar{z}} = \alpha(z,\lambda,\mu),\ z \in \overset{o}{\Delta}.$$

Proof of the lemma: The integral exists since $\dfrac{1}{z}$ is locally summable. Let δ be a closed square, centre 0, $\delta \subset \overset{o}{\Delta}$. It is sufficient to prove that $\dfrac{\partial f}{\partial \bar{z}} = \alpha$ for $z \in \overset{o}{\delta}$. Let $\beta(z)$ be a C^∞-function which is 1 in δ and $\beta(z) = 0$ in a neighbourhood of the boundary of Δ. (Such a $\beta(z)$ exists). Now $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 = \beta\alpha$, $\alpha_2 = (1-\beta)\alpha$, and we have

$$f(z) = f_1(z) + f_2(z),$$

where

$$f_1(z) = \frac{1}{2\pi i}\iint\limits_\Delta \frac{\alpha_1(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}, \quad f_2(z) = \frac{1}{2\pi i}\iint\limits_\Delta \frac{\alpha_2(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

55 and it is obvious that f_2 is holomorphic in z, $\dfrac{\partial f_2}{\partial \bar{z}} = 0$, if $z \in \delta^\circ$ and that f_2 is holomorphic in λ. Since $\alpha_1(z) = 0$ in a neighbourhood of the boundary of Δ, we can define $\alpha_1(z) = 0$ outside Δ and write

$$\begin{aligned} f_1(z,\lambda,\mu) &= \frac{1}{2\pi i}\iint \frac{\alpha_1(\zeta,\lambda,\mu)}{\zeta - z} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i}\iint \frac{\alpha_1(u+z,\lambda,\mu)}{u} \cdot du \wedge d\bar{u}, \end{aligned}$$

if we substitute $u = \zeta - z$ (integrals without limits being over the whole plane). From this form of the integral, it it clear that $f_1(z, \lambda, \mu)$ is differentiable in all the variables and holomorphic in λ. Also (writing $\alpha_1(u+z)$ for $\alpha_1(u + z, \lambda, \mu)$)

$$\begin{aligned}
\frac{\partial f_1}{\partial \overline{z}} &= \frac{1}{2\pi i} \iint \frac{\partial \alpha_1(u+z)}{\partial \overline{z}} \cdot \frac{1}{u} du \wedge d\overline{u} \\
&= \frac{1}{2\pi i} \iint \frac{\partial \alpha_1(u+z)}{\partial \overline{u}} \frac{1}{u} du \wedge d\overline{u} \\
&= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \iint_{|u| \geq \epsilon} \frac{\partial \alpha_1(u+z)}{\partial \overline{u}} \cdot \frac{1}{u}\, du \wedge d\overline{u} \\
&= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \iint\limits_{|u| \geq \epsilon} \frac{\partial(\alpha_1(u+z) \cdot \frac{1}{u})}{\partial \overline{u}} du \wedge d\overline{u}.
\end{aligned}$$

If $\Gamma_\varepsilon^+ (\Gamma_\varepsilon^-)$ is the positively (negatively) oriented circle $|u| = \varepsilon$, Riemann's formula applied to the last integral above gives

$$\begin{aligned}
\frac{\partial f_1}{\partial \overline{z}} &= \lim_{\epsilon \to 0} -\frac{1}{2\pi i} \int\limits_{\Gamma_\varepsilon^1} \alpha_1(u+z) \frac{du}{u} \\
&= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int\limits_{\Gamma_\varepsilon^+} \alpha_1(u+z) \frac{du}{u} = \alpha_1(z) = \alpha(z),
\end{aligned}$$

if $z \epsilon \delta^0$. This proves the lemma.

Proof of Grothendieck's Theorem: The proof will be given first for forms of type $(0, 1)$ to bring out the method clearly, and then it will be given in the general case.

The proof is by induction. Consider the following statement: **56**

For all forms ω of type $(0, 1)$ which are d''-closed and in which the coefficients of $d\overline{z}_{k+1}, \ldots, d\overline{z}_n$ are all 0, there exists an f such that $d'' f = \omega$.

The statement is trivially true when $k = 0$ for then $\omega = 0$ and we may take $f = 0$.

Suppose the statement true for all forms with k replaced by $k - 1$. Suppose that in ω the coefficients of $d\overline{z}_{k+1}, \ldots, d\overline{z}_n$ are zero. Then the coefficients of ω are holomorphic functions of $z_{k+1}, \ldots, z_n$ (for, if $\omega = \sum\limits_{j=1}^{k} a_j d\overline{z}_j$, $d\overline{z}_l (l > k)$ occurs as $\sum\limits_{j=1}^{k} \dfrac{\partial a_j}{\partial \overline{z}_l} d\overline{z}_l \wedge d\overline{z}_j$ in $d''\omega$ and, since ω is d''-closed $\dfrac{\partial a_j}{\partial \overline{z}_l} = 0$ for $l > k$). Now, by the lemma, there exists a function $g(z_1, \ldots, z_n)$, differentiable in all the variables $z_1, \ldots, z_n$ and holomorphic in $z_{k+1}, \ldots, z_n$ (in a neighbourhood of K) such that $\dfrac{\partial g}{\partial \overline{z}_k} = a_k$.

The problem is to find an f so that $d'' f = \omega$. If we put $f_1 = f - g$, the problem becomes that of finding f_1, so that $d'' f_1 = \omega_1 = \omega - d'' g$. Clearly $d''\omega_1 = 0$, and by the construction of g, the cofficient of $d\overline{z}_i$ in ω_1 is 0 if $l \geq k$. By inductive hypothesis there is an f_1 such that $d'' f_1 = \omega_1$ and the statement is true also for k. Grothendieck's theorem for forms of type $(0, 1)$ follows on taking $k = n$.

In the general case, the proof is the same.

Consider the following statement: For any d''-closed form ω of type (p, q) in which all terms in which $d\overline{z}_{k+1}, \ldots, d\overline{z}_n$ occur are zero, there exists a form π of type $(p, q-1)$ such that $d''\pi = \omega$.

57 The statement is tirvially true if $k = 0$. Suppose the statement is true when k is replaced by $k-1$, and let ω be a (p, q)-form (form of type (p, q)) such that all the terms in which $d\overline{z}_{k+1}, \ldots, d\overline{z}_n$ occur are zero. Then, in the same way as above, it is seen that all the coefficients of ω are holomorphic functions of $z_{k+1}, \ldots, z_n$. Suppose now that

$$\omega = d\overline{z}_k \wedge \overset{(p,q-1)}{\alpha} + \overset{(p,q)}{\beta}.$$

By the lemma, there exists a form ϕ of type $(p, q-1)$ differentiable in all the variables, such that its coefficients are holomorphic in $z_{k+1}, \ldots, z_n$ and such that $\dfrac{\partial \phi}{\partial \overline{z}_k} = \overset{(p,q-1)}{\alpha}$. (One has merely to apply the lemma to the coefficients of $\overset{(p,q-1)}{\alpha}$). As above, the problem reduces to finding a form π_1 such that $d''\pi_1 = \omega_1 = \omega - d''\phi$. Since $d''\omega_1 = 0$ and the terms in which $d\overline{z}_k, \ldots, d\overline{z}_n$ occur are zero by construction of ϕ, the existence of π_1 follows from inductive hypothesis and the theorem of Grothendieck follows on taking $k = n$.

It may be remarked that the theorem of Grothendieck is true also for open cubes and polydiscs, but the proof necessitates a limit process, and since this can be carried out for arbitrary "Stein manifolds," these special cases are not considered here.

4 Meromorphic functions

Let V be a complex analytic manifold, and let $a \epsilon V$. Let $\mathscr{O}_a$ denote the ring of germs of holomorphic functions at a. It can be easily verified that $\mathscr{O}_a$ is an integrity domain and we may therefore from the quotient field $\mathfrak{m}_a \cdot \mathfrak{m}_a$ is called the set of *germs of meromorphic functions at a*. 58
Let $\mathfrak{m} = \bigcup\limits_{a \in V} \mathfrak{m}_a$. A topology may be introduced on as follows. Let $a \in V$ and let $\dfrac{f_a}{g_a} = m_a \in \mathfrak{m}_a$. Let f_a and g_a be defined by holomorphic functions f, g in an open connected neighbourhood Ω of a. For every point $b \in \Omega$, m_b is defined to be $\dfrac{f_b}{g_b}$. If f', g' are two other holomorphic functions in Ω such that $\dfrac{f'_a}{g'_a} = m_a$, then $f'_a g_a - g'_a f_a = 0$ and $f' g - g' f = 0$ in a neighbourhood of a, and by the principle of analytic continuation, $f' g - g' f = 0$ in Ω so that $\dfrac{f'_b}{g_b} = \dfrac{f'_b}{g'_b}$ and the above definition is unique. A neighbourhood of $(a, m_a) \in \mathfrak{m}$ is now defined to be $\bigcup\limits_{b \in \Omega} (b, m_b)$, where Ω has the properties mentioned above. In this topology, $\mathfrak{m}$ is a sheaf over V.

A *meromorphic function* is now simply defined to be *a section of* $\mathfrak{m}$ *over* V, i.e, a continuous map $f : V \to \mathfrak{m}$ such that $f(a) \in \mathfrak{m}_a$ for every $a \in V$.

The weak principle of analytic continuation remains valid when holomorphic functions are replaced by meromorphic functions. Meromorphic functions may also be defined in terms of coverings and local quotients of holomorphic functions, with certain obvious consistency conditions.

Principle Parts.

A system of *principal parts* on V is a section of the quotient $\mathfrak{m}/\mathscr{O}$ ($\mathfrak{m}$ being the sheaf of additive groups of germs of meromorphic functions, $\mathscr{O}$ the sheaf of additive groups of germs of holomorphic functions.) But the following alternative definition is the one that will be used in Cousin's first problem.

59 A system of principal parts on the complex manifold V consists of an open covering $\{\Omega_i\}$ of V and meromorphic functions f_i in Ω_i; such that $f_i - f_j$ is holomorphic in $\Omega_i \cap \Omega_j$. (The meaning of this last statement is clear). Two systems $\{\Omega_i, f_i\}$, $\{\Omega'_j, f'_j\}$ define the same principal parts if $f_i - f'_j$ is holomorphic in $\Omega_i \cap \Omega'_j$ for every i, j. Here, and in what follows, properties like the above are assumed fulfilled when the intersections in question ar empty.

5 The first Cousin problem

The problem is the following: Suppose given a sytem of principal parts $\{\Omega_i, f_i\}$ on the complex manifold V. Then does there exist a meromorphic function f on V such that $f - f_i$ is holomorphic in Ω_i, i.e., when is the system of principal parts defined by one function?

This problem may be generalized to

The generalized first Cousin problem.

Let $\{\Omega_i\}$ be an open covering of the complex manifold V and suppose given a family of functions $\{c_{ij}\}$ such that c_{ij} is holomorhic in $\Omega_i \cap \Omega_j$ and having the following properties:

$$c_{ij} + c_{ji} = 0 \text{ in } \Omega_i \cap \Omega_j, \; c_{ij} + c_{jk} + c_{ki} = 0 \text{ in } \Omega_i \cap \Omega_j \cap \Omega_k.$$

Then, is it possible to find holomorphic functions c_i in Ω_i such that $c_{ij} = c_i - c_j$ in $\Omega_i \cap \Omega_j$?

A solution of this problem leads to a solution of the first cousin problem, for if we take $c_{ij} = f_i - f_j$ and $c_{ij} = c_i - c_j$, then $f_i - c_i = f_j - c_j$ in $\Omega_i \cap \Omega_j$, and if we define $f = f_i - c_i$ in Ω_i, it is easy to see that f solves the first Cousin problem.

60 **The first Cousin Problem fo the cube.**

The following therorem will now be proved.

Theorem 2. *Let K be a cube in C^n and let $\{\Omega_i, c_{ij}\}$ be a system such that $\{\Omega_i\}$ is an open covering of K and the c_{ij} have the properties given above. Then there exists a neighbourhood U of K such that the generalized Cousin problem has a solution for the system $\{U \cap \Omega_i, c_{ij}\}$.*

Proof. We assume first that $\{\Omega_i\}$ is a finite covering □

Step 1. There exists a neighbourhood U_1 of K and a system $\{\gamma_i\}$ of C^∞-functions γ_i in $\Omega_i \cap U_1$ such that $\gamma_i - \gamma_j = c_{ij}$.

Let $\{\phi_i\}$ be a differentiable partition of unity relative to the convering $\{\omega_i\}$ of K, i.e., ϕ_i is C^∞ and has compact support contained in Ω_i, $\phi_i \geq 0$ and $\sum \phi_i = 1$ in a neighbourhood U_1 of K. Such a partition of unity exists.

Consisder the following function γ_i on $\Omega_i \cap U_1$.

Let $z \in \Omega_i \cap U_1$; define $\gamma_i(z) = \sum\limits_{j \neq 1} \phi_j(z) c_{ij}(z)$. This sum is meaningful, for if $c_{ij}(z)$ is not defined, then $z \notin \Omega_j$ and so $\phi_j(z) = 0$, and we define $\phi_j(z) c_{ij}(z)$, for such z, to be zero. It is easily seen that γ_i isdifferentiable in $\Omega_i \cap U_1$. Now

$$\gamma_i - \gamma_j = \sum_{k \neq i,j} \phi_k (c_{ik} - c_{jk}) + \phi_j c_{ij} - \phi_i c_{ji}.$$

Also

$$c_{ik} - c_{jk} = c_{ik} + c_{kj} = -c_{ji} = c_{ij}.$$

Hence

$$\gamma_i - \gamma_j = \sum_k \phi_k c_{ij} = c_{ij} \text{ in } \Omega_i \cap \Omega_j \cap U_1.$$

Step 1 is completed.

Step 2. Solution of the generalized first Coursin problem.

In $\Omega_i \cap \Omega_j \cap U_1$ we have $d''\gamma_1 - d''\gamma_j = d''c_{ij} = 0$ since c_{ij} is **61**
holomorphic. Hence if we define a form α (of type $(0,1)$) by $\alpha = d''\gamma_i$ in $\Omega_i \cap U_1$ we have a well-defined form on U_1. Clearly $d''\alpha = 0$ and by Grothendieck's theorem, there is a $(0,0)$ form β, i.e., a function β such that $d''\beta = \alpha$ in a neighbourhood $U \subset U_1$ of K. If we set $c_i = \gamma_i - \beta$ in

$\Omega_i \cap U$, $d''c_i = d''\gamma_i - d''\beta = \alpha - \alpha = 0$ so that c_i is holomorphic in Ω_i while $c_i - c_j = \gamma_i - \gamma_j = c_{ij}$ in $\Omega_j \cap \Omega_j \cap U$. This commpletes the proof of the theorem when $\{\Omega_i\}$ is finite.

In the general case let $\Omega_1, \ldots, \Omega_p$ be a finite covering of the cube K, extracted from $\{\Omega_i\}$. By passing to suitable intersections, we may assume that each Ω_α is contained in $\Omega_1 \cup \ldots \cup \Omega_p$ while the functions $c_1, \ldots, c_p$ are defined everywhere in $\Omega_1, \ldots, \Omega_p$ respectively. Given α, we define $c_\alpha = c_i + c_{\alpha i}$ on $\Omega_\alpha \cap \Omega_i (i = 1, \ldots, p)$. On $\Omega_\alpha \cap \Omega_i \cap \Omega_j$, $c_i + c_{\alpha i} = c_j + c_{\alpha j}$, for $c_i - c_j = c_{ij} = c_{\alpha j} - c_{\alpha i}$. Since $\Omega_\alpha \subset \bigcup\limits_{i=1}^{p} \Omega_i$, c_α is defined on Ω_α and it is easily verified that the system $\{\Omega_\alpha, c_\alpha\}$ solves the generalized first Cousin problem.

Chapter 9

Holomorphic Regular Matrices

In IX and X, we shall prove analogues of the theorems of VIII for holomorphic functions whose *values are regular* (invertible) *matrices*, and give some applications of these generalizations. We begin by restating Grothendieck's theorem in a form which can be carried over. **62**

Let α be an arbitrary differentiable $(0,1)$ form in a neighbourhood of the cube K. The necessary and sufficient condition that there exists a C^∞-function f in a neighbourhood of K such that $f \neq 0$ at any point, and

$$f^{-1} d'' f = \alpha$$

is that $d''\alpha = 0$. (We have only to find g so that $d''g = \alpha$ and set $f = \exp g$).

In what follows, the functions or forms considered have values or coefficients in the space of $(m \times m)$ complex matrices or in the full linear group $GL(m, C)$ or regular matrices.

Our aim will be to generalize the above result to the case when α is a $(0,1)$ form whose coefficients are $(m \times m)$ complex matrices and f is replaced by a mapping in $GL(m, C)$. We shall need to generalize the lemma of VIII.

Theorem 1. *Let K be a rectangle in the C-plane, L, M compact sets in C^l, C^m respectively. Let $\alpha(z, \lambda, \mu)$ be a matrix valued function, defined in a neighbourhood of $K \times L \times M$ such that it is a differentiable function of allits variables and a holomorphic function of λ in this neighbourhood. Then there exists a C^∞-function $f(z, \lambda, \mu)$ in a neighbourhood of $K \times L \times M$ with values in $GL(M, C)$ which is differentiable in all its variables, is holomorphic in λ and is such that*

$$\frac{\partial f}{\partial \overline{z}} = f \cdot \alpha.$$

63 *We need some lemmas. The first two will not be proved here.*

Lemma 1. *Let B be a Banach space, $\mathscr{L}, \mathfrak{m}$ open sets in C^l, C^m respectively and $U(\lambda, \mu)$ a continuous linear operator $B \to B$ which has an inverse, for every $(\lambda, \mu) \in \mathscr{L} \times \mathfrak{m}$. Suppose that $U(\lambda, \mu)$ is a C^∞-function of λ and μ and is holomorphic in λ. Suppose, moreover, that $X(\lambda, \mu)$ is a differentiable function of λ, μ with values in B, which is holomorphic in λ.*

Then $U^{-1}(\lambda, \mu) X(\lambda, \mu)$ has also these properties.

Lemma 2. *Let $\mathscr{O}$ be an open set in the plane and H an open set in C^h. Let $f(z, \eta)$ be a continuous function of the set of all continuous functions of z in $\mathscr{O}$ with values in the space of differentiable functions of η in H. Suppose that the derivatives*

$$\frac{\partial k_f}{\partial \overline{z}^k}, \quad k = 0, 1, 2, \ldots$$

(in the sense of distributions) all exist and have the same properties.

Then $f(z, \eta)$ is an indefinitely differentiable function in $\mathscr{O} \times H$.

This is a particular case of a theorem on the "regularity in the interior" of solutions of elliptic partial differential equations. See, for example, Lions [5] (also exercise 1). The present situation involves vector functions with values in the space of differentiable functions, but the proof remains valid.

The proof of Theorem 1 will be in three parts.

64 **First Part: Proof of Theorem 1 in the particular case when α is "sufficiently near to zero".** [The last phrase means the following: if K', L', M' are compact neighbourhoods of K, L, M respectively such that $K' \times L' \times M'$ is contained in the domain of definition of α, then $\|\alpha\| < C(K', L', M')$ for a suitable $C(\|\alpha\|$ will denote, in what follows $\max\limits_{i,j} \|\alpha_{ij}\|_{K'\times L'\times M'}$ if $\alpha = (\alpha_{ij})$)].

The lemma that follows is the crucial step in the first part.

Lemma 3. *Suppose that $\alpha(z, \lambda, \mu)$ satisfies the hypothesis of Theorem 1 and that α is "sufficiently near zero". Then there exists a function $\beta(z, \lambda, \mu)$ in a neighbourhood of $K \times L \times M$ which is C^∞ in z, λ, μ and holomorphic in λ is such that*

$$\frac{\partial\beta}{\partial\bar{z}} + [\alpha, \beta] = \frac{\partial\alpha}{\partial z}.$$

($[\alpha, \beta]$ stands, as usual, for $\alpha\beta - \beta\alpha$).

Proof. Let $\gamma(z)$ be a C^∞-function which is 1 in an open neighbourhood of K and is zero near the boundary of K'. Consider the following integral equation (writing $[\alpha, \beta]$ for $[\alpha, \beta](\zeta, \lambda, \mu)$):

$$\beta(z, \lambda, \mu) + \frac{1}{2\pi i} \iint\limits_{K'} \gamma[\alpha, \beta] \frac{1}{\zeta - z}$$
$$d\zeta \wedge d\bar{\zeta} = \frac{1}{2\pi i} \iint\limits_{K'} \frac{\partial\alpha}{\partial\zeta} \frac{1}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

Consider the Banach space B of all continuous functions on K' whose values are $m \times m$ matrices, with norm $\|\beta\|$ (defined as for α) for $\beta \in B$. Let $A(\lambda, \mu)$ denote the operator defined by

$$A(\lambda, \mu)\beta(z) = \frac{1}{2\pi i} \iint\limits_{K'} \gamma(\zeta)$$
$$\{\alpha(\zeta, \lambda, \mu)\beta(\zeta) - \beta(\zeta)\alpha(\zeta, \lambda, \mu)\} \frac{1}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

for $z \in K'$. The integral equation can then be written

$$(I + A(\lambda, \mu))\beta = X(\lambda, \mu)$$

65 where

$$X(\lambda, \mu)(z) = \frac{1}{2\pi i} \iint_{K'} \gamma \cdot \frac{\partial \alpha}{\partial \zeta} \frac{1}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

Now, $X(\lambda, \mu)$ and $A(\lambda, \mu)$ are differentiable in λ, μ and holomorphic in λ. It is clear from the definition of $A(\lambda, \mu)$ that if α is sufficiently near zero, $\|A(\lambda, \mu)\| \leq \theta < 1$ for $(\lambda, \mu) \in L' \times M'$. Consequently, $I + A(\lambda, \mu)$ has an inverse for every $(\lambda, \mu) \in \overset{\circ}{L}{}' \times \overset{\circ}{M}{}'$, and so, by Lemma 1, $(I + A(\lambda, \mu))^{-1} X(\lambda, \mu)$ is differentiable in λ, μ and holomorphic in λ, and the integral equation has a solution $\beta(z, \lambda, \mu)$ which has the following properties:

(1) β is a differentiable function of $(\lambda, \mu) \in \overset{\circ}{L}{}' \times \overset{\circ}{M}{}'$ with values in B;

(2) β is holomorphic in λ.

From (1) it follows that β is a continuous function of $\zeta \in \overset{\circ}{K}{}'$ with values in the space of all differentiable function of (λ, μ) in $\overset{\circ}{L}{}' \times \overset{\circ}{M}{}'$.

Now, if $g(z) = \dfrac{1}{z}$, $\dfrac{1}{2\pi i} \iint_{K'} \gamma(\zeta) f(\zeta) \dfrac{1}{\zeta - z} d\zeta \wedge d\bar{\zeta} = \dfrac{1}{\pi} g^* \gamma f$ ($*$ being convolution). Since $\dfrac{\partial g}{\partial \bar{z}} = \pi \delta_o$ (δ_o is the Dirac distribution at 0; this is essentially the lemma proved before Grothendieck's theorem), it follows that

$$\frac{\partial \beta}{\partial \bar{z}} = -[\alpha, \beta] + \frac{\partial \alpha}{\partial z}$$

in $\mathscr{O} \times \overset{\circ}{L}{}' \times \overset{\circ}{M}{}'$ (in the sense of distributions). Since the terms on the right are continuous functions of z (with values in the space of differentiable functions in $\overset{\circ}{L}{}' \times \overset{\circ}{M}{}'$), so is $\dfrac{\partial \beta}{\partial \bar{z}}$ and so

$$\frac{\partial^2 \beta}{\partial \bar{z}^2} = -\left[\frac{\partial \alpha}{\partial \bar{z}}, \beta\right] - \left[\alpha, \frac{\partial \beta}{\partial \bar{z}}\right] + \frac{\partial^2 \alpha}{\partial z \partial \bar{z}}$$

66 has the same property. By iteration

$$\frac{\partial^k \beta}{\partial \overline{z}^k} \text{ is continuous for } k \geq 0,$$

and, by Lemma 2, β is C^∞ in a neighbourhood of $K \times L \times M$. This proves Lemma 3. □

Proof of Theorem 1 in the particular case. Let $\mathscr{O}$ be an open rectangle $\subset C$, $K \subset \mathscr{O}$ and let $\mathscr{L}$, $\mathfrak{m}$ be open neighbourhoods of L, M respectively, such that $\mathscr{O} \times \mathscr{L} \times \mathfrak{m}$ is contained in the domain of definition of α. We shall find a differentiable, regular matrix f such that

$$\frac{\partial f}{\partial z} = f \cdot \beta, \quad \frac{\partial f}{\partial \overline{z}} = f \cdot \alpha, \quad f(0) = I \text{ (unit matrix)}. \tag{1}$$

If we put $f(tz) = \phi_z(t)$, $\phi(t) = \phi_z(t)$ satisfies

$$\begin{cases} \phi(0) & = I \\ \dfrac{d\phi}{dt} & = z\phi(t)\beta(tz) + \overline{z}\phi(t)\alpha(tz) = \phi(t) \cdot A, \end{cases} \tag{2}$$

if f satisfies (1). By the classical theorems on systems of ordinary equations of the form (2), a solution $\phi(t)$ of (2) exists, is C^∞ in z, λ, μ, holomorphic in λ and is unique; thus, f, if it exists, is uniquely given by $f(z) = \phi_2(1)$.

Let now ϕ be the solution of (2); *define* f by $f(z) = \phi_z(1)$. Then f is C^∞ in $\mathscr{O} \times \mathscr{L} \times \mathfrak{m}$ and holomorphic as a function of λ. We shall show that

(i) $f(z)$ is a regular matrix;

(ii) f satisfies the equation (1).

Proof of (i): ϕ satisfies 67

$$\phi(0) = I,$$
$$\frac{d\phi}{dt} = \phi \cdot A.$$

Let ψ be the unique solution of

$$\psi(0) = I,$$
$$\frac{d\psi}{dt} = -A \cdot \psi.$$

Then $\dfrac{d(\phi \cdot \psi)}{dt} = 0$, $\phi\psi = \phi(0)\psi(0) = I$ so that, in particular, $f(z) \cdot \psi(1) = I$.

Proof of (ii): Let $g_z(t) = \partial\phi_z(t)/\partial\overline{z}$. We shall prove that

$$g_z(t) = t\phi_z(t)\alpha(tz) = h_z(t)$$

(the second equality is a definition) which implies (ii). Now, for $t = 0$, $g_z(0) = h_z(0) = 0$, and $g_z(t)$ satisfies the equation

$$\frac{dg_z}{dt} = g_z\{z\beta(tz) + \overline{z}\alpha(tz)\} + \phi_z\alpha(tz) + \phi_z\{zt\frac{\partial\beta}{\partial\overline{z}}(tz) + \overline{z}t\frac{\partial\alpha}{\partial\overline{z}}(tz)\} \tag{3}$$

Since $\dfrac{\partial\beta}{\partial\overline{z}} + [\alpha, \beta] = \dfrac{\partial\alpha}{\partial z}$, the equation (3) remains valied if g_z is replaced by h_z. Since $g_z(0) = h_z(0)(= 0)$, it follows from the uniqueness of a solution of an equation of the form (3) that $g_z(t) \equiv h_z(t)$ and this completes the proof of (ii). This concludes the proof of Theorem 1 in the particular case.

Corollary. *Under the hypothesis of Theorem 1, every point of K has an open neighbourhood U with a function $f(z, \lambda, \mu)$, C^∞ in all its variables and holomorphic in λ in a neighbourhood of $U \times L \times M$ such that*

$$\frac{\partial f}{\partial\overline{z}} = f\alpha \text{ in a neighbourhood of } U \times L \times M.$$

68 **Proof of the corollary.** It is enough to prove thsi for the point $0 \in K$.

Let

$$\alpha_t = \alpha(tz, \lambda, \mu), \quad t \geq 0.$$

It is clearly sufficient to find $t > 0$ such that for α_t there is a function f_t in a neighbourhood of $K \times L \times M$ with the required properties of regularity so that $\dfrac{\partial f_t}{\partial\overline{z}} = t f_t\alpha_t$ (by setting $f(z) = f_t\left(\dfrac{z}{t}\right)$ in a neighbourhood of $z = 0$).

It t is small enough, $t\alpha_t$ is near zero and by the particular case of Theorem 1, the matrix f_t exists.

Before continuing with the proof of Theorem 1, we shall deduce from the preceding results the following theorem of H. Cartan, which is all that will be required in the later theory.

Second Part: *Theorem on holomorphic regular matrices.*

Theorem 2. *Let K be a rectangle in the complex plane and L, M two compact sets in C^l, C^m respectively. Let H be the intersection of K with the line $\mathscr{R}z = 0$. Let $C(z, \lambda, \mu)$ be a C^∞-function in a neighbourhood of $H \times L \times M$ which is holomorphic in z and in λ with values in $GL(m, C)$. Let $K_1 = K \cap \{z \in C \big| \mathscr{R}z \geq 0\}$, $K_2 = K \cap \{z \in C \big| \mathscr{R}z \leq 0\}$. Then there exist functions $C_1(z, \lambda, \mu)$, $C_2(z, \lambda, \mu)$ in neighbourhoods of $K_1 \times L \times M$, $K_2 \times L \times M$ satisfying the same regularity conditions as C and such that, in a neighbourhood of $H \times L \times M$,*

$$C = C_1 C_2^{-1}.$$

Proof. The proof will be given first in the case when C differs little from the indentity matrix I in a sense which is obvious. Let H' be a rectangle with sides parallel to the coordinate axes in the plane containing H such **69**
that $H' \times L \times M$ is contained in the domain of definition of C. Then $logC$ is defined (as $\exp^{-1}(C)$) and is near zero if C is near I in $H' \times L \times M$. Let ϕ be a C^∞-function in a neighbourhood of K such that $\phi(z) = 1$ if $\mathscr{R}z \geq \epsilon$, $= 0$ if $\mathscr{R}z \leq -\epsilon$ (ϵ so chosen that the intersection of K with the strip $|\mathscr{R}_z| \leq \epsilon$ is contained in H').

Now define

$$\gamma_2^{-1} = \exp[\phi \log C]$$

and $\gamma_1 = C\gamma_2$ in a neighbourhood of $H' \times L \times M$. γ_2 is extended to a neighbourhood of $K_2 \times L \times M$ by setting $\gamma_2^{-1} = I$ for $\mathscr{R}z \leq -\epsilon$, and γ_1 to a neighbourhood of $K_1 \times L \times M$ by setting $\gamma_1 = I$ for $\mathscr{R}z \geq \epsilon$. Then we have

$$C = \gamma_1 \gamma_2^{-1}.$$

Also, if C is near I, γ_1, γ_2 are near I, while $\dfrac{\partial \gamma_1}{\partial \overline{z}}$, $\dfrac{\partial \gamma_2}{\partial \overline{z}}$ are near 0. Since

C is holomorphic, we have

$$C\frac{\partial\gamma_2}{\partial\overline{z}} = \frac{\partial\gamma_1}{\partial\overline{z}},$$

and

$$\gamma_1^{-1}\frac{\partial\gamma_1}{\partial\overline{z}} = \gamma_2^{-1}\frac{\partial\gamma_2}{\partial\overline{z}} = \alpha.$$

Since α is near 0 if C is near, I, there exists a C^∞-function $f(z,\lambda,\mu)$ holomorphic in λ in a neighbourhood of $K \times L \times M$ such that

$$f^{-1}\frac{\partial f}{\partial\overline{z}} = \alpha.$$

If

$$\begin{aligned} C_1 = \gamma_1 f^{-1},\ C_2 = \gamma_2 f^{-1},\ \frac{\partial C_1}{\partial\overline{z}} &= -\gamma_1 f^{-1}\frac{\partial f}{\partial\overline{z}} f^{-1} + \frac{\partial\gamma_1}{\partial\overline{z}} f^{-1} \\ &= -\gamma_1\alpha f^{-1} + \gamma_1\alpha f^{-1} = 0, \end{aligned}$$

so that C_1 and similarly C_2 are holomorphic in z and λ.

70 Clearly,

$$C = C_1 C_2^{-1}$$

in a neighbourhood of $H \times L \times M$.

To prove Theorem 2 in the general case, we proceed as follows. Let C be any holomorphic regular matrix in a neighbourhood of $H \times L \times M$. Then there exists a matrix C' holomorphic in z and λ in a neighbourhood of $K \times L \times M$ (and even in one of $C \times L \times M$, C being the complex plane) which approximates to C (one has only to approximate the entries of C), and so, there is a C' so that $C'^{-1}C$ is near I. By the particular case proved above, there exist holomorphic regular matrices C_1', C_2 in neighbourhoods of $K_1 \times L \times M$, $K_2 \times L \times M$ respectively so that

$$C'^{-1}C = C_1' C_2^{-1}$$

and so, we have

$$C = C_1 C_2^{-1},$$

where $C_1 = C'C_1'$. This concludes the proof of Cartan's theorem on holomorphic regular matrices. ◻

Third Part: Proof of Theorem 1 in the general case.

After the corollary to the particular case of Theorem 1, we can divide K into a finite number of closed rectangles K_i with sides parallel to the axes of coordinates and obtain functions f_i in neighbourhoods of K_i so that $\dfrac{\partial f_i}{\partial \bar{z}} = f_i\alpha$ and f_i has the required regularity properties. We require a certain function f in the whole of K. It is easy to see that it is enough to solve the following problem: given two functions f_1, f_2 in neighbourhoods of two adjacent rectangles K_1, K_2 such that in neighbourhoods of $K_1 \times L \times M$, $K_2 \times L \times M$, respectively,

$$\frac{\partial f_1}{\partial \bar{z}} = f_1\alpha, \quad \frac{\partial f_2}{\partial \bar{z}} = f_2\alpha$$

find a function f in a neighbourhood of $(K_1 \cup K_2) \times L \times M$ with $\dfrac{\partial f}{\partial \bar{z}} =$ 71
$f\alpha$. Now, since $f_1^{-1}\dfrac{\partial f_1}{\partial \bar{z}} = f_2^{-1}\dfrac{\partial f_2}{\partial \bar{z}}$, the function $c = f_1 f_2^{-1}$ is C^∞ in a neighbourhood of $H \times L \times M$ (H being the common side of (K_1 and K_2) and holomorphic in z and λ.) Consequently, by Theorem 2, there exist matrices c_1, c_2 in neighbourhoods of $K_1 \times L \times M$, $K_2 \times L \times M$ holomorphic in z and λ, so that $c = c_1 c_2^{-1}$ in a neighbourhood of $H \times L \times M$. Then $c = c_1 c_2^{-1} = f_1 f_2^{-1}$ and $c_1 f_1^{-1} = c_2 f_2^{-1}$ in a neighbourhood of $H \times L \times M$. If we define $f = c_1 f_1^{-1}$ in a neighbourhood of $K_1 \times L \times M, = c_2 f_2^{-1}$ in a neighbourhood of $K_2 \times L \times M$, then $\dfrac{\partial f}{\partial \bar{z}} = c_1^{-1}\dfrac{\partial f_1}{\partial \bar{z}} = c_1^{-1} f_1\alpha = f\alpha$ if z belongs to a neighbourhood of K_1, and the same equation holds also in a neighbourhood of $K_2 \times L \times M$. This completes the proof of Theorem 1 in the general case.

Chapter 10

Complementary Results

1 Generalization of Grothendieck's theorem

Let Ω be an open set in C^n, α a $(0,1)$ form in Ω whose coefficients are **72**
$m \times m$ (differentiable) matrices. We ask for a condition that there exist a differentiable mapping f of Ω in $GL(m, C)$ such that

$$f^{-1} d'' f = \alpha.$$

If $\alpha = \sum a_k d\bar{z}_k$ and $f^{-1} d'' f = \alpha$, then $\dfrac{\partial^2 f}{\partial \bar{z}_l \partial \bar{z}_k} = \dfrac{\partial f}{\partial \bar{z}_l} a_k + f \dfrac{\partial a_k}{\partial \bar{z}_l}$

$$= f a_l a_k + f \frac{\partial a_k}{\partial \bar{z}_l} = \frac{\partial^2 f}{\partial \bar{z}_k \partial \bar{z}_l} = f a_k a_l + f \frac{\partial a_l}{\partial \bar{z}_k},$$

$$\text{i.e.,} \qquad \frac{\partial a_l}{\partial \bar{z}_k} - \frac{\partial a_k}{\partial \bar{z}_l} + [a_k, a_l] = 0,$$

and if we write $[\alpha, \alpha] = \sum\limits_{k<l} [a_k, a_l] d\bar{z}_k \wedge d\bar{z}_l$, then we can write these equations as

$$d''\alpha + [\alpha, \alpha] = 0.$$

The following generalization of Grothendieck's theorem provides a converse in the case of a cube.

Theorem 1. *Let K be a cube in C^n, α a$(0,1)$ differentiable form in a neighbourhood of K. Suppose that*

$$d''\alpha + [\alpha, \alpha] = 0.$$

Then there exists a differentiable, regular matrix f in a neighbourhood U of K such that, in U,

$$f^{-1}d''f = \alpha.$$

Proof. We use induction as in Grothendieck's theorem. Consider the statement:

73 For every form α of type $(0,1)$ such that $d''\alpha + [\alpha, \alpha] = 0$ and for which the coefficients of $d\overline{z}_{k+1}, \ldots, d\overline{z}_n$ are zero, there exists f with values in $GL(m, C)$ so that $f^{-1}d''f = \alpha$. For $k = 0$, the statement is trivially true, since $\alpha = 0$ and we may take $f = I$. Suppose the statement true when k is replaced by $k-1$. Let $\alpha = \sum\limits_{j=1}^{k} a_j d\overline{z}_j$. Since $d''\alpha + [\alpha, \alpha] = 0$ and $[\alpha, \alpha]$ does not contain $d\overline{z}_l$, $l > k$, the a_j are holomorphic in $z_{k+1}, \ldots, z_n$. By 9, Theorem 1 there is a function g, holomorphic in $z_{k+1}, \ldots, z_n$ such that $\dfrac{\partial g}{\partial \overline{z}_k} = g \cdot a_k$. If we set $f = f' \cdot g$ then $f^{-1}d''f = g^{-1}f'^{-1}d''f'g + g^{-1}d''g$ and the problem reduces to the finding of an f' such that

$$f'^{-1}d''f' = g(\alpha - g^{-1}d''g)g^{-1} = \beta$$

say. It is easily verified that $d''\beta + [\beta, \beta] = 0$ and clearlyl the coefficients of $d\overline{z}_l$, $l \geq k$, in β are zero, and by inductive hypothesis f' exists. ◻

2 Linear bundles

Let V be a topological space, $\{\mathscr{O}_i\}_{i \in \mathfrak{I}}$ an open covering of V. Suppose that in every $\mathscr{O}_i \cap \mathscr{O}_j$ is defined a continuous function c_{ij} with values in $GL(m, C)$ such that the set $\{c_{ij}\}$ satisfies

(i) $c_{ij}c_{ji} = I$ in $\mathscr{O}_i \cap \mathscr{O}_j$,

(ii) $c_{ij}c_{jk}c_{ki} = I$ in $\mathscr{O}_i \cap \mathscr{O}_j \cap \mathscr{O}_k$.

Consider the set $\bigcup\limits_{i\in\mathscr{I}}(\mathscr{O}_i\times C^m)$. Let $(x,y)\in\mathscr{O}_i\times C^m$, $(x',y')\in\mathscr{O}_j\times C^m$. We identify (x,y) and (x',y') if $x=x'$ in V and $y'=c_{ij}(x)y$. The quotient is denoted by E.

Definition. E together with the given system $\{\mathscr{O}_i,c_{ij}\}$ is called a *linear bundle over V* (with fibre C^m). V is said to be the base of E.

E is a topological space with the following properties: 74

(a) There is a canonical mapping $p:E\to V$ which is continuous and onto V.

(b) $p^{-1}(a)\simeq C^m$ for every $a\in V$ (topologically and as a vector space over C).

(c) Every point $a\in V$ has a neighbourhood $\mathscr{O}$ such that $p^{-1}(\mathscr{O})\simeq\mathscr{O}\times C^m$ the isomorphism being topological and compatible with (a) and (b) in an obvious sense.

Suppose E' is another linear boundle over V, defined by the system $\{\mathscr{O}'_\alpha,c'_{\alpha\beta}\}_A$. Suppose p' is the p corresponding to E' and that there is a homeomorphism of E onto E' compatible with (a), (b) and (c). Then it is easily shown that $\{\mathscr{O}_i,c_{ij}\}_{\mathscr{I}}$, $\{\mathscr{O}'_\alpha,c'_{\alpha\beta}\}_A$ are related by a finite number of applications of the following two operations:

(1°) Passage to refinements or the converse. $\{\mathscr{O}'_\alpha,c'_{\alpha\beta}\}_A$ is a refinement of $\{\mathscr{O}_i,c_{ij}\}$ if there is a mapping $\phi:A\to\mathfrak{J}$ such that $\mathscr{O}'_\alpha\subset\mathscr{O}_{\phi(\alpha)}$ and $c'_{\alpha\beta}=c_{\phi(\alpha)\phi(\beta)}$.

(2°) The covering $\{\mathscr{O}_i\}_{i\in\mathfrak{J}}$ being the same, one passes to new functions c'_{ij} by defining $c'_{ij}=c_ic_{ij}c_j^{-1}$ where c_i is continuous in $\mathscr{O}_i$.

(Moreover, it can be shown easily that if, in defining E and E' the same covering is used, then only one application, namely of operation (2°), is necessary). If E and E' are related by such a homeomorphism, we say that they are in the *same clases*.

The *trivial class* is defined as the class containing the bundle defined by taking for the covering, just V. With respect to a covering $\{\mathscr{O}_i\}_{i\in\mathfrak{J}}$ this class can be defined by taking for the functions c_{ij} the unit matrix I or, 75 generally, any functions $c_i c_j^{-1}$ where c_i is continuous in $\mathscr{O}_i$. A bundle is *trivial* if it is in the trivial class.

If V were a differentiable (complex analytic) manifold, then we define a *differentiable (analytic) linear bundle* over V in the same way, but now requiring the c_{ij} to be differentiable (analytic).

The differentiable or analytic class of a differentiable or analytic bundle can be defined in the obvious way and we speak of differentiable or *analytic equivalence* and *triviality*.

The following important theorem holds.

Theorem 2. *Let K be a cube in C^n, E an analytic bundle over a neighbourhood of K. Then E is trivial over a neighbourhood of K.*

Proof. By dividing K into smaller (closed) cubes K_i with faces parallel to the coordinate hyperplanes, we obtain holomorphic regular matrices c_{ij} in neighbourhoods of $K_i \cap K_j$ respectivelyl. Also, if the bundle is defined by $\{\mathscr{O}_i, c_{ij}\}$ and is trivial over $\mathscr{O}_1$ and $\mathscr{O}_2$, we may replace $\mathscr{O}_1, \mathscr{O}_2$ by their union and modify the c_{ij} to obtain an equivalent bundle (if $c_{12} = c_1 c_2^{-1}$ set $\mathscr{O}'_1 = \mathscr{O}_1 \cup \mathscr{O}_2$, $\mathscr{O}'_i = \mathscr{O}_i$ if $i \neq 1, 2$, $c'_i = c_{ij}$ if $i, j \neq 1$, $c'_{1j} = c_1^{-1} c_{1j} (= c_2^{-1} c_{2j})$. Then $\{\mathscr{O}'_i, c_{ij}\}$ defines an equivalent bundle). We have thus only to prove the following result: given two adjacent cubes K_1, K_2 and a holomorphic regular matrix c in a neighbourhood of their common face, we can write $c = c_1 c_2^{-1}$ in a neighbourhood of $K_1 \cap K_2$, c_1, c_2 being holomorphic regular matrices in neighbourhoods of K_1, K_2 respectively. But this follows at once from Cartan's theorem on holomorphic regular matrices (9, Theorem 2). □

3 Application to the second Cousin problem

76 **Divisor:** A *divisor* can be defined in two ways similar to the two definitions of meromorphic functions and of principal parts.

(a) Let $\mathfrak{m}^*$ be the sheaf of multiplicative groups of germs of meromorphic functions $\not\equiv 0$ on the complex manifold V, $\mathscr{H}$ the sheaf of multiplicative groups of germs of invertible holomorphic functions. A divisor is a section of $\mathfrak{m}^*/\mathscr{H}$ over V.

(b) Let $\{\mathscr{O}_i\}$ be an open covering of V and let g_i be a meromorphic function in $\mathscr{O}_i$. The system $\{\mathscr{O}_i, g_i\}$ defines a divisor if the function $g_i g_j^{-1}$ and its reciprocal are holomorphic in $\mathscr{O}_i \cap \mathscr{O}_j$. Two systems $\{\mathscr{O}_i, g_i\}$ and $\{\mathscr{O}'_j, g'_j\}$ define the same divisor if $g_i g'^{-1}_j$ is holomorphic and invertible in $\mathscr{O}_i \cap \mathscr{O}'_j$.

The second Cousin problem is as follows:

Given a divisor $\{\mathscr{O}_i, g_i\}$ on V, does there exist a meromorphic function f on V such that $f = \gamma_i g_i$ in $\mathscr{O}_i$, where γ_i and γ_i^{-1} are holomorphic in $\mathscr{O}_i$, i.e., does there exist one meromorphic function f on V which defines the same divisor as $\{\mathscr{O}_i, g_i\}$? As in the case of the first Cousin problem, this problem can be generalized.

Let $\{\mathscr{O}_i\}$ be an open covering of V and let there be given a holomorphic invertible function γ_{ij} in $\mathscr{O}_i \cap \mathscr{O}_j$ such that

$$\gamma_{ij}\gamma_{ji} = 1 \text{ in } \mathscr{O}_i \cap \mathscr{O}_j,$$
$$\gamma_{ij}\gamma_{jk}\gamma_{ki} = 1 \text{ in } \mathscr{O}_i \cap \mathscr{O}_j \cap \mathscr{O}_k.$$

Then, does there exist a holomorphic invertible function γ_i in $\mathscr{O}_i$ such that

$$\gamma_i \gamma_j^{-1} = \gamma_{ij} \text{ in } \mathscr{O}_i \cap \mathscr{O}_j?$$

It is easily seen that a solution of this problem leads to a solution of the second Cousin problem: for if we define $\gamma_{ij} = g_i g_j^{-1}$ in $\mathscr{O}_i \cap \mathscr{O}_j$ ($\{\mathscr{O}_i, g_i\}$ is the second Cousin datum) and if $\gamma_{ij} = \gamma_i \gamma_j^{-1}$ (also in $\mathscr{O}_i \cap \mathscr{O}_j$), then $\gamma_i^{-1} g_i = \gamma_j^{-1} g_j$ in $\mathscr{O}_i \cap \mathscr{O}_j$, and the meromorphic function f defined by $f = \gamma_i^{-1} g_i$ in $\mathscr{O}_i$ solves the second Cousin problem. **77**

Theorem 3. *The generalized second Cousin problem is always solvable for a neighbourhood of a cube.*

Proof. Given the system $\{\mathscr{O}_i, \gamma_{ij}\}$ in a neighbourhood of the cube K, the system defines an *analytic line bundle* over a neighbourhood of K (linear boundle with fibre C). The solubility of the second Cousin problem is precisely the triviality of this line bundle over a neighbourhood of K, and this has been proved in Theorem 2. □

Eexercise

1. Let $\alpha(z)$ be a C^∞-function with compact support in the plane. Let 78

$$f(z) = -\frac{1}{2\pi i}\iint \alpha(\zeta)\frac{\overline{z}-\overline{z}}{\zeta - z}d\zeta \wedge d\overline{\zeta}.$$

   Prove that $\dfrac{\partial^2 f}{\partial \overline{z}^2} = \alpha(z)$. If α is only a distribution with compact support, prove that this equation holds in the sense of distributions. Deduce that if f is a distribution such that $\dfrac{\partial^2 f}{\partial \overline{z}^2}$ is a continuous functions, then so are $f, \dfrac{\partial f}{\partial z}, \dfrac{\partial f}{\partial \overline{z}}$.

2. Let V be a complex analytic manifold, and suppose that the generalized first Cousin problem is always solvable on V. Prove that the generalized second Cousin problem is solvable, if it is "differentiably solvable" (in an obvious sense).

   Prove also that on the Riemann sphere, the first Cousin problem is always solvable, while the second is not.

3. Let K be a compact set in C. Prove that an analytical bundle differentiably trivial over a neighbourhood of K is analytically trivial over a neighbourhood of K.

4. Let V be a complex analytic manifold and let $\{\mathscr{O}_i, c_{ij}\}$ define an analytic bundle over V which is differentiably trivial. Let γ_i be a C^∞-function in $\mathscr{O}_i$ with $c_{ij} = \gamma_i\gamma_j^{-1}$ in $\mathscr{O}_i \cap \mathscr{O}_j$. Let $\alpha_i = \gamma_i^{-1}d''\gamma_i$.

Show that the α_i define a form α of type $(0, 1)$. What relation does α satisfy?

Given a form α of type $(0, 1)$ with $d''\alpha + [\alpha, \alpha] = 0$, show that it defines a class of analytic bundles over V which is differentiably
79 trivial. When do two such forms α and β define the same (analytic) class of analytic bundles?

5. Generalize the results of exercise 4 to the case of a nontrivial class of differentiable bundles on V.

   As an application, prove that if V is a Riemann surface and E and an arbitrary differentiable bundle on V, there always exists an analytic bundle which is differentiably equivalent with E (use a device similar to that used in Step 1 in 8, Theorem 2.).

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Part III

Coherent Analytic Sheaves

Chapter 11

Sheaves

Definition. Let X and $\mathscr{F}$ be topological spaces and π a mapping $\mathscr{F} \to X$ **81**
such that

(i) π is onto;

(ii) π is a local homeomorphism.

Then we call $\mathscr{F}$ (with the mapping π) a *sheaf* on X. π will be called the *projection* of $\mathscr{F}$ on X.

A *section* of the sheaf $\mathscr{F}$ (over X) is a continuous mapping $s : X \to \mathscr{F}$ such that $\pi \circ s =$ identity.

If W is any subset of X, $\pi^{-1}(W)$ is a sheaf on W in a natural way

A section of the sheaf $\mathscr{F}$ over a subset W of X is a section of the sheaf $\pi^{-1}(W)$ over W.

We shall sometimes say 'section' for the image of the section s in $\mathscr{F}$.

If $x \in X$, $\mathscr{F}_x$ will stand for $\pi^{-1}(x)$.

Proposition 1. *A section over an open set $U \subset X$ is an open set in $\mathscr{F}$.*

Proof. Let $s : U \to \mathscr{F}$ be a section over U. Let $a \in s(U)$ and let $\mathscr{O}$ be an open set in $s(U)$ such that $\mathscr{O} = \mathscr{O}' \cap s(U)$ where $\mathscr{O}'$ is open in $\mathscr{F}$ and π restricted to $\mathscr{O}'$ is a homeomorphism of $\mathscr{O}'$ onto an open subset of X. Since s is continuous, $s^{-1}(\mathscr{O})$ is open in X, i.e., $\pi(\mathscr{O})$ is open in X and so in $\pi(\mathscr{O}')$. Since π is a homeormphism in $\mathscr{O}'$, $\mathscr{O}$ is an open set in $\mathscr{O}'$ and so in $\mathscr{F}$.

82 This implies, in particular, that two sections which coincide at a point, coincide in a neighbourhood of this point. □

Proposition 2. *If $\mathscr{F}$ is a Hausdorff space, a section over a closed set is closed.*

Proof. Let W be a closed set in X, and let $a \in \overline{s(W)}$ (s is the given section). Let $x = \pi(a)$. Let Ω_a be an open neighbourhood of a, homeomorphic with its projection. a belongs to the closure of $\Omega_a \cap s(W)$ in Ω_a and so $\pi(a)$ belongs to the closure of $\pi(\Omega_a \cap s(W)) \subset \pi \circ s(W) = W$ so that, since W is closed, $\pi(a) \in W$. Suppose next that $s(x) = b \neq a$ and let Ω_a, Ω_b be disjoint neighbourhoods of a, b, homeomorphic with their projections. If y is near enough to x, then $s(y) \in \Omega_b$ and consequently Ω_a does not meet $\overline{s(W)}$, a contradiction.

Proposition 2 is not true if $\mathscr{F}$ is not a Hausdorff space. □

Examples of sheaves.

1°) If V, W are manifolds and W is spread on V by a mapping onto V, W is a sheaf on V.

2°) X being a topological space, Y an arbitrary set, the set of all mappings $X \to Y$ give rise to a sheaf, the sheaf of germs of the mappings $X \to Y$ in the following way: two mappings of a neighbourhood of $x \in X$ are identified if they coincide in a neighbourhood of x. The set of equivalence classes at x is $\mathscr{F}_x$ and $\mathscr{F} = \bigcup\limits_{x \in X} \mathscr{F}_x$. The topology on $\mathscr{F}$ is obtained by a method similar to that used in the case of the sheaf of germs of holomorphic function in IV.

3°) In the same way, X, Y, being topological spaces, one defines the sheaf of germs of continuous mappings of X in Y.

83 4°) If V, W are complex analytic manifolds, we can define the sheaf of germs of analytic mappings of V in W. This sheaf is a Hausdorff space, by reasoning similar to that used in IV when $W = C$.

Diagonal product of two sheaves.

Let $(\mathscr{F}, \pi)$, $(\mathscr{G}, \pi')$ be sheaves on X. The set of $(f, g) \in \mathscr{F} \times \mathscr{G}$ such that $\pi(f) = \pi'(g)$ can be topologized in a natural way. This set then forms a sheaf on X called the *diagonal product* $\mathscr{F}_V\mathscr{G}$ of the sheaves $\mathscr{F}$ and $\mathscr{G}$.

Sheaf of groups. Let $\mathscr{F}$ be a sheaf on X. $\mathscr{F}$ is a *sheaf of groups* if

1°) for ever $x \in X$, $\mathscr{F}_x$ is a group;

2°) the mapping $a \to a^{-1}$ of $\mathscr{F}(a, a^{-1} \in \mathscr{F}_x)$ in $\mathscr{F}$ is continuous.

3°) the mapping $(f, g) \to fg$ of $\mathscr{F}_V\mathscr{F}$ to $\mathscr{F}$ is continuous.

Example. The sheaf of germs of continuous mappings of X in a topological group is a sheaf of groups.

Sheaf of rings.

Let $\mathscr{F}$ be a sheaf on X. $\mathscr{F}$ is a *sheaf of rings* if

1°) for every $x \in X$, $\mathscr{F}_x$ is a ring;

2°) $\mathscr{F}$ is a sheaf of (additive, abelian) groups;

3°) the mapping $(f, g) \to fg$ of $\mathscr{F}_V\mathscr{F} \to \mathscr{F}$ is continuous.

Let $\mathfrak{a}$ be a sheaf of rings and $\mathfrak{m}$ a sheaf on the same space $X \cdot \mathfrak{m}$ is called a *sheaf of* $\mathfrak{a}$ *modules* if

1°) $\mathfrak{m}$ is a sheaf of abelian groups (additive);

2°) for every $x \in X$, $\mathfrak{m}_x$ is an $\mathfrak{a}_x$-module; **84**

3°) the mapping $(\alpha, m) \to \alpha m$ of $\mathfrak{a}_V\mathfrak{m} \to \mathfrak{m}$ is continuous (it being assumed that $\mathfrak{m}_x$ is a left $\mathfrak{a}_x$ module).

The following is an important example.

Let $\{\mathscr{O}, c_{ij}\}$ cefine a linear bundle E on the space V (the c_{ij} are continuous mappings of $\mathscr{O}_i \cap \mathscr{O}_j$ into $GL(m, C)$). Let p be the projection of E onto V. A *cross-section* of the bundle is a continuous map $s : V \to E$ such that $p \circ s$ is the identity.

We can now define the *sheaf of germs of sections of the bundle* in the usual way. This sheaf is a sheaf of $\mathscr{C}$-modules, where $\mathscr{C}$ is the sheaf of germs of continuous, complex valued functions in V.

It is of importance to decide when a sheaf of modules over $\mathscr{C}$ can be obtained from a bundle by the method given above.

Suppose that the sheaf $\mathscr{F}$ can be so obtained. Then, since every $a \in V$ has a neighbourhood U such that $p^{-1}(U) \simeq U \times C^m$ (as linear bundles) locally, the contraction of $\mathscr{F}$ to U is isomorphic (as a sheaf) to $\mathscr{C}^m_U$ where $\mathscr{C}_U$ is the sheaf of germs of continuous functions in U.

Suppose, conversely, that the sheaf $(\mathscr{F}, \pi)$ has the above property. Let $\{\mathscr{O}_i\}$ be a covering of V such that $\pi^{-1}(\mathscr{O}_i) \simeq \mathscr{C}^m_{\mathscr{O}_i}$. Then $\mathscr{O}_i \cap \mathscr{O}_j$, being an open subset of both $\mathscr{O}_i$ and $\mathscr{O}_j$, gives rise to an automorphism ϕ of $\mathscr{C}^m_{\mathscr{O}_i \wedge \cap \mathscr{O}_j}$ as a sheaf of modules over $\mathscr{O}_{\mathscr{O}_i \cap \mathscr{O}_j}$. Let $e_p \in \mathscr{C}^m_{\mathscr{O}_i \cap \mathscr{O}_j}$ be the element defined by $(0, \ldots, 0, 1, 0, \ldots, 0)$ where all but the p-th place contain 0. Then $\phi(e_p) = \sum a_{pq} e_q$, (since ϕ is an automorphism of the
85 sheaf $\mathscr{C}^m_{\mathscr{O}_i \cap \mathscr{O}_j}$ as a sheaf of modules over $\mathscr{C}_{\mathscr{O}_i \cap \mathscr{O}_j}$) and we can define the matrix $c_{ij} = (a_{pq})$. Since ϕ is an automorphism, c_{ij} is invertible. It is easy to see that the bundle defined by $\{\mathscr{O}_i, c_{ij}\}$ gives rise to the sheaf $\mathscr{F}$.

This leads to a one-one correspondence between classes of linear bundles and sheaves locally isomorphic with $\mathscr{C}^m$.

Also there is a one-one correspondence between cross-sections of a bundle and sections of the sheaf defined by the bundle.

Chapter 12

General properties of Coherent Analytic Sheaves

1 Analytic Sheaves

Let $\underline{\mathscr{F}}$ be a sheaf on the base space X. The concept of a *subsheaf* is **86**
defined in the obvious way (as a subset of $\underline{\mathscr{F}}$ which is made into a sheaf by the restriction of the projection to the subset). It is clear that if $\underline{\mathscr{H}}$, $\underline{\mathscr{G}}$ are subsheaves of $\underline{\mathscr{F}}$, so is $\underline{\mathscr{H}} \cap \underline{\mathscr{G}}$.

Let now $\underline{\mathscr{F}}$, $\underline{\mathscr{G}}$ be two sheaves of groups on X with projections π_f, π_g. Let ϕ be a mapping $\underline{\mathscr{F}} \to \underline{\mathscr{G}}$. ϕ is called a (*sheaf*) *homomorphism* if

(i) ϕ is continuous;

(ii) $\pi_f = \pi_g \circ \phi$;

(iii) the restriction of ϕ to $\mathscr{F}_x (= \pi_f^{-1}(x), x \in X)$ is a homomorphism of the group $\mathscr{F}_x$ in $\mathscr{G}_x (= \pi_g^{-1}(x))$.

There are corresponding definitions of subsheaves of sheaves of algebraic structures and of homomorphisms between such sheaves. The concepts of a one-one mapping (into), mapping onto, *image* of a homomorphism, *kernel* of a homomorphism are defined in the obvious way.

A sequence

$$\cdots \to \underline{\mathscr{F}}_p \xrightarrow{d_p} \underline{\mathscr{F}}_{p+1} \xrightarrow{d_{p+1}} \underline{\mathscr{F}}_{p+2} \to \cdots$$

of sheaves $\underline{\mathscr{F}}_p$ of groups (or other algebraic structures) and homomorphisms $d_p : \underline{\mathscr{F}}_p \to \underline{\mathscr{F}}_{p+1}$ is said to be *exact at* $\underline{\mathscr{F}}_p$ if the kernel of d_{p+1} = image of d_p; it is *exact* if it is exact at $\underline{\mathscr{F}}_p$ for all p.

Quotient Sheaves.

Let $\underline{\mathscr{G}}$ be a sheaf of groups on the topological space X, $\mathscr{F}$ a subsheaf
87 of $\underline{\mathscr{G}}$ such that for every $x \in X$, $\mathscr{G}_x$ is a normal subgroup of $\mathscr{G}_x$. Then there is precisely one sheaf $\underline{\mathscr{H}}$ on X such that $\mathscr{H}_x = \mathscr{G}_x/\mathscr{F}_x$ and the mapping $\eta : \underline{\mathscr{G}} \to \underline{\mathscr{H}}$ ($\eta_x : \mathscr{G}_x \to \mathscr{H}_x$ is the natural homomorphism) is a sheaf homomorphism. We have only to set $\underline{\mathscr{H}} = \bigcup\limits_{x \in X} (\mathscr{G}_x/\mathscr{F}_x)$ and put quotient topology on $\underline{\mathscr{H}}$. It is clear the the conditions on $\underline{\mathscr{H}}$ above determine uniquely the topology on it.

Analytic Sheaves.

Let V be a complex analytic manifold, and $\underline{\mathscr{O}} = \underline{\mathscr{O}}_V$ the sheaf of germs of holomorphic functions on V.

Definition. An *analytic sheaf* on V is a sheaf of $\underline{\mathscr{O}}$-modules.

One can then define *(analytic) subsheaves* of an analytic sheaf $\underline{\mathscr{F}}$ in the obvious way. Clearly, the intersection of analytic subsheaves of $\underline{\mathscr{F}}$ is analytic.

Notations. In what follows, $\mathscr{O}$ will denote the ring of holomorphic funtions on the complex manifold V, $\underline{\mathscr{O}}$ the sheaf of germs of holomorphic functions on V; $\mathscr{O}^m, \underline{\mathscr{O}}^m$ will stand for the m-th powe of $\mathscr{O}$, $\underline{\mathscr{O}}$ respectively.

If $\underline{\mathscr{F}}$ is a sheaf on the space X, and U is a subset of X, $\underline{\mathscr{F}}_U$ will denote the restriction of $\underline{\mathscr{F}}$ to U. $\mathscr{F}_U$ or $\Gamma(U, \mathscr{F})$ will stand for the *sections* of $\mathscr{F}$ over U.

Examples. 1°) $\underline{\mathscr{O}^m}$ is an analytic sheaf;

2°) Let $\mathfrak{m}$ be a finitely generated submodule of $\mathscr{O}^m$ (generated by $h_1, \ldots, h_p$ say). For $a \in V$, let $\mathfrak{m}_a$ be the submodule of $\mathscr{O}_a^m$ generated over $\mathscr{O}_a$ by $(h_1)_a, \ldots, (h_p)_a$. Then $\underline{\mathfrak{m}} = \bigcup\limits_{a \in V} \mathfrak{m}_a$ is an analytic sheaf on V. To prove this, one has only to show that if f is a section of $\underline{\mathscr{O}}^m$ over an open neighbourhood U of $a \in V$, then there is a neighbourhood $U_1 \subset U$ of a such that $f_a \in \mathfrak{m}_a$ implies $f_b \in \mathfrak{m}_b$ for $b \in U_1$.

Now, $f_a = \sum\limits_{i=1}^{p} (\lambda_i)_a (h_i)_a$, $(\lambda_i)_a \in \mathscr{O}_a$, and there is an open neigh- **88**
bourhood U_1 of a, and functions f, λ_i in U_1 such that $f = \sum\limits_{i=1}^{p} \lambda_i h_i$ in U_1 , which implies that $(f)_b = \sum\limits_{i=1}^{p} (\lambda_i)_b (h_i)_b \in \mathfrak{m}_b$ for $b \in U_1$.

3°) The *sheaf of relations* between q elements of $\mathscr{O}^m$. Let $h_1, \ldots, h_q \in \mathscr{O}^m$. Let $\mathscr{R}_a$ be the submodule of $\mathscr{O}_a^q$ consisting of the q-tuples $((c_1)_a, \ldots, (c_q)_a) \cdot (c_i)_a \in \mathscr{O}_a$ such that $(c_1)_a (h_1)_a + \cdot + (c_q)_a (h_q)_a = 0$. Then it is easily verified that $\underline{\mathscr{R}} = \underline{\mathscr{R}}(h_1, \ldots, h_q) = \bigcup\limits_{a \in V} \mathscr{R}_a$ is an analytic sheaf, called the *sheaf of relations* between $h_1, \ldots, h_q$.

2 Coherent analytic subsheaves of $\mathscr{O}^m$

Definition. Let $\underline{\mathscr{F}}$ be an analytic subsheaf of $\underline{\mathscr{O}}^m$. $\underline{\mathscr{F}}$ is said to be *coherent* if the following is true: for every $a \in V$ there is a neighbourhood U of a, and a *finite number of sections of $\mathscr{F}$ over U*, $f_1, \ldots, f_q$, such that for every $b \in U$, $\mathscr{F}_b$ is $\mathscr{O}_b$-generated by $(f_1)_b, \ldots, (f_q)_b$.

The following important theorem holds, but we shall not prove it here. The theorem is due to *K*. Oka 7. For the proof see Cartan [2], [3] Lecture XV.

Theorem of Oka. Let $h_1, \ldots, h_q \in \mathscr{O}^m$. Then, the sheaf of relations $\underline{\mathscr{R}}(h_1, \ldots, h_q)$ is a coherent analytic sheaf.

Corollary. *Let $\underline{\mathscr{F}}, \underline{\mathscr{G}}$ be coherent analytic subsheaves of $\underline{\mathscr{O}}^p$. Then $\underline{\mathscr{F}} \cap \underline{\mathscr{G}}$ is a coherent analytic sheaf.*

Proof of the Corollary. Let U be a neighbourhood of $a \in V$, $f_1, \ldots, f_k$; $g_1, \ldots, g_m$ sections of $\underline{\mathscr{F}}$, $\underline{\mathscr{G}}$ over U such that $(f_1)_b, \ldots, (f_k)_b$ (resp. $(g_1)_b, \ldots, (g_m)_b$) $\mathscr{O}_b$-generate $\mathscr{F}_b$ (resp. $\mathscr{G}_b$) for every $b \in U$. Consider the sheaf $\underline{\mathscr{R}}(f_1, \ldots, f_k, g_1, \ldots, g_m) = \underline{\mathscr{R}}$ on U.

89 It is coherent, by Oka's theorem. We define a map ϕ of $\underline{\mathscr{R}}$ onto $\underline{\mathscr{F}}_U \cap \underline{\mathscr{G}}_U$, as follows: For $x \in U$, let $r = ((c_1)_x, \ldots, (c_k)_x, (c'_1)_x, \ldots, (c'_m)_x) \in \underline{\mathscr{R}}$. i.e., $\sum (c_i)_x (f_i)_x + \sum (c'_j)_x (g_j)_x = 0$. The image $\phi(r)$ of this point is, by definition, $\sum (c_i)_x (f_i)_x \in \mathscr{F}_x$. By the above relation, $\phi(r) \in \underline{\mathscr{G}}_x$ and so $\phi(r) \in \mathscr{F}_x \cap \mathscr{G}_x$. ϕ is clearly a homomorphism of $\underline{\mathscr{R}}$ in $\underline{\mathscr{F}}_U \cap \underline{\mathscr{G}}_U$. Also, if $f_x = \sum (c_i)_x (f_i)_x = \sum (c_j)_x (g_j) \in \mathscr{F}_x \cap \mathscr{G}_x$, then $\phi(r) = f_x$ where $r = ((c_1)_x, \ldots, (c_k)_x, (-c'_1)_x, \ldots, (-c'_m)_x)$ and so ϕ is a homomorphism of $\underline{\mathscr{R}}$ onto $\underline{\mathscr{F}}_U \cap \underline{\mathscr{G}}_U$. It follows easily that $\underline{\mathscr{F}}_U \cap \underline{\mathscr{G}}_U$ is coherent and the result follows.

We give some examples of non-coherent, analytic sheaves.

(a) Let $\underline{\mathscr{F}}$ be a sheaf of ideals on V, i.e., $\mathscr{F}_a$ is an ideal of $\mathscr{O}_a$ (so an $\mathscr{O}_a$-module). Suppose that $\underline{\mathscr{F}}$ is coherent and let, in a neighbourhood U of a, $f_1, \ldots, f_p$ generate $\mathscr{F}_b$ over $\mathscr{O}_b$. Then, the necessary and sufficient condition that $\mathscr{F}_b = \mathscr{O}_b$ is that at least one $f_i(b) \neq 0$. Hence the set of b with $\mathscr{F}_b \neq \mathscr{O}_b$ is precisely the set of common zeros of $f_1, \ldots, f_p$. Hence the set of b wiht $\mathscr{F}_b \neq \mathscr{O}_b$ is an *analytic subset* of V [i.e., locally in V, it is the set of common zeros of a finite number of holomorphic functions].

The complement of an open ball, S, in C^n is not an analytic subset. Hence if we get $\mathscr{F}_a = \mathscr{O}_a$ for $a \in S$, $\mathscr{F}_a = 0$ for $a \notin S$, the analytic sheaf $\underline{\mathscr{F}} = \bigcup\limits_{a \in C^n} \mathscr{F}_a$ is not coherent.

(b) Let $\Omega \neq V$ be an open subset of the complex manifold V. Let $\mathscr{F}_a = \mathscr{O}_a$, $a \notin \Omega$; $\mathscr{F}_a = 0$, $a \notin \Omega$. The sheaf $\underline{\mathscr{F}} = \bigcup\limits_{a \in V} \mathfrak{F}_a$ is analytic but not coherent (the definition is violated at a point on the boundary of Ω).

Let $\underline{\mathscr{F}}$ be a coherent analytic sheaf on a (connected) complex manifold V. Then $\mathscr{F}_a \neq 0$ at any point $a \in V$ unless $\mathscr{F}_a = 0$ for all $a \in V$: if
90 $\mathscr{F}_a = 0$ and $(f_1)_b, \ldots, (f_p)_b \mathscr{O}_b$-generate $\mathscr{F}_b$ for $b \in U$ where U is a connected neighbourhood of a, then $f_1, \ldots, f_p$ are zero in a neighbourhood

of a since $\mathscr{F}_a = 0$ and $\mathscr{F}_b = 0$ for $b \in U$ by the principle of analytic continuation. Hence the set of a with $\mathscr{F}_a = 0$ is open. It is easily proved in the same way that this set is closed and the result follows.

3 General coherent analytic sheaves on a complex analytic manifold

Definition. Let V be a complex analytic manifold and $\underline{\mathscr{F}}$ an analytic sheaf on V. $\underline{\mathscr{F}}$ is said to be *coherent* if every $a \in V$ has an open neighbourhood Ω such that $\underline{\mathscr{F}}_\Omega \simeq \underline{\mathscr{O}}^p_\Omega / \underline{\mathscr{N}}$, where $\underline{\mathscr{N}}$ is a *coherent subsheaf of* $\underline{\mathscr{O}}^p_\Omega$ (in the first sense).

For subsheaves of $\underline{\mathscr{O}}^m$, the two definitions of coherence coincide. It is clear that given a subsheaf of $\underline{\mathscr{O}}^m$, which, locally, is an *arbitrary* quotient of an $\underline{\mathscr{O}}^p$, there is a natural homomorphism of $\underline{\mathscr{O}}^p_\Omega$ onto $\underline{\mathscr{F}}_\Omega$ (Ω being an open neighbourhood of a given point $a \in V$) and $\underline{\mathscr{F}}$ is coherent in the first definition. Converselyl, suppose that $\underline{\mathscr{F}} \subset \underline{\mathscr{O}}^m$ and that to every $a \in V$, there is a neighbourhood Ω such that in Ω, p elements $g_1, \ldots, g_p$ of $\mathscr{O}^p_\Omega$ generate $\underline{\mathscr{F}}$. We have a homomorphism

$$((\phi_1)_b, \ldots, (\phi_p)_b)\{\in \mathscr{O}^p_b\} \to (\phi_1)_b(g_1)_b + \cdot + (\phi_p)_b(g_p)_b$$

of $\underline{\mathscr{O}}^p_\Omega$ onto $\underline{\mathscr{F}}_\Omega$, of which the kernel is the sheaf of relations $\underline{\mathscr{R}}_\Omega(g_1, \ldots, g_p)$ which is coherent by the theorem of Oka. Hence the condition of the second definition is fulfilled.

An example of a coherent sheaf which is not a subsheaf of some $\underline{\mathscr{O}}^m$ is the sheaf of germs of sections of an analytic, non-trivial, bundle. This sheaf is locally isomorphic to an $\underline{\mathscr{O}}^m$.

Proposition 1. *Let $\underline{\mathscr{F}}$ be a coherent analytic sheaf. Let $f_1, \ldots, f_q$ be a finite number of sections of $\underline{\mathscr{F}}$. Then the sheaf $\underline{\mathscr{R}}(f_1, \ldots, f_q)$ of relations between $f_1, \ldots, f_q$ is coherent.* **91**

Proof. Let $a \in V$. There are p sections $g_1, \ldots, g_p$ over a neighbourhood Ω of a such that

(i) $(g_1)_b, \ldots, (g_p)_b \mathscr{O}_b$-generate $\mathscr{F}_b$ for $b \in \Omega$.

(ii) $\overline{\mathscr{R}}_\Omega(g_1, \ldots, g_p)$ is a coherent subsheaf of $\mathscr{O}^p_\Omega$.

[This is just a reformulation of the definition of coherence]. We have $(f_i)_a = \sum\limits_{j=1}^{p} (\lambda^j_i)_a (g_j)_a ((\lambda^j_i)_a \in \mathscr{O}_a)$ so that there is an open neighbourhood $\Omega' \subset \Omega$ of a such that $f_i = \sum \lambda^j_i g_j$ in Ω'. Now suppose $b \in \Omega'$ and that $((\mu^1)_b, \ldots, (\mu^q)_b) \in \mathscr{R}_b(f_1, \ldots, f_q)$. Then, we have $\sum\limits_{i=1}^{q} (\mu^i)_b (f_i)_b = 0$, so that $\sum\limits_{i,j} (\mu^i)_b (\lambda^j_i)_b (g_j)_b = 0$. This implies that $(\sum\limits_i (\mu^i)_b (\lambda^1_i)_b, \ldots, \sum\limits_i (\mu^i)_b (\lambda^q_i)_b) \in \mathscr{R}_b(g_1, \ldots, g_p)$. Since by hypothesis $\underline{\mathscr{R}}_\Omega(g_1, \ldots, g_p)$ is coherent, there exist functions h^j_k, $j = 1, \ldots p$, $k = 1, \ldots, r$ and a neighbourhood $\Omega'' \subset \Omega'$ of a such that

$$\sum_i (\mu^i)_b (\lambda^j_i)_b = \sum_{k=1}^{r} (\nu^k)_b (h^j_k)_b$$

for $b \in \Omega''$, where $(\nu^k)_b \in \mathscr{O}_b$. This implies that

$$(\mu^1, \ldots, \mu^q, -\nu^1, \ldots, -\nu^r) \in \underline{\mathscr{R}}_{\Omega''}(\lambda^j_i, h^j_k).$$

By Oka's theorem $\underline{\mathscr{R}}_{\Omega''}(\lambda^j_i, h^j_k)$ is coherent and the systems $(\mu^1, \ldots, \mu^q) \in \underline{\mathscr{R}}_{\Omega''}(f_1, \ldots, f_q)$ form a quotient of $\underline{\mathscr{R}}_{\Omega''}(\lambda^j_i, h^j_k)$ and, this being $\subset \underline{\mathscr{O}}^q_{\Omega''}$, $\underline{\mathscr{R}}_{\Omega''}(f_1, \ldots, f_q)$ is coherent. This proves Proposition 1. □

92 **Theorem 1.** *Let $\underline{\mathscr{G}}$ and $\underline{\mathscr{F}}$ be two coherent analytic sheaves. Let ϕ : $\underline{\mathscr{G}} \to \underline{\mathscr{F}}$ be a homomorphism (as analytic sheaves) of $\mathscr{G}$ in $\mathscr{F}$. Then the kernel, the image, the cokernel and the coimage of ϕ are coherent analytic sheaves.*

(The cokernel is $\underline{\mathscr{F}}/\phi(\underline{\mathscr{G}})$, the coimage is $\underline{\mathscr{G}}/\phi^{-1}(\underline{\mathscr{O}})$.)

Proof. 1) **The image of ϕ.** Let $g_1, \ldots, g_q$ be sections of $\underline{\mathscr{G}}$ over an open set Ω such that $(g_1)_a, \ldots, (g_q)_a$ $\mathscr{O}_a$-generate $\mathscr{G}_a$ for $a \in \Omega$. Then $\phi(g_1), \ldots, \phi(g_q)$ are sections of $\underline{\mathscr{F}}$ over Ω and they $\mathscr{O}_a$-generate $\phi(\underline{\mathscr{G}})_\Omega$ for $a \in \Omega$. Hence on Ω, $\phi(\underline{\mathscr{G}})$ is isomorphic with $\underline{\mathscr{O}}^q_\Omega/\underline{\mathscr{N}}$ where $\underline{\mathscr{N}} = \underline{\mathscr{R}}_\Omega(\phi(g_1), \ldots, \phi(g_q))$. The result follows from Proposition 1 and the definition.

2) **The kernel of ϕ.** Let $g_1, \ldots, g_q \in \mathscr{G}_\Omega$ generate $\underline{\mathscr{G}}_\Omega$. Now, the sheaf $\underline{\mathscr{R}}_\Omega(\phi(g_1), \ldots, \phi(g_q))$ is coherent. We define a mapping of $\mathscr{R}_\Omega$ to the kernel of ϕ as follows: if $((c_1)_a, \ldots, (c_q)_a) \in \mathscr{R}_a$, map this point on $(c_1)_a(g_1)_a + \cdots + (c_q)_a(g_q)_a$. This gives us a homomorphism of $\underline{\mathscr{R}}_\Omega$ onto the kernel of ϕ (restricted to Ω) and by 1) the kernel is coherent.

3) and 4)**The cokernel and the coimage of ϕ.** After 1) and 2) it is clearly sufficient to prove the following statment:

If $\underline{\mathscr{G}}$ is a coherent analytic sheaf, $\underline{\mathscr{F}}$ a coherent analytic subsheaf of $\underline{\mathscr{G}}$, then $\underline{\mathscr{G}}/\underline{\mathscr{F}}$ is coherent. To every $a \in V$ corresponds an open set Ω, $a \in \Omega$ with $\underline{\mathscr{G}}_\Omega \simeq \underline{\mathscr{O}}^p_\Omega/\underline{\mathscr{N}}$ where $\underline{\mathscr{N}}$ is coherent. Let $f'_1, \ldots, f'_q \in \mathscr{F}_\Omega$ generate $\underline{\mathscr{F}}_\Omega$. Since $\underline{\mathscr{F}}_\Omega \subset \underline{\mathscr{G}}_\Omega$, there are elements $f_1, \ldots, f_q \in \mathscr{O}^p_\Omega$ which go into $f'_1, \ldots, f'_q$. Let $\underline{\mathscr{R}}$ be the analytic sheaf on Ω generated by $f_1, \ldots, f_p$; this sheaf is clearly coherent. One has $\underline{\mathscr{F}}_\Omega \simeq (\underline{\mathscr{R}} + \underline{\mathscr{N}})/\underline{\mathscr{N}}$, and consequently **93**
$\underline{\mathscr{G}}_\Omega/\underline{\mathscr{F}}_\Omega \simeq \underline{\mathscr{O}}^q_\Omega/(\underline{\mathscr{R}} + \mathscr{N})$. Since obviously $\underline{\mathscr{R}} + \underline{\mathscr{N}}$ is coherent, the result follows.

□

4 Coherent analytic sheaves on subsets of a complex analytic manifold

Let X be a subset of the complex manifold V. An *analytic* sheaf on X is defined to be a sheaf of $\underline{\mathscr{O}}_X$-modules ($\underline{\mathscr{O}}_X$ is, of course, the restriction of $\underline{\mathscr{O}}$ to X). The definition of a *coherent* analytic sheaf $\underline{\mathscr{F}}$ on X is the same as before: if $\underline{\mathscr{F}} \subset \underline{\mathscr{O}}^p_X$, then $\underline{\mathscr{F}}$ is coherent if to every $a \in X$ exist a neighbourhood U in X and elements $f_1, \ldots, f_p \in \mathscr{F}_U$ such that $(f_1)_b, \ldots, (f_p)_b$ $\mathscr{O}_{Xb}$-generate $\mathscr{F}_b$ for $b \in U$. An analytic sheaf $\underline{\mathscr{F}}$ on X is *coherent*, if every $a \in X$ has an open neighbourhood $U \subset X$ such that $\underline{\mathscr{F}}_U \simeq \underline{\mathscr{O}}^p_U/\underline{\mathscr{N}}$ where $\underline{\mathscr{N}}$ is a coherent analytic subsheaf of $\underline{\mathscr{O}}^p_U$. The result of 3 generalize to the sheaves $\mathscr{O}^p_X$. The following theorem will be proved.

Theorem 2. *Let X be a compact subset of the complex manifold V, and $\mathscr{F}$ a coherent analytic sheaf on X. Then there is an open set $\Omega \supset X$ (open in V) and a coherent analytic sheaf $\underline{\mathscr{G}}$ on Ω such that $\underline{\mathscr{G}}_X \simeq \underline{\mathscr{F}}$.*

We begin the proof with a remark, which follows at once from the fact that a section of sheaf is an open mapping and the definition of coherence.

Remark. Let $\underline{\mathscr{F}}, \underline{\mathscr{G}}$ be two coherent analytic sheaves on a subset Y of the complex manifold V, f, g two homomorphisms: $\underline{\mathscr{F}} \to \underline{\mathscr{G}}$. Then, the set of $y \in Y$ with $f_y = g_y$ (f_y, g_y are the homomorphisms $\mathscr{F}_y \to \mathscr{G}_y$ determined by f, g respectively) is open in Y.

For the proof of Theorem 2, we require two lemmas.

94 **Lemma 1.** *Let $\underline{\mathscr{F}}, \underline{\mathscr{G}}$ be coherent analytic sheaves on a subset $Y \subset V$. Let $a \in Y$ and suppose that there is a homomorphism $\phi_a : \mathscr{F}_a \to \mathscr{G}_a$. Then there is a neighbourhood Ω of a in Y such that ϕ_a can be continued to a homomorphism $\phi : \underline{\mathscr{F}}_\Omega \to \underline{\mathscr{G}}_\Omega$ (in an obvious sense).*

Proof. Suppose that in a neighbourhood Ω' of a, $f_1, \ldots, f_q \in \mathscr{F}_\Omega$, generate $\underline{\mathscr{F}}_{\Omega'}$; let $g_1, \ldots, g_p \in \underline{\mathscr{G}}_{\Omega'}$ define respectively the germs $(g_1)_a = \phi_a((f_1)_a), \ldots, (g_p)_a = \phi_a((f_p)_a)$. The sheaf of relations between $f_1, \ldots, f_p$ is coherent, and so, if Ω' is small enough, is generated in Ω' by functions (λ_i^k) and $\sum_i (\lambda_i^k)_a (f_i)_a = 0$ so that we have $\sum_i (\lambda_i^k)_a (g_i)_a = 0$. Let $\Omega \subset \Omega'$ be such that $\sum_i \lambda_i^k f_i = 0$, $\sum_i \lambda_i^k g_i = 0$ in Ω. Let $b \in \Omega$ and let $\sum_i (\mu_i)_b (f_i)_b = 0$. Then $(\mu_i)_b = \sum_k (a_k)_b (\lambda_i^k)_b$ and $\sum_i (\mu_i)_b (g_i)_b = \sum_k (a_k)_b \sum_i (\lambda_i^k)_b (g_i)_b = 0$. Hence, if $b \in \Omega$, $\sum_i (\mu_i)_b (f_i)_b = 0$ implies $\sum_i (\mu_i)_b (g_i)_b = 0$ and the homomorphism ϕ on Ω can be defined by $\phi_b((f_i)_b) = (g_i)_b$ for $b \in \Omega$. □

Lemma 2. *Let X be a compact set in V, $\underline{\mathscr{F}}, \underline{\mathscr{G}}$, coherent analytic sheaves on a neighbourhood of X. Let ϕ be a homomorphism $\underline{\mathscr{F}}_X \to \mathscr{G}_X$. Then ϕ can be continued to a homomorphism $\underline{\mathscr{F}}_U \to \underline{\mathscr{G}}_U$, U being a suitable neighbourhood of X.*

Proof. By Lemma 1, for every $a \in X$, ϕ_a can be extended to a homomorphism $\phi_{\Omega_a} : \underline{\mathscr{F}}_{\Omega_a} \to \underline{\mathscr{G}}_{\Omega_a}$ in a neighbourhood Ω_a of a. Now ϕ and ϕ_{Ω_a} determine the same homomorphism ϕ_a of $\mathscr{F}_a \to \mathscr{G}_a$. By the remark before the proof of Lemma 1, we may suppose that $\phi = \phi_{\Omega_a}$ in $\Omega_a \cap X$. 95
Since X is compact, we obtain a finite covering $\{\Omega_i\}$ of X and homomorphisms $\psi_i : \underline{\mathscr{F}}_{\Omega_i} \to \underline{\mathscr{G}}_{\Omega_i}$ such that $\psi_i = \psi_j = \phi$ in $\Omega_i \cap \Omega_j \cap X$. Again, by the remark before Lemma 1 and the compactness of X, we may assume that $\psi_i = \psi_j$ in $U_i \cap U_j$ where U_i is an open set containing $\Omega_i \cap X$. It is clear that there is an open set $U \supset X$ such that $\psi_i = \psi_j$ in $\Omega_i \cap \Omega_j \cap U$ and $\psi_i = \phi$ on $\Omega_i \cap X$. Lemma 2 follows. $\square$

Proof of Theorem 2. From the definition of coherent analytic sheaves (as locally isomorphic to quotient of $\underline{\mathscr{O}}^p_X$, $\underline{\mathscr{O}}_X$ being the restriction of $\underline{\mathscr{O}}_V$) it follows that if $a \in X$, there is an open set Ω_a, $a \in \Omega_a$ and a coherent analytic sheaf $\underline{\mathscr{G}}^a$ on Ω_a such that $\underline{\mathscr{G}}^a_{\Omega_a \cap X} \simeq \underline{\mathscr{F}}_{\Omega_a \cap X}$. Since X is compact, there are finitely many coherent analytic sheaves $\underline{\mathscr{G}}^1, \ldots, \underline{\mathscr{G}}^r$ on open sets $\Omega_1, \ldots, \Omega_r$ respectively $(\bigcup\limits_{i=1}^{r} \Omega_i \supset X)$ such that $\underline{\mathscr{G}^i_{\Omega_i \cap X}} \simeq \underline{\mathscr{F}}_{\Omega_i \cap X}$. Let this isomorphism be given by a mapping $c_i : \underline{\mathscr{G}}^i_{\Omega_i \cap X} \to \underline{\mathscr{F}}_{\Omega_i \cap X}$. On $\Omega_i \cap \Omega_j \cap X$, there is thus a homomorphism $c_{ij} : \underline{\mathscr{G}}^i_{\Omega_i \cap \Omega_j \cap X} \to \underline{\mathscr{G}}^j_{\Omega_i \cap \Omega_j \cap X}$ (where $c_{ij} = c_j^{-1} c_i$). Also c_{ii} = identity, $c_{ij} c_{jk} c_{ki}$ = identity on $\Omega_i \cap \Omega_j \cap \Omega_k \cap X$. By Lemma 2, there is a neighbourhood U_{ij} of $\Omega_i \cap \Omega_j \cap X$ (with $U_{ij} = U_{ji}$) and a homomorphism

$$\gamma_{ij} : \underline{\mathscr{G}}^i_{\Omega_i \cap \Omega_j \cap U_{ij}} \to \underline{\mathscr{G}}^j_{\Omega_i \cap \Omega_j \cap U_{ij}}.$$

Also, by the remark before Lemma 1, we may suppose that U_{ij} is such that γ_{ii} = identity on $\Omega_i \cap U_{ii}$, $\gamma_{ij} \gamma_{jk} \gamma_{ki}$ = identity on $\Omega_i \cap \Omega_j \cap \Omega_k \cap U_{ij} \cap U_{jk} \cap U_{ki}$; in particular, γ_{ij} is an isomorphism of $\underline{\mathscr{G}}^i_{\Omega_i \cap \Omega_j \cap U_{ij}}$ onto $\underline{\mathscr{G}}^i \Omega_i \cap \Omega_j \cap U_{ij}$. Since X is compact, there is a neighbourhood U of X 96
such that γ_{ij} is an isomorphism of $\underline{\mathscr{G}}^i_{\Omega_i \cap \Omega_j \cap U}$ onto $u\mathscr{G}^j_{\Omega_i \cap \Omega_j \cap U}$, and γ_{ii} = identity, $\gamma_{ij} \gamma_{jk} \gamma_{ki}$ = identity (on $\Omega_i \cap U$, $\Omega_i \cap \Omega_j \cap \Omega_k \cap U$ respectively). These sheaves and isomorphism give rise to a coherent analytic sheaf $\underline{\mathscr{G}}$ on a neighbourhood of X. Also, by the definition of the γ_{ij} it is clear that $\underline{\mathscr{G}}_X \simeq \underline{\mathscr{F}}$ and the proof of Theorem 2 is complete.

If the manifold V is paracompact, X may be replaced by any closed set in Theorem 2. For the details of proof, see Cartan [4] or Dowker [6].

Chapter 13

Cohomology with coefficients in a sheaf

1 Cohomology of a covering

Let X be a topological space, $\underline{\mathscr{F}}$ a sheaf of abelian groups on X. Let **97**
$\mathscr{O} = \{\mathscr{O}_i\}_{i\in I}$ be an open covering of X. We shall denote by $\mathscr{O}_{i_\circ,\ldots,i_p}$ the set $\mathscr{O}_{i_\circ} \cap \ldots \cap \mathscr{O}_{i_p}$ and, U being an open set in X, by $\Gamma(U, \underline{\mathscr{F}}) = \underline{\mathscr{F}}_U$ the sections of $\underline{\mathscr{F}}$ over U. (If U is empty, we set $\Gamma(U, \underline{\mathscr{F}}) = 0$).

Definition. A *p-cochain* of $\mathscr{O}$ is a mapping c of I^{p+1} such that $c_{j_\circ\ldots j_p} \in \Gamma(\mathscr{O}_{i_\circ\cdot i_p}, \underline{\mathscr{F}})$ and which is, moreover, *alternate*, (i.e., $c_{j_\circ\ldots j_p} = \varepsilon c_{i_\circ\ldots i_p}$ if $(j_\circ, \ldots, j_p)$ is a permutation of $(i_\circ, \ldots, i_p)$ and $\varepsilon = \pm 1$ according as this permutation is even or odd).

$\mathscr{C}^p(\mathscr{O}, \underline{\mathscr{F}})$ will denote the abelian group of the p-cochains of $\mathscr{O}$, $\mathscr{C}(\mathscr{O}, \underline{\mathscr{F}}) = \sum_{p\geq 0} \mathscr{C}^p(\mathscr{O}, \underline{\mathscr{F}})$ the direct sum of the $\mathscr{C}^p(\mathscr{O}, \underline{\mathscr{F}})$ for $p \geq 0$.

The *coboundary operator* $\delta^p : \mathscr{C}^p(\mathscr{O}, \underline{\mathscr{F}}) \to \mathscr{C}^{p+1}(\mathscr{O}, \underline{\mathscr{F}})$ is defined as follows: if $c \in \mathscr{C}^p$,

$$(\delta^p c)_{i_\circ\ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j c_{i_\circ\ldots\hat{i}_j\ldots i_{p+1}}$$

where $i_\circ, \ldots \hat{i}_j \ldots i_{p+1}$ signifies that the j-th index i_j is to be omitted. The δ^p give rise to a coboundary operator $\delta : \mathscr{C}(\mathscr{O}, \underline{\mathscr{F}}) \to (\mathscr{O}, \underline{\mathscr{F}})$. It can be verified that

$$\delta^p \circ \delta^{p-1} = 0 \text{ i.e., } \delta \circ \delta = 0.$$

Now, the elements c of $\mathscr{C}^\circ(\mathscr{O}, \underline{\mathscr{F}})$ such that $\delta c = 0$ are precisely those
98 elements with $c_{i_1} - c_{i_\circ} = 0$ in $\mathscr{O}_{i_\circ i_1}$ (by the definition of δ) and so, since $c_{i_\circ} = c_{i_1}$ in $\mathscr{O}_{i_\circ} \cap \mathscr{O}_{i_1}$ they correspond to sections over the whole of X, i.e., to elements of $\Gamma(X, \underline{\mathscr{F}})$. Hence we have the following compelx:

$$0 \to \Gamma(X, \underline{\mathscr{F}}) \xrightarrow{i} \mathscr{C}^\circ(\mathscr{O}, \underline{\mathscr{F}}) \xrightarrow{\delta^\circ} \mathscr{C}^1(\mathscr{O}, \underline{\mathscr{F}}) \xrightarrow{\delta^1} \ldots$$

and $\delta^p \circ \delta^{p-1} = 0$. Setting $z^p(\mathscr{O}, \underline{\mathscr{F}}) =$ kernel of δ^p, $B^p(\mathscr{O}, \underline{\mathscr{F}}) =$ image of δ^{p-1}, we define the p-th *cohomology group $H^p(\mathscr{O}, \underline{\mathscr{F}})$ of the covering $\mathscr{O}$ with coefficient sheaf $\underline{\mathscr{F}}$ by*

$$H^p(\mathscr{O}, \underline{\mathscr{F}}) = Z^p(\mathscr{O}, \underline{\mathscr{F}})/B^p(\mathscr{O}, \underline{\mathscr{F}}), \; p > 0$$
$$H^\circ(\mathscr{O}, \underline{\mathscr{F}}) = \Gamma(X, \underline{\mathscr{F}}).$$

2 Cohomology of the space X

Let $\mathscr{O} = \{\mathscr{O}_i\}_{i \in I}$, $\Omega = \{\Omega_\alpha\}_{\alpha \in A}$ be two (indexed) coverings of X and suppose that Ω is a refinement of $\mathscr{O}$, i.e., there is a mapping $\phi : A \to I$ such that $\Omega_\alpha \subset \mathscr{O}_{\phi(\alpha)}$. (We do not consider ϕ as given once for all, but merely require its existence). The mapping ρ of $\mathscr{C}^p(\mathscr{O}, \underline{\mathscr{F}})$ in $\mathscr{C}^p(\Omega, \underline{\mathscr{F}})$ defined by

$$(\rho c)_{\alpha_\circ \ldots \alpha_p} = \text{ restriction of } c_{\phi(\alpha_\circ) \ldots \phi(\alpha_p)} \text{ to } \Omega_{\alpha_\circ, \ldots, \alpha_p}$$

induces a mapping ρ^* of $H^p(\mathscr{O}, \underline{\mathscr{F}}) \to H^p(\Omega, \underline{\mathscr{F}})$ (this is easy to verify).

Proposition 1. *ρ^* does not depend on ϕ.*

Proof. Let ϕ, ψ be two mappings $A \to I$ such that $\Omega_\alpha \subset \mathscr{O}_{\phi(\alpha)} \cap \mathscr{O}_{\psi(\alpha)}$. Suppose that A is totally ordered. For $p = 0$, the result is obvious since

$H^\circ(\mathscr{O}, \underline{\mathscr{F}}) = \Gamma(X, \underline{\mathscr{F}})$ for every $\mathscr{O}$. If $p \geq 1$, we define a mapping k ("homotopy operator"): $\mathscr{C}^{p+1}(\mathscr{O}, \underline{\mathscr{F}}) \to \mathscr{C}^p(\Omega, \mathscr{F})$ by

$$(kc)_{\alpha_\circ \ldots \alpha_p} = \sum_{j=0}^{p} (-1)^j c_{\phi(\alpha_\circ)\ldots\phi(\alpha_j)\psi(\alpha_j)\ldots\psi(\alpha_p)},$$

if $\alpha_\circ < \alpha_1 < \ldots < \alpha_p$ in the total order of A and then define kc uniquely **99**
to be an (alternate) cochain. If the cochains corresponding to the maps ϕ, ψ are c', c'' in $\mathscr{C}^p(\Omega, \mathscr{F})$ respectively, it can be verified that

$$(k\delta + \delta k)c = c' - c''.$$

Consequently, if c is a cocycle (i.e., $\delta c = 0$), then $c' - c''$ is a coboundary, $c' - c'' = \delta(kc)$ and the mappings ϕ and ψ induce the same homomorphism of $H^p(\mathscr{O}, \underline{\mathscr{F}}) \to H^p(\Omega, \underline{\mathscr{F}})$. This proves Proposition 1.

This homomorphism is denoted by $\sigma(\mathscr{O}, \Omega)$. It satisfies certain obvious transitivity properties (as a functions of $\mathscr{O}$, Ω).

If $p = 0$, σ is always an isomorphism as observed above. □

Proposition 2. *If $p = 1$, σ is a monomorphism.*

We have to show that if $\phi : A \to I$ is such that $\Omega_\alpha \subset \mathscr{O}_{\phi(\alpha)}$ and $c = 0$ in $H^1(\Omega, \underline{\mathscr{F}})$, then $c = 0$ in $H^1(\mathscr{O}, \mathscr{F})$. Let c be a cochain with $c_{\phi(\alpha)\varphi(\beta)} = \gamma_\beta - \gamma_\alpha$ in $\Omega_{\alpha\beta}$. For every i and $x \in \mathscr{O}_i$ if $x \in \Omega_\alpha$, set $c_i(x) = \gamma_\alpha(x) + c_{\phi(\alpha)i}(x)$. If x is also in Ω_β, then $\gamma_\beta(x) + c_{\phi(\beta)i}(x) = \gamma_\alpha(x) + c_{\phi(\alpha)i}(x)$ since $\gamma_\beta - \gamma_a = c_{\phi(\alpha)\phi(\beta)} = c_{\phi(\alpha)i} - c_{\phi(\beta)i}$. Hence this defines a section c_i on $\mathscr{O}_i$ and clearly $c_{ij} = c_i - c_j$ in $\mathscr{O}_{ij}$, which proves the proposition.

The homomorphism $\sigma(\mathscr{O}, \Omega)$ defined above depends only on the coverings $\mathscr{O}, \Omega$. If $\dot{\mathscr{O}}, \Omega$ are refinements of one another, $\sigma(\mathscr{O}, \Omega)$ is an isomorphism. Hence we identify all coverings which are two by two refinements of one another, and consider the class of all indexed coverings modulo this identification. It is clear that this quotient can be put in one-one correspondence with a subclass of the power set of X and so is a set. It is clearly a directed set and we have a directed system

$$\{H^p(\mathscr{O}, \underline{\mathscr{F}}), \sigma(\mathscr{O}, \Omega)\}_{\mathscr{O}}$$

for every p. The direct limit of this system is called the p-th cohomology **100**
group of X with coefficient sheaf $\underline{\mathscr{F}}$ and is denoted $H^p(X, \underline{\mathscr{F}})$. It is obvious that $H^\circ(X, \underline{\mathscr{F}}) = \Gamma(X, \underline{\mathscr{F}})$.

3 The exact cohomology sequence

Let
$$0 \to \underline{\mathscr{F}} \xrightarrow{i} \underline{\mathscr{G}} \xrightarrow{\eta} \underline{\mathscr{H}} \to 0$$
be an exact sequence of sheaves. This evidently gives rise to an exact sequence
$$0 \to \mathscr{C}(\mathscr{O}, \underline{\mathscr{F}}) \to \mathscr{C}(\mathscr{O}, \underline{\mathscr{G}}) \to \mathscr{C}(\mathscr{O}, \underline{\mathscr{H}})$$
but the last mapping is not in general onto. If we denote by $\mathscr{C}_a(\mathscr{O}, \underline{\mathscr{H}})$ the image $\mathscr{C}(\mathscr{O}, \underline{\mathscr{G}})$ (group of "cochaînes ascensionelles") then we obtain the exact sequence
$$0 \to \mathscr{C}(\mathscr{O}, \underline{\mathscr{F}}) \to \mathscr{C}(\mathscr{O}, \underline{\mathscr{G}}) \to \mathscr{C}_a(\mathscr{O}, \underline{\mathscr{H}}) \to 0.$$
It Z_a^p is the group of cochains c in $\mathscr{C}_{\underline{a}}^p(\mathscr{O}, \underline{\mathscr{H}})$ with $\delta c = 0$ and B_a^p is the group of cochains δc, $c \in \mathscr{C}_a^{p-1}(\mathscr{O}, \mathscr{H})$, we define the group $H_a^p(\mathscr{O}, \underline{\mathscr{H}})$ by
$$H_a^p(\mathscr{O}, \underline{\mathscr{H}}) = Z_a^p / B_a^p.$$
We now define a mapping $d^* : H_a^p(\mathscr{O}, \underline{\mathscr{H}}) \to H^{p+1}(\mathscr{O}, \underline{\mathscr{F}})$ in the following way. Let $h \in Z_a^p$ and let $h_{i_\circ \ldots i_p} = \eta(g_{i_\circ \ldots i_p})$ where $g \in \mathscr{C}^p(\mathscr{C}, \underline{\mathscr{G}})$; also, since clearly η and δ commute, $\eta\{(\delta_g)_{i_\circ \ldots i_{p+1}}\} = (\delta h)_{i_\circ \ldots i_{p+1}} = 0$ since $h \in Z_a^p \cdot (\delta_g)_{i_\circ, \ldots i_{p+1}}$ being a section over $\mathscr{O}_{i_\circ \ldots i_{p+1}}$ which goes to 0 under η, $\delta g \in \mathscr{C}^{p+1}(\mathscr{O}, \underline{\mathscr{F}})$ (since $\underline{\mathscr{F}}$ is the kernel of η). It is easy to see that
101 the class of δg in $H^{p+1}(\mathscr{O}, \underline{\mathscr{F}})$ remains unchanged if g is replaced by another cochain g' with $\eta g' = h$ and if h is replaced bya cohomologous cocycle. This defines d^*.

It is clear that i, η induce homomorphisms
$$i^*, \eta^* : H^p(\mathscr{O}, \underline{\mathscr{F}}) \xrightarrow{i^*} H^p(\mathscr{O}, \underline{\mathscr{G}}) \xrightarrow{\eta^*} H^p(\mathscr{O}, \underline{\mathscr{H}})$$
and it can be verified that the following sequence is exact:
$$0 \to H^\circ(\mathscr{O}, \underline{\mathscr{F}}) \xrightarrow{i^*} H^\circ(\mathscr{O}, \underline{\mathscr{G}}) \xrightarrow{\eta^*} H_a^\circ(\mathscr{O}, \underline{\mathscr{H}}) \xrightarrow{d^*} H^1(\mathscr{O}, \underline{\mathscr{F}}) \xrightarrow{i^*} \ldots$$
$$\ldots \xrightarrow{d^*} H^p(\mathscr{O}, \underline{\mathscr{F}}) \xrightarrow{i^*} H^p(\mathscr{O}, \underline{\mathscr{G}}) \xrightarrow{\eta^*} H_a^p(\mathscr{O}, \underline{\mathscr{H}}) \xrightarrow{d^*} H^{p+1}(\mathscr{O}, \underline{\mathscr{F}}) \xrightarrow{i^*} \ldots$$

We can now define the groups $H^p_a(X, \underline{\mathscr{H}})$ by taking direct limits as the covering becomes finer, as for the groups $H^p(X, \underline{\mathscr{H}})$. Also there is a canonical mapping $H^p_a(\mathscr{O}\underline{\mathscr{H}}) \to H^p(\mathscr{O}, \underline{\mathscr{H}})$ and so a canonical homomorphism $H^p_a(X, \underline{\mathscr{H}}) \to H^p(X, \underline{\mathscr{H}})$. Since the operation of taking direct limits commutes with exact sequences we obtain the exact sequence

$$\ldots \xrightarrow{d^*} H^p(X, \underline{\mathscr{F}}) \xrightarrow{i^*} H^p(X, \underline{\mathscr{G}}) \xrightarrow{\eta^*} H^p_a(X, \underline{\mathscr{H}}) \xrightarrow{d^*} H^{p+1}(X, \underline{\mathscr{F}}) \xrightarrow{i^*} \ldots$$

It is of interest to decide when $H^p_a(X, \underline{\mathscr{H}}) = H^p(X, \underline{\mathscr{H}})$. This is so in the case when X is paracompact (i.e., a Hausdorff space in which every covering admits a locally finite refinement).

Theorem. *If X is paracompact and*

$$0 \to \underline{\mathscr{F}} \xrightarrow{i} \underline{\mathscr{G}} \xrightarrow{\eta} \underline{\mathscr{H}} \to 0$$

an exact sequence of sheaves on X, then the canonical homomorphism

$$H^p_a(X, \underline{\mathscr{H}}) \to H^p(X, \mathscr{H})$$

is an isomorphism.

The theorem follows at once from the following

Lemma. *If $\mathscr{O} = \{\mathscr{O}_i\}_{i\in I}$ is a covering of X and $c \in \mathscr{C}^p(\mathscr{O}, \underline{\mathscr{H}})$, there exists a covering $\Omega = \{\Omega_\alpha\}_{\alpha\in A}$ and a mapping $\phi : A \to I$ with $\Omega_\alpha \subset \mathscr{O}_{\phi(\alpha)}$* **102**
such that the induced homomorphism $\phi^* : \mathscr{C}^p(\mathscr{O}, \underline{\mathscr{H}}) \to \mathscr{C}^p(\Omega, \underline{\mathscr{H}})$ take c to a cochain $\phi^*(c) \in \mathscr{C}^p_a(\Omega, \underline{\mathscr{H}})$.

Proof of the lemma: Since X is paracompact, we may suppose $\mathscr{O}$ locally finite. Since X is *normal*, (see Dieudonné [5]) there is an open covering $\{\mathscr{O}'_i\}_{i\in I}$ such that $\overline{\mathscr{O}'_i} \subset \mathscr{O}_i$. For every $x \in X$, we choose an open neighbourhood Ω_x of x such that

(i) $x \in \mathscr{O}_i$ (respectively $\mathscr{O}'_i$) implies $\Omega_x \subset \mathscr{O}_i$ (respectively $\mathscr{O}'_i$).

(ii) $\Omega_x \cap \mathscr{O}'_i \neq 0$ implies $\Omega_x \subset \mathscr{O}_i$.

(iii) If $x \in \mathscr{O}_{i_\circ \dots i_p}$, there is a section S of $\underline{\mathscr{G}}$ over Ω_x such that $\eta(s) = c_{i_\circ \dots i_p}$ on Ω_x.

Since $\mathscr{O}$ is locally finite, it follows from the definition of quotient sheaf that (iii) can be fulfilled; and (i) and (ii) are then ensured if we choose the Ω_x small enough. This gives us a covering $\{\Omega_x\}_{x \in X} = \Omega$ of X; we choose a mapping $\phi : X \to I$ such that $\Omega_x \subset \mathscr{O}'_{\phi(x)}$. It is then easy to verify that Ω and ϕ have the property stated in the lemma.

Chapter 14

Coherent analytic sheaves on a cube

1 The abstract de Rham Theorem

Let X be a paracompact topological space, $\underline{\mathscr{F}}$ a sheaf of abelian groups 103
on X, and suppose that

$$0 \to \underline{\mathscr{F}} \xrightarrow{i} \underline{\mathscr{G}_\circ} \xrightarrow{d_\circ} \underline{\mathscr{G}_1} \xrightarrow{d_1} \dots \xrightarrow{d_{k-1}} \underline{\mathscr{G}_k} \xrightarrow{d_k} \dots$$

is an exact sequence of sheaves on X and that $H^p(X, \underline{\mathscr{G}}_k) = 0$ for $p \geq 1$, $k \geq 0$. Consider the sequence

$$0 \to \Gamma(X, \underline{\mathscr{F}}) \xrightarrow{i^*} \Gamma(X, \underline{\mathscr{G}}_\circ) \xrightarrow{d_\circ^*} \dots \xrightarrow{d_{k-1}^*} \Gamma(X, \underline{\mathscr{G}}_k) \xrightarrow{d_k^*} \dots$$

with the induced homomorphisms d_k^* (this is not in general exact).

Then

$$H^k(X, \underline{\mathscr{F}}) \simeq \text{ kernel } d_k^* / \text{ image } d_{k-1}^* \text{ for } k \geq 1.$$

Proof. Consider the exact sequence

$$\underline{\mathscr{G}}_{k-1} \xrightarrow{d_{k-1}} \underline{\mathscr{G}}_k \xrightarrow{d_k} \underline{\mathscr{G}}_{k+1}$$

and let $\underline{\mathscr{H}}_k$ = kernel d_k = image d_{k-1}. Then we have an exact sequence

$$0 \to \underline{\mathscr{H}}_k \to \underline{\mathscr{G}}_k \to \underline{\mathscr{H}}_{k+1} \to 0$$

and X being paracompact, we obtain, for $q > 0$, the exact sequence

$$H^q(X, \underline{\mathscr{G}}_k) \to H^q(X, \underline{\mathscr{H}}_{k+1}) \to H^{q+1}(X, \underline{\mathscr{H}}_k) \to H^{q+1}(X, \underline{\mathscr{G}}_k)$$

and since $H^p(X, \underline{\mathscr{G}}_k) = 0$, if $p \geq 1$,

$$H^q(X, \underline{\mathscr{H}}_{k+1}) \simeq H^{q+1}(X, \underline{\mathscr{H}_k})$$

for $q \geq 1$. By iteration

$$H^p(X, \underline{\mathscr{F}}) \simeq H^{p-1}(X, \underline{\mathscr{H}}_1) \simeq \ldots \simeq H^1(X, \underline{\mathscr{H}}_{p-1}). \tag{1}$$

Also we have the exact sequence

$$0 \to \underline{\mathscr{H}}_{p-1} \to \underline{\mathscr{G}}_{p-1} \to \underline{\mathscr{H}}_p \to 0$$

104 and the induced exact sequence

$$H^\circ(X, \mathscr{G}_{p-1}) \to H^\circ(X, \underline{\mathscr{H}}_p) \to H^1(X, \underline{\mathscr{H}}_{p-1}) \to H^1(X, \underline{\mathscr{G}}_{p-1}).$$

Since the last term is 0 by hypothesis,

$$H^1(X, \bar{H}_{p-1}) \simeq H^\circ(X, \underline{\mathscr{H}}_p) / \text{ image } H^\circ(X, \underline{\mathscr{G}}_{p-1}).$$

It is easy to see that $H^\circ(X, \underline{\mathscr{H}}_p) \simeq$ kernel d^*_k, while image $H^\circ(X, \mathscr{G}_{p-1}) =$ image d^*_{k-1} and the result follows from (1). ◻

Applications.

a) **de Rham's Theorem.** Let $X = V$ be a paracompact differentiable manifold and $\underline{\mathscr{E}}^p$ the sheaf of germs of differentiable p-forms on V. Then we have a sequence

$$0 \to \underline{C} \xrightarrow{i} \underline{\mathscr{E}}^\circ \xrightarrow{d} \cdots \xrightarrow{d} \underline{\mathscr{E}}^p \xrightarrow{d} \underline{\mathscr{E}}^{p+1} \xrightarrow{d} \cdots$$

(C is a constant sheaf, C being the group of complex numbers). This sequence is exact by the local form of Poincaré's theorem (VIII). By the method given in Step 1 of the solution of the generalized first Cousin problem (VIII) it is seen that $H^k(V, \underline{\mathscr{E}}^p) = 0$ if $k \geq 1$, $p \geq 0$. Hence, the above theorem shows that

$$H^p(V, \underline{C}) \simeq$$

(group of closed p-forms)/(the differentials of $(p-1)$ forms), i.e., $H^p(V, C)$ is the same as the p-dimensional d-cohomology of V. This is the theorem of de Rham.

b) **Dolbeault's theorem.** Let V: a paracompact complex manifold and $\underline{\mathscr{E}}^{o,p}$ the sheaf of germs of forms of type $(0, p)$ on V. Then, the local form of Grothendieck's theorem shows that the sequence

$$0 \to \underline{\mathscr{O}} \xrightarrow{i} \underline{\mathscr{E}}^{o,o} \xrightarrow{d''} \underline{\mathscr{E}}^{0,1} \xrightarrow{d''} \ldots$$

is exact ($\underline{\mathscr{O}}$ is the sheaf of germs of holomorphic functions on V). Again, by the method of the generalized first Cousin problem, we **105**
prove that

$$H^k(V, \underline{\mathscr{E}}^{o,p}) = 0, \quad p \geq 0, \quad k \geq 1,$$

and we obtain $H^p(V, \underline{\mathscr{O}}) \simeq$ (d''-closed $(0, p)$ forms)/(d''-differentials of $(0, p-1)$ forms), i.e., $H^p(V, \underline{\mathscr{O}})$ is the same as the p - dimensional d''-cohomology of the $(0, r)$ forms, $r = 0, 1, \ldots$ on V.

Similar reasoning proves that the p-th cohomology of V with coefficients in the sheaf of germs of holomorphic $(q, 0)$-forms is the p-th d''-cohomology of the (q, r)-forms, $r = 0, 1, \ldots$ on V. This is a particular case of Dolbeault's theorem.

c) Let K be a cube imbedded in C^n, i.e., K is a subset of a fixed C^n, consisting of the points $z \in C^n$ with

$$|\mathscr{R} z_i| \leq a_i |\Im z_i| \leq b_i, \ a_i, \ b_i \geq 0.$$

Consider the sheaf $\underline{\mathscr{O}} = \underline{\mathscr{O}}_K$ (in the sense of XII, i.e., the restriction to K of the sheaf of germs of holomorphic functions in C^n).

We define the sheaves $\underline{\mathscr{E}}^{o,p}$ in the same way as the restriction to K of the sheaf of germs of $(0,p)$-forms in C^n. As before we have the exact sequence

$$0 \to \underline{\mathscr{O}} \xrightarrow{i} \underline{\mathscr{E}}^{o,o} \xrightarrow{d''} \underline{\mathscr{E}}^{o,1} \xrightarrow{d''} \ldots$$

and $H^p(K, \underline{\mathscr{O}}) \simeq$ (d''-closed $(0,p)$-forms)/(d''-differentials of $(0, p-1)$-forms). Grothendieck's theorem shows that this quotient is zero. (The theorem was proved only when a_i, $b_i > 0$, but it is immediate that the theorem holds also when some of the a_i, b_i are 0). This proves the following

Theorem 1. *If K is a closed cube in C^n, $H^p(K, \underline{\mathscr{O}}) = 0$ for $p \geq 1$.*

106 **Corollary.** *If $p, q \geq 1$, $H^p(K, \underline{\mathscr{O}}^q) = 0$.*

(One has only to apply Theorem 1 to the components of an element c of $Z^p(K, \underline{\mathscr{O}}^q)$ to see that c is a coboundary).

It is instructive to compare the above proof with the solution of the generalized first Cousin problem for a cube which implies that $H^1(K, \underline{\mathscr{O}}) = 0$. (This is exactly the generalization of that proof to the more general setting here).

2 Coherent analytic sheaves on a cube

Let K be a cube in C^n and $\underline{\mathscr{F}}$ a coherent analytic sheaf on K. *Fundamental Theorem. (Oka-Cartan-Serre).*

A) $\mathscr{F}$ **is (globally) a quotient of a sheaf** $\underline{\mathscr{O}}^N$ (This can also be expressed by saying that there are N (global) sections $f_1, \ldots, f_N$ of $\underline{\mathscr{F}}$ which $\mathscr{O}_b$-generate $\mathscr{F}_b$ for every $b \in K$.)

B) For $p \geq 1$, $H^p(K, \underline{\mathscr{F}}) = 0$.

We introduce the following statement.

A') $\mathscr{F}$ is (globally) a quotient of a coherent analytic sheaf locally isomorphic to $\underline{\mathscr{O}}^N$.

The proof of the fundamental theorem now divides into two parts, the proof that A') implies A) and B) for a cube and the proof of A') for a cube.

Step 3. The truth of A') for every $\underline{\mathscr{F}}$ implies the truth of A) and B) for every $\underline{\mathscr{F}}$.

(i) Suppose A') true for all $\underline{\mathscr{F}}$. Then $\underline{\mathscr{F}}$ is a quotient of a sheaf $\underline{\mathscr{G}}$ locally isomorphic to $\underline{\mathscr{O}}^N$. Hence $\underline{\mathscr{G}}$ defines a class of analytic bundles over a neighbourhood of K (end of XI; the results proved there relate to topological bundles, but they remain valid with obvious modifications for analytic bundles). By Theorem 2 of X, **107**
this class is the trivial class (on some neighbourhood of K) and so $\underline{\mathscr{G}}$ is $\simeq \underline{\mathscr{O}}^N$ and A) is proved.

(ii) According to A), we have an exact sequence

$$\underline{\mathscr{O}}^{N_1} \to \underline{\mathscr{F}} \to 0$$

and if $\underline{\mathscr{G}}_1$ is the kernel of this mapping, $\underline{\mathscr{G}}_1$ is coherent analytic by Theorem 1 of XII, and we have the exact sequence

$$0 \to \underline{\mathscr{G}}_1 \to \underline{\mathscr{O}}^{N_1} \to \underline{\mathscr{F}} \to 0.$$

Since A) is supposed to hold for all coherent analytic sheaves, we obtain exact sequences

$$\begin{gathered} 0 \to \underline{\mathscr{G}}_2 \to \underline{\mathscr{O}}^{N_2} \to \underline{\mathscr{G}}_1 \to 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \to \underline{\mathscr{G}}_k \to \underline{\mathscr{O}}^{N_k} \to \underline{\mathscr{G}}_{k-1} \to 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{gathered}$$

This leads to the exact sequence

$$H^p(K, \underline{\mathscr{O}}^{N_k}) \to H^p(K, \underline{\mathscr{G}}_{k-1}) \to H^{p+1}(K, \underline{\mathscr{G}}_k) \to H^{p+1}(K, \underline{\mathscr{O}}^{N_k}).$$

If $p \geq 1$, the first and last terms are zero by Theorem 1 above and consequently,

$$H^p(K, \underline{\mathscr{G}}_{k-1}) \simeq H^{p+1}(K, \underline{\mathscr{G}}_k) \tag{2}$$

Now, there is an integer m such that every covering of K has a refinement in which the intersection of any $m+1$ sets (of the refinement) is empty. [This is seen for example by subdividing K

into smaller cubes by hyperplanes parallel to the coordinate hyperplanes. The stetement is, however, essentially of dimension 108
theoretic character; classical dimension theory shows that te best possible m is $2n + 1$ and that this is not special to cubes. The above property is the starting point of the more modern theory of dimension]. Hence $H^{m+p}(K, \underline{\mathscr{H}}) = 0$ for every sheaf $\underline{\mathscr{H}}$ over K and $p \geq 1$. By iterating (2), we obtain

$$H^p(K, \underline{\mathscr{F}}) \simeq H^{p+1}(K, \underline{\mathscr{G}}_1) \simeq \ldots \simeq H^{m+p}(K, \underline{\mathscr{G}}_m) = 0$$

and B) is proved.

Step 4. Proof of A') **for a cube.** The proof will be by induction on the real dimension of the cube K. If K has dimension $0, A'$) is just the definition of a coherent analytic sheaf. Suppose A') true for all cubes K' of real dimension p and all coherent analytic sheaves $\underline{\mathscr{F}}$ on K'. Then A) and B) are also true for K' and $\underline{\mathscr{F}}$.

Let now K be a cube of real dimension $p + 1$. We find a coordinate hyperplane such that the intersection of K with any hyperplane parallel to it is of dimension p (if it is non-empty). The restriction of $\underline{\mathscr{F}}$ to each such intersection is coherent analytic, and by inductive hypothesis, $\simeq$ a quotient of $\underline{\mathscr{O}}^N$. The extension theorem of XII shows that there is a neighbourhood of each intersection in which $\underline{\mathscr{F}}$ induces a coherect analytic sheaf which is isomorphic to a quotient of same $\underline{\mathscr{O}}^N$. By the Borel-Lebesgue lemma, it follows that we have only to prove the following result:

Given two adjacent cubes K_1, K_2 of dimension $p + 1$ such that $P = K_1 \cap K_2$ is of dimension p, and a coherent analytic sheaf $\underline{\mathscr{F}}$ on $K_1 \cup K_2$ such that $\underline{\mathscr{F}}$ is a quotient of a sheaf $\underline{\mathscr{O}}^{N_1}$ on K_1, and a quotient of a sheaf $\underline{\mathscr{O}}^{N_2}$ on K_2, then $\underline{\mathscr{F}}$ is a quotient of a sheaf locally isomorphic to $\underline{\mathscr{O}}^{N_1+N_2}$ on $K_1 \cup K_2$.

109 Let $(f) = \begin{pmatrix} f_1 \\ \vdots \\ f_{N_1} \end{pmatrix}$, $f_1, \ldots, f_{N_1}$ being sections of $\underline{\mathscr{F}}$ over K_1 which $\mathscr{O}_a$-

generate $\mathscr{F}_a$ for $a \in K_1$, $(g) = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}$, $g_1, \ldots, g_{N_2}$ being sections of $\underline{\mathscr{F}}$ over K_2 which $\mathscr{O}_b$-generate $\mathscr{F}_b$ for $b \in K_2$. It is easily seen that it is enough to find a holomorphic regular matrix c on P such that

$$c \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

To find c, we use the following lemma, which is of interest by itself.

Lemma. *If ϕ is a section of $\underline{\mathscr{F}}$ over P, there are N_1 holomorphic functions $\lambda_1, \ldots, \lambda_{N_1}$ on P such that*

$$\phi = \lambda_1 f_1 + \cdots + \lambda_{N_1} F_{N_1}.$$

Proof of the lemma: Consider the exact sequence

$$0 \to \underline{\mathscr{G}'} \to \underline{\mathscr{O}}^{N_1} \to \underline{\mathscr{F}} \to 0$$

(these being sheaves on $P = K_1 \cap K_2$; $\underline{\mathscr{G}'}$ is the kernel of the homomorphism $\underline{\mathscr{O}}^{N_1} \to \underline{\mathscr{F}}$). This gives us the exact sequence

$$H^\circ(P, \underline{\mathscr{O}}^{N_1}) \to H^\circ(P, \underline{\mathscr{F}}) \to H^1(P, \underline{\mathscr{G}'})$$

and by inductive hypothesis, $H^1(P, \underline{\mathscr{G}'}) = 0$ since P has dimensions p. Hence the mapping

$$H^\circ(P, \underline{\mathscr{O}}^{N_1}) \to H^\circ(P, \underline{\mathscr{F}})$$

is onto and the lemma follows.

Construction of c. The lemma proves that there is a matrix γ_1 of N_1 columns and N_2 rows such that

$$\gamma_1 \begin{pmatrix} f_1 \\ \vdots \\ f_{N_1} \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_{N_2} \end{pmatrix}$$

i.e., $\gamma_1(f) = (g)$. In the same way there is a matrix γ_2 of N_2 columns 110
and N_1 rows such that

$$\gamma_2(g) = (f).$$

If we set

$$c = \begin{pmatrix} -I & \gamma_2 \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ \gamma_1 & I \end{pmatrix}$$

it is clear that c is regular. Also $\begin{pmatrix} I & 0 \\ \gamma_1 & I \end{pmatrix}$ takes $\begin{pmatrix} f \\ 0 \end{pmatrix}$ to $\begin{pmatrix} f \\ g \end{pmatrix}$, $\begin{pmatrix} -I & \gamma_2 \\ 0 & I \end{pmatrix}$ takes $\begin{pmatrix} f \\ g \end{pmatrix}$ to $\begin{pmatrix} 0 \\ g \end{pmatrix}$. This concludes Step 2 and with it the proof of the fundamental theorem.

Remark. All reference to bundles can be avoided if after the construction of c, one applies Cartan's theorem on holomorphic, regular matrices to prove directly A) and B) without introducing A').

The statement A') is true for a much wider class of sets than is A). Actually the problem of classifying the compact sets for which A') is true is still open (even in C^n).

Chapter 15

Stein Manifolds: preliminary results

1 Theorems A) and B) for closed polydiscs in C^n

Let V be a complex analytic manifold, K a compact subset of V. We say that Theorems A) and B) are true for K if A) every coherent analytic sheaf $\underline{\mathscr{F}}$ on K is a quotient of a sheaf $\simeq \underline{\mathscr{O}}^N$ and B)$H^p(K, \underline{\mathscr{F}}) = 0$ for $p \geq 1$. 111

Proposition 1. *Theorems A) and B) are true for (closed) polydiscs in C^n.*

Proof. Let P be the given polydisc. P has a fundamental system of neighbourhoods each of which is analytically isomorphic to a closed cube: P has a fundamental system of neighbourhoods which are open polydiscs $\prod$; $\prod$, being a product of open discs, is isomorphic to an open cube $\prod_1$ and the image of P in $\prod_1$ is contained in a closed cube contained in $\prod_1$ whose inverse image in $\prod$ is a neighbourhood isomorphic to a closed cube. Since Theorems A) and B) are true for closed cubes, it follows that Theorems A) and B) are ture for a fundamental system of neighbourhoods of P and after the extension theorem of XII it is easily seen that A) is true for P, and B) follows from $\square$

Lemma 1. *Let X be a paracompact topological space, $\underline{\mathscr{F}}$ a sheaf of abelian groups on X. Let Y be a closed set in X with a fundamental system of closed neighbourhoods L. Then $H^p(Y, \underline{\mathscr{F}})$ is the direct limit of $H^p(L\underline{\mathscr{F}})$ as L shrinks to Y.*

Proof. For $p = 0$, this follows from the fact that every section over Y can be extended to a section over an open neighbourhood of Y and the fact that the set of points at which two sections coincide is open. For $p > 0$, we construct an exact sequence

$$0 \to \underline{\mathscr{F}} \xrightarrow{i} \underline{\mathscr{G}}_\circ \xrightarrow{d_\circ} \underline{\mathscr{G}}_1 \xrightarrow{d_1} \dots \xrightarrow{d_{k-1}} \underline{\mathscr{G}}_k \xrightarrow{d_k} \dots \tag{1}$$

112 such that $H^p(E, \underline{\mathscr{G}}_1) = 0$ if $p > 0$, $l \geq 0$ for all subsets E of X. To do this it is clearly sufficient to construct an exact sequence

$$0 \to \underline{\mathscr{F}} \to \underline{\mathscr{G}} \text{ with } H^p(E, \underline{\mathscr{G}}) = 0 \text{ for } p > 0$$

(for the construction can be repeated with the quotient $\underline{\mathscr{G}}/\underline{\mathscr{F}}$ and the process continued). We define $\underline{\mathscr{G}}$ to be the sheaf of germs of all mappings $f : X \to \underline{\mathscr{F}}$ with $f(x) \in \mathscr{F}_x$ for $x \in X$. Then clearly we have an exact sequence

$$0 \to \underline{\mathscr{F}} \to \underline{\mathscr{G}}$$

and $H^p(E, \underline{\mathscr{G}}) = 0$ for $p > 0$ by the theorem in the appendix

Having constructed the exact sequence (1) we consider the associated sequence

$$0 \to \Gamma(E, \underline{\mathscr{F}}) \xrightarrow{i^*} \Gamma(E, \underline{\mathscr{G}}_\circ) \xrightarrow{d^*_\circ} \dots \xrightarrow{d^*_{k-1}} \Gamma(E, \underline{\mathscr{G}}_k) \xrightarrow{d^*_k} \dots$$

and by the abstract de Rham theorem, we have

$$H^p(E, \underline{\mathscr{F}}) \simeq \text{ kernel } d^*_p / \text{ image } d^*_{p-1} (p \geq 1). \tag{2}$$

As in the case when $p = 0$, as L shrinks to Y, kernel d^*_k, image d^*_{k-1} (with E replaced by L) have as their direct limit, the kernel and image of the mappings

$$\gamma(Y, \underline{\mathscr{G}}_k) \to \Gamma(Y, \underline{\mathscr{G}}_{k+1}) \text{ and } \Gamma(Y, \underline{\mathscr{G}}_{k-1}) \to \Gamma(Y, \underline{\mathscr{G}}_k)$$

respectively and an application of (2) with $E = Y$ establishes the lemma. $\square$

2 Coherent analytic sheaves on an analytic submanifold

Let X be a topological space, Y a closed subset of X, $\underline{\mathscr{F}}$ a sheaf of abelian groups on Y. We define a sheaf $\underline{\tilde{\mathscr{F}}}$ on X by setting $\tilde{\mathscr{F}}_a = \mathscr{F}_a$ if $a \in Y$, $=$ the group 0 if $a \notin Y$. Clearly, this defines a sheaf on X. Then one has **113**

Proposition 2. *For* $p = 0, 1, \ldots$

$$H^p(Y, \underline{\mathscr{F}}) \simeq H^p(X, \underline{\tilde{\mathscr{F}}}).$$

Proof. If $\mathscr{O} = \{\mathscr{O}_i\}_{i \in I}$ is an open covering of X, $\mathscr{O}'_i = \mathscr{O}_i \cap Y$, then $\{\mathscr{O}'_i\} = \mathscr{O}'$ is an open covering of Y and clearly

$$H^p(\mathscr{O}, \underline{\tilde{\mathscr{F}}}) \simeq H^p(\mathscr{O}' \underline{\mathscr{F}}).$$

Also given an open covering $\mathscr{O}' = \{\mathscr{O}'_i\}$ of Y, if $\mathscr{O}_i$ is such that $\mathscr{O}_i \cap Y = \mathscr{O}'_i$, then the open covering $\{\mathscr{O}_i, X - Y\}$ of X gives rise to $\mathscr{O}'$ in the above way and Proposition 2 follows. □

Definition. Let V^n be a complex analytic manifold of complex dimension n. Let W^m be a closed subset of V^n. W^m is called an *analytic submanifold of dimension m*, if, for $a \in W^m$, the local coordinates $(z_1, \ldots, z_n)$ at a on V^n (in an open set $U \subset V^n$) can be so chosen that $W^m \cap U = \{z \in U \big| z_{m+1} = \ldots = z_n = 0\}$.

An application of the implicit function theorem shows that if W is a complex analytic manifold of dimension m and i an analytic one-one mapping of W into V^n, $i(W)$ is an analytic submanifold of V^n if and only if

1) i is proper: the inverse image of a compact subset of V^n is a compact subset of W;

2) i has rank m (i.e., the Jacobian matrix of i has rank m at every point of W).

Let V be a complex manifold and W a submanifold of V. Let $X \subset V$ and $Y = X \cap W$. We shall denote by ${}_V\underline{\mathscr{O}}$, ${}_W\underline{\mathscr{O}}$ the sheaves of germs of holomorphic functions on V and W respectively, considered as complex manifolds *in their own rights.*

114 Let $\underline{\mathscr{F}}$ be a coherent ${}_W\underline{\mathscr{O}}$-analytic sheaf on Y, and $\underline{\tilde{\mathscr{F}}}$ the sheaf which continues $\underline{\mathscr{F}}$ to X by 0 outside Y. Then $\underline{\tilde{\mathscr{F}}}_a$ has a structure of ${}_V\underline{\mathscr{O}}$-analytic sheaf on X: if $a \notin Y$, $\tilde{\mathscr{F}}_a$ is an ${}_V\mathscr{O}_a$-module ($\tilde{\mathscr{F}}_a$ being 0); if $a \in Y$, $f_a \in \tilde{\mathscr{F}}_a = \mathscr{F}$ and $h_a \in {}_V\mathscr{O}a$, $h_a f_a$ is defined to be $h_{aW} f_a$, where h_{aW} is the restriction of h_a to W.

Let $\underline{\mathfrak{I}}(W)$ be the subsheaf of ${}_V\underline{\mathscr{O}}$ consisting of those germs which vanish on W. Then we have the isomorphism

$$ {}_V\underline{\mathscr{O}}/\underline{\mathfrak{I}}(W) \simeq W\underline{\tilde{\mathscr{O}}}; $$

by a theorem of Cartan [2], [3, lecture XVI], $\underline{\mathfrak{I}}(W)$ is coherent.

Proposition 3. *If $\underline{\mathscr{F}}$ is a coherent ${}_W\underline{\mathscr{O}}$-analytic sheaf on Y, then $\underline{\tilde{\mathscr{F}}}$ is a coherent ${}_V\mathscr{O}$-analytic sheaf on X.*

Proof. If $\underline{\mathscr{F}}$ is the sheaf ${}_W\underline{\mathscr{O}}_Y$ (restriction of ${}_W\underline{\mathscr{O}}$ to Y), then, by what we observed above,

$$ {}_V\underline{\mathscr{O}}_X/\underline{\mathfrak{I}}(W)_X \simeq {}_W\underline{\tilde{\mathscr{O}}} $$

and so $\underline{\tilde{\mathscr{O}}}$ is a coherent ${}_V\underline{\mathscr{O}}$-analytic sheaf.

In the general case, let $\underline{\mathscr{F}}$ be a sheaf on Y, $a \in Y$, Ω an open neighbourhood of a in Y such that

$$ \underline{\mathscr{F}}_\Omega \simeq {}_W\underline{\mathscr{O}}^N_\Omega/\underline{\mathscr{R}}, $$

where $\underline{\mathscr{R}}$ is a coherent analytic subsheaf of ${}_W\underline{\mathscr{O}}^N$.

Let Ω be chosen so small that there are N_1 sets of N holomorphic functions on W which ${}_W\mathscr{O}_b$-generate $\mathscr{R}_b$ at every point of Ω. If Ω is again sufficiently small, there is a neighbourhood Ω' of a in V such that $\Omega' \cap Y = \Omega$ and these functions are restrictions of holomorphic func-
115 tions in Ω' to Ω. Let $\mathscr{R}'$ be the subsheaf of ${}_V\underline{\mathscr{O}}^N_{\Omega'}$ generated by these N_1 elements of ${}_V\underline{\mathscr{O}}^N_{\Omega'}$. Then $\underline{\mathscr{R}}'$ is a coherent analytic subsheaf of ${}_V\underline{\mathscr{O}}^N_{\Omega'}$, while clearly $\tilde{\mathscr{F}}_{\Omega'\cap X} \simeq {}_V\underline{\mathscr{O}}^N_{\Omega'\cap X}/\underline{\mathscr{R}}'_X + \underline{\mathfrak{I}}^N(W)_X$ and the proposition follows. $\square$

Proposition 4. *Let P be a closed polydisc in C^n, $\prod$ an open polydisc $\supset P$. Let W be an analytic manifold which is a submanifold of $\prod$. Then, Theorems A) and B) are true for $W \cap P$ (considered as subset of W). This follows at once from Propositions 1, 2 and 3.*

3 Stein Manifolds

Definition . A complex analytic manifold V of dimension n which is countable at infinity is said to be a *Stein manifold* if

(α) V is holomorph-convex (VII, 3);

(β) for any two points $a \neq b$ on V, there exists a holomorphic f on V, such that $f(a) \neq f(b)$.

(γ) if $a \in V$, there are n functions holomorphic in V which form a system of local coordinates at a.

Examples of Stein manifolds.

1. Univalent domains of holomorphy in C^n (see VII, Prop. 1)

2. Any open connected Riemann surface. [A Riemann surface is countable at infinity by Rado's theorem; (α), (β) and (γ) follow from Runge's theorem. For the details of proof, see H. Behnke and K. Stein: Entwicklung analytischer Funktionen auf Riemannschen Flächen, Math. Ann., 120 (1948), 430-461, and B. Malgrange: Existence at approximation des solutions des équations aux déxivées partielles et des équations de convolution Thèse, Paris, 1956 (Chap.III, §4)].

3. Analytic submanifolds of C^n. In particular, algebraic varieties over C which have no singularities. **116**

Lemma 2 (on Stein manifolds)**.** *Let V be a Stein manifold, K a compact subset of V such that $K = \hat{K}$ ($\hat{K}$ is the $\mathscr{H}_V$-envelope of K; see VI). Then K has a fundamental system $\{L\}$ of compact neighbourhoods L having the following properties:*

To every L correspond an open set $\wedge \supset L$, a closed polydisc $P \subset C^N$ an open polydisc $\prod \supset P$ and a finite number N of holomorphic functions $f_1, \ldots, f_N$ on V such that the restrictions to $\wedge$ of the f_i realize $\wedge$ as an analytic submanifold of $\prod$ and such that $\phi(L) = P \cap \phi(\wedge)$ where ϕ is the mapping $(f_1, \ldots, f_N)$ of V in C^N.

Proof. Let Ω be a relatively compact neighbourhood of K, and let F be the boundary of Ω. For every $a \in F$, there is an f such that $|f(a)| > 1$, $\|f\|_K < 1$. Since F is compact and the set of a with $|f(a)| > 1$ is open, there are a finite number, $f_1, \ldots, f_{N'}$ of holomorphic functions such that $\|f_i\|_K \leq \theta < 1$ while $\max\limits_i |f_i(a)| > 1$ for $a \in F$. Let Ω' be the set of $a \in \Omega$ with $|f_i(a)| < 1$ for $i = 1, \ldots, \underline{N'}$. Also, the set of $a \in \Omega$ with $|f_i(a)| \leq \rho$, $\theta < \rho < 1$ is compact since $\overline{\Omega'}$ is compact and the closure of this set does not intersect F. This shows that the mapping of Ω' in $C^{N'}$ defined by $(f_1, \ldots, f_{N'})$ is proper.

Set $\wedge = \Omega'$. By adjoining a finite number of functions $f_{N'+1}, \ldots, f_N$ to $f_1, \ldots, f_{N'}$ we can ensure that points of Ω are separated by the mapping $\phi = (f_1, \ldots, f_N)$ and such that ϕ is of maximal rank {this follows from properties (β), (γ) of Stein manifolds and the compactness of $\overline{\Omega}$}.

117 If $1 > \rho > \theta$ and $\|f_{N'+1}\|_K < A, \ldots, \|f_N\|_K < A$, we thke P to be the polydisc $|z_i| \leq \rho$, $i \leq N'$, $|z_i| \leq A$, $i \geq N' + 1$ in C^N and L to be the inverse image in $\wedge$ of $P \cap \phi(\wedge)$ under the mapping ϕ. Since K has a fundamental system of relatively compact neighbourhoods Ω, Lemma 2 is proved. ◻

Theorem. *Let V be a Stein manifold, K a compact set $\subset V$ such that $K = \hat{K}$. Then*

1) *Theorems A) and B) are true for K.*

2) *Every holomorphic function on K can be approximated, uniformly on K by holomorphic funtions on V.*

Proof. 1) follows from Proposition 4 and Lemmas 1 and 2. To prove 2), let g be a holomorphic function on K, and L a neighbourhood of K having the properties of Lemma 2, such that g is holomorphic on L.

Now, by Proposition 4, we have the exact sequence

$$_{C^N}\underline{\mathscr{O}}_P \to {}_V\underline{\tilde{\mathscr{O}}}_L \to 0$$

(L is considered as a subset of $\prod$), and if $\underline{\mathfrak{I}}$ is the kernel of the first mapping, the sequence

$$0 \to \underline{\mathfrak{I}} \to {}_{C^N}\underline{\mathscr{O}}_P \to {}_V\underline{\tilde{\mathscr{O}}}_L \to 0$$

is exact. The associated exact cohomology sequence

$$H^\circ(P, {}_{C^N}\underline{\mathscr{O}}_P) \to H^\circ(P, {}_V\underline{\tilde{\mathscr{O}}}_L) \to H^1(O, \underline{\mathfrak{I}})$$

shows, since $H^1(P, \underline{\mathscr{I}}) = 0$ by Theorem B) for a polydisc, that every element of $H^\circ(P, {}_V\underline{\tilde{\mathscr{O}}}_L)$ is the image of an element of $H^\circ(P, {}_{C^N}\underline{\mathscr{O}}_P)$ and g is the restriction to L of a holomorphic function on P; hence g can be expanded in a power series in the $f_1, \ldots, f_N$ which converges uniformly on $K \subset \overset{\circ}{L}$. Since the partial sums of this power series, being polynomials **118**
in $f_1, \ldots, f_N$ are holomorphic on V, the theorem is proved. $\square$

Appendix

Theorem. *Let X be a topological space, $\underline{\mathscr{F}}$ a sheaf of abelian groups on X which is such that any section of $\underline{\mathscr{F}}$ over an open set of X can be extended to a section of $\underline{\mathscr{F}}$ over X. Then, for any open covering $\mathscr{O} = \{\mathscr{O}_i\}_{i \in I}$ of X, $H^p(\mathscr{O}, \underline{\mathscr{F}}) = 0$ for $p > 0$, and in particular $H^p(X, \underline{\mathscr{F}}) = 0$ for $p > 0$.*

Proof. The proof is by induction on p.

a) $p = 1$. Let c be a 1-cocycle of the covering $\mathscr{O} = \{\mathscr{O}_i\}_{i \in I}$. Suppose J is a subset of the indexing set I such that there is a 0-cochain γ of $\mathscr{O}$ with $\gamma_i - \gamma_j = c_{ij}$ for $i, j \in J$. Let $\alpha \in I$, $\alpha \notin J$. We define a 0-cochain γ' as follows: $\gamma'_i = \gamma_i$ if $i \neq \alpha$, $\gamma'_\alpha = \gamma_i + c_{\alpha i}$ on $\mathscr{O}_\alpha \cap \mathscr{O}_i$, $i \in J$. Then on $\mathscr{O}_\alpha \cap \mathscr{O}_i \cap \mathscr{O}_j$, $\gamma_i + c_{\alpha i} = \gamma_j + c_{\alpha j}$ since $\gamma_i - \gamma_j = c_{ij} = c_{\alpha j} - c_{\alpha i}$ (c being alternate). Hence γ' is defined uniquely on $\bigcup_{i \in J}(\mathscr{O}_\alpha \cap \mathscr{O}_i)$. By hypothesis, γ'_α can be extended to a section of $\underline{\mathscr{F}}$ over $\mathscr{O}_\alpha$ and the cochain γ' is defined completely. Also $\gamma'_i - \gamma'_j = c_{ij}$ if $i, j \in J \cup \{\alpha\}$. It is clear that J is non-empty (since c is alternate) and the theorem for $p = 1$ follows by an application of Zorn's lemma.

b) $p > 1$. Suppose the theorem true with p replaced by $p - 1$ for all spaces X, all coverings $\mathscr{O}$ of X and all $(p-1)$-cocycles of $\mathscr{O}$. Let c be a p-cocycle and let J be a subset of I such that there is a $(p-1)$-cochain γ with $(\delta\gamma)_{i_\circ \ldots i_p}$ for $i_\circ, \ldots, i_p \in J$. Let $\alpha \in I$, $\alpha \notin J$.

119 For every $i_\circ, \ldots, i_{p-2} \in J$, we determine a section $\gamma'_{i_\circ \ldots i_{p-2}\alpha}$ of $\underline{\mathscr{F}}$ over $\mathscr{O}_{i_\circ \ldots i_{p-2}\alpha}$ such that

$$c_{i_\circ \ldots i_{p-1}\alpha} = \sum_{k=0}^{p-1} (-1)^k \gamma'_{i_\circ \ldots \hat{i}_k \ldots i_{p-1}\alpha} + (-1)^p \gamma_{i_\circ \ldots i_{p-1}}$$

over $\mathscr{O}_{i_\circ \ldots i_{p-1}\alpha}$. This is possible: it is easily seen, from the definition of γ, that the $(p-1)$-cochain c' defined by

$$c'_{i_\circ \ldots i_{p-1}} = c_{i_\circ \ldots i_{p-1}\alpha} + (-1)^{p-1} \gamma_{i_\circ \ldots i_{p-1}}$$

is a cocycle of the covering $\{\mathscr{O}_\alpha \cap \mathscr{O}_i\}_{i \in J}$ of the space $Y = \bigcup_{i \in J} (\mathscr{O}_\alpha \cap \mathscr{O}_i)$ (since c is a cocycle) and the existence of the $\gamma'_{i_\circ \ldots i_{p-2}\alpha}$ follows from inductive hypothesis.

We new define a $(p-1)$-cochain γ_1 as follows: if $i_\circ, \ldots, i_{p-2} \in J$, $(\gamma_1)_{i_\circ \ldots i_{p-2}\alpha} = \gamma'_{i_\circ \ldots i_{p-2}\alpha}$; γ_1 is defined by the condition that it is alternate for other p-tuples of indices of $J \cup \{\alpha\}$ which contain α and $(\gamma_1)_t = \gamma_t$ if $t \in J^p$. γ_1 has the property that

$$(\delta\gamma_1)_{j_\circ \ldots j_p} = c_{j_\circ \ldots j_p} \text{ for } j_\circ, \ldots, j_p \in J \cup \{\alpha\}$$

while $\gamma_1 = \gamma$ on J^p. If we partially order the pairs (J, γ) by setting $(J, \gamma) < (J', \gamma')$ if $J \subset J'$ and $\gamma' = \gamma$ on J^p, the theorem follows by an application of Zorn's lemma. $\square$

Chapter 16

Coherent analytic sheaves on a Stein manifold

1. We shall prove here the fundamental theorem of Oka-Cartan-Serre on Stain manifolds. **120**

Fundamental Theorem. Let V be a Stein manifold and $\underline{\mathscr{F}}$ a coherent analytic sheaf on V. Then

A) For every $a \in V$, $H^\circ(V, \underline{\mathscr{F}})$ $\mathscr{O}_a$-generates $\mathscr{F}_a$.

B) For $p \geq 1$, $H^p(V, \underline{\mathscr{F}}) = 0$.

(It is clear that for compact subsets of V, Theorem A) as formulated in XV is equivalent to the theorem as formulated above).

The following two results will be required, the first will not be proved here. For the proof, see Cartan [1].

Theorem 1. *Let V be a complex analytic manifold and let $a \in V$. Let $\mathfrak{M}$ be a submodule of $\mathscr{O}_a^p$ {as an $\mathscr{O}_a$-module} and let $f = (f_1, \ldots, f_p) \in \mathscr{O}^p$ ($\mathscr{O} = \mathscr{H}_V$ is the space of all holomorphic funtions on V). Suppose that f is the limit in $\mathscr{O}^p$ of functions $f_i \in \mathscr{O}^p$ such that $(f_i)_a \in \mathfrak{m}$. Then $f_a \in \mathfrak{m}$.*

Lemma. *Let K be a compact subset of the Stein manifold V such that $K = \hat{K}$ ($\hat{K}$ is the $\mathscr{H}_V$-envelope of K) and $\underline{\mathscr{F}}$ a coherent analytic sheaf on K. Let $f_1, \ldots, f_m \in H^\circ(K, \underline{\mathscr{F}})$ and suppose that for every $a \in K$,*

$f_1, \ldots, f_m$ $\mathscr{O}_a$-generate $\mathscr{F}_a$. Then $f_1, \ldots, f_m$ $H^\circ(K, \underline{\mathscr{O}})$-generate $H^\circ(K, \underline{\mathscr{F}})$.

This lemma is proved, using Theorems A) and B) for K in exactly the same way as was the lemma in the proof of Theorems A) and B) for a cube in XIV.

2 Topology on $H^\circ(V, \underline{\mathscr{F}})$

121 Let $\{K_p\}$ be a sequence of compact subsets of V such that $K_p \subset \overset{\circ}{K}_{p+1}$, $\bigcup\limits_1^\infty K_p = V$ and $K_p = \hat{K}_p$ (such a sequence exists since V is eountable at infinity and, for any compact set K, $(\hat{\hat{K}}) = \hat{K}$).

For an integer $N \geq 1$, we introduce a norm in $H^\circ(K_p, \underline{\mathscr{O}}^N)$ by setting the norm of $f = (f_1, \ldots, f_N) \in H^\circ(K_p, \underline{\mathscr{O}}^N)$ to the equal to the greatest of the suprema of $|f_1|, \ldots, |f_N|$ on K_p. We then introduce a seminorm $\|\ldots\|_p$ on $H^\circ(K_p, \underline{\mathscr{F}})$ as follows: by Theorem A) for K_p, $\underline{\mathscr{F}}_{K_p} \simeq \underline{\mathscr{O}}^N / \underline{\mathscr{R}}$ and $\|\ldots\|_p$ is defined to be the quotient seminorm of the norm on $\underline{\mathscr{O}}^N$. It is easy to verify that two isomorphisms $\underline{\mathscr{F}}_{K_p} \simeq \underline{\mathscr{O}}^{N_1} / \underline{\mathscr{R}}_1 \simeq \underline{\mathscr{O}}^{N_2} / \underline{\mathscr{R}}_2$ give rise to equivalent seminorms. Also, for every p, there is a canonical mapping $H^\circ(V, \underline{\mathscr{F}})$ $H^\circ(K_p, \underline{\mathscr{F}})$ (namely, restriction to K_p). On $H^\circ(V, \underline{\mathscr{F}})$ we put the weakest topology for which these mappings are continuous in these seminorms (which may also be described by saying that $f \in H^\circ(V, \underline{\mathscr{F}})$ tends to zero if $\|f\|_p \to 0$ for every p). Also, it is easily seen that the topology induced by $\|\ldots\|_{p+1}$ on $H^\circ(K_p, \underline{\mathscr{F}})$ is finer than that given by $\|\ldots\|_p$.

The next results will show that $H^\circ(V, \underline{\mathscr{F}})$ is a Fréchet space. One has only to show that it is Hausdorff and complete.

(a) If $f_{p+1} \in H^\circ(K_{p+1}, \underline{\mathscr{F}})$ and $\|f_{p+1}\|_{p+1} = 0$, then the restriction of f_{p+1} to K_p is zero. (As a consequence, the topology of $H^\circ(V, \underline{\mathscr{F}})$ is Hausdorff).

Proof. If $\phi_1, \ldots, \phi_{N_{p+1}}$ $\mathscr{O}_a$-generate $\mathscr{F}_a$ for $a \in K_{p+1}$ (Theorem A) for

K_{p+1}) and

$$f_{p+1} = \sum_{i=1}^{N_{p+1}} c_i\phi_i \quad \text{(lemma on p.120)}$$

(the c_i are holomorphic functions on K_{p+1}) it follows from the definition **122**
of the seminorm $\|\ldots\|_{p+1}$ and the fact that $\|f_{p+1}\|_{p+1} = 0$ that, given $\epsilon > 0$ there are holomorphic functions $c_1^\epsilon, \ldots, c^\epsilon_{N_{p+1}}$ on K_{p+1} such that

$$f_{p+1} = \sum_{i=1}^{N_{p+1}} c_i^\epsilon\phi_i$$

and $\sup\limits_{i,a\epsilon K_{p+1}} |c_i(a)| < \epsilon$. If $\gamma_i^\epsilon = c_i - c_i^\epsilon$, then $(c_1, \ldots, c_{N_{p+1}})$ is uniformly approximated on K_{p+1} by $(\gamma_1^\epsilon, \ldots, \gamma^\epsilon_{N_{p+1}})$ where $(\gamma_1^\epsilon, \ldots, \gamma^\epsilon_{N_{p+1}})$ is an element of the sheaf $\underline{\mathscr{R}}$ of relations between $\phi_1, \ldots, \phi_{N_{p+1}}$. It follows from Theorem 1 (stated on page 120) that on $\overset{\circ}{K}_{p+1} \supset K_p(c_1, \ldots, c_{N_{p+1}}) \in \underline{\mathscr{R}}$ and (a) is proved.

(b) If $f_1, \ldots, f_, \ldots$ is a sequence of elements of $H^\circ(K_{p+1}, \underline{\mathscr{F}})$ such that

$$\sum_{k=1}^{\infty} \|f_k\|_{p+1} < +\infty,$$

then the sequence $\{\sum\limits_{k=1}^{N} f_k\}$ has a limit point in $H^\circ(K_p, \underline{\mathscr{F}})$. The restrictions to K_{p-1} of two such limit points coincide. ◻

Proof. Let $\phi_1, \ldots, \phi_{N_{p+1}}$ $\mathscr{O}_a$-generate $\mathscr{F}_a$ for $a \in K_{p+1}$ and let

$$f_k = \sum_{i=1}^{N_{p+1}} c_i^{(k)}\phi_i.$$

Then, since $\sum\limits_{k=1}^{\infty} \|f_k\|_{p+1} < +\infty$, the $c_i^{(k)}$ can be so chosen that $\sum\limits_k \max\limits_i \|c_i^{(k)}\|_{K_{p+1}} < +\infty$ (by the definition of $\|\ldots\|_{p+1}$, one may, for example, take the $c_i^{(k)}$ such that $\|c_i^{(k)}\|_{K_{p+1}} < 2\|f_k\|_{p+1}$). Then, for each i, $\sum_k c_i^{(k)}$ converges to a holomorphic function c_i on K_p, and it is clear that

$\|\sum\limits_{k=1}^{N} f_k - \sum\limits_{i=1}^{N_{p+1}} c_i\phi_i\|_p \to 0$ as $N \to \infty$. This proves the existence of the limit point. The uniqueness on K_{p-1} follows at once from a).

123 (c) $H^\circ(V, \underline{\mathscr{F}})$ is a Fréchet space.

Given a Cauchy sequence $\{s_k\}$, $\|s_k - s_l\|_p \to 0$ as $k, l \to \infty$ for every p. If we choose a sequence $\{n_k\}$ of integers such that $\|s_m - s_{n_k}\|_p < 1/2^k$ for $p \leq k$ and $m \geq n_k$ (with $n_{k+1} > b_k$) then $\sum\limits_k \|s_{n_{k+1}} - s_{n_k}\|_p < +\infty$ for every p. It follows at once from (b) that $\{s_k\}$ has a limit in $H^\circ(V, \underline{\mathscr{F}})$ which is unique since $H^\circ(V, \underline{\mathscr{F}})$ is Hausdorff.

(d) (Approximation property). Given $f_p \in H^\circ(K_p, \underline{\mathscr{F}})$ and $\epsilon > 0$ there is a section $f \in H^\circ(V, \underline{\mathscr{F}})$ such that $\|f_p - f\|_p < \epsilon$. □

Proof. If $\phi_1, \ldots, \phi_{N_{p+1}} \mathscr{O}_a$-generate $\mathscr{F}_a$ for $a \in K_{p+1}$, then their restrictions to K_p clearly $\mathscr{O}_a$-generate $\mathscr{F}_a$ for $a \in K_p$. Hence, by the lemma,

$$f_p = \sum_{i=1}^{N_{p+1}} c_i \phi_i$$

where the c_i are holomorphic on K_p. By the theorem of XV, the c_i can be approximated uniformly on K_p by holomorphic functions on V. This shows that f_p can be approximated on K_p (in $\|\ldots\|_p$) by a section $f_{p+1} \in H^\circ(K_{p+1}, \underline{\mathscr{F}})$. Approximating f_{p+1} on K_{p+1} by $f_{p+2} \in H^\circ(K_{p+2}, \underline{\mathscr{F}})$ in $\|\ldots\|_{p+1}$ and continuing this process, we construct a sequence $f_{p+1}, f_{p+2}, \ldots$ such that $\|f_{p+k+1} - f_{p+k}\|_{p+k} \leq \epsilon_k$. If the ϵ_k are small enough, it is seen that $\sum\limits_{m=k}^{\infty} \|f_{p+m+1} - f_{p+m}\|_{p+m} < +\infty$ (since $\|\ldots\|_{m+1}$ is finer than $\|\ldots\|_m$ on K_m) and so $f'_{p+k} = f_{p+k} + \sum\limits_{k}^{\infty} (f_{p+m+1} - f_{p+m})$ is defined uniquely in K_{p+k-2}. It is clear that $f'_{p+k+1} = f'_{p+k}$ on K_{p+k-2} and so there is an f $H^\circ(V, \underline{\mathscr{F}})$ with $f = f'_{p+k}$ on K_{p+k-2}. If the ϵ_k are small enough, f approximates to f_p in $\|\ldots\|_p$.

124 [If we say that a sequence $\{s_m\}$, where $s_m \in H^\circ(K_m, \underline{\mathscr{F}})$, converges to $s \in H^\circ(V, \underline{\mathscr{F}})$ if $\|s - s_m\|_m \to 0$ as $m \to \infty$, the above proof may be rephrased by saying simply that $f_{p+m} \to f$ as $m \to \infty$.]. □

3 Proof of the fundamental theorem

Proof of Theorem A):

Let $a \in V$ and suppose that $a \in \overset{\circ}{K}_p$. Let $\phi_1, \ldots, \phi_{N_p}$ $\mathscr{O}_b$-generate $\mathscr{F}_b$ for $b \in K_p$. Theorem A) asserts that the set of the N_p-tuples $(a_1, \ldots, a_{N_p})$, where $a_i \in \mathscr{O}_a$, such that $\sum_i \phi$ belongs to the submodule of $\mathscr{F}_a$ generated by $H^\circ(V, \underline{\mathscr{F}})$ is $\mathscr{O}_a^{N_p}$. Such N_p-tuples from a submodule $\mathfrak{m}$ (over $\mathscr{O}_a$) of $\mathscr{O}_a^{N_p}$ and after Theorem 1, we have only to prove that on a certain fixed open nieghbourhood U of a, every N_p-tuple of holomorphic functions on U is the uniform limit of N_p-tuples $(b_1, \ldots, b_{N_p})$ such that $\sum\limits_1^{N_p} b_i \phi_i$ induces at a an element of $\mathscr{F}_a$. But this follows at once from the approximation property.

Proof of Theorem B): We prove first that $H^p(V, \mathscr{F}) = 0$ if $p > 1$. Let α be a p-cocycle of V. On K_m we have $\alpha = \delta\beta_m$, β_m a cochain of K_m, by Theorem B) for $K_m (m = 1, 2, \ldots)$. Also, on K_m, $\delta(\beta_{m+1} - \beta_m) = 0$, so that $\beta_{m+1} - \beta_m = \delta\gamma'_m$ where γ'_m is a $(p-2)$-cochain of K_m. By the definition of cochain, we may suppose that γ'_m is the restriction to K_m of a $(p-2)$-cochain γ_m of V, and so $\beta_m = \beta_{m+1} - \delta\gamma_m$ on K_m, while $\delta(\beta_{m+1} - \delta\gamma_m) = \alpha$ on K_{m+1}. It is clear that by repeating this process with $m = 1, 2, \ldots$ we obtain a $(p-1)$-cochain β of V such that $\delta\beta = \alpha$ and so $H^p(V, \underline{\mathscr{F}}) = 0$.

Finally, we turn to the proof that $H^1(V, \underline{\mathscr{F}}) = 0$. Let α be a 1-cocycle of V and let $\alpha = \delta\beta'$ on K_p, where β'_p is a 0-cochain of K_p. Again, $\beta'_{p+1} - \beta'_p \in H^\circ(K_p, \underline{\mathscr{F}})$, i.e., is a cocycle.

By the approximation property, there is a cocycle $c'_{p+1} \in H^\circ(V, \underline{\mathscr{F}})$ **125**
such that $\|c'_{p+1} + \beta'_{p+1} - \beta'_p\|_p \leq \epsilon_p$, where ϵ_p can be chosen arbitrarily small. It is clear that we find thus a cochain β_p of K_p with $\delta\beta_p = \alpha$ on K_p and $\|\beta_{p+1} - \beta_p\|_p \leq \epsilon_p$ for every $p \geq 1$.

If we say that a sequence of cochains, $\{\beta_p\}$, where β_p is a cochain of K_p, tends to a cochain β of V if $\beta - \beta_p \in H^\circ(K_p, \underline{\mathscr{F}})$ and $\|\beta - \beta_p\|_p \to 0$ as $p \to \infty$, a repetition of the proof of the approximation property (with trivial modifications) shows that the sequence $\{\beta_p\}$ defined above tends to a cochain β of V if the ϵ_p are small enough and it is clear that $\delta\beta = \alpha$.

This proves Theorem B) for $p = 1$ and the proof of the fundamental theorem is complete.

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Supplementary References

In the papers listed below, the reader will find applications of the theorems proved in these lectures and several important results that could **127**

not be treated here. For further references, see the Scientific report on the second summer Institute: Several Complex Variables, by W. T. Martin, S. S. Chern and O. Zariski, Bull. Amer. Math. Soc., 62 (1956), 79-141.

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