Lectures On Levi Convexity Of Complex Manifolds And Cohomology Vanishing Theorems

By

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Preface

These are notes of lectures which I gave at the Tata Institute of Fundamental Research in the Winter 1965.

Most of the material of these notes may be found in the papers [1], [2] and [16] listed in the Bibliography at the end of this book.

My thanks are due to M.S. Raghunathan for the preparation of these notes and to R. Narasimhan and M.S. Narasimhan for many helpful suggestions.

E. Vesentini

Contents

Chapter 0

Prerequisites

In this chapter we collect together some well known results which will 1
be used in the course of these lectures.

1. Let X be a connected, paracompact complex manifold of (complex) dimension we denote by Ω The sheaf of germs of homomorphic p-foxes end by $\wedge^{x,q}$ the sheaf of germs of 0^{∞} forms of type $(P, \mathcal{F})$. Further $\{overset{-}\partial$ will denote the exterior differentiation with respect to $\bar{\partial}$ Then we have one.

Proposition. *The sequence*

$$0 \to \Omega^P \to A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \ldots \xrightarrow{\bar{\partial}} A^{p,n} \to 0$$

is exact. Similarly if $K^{p,q}$ denotes the sheaf of germs of (p,q) currents, the sequence

$$0 \to \Omega^p \to K^{p,c} \xrightarrow{\bar{\partial}} K^{p,1} \to \ldots \xrightarrow{\bar{\partial}} K^{p,n} \to 0$$

is exact.

We point out that the exactness of the first step of the second sequence:

$$0 \to \Omega^p \to K^{po}$$

is equivalent to the fact that every $\bar{\partial}$ closed distribution is a holomorphic function[1]

2 (For a proof see for instance [5], Exposé XVIII, [22])

The sheaves $A^{p,q}$ being fine, the cohomology group $H^q(X, \Omega^p)$ is canonically isomorphic to the qth cohomology group of either one of the complexes

$$0 \longrightarrow H^\circ(X, A^{p,o}) \xrightarrow{\bar{\partial}} H^\circ(X, A^{p,1}) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} H^\circ(X, A^{p,n}) \longrightarrow 0$$

$$0 \longrightarrow H^\circ(X, K^{p,o}) \xrightarrow{\bar{\partial}} H^\circ(X, K^{p,1}) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} H^\circ(X, K^{p,n}) \longrightarrow 0.$$

2. We need the following theorem due to Leray.

Let $\mathfrak{G} = \{U_i\}_{i \in I}$ be a locally finite open covering of a paracompact space X. Let $\mathcal{F}$ be a sheaf of abelian groups on X such that for every $q > 0$, $H^q(U_{i_o} \cap \ldots \cap U_{i_p}, \mathcal{F}) = 0$ for every set $(i_0, \ldots i_p)$ of p-elements in I. Then the canonical map

$$H^q(\mathfrak{G}, \mathcal{F}) \to H^q(x, \mathcal{F})$$

is an isomorphic for all $q \geq o$

For a proof see for instance [13] 209-210

3. The following result enables us to apply the above theorem to locally free sheaves over homomorphic function on a complex manifold.

Let X be a (para compact) complex manifold of complex dimension n. Given a vector bundle E an X and any covering $\mathfrak{G} = \{U_i\}_{i \epsilon I}$ there is a refinement $\{V_j\}_{j \epsilon J}$ such that for $j_1, \ldots, j_k$ with $V_{j1} \cap \ldots \cap V_{jk} \neq \phi$, for $0 \leq p \leq n$, the sequence

[1]The idea of the proof is the following. Let α be any compactly supported function of class C^k with $k > 0$. Since $\bar{\partial}(T * \alpha) = 0$, $T * \alpha$ is a homomorphic function hence it is C^∞. Then T itself is a C^∞ function [30, Vol. II, Theorem XXI, 50]. But $\bar{\partial} T = 0$. Then T is holomorphic.

$$0 \to H^0(V_{j1,\dots jk}, \Omega^p \underset{\mathcal{O}}{\otimes} \underline{E} \to H^0(V_{j1\dots jk}\underline{A}^{p1} \underset{\mathcal{O}}{\otimes} \underline{E}) \xrightarrow{\bar{\partial}\otimes 1} \dots$$

$$\dots \xrightarrow{\bar{\partial}\mathcal{O}1} H^0\ (V_{j1\dots jk}, A^{pn} \underset{\mathcal{O}}{\otimes} \underline{E}) \to 0$$

where $\mathcal{O}$ is the structure sheaf, is exact, and $V_{j1\dots jk}$ is the intersection $V_{j1} \cap \dots \cap V_{jk}$*and* $\underline{\mathrm{E}}$ is the locally free $\mathcal{O}$ sheaf associated to E. (Since the sheaves, $A^{pi} \underset{\mathcal{O}}{\otimes} E$ are fine sheaves, this implies that $H^p(V_{ji}\dots{}_{jk'}\ \Omega^p \underset{\mathcal{O}}{\otimes} \underline{\mathrm{E}}) = 0$ **3**

4. Let $\Pi : E \to X$ be a holomorphic vector bundle on the complex manifold X. Let $\mathfrak{G} = (U_i)_{i\in I}$ be a covering of X and $e_{ij} : U_i \cap U_j \to G = GL(n, \mathbb{C})$ be transition functions with respect to $\mathfrak{G}'$ such that $\{\mathfrak{G}, e_{ij}\}$ define E. This means that we are given isomorphisms

$$\varphi_i : E/U_i \to U_i \times \mathbb{C}^n,$$

for $i \in I$ such that e_{ij} are the maps defined by

$$(\varphi_j \circ \varphi_i^{-1})(x, v) = (x, e_{ij}(x)(v))$$

for $x \in U_i \cap U_j$. In particular, we have for $i, j, k \in I$

$$e_{ik}(x) = e_{ij}(x)e_{jk}(x)$$

provided that $x \in U_i \cap U_j \cap U_k$.

For a vector bundle $\pi : E \to X$, we denote, as before by $\underline{\mathrm{E}}$ or $\Omega(E)$ the sheaf of germs of holomorphic section of E. $\Omega(E)$ is a locally free sheaf over the sheaf of germs of holomorphic functions on X (we denote this latter sheaf by $\mathcal{O}$). Then, in the above notation,

$$\Omega(E)\Big|_{U_i} \simeq \mathcal{O}^n$$

We fix moreover the following notation:

$$\Omega^p(E) = \underline{\mathrm{E}} \otimes_{\mathcal{O}} \Omega^p, \quad \text{in particular} \quad \Omega^\circ(E) = \Omega(E);$$
$$\underline{\mathrm{A}}^{pq}(E) = \underline{\mathrm{E}} \otimes_{\mathcal{O}} A^{pq};$$
$$\underline{\mathrm{K}}^{pq}(E) = \underline{\mathrm{E}} \otimes_{\mathcal{O}} K^{pq}.$$

$\Omega^p(E)$ (resp. $\underline{A}^{pq}(E)$, resp. $\underline{K}^{pq}(E)$) is the sheaf of germs of holo- **4**
morphic E-valued p-forms (resp. differentiable (p,q) forms with values in E, resp. (p,q) currents with values in E ;

$\Gamma(X, \underline{A}^{pq}(E)) =$ Global$(p,q)C^\infty$ forms with values in E;

$\Gamma(X, \underline{K}^{pq}(E)) =$ Global(P,q) currents with values in E.

There is a one-one correspondence between global sections of $\underline{A}^{pq}(E)$ and collections $(\varphi_i)_{i\in I}$ where each φ_i is a vector valued form,

$$\varphi_i = \begin{pmatrix} \varphi_i' \\ \vdots \\ \varphi_i^m \end{pmatrix}$$

each φ_i^k being a scalar C^∞ (p,q) forms on U_i such that for $x \in U_i \cap U_j$ we have $\varphi_i(x) = e_{ij}(x)\varphi_j(x)$. One can set up a similar one-one correspondence between sections of $K^{pq}(E)$ and families $(\varphi_i)_{i\in I}$ of currents, each $\{\varphi_i^k\}_i$ being defined on U_i, satisfying $e_{ij}\varphi_j = \varphi_i$ on $U_i \cap U_j$.

We denote by $\overline{\partial}$ the operators

$$1 \otimes \overline{\partial} : \underline{E} \underset{\mathcal{O}}{\otimes} A^{pq} \longrightarrow \underline{E} \otimes A^{p,q+1}$$

and

$$1 \otimes \overline{\partial} : \underline{E} \underset{\mathcal{O}}{\otimes} K^{pq} \longrightarrow \underline{E} \otimes K^{p,q+1}.$$

Then the sequences

$$0 \to \Omega^p(E) \to \underline{A}^{p,o}(E) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \underline{A}^{p,n}(E) \to 0$$

and

$$0 \to \Omega^p(E) \to \underline{K}^{p,o}(E) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \underline{K}^{p,n}(E) \to 0$$

are exact. Moreover the sheaves $\underline{A}^{pq}(E)$ and $\underline{K}^{pq}(E)$ are fine.

5 Hence if Φ denotes either the family of all closed subsets of X or the family of all compact subsets of X, (more generally, any paracompactifying family) and we set,

$$\Gamma_\Phi(X, \underline{A}^{pq}(E)) = \{\sigma | \sigma \text{ a section of } A^{pq}(E) \text{ over } X, \operatorname{Supp}\sigma \in \Phi\},$$

then

$$H^q_\Phi(X, \Omega^p(E)) \simeq \left\{\varphi | \varphi \in \Gamma_\Phi(X, A^{p,q}(E)); \overline{\partial}\varphi = 0\right\} | \left\{\overline{\partial}\Gamma_\Phi(X, A^{p,q-1}(E))\right\}$$

and similarly with the obvious notation,

$$H^q_\Phi(X, \Omega^p(E)) \simeq \\ \left\{\varphi|\varphi \in \Gamma_\Phi(X, \underline{\mathrm{K}}^{p,q}(E)); \bar{\partial}\varphi = 0\right\} | \left\{\bar{\partial}\Gamma_\Phi(X, K^{p,q-1}(E))\right\}$$

Chapter 1

Vanishing theorems for hermitian manifolds

1 The space $\mathcal{L}^{p,q}$

All manifolds considered are assumed to be connected and paracompact. **6**

Let $\pi : E \to X$ be a holomorphic vector bundle on a paracompact complex manifold X of complex dimension n.

Let $\mathfrak{G} = \{U_i\}_{i\epsilon I}$ be a locally finite open coordinate covering of X and $\{e_{ij} : U_i \cap U_j \to GL(m\mathbb{C})\}_{ij\in I\times I}$ transition functions defining E. A hermitian metric along the fibres of E is a collection $\{h_i : U_i \to GL(m,\mathbb{C})\}_{i\epsilon I}$ of C^∞ -maps such that for $x\epsilon U_i, h_i(x)$ is a positive definite hermitian matrix and for $x\epsilon U_i \cap U_j, h_j(x) =^t \overline{e_{ij}}(x)h_i(x)e_{ij}(x)$.

Lemma 1.1. *Every holomorphic vector bundle on a complex manifold admits a hermitian metric.*

Proof. Let $\{\rho k\}_{k\in I}$ be a C^∞-partition of unity subordinate to $\{U_k\}_{k\in I}(= \mathfrak{G})$. Let $\{h_i^o : U_i \to GL(m,\mathbb{C})\}_{i\in I}$ be any family of C^∞-functions such that for $i \in I$, $x \in U_i$, $h_i^o(x)$ is a positive definite hermitian matrix. Then the family $\{h_i^o\}_{i\in I}$ defined by

$$h_i(x) = \sum_k \rho_k(x)^t\overline{e_{ki}}(x)h_k^o(x)e_{ki}(x)$$

is a metric along the fibres of E. □

7 In particular, the holomorphic tangent bundle Ⓗ $\rightarrow X$ admits a hermitian metric. Let $\mathfrak{G} = \{U_i\}_{i\in I}$ be a covering of X by means of coordinate open sets. Let $Z_i^1, \ldots, Z_i^n$ be any system of coordinates in U_i. Then Ⓗ $\rightarrow X$ is defined with respect to $\mathfrak{G}$ by means of the transition functions $J_{ij} : U_i \cap U_j \rightarrow GL(n, \mathbb{C})$ defined by

$$J_{ij}(x) = \left(\frac{\partial Z_j^\alpha}{\partial Z_l^\rho \beta}(x)\right)_{\alpha\beta} \quad \text{for } x \in U_i \cap U_j.$$

A hermitian metric on Ⓗ is then a family of C^∞-hermitian -positive definite matrix-valued functions $g_i : U_i \rightarrow GL(n, \mathbb{C})$ such that

$$g =^t \overline{J}_{ij} g_i J_{ij}$$

Identifying hermitian matrices with hermitian bilinear forms on the tangent spaces through the basis $\dfrac{\partial}{\partial Z^1} \cdots \dfrac{\partial}{\partial Z^n}$ of the holomorphic tangent space, we may write g_i as

$$\sum_{\alpha\beta} g_{i\alpha\overline{\beta}} dZ_i^\alpha d\overline{Z}_\alpha^\beta.$$

A hermitian metric on X defines on X (regarded as a differentiable manifold), a Riemannian metric: in fact, if $z_i^\alpha = x_i^\alpha + iy_i^\alpha$, then we have

$$ds^2 = \sum g_{i\alpha\overline{\beta}}\, dZ_i^\alpha \cdot dI_i^\beta = \sum Re \quad g_{i\alpha\overline{\beta}}(dx_i^\alpha dx_i^\beta + dy_i^\alpha dy_i^\beta)$$

in the coordinate open set U_i, in terms of the local real coordinates x_i, y_i. Since X carries a complex structure, it has a canonical orientation defined by this structure. Hence X has a canonical structure of an oriented Riemannian manifold. We denote by ℓ the volume form on X with respect to this oriented Riemannian structure.

8 Let $C^r(X)$, $0 \leq r \leq 2n$ (n= complex dimension of X) denote the space of complex valued exterior differential forms of degree r. It is a module over the algebra, F, (over the complex numbers) of complex valued C^∞ function on X

Definition 1.1. *The "star operator" $* : C^r(X) \to C^{m-r}(X)$ on an oriented Riemannian manifold X of (real) dimension m is defined by the formula*

$$(*\varphi)(t_1, \ldots, t_{m-r}).\ell = \varphi \wedge \tau(t_1) \wedge \ldots \wedge \tau(t_{m-r)}$$

*where $\varphi \in C^r(X), t_1, \ldots, t_{m-r}$ are tangent vectors to X at a point P and τ is the canonical isomorphism of the F-module of tangent vector fields on X onto the F-module of 1-forms. Clearly * is a real operator, i.e. $\overline{*\varphi} = \overline{*\varphi}$ (the bar denoting conjugation of complex numbers). Furthermore we have*

Lemma 1.2. *Let (X, g) be an oriented Riemannian manifold of dimension m. Then*

$$* * \varphi = (-1)^{r(m-1)}\varphi \text{ for } \varphi \in C^r(X). \qquad (1 \quad \mathrm{I})$$

Proof. ([18]) Let $x \in X$ and $t_1, \ldots, t_m$ be an orthonormal basis of the tangent space at x with respect to the Riemannian metric. Then it is enough to check that

$$* * (\tau(t_{i_1}) \wedge \ldots \wedge \tau(t_{ir})) = (-1)^{r(m-1)}\tau(t_{i_1}) \wedge \ldots \wedge \tau(t_{i_r})$$

for every $0 < i_1 < i_2 < \ldots < i_r \leq m$, or again that for every sequence $0 < j_1 < \ldots < j_r \leq m$,

$$(**)\tau\tau(t_{i_1}) \wedge \ldots \wedge \tau(t_{i_r})(t_{j_1}, \ldots t_{j_r}) = \left\{\tau(t_{i_1}) \wedge \ldots \wedge \tau(t_{i_r})\right\}(t_{j_1}, \ldots t_{j_r}) \cdot (-1)^{r(m-1)}.$$

□

Suppose now that $(j_1, ..., j_r) \neq (i_1, ..., i_r)$. then **9**

$$(**)(\tau(t_{i_1}) \wedge ..., \wedge\tau(t_{i_r})(t_{j_1}, ..., t_{j_r}).l = * \{\tau(t_{i_1}) \wedge ... \wedge \tau(t_{i_r})\} \wedge \tau(t_{j_1}) \wedge \cdots \wedge \tau(t_{j_r})$$

and the right hand side is zero because,

$$\{\tau(t_{i_1}) \wedge \cdots \wedge \tau(t_{i_r})\} (t_{\lambda_1}, \ldots, t_{\lambda_{m-r}})l$$

$$= \tau(t_{i_1}) \wedge ... \wedge \tau(t_{i_r}) \wedge \tau(t_{\lambda_1}) \wedge ... \wedge \tau(t_{\lambda_{m-r}})$$

and here the right side is zero unless $\{i_1, .., i_r, \lambda_1, .., \lambda_{m-r}\} = \{1, 2, ..., m\}$.

$t_1, ..., t_m$ are so chosen that $\tau(t_1) \wedge ... \wedge \tau(t_m)$ give the natural orientation,

$$*(\tau(t_{i_1}) \wedge ... \wedge \tau(t_{i_r}))(t_{\lambda_1}, ..., t_{\lambda_{m-r}}) = 0$$

if $(i_1, ..., i_r, \lambda_1, ..., \lambda_{m-r}) \neq (1, 2, ..m)$ and

$$\begin{aligned} &*(\tau(t_{i_1}) \wedge ... \wedge \tau(t_{i_r}))(t_{\lambda_1}, ...t_{m-r})l = \\ &\qquad = \tau(t_{i_1}) \wedge \tau(t_{i_r}) \wedge \tau(t_{\lambda_1})...(t_{\lambda_{m-r}}). \end{aligned}$$

Now, the right side is clearly $\varepsilon \cdot l$ where ε is the signature of the permutation

$$(1, 2, ..., n) \rightsquigarrow (t_{i_1}, ..., t_{i_r}, t_{\lambda_1}, ...t_{\lambda_{m-r}}).$$

It follows that

$$*(\tau(t_{i_1}) \wedge ... \wedge \tau(t_{i_r})) = \varepsilon\tau(t_{\mu_1}) \wedge ... \wedge \tau(t_{\mu_{m-r}})$$

10 where $0 < \mu_1 < \ldots < \mu_{m-r} \leq m$ are so chosen that

$$\{t_{i_1}, ..., t_{i_r}, t_{\mu_1}, .., t_{\mu_{m-r}}\} = (1, 2, .., m).$$

Hence

$$\begin{aligned} &(**)(\tau(t_{i_1})\wedge), \wedge (t_{i_1}, ..., t_{i_r}) \\ &\qquad = \varepsilon\tau(t_{\mu_1}) \wedge .. \wedge \tau(t_{\mu_r}) \wedge \tau(t_{i_1}) \wedge ... \wedge \tau(t_{i_r}) \\ &\qquad = \varepsilon\varepsilon' \cdot \ell \end{aligned}$$

where ε' is the signature of the permutation

$$(1, 2, ..., m) \rightsquigarrow (t_{\mu_1}, .., t_{\mu_r}, t_{i_1}, .., t_{i_r}).$$

Hence $\varepsilon\varepsilon' = (-1)^{r(m-r)} = (-1)^{r(m-1)}$. It follows that

$$(**)(\tau(t_{i_1}) \wedge ... \wedge \tau(t_i)(t_{i_1}, .., t_{i_r}) = (-1)^{r(m-1)} \{\tau(t_{i_1}) \wedge ... \wedge \tau(t_{i_r})\} (t_{i_1}, ..., t_{i_r}).$$

For $(j_1, j_2, .., j_r) \neq (i_1, .., i_r)$ we have already seen

$$(**)\tau(t_{i_1}) \wedge .. \wedge \tau(t_i))(t_{j_1}, ..., t_{j_r}) = 0.$$

Evidently,

$$\{\tau(t_{i_1}) \wedge ... \wedge \tau(t_i)\}(t_{j_1}, .., t_{j_r}) = 0.$$

Hence the lemma.

The isomorphisms τ between the F-module of C^∞ tangent vector fields on X onto the E-module C^1 extends to an isomorphism between the r-th exterior power of these modules.

Let U be a coordinate open set on the differentiable manifold X. Let $x^1, ..., x^m$ be a coordinate system on U. Then for the fixed basis $dx^1, ..., dx^m$ for the space of differentials at every point of U, any r-form 11
$\varphi \in C^r$ can be represented in U by

$$\phi = \phi_I dx^I,$$

where ϕ_I is a C^∞ function on U, $I = (i_1, ..., i_r)$ is an r-tuple of indices $1 \leq, i_1 < \ldots < i_r \leq m = \dim_{\mathbb{R}} X$, and $dx^I = dx^{1_i} \wedge \ldots \wedge dx^{1_r}$.

Let g_{ij} be the metric tensor in U, and let $g^{ij}(i, j = 1, ..., m)$ be the elements of the inverse matrix $g^{-1} = (g_{ij})^{-1}$. Then $\tau^{-1}\phi$ is defined in U by its components (with respect to the basis $\dfrac{\partial}{\partial x^1}, ..., \dfrac{\partial}{\partial x^m}$)

$$\varphi^{i_1 \ldots i_r} = g^{i_1 k_1} \ldots g^{i_r k_r} \varphi_{k_1 \ldots k_r}.$$

If

$$(*\varphi)_J \; dx^J$$

is the local representation in U of the $(m-r)$-form $*\varphi$, J being the multi-index $J = (j_1, \ldots, j_{m-r})(1 \leq j_1 < \ldots < j_{m-r} \leq m)$, then

$$(*\varphi)_J = \ell_{IJ} \varphi^I.$$

where $\ell = \ell_{1 \ldots m} dx' \wedge ... \wedge dx^m$ is the representation of the volume element in $U(\ell_1 \ldots_m = \sqrt{det(g_{ij})})$ and $\ell_{IJ} = \varepsilon \ell'_{1 \ldots m} \varepsilon$ being the sign of the summation $(1, ...m) \rightsquigarrow (IJ)$.

The proof is straightforward. It follows that for $\varphi, \psi \in C^r$,

$$\varphi \wedge *\psi = \frac{1}{r!}\left\{\sum_I \varphi_I \psi^I\right\}\ell \tag{1, II}$$

the summation being over all multi-indices $I = (i_1, ..., i_r)$ or

$$\varphi \wedge *\psi = \sum \varphi_I \psi^I$$

if we extend the summation only to the multi-indices $I = (i_1, ..., i_r)$, $l \leq i_1 < \ldots < i_r \leq m$ clearly,

$$\left.\begin{array}{ll} \text{clearly,} & \varphi \wedge *\psi = \psi \wedge *\varphi \\ \text{and} & *\varphi \wedge **\psi = \varphi \wedge *\psi \end{array}\right\} \quad \ldots \text{(1, III)}$$

12 We now revert to the situation when the manifold X is a complex manifold and the Riemannian structure is the one canonically associated to a hermitian structure on X. Let $C^{pq}(X)$ be the complex vector space of C^∞ forms of type (p, q) on X.

Lemma 1.3. *The $*$-operator maps $C^{pq}(X)$ into $C^{n-q,n-p}(X)$.*

Proof. The isomorphism τ of the (real) tangent space to X at $x \epsilon X$ onto space of differentials extends to a complex linear isomorphism of the "complexified" tangent space onto the space of complex valued differentials. Denoting the extension again by τ and noting the fact that every complex valued differential form may be regarded as a multilinear form on the complexified tangent space, we see that the formula

$$(*\varphi)(t_1, \ldots, t_{m-r})\ell = \varphi \wedge \tau(t_1) \wedge \cdots \wedge \tau(t_{m-r})$$

holds for complex tangent vectors $(t_1, \ldots, t_{m-r})$ as well. Moreover from the definitions of the associated Riemannian structure, we see easily that for every $\alpha, \tau\left(\frac{\partial}{\partial z_i^\alpha}\right)$ is a linear combination of the dz_i^β while $\tau\left(\frac{\partial}{\partial \overline{z}^\alpha \alpha}\right)$ is a linear combination of the dz_i^β. Our assertion now follows from the definitions of forms of type (p, q). $\square$

Now, let E be a holomorphic vector bundle on X defined by ($\mathfrak{G} = \{U_i\}_{i\in I}$; $e_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})$).

Let $h = (h_i)_{i\in I}$ be a hermitian metric along the fibres. Let $C^{p,q}(X, E)$ denote the space of C^∞ - forms of type (p, q) on X with values in E . The operator * of the associated Riemannian structure defines also a canonical isomorphism

$$* : C^{p,q}(X, E) \to C^{n-p,n-p}(X, E)$$

as follows. If a form φ in $C^{p,q}(X, E)$ is represented in U_i by a column **13**
$\begin{pmatrix}\varphi_i^1\\ \vdots\\ \varphi_i^m\end{pmatrix}$ where each φ_i^k is a scalar (p, q) form, then $*\varphi$ is represented by $\begin{pmatrix}*\varphi_i^1\\ \vdots\\ *\varphi_i^m\end{pmatrix}$ in U_i. Next, the hermitian metric h enables us to define an operator

$$\# : C^{p,q}(X, E) \to C^{q,p}(X, E^*),$$

where E^* is the dual-bundle to E.(E^* is defined by the transition functions $\{x \rightsquigarrow t^{-1}_{e_i(x)}\}$ with respect to the covering $\mathfrak{G} = \{U_i\}_{i\in I}$). In fact if we represent a (p, q)-form φ in U_i by a column $\begin{pmatrix}\varphi_i^1\\ \vdots\\ \varphi_i^m\end{pmatrix}$, where each φ_i^k is a scalar form of type (p, q), then $\#\varphi$ in U_i is given by the column of (p, q)-forms

$$(\#\varphi)_i = \begin{pmatrix}h_i\varphi_i^1\\ h_i\varphi_i^2\\ h_i\varphi_i^m\end{pmatrix}$$

That $\{(\#\varphi)_i\}_{i\in I}$ define forms with values in E^* follows from the fact

$$\begin{aligned}\overline{h_i\varphi_i} = (t_{\overline{e}_{ji}} h_j e_{ji} e_{ij} \varphi_j) &= (t_{\overline{e_{ji}h_j}}\varphi_j)^-\\ &= e_{i_j}^{t-1}\overline{h_j\varphi_j}\end{aligned}$$

Remark . $C^{p,q}(X, E)$ has a structure of an F-module where F is the **14**
ring of global complex-valued C^∞-functions. For this structure of F-modules on the $C^{p,q}(X, E)$

$$* : C^{p,q}(X, E) \to C^{n-q,n-p}(X, E)$$

is an F-linear isomorphism, while the operator

$$\# : C^{p,q}(X, E) \to C^{q,p}(X, E^*)$$

satisfies the condition

$$\#(f.\varphi) = \overline{f}(\#\varphi) \text{ where } f \in F, \varphi \in C^{p,q}(X, E).$$

It follows that the operators * and # define actually homomorphisms

$$* : A^{p,q}(E) \to A^{n-q,n-p}(E) \; (resp. \;\; A^{p,q}(E) \overset{\sharp}{\to} A^{q,p}(E^*)$$

where $$A^{rs}(E)(resp. \;\; A^{rs}(E^*))$$

is the vector-bundle of exterior differential forms of type (r, s) with values in E (resp. E^*); here * is $\mathbb{C}$-linear while # is anti linear.

In the special case when E is trivial bundle, the map $\# : C^{\circ\circ}(X, E) \to C^{\circ\circ}(X, E^*)$ is simply the map

$$f \rightsquigarrow \overline{-f}.$$

Let $\alpha \in C^{p,q}(X, E)$ and $\beta \in C^{rs}(E^*)$. Then $(\alpha \wedge \beta)$ is defined as a scalar form of type $(p + r, q + s)$. For $\varphi, \psi \in C^{pq}(X, E)$ we set

$A_E(\varphi, \psi)l = \varphi \wedge (\# * \psi)$. (We note that $\#* = *\#$.) If $(\operatorname{supp} \varphi \cap \operatorname{supp} \psi)$ is compact, then

$$\int_X |\varphi \wedge *\#\psi| = \int_X |A_E(\varphi, \psi)| < \infty$$

Let U be an open coordinate set in X and let φ, ψ be represented in U by

$$\varphi = \left\{ \frac{1}{p!q!} \varphi^a_{A\overline{B}} dz^A \wedge \overline{dz^B} \right\}, \psi = \left\{ \frac{1}{p!q!} \psi^a_{A\overline{B}} dz^A \wedge \overline{dz^B} \right\},$$

15 where: $a = 1, \ldots, m = rankE$, $A = (\alpha_1, \ldots, \alpha_p)$, $B = (\beta_1, \ldots, \beta_q)$, $1 \leq \alpha_i, \beta_i \leq n$, $dz^A = dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_p}$, $dz^\beta = dz^{\beta_1} \wedge \ldots \wedge dz^{\beta_q}$.

Then it follows from (1 II) that

$$A(\varphi, \psi) = \frac{1}{p!\,q!} h_{\overline{b}a} \varphi^a_{A\overline{B}} \phi^{\overline{\psi^{b\overline{A}B}}},$$

$(h_{\overline{b}a})_{a,b=1,\dots,m}$ being the local representation in U of the metric along the fibres of E. We shall call $A(\varphi,\varphi)^{\frac{1}{2}}$ the "length of the form φ.

Sometimes it will be denoted also by $|\varphi|$.

Let $\vartheta^{p,q}(X,E)$, be the space of $C^\infty-(p-q)-$ forms with compact support. Then for $\varphi,\psi\epsilon\mathscr{D}^{pq}(X,E)$ we can define a scalar product:

$$(\varphi,\psi)=\int_X \varphi\wedge(*\#\psi)=\int_X A_E(\varphi,\psi).$$

Also, we denote (φ,φ) by $||\varphi||^2$; $||\quad||$ is in fact a norm: the hermitian bilinear form (φ,ψ) defines on $\mathscr{D}^{p,q}(X,E)$ the structure of a prehilbert space over $\mathbb{C}$. The completion of $\mathcal{D}^{p,q}(X,E)$ is denoted $\mathcal{L}^{p,q}(X,E)$. This latter space is referred to in the sequel also as the space of square summable forms. Because of the Riesze-Fisher theorem $\mathcal{L}^{pq}(X,E)$ can be identified with the space of square forms of type (p,q)-i.e. the space of φ such that $\int_X A(\varphi,\varphi)<\infty$.

2 The space $W^{p,q}$

The operator $\overline{\partial}:\underline{A}^{p,q}(E)\to\underline{A}^{p,q+1}(E)$ defines homomorphisms (again denoted by the same symbol)

$$\overline{\partial}:C^{p,q}(X,E)\to C^{p,q+1}(X,E)$$
$$\overline{\partial}:\mathscr{D}^{p,q}(X,E)\to\mathscr{D}^{p,q+1}(X,E).$$

We can now define the adjoint of $\overline{\partial}$ with respect to hermitian metric on X and a hermitian metric along the fibres of the vector bundle $\pi:E\to X$, i.e. an operator **16**

$$\vartheta:C^{p,q}(X,E)\to C^{p,q-1}(X,E)$$

satisfying

$$(\overline{\partial}\varphi,\psi)=(\varphi,\vartheta\psi)\tag{1.1}$$

for $\varphi\epsilon\mathscr{D}^{p,q}(X,E)$, $\psi\epsilon\mathscr{D}^{p,q+1}(X,E)$. Such an operator, if it exists is immediately seen to be unique. For the existence we have the

Lemma 1.4. *If $\varphi \epsilon \mathscr{D}^{p,q}(X,E)$, $\psi \epsilon \mathscr{D}^{p,q+1}(X,E)$ are Lipschitz continuous forms with compact support, then $(\overline{\partial}\varphi,\psi) = (\varphi, -*\#^{-1}\overline{\partial}*\#\psi)$.*

Proof. We note first that the scalar products on both sides are defined. Firstly, the $*$ and $\#$ being $\mathbb{C}$-linear (resp. $\mathbb{C}$-anti linear) bundle homomorphisms, these can also be defined on arbitrary forms - that is arbitrary sections of the vector bundle $A^{p,q}(E)$. Secondly, since φ and ψ are Lipschitz continuous $\overline{\partial}\varphi$ and $-{}^*\#^{-1}\overline{\partial}*\#$ are defined almost every where and are bounded measurable forms with compact support. We have moreover,

$$\begin{aligned} d(\varphi \wedge *\#\psi) &= \overline{\partial}(\varphi \wedge *\#\psi) \\ &= \left\{\overline{\partial}\varphi \wedge *\#\psi\right\} + \{(-1)^{p+q}\varphi \wedge \overline{\partial} * \#\psi\} \\ &= \left\{\overline{\partial}\varphi \wedge *\#\psi\right\} - \{\varphi \wedge *\#\vartheta\psi\} \end{aligned}$$

since $\quad ** = (-1)^{(p+q)(2n-p-q)} = (-1)^{p+q}$

(here n is the complex dimension of X). Applying Stoke's formula, we obtain (since φ and ψ are compact support),

$$0 = \int\limits_X (\overline{\partial}\varphi \wedge *\#\psi) - \int\limits_X \varphi \wedge *\#\vartheta\psi$$

i.e. $(J\varphi,\psi) = (\varphi,\vartheta\psi)$, which proves the lemma. $\square$

17 The operator ϑ depends on the hermitian metric on X and on the hermitian metric along the fibres of E. To emphasise this fact, we may write sometimes ϑ_E for ϑ.

We have already introduced the norm $\|\ \|$ on $\mathscr{D}^{p,q}(X,E)$ and denoted the corresponding completion by $\mathcal{L}^{p,q}(X,E)$. We will now introduce another norm on $\mathscr{D}^{p,q}(X,E)$.

Definition 1.2. *For $\varphi,\psi \epsilon \mathscr{D}^{p,q}(X,E)$.*

$$a(\varphi,\psi) = (\overline{\partial}\varphi,\overline{\partial}\psi) + (\vartheta\psi,\vartheta\psi) + (\varphi,\psi).$$

and $\quad N(\varphi)^2 = a(\varphi,\varphi)$.

As before we see that the hermitian scalar product $a(\varphi,\psi)$ defines a complex prehilbert-space structure on $\mathscr{D}^{p,q}(X,E)$. We denote by $W^{p,q}$

(X, E) the corresponding completion. Clearly, we have for $\varphi, \psi \epsilon \mathscr{D}^{p,q}$ (X, E) the inequality $||\varphi|| \leq N(\varphi)$, so that the identity

$$\mathscr{D}^{p,q}(X, E) \to \mathscr{D}^{p,q}(X, E)$$

extends uniquely to a continuous linear map,

$$i : W^{p,q}(X, E) \to \mathcal{L}^{p,q}(X, E).$$

Proposition 1.1. *i is injective.*

Proof. Let $\varphi_\nu \epsilon \mathscr{D}^{p,q}(X, E)$ be a Cauchy sequence in N and assume that $||\varphi_\nu|| \to 0$. Since φ_ν is a Cauchy-sequence in N, $\bar{\partial}\varphi_\nu$ and $\vartheta\varphi_\nu$ are Cauchy sequences in the norm $||\quad||$. Hence $\bar{\partial}\varphi_\nu$ and $\vartheta\varphi_\nu$ tend to a limit in $\mathcal{L}^{p,q+1}(X, E)$ and $\mathcal{L}^{p,q-1}(X, E)$ respectively.

We denote these limits by u and v. By our identification of $\mathcal{L}^{p,q}$ (X, E) as the space of square summable E-valued forms, u and v may be regarded as forms on X with values on E. □

Now for any $\psi \in \mathscr{D}^{p,q+1}(X, E)$ **18**

$$(u, \psi) = \lim_{\nu\to\infty}(\bar{\partial}\varphi_\nu, \psi) = \lim_{\nu\to\infty}(\varphi_\nu, \vartheta\psi) = 0$$

since $||\varphi_\nu|| \to 0$. Since ψ is arbitrary in $\mathscr{D}^{p,q+1}(X, E), U = 0$. Similarly $v = 0$. That is, $\bar{\partial}\varphi_\nu \to 0$ and $\vartheta\varphi_\nu \to 0$ in $\mathcal{L}^{p,q+1}(X, E)$ and $\mathcal{L}^{p,q-1}(X, E)$ respectively. Thus $N(\varphi_\nu) \longrightarrow 0$ as $\nu \longrightarrow \infty$, hence the proposition.

By continuity we obtain from (1.1) that, if $\varphi \in W^{p,q}(X, E), \psi \in W^{p,q+1}(X, E)$, then

$$(\bar{\partial}\varphi, \psi) = (\varphi, \vartheta\psi)$$

Clearly, we may now regard $W^{p,q}(X, E)$ also as a space of measurable E-valued forms on X. From the definitions, it is moreover clear that if $f \in W^{p,q}(X, E)$, $\bar{\partial} f$ and ϑf in the distribution sense are currents representable by square summable $(p, q+1)$ and $(p, q-1)$ forms respectively. In general $W^{p,q}(X, E)$ is not the space of all square summable forms ω of type (p, q) whose $\bar{\partial}$ and ϑ (in the sense of distributions) are again square summable. We have however, the

Theorem 1.1. *If the Riemannian metric (associated to the hermitian metric) on X is complete, then*

$$W^{p,q}(X,E) = \left\{\varphi \mid \varphi \in \mathcal{L}^{p,\wedge q}(X,E); \overline{\partial}\varphi \in \mathcal{L}^{p,q+1}(X,E); \vartheta\varphi \in \mathcal{L}^{p,q-1}(X,E)\right\}$$

Proof. We will first establish three lemmas which are needed for the proof. □

Lemma A. *There exists $C_\circ > 0$ depending only on the dimension of X such that for any scalar form u and any $v \in C^{p,q}(X,E)$,*

$$A_E(u \wedge v, u \wedge v)(x) \leq \{C_\circ \mid u \mid^2 A_E(v,v)\}(x)$$

19 *where $\mid u \mid^2$ denotes the length of the scalar form a). (We recall that $A_E(\varphi,\psi)\ell = \varphi \wedge *\#\psi$). This is simply a lemma on finite dimensional vector space with scalar products. We omit the proof.*

Lemma B. *Let $p_\circ \in X$ be any point. The function $\rho(x) = d(p_\circ, x) =$ distance from 0 of x, is locally Lipschitz continuous and wherever ρ has partial derivatives, $\mid d\rho \mid^2 \leq 2n$.*

Proof. We have by the triangle inequality

$$\mid \rho(x) - \rho(y) \mid \leq d(x,y).$$

Now if U is a coordinate open set with coordinates $(X^1, \ldots, X^m)$ (X considered as a differentiable manifold of dimension $m = 2n$) and the Riemannian metric is given on U by $\sum g_{ij} dx^i x^j$ then for any $V \subset 0 \subset U$, there exists λ and μ such that

$$\lambda \sum dx^{i^2} \leq \sum g_{ij} dx^i dx^j \leq \mu \sum dx^{i^2}$$

so that if V is a ball about the origin in U, there exist constants $C_1 C_2 > 0$ such that

$$C_1 \mid x - y \mid \leq d(x,y) \leq C_2 \mid x - y \mid \text{ for } x, y \in V.$$

Thus $\mid \rho(x) - \rho(y) \mid < C_2 \mid x - y \mid$ for $x, y \in V$. Hence the first assertion. □

For the second assertion consider about a point p a neighbourhood U in which we can introduce geodesic polar-coordinates. Let $(x^1, \ldots, x^m)$ be a coordinate system such that $x^i(p) = 0$ for $1 \leq i \leq m$.

For $\varepsilon > 0$ we can choose U so small that $| g_{ij} | < \delta_{ij} + \varepsilon$, i.e. $d(x,p)^2 \leq (1+\varepsilon)\sum x_i^2$. In such a coordinate system, we have setting $e_i = (0, \ldots, 1 \ldots, 0)$ (1 at the i^{th} place)

$$\frac{\partial \varrho}{\partial x^i}(\rho) = \underset{h\to 0}{Lt} \frac{\rho(he_i) - \varrho(0)}{h}$$

On the other hand we have 20

$$| \rho(he_i) - \rho(0) | \leq d(0, h, e_i) \leq (1+\varepsilon)h$$

Since we are working with a coordinate system such that $d(x,0) \leq (1+\varepsilon\left(\sum x_i^2\right)^{\frac{1}{2}}$. Hence ρ is Lipschitz continuous and, if the derivatives exist, $\dfrac{\partial \rho}{\partial x^i}(p) \leq 1$. Now, by definition $| d\rho |^2 = \sum g_{ij} \frac{\partial \rho}{\partial X^i} \frac{\partial \rho}{\partial X^j}$ and in the coordinate system introduced above, $g_{ij}(0) = \delta_{ij}$ so that $| \rho |^2 = \sum\limits_{1=1}^{m} \left(\frac{\partial \rho}{\partial x^i}\right)^2 \leq m$. When X is a complex manifold of complex dimension n, we have,

$$| d\rho |^2 \leq 2n.$$

Lemma C. *Let*

$$\dot{\mathscr{D}}^{p,q}_{\overline{\partial}}(X,E) = \left\{\varphi \mid \varphi \in \mathcal{L}^{p,q}(X,E), \overline{\partial}\varphi \in \mathcal{L}^{p,q+1}(X,E), \operatorname{Supp}\varphi \subset\subset X\right\}$$

$$\begin{aligned}\dot{\mathscr{D}}^{p,q}_{\vartheta}(X,E) = \{&\varphi \mid \varphi \in \mathcal{L}^{p,q}(X,E),\\ &\vartheta\varphi \in \mathcal{L}^{p,q}(X,E), \vartheta\varphi \in \mathcal{L}^{p,q-1}(X,E), \operatorname{supp}\varphi \subset X\right\}\end{aligned}$$

(where $\overline{\partial}$ and ϑ are in the sense of distributions) and finally

$$\dot{\mathscr{D}}^{p,q}(X,E) = \dot{\mathscr{D}}^{p,q}_{\overline{\partial}}(X,E) \cap \dot{\mathscr{D}}^{p,q}_{\vartheta}(X,E).$$

Then

(i) *$\mathscr{D}^{p,q}(X,E)$ is dense in $\dot{\mathscr{D}}^{p,q}_{\overline{\partial}}(X,E)$ w.r.t. the norm*

$$P(\varphi) : P(\varphi)^2 = ||\varphi||^2 + ||\overline{\partial}\varphi||^2$$

and

(ii) *$\mathscr{D}^{p,q}(X,E)$ is dense in $\dot{\mathscr{D}}^{p,q}_{\vartheta}(X,E)$ with the norm*

$$Q(\varphi) : Q(\varphi)^2 = ||\varphi||^2 + ||\vartheta\varphi||^2$$

(iii) *$\mathscr{D}^{p,q}(X,E)$ is dense in $\dot{\mathscr{D}}^{pq}(X,E)$ with the norm*

$$N : N(\varphi)^2 = ||\varphi||^2 + ||\overline{\partial}\varphi||^2 + |\vartheta\varphi||^2.$$

21 *Proof.* We will prove (i) ; the proofs for the other two cases are similar. Let $\mathfrak{G} = \{U_i\}_{i\in I}$ be a locally finite covering of X such that $E\mid_{U_i}$ is trivial. Let m be the rank of E. Assume further that each U_i is a coordinate open set. Let $\{V_i \mid V_i \subset\subset U_i\}_{i\in I}$ be a shrinking of U_i. Let $(\rho_i)_{i\in I}$ be a partition unity subordinate to $\{V_i\}_{i\in I}$. Let $\varphi \in \dot{mathscrD}^{p,q(X,E)}_{\overline{\sigma}}$. $E\mid_{U_i}$ being trivial, we may take $\rho_i\varphi_i$ to be a $\mathbb{C}^m$-valued from on $\mathbb{C}^n$ with compact support in V_i. For any $\varepsilon > 0$, we can find a $\mathbb{C}^m$ valued C^∞ from ψ_i with compact support in $V_i \subset \mathbb{C}^n$, such that $\|\ \overline{\partial}\rho_i\varphi_i - \overline{\partial}\psi_i\ \| < \varepsilon$ and $||\rho_i\varphi_i - \psi_i|| < \varepsilon$. This can be secured by the usual regularisation methods. Since the support of φ is compact, we may take $\psi_i = 0$ except for a finite number of i. Since $E\mid_{U_i}$ is trivial and $\rho_i\varphi_i$ were regarded as scalar forms through suitable trivialisations, we may now revert the process and consider ψ_i as E-valued C^∞-forms with support in V_i. It follows that $\psi = \sum\psi_i$ is a C^∞-form with compact support and

$$||\varphi - \psi|| = ||\sum\rho_i\varphi_i - \sum\psi_i|| \leq \sum_i ||\rho_i\varphi_i - \psi_i|| < M\varepsilon$$

and similarly $||\overline{\partial}\varphi - \overline{\partial}\psi|| < M\varepsilon$ where M is the number of indices of the finite set

$$\{i \mid i \in I, U_i \cap \quad \text{support} \quad \varphi \neq \phi\}.$$

Since M is fixed for a given φ and ε is at our choice, the lemma is proved. □

Lemma C enables us to prove the following more general statement.

22 **Theorem 1.1.** *If the Riemannian metric (associated to the hermitian metric) of X complete, then*

i) $\mathscr{D}^{p,q}(X,E)$ *is dense in the space*

$$\left\{\varphi \mid \varphi \varepsilon \mathcal{L}^{p,q}(X,E), \overline{\partial}\varphi \in \mathcal{L}^{p,q+1}(X,E)\right\},$$

with respect to the norm $P(\varphi)$;

ii) $\mathscr{D}^{p,q}(X,E)$ *is dense in the space*

$$\left\{\varphi \mid \varphi \in \mathcal{L}^{p,q}(X,E), \vartheta\varphi \in \mathcal{L}^{p,q-1}(X,E)\right\},$$

with respect to the norm $Q(\varphi)$;

iii) $\mathscr{D}^{p,q}(X,E)$ *is dense in the space*

$$\left\{\varphi \mid \varphi \in \mathcal{L}^{p,q}(X,E), \overline{\partial}\varphi \in \mathcal{L}^{p,q+1}(X,E), \vartheta\varphi \in \mathcal{L}^{p,q-1}(X,E)\right\},$$

with respect to the norm $N(\varphi)$.

Proof. We will prove i). The proofs of ii) and iii) are similar.

In view of Lemma C it is sufficient to prove that every distributive form φ in $\mathcal{L}^{p,q}(X,E)$ such that $\overline{\partial}\varphi \in \mathcal{L}^{p,q+1}(X,E)$ can be approximated as closely as we want by forms ψ such that $\psi, \overline{\partial}\psi$ are square summable and supp $\psi, \subset\subset X$.

Let $\mu :| R^1 \to [0,1]$ be a C^∞ function such that

(i) $\mu(t) = 1$ if $t < 1$ and (ii) $\mu(t) = 0$ if $t > 2$.

Let $M = \operatorname*{Supp}_t \mid \frac{d\mu}{dt} \mid$. Let $d(x,y)$ be the distance function defined by the complete Riemannian metric of X. Then we fix a point $p_\circ \in X$ and 23
set $\rho(x) = d(x, p_\circ)$ for $x \in X$, the function

$$\omega_\nu(x) = \mu\left(\frac{\rho(x)}{\nu}\right) \nu > 0$$

is locally Lipschitz, and where the derivatives $\frac{\partial\rho}{\partial x^i}$ exist, then

$$\mid d\omega_\nu \mid^2 \leq \mid \frac{1}{\gamma}\frac{\partial\mu}{\partial t}\left(\frac{\rho(x)}{\nu}\right) d\rho \mid^2 \leq \frac{2nM^2}{\nu^2}$$

in view of Lemma B. Since the metric is complete the ball of centre p and radius c,

$$B_c = \{p \mid d(p, p_\circ) < c\}$$

is relatively compact in X for all $c > 0$.

Consider now the form $\omega_\nu.\varphi = \varphi_\nu$; φ_ν has compact support. It is easily seen that $\omega_\nu.\varphi = \varphi_\nu$ is in $\dot{\mathscr{D}}^{p,q}(X, E)$. We will now prove that $\varphi_\nu \to \varphi$ in the norm N as $\nu \to \infty$. In fact, we have first of all

$$||\varphi - \varphi_\nu|| = ||(1 - \omega_\nu)\varphi||_{X-B_\nu} \leq ||\varphi||_{X-B_\nu} \to 0 \text{ as } \nu \to \infty.$$

Secondly

$$\begin{aligned} ||\overline{\partial}\varphi - \overline{\partial}\varphi_\nu|| &= ||\overline{\partial}\varphi - \omega_\nu\overline{\partial}\varphi - \overline{\partial}\omega_\nu \wedge \varphi|| \\ &\leq ||\overline{\partial}\varphi||_{X-B_\nu} + ||\overline{\partial}\omega_\nu \wedge \varphi|| \end{aligned}$$

Now,

$$\begin{aligned} ||\overline{\partial}\omega_\nu \wedge \varphi|| &= A(\overline{\partial}\omega_\nu \wedge \varphi, \overline{\partial}\omega_\nu \wedge \varphi) \\ &\leq \mid \overline{\partial}\omega_\nu \mid^2 \cdot A(\varphi, \varphi) \qquad \text{by Lemma A.} \end{aligned}$$

On the other hand, one checks easily that almost everywhere

$$\mid \overline{\partial}\omega_\nu \mid^2 \leq \mid d\omega_\nu \mid^2 \leq \frac{2nM^2}{\nu^2},$$

so that, we obtain

$$||\overline{\partial}\varphi_\nu - \overline{\partial}\varphi|| \leq ||\overline{\partial}\varphi||_{X-B_\nu} + \frac{C'}{\nu}||\varphi||.$$

24 Hence $\| \overline{\partial}\varphi_\nu - \overline{\partial}\varphi \| \to 0$ as $\nu \to \infty$. This completes the proof of Theorem 1.1 (i). $\square$

Definition 1.3. $\square : C^{p,q}(X, E) \to C^{p,q}(X, E)$ *is the operator* $\overline{\partial}\vartheta + \vartheta\overline{\partial}$. *The following result is easily checked.*

Lemma 1.5. *The operator $\square$ defined above is strongly elliptic.*

The operator $\square$ depends on the metric on X and on the metric along the fibres of E. To emphasize this fact we may write $\square_E$ for $\square$.

For the operator $\square$ we have the following result.

Theorem 1.2 (Stampacchia Inequality). *We assume the Riemannian metric on X (associated to the hermitian structure) to be complete. Let $p_o \in X$ and for $\nu > 0$, let $B_\nu = \{x \mid d(x, p_o) < \nu\}$. Then there exists a constant $A > 0$ such that for every $\sigma > 0$, for every choice $r < R$ of a pair of positive reals r, R and every $\varphi \in C^{p,q}(X, E)$,*

$$\| \overline{\partial}\varphi \|^2_{B_r} + \| \partial\varphi \|^2_{B_r} \leq \sigma \| \square\varphi\|^2_{B_r} + \left(\frac{1}{\sigma} + \frac{A}{(R-r)^2}\right) ||\varphi||^2_{B_R}$$

Proof. Let us choose as before a C^∞ function $\mu : \mathbb{R}^1 \to [0, 1]$ such that $\mu(t) = 1$ for $t < 1$ and $\mu(t) = 0$ for $t > 2$.

Let $M = sup|\frac{d\mu}{dt}|$. consider the function defined by

$$W(x) = \mu\left\{\frac{\rho(x) + R - 2r}{R - r}\right\}$$

where $\sigma(x) = d(p_o, x)$. Evidently then, ω has support in B_R and $w \equiv 1$ on $B_r(B_R \subset\subset X)$. □

Moreover, $|d\omega|^2 = |\frac{1}{R-r}\frac{\partial\mu}{\partial t}\left\{\frac{\rho(x)+R-2r}{R-r}\right\} \cdot d_\rho|^2$. It follows that $|d\omega|^2 \leq$ **25**
$\frac{2nM^2}{(R-r)^2}$.

Suppose now that $\varphi \in C^{pq}(X, E)$ and ψ is any Lipschitz continuous form with compact support in B_R, then

$$(\overline{\partial}\varphi, \overline{\partial}\psi)_{B_R} + (\vartheta\varphi, \vartheta\psi)_{B_R} = (\square\varphi, \psi)_{B_R}.$$

Set $\psi = \omega^2\varphi$. We have then

$$\begin{aligned}\overline{\partial}\psi &= \omega^2\overline{\partial}\varphi + 2\omega\overline{\partial}\omega\Lambda\varphi\\ \vartheta\psi &= \omega^2\vartheta\varphi - *(2\omega\partial\omega\Lambda * \varphi).\end{aligned}$$

This leads to

$$\begin{aligned}&(\omega\overline{\partial}\varphi, \omega\overline{\partial}\varphi)_{B_R} + (\omega\vartheta\varphi, \omega\vartheta\psi)_{B_R}\\ &\qquad = (\square\varphi, \omega^2\varphi) - (\omega\overline{\partial}\varphi, 2\overline{\partial}\omega\Lambda\varphi) + (\omega\vartheta\varphi, *(2\partial\omega\Lambda * \varphi)).\end{aligned}$$

Now, by Schuarz inequality,

$$| (\square\varphi, \omega^2\varphi)_{B_R} |\leq \frac{1}{2}\sigma||\square\varphi||^2_{B_R} + \frac{1}{2\sigma}||\omega^2\varphi||_{B_R} \leqslant \frac{\sigma}{2}||\square\varphi||^2_{B_R} + \frac{1}{2\sigma}||\varphi||^2_{B_R}$$

$$|\ (\omega\bar{\partial}\varphi, 2\bar{\partial}\omega\Lambda\varphi)_{B_R}\ | \leqslant \frac{1}{2}||\omega\bar{\partial}\varphi||^2_{B_R} + \frac{4c_o nM^2}{(R-r)^2}||\varphi||^2_{B_R}$$

and $$|\ (\omega\vartheta\varphi, *(2\partial\omega\Lambda * \varphi))\ | \leqslant \frac{1}{2}||\omega\vartheta\varphi||^2_{B_R} + \frac{4c_o nM^2}{(R-r)^2}||\varphi||^2_{B_R}$$

(where c_o is the positive constant which has been introduced in lemma A)

26 If follows that

$$||\omega\bar{\partial}\varphi||^2_{B_R} + ||\omega\vartheta\varphi||^2_{B_R} \leq \sigma||\square\varphi||^2_{B_R} + \sigma||\varphi||^2_{B_R} + \frac{8c_o nM^2}{(R-r)^2}||\varphi||^2_{B_R}.$$

The inequality follows now from

$$||\omega\bar{\partial}\varphi||^2_{B_R} \geqslant ||\bar{\partial}\varphi||^2_{B} \text{ and } ||\omega\vartheta\varphi||^2_{B_R} \geqslant ||\vartheta\varphi||^2_{B_R}$$

This completes the proof of theorem 1.2.

Corollary 1. *For $\varphi_2 \in C^{p,q}(X,E)$ and for any $\sigma > 0$,*

$$||\bar{\partial}\varphi||^2 + ||\vartheta\varphi||^2 \leq \sigma||\square\varphi||^2 + \frac{1}{\sigma}||\varphi||^2.$$

Proof. Set $R = 2r$ in stampachia inequality and let $r \to \infty$. □

Corollary 2. *If $\varphi \in C^{p,q}(X,E)$, $||\varphi|| < \infty$ and $||\square\varphi|| < \infty$, then $||\bar{\partial}\varphi|| < \infty$, $||\vartheta\varphi|| < \infty$. If square$\varphi = 0$, then $\bar{\partial}\varphi = \vartheta\varphi = 0$.*

Proof. The first assertion follows from corollary 1. The second again from corollary 1 since σ is arbitrary. □

Remark . On any (paracompact) complex manifold X there exists a complete hermitian metric. More exactly we shall prove that, given any hermitian metric ds^2 on X, there exists a C^∞ function $F : X \to \mathbb{R}$ such that $F \cdot ds^2$ is a complete metric.

Proof. Let $\{B_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of compact sets such that

$$B_\nu \subset \overset{o}{B}_{\nu+1}, \cup B_\nu = X.$$

27 Let $f_\nu : X \to \mathbb{R}$ be a C^∞ function on X, satisfying the following conditions:

$$\begin{aligned}
&0 \leqslant f_\nu \leqslant 1,\\
&f_\nu = 1 \text{ on } B_{\nu+1} - B_\nu,\\
&\text{Supp} \quad f_\nu \subset \overline{B_{\nu+2} - B_{\nu-1}}
\end{aligned}$$

Let $d(x,y)$ $(x, y \in X)$ be the distance function determined by the hermitian ds^2, and let

$$\varepsilon_\nu = \inf_{\substack{x \in B_\nu \\ y \in \partial B_{\nu+1}}} d(x,y).$$

Then $\varepsilon_\nu > 0$. Consider the positive C^∞ function

$$F(x) = \sum_{\nu=0}^{+\infty} \frac{f_\nu(x)}{\varepsilon_\nu}$$

and the hermitian metric $\widetilde{ds}^2 = F(x) ds^2$. Let $\tilde{d}(x,y)$ be the distance function determined by ds^2. We have

$$\begin{aligned}
\breve{d}(\overline{B}_\nu, B_{\nu+1}) \geqslant &\\
&\geqslant \quad x \in \frac{\inf}{B_{\nu+1} - B_\nu} F(x) d(\overline{B}_\nu, B_{\nu+1})\\
&\geqslant \quad \frac{1}{\varepsilon_\nu} \varepsilon_\nu = 1.
\end{aligned}$$

This implies that every Cauchy sequence for the distance $\tilde{d}(x,y)$ converges. Hence $\widetilde{ds}^2$ is a complete hermitian metric. □

3 W-ellipticity and a weak vanishing theorem

We now introduce a seminorm on $W^{p,q}(X,E)$ as follows: for $\varphi, \psi \epsilon W^{p,q}$ (X,E), let $b(\varphi,\psi) = (\overline{\partial}\varphi, \overline{\partial}\psi) + (\vartheta\varphi, \vartheta\psi)$; then b is a positive semi definite form and $b(\varphi\varphi)$ defines a semi-norm on $W^{p,q}(X,E)$. the map 28

$$j : W^{p,q}(X,E) \to W_b^{p,q}(X,E)$$

where the latter space is $W^{p,q}(X, E)$ provided with the (not necessarily Hausdorff) topology defined by the seminorm $b(\varphi, \varphi)$ and j is the identity is clearly continuous: in fact, $b(\varphi, \varphi) \leqslant N(\varphi)$.

Definition 1.4. *We say that E is $W^{p,q}$ elliptic with respect to the hermitian metric on X and the and the hermitian metric along the fibres of E, if j admits a continuous inverse.*

In particular, the $W^{p,q}$-ellipticity of E implies that $b(\varphi, \varphi)^{\frac{1}{2}}$ is actually a norm. In fact, there is a $K > 0$ such that $N(\varphi)^2 \leqslant Kb(\varphi, \varphi)$. Since by definition, $N(\varphi) = b(\varphi, \varphi) + ||\varphi||^2$, W^{pq}-ellipticity is equivalent to following: there is a constant $C > 0 (= K - 1$ when K is as above) such that

$$||\varphi||^2 \leqslant Cb(\varphi, \varphi) = C(||\overline{\partial}\varphi||^2 + ||\vartheta\varphi||^2). \tag{1.2}$$

We shall call C a $W^{p,q}$-ellipticity constant.

Proposition 1.2. *Assume given a hermitian metric on X and a hermitian metric along the fibres of $\pi : E \to X$. Suppose further that E is W^{pq} -elliptic with reference to these hermitian metrics. Then there is a linear map $G : \mathcal{L}^{p,q}(X, E) \to W^{p,q}(X, E)$ such that, for $f \in \mathcal{L}^{p,q}(X, E), \varphi \in W^{p,q}(X, E)$, we have*

$$(f, \varphi) = (\overline{\partial}\, Gf,\ \overline{\partial}\varphi) + (\vartheta\, Gf,\ \varphi).$$

29 The linear map G is continuous more exactly:

$$b(Gf, Gf) \leqslant C||f||^2 \tag{1.3}$$

C being the constant which appears in (1.2). Moreover Gf is uniquely determined by the above formula.

Proof. $\varphi \rightsquigarrow (\varphi, f)$ defines a linear form on $W^{p,q}(X, E)$ which is evidently continuous. By the Riesz representation theorem (since E is $W^{p,q}$ elliptic, $b(\varphi, \psi)$ defines a Hilbert space structure on $W^{p,q}(X, E)$ equivalent to that defined by N), there is a unique element $Gf \in W^{p,q}(X, E)$ such that

$$(f, \varphi) = b(Gf,\ \varphi) = (\overline{\partial}\, Gf,\ \overline{\partial}\varphi) + (\vartheta Gf, \vartheta\varphi).$$

We have, now

$$\begin{aligned} b(Gf, Gf) &= (\bar\partial Gf, \bar\partial Gf) + (\vartheta Gf, \vartheta Gf). \\ &= (f, Gf) \text{ by the above equation.} \end{aligned}$$

Hence $b(Gf, Gf)^2 \leqslant ||f||^2.||Gf||^2 \leqslant C||f||^2 b(Gf, Gf)$ in view of W^{pq}-ellipticity. That is,

$$b(Gf, Gf) \leqslant C||f||^2.$$

We obtain

$$N(Gf)^2 \leqslant (C+1)||f||^2.$$

This completes the proof of Proposition 1.2. □

Corollary. *$Gf = f$ (in the sense of distributions).*

Proof. For $u \in \mathscr{D}^{p,q}(X, E)$, we have $(f, u) = (\bar\partial Gf, \bar\partial u) = (\vartheta Gf, \Box u)$.

This proves the lemma. □

Proposition 1.3. *Let E be $W^{p,q}$-elliptic with respect to a given metric along the fibers and to a complete hermitian metric on X. In the notation of Proposition 1.2 we have the following.* **30**

(i) *If $f \in \mathcal{L}^{p,q}(X, E)$ and $\bar\partial f \in \mathcal{L}^{p,q+1}(X, E)$, then*

$$\bar\partial Gf \in W^{p,q+1}(X, E); \Box\bar\partial Gf = \bar\partial f;$$
$$||\vartheta\bar\partial Gf||^2 \leqslant \frac{1}{\sigma}||\overline{\bar\partial f}||^2 + \sigma||\bar\partial Gf||^2 \text{ for } \sigma > 0.$$

(ii) *If $f \in \mathcal{L}^{p,q-1}(X, E)$ and $\vartheta f \in \mathcal{L}^{p,q+1}(X, E)$, then*

$$\vartheta Gf \in W^{p,q-1}(X, E); \Box\vartheta Gf = \vartheta f;$$
$$||\bar\partial\vartheta Gf\,||^2 \leqslant \sigma||\vartheta\ f||^2 + \sigma||\vartheta\ Gf\,||^2 for \sigma > 0.$$

Proof. Since the metric is complete, by Theorem 1.1' $\mathcal{D}^{p,q}(X, E)$ is dense in the space $\left\{\varphi \mid \varphi \in \mathcal{L}^{p,q}(X, E), \bar\partial\varphi \in \mathcal{L}^{p,q+1}(X, E)\right\}$ provided with the norm $P(\varphi)^2 = ||\varphi||^2 + ||\bar\partial\varphi||^2$. Let $f \in \mathcal{L}^{p,q}(X, E)$ and $\bar\partial f \in \mathcal{L}^{p,q+1}(X, E)$. Then there is a sequence $f_n \in \mathcal{D}^{p,q}(X, E)$ such that $||f_n - f|| \to$

$0, \|\overline{\partial} f_n - \overline{\partial f}\| \to 0$ as $n \to \infty$. We have moreover, from Proposition 1.2, that

$N(G\varphi)^2 \leqslant K\|\varphi\|^2$ for $\varphi \in \mathscr{L}^{p,q}(X, E)$. Hence

$$\|G f_n - G f_m\| \leqslant K\| f_n - f_m\|$$

and $$\|\overline{\partial} G f_n - \overline{\partial} G f_n\| + \|\vartheta G f_n - \vartheta G f_n\| \leqslant K\| f_n - f_m\|.$$

Applying now Stampachia inequality (Corollary 2, Theorem 1.2) to $\overline{\partial} G f_n$, and taking into account the fact that $\Box$ and $\overline{\partial}$ commute, we obtain, for $\sigma > 0$,

$$\|\vartheta\overline{\partial}(G f_n - G f_m)\|^2 \leqslant \frac{1}{\sigma}\overline{\partial}(f_\nu - f_\mu)\|^2 + \sigma\|(\overline{\partial} G(f_\nu - f_\mu)\|^2$$

Since the right hand side tends to 0 as $n, m \to \infty$, it follows that
31 $\left\{\vartheta\overline{\partial} f_n\right\}$ is a Cauchy-sequence in $\mathcal{L}^{p,q}(X, E)$. It is immediate then that $\vartheta\overline{\partial} G f = \lim\limits_{n\to\infty} \vartheta\overline{\partial} G f_n \in \mathcal{L}^{p,q}(X, E)$. On the other hand $\overline{\partial}(\overline{\partial} G f) = 0$. Hence $\overline{\partial} G f \in W^{p,q+1}(X, E)$. The equation $\Box\overline{\partial} G f = \overline{\partial} f$ follows from the fact that $\overline{\partial}$ and $\Box$ commute and the fact that $\Box G f = f$. The last inequality is the Stampachia inequality applied to $\overline{\partial} G f$. The proof of Part (ii) of the proposition is entirely analogous. $\Box$

Theorem 1.3. *Let $\pi : E \to X$ be a holomorphic vector-bundle which is $W^{p,q}$-elliptic with respect to a complete hermitian metric on X and a hermitian metric along the fibres of E. Then if $q > 0$, given $f \in \mathcal{L}^{pq}(X, E)$ with $\overline{\partial} f = 0$, there is a unique $x \in W^{pq}(X, E)$ such that $f = \overline{\partial}\vartheta x$ and $\overline{\partial} x = 0$. Moreover, we have $\|\vartheta x\|^2 \leqslant C\|f\|^2$, C being a $W^{p,q}$-ellipticity constant.*

Proof. We set $x = G f$. We have then $\Box G f = f$. (Corollary to Proposition 1.2). Clearly $\overline{\partial} f = 0 \in \mathcal{L}^{p,q+1}(X, E)$. Hence by (i) of proposition 1.3, we have $\overline{\partial} G f \in W^{p,q+1}(X, E)$ and further $\|\vartheta\overline{\partial} G f\|^2 \leqslant \sigma\|\Box\overline{\partial} G f\|^2 + \frac{1}{\sigma}\|\overline{\partial} G f\|^2$. On the other hand since $\Box$ and $\overline{\partial}$ commute, it follows again from the Corollary to Proposition 1.2, that $\Box\overline{\partial} G f = 0$. Since σ is arbitrary, $\vartheta\overline{\partial} x = 0$. It follows that $(\overline{\partial} x, \overline{\partial} x) = (x, \vartheta\overline{\partial} x) = 0$ (note that $\overline{\partial} x \in W^{p,q+1}(X, E)$: Proposition 1.3). Hence $\overline{\partial} x = 0$. Now $\Box = \overline{\partial}\vartheta + \vartheta\overline{\partial}$ so that $\Box G f = f$ leads to $(\overline{\partial}\vartheta + \vartheta\overline{\partial}) G f = f$ that is, $\overline{\partial}\vartheta G f = f$. $\Box$

Finally (1.3) yields **32**

$$\|\vartheta x\|^2 \leqslant C\|f\|^2.$$

The uniqueness of x satisfying $\bar{\partial}\vartheta x = f$ and $\bar{\partial} x = 0$ is easily checked.

This completes the proof of the theorem.

It follows from the regularity theorem for elliptic systems that , if f $\in C^{p,q}$ $(X, E) \cap \mathcal{L}^{p,q}$ then Gf (can be modified on a null set so that it) will be of class C^∞ on X.

In particular we have the

Corollary. *If $f \in \mathcal{D}^{p,q}(X, E)$ and $\bar{\partial} f = 0$ then there exists $\psi \in C^{p,q-1}$ (X, E) such that $\bar{\partial}\psi = f$*

Remark. The corollary above clearly implies the following.

If E is W^{pq}-elliptic with respect to a complete hermitian metric, then the natural map

$$H^q_k(X, \Omega^p(E)) \to H^q(X, \Omega^p(E))$$

where the left-side stands for the q^{th} cohomology with compact supports of X with values in $\Omega^p(E)$, is the trivial map $\alpha \rightsquigarrow 0$ for every $\alpha \in H^q_k(X, \Omega^p(E))$.

Remark. Let $\mathfrak{G} = \left\{U_i\right\}$ be a covering of X such that $\pi : \ E \to \ X$ is defined with respect to $\mathfrak{G}$ by holomorphic transitions $e_{i_j} : U_i \cap U_j \to GL(m, \mathbb{C})(m =$ rank of $E)$.

Let $\left\{h_i\right\}_{i \in I}$ be a hermitian metric along the fibers of E; then h_i is a C^∞ function on U_i whose values are positive definite hermitian matrices.

We have on $U_i \cap U_j$ **33**

$$h_i =^t {}_{\bar{e}_{ji}} h_j e_{ji},$$

and therefore $$t_{h_i^{-1}} =^t (\overline{{}^t e_{ji}^{-1}})^t h_j^{-1} \, {}^t e_{ji}^{-1}$$

That means that $\left\{{}^t h_i^{-1}\right\}$ defines a matric along the fibres of the dual bundle E^*.

We note here for future reference the following identities, holding with respect tot he metrics $\{h_i\}$ on E and $\{t_{h_i^{-1}}\}$ on E^*. They follow immediately from the definitions of the operators involved.

$$A_{E^*}(*\#\varphi, *\#\psi) = A_E(\varphi, \psi) \tag{1.4}$$

$$\overline{\partial} * \#\varphi = (-1)^{p+q} * \#\vartheta_E \varphi, (\varphi, \psi \in C^{pq}(X, E)) \tag{1.5}$$

$$\vartheta_{E^*} \; * \#\varphi = (-1)^{p+q+1} * \#\overline{\partial}\psi, \tag{1.6}$$

$$\Box_{E^*} * \#\varphi = *\#\Box_E \varphi. \tag{1.7}$$

As a corollary we have that $*\#$ defines an isometry of $\mathscr{L}^{pq}(X, E)$ onto $\mathscr{L}^{n-p\; n-q}(X, E)$ which maps W^{pq} isometrically onto $W^{n-p\; n-q}(X, E^*)$.

Furthermore, if E is W^{pq}-elliptic with respect to the metric $\{h_i\}$ on E, then E^* is $W^{n-p\; n-q}$- elliptic with respect to the metric $\{t_{h_i^{-1}}\}$ on E^* the W-ellipticity constants being the same.

4 Carleman inequalities

34 We will now formulate certain further conditions on the vector bundles and show how these more stringent conditions lead to stronger vanishing theorems than the one above.

We assume always that the hermitian metric denoted ds^2 on X is complete. Let the given metric on E be denoted by h.

C_1. There is given a C^∞ function $\phi : X \to \mathbb{R}^+$.

C_2. For every non decreasing convex C^∞-function $\lambda : \mathbb{R} \to \mathbb{R}$.

E is W^{pq}-elliptic with respect to $(ds^2, e_h^{\lambda\phi})$.

C_3. The W^{pq}-ellipticity constant is independent of.

That is, there is $C > 0$ independent of λ such that

$\| f \|_\lambda^2 \leqslant C\left\{\| \overline{\partial} f \|_\lambda^2 + \| \vartheta_\lambda f \|_\lambda^2\right\}$ for $f \in \mathscr{D}^{pq}(X, E)$ where $\| \; \|_\lambda$ stands for the norm with respect to $(ds^2, e^{\lambda(\phi)}. h)$, and ϑ_λ denotes the ϑ-operator with respect to these metrics.

Conditions C_1, C_2, C_3 imply C_1', C_2', C_3' below. This weaker set of conditions are sufficient for the "vanishing theorems" we will now prove.

C_1'. There is given a C^∞ function $\phi : X \to \mathbb{R}^+$ (this is the same as C_1).

C_2'. Given $C_o > 0$, there is a non-decreasing C^∞-function $\lambda : \mathbb{R} \to \mathbb{R}$ that $\lambda(t) = 0$ for $t \leqslant C_o$ and $\lambda(t) > 0$ for $t > C_o$, such that, E is W^{pq} elliptic with respect to $(ds^2, e^{\nu\lambda(\phi)}h)$ for every positive integer ν.

C_3'. The W^{pq} ellipticity constant is independent of ν that is, there is a **35**
$C > 0$ such that for $f \in \mathscr{D}^{pq}(X, E)$

$$\| f \|_\nu^2 \leq C \left\{ \| \overline{\partial} f \|_\nu^2 + \| \vartheta_\nu f \|_\nu^2 \right\},$$

where $\| \, \|_\nu$ stands for $\| \, \|_{\nu\lambda}$ and ϑ_ν for $\vartheta_{\nu\lambda}$.

Lemma 1.6. *Assume given a constant $C_o > 0$ and that the condition C_1', C_2', C_3' above are satisfied (C_o in condition C_2' is taken as the above constant). For λ as in condition C_2', let $\mathcal{L}_\nu^{p,q}(X, E)$ denote the space of E-valued forms on X which are square-summable with respect to $(ds^2, e^{\nu\lambda(\phi)}.h)$. Then for $f \in \bigcap\limits_\nu \mathcal{L}_\nu^{p,q}(X, E)(q > 0)$ such that $\overline{\partial} f = 0$, there exist Ψ_ν for every integer $\nu \geq 0$ such that $\Psi\nu \in \mathcal{L}_\nu^{p,q-1}(X, E)$, $\overline{\partial}\Psi_\nu = f$ and $\| \Psi_\nu \|_\nu \leq C \| f \|_\nu$ (We assume that ds^2 is complete).*

Proof. This follows from Theorem 1.3. □

Theorem 1.4. *Assume that conditions C_1', C_2', C_3' are satisfied. Then for every square-summable form f of type $(p, q)(q > 0)$ having compact support such that $\overline{\partial} f = 0$, there exists $\Psi \in \mathcal{L}^{p,q-1}(X, E)$ such that, $f = \overline{\partial}\Psi$, $\| \Psi \|^2 \leq C \| f \|^2$ and (support of Ψ)*

$$\subset \{x | \phi(x) < (\sup \phi(y)) \ y \in \textit{ support } f)\}.$$

Proof. Let $C_o = \sup \phi(x)$, $x \in$ support f. Then we have

$$\| f \|_\nu^2 = \int_X e^{\nu\lambda(\phi)} A(f, f) dx = \int\limits_{\text{Supp}.\, f} e^{\nu\lambda(\phi)} A(f, f) dX$$

$$= \int_{\text{Supp } f} A(f, f) dx = \int_X A(f, f) dX = \| f \|^2$$

36 Since $\nu\lambda(\varphi(x)) = 0$ on the support of f. It follows that f $\epsilon \bigcap_\nu \mathcal{L}^{p,q}_\nu(X, E)$ for every integer $\nu > 0$. Hence by Lemma 1.6, we can find $\psi_\nu \epsilon \mathcal{L}^{p,q-1}_{\nu_2}(X, E)$ for every $\nu \epsilon \mathbb{Z}^+$ such that $\bar{\partial}\psi_\nu = f$ and $\| \psi \|^2_\nu \leq C \| f \|^2$. On the other hand, we have $\| \psi_\nu \|_\nu \geq \| \psi_\nu \|$ so that, $\| \psi_\nu \|^2 \leq \| \psi_\nu \|^2_\nu \leq C \| f \|^2 = C_1$ say. It follows that we can assume (by passing to a subsequence if necessary) that ψ_ν converges weakly in $\mathcal{L}^{p,q-1}(X, E)$ to a limit ψ. We have $\| \psi \| < C \| f \|$. On the other hand, for every $\varepsilon > 0$,

$$\int_{\phi(x) > c_0 + \varepsilon} e^{\nu\lambda(\phi)} A(\psi_\nu, \psi_\nu) dX \leq C_1$$

and since λ is non-decreasing, we have,

$$e^{\nu\lambda(C_0+\varepsilon)} \int_{\varphi \geq C_0 + \varepsilon} A(\psi_\nu, \psi_\nu) dX \leq C_1.$$

It follows that $\int_{\varphi \geq C_0 + \varepsilon} A(\psi_\nu, \psi_\nu) dX$ tends to zero and hence $\psi_\nu \to 0$ almost everywhere in $\varphi \geq C_0 + \varepsilon$. (The integrand is positive for all ν.) Hence $\psi = 0$ on $\{x \mid \varphi(x) \geq C_0 + \varepsilon\}$ for every ε. Hence support $\psi \subset \{x \mid \phi'(x) \leq C_0\}$. Finally for

$$\begin{aligned} u \in \mathscr{D}^{n-p,n-q}(X, E^*), (-1)^{p+q} \langle \psi, \bar{\partial} u \rangle &= (-1)^{p+q} \int \psi \wedge \bar{\partial} u \\ &= Lt (-1)^{p+q} \int \psi_\nu \wedge \bar{\partial} u \\ &= \langle f, u \rangle. \end{aligned}$$

Hence $\bar{\partial}\psi = f$ in the sense of distributions. This completes the proof of the theorem. □

37 **Remark.** The proof contains the following lemma (which has no connection with $W^{p,q}$-ellipticity.

Lemma 1.7. *Suppose given a C^∞ non-decreasing function $\lambda : \mathbb{R} \to \mathbb{R}$ such that $\lambda(t) = 0$ for $t \leq C_0, \lambda(t) > 0$ for $t > C_0$, and a C^∞ function $\varphi : X \to \mathbb{R}$. Suppose further that there is given an $f \in \mathscr{L}^{pq}(X, E)$ such that sup $\varphi(x)$ on the support of f is C_0. If for every integer $\nu > 0$ there exists ψ_ν such that $\bar{\partial}\psi_\nu = f$ and $\| \psi_\nu \|_\nu^2 \leq C \| f \|^2$ for a constant C independent of ν, then there exists $\psi \in \mathscr{L}^{p,q-1}(X, E)$ such that $\bar{\partial}\psi = f, \| \psi \|^2 \leq C \| f \|^2$ and support $\psi \subset \{x \mid \varphi(x) \leq C_0\}$. (Here by) $\| \psi_\nu \|_\nu^2$ we mean $\int e^{\nu\lambda(\phi)} A(\psi_\nu, \psi_\nu) dX$).*

Suppose now that the function ϕ in C_1' satisfies the following additional condition.

C_4'. For every $C > 0$, the set $\{x \mid \phi(x) < C\} \subset\subset X$.

Remark. If C_4' is satisfied in addition to C_1', C_2' and C_3' then in Theorem 1.4, we can assert that the support of ψ is compact.

Here we can state, as a corollary to Theorem 1.4 the following Theorem 1.4$'$. If the hypothesis of Theorem 1.4 are fulfilled and if C_4' holds, then

$$H_k^q(X, \Omega^p(E)) = 0.$$

Lemma 1.8. *Let $V \subset \mathscr{L}^{p,q}(X, E)$ be the set $V = V^{p,q}(X, E) = \{\varphi \in \mathscr{L}^{pq}(X, E) \Big|$ there exists $\psi \in \mathscr{L}^{p,q-1}(X, E)$ such that $\bar{\partial}\psi = \varphi\}$ and $N =$* 38
$N^{p,q}(X, E) = \{\varphi \in \mathscr{L}^{p,q}(X, E) \mid \vartheta\varphi = 0\}$. Then N is the orthogonal complement of V provided that the metric is complete.

Proof. Let $\rho \in \mathscr{L}^{p,q}(X, E)$. Then $\rho \in \{$ orthogonal complement of $V\}$ if and only if $(\rho, \bar{\partial}\psi) = 0$ for every $\psi \in \mathscr{L}^{p,q-1}(X, E)$ with $\bar{\partial}\psi \in \mathscr{L}^{p,q-1}(X, E)$. Since the metric is complete, this is equivalent to $(\rho, \bar{\partial}\psi) = 0$ for every $\psi \in \mathscr{D}^{p,q-1}(X, E)$. Hence the lemma. ☐

Theorem 1.5. *Assume conditions C_1', C_2', C_3' to hold for $\pi : E \to X$. Let $f \in \mathscr{L}^{p,q+1}(X, E)$ be such that sup $\varphi(x)$ on the support of f is C_0. Suppose further that $(f, g) = 0$ for all g which belong to some $\mathscr{L}_\nu^{p,q+1}$, and are such that $\vartheta_\nu g = 0 (\nu = 1, 2, ...)$[1]. Then there exists $\psi \in \mathscr{L}^{p,q}(X, E)$ such that $\bar{\partial}\psi = f$ and support $\psi \subset \{X \mid \varphi(X) \leq C_0\}$.*

[1]In view of the choice of λ, we have $(f, g) = (f, g)_\nu$.

Let $V_\nu^{p\ q+1}$ be the space $V^{p\ q+1}$ constructed with respect to the metrics
$(e^{\nu\lambda(\phi)}h, ds^2)$. Since $\lambda = 0$ on supp f, then $f \epsilon \mathcal{L}_\nu^{pq}(X, E)$ for all $\nu > 0$.

By the previous lemma

$$f \in \overline{V_\nu^{p,q+1}} \quad \text{for} \quad \nu = 1, 2, \ldots$$

First assume that

$$f \in V_\nu^{p\ q+1} \text{ for a particular } \nu.$$

In this case, there exists $\varphi_\nu \in \mathcal{L}_\nu^{p,q}(X, E)$ such that $f = \bar{\partial}\varphi_\nu$ Now by Proposition 1.3, there is an $X_\nu = G_\nu\varphi_\nu \in W_\nu^{p,q}(X, E)$.

39 Such that $\square_\nu x_\nu = \varphi_\nu$. We now set $\psi_\nu = \vartheta_\nu\bar{\partial}$. By Proposition 1.3 $\psi_\nu \in \mathcal{L}_\nu^{pq}(X, E)$. Further we have $\bar{\partial}\psi_\nu = \bar{\partial}\vartheta_\nu\bar{\partial}x_\nu = \bar{\partial}(\bar{\partial}\vartheta_\nu + \vartheta\bar{\partial})x_\nu = \bar{\partial}\varphi_\nu = f$. Since $\vartheta_\nu\psi_\nu = 0$, then by Theorem 1.1 $\psi_\nu \in W_\nu^{pq}(X, E)$. On the other hand, in view of $W_\nu^{p,q}$-ellipticity,

$$\|\psi_\nu\|_\nu^2 \leq C(\|\bar{\partial}\psi_\nu\|_\nu^2 + \|\vartheta_\nu\psi_\nu\|_\nu^2) = C\|f\|_\nu^2 = C_1$$

say C_1 is independent of ν since $\lambda(\varphi(x)) = 0$ for $x \in$ support f.

Suppose now $f \in \overline{V_\nu^{pq+1}}$, then there exists a sequence $f_\nu^i \in V_\nu^{pq+1}$ such that $\|f_\nu^i - f\|_\nu \to 0$. Then choosing $\psi_\nu^i \in W_\nu^{pq}(X, E)$ is above for each $f_\nu^i (\in V_\nu^{pq+1})$ we have

$$\bar{\partial}\psi_\nu^i = f_\nu^i$$
$$\|\psi_\nu^i\psi_\nu^2 \leqslant C\|f_\nu^i\|_\nu^2 \ \text{ and } \ \|\psi_\nu^i - \psi_\nu^j\|_\nu^2 \leqslant C\|f_\nu^i - f_\nu^l\|_\nu^2.$$

Hence ψ_ν^i converge to a limit ψ_ν in $\mathcal{L}_\nu^{p,q}(X, E)$. Clearly $\bar{\partial}\psi_\nu = F$. Further, from $\|\psi_\nu^i\|_\nu^2 \leqslant C\|f_\nu^i\|_\nu^2$, we deduce that $\|\psi_\nu\|_\nu^2 \leqslant C\|f\|^2$. Now the proof follows from Lemma 1.7.

The conditions C_1^* , C_2^* , C_3^* below are dual to C_1' , C_2' , C_3' : that is if C_1^* , C_2^* , C_3^* hold for E-valued (p, q)-forms, then C_1', C_2' , C_3' hold for E^* valued $(n - p, n - q)$ forms.

C_1^*. $(= C_1' = C_1)$ There is given a C^∞ function $\phi : X \to \mathbb{R}^+$.

C_2^*. For every $C_o \geqslant 0$, there exists a non-decreasing C^∞-function, $\lambda : \mathbb{R} \to \mathbb{R}$ such that $\lambda(t) = 0$ if $t \leqslant C_0$ and $\lambda(t) > 0$ for $t > C_0$ and E is W^{pq}-elliptic with respect to $(ds^2, e^{\nu\lambda(\phi)}h)$

C_3^*. The $W^{p,q}$ ellipticity constant is independent of $\nu = 1, 2, \ldots$

Then we have analogous to Theorem 1.5, the following **40**

Theorem 1.6. *Assume C_1^*, C_2^*, C_3^*. Then if $f \in \mathcal{L}^{p,q-1}(X, E)$, is such that* $\sup \phi(X)$ *on support of $f = C_0$, and if further $(f, u) = 0$ for every $u \in \mathcal{L}_{-\nu}^{p,q-1}(X, E)$, such that $\partial u = 0$, then there exists $\chi \in \mathcal{L}^{p,q}(X, E)$ such that $f = \vartheta\chi$ and support $\chi \subset \{X | \phi(X) \leq C\}$.*

Theorem 1.7. *Let $T = T^{p,q-1}$ be a distribution valued form type $(p, q-1)$ (that is a current of type $(p, q-1)$). Suppose further that $\partial T \in C^{p,q}(X, E)$ and let $K =$ support of T. Then for any neighbourhood U of K, there is a form $\eta \in C^{p,q-1}(X, E)$ such that $\bar{\partial}T = \partial\eta$ and support $\eta \subset V$. In particular if K is compact, we can find an η with the above property.*

Proof. We recall (Prerequisites, 4) that we have fine resolutions

$$0 \to \Omega^p \to A^{p,o} \xrightarrow{\bar{\partial}} A^{p,1} \to \ldots \xrightarrow{\bar{\partial}} A^{p,n} \to 0$$

and

$$0 \to \Omega^p \to K^{p,o} \xrightarrow{\bar{\partial}} K^{p,1} \to \ldots \xrightarrow{\bar{\partial}} K^{p,n} \to 0.$$

Further there is canonical injection $A^{p,q} \to K^{p,q}$ which is compatible with the operator $\bar{\partial}$. Hence by a standard result on cohomology of sheaves the induced map of complexes

$$\sum_{q \geqslant 0} \Gamma_\Phi(X, A^{p,q}) \to \sum \Gamma_\Phi(X, K^{p,q})$$

is a homotopy equivalence of complexes for any para compactifying family Φ of closed sets on X. We consider in particular the family of all closed sets of X which are contained in U. This is evidently a para-compactifying family and theorem follows from the general result stated above (for a more detailed proof, see [2], 97-99). □

Theorems 1.5 and 1.7 together enable us to prove **41**

Theorem 1.8. *Let $\pi : E \to K$ be a holomorphic vector bundle. Assume that for a suitable complete hermitian metric ds^2 on X and a suitable hermitian metric h along the fibres of E, conditions C'_1, C'_2, C'_3 and C'_4 hold. Then the image $\bar\partial\mathscr{D}^{pq}(X,E)$ of $\bar\partial$ in $\mathscr{D}^{p,q+1}(X,E)$ is a closed subspace for the usual topology on $\mathscr{D}^{pq}(X,E)$. Hence $H_k^{q+1}(X,\Omega^p(E))$ has a structure of a separated topological vector space.*

Proof. The "usual" topology on $\mathscr{D}^{pq}(H,E)$ may be described as follows. Let $\{K_\nu\}$ be an increasing sequence of compact sets such that $\overline{K_\nu} \subset \overset{o}{K}_\nu + 1$ and $UK_\nu = K$. Further, let $\mathfrak{G} = \{U_i\}_{i\in I}$ be a locally finite covering of X by open sets such that each U_i is a relatively compact open subset of a coordinate open set V_i on X. Let $\mathfrak{G}_\nu = \{U_i | U_i \cap K_\nu \neq \phi\}$. We topologize each $\mathscr{D}^{pq}(K_\nu,E) = \{\varphi | \varphi \in \mathscr{D}^{pq}(X,E), \operatorname{supp}\varphi \subset_\nu\}$ as follows: for $U_i \in \mathfrak{G}_\nu, \varphi|U_i$ may be regarded as a C^∞ vector-valued function φ_i on U_i ; a fundamental system of neighbourhoods of zero in $\mathscr{D}^{pq}(K_\nu,E)$ is given by,

$$\left\{\varphi \in \mathscr{D}^{pq}(K_\nu,E)\Big|\left|^{\bar\alpha}\varphi_i\right| < \varepsilon_\alpha, \ \text{ for every }\ U_i \in \mathfrak{G}_\nu\right\}$$

where $\{\varepsilon_\alpha\}$ is an arbitrary family of positive reals.

$\mathscr{D}^{pq}(K_\nu,E)$ is a Frechet space. There is a natural injection

$$\mathscr{D}^{pq}(K_\nu,E) \to \mathscr{D}^{pq}(K_{\nu+1},E).$$

$\square$

42 The image of $\mathscr{D}^{pq}(K_\nu,E)$ is a closed subset of $\mathscr{D}^{pq}(K_{\nu+1},E)$. The induced topology on the image coincides with the topology of $\mathscr{D}^{pq}(K_\nu,E)$. This shows that $\mathscr{D}^{pq}(X,E)$ is a strict inductive limit of the Frechet spaces $\mathscr{D}^{pq}(K_\nu,E)$ [8] 66-67; [9] (225-227). The "usual" topology of $\mathscr{D}^{pq}(X,E)$ is the inductive limit topology.

A subset of $\mathscr{D}^{pq+1}(X,E)$ is closed if, and only if, it is sequentially closed ([19], 228). Therefore it will be sufficient to prove that $\bar\partial\mathscr{D}^{pq}$ (X,E) is sequentially closed in $\mathscr{D}^{pq+1}(X,E)$.

Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a sequence of (p,q)- forms of $\mathscr{D}^{pq}(X,E)$. Such that the sequence $\{\bar\partial\varphi_i\}_{i\in\mathbb{N}} \subset \bar\partial\mathscr{D}^{pq}(X,E)$ converges to an element φ of $\mathscr{D}^{pq+1}$ (X,E). The sequence $\{\xi\varphi_i\}$ is a bounded set in $\mathscr{D}^{pq+1}(X,E)$.

Hence there exists a compact K_s such that $\bar{\partial}\varphi_i \in \mathscr{D}^{pq+1}(K_s, E)$ for $i = 1, 2, \ldots$, and $\varphi \in \mathscr{D}^{pq+1}(K_s, E)$ ([8], 70; [19], 226).

Let $c_o = \sup \phi(x)$ on K_s. Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a non decreasing C^∞ function such that

$$\lambda(t) = 0 \text{ for } t \leqslant c_o,$$
$$\lambda(t) > 0 \text{ for } t > c_o,$$

and let us consider the spaces $\mathscr{L}^{pq+1}_\nu(X, E) = \mathscr{L}^{pq+1}_{\nu\lambda(\varphi)}(X, E)$ for $\nu = 1, 2, \ldots$.

Let $g \in \mathscr{L}^{pq+1}_\nu(X, E)$ be such that

$$\vartheta_\nu g = 0.$$

Then we have

$$(\varphi, g) = (\varphi, g)_\nu = \lim(\bar{\partial}\varphi_i, g)_\nu = \lim(\varphi_i, \vartheta_\nu g) = 0.$$

Thus by theorem 1.5 there exists a $\psi \in \mathscr{L}^{pq}(X, E)$ with compact support (since we have assumed c'_4), such that **43**

$$\varphi = \bar{\partial}\psi.$$

Now, since in addition φ is C^∞, ψ may be assumed to be a C^∞ form, i.e. $\psi \in \mathscr{D}^{pq}(X, E)$. Hence $\in \bar{\partial}\mathscr{D}^{pq}(X, E)$. This completes the proof of theorem 1.8.

Chapter 2

W-ellipticity on Riemannian manifolds

5 W-ellipticity on Riemannian manifolds

We carry over the results proved for $\overline{\partial}$ on a complex manifold to the **44**
operator d-exterior differentiation of ordinary forms. In the sequel until further notice we consider only real valued forms. Through all this chapter X will be a connected orientable Riemannian manifold. We choose an orientation for X, and we denote by $C^q = C^q(X)$ (resp. $\mathscr{D}^q(X) = \mathscr{D}^q$) the space of C^∞ q-forms (resp. C^∞ forms with compact support). We defined in Chapter 1 the operator

$$* : C^q \to C^{n-q}$$

and the scalar product,

$$(\varphi, \psi) = \int A(\varphi, \psi) dX = \int\limits_X \varphi \wedge *\psi$$

of two forms φ, ψ in $\mathscr{D}^q(X)$.

We define the operator

$$\delta : C^q \longrightarrow C^{q-1}$$

as the linear differential operator

$$\varphi \rightsquigarrow (-1)^q *^{-1} d(*\varphi)$$

($\varphi \in C^q(X)$). If $\varphi \in C^q(X), \psi \in C^{q+1}(X)$, and if $\operatorname{supp}\varphi \cap \operatorname{supp}\psi$ is compact, then $(d\varphi, \psi) = (\varphi, \delta\psi)$. We also introduce the Laplace operator

$$\triangle = d\delta + \delta d;$$

$\triangle$ is a strongly elliptic operator.

45 We have then

$$(\triangle\varphi, \psi) = (d\varphi, d\psi) + (\delta\varphi, \delta\psi)$$

for $\varphi, \psi \in \mathscr{D}^q(X)$.

On the space $\mathscr{D}^q(X)$, we define another scalar product: for $\varphi, \psi \in \mathscr{D}^q(X)$,

$$a(\varphi, \psi) = (\varphi, \psi) + (d\varphi, d\psi) + (\delta\varphi, \delta\psi).$$

Then $N(\varphi) = a(\varphi, \varphi)^{\frac{1}{2}}$ defines a norm on $\mathscr{D}^q(X)$. We denote the completion of $\mathscr{D}^q(X)$ under N by W^q. If $\mathcal{L}^q$ denotes the completion of $\mathscr{D}^q(X)$ with respect to the norm, $\| \varphi \|^2 = (\varphi, \varphi)$, then the identity extends canonically to a continuous map

$$i : W^q \longrightarrow \mathcal{L}^q.$$

We have then the following result.

Proposition 2.1. *i is an injection.*

The proof is analogous to that of proposition 1.1. As in theorem 1.1 (Chapter 1), we have the following characterisation of W^q (rather $i(W^q)$) when the Riemannian metric on X is complete.

Theorem 2.1. *If the Riemannian metric on X is complete, then*

$$W^q(X) = \{\varphi \mid \varphi \in \mathcal{L}^q(X), d\varphi \in \mathcal{L}^{q+1}(X),\ \delta\varphi \in \mathcal{L}^{q-1}(X)\}.$$

Moreover $\mathscr{D}^q(X)$ is dense in the space $\{\varphi \mid \varphi \in \mathcal{L}^q, d\varphi \in \mathcal{L}^{q-1}\}$, with respect to the norm $(\| \varphi \|^2 + \| d\varphi \|^2)^{\frac{1}{2}}$, and in the space $\{\varphi \mid \varphi \in \mathcal{L}^q,\ \delta\varphi \in \mathcal{L}^{q+1}\}$ with respect to the norm $(\| \varphi \|^2 + \| \delta\varphi \|^2)^{\frac{1}{2}}$.

Once again, we omit the proof which is analogous to that of Theorem 1.1 (Chapter 1). We state also, without proof, the analogue of Corollary 1 to Theorem 1.2 of Chapter 1.

Proposition 2.2. *For $\varphi \in C^q$ and for any $\sigma > 0$* 46

$$\| d\varphi \|^2 + \| \delta\varphi \|^2 \leqslant \sigma \| \Delta\varphi \|^2 + \frac{1}{\sigma} \| \varphi \|^2 .$$

Definition 2.1. *The Riemannian manifold X is W^q-elliptic if there is a constant $C > 0$ such that*

$$\| \varphi \|^2 \leqslant C \left\{ \| d\varphi \|^2 + \| \delta\varphi \|^2 \right\}.$$

We shall call such a constant C a W^q-ellipticity constant.

In further analogy to Chapter 1, we have finally the following

Theorem 2.2. *If X is W^q elliptic, then for $f \in \mathscr{L}^q$, there is a unique $x \in W^q$ such tht $f = \triangle x$ (in the sense of distributions). Further,*

$$\| x \| \leqslant c \| f \|$$
$$\| dx \|^2 + \| \delta x \|^2 \leqslant C \| f \|^2,$$

C being a W^q- ellipticity constant. If f is C^∞, then $x \in C^q \cap W^q$. If the metric on X is complete and $\| df \| < \infty$, then, for $\sigma > 0$,

$$\| \delta dx \|^2 \leq \frac{1}{\sigma} \| df \|^2 + \sigma \| dx \|^2 .$$

In particular, if $df=0$, $f=d\delta x$ and $dx=0$.

Except for obvious modifications, the proof is contained in Propositions 1.2 and 1.3 and Theorem 1.3 (Chapter 1).

Before we give criteria for W^{pq}-ellipticity, on a complex manifold, we give sufficient conditions for W^q-ellipticity, on an orientable Riemannian manifold X. In order to do this we first write down explicitly in 47
a coordinate open set the effect of the Laplacian Δ on a $C^\infty-p-$form. Let $(x, \ldots x^n)$ be a coordinate system on an open set U in X. Let the Riemannian metric be

$$\sum g_{ij}\, dx^i dx^j$$

in this open set. We denote $\triangledown$ the covariant derivation with respect to the Riemannian metric. For a tensor φ, $\triangledown_\alpha \varphi$ denotes the tensor

$$(\triangledown\varphi)(\frac{\partial}{\partial x^\alpha}).$$

($\nabla\varphi$ is regarded as a 1-form with values in the tensor bundle). In the local-coordinate system, a form $\varphi \in C^q(X)$ may be written in the form.

$$\varphi = \sum_1 \frac{1}{q!}\varphi_I dx^I$$

where I runs over all q-tuples $(i_1, \ldots, i_q)$ of integers with $0 < i_j \leqslant n$ and if $I = (i_1, \ldots, i_q)$ is a permutation $(\sigma(j_i), \ldots, \sigma(j_q))$ of $I' = (j_1, \ldots, j_q)$, then

$$\varphi_I = \varepsilon_\sigma \varphi_{I'}.$$

(In particular, if I has repeated indices, $\varphi_I = 0$). We have

$$A(\varphi, \psi) = \frac{1}{q!}\varphi_I \psi^I (\varphi, \psi \in C^q)$$

For $\varphi \in C^q$ and a q-tuple $I = (i_1, \ldots, i_q)$,

$$\begin{aligned} d\varphi_{iI} &= \frac{\partial \varphi_I}{\partial x^i} + \sum_r (-1)^r \frac{\partial \varphi_i}{\partial x^{i_r}} i_1, \ldots, i_r, \ldots i_q \\ &= \nabla_i \varphi_I + \sum_r (-1)^r \nabla_{ir} \varphi_{ii_1}, \ldots i_r \ldots, i_q \end{aligned} \tag{2.1}$$

48 *as is seen by a direct computation. Similarly we have*

$$\delta\varphi_J = -\Sigma \nabla_i \varphi^i_J \tag{2.2}$$

where J is a $(q-1)$-tuple and φ^i is the $(q-1)$-form defined by

$$\varphi^i_J = \sum g^{ij} \varphi_{iJ}$$

for any $(q-1)$-tuple $J = (j_1, \ldots j_{q-l})$.

From (2.1) and (2.2), it follows that

$$\begin{aligned} \delta d\varphi_I &= -\sum_{i=1}^n \nabla_i (d\varphi)^i_I \\ &= -\sum_{i=1}^n \nabla_i \nabla^i \varphi_I - \sum_{i=1}^q (-1)^r \nabla_i (\nabla_{ir} \varphi^i) i_i \ldots \hat{i}_r \ldots i_q \end{aligned} \tag{2.3}$$

(where $\nabla^i = \sum_j g^{ij}\nabla_j$*) and*

$$d\delta\varphi_I = \sum_{r=I}^{q}\sum_{i=I}^{n}(-1)^r\nabla_{i_r}\nabla_i\varphi^i_{i_1\ldots\hat{i}_r\ldots i_q} \tag{2.4}$$

Hence
$$(\nabla\delta)_I = -\sum\nabla_i\nabla^i\varphi_I + (\kappa\varphi)_I \tag{2.5}$$

In the equation (2.5)

$$\begin{aligned}(\kappa\varphi)_I &= \sum_i\sum_r(-1)^r(\nabla_{i_r}\nabla_i - \nabla_i\nabla_{i_r})\varphi^i_{i_1\ldots\hat{i}_r\ldots i_q}\\ &= \sum_{r=1}^{q}\sum_j(-1)^{r-1}R^j_{i_r}\varphi_{ji_1\ldots\hat{i}_r\ldots i_q} + \sum_{r,s=1}^{q}(-1)^{r+s}R^{ij}_{i_r i_s}\varphi_{i.j.i_{\cdot}.\hat{i}_r.\hat{i}_s}\end{aligned}$$

In the special case when φ *is a function,*

$$\nabla\varphi = -\nabla_i\nabla^i\varphi \tag{2.6}$$

Lemma 2.1. *For every* $\varphi \in \mathscr{D}^q$ *we have* **49**

$$\|\nabla\varphi\|^2 + (K\varphi,\varphi) = \|d\varphi\|^2 + \|\delta\varphi\|^2 \tag{2.7}$$

Proof. From (2.6) above, we have setting $|\varphi|^2 = A(\varphi,\varphi) = \frac{1}{\varphi}\varphi_I\varphi^I$,

$$\begin{aligned}\Delta(|\varphi|^2) &= -\sum\nabla_i\nabla^i|\varphi|^2 = -\sum\frac{1}{q!}\nabla_i\nabla^i\varphi_I\varphi^I\\ &= -\frac{2}{q!}\sum\nabla_i(\nabla^i\varphi_I.\varphi^I)\\ &= -\frac{-2}{q!}\sum\nabla_i\nabla^i\varphi_I.\varphi^I - 2|\nabla\varphi|^2\\ &= \sum_I\left\{\frac{2}{q!}\nabla\varphi_I\varphi^I - \frac{2}{q!}(K\varphi)_I\varphi^I\right\} - 2|\nabla\varphi|^2\end{aligned}$$

That is
$$\nabla|\varphi|^2 = 2A(\nabla\varphi,\varphi) - 2A(K\varphi,\varphi) - 2A(\nabla\varphi,\nabla\varphi)\ldots$$

If φ has compact support,

$$\int\nabla|\varphi|^2dx = \int d\delta|\varphi|^2dX = 0$$

by Stokes formula.

On the other hand,

$$\int A(\nabla\varphi, \varphi)dX = \| d\varphi \|^2 + \| \delta\varphi \|^2 .$$

Hence

$$(K\varphi, \varphi) + \| \nabla\varphi \|^2 = \| d\varphi \|^2 + \| \delta\varphi \|^2 : (\varphi \in \mathscr{D}^q(X)).$$

If there exists a positive constant C such that
$A(K\varphi, \varphi) \geq C|\varphi|^2$ for every $\varphi \in C^q$ and at each point of X, then

$$\|\varphi\|^2 \leq C'(K\varphi, \varphi) \leq C' \left\{\|d\varphi\|^2 + \|\delta\varphi\|^2\right\}$$

where $C' = \dfrac{1}{C}$. Hence the following. □

50 **Lemma 2.2.** *If there is a positive constant C such that*

$$A(K\varphi, \varphi) \geq C\, A(\varphi, \varphi)$$

for every $\varphi \in C^q$ and at each point of X, then the Riemannian manifold X is W^q-elliptic.

6 A maximum principle

In the sequel we consider weaker condition on the expression $A(\kappa\varphi, \varphi)$. We shall deal with the case where the quadratic from $A(\kappa\varphi, \varphi)$ is positive semi- definite outside of a compact.

From now on we shall always assume X to be oriented and connected, and the Riemannian metric of X to be complete.

Lemma 2.3. *Assume $A(\kappa\varphi, \varphi) \geq 0$ (i.e. $A(\kappa\varphi, \varphi)$ positive semi-definite), outside a compact set K in X. Then for $\varphi \in W^q$, $\| \nabla\varphi \|^2 < \infty$ and (2.7) holds.*

Proof. By Lemma 2.1 we have for $\varphi \in \mathcal{D}^q(X)$,

$$\| \nabla\varphi \|^2 + (\kappa\varphi, \varphi) = (d\varphi\ , d\varphi)\ + (\delta\varphi, \delta\varphi).$$

Since $\mathcal{K}$ is compact, there is $C \geq 0$ such that $(\mathcal{K}\varphi, \varphi) > -CC|\varphi|^{-2}$ for every $\varphi\epsilon C^q$ and each point of $\mathcal{K}$. Hence,

$$\| \nabla\varphi \|^2 + \int_X A(k\varphi, \varphi) \leq C \| \varphi \|_k^2 + \| d\varphi \|^2 + \| \delta\varphi \|^2\ .$$

Since $A(\kappa\varphi, \varphi) \geq 0$ on $X - \mathcal{K}$ we have for $\varphi \in \mathcal{D}^q$,

$$\| \nabla\varphi \|^2 \leq C \left\{ \| \varphi \|^2\ + \| d\varphi \|^2 + \| \delta\varphi \|^2 \right\}$$

The lemma follows now from the fact that $\mathcal{D}^q$ is dense in W^q. (Theorem 2.1.) □

Remark. We have proved more: there is a $C > 0$ such that 51

$$\|\nabla\varphi\|^2 \leqslant C \left\{ \|\varphi\|\ _K^2\ + \|d\varphi\|^2\ + \|\delta\varphi\|^2 \right\}$$

Let X be a manifold as in the above lemma.

Let $\varphi \in C^q$. Identity (2.8) shows that $\triangle|\varphi|^2 \leqslant 0$ at each point of the set Y where $\triangle\varphi = 0$ and $A(K\varphi, \varphi) \geqslant 0$

Applying a classical lemma of E. Hopf ([36], 26-30) we see that $|\varphi|^2$ cannot have a relative maximum at any point of Y. Thus, if X is compact, $|\varphi|^2$ takes its, maximum in the set Supp $f \cup K$.

If X is not compact we cannot draw the same conclusion. However the following proposition will provide an estimate of $|\varphi|^2$ on X in terms of Sup $|\varphi|^2$ on Supp $f \cup \mathcal{K}$.

Proposition 2.3. *Let X be a connected oriented and complete Riemannian manifold. Assume give a compact set $\mathcal{K}$ on X such that for $X \notin \mathcal{K}, A(\mathcal{K}\varphi, \varphi)(x) \geq 0$ for every $\varphi \in C^q$. Suppose that $\varphi \in C^q \cap \mathcal{L}^q$ is such that $\triangle\varphi\ =\ f \in \mathcal{L}^q$. Then $\varphi \in W^q$ and $\sup_{x \in X} |\varphi(x)|^2 \leqslant C_0$ where*

$$C_0\ \ =\ \ \operatorname*{Sup}_{x \in \text{supp } f \cup \mathcal{K}} |\varphi(x)|^2.$$

For the proof of this proposition we utilise the following result

Lemma 2.4 (Gaffney [12]). *Let $\mathcal{K}$ be an oriented complete Riemannian manifold and ω a C^∞ 1-form. Then if $\int_X |\omega| dX < \infty$ and $\int_X |\delta\omega| dX < \infty$, we have*

$$\int_X \delta\omega dX = 0.$$

52 *Proof.* Let λ be a real value C^∞ function on $\mathbb{R}$ such that $\lambda(t) = 1$ for $t \leqslant 0$ and $\lambda(t) = 0$ for $t \geqslant 1$. Let $p_0 \in X$ be a fixed point and $\rho = \rho(x)$ denote the distance function (from p_0). Then $\rho(x)$ is locally Lipchitz as was remarked earlier (Lemma B of Chapter 1). Let $0 < r < \mathcal{R}$ and let us consider the form

$$\lambda\left\{\frac{\rho(x) - r}{\mathcal{R} - r}\right\} \cdot \omega$$

This form is locally Lipchitz and has compact support in the ball

$$\{x|\rho(x) \leqslant R\}.$$

Hence by Stoke's formula (which is applicable to Lipchitz continuous form),

$$\int_X d * \lambda\left(\frac{\rho(x) - r}{R - r}\right) \cdot \omega = 0;$$

We have,

$$\int_X \frac{1}{R - r}\lambda\left(\frac{\rho(x) - r}{-r}\right) \cdot d\rho\Lambda * \omega + \int_X \lambda\left(\frac{\rho(x) - r}{R - r}\right) \cdot d * \omega = 0.$$

Now set $R = 2r$. Then

$$\int_X \frac{1}{r}\lambda'\left(\frac{\rho(x) - r}{r}\right) d\rho\Lambda * \omega + \int_X \lambda\left(\frac{\rho(x) - r}{r}\right) \cdot d * \omega = 0$$

Now $|d\rho|^2 \leqslant$ (lemma B of Chapter 1).

53 Hence, since $*\omega$ is integrable, we have

$$\int_X |d\rho\Lambda * \omega| dX < \infty.$$

$\square$

Hence, using the fact that λ' is bounded, we see that

$$\frac{1}{r}\int_X \lambda'\left(\frac{\rho(x)-r}{r}\right)d\rho\Lambda * \omega \to 0 \text{ as } \to \infty.$$

It follows that

$$r \overset{Lt}{\to} \infty \quad \int_X \lambda\left(\frac{\rho(x)-r}{r}\right)d * \omega = 0$$

On the other hand, since $\int |d * \omega| dX < \infty$ and $\lambda\left(\frac{\rho(x)-r}{r}\right)$ is bounded and tends to 1 as $r \to \infty$,

$$r \overset{Lt}{\to} \infty \quad \int \lambda\left(\frac{\rho(X)-r}{r}\right)d * \omega = \int d * \omega.$$

This proves the lemma.

Proof (of Proposition 2.3) Let C_0 and C_1 be two positive constants, $0 < C_0 < C_1$, and let $\lambda : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that

$$\begin{aligned}
&\lambda(t) \geqslant 0, \quad \lambda'(t) \geqslant 0, \quad \lambda''(t) \geqslant 0,\\
&\lambda(t) = 0, \quad \text{for } t \leqslant C_0\\
&\lambda'(t) > 0, \quad \text{for } t > C_0\\
&\lambda''(t) = 0, \quad \text{for } t < C_0 \text{ and for } t > C_1,
\end{aligned}$$

and such that $\lambda'(t)$ be bounded. Since $\lambda''(t)$ has compact support, it is bounded. We have then for φ as in the proposition,

$$\begin{aligned}
\Delta\lambda(|\varphi|^2) &= \delta d(\lambda(|\varphi|^2))\\
&= \delta\left\{\lambda'(|\varphi|^2).d|\varphi|^2\right\}\\
&= - * d * \left\{\lambda'(|\varphi|^2)d|\varphi|^2\right\}\\
&= - * d\left\{\lambda'(|\varphi|^2) * d|\varphi|^2\right\}\\
&= - * \left\{d\lambda'(|\varphi|^2) \wedge * d|\varphi|^2 + \lambda'(|\varphi|^2)d * d|\varphi|^2\right\}\\
&= - * \left\{\lambda''(|\varphi|^2)d|\varphi|^2 \wedge * d|\varphi|^2 + \lambda'|\varphi|^2 d * d|\varphi|^2\right\}\\
&= -\lambda''|\varphi|^2|d|\varphi|^2|^2 + \lambda'|\varphi|^2 .\Delta(|\varphi|^2).
\end{aligned}$$

In the view of (2.8), **54**

$\triangle(\lambda|\varphi|^2) = -\lambda''(|\varphi|^2)|d|\varphi|^2|^2 + 2\lambda'|\varphi|^2\{A(\triangle\varphi,\varphi) - A(\mathbb{K}\varphi,\varphi) - A(\nabla\varphi,\nabla\varphi)\}$

On the other hand by proposition 2.2 we have for $\varphi \in C^q$,

$$\|\partial\varphi\|^2 + \|\delta\varphi\|^2 \leqslant \|\Delta\varphi\|^2 + \|\varphi\|^2$$

In particular, if $\Delta\varphi \in \mathcal{L}^q$ and $\varphi \in C^q \cap \mathcal{L}^q$ then $\|d\varphi\|^2 < \infty$ as also $\|\delta\varphi\|^2 < \infty$. In the view of theorem 2.1 this proves that $\varphi \in W^q$.

Now

$$\int_x \Delta\lambda(|\varphi|^2)dX = \int_X *d * d\lambda(|\varphi|^2)$$

We shall show that these integrals vanish.

Using Schwartz inequality, one checks that there exists a positive constant C_2 such that

$$|d|\varphi|^2| \leqslant C_2|\varphi||\nabla\varphi| \text{ at each point of } X. \tag{2.10}$$

Hence, since $|\lambda'(t)|$ is bounded it follows that

$$\int_X |d\lambda(|\varphi|^2)|dX < \infty.$$

We will prove now that

$$\int_X |\Delta\lambda(|\varphi|^2)|dX < \infty.$$

Letting $C_3 = \text{Sup } \lambda''(t)$, we have, by (2.10),

$$\int_X \lambda''(|\varphi|^2)|d|\varphi|^2|^2 dX = \int_{C_\circ<|\varphi|^2<C_1}^{\lambda:(|\varphi|_2^2)|d|\varphi|^2|^2} dX \leqslant C_3 \int_{C_\circ<|\varphi|^2<C_1}^{|d|\varphi|^2|^2} |d|\varphi|^2|^2 dX$$
$$\leqslant C_2Cc_3 \int_{C_\circ<|\varphi|^2<C_1}^{|\varphi|^2|\nabla\varphi|^2} dX \leqslant C_1C_2C_3 \int_X |\nabla|^2 dX = C_1C_2C_3\|\nabla\varphi\|^2$$

55 Since $\varphi \in W^q$, then by Lemma 2.3 $\|\nabla\varphi\|^2 < \infty$, and therefore

$$\int_X \lambda''(|\varphi|^2)|d|\varphi|^2|^2 dX < \infty$$

Since K is compact, there is a constant $C_4 > 0$ such that

$$|A(\mathcal{K}u, u)| \leqslant C_4 A(u, u)$$

for any $u \in C^q$ and at each point of $\mathcal{K}$. Then we have

$$\begin{aligned}\int_X |A(K\varphi, \varphi)| dX &= \int_{X-\mathcal{K}} A(\mathcal{K}\varphi, \varphi) dX + \int_{\mathcal{K}} |A(\mathcal{K}\varphi, \varphi)| dX \\ &\leqslant \int_X A(\mathcal{K}\varphi, \varphi) dX + 2 \int_{\mathcal{K}} |A(\mathcal{K}\varphi, \varphi)| dX \\ &\leqslant (\mathcal{K}\varphi, \varphi) + 2C||\varphi||^2 \leqslant 2C||\varphi||^2 + ||d\varphi||^2 + ||\delta\varphi||^2 < \infty.\end{aligned}$$

Finally we have

$$\int_X |A(\triangle\varphi, \varphi)| dX \leq \frac{1}{2} \int_X A(\triangle\varphi, \triangle\varphi) dX + \frac{1}{2} \int_X A(\varphi, \varphi) dX < \infty.$$

Then, it follows from (2.9) and from the fact that $|\lambda'|$ and $|\lambda''|$ are bounded, that

$$\int_X |\triangle\lambda(|\varphi|^2)| dX < \infty.$$

Hence, by Lemma 2.4,

$$\int_X \triangle\lambda(|\varphi|^2) dX = 0, \textit{i.e. by} \tag{2.9}$$

$$\int_X \lambda''(|\varphi|^2)^2 (d|\varphi|^2)^2 dX + 2 \int_X \lambda'(|\varphi|^2).A(\mathcal{K}\varphi, \varphi) dX + 2 \int_X \lambda'(|\varphi|^2) |\nabla\varphi|^2 dX = \int_X 2\lambda'(|\varphi|^2) A(\triangle\varphi, \varphi) dX. \tag{2.11}$$

Let us consider now $\sup |\varphi|$ on Supp $f \cup k$. If it is not finite there is 56
nothing to prove. If it is finite, we set $C_0 = \operatorname{Sup} |\varphi|^2$ on $K \bigcup$ Support f. In view of our choice of the function λ all the terms on the left hand side of (2.11) are non negative. We obtain

$$0 \leqslant 2 \int_X \lambda'(|\varphi|^2) |\nabla\varphi|^2 dX \leqslant \int_X 2\lambda'(|\varphi|^2) A(\triangle\varphi, \varphi)$$

$$= 2\int_X \lambda'(|\varphi|^2).A(f,\varphi).$$

Once again since $|\varphi|^2 \leqslant C_0$ on support f, the integral on the right is zero. Hence

$$\int_X \lambda'(|\varphi|^2)|\nabla\varphi|^2 dX = 0$$

By consequences we have also $d|\varphi|^2 = 0$, i.e. $|\varphi|$ is a constant, on the open set $\{|\varphi|^2 > C_o\}$. Hence $|\varphi|^2 \leq C_o$ on X. This concludes the proof.

Corollary. *If $\varphi \in \mathcal{L}^q \cap C^q$ and $\triangle\varphi \in \mathscr{D}^q$ (in particular, if $\triangle\varphi = 0$), then $|\varphi|$ is bounded on X.*

7 Finite dimensionality of spaces of harmonic forms

Let γ be a real number and let $\mathfrak{M}^q_\gamma$ be the vector space

$$\mathfrak{M}^q_\gamma = \{\varphi|\varphi \in \mathcal{L}^q, \triangle\varphi = \gamma\varphi\}.$$

we set $\mathfrak{M}^q = \mathbb{H}^q$

Lemma 2.5. $\mathfrak{M}_\gamma \subset W^q$ *and* $\gamma \geqslant 0$.

Proof. If $\varphi \in \mathfrak{M}^q_\gamma$, then $\triangle\varphi \in \mathcal{L}^q$. Thus, by the first part of Proposition
57 2.3, $\varphi \in W^q$. Since φ is a solution of the elliptic equation $\Delta - \gamma$, we may assume $\varphi \in C^q$. We have moreover (since the metric is complete)

$$\begin{aligned}
||\delta d\varphi||^2 &\leqslant\ ||\Delta d\varphi||^2 + ||d\varphi||^2 \leqslant (\gamma^2+1)||d\varphi||^2,\\
||d\delta\varphi||^2 &\leqslant\ ||\Delta\delta\varphi||^2 + ||\delta\varphi||^2 \leqslant (\gamma^2+1)||\delta\varphi||^2.
\end{aligned}$$

Hence $d\varphi \in W^{q+1}, \delta\varphi \in W^{q-1}$ and therefore

$$\gamma||\varphi||^2 = (\Delta\varphi,\varphi) = ||d\varphi||^2 + ||\delta\varphi||^2 \tag{2.12}$$

This proves that $\gamma \geqslant 0$ □

It follows from (2.12) that, if $\gamma = 0$, i.e. if $\varphi \in \mathbb{H}^q$, then $d\varphi = 0$, $\delta\varphi = 0$.

Theorem 2.3. *Let X be a connected, oriented and complete Riemannian manifold and assume that $A(\mathcal{K}\varphi, \varphi) \geqslant C|\varphi|^2$ for all $\varphi \in C^q$ for some $C > 0$, outside a compact set $K \subset X$. Then for $\gamma < C$, $\mathfrak{M}^q_\gamma$ is finite dimensional. In particular dim $\mathbb{H}^q < \infty$.*

Proof. Let $\varphi \in \mathfrak{M}^q_\gamma$. Since $\varphi \in W^q$, then by Lemma 2.3

$$||\nabla\varphi||^2 + (\mathcal{K}\varphi, \varphi) = ||d\varphi||^2 + ||\delta\varphi||^2$$

and by (2.12)

$$||\nabla\varphi||^2 + (\mathcal{K}\varphi, \varphi) = ||d\varphi||^2 + ||||\delta\varphi||^2 = \gamma||\varphi||^2$$

Now $\mathcal{K}$ being compact, there is a constant $C_2 > 0$ such that $(\mathcal{K}u, u) \geqslant -C_2|u|^2$ for every $u \in C^q$ and at each point of $\mathcal{K}$. Since $(\mathcal{K}\varphi, \varphi) \geqslant C_2|\varphi|^2$ on the complement $X - \mathcal{K}$ we have

$$||\nabla\varphi||^2 + C||\varphi||^2_{X-\mathcal{K}} \leqslant C_2||\varphi||^2_{\mathcal{K}} + \gamma||\varphi||^2$$

or again,

$$||\nabla\varphi||^2 + C||\varphi||^2 \leqslant (C + C_2)||\varphi||^2_{\mathcal{K}} + \gamma||\varphi||^2$$

Hence, **58**

$$(C - \gamma)||\varphi||^2 \leqslant (C + C_2)||\varphi||^2_K \tag{2.13}$$

Suppose now that $\mathfrak{M}^q_\gamma$ is infinite dimensional for $\gamma < C$, then the evaluation map

$$\omega \rightsquigarrow \omega(x) \ (x \in K)$$

(which associates to a form ω, the element of the q^{th} exterior power of the tangent space at x which is defined by ω) is a linear map into a finite dimensional space. Hence the kernel of this map is of finite codimension. Since the finite intersection of subspaces of finite codimension is non-zero (over an infinite field !) it follows that given any sequence $x_1, \ldots, x_\nu, \ldots$ of points of K, we can find forms $\varphi_\in \in \mathfrak{M}^q_\gamma$ such that $\varphi_\nu(x_i) = 0$ for $i \leqslant \nu$. We may further assume that $||\varphi_\nu|| = 1$. It follows then that we can choose a subsequence $\psi_k = \varphi_{\nu_k}$ of $\{\varphi_\nu\}$ such that ψ_k converges weakly to a limit ψ in $\mathcal{L}^q$. Now this implies that $\psi \in \mathfrak{M}^q_\gamma$, since for any $\varphi \in \mathscr{D}^q$,

$$(\psi, \Delta\varphi) = \mathrm{Lt}(\psi_k, \Delta\varphi) = \mathrm{Lt}(\Delta\psi_k, \varphi) = \gamma(\psi, \varphi).$$

Hence we can assume $\psi \in C^q$. Now the ψ_k are solutions of the elliptic equation $\Delta - \gamma = 0$. Hence the weak convergence of ψ_k implies that ψ_k converge uniformly together with all partial derivatives on every compact set. Clearly we have $\mathrm{Lt}\,\psi_k(x_\nu) = 0$ for every ν. If we choose x_ν to be dense in K then $\psi \equiv$ on K since $\psi \in C^q$. Hence it follows that

$$|\psi_k| < \varepsilon \text{ on } K.$$

59 for any given $\varepsilon > 0$ and all large k. We conclude that

$$||\psi_k||_K^2 \longmapsto 0$$

as $k \to \infty$. On the other hand since $\psi_k \in \mathfrak{M}_\gamma^q$, we have by (2.13),

$$(C - \gamma)||\psi_k||^2 \leqslant (C + C_2)||\psi_K||_K^2.$$

Since $C - \gamma > 0$ and $||\psi_k||^2 = 1$, we arrive at contradiction. Hence $\mathfrak{M}_\gamma^q$ is finite dimensional for every $\gamma < C$. This proves Theorem 2.1 completely. □

Theorem 2.4. *Under the hypothesis of Theorem 2.3 we have the following. Let S^q denote the orthogonal complement of $\mathbb{H}^q$ in the Hilbert space W^q. Then there is a $C > 0$ such that*

$$||\varphi||^2 \leqslant C\{||d\varphi||^2 + ||\delta\varphi||^2\} \textit{ for } \forall\varphi \in W^q$$

For the proof of the theorem we need the following result due to Rellich.

Lemma 2.6. *Let X be a Riemannian manifold and $\Omega \subset\subset X$ Let $\varphi_\nu \in \mathcal{L}^q(\Omega)$ be a sequence such that*

$$||\varphi_\nu||^2 < \infty \textit{ and } ||\nabla\varphi_\nu||_\Omega^2 < \infty$$

(We denote $\int_\Omega |\nabla\varphi_\nu|^2 dX$ by $||\nabla\varphi_\nu||_\Omega^2$). Suppose that $||\varphi_\nu||_\Omega^2 + ||\nabla\varphi_\nu||_\Omega^2 \leqslant M$ for a fixed constant $M > 0$. Then if $\partial\Omega$ is smooth, we can find a subsequence $\psi_k = \varphi_{\nu_k}$ of φ_ν such that $||\psi_k - \psi_{k'}||_\Omega \to 0$ as $k, k' \to \infty$.

60 If $\varphi_\nu \epsilon \mathcal{L}_2^q(X)$ is a sequence such that $\|\varphi_\nu\|^2 + \|\nabla\varphi_\nu\|^2 \leq M$ for a fixed constant $M > 0$, then we can find a subsequence $\psi_k = \varphi_{\nu_k}$ such that for each compact $K' \subset X$,

$$\|\psi_k - \psi_{k'}\|_K \to 0 \text{ as } k, k' \to \infty.$$

Proof. The set Ω being relatively compact, it is sufficient to prove the lemma in the case $q = 0$, i.e. in the case of L_2 functions. For this proof see e.g. [7], 339. $\square$

Proof of Theorem 2.4. Suppose that the theorem is false. Then we can find a sequence $\varphi_\nu \epsilon W^q$ such that

$$\| d\varphi_\nu \|^2 + \| \delta\varphi_\nu \|^2 + \|\varphi_\nu\|^2 = 1$$

while $\|d\varphi_\nu\| + \|\delta\varphi_\nu\|^2 \leq \frac{1}{\nu}\|\varphi_\nu\|^2$. In view of Remark under Lemma 2.3 this implies that

$$\|\nabla\varphi_\nu\|^2 \leq C_1$$

for some $C_1 > 0$. By Rellich's Lemma (Lemma 2.6) we can find a $\varphi\epsilon\mathcal{L}^q$ (by passing of a subsequence if necessary) such that

$$\|\varphi_\nu - \varphi\|_\Omega \to 0$$

for every relatively compact $\Omega \subset X$.

In particular, we have

$$\|\varphi_\nu - \varphi\|_K^2 \to 0.$$

We assert that φ is harmonic, that is $\triangle\varphi = 0$. In fact for $u \in \mathcal{D}^q$, we **61**
have

$$\begin{aligned}
(\varphi, \triangle u) &= \mathrm{Lt}(\varphi_\nu, \triangle u) \\
&= \mathrm{Lt}(d\varphi_\nu, du) + (\delta\varphi_\nu, \delta u) \\
&\leq \mathrm{Lt}(\|d\varphi_\nu\| \, \|du\| + \|\delta\varphi_\nu\|.\|\delta u\|) \\
&\leq \lim \frac{1}{\sqrt{2}}\|\varphi_\nu\|(\|du\| + +\|\delta u\|) = 0
\end{aligned}$$

since $\|\varphi_\nu\|^2$ is bounded. Hence $\varphi\epsilon W^q \cap \mathbb{H}^q$.

Now we have for $f\epsilon W^q$,

$$\|\nabla f\|^2 + (\mathcal{K}f, f) = \|\mathrm{df}\|^2 + \|\delta f\|^2.$$

We have $(\mathcal{K}f, f) \geq C|f|^2$ on $X - K$. On the other hand there is a $C_2 \geq 0$ such that $(\mathcal{K}f, f) > -C_2|f|^2$ on K. Since $||\triangledown f||^2 \geq o$, we obtain,

$$C||f||^2 \leq (C + C_2)||f||_k^2 + ||df||^2 + ||\delta f||^2.$$

Setting $f = \varphi - \varphi_\nu$, we obtain,

$$C||\varphi - \varphi_\nu||^2 \leq (C + C_2)||\varphi - \varphi_\nu||_k^2 + ||d(\varphi - \varphi_\nu)||^2 + ||\delta(\varphi - \varphi_\nu)||^2.$$

Now $\varphi \epsilon \mathbb{H}^q \cap C^q$ so that

$$(d\varphi, d\varphi) + (\delta\varphi, \delta\varphi) = (\triangle \varphi, \varphi) = 0.$$

Hence

$$\begin{aligned} C||\varphi - \varphi_\nu||^2 &\leq (C + C_2)||\varphi - \varphi_\nu||_k^2 + ||d\varphi_\nu||^2 + ||\delta\varphi_\nu||^2 \\ &\leq (C + C_2)||\varphi - \varphi_\nu||_k^2 + \frac{1}{\nu}||\varphi_\nu||^2. \end{aligned}$$

62 It follows that $\varphi_\nu \to \varphi$ in $\mathcal{L}^q$ on the whole of X. On the other hand since

$$||d\varphi_\nu||^2 + ||\delta\varphi_\nu||^2 \leq \frac{1}{\nu}||\varphi_\nu||^2$$

$d\varphi_\nu$ and $\delta\varphi_\nu$ converge to zero. Hence φ_ν converges in W^q to a limit φ satisfying $darphi = 0$, $\delta\varphi = 0$, i.e. $\varphi \epsilon \mathbb{H}^q$. But $\varphi_\nu \epsilon S^q$ and S^q is a closed subspace. Hence $\varphi \epsilon S^q$ and therefore $\varphi = 0$. But then $\varphi_\nu \to 0$ in W^q, and this is absurd, since $||\varphi_\nu||^2 + ||d\varphi_\nu||^2 + ||\delta\varphi_\nu||^2 = 1$. It follows that there exists $C' > 0$ such that for $\varphi \epsilon S^q$

$$C^{'}\{||\mathrm{d}\varphi||^2 + ||\delta\varphi||^2\} \geq ||\varphi||^2$$

Remark. Since for $\varphi \epsilon \mathbb{H}^q$, $d\varphi = 0$, $\delta\varphi = 0$, S^q is also the orthogonal complement of $\mathbb{H}^q$ with respect to the scalar product on W^q induced by $\mathcal{L}^q$.

The hypothesis of Theorem 2.3 can be weakened, further the space $\mathbb{H}^q$.

Theorem 2.5. *Let X be an oriented, connected and complete Riemannian manifold and let $A(\mathcal{K}\varphi,\varphi) \geq 0$ outside a compact set $\mathcal{K}$ for every $\varphi \in C^q$. Then* $\dim \mathbb{H}^q < \infty$.

Proof. If $\dim \mathbb{H}^q = \infty$, then given any dense sequence $\{x_\nu\}$ of points $x_\nu \epsilon X$, we can find a sequence $\{\varphi_\nu\}$ of forms $\varphi_\nu \epsilon \mathbb{H}^q$ such that

$$\varphi_\nu(x_i) = 0 \text{ for i } \leq \nu, \|\varphi_\nu\|^2 = 1.$$

We assert now that $\|\varphi_\nu\|_\Omega > 0$ for any relatively compact open subset $\Omega \supset K$. In fact, we have

$$\|\nabla\varphi_\nu\|^2 + (\mathcal{K}\varphi_\nu\varphi_\nu) = \|d\varphi_\nu\|^2 + \|\delta\varphi_\nu\|^2 = o \text{ since } \varphi_\nu \epsilon \mathbb{H}^q.$$

Hence from the Remark under Lemma 2.3 it follows that there is a constant $C > 0$ such that **63**

$$\| \nabla\varphi_\nu \|^2 < C \| \varphi_\nu \|_K^2 \tag{2.14}$$

Hence if $\| \varphi_\nu \|_\Omega = 0$, then $\| \varphi_\nu \|_K^2 = 0$ so that $\nabla\varphi_\nu = 0$. But in the case, $| \varphi_\nu |^2$ is a constant. Hence $\| \varphi_\nu \|_\Omega > 0$, a contradiction.

We may therefore assume that for a fixed $\Omega \supset K$ $\| \varphi_\nu \|_\Omega = 1$. From (2.14),$\| \varphi_\nu \|_\Omega^2 + \| \nabla\varphi_\nu \|_\Omega^2 < C'$ for some $C' > 0$. Hence by Rellich's Lemma (Lemma 2.6), we can, by passing to a subsequence, if necessary, assume that $\| \varphi_\nu - \varphi_\mu \|_\Omega^2 \to 0$ as $\nu, \mu \to \infty$ provided that $\partial\Omega$ is smooth. Hence $\{\varphi_\nu\}$ converges to a limit φ in $\mathcal{L}^q(\Omega)$.

But since $\{\varphi_\nu\}$ is a sequence of solutions of an elliptic operator, φ_ν converges to φ uniformly on every compact subset of Ω. Since $\varphi_\nu(x_i) = 0$ for $i \leqslant \nu$, we see that $\varphi \equiv 0$ on Ω. Hence $\|\varphi_\nu\|_\Omega \to 0$ as $\nu \to \infty$, a contradiction. It follows that $\mathbb{H}^q$ is finite dimensional. Hence the theorem. An estimate for the dimension of $\mathbb{H}^q$ is provided by the following. □

Proposition 2.3. *Assume that, under the same hypotheses for X, $\mathcal{K} \equiv 0$ on X. Then for $\varphi \epsilon \mathbb{H}^q$, $\nabla\varphi = 0$. Hence,* $\dim \mathbb{H}^q \leqslant \binom{n}{q}$. *Moreover, if* $\dim \mathbb{H}^q > 0$*, then vol $X < \infty$.*

Proof. Since now 64

$$\| \nabla\varphi \|^2 = \|d\varphi\|^2 + \|\delta\varphi\|^2 = 0 \text{ for } \varphi\epsilon\mathbb{H}^q$$

we have $\nabla\varphi = 0$ for $\varphi\epsilon\mathbb{H}^q$. A form invariant under parallel translation is determined by its value at one point. Hence dim $\mathbb{H}^q \leqslant \binom{n}{q}$. If $\varphi\epsilon\mathbb{H}^q$ is non-zero, then $\|\varphi\|^2 < \infty$; on the other hand since $\nabla\varphi = 0$, $|\varphi|^2$ is a constant and the last assertion follows. □

Lemma 2.7. *Under the hypotheses of Theorems 2.3 and 2.4, for every $s\epsilon S^q$ there exists a unique $x\epsilon S^q$ such that*

$$s = \triangle x$$

in the sense of distributions. Moreover

$$\| x \| \leqslant C' \| s \|,$$
$$\|dx|^2 + \|\delta x\|^2 \leq C' \| s \|^2$$

C' being the constant which appears in Theorem 2.4. If $s\,\epsilon C^q \cap S^q$, then (x can be modified on a null set in sch a way that) $x\epsilon C^q \cap S^q$ and the equation $\triangle x = s$ holds in the ordinary sense.

Proof. S^q as a (closed) subspace of the Hilbert space W^q is a Hilbert space. By theorem 2.4 the norm N is equivalent on S^q to the norm $(\|d\|^2 + \|\delta\|^2)^{\frac{1}{2}}$. $f \rightsquigarrow (s, f)$ is a continuous linear form on S^q. Thus there is a unique $x \in S^q$ such that

$$(s, f) = (dx, df) + (\delta x, \delta f) \text{for all } f\epsilon S^q.$$

Let now φ be any element of W^q and let h and ψ be its orthogonal projections into $\mathbb{H}^q$ and S^q. They are uniquely defined, and furthermore

$$\varphi = h + \psi$$

65 Since h is orthogonal to s and $dh = 0$, $\delta h = 0$, then

$$(s, \varphi) = (s, \psi) = (dx, d\psi) + (\delta x, \delta\psi) = (dx, d\varphi) + (\delta x, \delta\varphi).$$

This proves the lemma. □

Since

$$W^q = \mathbb{H}^q \oplus S^q$$

we can state the following.

Proposition 2.4. *Under the hypotheses of Theorem 2.3, for every $\varphi \epsilon W^q$ there exists a unique $h \epsilon \mathbb{H}^q$ and a unique $x \epsilon S^q$ such that*

$$\varphi = h + \triangle x$$

(in the sense of distributions). Moreover

$$||x|| \leqslant C'||\varphi||$$
$$||dx||^2 + ||\delta x||^2 \leqslant C'||\varphi||^2$$

8 Orthogonal decomposition in $\mathcal{L}^q$

Let X be an oriented, complete and connected Riemannian manifold. Assume that there exists a compact set $K \subset X$ and a positive constant C such that

$$A(\mathcal{K}\varphi, \varphi) \geqslant CA(\varphi, \varphi)$$

outside K, for each $\varphi \epsilon C^q$.

The space $\mathcal{L}^q$ can be decomposed as direct sum of orthogonal subspaces

$$\mathcal{L}^q = \mathbb{H}^q \oplus \overline{d\mathscr{D}^{q-1}} \oplus \overline{\delta\mathscr{D}^{q+1}},$$

where $\overline{d\mathscr{D}^{q-1}}$ and $\overline{\delta\mathscr{D}^{q+1}}$ are the closure of d $\mathscr{D}^{q-1}$ and $\mathcal{L}\mathcal{D}^{q+1}$ with **66**
respect to the norm $||\quad||$.

Let $\varphi \in \mathcal{L}^q$; - if $d\varphi = 0$, then the orthogonal projection of φ into $\overline{\delta\mathcal{D}^{q+1}}$ is zero; if $\delta\varphi = 0$, then the orthogonal projection into $\overline{d\mathcal{D}^{q-1}}$ is zero (see [17] 602-605; [29], 165).

Lemma 2.8. *For every $\phi \in \overline{d\mathcal{D}^{q-1}}$ there is a unique form $x \in S^q$ such that*

$$\phi = d\delta x, \quad dx = 0.$$

Moreover

$$\| x \| \leq C' \| \phi \|,$$
$$\| \delta x \|^2 \leq C' \| \phi \|^2$$

(C' being the constant introduced in Theorem 2.4)

Proof.

$$s \rightsquigarrow (\phi, s)$$

is a continuous form on S^q. Then there exists a unique $x \in S^q$ such that

$$(\phi, s) = (dx, ds) + (\delta x, \delta s) \text{ for all } s \in S^q.$$

Moreover, by Theorem 2.4,

$$\| x \|^2 \leq C'(\| dx \|^2 + \| \delta x \|^2) = C'(\phi, x) \leq C' \| \phi \| . \| x \|,$$
$$\| dx \|^2 + \| \delta x \|^2 = (\phi, x) \leq \| \phi \| . \| x \| \leq C' \| \phi \|^2 .$$

Let now $u \in \mathcal{D}^q \cdot u$ can be written

$$u = h + s, \quad h \in \mathbb{H}^q, \quad s \in S^q.$$

67 Since $h \perp \overline{d\mathcal{D}^{q-1}}$ in $\mathcal{L}^q$ and $dh = 0$, $\delta h = 0$, then $(\varphi, u) = (\varphi, s) = (dx, ds) + (\delta x, \delta s) = (dx, du) + (\delta x, \delta u) = (x, \triangle u)$ i.e.

$$\varphi = \triangle x$$

in the sense of distributions. $\square$

Let $\phi \in C^q \cap \overline{d\mathcal{D}^{q-1}}$. By the regularity theorem, we can modify x on a null set in such a way that $x \in C^q \cap S^q$, and that the latter equation holds in the ordinary sense. Thus, by proposition 2.2

$$\| \delta dx \|^2 \leq \frac{1}{\sigma} \| d\phi \|^2 + \sigma \| dx \|^2 = \sigma \| dx \|^2 \quad \text{for all } \sigma > 0.$$

Hence

$$\delta dx = 0, \quad \varphi = d\delta x.$$

But then, by Theorem 2.1, $\delta x \in W^{q-1}$, and therefore

$$\| dx \|^2 = (x, \delta dx) = 0 \quad \text{i.e.} \quad dx = 0.$$

If φ is not C^∞, there exists a sequence $\{u_\nu\}$ of forms $u_\nu \in \mathscr{D}^{q-1}$ such that $\| du_\nu - \varphi \| \longrightarrow 0$. Clearly $du_\nu \perp \mathbb{H}^q$; hence $du_\nu \in S^q$. Setting $\varphi_\nu = du_\nu$ and applying to φ_ν the above argument, we can find, for each φ_ν, a unique $x_\nu \in S^q \cap C^q$ such that

$$\varphi_\nu = d\delta x_\nu, dx_\nu = 0$$
$$\| x_\nu \| \leq C' \| \varphi_\nu \|,$$
$$\| \delta x_\nu \|^2 \leq C' \| \varphi_\nu \|^2$$

Hence $\{x_\nu\}$ is a Cauchy sequence in S^q. Let 68

$$x = \lim \ x_\nu.$$

Then

$$dx = 0,$$
$$\| x \| \leq C' \| \varphi \|,$$
$$\| \delta x \|^2 \leq C' \| \varphi \|^2,$$

and finally

$$\varphi = d\delta x,$$

in the sense of distributions. This proves the lemma.

An analogous argument yields

Lemma 2.9. *For any $\psi \in \overline{\delta \mathscr{D}^{q+1}}$ there is a unique form $y \in S^q$ such that*

$$\psi = \delta dy \quad \delta y = 0$$

Moreover

$$\| y \| \leq C' \| \psi \|$$
$$\| dy \|^2 \leq C' \| \psi \|^2$$

Theorem 2.6. *Every current $f \in \mathcal{L}^q$ can be decomposed as the sum of three currents*

$$f = h + d\varphi_1 + \delta\psi_1$$

with $h \in \mathbb{H}^q, \varphi_1 \in W^{q-1}, \psi_1 \in W^{q+1}$. Moreover

$$N(h)^2 = \| h \|^2 \leq \| f \|^2, N(\varphi_1)^2 \leq (C'+1) \| \varphi \|^2, N(\psi_1)^2 \leq (C'+1) \| \psi \|^2$$

Proof. First of all any element $f \in \mathcal{L}^q$ can be expressed in a unique way as the sum

$$f = h + \varphi + \psi$$

69 of three forms $h \in \mathbb{H}^q$, $\varphi \epsilon \overline{d\mathscr{D}^{q-1}}$, $\psi \epsilon \overline{\delta\mathscr{D}^{q+1}}$

$$\|f\|^2 = \|h\|^2 + \|\varphi\|^2 + \|\psi\|^2.$$

Next we apply to φ and ψ Lemma 2.8 and 2.9 setting then

$$\varphi = \delta\varphi_1, \quad \psi = d\psi_1;$$
$$\varphi_1 \in W^{q-1} \quad \psi_1 \in W^{q+1}.$$

Moreover

$$N(\varphi_1)^2 = \|\varphi_1\|^2 + \|\varphi\|^2 \leq (C' + 1)\|\varphi\|^2 \leq (C' + 1)\|f\|^2$$
$$N(\psi_1)^2 = \|\psi_1\|^2 + \|\psi\|^2 \leq (C' + 1)\|\psi\|^2 \leq (C' + 1)\|f\|^2$$

□

Remark. If the hypotheses of Theorem 2.3 are satisfied, not only for the forms of degree q, but also for those degree $q - 1$ and $q + 1$ then spaces $\mathbb{H}^{q-1}$, $\mathbb{H}^{q+1}$, S^{q-1} and S^{q+1} can be introduced and one checks that

$$\varphi_1 \perp \mathbb{H}^{q-1}, \qquad \text{i.e.} \qquad \varphi_1 \in S^{q-1}$$
$$\varphi_1 \perp \mathbb{H}^{q+1}, \qquad \text{i.e.} \qquad \psi_1 \in S^{q+1}.$$

Chapter 3

Local expressions for $\square$ and the main inequality

9 Metrics and connections

We now go back to the case of holomorphic vector bundles on a complex 70
manifold.

Let x be a complex manifold and $\pi : E \to X$ a holomorphic vector bundle. Let $\mathcal{U} = \{U_i\}_{i \epsilon I}$ be a covering of X such that $\pi : E \to X$ is defined with respect to $\mathcal{U}$ by transition functions $e_{ij} : U_i \cap U_j \to GL$ $(m, \mathbb{C})$ (E is of rank m).

Let $(h_i)_{i \epsilon I}$ be a hermitian metric along the fibres of E: then h_i are C^∞ functions U_i whose values are positive definite hermitian matrix such that.

$$h_i = {}^t\bar{e}_{ji} h_j e_{ji} = {}^t\overline{e}_{ij}^{-1} h_j e_{ij}^{-1} \text{ on } U_i \cap U_j.$$

Consider the 1-form $l_i = h_i^{-1} \partial h_i$ where ∂ is the exterior differentiation with respect holomorphic coordinates. We have for this family of forms the following.

Lemma 3.1. *If $l_i = h_i^{-1} \partial h_i$, then l_i are $(1, 0)$ forms with values in $M_m(\mathbb{C})$ and on $U_i \cap U_j$ we, have*

$$l_i = e_{ji}^{-1} l_j e_{ji} + e_{ji}^{-1} \partial e_{ji}.$$

71 In other words l_i are the local representations of a connection of type $(1,0)$ on the principal bundle associated to E.

Proof. We have

$$e_{ji}^{-1}\; l_j\; e_{ji} + \; e_{ji}^{-1}\,\partial e_{ji} \; = e_{ji}^{-1}\;\partial h_j e_{ji} \; + e_{ji}^{-1}\;\partial e_{ji}$$

Now $h_i = {}^t\overline{e}_{ji}\; h_j\; e_{ji}$, so that

$$h_i^{-1}\partial h_i = e_{ji}^{-1}\; h_j^{-1}\; {}^t\overline{e}_{ji}^{-1} \left\{\partial {}^t\overline{e}_{ji}\cdot\; h_j e_{ji} \; + {}^t\overline{e}_{ji}\;\partial h_j e_{ji} \; + {}^t\overline{e}_{ji}\; h_j \partial e_{ji}\right\}.$$

Since the e_{ij} are holomorphic functions,

$$\partial {}^t\overline{e}_{ji} \; = 0.$$

Hence $\quad h_i^{-1}\partial h_i = e_{ji}^{-1}\; h_j^{-1}\;\partial h_j e_{ji} + e_{ji}^{-1}\partial e_{ij}.$ □

The principal bundles $\widehat{\omega} : P \longrightarrow X$ associated to E is defined, with respect to the covering $\mathcal{U} = \{U_i\}$, by the transition function $\{e_{ij}\}$ acting on the fibre $GL(m,\mathbb{C})$ as follows:

$$Z_i = e_{ij} Z_j \qquad (Z_i, Z_j \in GL(m,\mathbb{C})).$$

The computations above shows that

$$Z_i^{-1}(l_i Z_i + dZ_i) = Z_j^{-1}(l_j Z_j + dZ_j) \;\text{ on }\; \widehat{\omega}^{-1}(U_i \cap \widehat{\omega}(U_j)$$

Hence $\omega_i = Z_i^{-1}(l_i Z_i + dZ_i)$ is the local representation on $\widehat{\omega}^{-1}(U_i)$ of a global 1-form ω with values in the Lie algebra of $GL(m,\mathbb{C})$.

72 Let P_ξ be the tangent space to P at a point $\xi \in P$. Let $Q_\xi \subset P_\xi$ be the subspace annihilated by ω. The family of these subspaces defines a connection in P [26]. This proves the lemma.

The covariant derivation associated to the connection defined above is a map

$$\nabla : \Gamma(X, \mathcal{A}(E)) \to \Gamma(X, \mathcal{A}(E \otimes \Theta^*)$$

where for a vector-bundle F, $\mathcal{A}(F)$ denotes the sheaf of germs of differentiable sections of F, and Θ^* is the bundle of 1-forms with value in $\mathbb{C}$.

(In the above we regard E as a differentiable vector bundle). $\triangledown$ is defined in an open set $\{U_i\}$ of the covering $\mathcal{U}$ as follows.

Let $z^1, \ldots, z^n$ be a system of local coordinates on U_i. Since we are given a local trivialisation over U_i of E, a section σ over U_i of E may be regarded as function on X with values in $\mathbb{C}^m$ (rank $E = m$). We then have

$$\triangledown_\alpha(\sigma) = \triangledown(\sigma)\left(\frac{\partial}{\partial z^\alpha}\right) = \partial\sigma\left(\frac{\partial}{\partial z^\alpha}\right) + l_i\left(\frac{\partial}{\partial z^\alpha}\right)(\sigma)$$

and
$$\triangledown_{\overline{\alpha}}(\sigma) = \triangledown(\sigma)\left(\frac{\partial}{\partial \overline{z}^\alpha}\right) = \overline{\partial}\sigma\left(\frac{\partial}{\partial z^\alpha}\right).$$

It is easy to check that these local representation defined a global map $\triangledown$ as above.

We consider next the curvature form of the connection.

Let Ω be the curvature form of the connection form ω in the principal bundle P associated to E. The values of Ω on a pair, u_1, u_2, of tangent vector fields to P is given by the structure formula ([26], 34-35): 73

$$\Omega(u_1, u_2) = d\omega(u_1, u_2) + \frac{1}{2}[\omega(u_1), \omega(u_2)].$$

Hence the component of type (1,1) of Ω is given on $\overline{\omega}^{-1}(U_i)$ by

$$\overline{\partial}\omega_i = Z_i^{-1} s_i Z_i,$$

where
$$s_i = \overline{\partial} l_i.$$

We shall call s the curvature form of the (connection $\{l_i\}$ defined by the) metric $\{h_i\}$ along the fibers of E. On $U_i \cap U_j$

$$s_i = e_{ij}\, s_j\, e_{ji}.$$

Hence we have

Lemma 3.2. *The curvature form of the connection defined above is given in U_i by the matrix valued 2-form*

$$(s_t^a)_i = s_i = \overline{\partial} l_i$$

Hence it is a form of type (1,1) with values in the "adjoint" bundle End (E).

The bundle Θ^* decomposes canonically as a direct sum

$$\Theta^* \simeq \Theta^*_\circ \oplus \overline{\Theta}^*_\circ$$

74 where $\Theta_\circ$ (resp. $\overline{\Theta}_\circ$) is the holomorphic tangent bundle (resp. anti holomorphic tangent bundle) of X. (The two bundles in the above decomposition are regarded as differentiable bundles. As a differentiable bundle $\Theta^*_\circ$ (resp. $(\overline{\Theta}^*_\circ)$, is simply the bundle of $C^\infty-$ forms on X of type (1,0) (resp. type (0,1)). The decomposition above gives a direct-sum representations

$$\mathcal{A}(\Theta^*) \simeq \mathcal{A}(\Theta^*_\circ) \oplus \mathcal{A}(\overline{\Theta}^*_\circ),$$

and for any holomorphic vector bundle E on X,

$$\Gamma(X, \mathcal{A}(E \otimes \Theta^*)) \simeq \Gamma(X, \mathcal{A}(E \otimes \Theta^*_\circ)) \oplus \Gamma(X, \mathcal{A}(E \otimes \Theta^*_\circ)).$$

Now if we are given a connection on E, it defines as we have remarked a covariant derivation

$$\nabla : \Gamma(X, \mathcal{A}(E)) \to \Gamma(X, \mathcal{A}(E \otimes \Theta^*));$$

composing with the natural projections defined by the direct sum decomposition above we obtain maps

$$\nabla' : \Gamma(X, \mathcal{A}(E)) \to \Gamma(X, \mathcal{A}(E \otimes \Theta^*_\circ))$$

and $$\nabla'' : \Gamma(X, \mathcal{A}(E)) \to \Gamma(X, \mathcal{A}(E \otimes \overline{\Theta}^*_\circ)).$$

Remarks. **(1)** As it has been remarked at the end of §3, the metric $\{h_i\}$ on E induces a metric on the dual E^*: this is simply given by the family of positive definite matrix-valued functions $\left\{{}^t h_i^{-1}\right\}_{i \in I}$. The
75 corresponding connection is given locally on U_i by the form $-{}^t l_i$. It is also easy to see that the curvature form of this connection is given by $\{-{}^t s_i\}_{i \in I}$

(2) Let $\overline{E}$ denote the "anti holomorphic" bundle associated to E. $\overline{E}$ is the bundle with transition functions $\overline{e}_{ij}$. Then ${}^t h_i = \overline{h}_i$ define a metric on $\overline{E}$ as well. In this case, the local forms

$$\overline{l_i} = \overline{h}_i^{-1} \ \overline{\partial} \ \overline{h}_i$$

define a connection of type (0,1). When we speak of a connection on $\overline{E}$ without further comment, it will always be of this connection (of course, it is necessary to assume given a metric or at least a connection on E.)

(3) Suppose that E and F are holomorphic vector bundles on X and that we given hermitian metrics $\{h_i^1\}_{i\in I}$ and $\{h_i^2\}_{i\in I}$ on E and F respectively. (We assume here, as we may, that E and F are defined by means of transition function $\{e_{ij}\}\{f_{ij}\}$ with respect to the same covering $\mathcal{U} = \{U_i\}_{i\in I}$). Then the family of matrix valued functions $\{h_i^1 \otimes h_i^2\}$ defines a metric $\{h_i\}$ along the fibres of $E \otimes F$. We have then

$$\partial(h_i) = \partial(h_i^1 \otimes h_i^2) = \partial h_i^1 \otimes h_i^2 + h_i^1 \otimes \partial h_i^2$$

so that $$l_i = h_i^{-1}\partial h_i = h_i^{-1}\partial h_i^1 \otimes I_{r_2} + I_{r_1} \otimes h_i^{2^{-1}}\partial h_i^2$$

where r_1 (resp. r_2) is the rank of E (resp. rank F) and for an integer m, I_m denotes the $(m \times m)$ identity matrix. One sees immediately then that the curvature form s_i is given by the formula **76**

$$s_i = \overline{\partial} l_i = s_i^1 \otimes I_{r_2} + I_{r_1} \otimes s_i^2.$$

It is clear that the above considerations can be carried over to the case when one or both of E and F are anti holomorphic.

In the sequel, we call a connection on a (differential) vector bundle (with complex fibres) a $\partial-$connection (resp. $\overline{\partial}$- connection) if the associated connection form is of type (1,0) (resp. type (0,1)). The connection l_i on a holomorphic bundle E defined with respect to a metric h_i is then a $\partial-$ connection while that on the conjugate bundle $\overline{E}$ is a $\overline{\partial}-$ connection.

Now if $\{h_i\}_{i\in I}$ is a hermitian metric along the fibres of E, then it can be regarded as a section h of the bundle $E^* \otimes \overline{E}^*$. We have also a canonical connection on $E^* \otimes \overline{E}^*$. We denote the covariant derivative on section of $E^* \otimes \overline{E}^*$ again by $\triangledown$:

$$\triangledown : \Gamma(X, \mathcal{A}(E^* \otimes \overline{E}^*)) \to \Gamma(X, \mathcal{A}(E^* \otimes \overline{E}^* \otimes \Theta^*)).$$

Proposition 3.1. $\nabla h = 0$

Proof. Let $z^1, .., z^n$ be a coordinate system on U_1. In this proof we write l for l_i and h for h_i.

77 We have

$$\begin{aligned}
\nabla_\alpha h = (\nabla h)\left(\frac{\partial}{\partial z^\alpha}\right) &= \partial_\alpha h_{\overline{a}b} - \sum_c l^c_{b\alpha} h_{\overline{a}c} \\
&= \partial_x h_{\overline{a}b} - \sum (h^{-1})^{c\overline{d}} \partial_\alpha h_{\overline{d}b} h_{\overline{a}c} \\
&= \partial_\alpha h_{\overline{a}b} - \sum \delta^d_a \partial_\alpha h_{\overline{d}b} = \partial_\alpha h_{\overline{a}b} - \partial_\alpha h_{\overline{a}b} = 0.
\end{aligned}$$

Similarly one shows that

$$\nabla_{\overline{\alpha}} h = \nabla h(\frac{\partial}{\partial \overline{z}^\alpha}) = 0.$$

Hence the proposition.

(Here h is regarded as a function from U_i into $\left\{\mathbb{C}^m \otimes \overline{\mathbb{C}}^m\right\}^*$ using the given trivialisation of $E \mid U_i$. A vector $t \in \mathbb{C}^m$ is an m-tuple denoted $(t^a)_{1\leq a\leq m}$ and $\overline{t}\epsilon\overline{\mathbb{C}}^n$ is again an m-tuple but is denoted $(t^{\overline{a}})_{1\leq a\leq m}$. Further conventions are as follows: a vector in the dual of $\mathbb{C}^m$ is an m-tuple but with subscripts instead of super scripts. In other words we write

$$\begin{aligned}
t &= \sum t^a e_a \quad \text{for } t \in \mathbb{C}^m \\
t &= \sum t^{\overline{a}} \overline{e}_a \quad \text{for } t \in \overline{\mathbb{C}}^m \\
t &= \sum t_a e^*_a \quad \text{for } t \in \mathbb{C}^{m^*} \\
t &= \sum t_{\overline{a}} \overline{e}^*_a \quad \text{for } t \in \overline{\mathbb{C}^{m^*}}
\end{aligned}$$

where $\{e_a\}$ (resp. $\{e_a\}$, $\{e^*_a\}$, $\{\overline{e}^*_a\}$ is the canonical (resp. conjugate of the canonical, dual of the canonical, conjugate dual of the canonical) basis of $\mathbb{C}^m$). $\square$

78 **Proposition 3.2.** *Let $z^1, \ldots, z^n$ be a coordinate system on U_i. Then in U_i, the curvature form can be written as*

$$a_i = (s^a_{b\overline{\beta}\alpha})_{ab} d\overline{z}_\beta \wedge dz_{\alpha,}$$

where

$$(s^a_{b\bar{\beta}\alpha})_{ab} = \nabla_{\bar{\beta}}\nabla_\alpha - \nabla_\alpha\nabla_{\bar{\beta}} \tag{3.1}$$

Proof. Let $t = (t^a)$ be a section of the bundle over U_i.

Then

$$\nabla_{\bar{\beta}}t = \partial_{\bar{\beta}}t = (\partial_{\bar{\beta}}t^a)$$

Hence

$$(\nabla_\alpha\nabla_{\bar{\beta}}t)^a = \partial_\alpha\partial_{\substack{t^a \\ \bar{\beta}}} + \sum_b l^a_{b\alpha}\partial_{\bar{\beta}}t^b$$

where $l = \sum l^a_{b\beta}\, dz_\alpha$ is the connection form.

$$\nabla_\alpha t = \partial_\alpha t + (l^a_{b\alpha})(t) = \partial_\alpha t^a + \sum_b l^a_{b\alpha}\, t^b$$

so that

$$\begin{aligned}\nabla_{\bar{\beta}}\nabla_\alpha t &= \partial_{\bar{\beta}}\partial_\alpha t^a + \sum_b \partial_{\bar{\beta}} l^a_{b\alpha} t^b + \sum_b l^a_{b\alpha}\partial_{\bar{\beta}}t^b \\ &= \partial_{\bar{\beta}}\partial_\alpha t^a + s^a_{b\bar{\beta}\alpha} t^b + \sum_b l^a_{b\alpha}\partial_{\bar{\beta}}t^b.\end{aligned}$$

Hence $-(\nabla_\alpha\nabla_{\bar{\beta}}t - \nabla_{\bar{\beta}}\nabla_\alpha t)^a = \sum_b s^a_{b\bar{\beta}\alpha} t^b$.

This proves the proposition.

The equation (3.1) is known as the *Bianchi Identity*.

(We remark that s is a form of type (1,1).).

The following lemma is easy to prove. □ **79**

Lemma 3.3. $\nabla\# = \#\nabla$.

Next we specialize the preceding considerations to the case $E = \Theta_o$ the homomorphic tangent bundle. Let $z^1, \ldots, z^n$ be a local coordinate system in an open set $U \subset X$. Let

$$g = \sum g_{\bar{\alpha}\beta} dz^{-\alpha} dz^\beta$$

be the expression for the hermitian metric in this coordinate system. The associated ∂-connection is then given in U by the (1,0)- form

$$\sum C^\alpha_{\beta\gamma} dz^\gamma$$

where $$C^{\alpha}_{\beta\gamma} = \sum g^{\bar{\tau}\alpha} \frac{\partial g_{\bar{\tau}\beta}}{\partial z^{\gamma}}.$$

(here, as is usual, $g^{\bar{\alpha}\beta}$ is defined by $(g^{\bar{\alpha}\beta})(g_{\bar{\alpha}\beta}) = Identity$).

That means that

$$\begin{cases} \nabla_{\beta}\left(\frac{\partial}{\partial z^{\alpha}}\right) = \sum C^{\alpha}_{\beta\gamma} \frac{\partial}{\partial z^{\gamma}} \\ \nabla_{\bar{\beta}} \frac{\partial}{\partial z^{\alpha}} = 0 \end{cases} \tag{3.2}$$

where

$$\nabla : \Gamma(X, \mathcal{A}(\Theta)) \to \Gamma(X, \mathcal{A}(\Theta_o \otimes \Theta^*))$$

is the covariant derivation.

80 The ∂- connection defined above is not in general symmetric: $C^{\gamma}_{\alpha\beta} \neq C^{\gamma}_{\beta\alpha}$; in fact this connection is symmetric if and only if the hermitian metric g is Kahler. We set then

$$S = \sum \frac{1}{2}(C^{\gamma}_{\beta\alpha} - C^{\gamma}_{\alpha\beta}) dz^{\beta} \wedge dz^{\alpha}.$$

Clearly S is an alternating (2,0) form with values in the tangent bundle Θ_0. It is called the torsion form of the ∂- connection. Its vanishing characterises the Kahler metrics.

A hermitian metric on the holomorphic tangent bundle defines a Riemannian metric as well on X. We have corresponding to this metric a Riemannian connection on X. The associated covariant derivative is a map

$$D : \Gamma(X\mathcal{A}(\Theta)) \to \Gamma(X, \mathcal{A}(\Theta \otimes \Theta^*)).$$

In particular, we have

$$D : \Gamma(X\mathcal{A}(\Theta_0)) \to \Gamma(X, \mathcal{A}(\Theta_0 \otimes \Theta^*)).$$

Denoting, as is usual, the Riemann-Christofel symbols by $\Gamma^{\gamma}_{\alpha\beta}, \Gamma^{\gamma}_{\alpha\beta}$,.. etc, one can prove easily that these are related to the $C^{\gamma}_{\alpha\beta}$ as follows:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}(C^{\alpha}_{\beta\gamma} + C^{\alpha}_{\gamma\beta})$$

$$\sum_{\gamma} \Gamma^{\bar{\gamma}}_{\beta\bar{\gamma}} = -\sum_{r} S^{\gamma}_{\beta\gamma}$$

Since $\Theta \simeq \Theta_\circ \oplus \overline{\Theta}_\circ$ as a differentiable vector bundle, we have a direct sum decomposition

$$\Gamma(X, \mathcal{A}(\Theta \otimes \Theta^*)) \simeq \Gamma(X, \mathcal{A}(\Theta \otimes \Theta^*_\circ)) \otimes \Gamma(X, \mathcal{A}(\Theta \otimes \Theta^*_\circ))$$

and hence a natural projection **81**

$$\Gamma(X, \mathcal{A}(\Theta \otimes \Theta^*)) \to \Gamma(X, \mathcal{A}(\Theta \otimes \Theta^*));$$

composing this projection with D and $\triangledown$, we obtain two linear maps

$$\triangledown'' : \Gamma(X, \mathcal{A}(\Theta_o)) \to \Gamma(X, \mathcal{A}(\Theta_o \otimes \overline{\Theta}^*_o))$$

and $$D'' : \Gamma(X, \mathcal{A}(\Theta_o)) \to \Gamma(X, \mathcal{A}(\Theta_o \otimes \overline{\Theta}^*_o)).$$

If the metric is Kahler, these two maps coincide. In a similar way, composing the natural projection

$$\Gamma(X, \mathcal{A}(\Theta \otimes \Theta^*)) \to \Gamma(X, \mathcal{A}(\Theta \otimes \Theta^*_o))$$

with $\triangledown$ and D we define two linear maps,

$$\begin{aligned} \triangledown' &: \Gamma(X, \mathcal{A}\Theta_o)) \to \Gamma(X, \mathcal{A}(\Theta_o \otimes {\Theta_o}^*)) \\ D' &: \Gamma(X, \mathcal{A}\Theta_o)) \to \Gamma(X, \mathcal{A}(\Theta_o \otimes \Theta^*_o)) \end{aligned}$$

which coincide if the hermitian metric of X is a Kahler metric.

Suppose now that we are given a vector bundle E on X and hermitian metrics on X and along the fibres of E. We then have canonical ∂ -connections on the bundle E and Θ_o and a ∂- connection on Θ_o. These connections extend canonically to connections on each of the bundles

$$\overset{p}{\Lambda}\Theta^*_o \otimes \overset{q}{\Lambda}\overline{\Theta}^*_o \otimes E$$

We denote by $\triangledown$ the covariant derivation in any of these bundles: **82**

$$\triangledown : \Gamma(X, \mathcal{A}(\overset{p}{\wedge}\Theta^*_o \otimes \overset{q}{\wedge}\overline{\Theta}^*_o \otimes E)) \to \Gamma(X, \mathcal{A}(\overset{p}{\wedge}\Theta^*_o \otimes \overset{q}{\wedge}\overline{\Theta}^* \otimes E \otimes \Theta^*))$$

Once again, composing with the natural projections, we can define

$$\triangle'' : \Gamma(X, \mathcal{A}(\overset{p}{\wedge}\Theta_o^* \otimes \overset{q}{\wedge}\overline{\Theta}_o \overset{*}{\otimes} E)) \to \Gamma(X, \mathcal{A}(\overset{p}{\wedge}\Theta_o^* \otimes \overset{q}{\wedge}\overline{\Theta}_o^* \otimes E \otimes \overline{\Theta}_o^*)),$$

$$\triangle' : \Gamma(X, \mathcal{A}(\overset{p}{\wedge}\Theta_o \overset{q}{\wedge} \overline{\Theta}_o \otimes E)) \to \Gamma(X, \mathcal{A}(\overset{p}{\wedge}\Theta_o \overset{q}{\wedge} \Theta_o \otimes E \otimes E \otimes \Theta_o^*),$$

Let $s = \sum\limits_{\alpha,\beta} s^a_{b\bar{\alpha}\beta} d\bar{z}^\alpha \wedge dz^\beta$ be the curvature form of the ∂ -connection on E in a co-ordinate open set Y over which the bundle E is trivial, the complex analytic coordinates in U being $(z^1, \ldots, z^n)$: if (rank E)= m, a, b run through 1 to m and for fixed α, β, $(s^a_{b\overline{\alpha}\beta})$ is an $(m \times m)$-matrix.

In the same coordinate neighborhood, let

$$L = \sum_{\sigma,\tau} L^\alpha_{\beta\bar{\sigma}\tau} d\overline{z}^\sigma \wedge dz^\tau$$

be the curvature form of the ∂-connection on the holomorphic tangent bundle Θ_o. For fixed σ, τ, $(L^\alpha_{\beta\bar{\sigma}\tau})$ is an $(n \times n)$-matrix.

10 Local expressions for $\overline{\partial}$, ϑ and □

We will now obtain a local expression for the operator

$$\square = \overline{\partial}\vartheta + \vartheta\overline{\partial}$$

in terms of $\triangledown_\alpha$, $\overline{\triangledown}_\alpha$ and the forms L and s above.

83 We adopt the following notation : for a p- tuple $A = (\alpha_1, \ldots \alpha_p)$, we set $dz^A = dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_p}$ (resp. $dz^{-A} = d\overline{z}^{\alpha_1} \wedge \ldots \wedge d\overline{z}^{\alpha_p}$). Then if for $\phi \epsilon C^{pq(X,E)}$, we set

$$\varphi\left(\frac{\partial}{\partial z^{\alpha 1}}, \ldots, \frac{\partial}{\partial z^{\alpha p}}, \frac{\partial}{\partial \overline{z}^{\beta 1}}, \ldots, \ldots \frac{\partial}{\partial \overline{z}^{\beta q}}\right) = \varphi^a_{A\overline{B}}$$

we have
$$\varphi = \sum_{A,B} \frac{1}{p!q!} \varphi^a_{A\overline{B}} dz^A \wedge d\overline{z}^B$$

where $A = (\alpha_1, \ldots, \alpha_p)$ and $B = (\beta_1, \ldots, \beta_q)$ for a p-tuple A and a $(q+1)$-tuple $(\beta_1, \ldots, \beta_{q+1})$ we have

$$(\overline{\partial}\varphi)^a_{A\overline{\beta}_1 \ldots \overline{\beta}_{q+1}} = (-1)^p \sum_{r=1}^{q+1} (-1)^{r-1} \overline{\partial}_{\overline{\beta}_r} \varphi^a_{A\overline{\beta}_1 \ldots \beta_r \ldots \beta_{q+1}}.$$

On the other hand

$$\nabla_{\bar\beta_r}\varphi^a_{\bar\beta_1\ldots\hat{\bar\beta}_r\ldots\bar\beta_{q+1}} = \partial_{\bar\beta_r}\varphi^a_{A\bar\beta_1\ldots\hat{\bar\beta}_r\ldots\bar\beta_{q+1}} - \sum_{i\neq r} C^{\bar\alpha}_{\bar\beta_i\bar\beta_r}\varphi^a_{A\bar\beta_1\ldots\bar\beta_{i-1}\bar\alpha\bar\beta_{i+1}\ldots\hat{\bar\beta}_r\ldots\bar\beta_{q+1}}$$

$$= \partial_{\bar\beta_r}\varphi^a_{A\bar\beta_1\ldots\hat{\bar\beta}_r\ldots\bar\beta_{q+1}} - \sum_{i\neq r}\overline{B^{\alpha}_{\beta_i\beta_r}} + \overline{S}^{\alpha}_{\beta_i\beta_r}\varphi^a_{A\bar\beta_1\ldots\bar\beta_{i-1}\bar\alpha\beta_{i+1}\ldots\hat{\bar\beta}_r\ldots\beta_{q+1}}$$

where $C^{\gamma}_{\alpha\beta}$ has been defined before and **84**

$$B^{\gamma}_{\alpha\beta} = \frac{1}{2}(C^{\gamma}_{\alpha\beta} + C^{\gamma}_{\beta\alpha})$$

while $$S^{\gamma}_{\alpha\beta} = \frac{1}{2}(C^{\gamma}_{\alpha\beta} - C^{\gamma}_{\beta\alpha})$$

so that $$S = \sum S^{\gamma}_{\alpha\beta} dz^{\alpha}\wedge dz^{\beta}$$

is the torsion of the connection on $\Theta_\circ$.

We have therefore

$$(\bar\partial\varphi^a)_{A\bar\beta_1\ldots\bar\beta_{q+1}} = (-1)^p\sum(-1)^{r-1}\nabla\bar\beta_r\varphi^a_{A\bar\beta_1\ldots\hat{\bar\beta}_r\ldots\bar\beta_{q+1}} + (-1)^p\sum_{i\neq r}(-1)^{r-1}\overline{S}^{\alpha}_{\beta_i\beta_r}\varphi^a_{A\bar\beta_1\ldots(\alpha)_1\ldots\hat{\bar\beta}_r\ldots\bar\beta_{q+1}}.$$

Let **85**

$$S : C^{pq}(X,E)\to C^{p,q+1}(X,E)$$

be the operator defined by

$$(S\varphi)^a_{A\bar\beta_1\ldots\bar\beta_{q+1}} = (-1)^p\sum_{i\neq r}(-1)^{r-1}\overline{S}^{\alpha}_{\beta_i\beta_r}\varphi^a_{A\beta_1\ldots\beta_{i-1}\alpha\beta_{i+1}\beta_\gamma\bar\beta_{--q-1}} \tag{3.4}$$

and set

$$\bar\partial = \tilde\partial + S, \tag{3.5}$$

so that

$$(\tilde\partial\varphi)^a_{A\bar\beta_1\ldots\bar\beta_{q+1}} = (-1)^p\sum(-1)^{r-1}\nabla_{\bar\beta_r}\varphi^{\alpha}_{A\bar\beta_1\ldots\hat{\bar\beta}_r\ldots\beta_{q-1}} \tag{3.6}$$

Let

$$\tilde\vartheta = -*\#^{-1}\tilde\partial*\# : C^{pq}(X,E)\to C^{p,q-1}(X,E) \tag{3.7}$$

and
$$T = - * \#^{-1} S * \# : C^{pq}(X, E) \to C^{p,q-1}(X, E) \tag{3.8}$$
so that
$$\vartheta = \tilde{\theta} + T. \tag{3.9}$$

Finally let
$$\tilde{\square} = \tilde{\partial}\tilde{\vartheta} + \tilde{\vartheta}\tilde{\partial}$$

86 If the hermitian metric on X is a Kahler metric, then
$$\overline{\partial} = \tilde{\partial}, \quad \vartheta = \tilde{\vartheta}, \quad \square = \tilde{\square}.$$

For $\varphi \in C^{pq}(X, E)$
$$\left(\tilde{\vartheta}\varphi\right)^{a}_{\overline{AB'}} = (-1)^{p-1} \nabla_{\alpha} \varphi^{a}_{A} \alpha_{\overline{B'}}, \tag{3.10}$$
so that, exactly as in the case of the Laplacian $\triangle$ in Chapter 2, we have
$$\left(\tilde{\square}\varphi\right)^{a}_{\overline{AB}} = -\nabla_{\alpha}\nabla^{\alpha}\varphi^{a}_{\overline{AB}} + \sum_{r=1}^{q} (-1)^{r-1} (\nabla_{\alpha}\nabla_{\overline{\beta_r}} - \nabla_{\overline{\beta_r}}\nabla_{\alpha}) \varphi^{a}_{A} \alpha_{\overline{B'_r}} \tag{3.11}$$
where
$$\nabla^{\alpha} = g^{\alpha\overline{\beta}} \nabla_{\overline{\beta}},$$
and $A = (\alpha_1, \ldots\ldots\alpha_p), B = (\beta_1, \ldots\ldots\beta_q), B'_r = (\beta_1, \ldots\ldots, \hat{\beta}_r, \ldots\ldots, \beta_q)$.

In view of the Ricci identity, the summand of (3.11) can be expressed by
$$\sum_{r=1}^{q} (-1)^{r-1} (\nabla_{\alpha}\nabla_{\overline{\beta_r}} - \nabla_{\overline{\beta_r}}\nabla_{\alpha}) \varphi^{a}_{A} \alpha_{\overline{B'_r}} = (\tilde{\mathcal{K}}\varphi)^{a} AB \tag{3.12}$$
where $\tilde{\mathcal{K}}$ is a mapping
$$\tilde{\mathcal{K}} : C^{pq}(X, E) \to C^{pq}(X, E),$$
which is linear over C^{∞} functions, whose local expression involves lin-
87 early (with integral coefficients) only the coefficients of the curvature

forms, s and L, of E and $\Theta_\circ$.

By Remark (3) after Lemma 3.2 we have

$$\left(\tilde{\mathcal{K}}\varphi\right)^a_{A\overline{B}} = \sum_{r=1}^{q} (-1)^r s^a_{b\overline{\beta}_r\alpha}\varphi^b_A \alpha_{\overline{B'_r}} + \left(\tilde{\mathcal{K}}_\circ\varphi\right)^a_{A\overline{B}}, \tag{3.13}$$

where $\tilde{\mathcal{K}}_\circ$ involves only the curvature tensor of $\Theta_\circ$, and is completely independent of E.

Formula (3.11) can be also written as

$$(\tilde{\square}\varphi)^a_{A\overline{B}} = -\nabla_\alpha\nabla^\alpha\varphi^a_{A\overline{B}} + (\tilde{\mathcal{K}}\varphi)^a_{A\overline{B}}. \tag{3.14}$$

Now

$$\begin{aligned}\square &= (\bar{\partial}\vartheta + \vartheta\bar{\partial}) = (\bar{\partial} + S)(\tilde{\vartheta} + T) + (\tilde{\vartheta} + T)(\tilde{\partial} + S)\\ &= \tilde{\square} + \tilde{\partial}T + T\tilde{\partial} + \tilde{\vartheta}S + S\tilde{\vartheta} + ST + TS.\end{aligned}$$

It follows that

Lemma 3.4. *For any $\varphi\epsilon C^{pq}(X,E)$*

$$(\square\varphi)^a_{A\overline{B}} = (\tilde{\square}\varphi)^a_{A\overline{B}} + (F_1\varphi)^a_{A\overline{B}} + (F_2\nabla'\varphi)^a_{A\overline{B}} + (F_3\nabla''\varphi)^a_{A\overline{B}}$$

where

$$\begin{aligned}&F_1 : C^{pq}(X,E) \to C^{pq}(X,E),\\ &F_2 : C^{pq}(X,E\otimes\Theta^*_o) \to C^{pq}(X,E),\\ &F_3 : C^{pq}(X,E\otimes\Theta^*_o) \to C^{pq}(X,E),\end{aligned}$$

are linear over C^∞ functions. Their local expression involves the tensor **88**
tensor and its first derivatives.

If the metric on X is a Kahler metric, then $F_1 \equiv 0, F_2 \equiv 0, F_3 \equiv 0$.

Applying proposition 3.1 to the hermitian metric on X we have that

$$\nabla * = *\nabla.$$

Hence, by (3.14),

$$\tilde{\square} - *^{-1}\tilde{\square}* = \tilde{\mathcal{K}} - *^{-1}\tilde{\mathcal{K}}* . \tag{3.15}$$

If the metric on X is Kahler, this identity becomes

$$\square - *^{-1}\square* = \tilde{\mathcal{K}} - *^{-1}\tilde{\mathcal{K}}* .$$

A direct computation shows that, in this case, $\tilde{\mathcal{K}} - *^{-1}\tilde{\mathcal{K}}*$ does not depend on the curvature form of the Kahler metric. We have in fact (see [2], 103)

$$((\tilde{\mathcal{K}} - *^{-1}\tilde{\mathcal{K}}*))^{a}_{A\overline{B}} = \sum_{i=1}^{q}(-1)^{i} s^{a}_{b\overline{\beta}_{i}\beta}\varphi^{b\beta}_{A\overline{B'_{i}}} + \sum_{j=1}^{p}(-1)^{j} s^{a}_{b\overline{\alpha}\alpha_{j}}\varphi^{b\overline{\alpha}}_{A'_{j}\overline{B}} + s^{a}_{b\overline{\beta}}\overline{\beta}\varphi^{b}_{A\overline{B}}$$

where $A'_j = (\alpha_1, \ldots, \hat{\alpha}_j, ..., \alpha_p)$.

Starting from this formula, it is easy to check that

$$\tilde{\mathcal{K}} - *^{-1}\tilde{\mathcal{K}}* = e(s)\Lambda - \Lambda e(s),$$

89 where

$$e(s) : C^{pq}(X, E) \to C^{p+1,q+1}(X, E)$$

is the linear mapping locally defined by

$$(e(s)\varphi)_i = \sqrt{-1} s^{a}_{ib} \wedge \varphi^{b},$$

and Λ is the classical operator of the Kahler geometry (see e.g. [35], 42). Thus we have, in the Kahler case,

$$\square - *^{-1}\square* = e(s)\Lambda - \Lambda e(s)$$

which, by (1.7), can also be written

$$\square_E - \#^{-1}\square_E * \# = e(s)\Lambda - \Lambda e(s).$$

This formula, which was first obtained in [4], 483, yields W-ellipticity conditions on Kähler manifolds (see [2],).

If the metric on X is not Kahler, then it follows from Lemma 3.4 and from (3.15) that

$$\square - *^{-1}\square * = \tilde{K} - *^{-1}\tilde{K} * + G_1\varphi + G_2\nabla'\varphi + G_3\nabla''\varphi$$

where

$$\begin{aligned} G_1 &: C^{pq}(X, E) \to C^{pq}(X, E), \\ G_2 &: C^{pq}(X, E \otimes \Theta^*_\circ) \to C^{pq}(X, E), \\ G_3 &: C^{pq}(X, E \otimes \Theta^*_\circ) \to C^{pq}(X, E), \end{aligned}$$

are linear over C^∞ functions. Their local expressions involve the torsion tensor and its first covariant derivatives. If the hermitian metric is a 90
Kähler metric, then $G_1 \equiv 0$, $G_2 \equiv 0$, $G_3 \equiv 0$.

11 The main inequality

We shall now establish an integral inequality which, under convenient hypotheses, yields a sufficient condition for the W^{pq}- ellipticity of E.

Let $\varphi \in C^{pq}(X, E)(q > 0)$. Let ξ and η be two tangent vector fields to X, defined by

$$\begin{aligned} \xi &= (\xi^\beta = h_{\bar{b}a}\nabla_{\bar\gamma}\varphi^a_A\beta_{\bar{B}'}\overline{\varphi^{b\bar{A}\gamma B'}}, \xi^{\bar\beta} = 0) \\ \eta &= (\eta^\gamma = 0, \eta^{\bar\gamma} = h_{\bar{b}a}\nabla_\beta\varphi^a_A\beta_{\bar{B}'}.\overline{\varphi^{b\bar{A}\gamma B'}}) \end{aligned}$$

We have (η being the complex dimension of X)

$$div\xi = \sum_{i=1}^{2n} D_i\xi^i = \sum_{\beta=1}^{n} \nabla_\beta\xi^\beta - \sum_{\alpha,\beta} S^\beta_\alpha\beta\xi^\alpha,$$

where

$$\begin{aligned} \nabla_\beta\xi^\beta &= h_{\bar{b}a}\nabla_\beta\nabla_{\bar\gamma}\varphi^a_A\beta_{\bar{B}'}.\overline{\varphi^{b\bar{a}\gamma B'}} + h_{\bar{b}a}\nabla_{\bar\gamma}\varphi^a_A\beta_{\bar{B}'}.\overline{\nabla_{\bar{B}}\varphi^{b\bar{A}\gamma B'}} \\ &= h_{\bar{b}a}\nabla_{\bar\gamma}\nabla_\beta\varphi^a_A\beta_{\bar{B}'}.\overline{\varphi^{b\bar{a}\gamma B'}} + h_{\bar{b}a}\nabla_{\bar\gamma}\varphi^a_A\beta_{\bar{B}'}.\overline{\nabla_{\bar{B}}\varphi^{b\bar{A}\gamma B'}} \\ &\qquad + h_{\bar{b}a}(\nabla_\beta\nabla_{\bar\gamma} - \nabla_{\bar\gamma}\nabla_\beta)\varphi^a_A\beta_{\bar{B}'}.\overline{\varphi^{b\bar{A}\gamma B'}}. \end{aligned}$$

The last summand can be evaluated using the Ricci identity. We have, by (3.12). 91

$$h_{\bar{b}a}(\nabla_\beta\nabla_{\bar{\gamma}}-\nabla_{\bar{\gamma}}\nabla_\beta)\varphi^a_A\beta_{\bar{B}'}.\overline{\varphi^{b\bar{A}\gamma B}} = \frac{1}{q}h_{ba}(\tilde{K}\varphi)^a_{A\bar{B}}\overline{\varphi^{bAB}} = p!(q-1)!A(\tilde{K}\varphi,\varphi).$$

Furthermore a direct computation, starting from (3.6), shows that

$$A(\tilde{\partial}\varphi,\tilde{\partial}\varphi) = A(\nabla''\varphi,\nabla''\varphi) - \frac{1}{p!(q-1)!}h_{\bar{b}a}\nabla_{\bar{\gamma}}\varphi^a_A\beta_{\bar{B}'}.\overline{\nabla_{\bar{\beta}}\varphi^{b\bar{A}\gamma B'}}.$$

Hence we have

$$\nabla_\beta\xi^\beta = h_{\bar{b}a}\nabla_{\bar{\gamma}}\nabla_\beta\varphi^a_A\beta_{\bar{B}'}.\overline{\varphi^{b\bar{A}\gamma B'}} + p!(q-1)! \left\{A(\tilde{K}\varphi,\varphi) + A(\nabla''\varphi,\nabla''\varphi) - A(\tilde{\partial}\varphi,\tilde{\partial}\varphi)\right\}.$$

Now

$$\operatorname{div}\eta = \sum_{i=1}^{2n} D_i\eta^i = \sum \nabla_{\bar{\gamma}}\eta^{\bar{\gamma}} - 2\sum_{\gamma,\beta}\overline{S^\beta_{\gamma\beta}}\eta^{\bar{\gamma}},$$

$$\begin{aligned}\nabla_{\bar{\partial}}\eta^{\bar{\gamma}} &= h_{\bar{b}a}\nabla_{\bar{\gamma}}\nabla_\beta\varphi^a_A\beta_{\bar{B}'}.\overline{\varphi^{b\bar{A}\gamma B'}} + h_{\bar{b}a}\nabla_\beta\varphi^a_A\beta_{\bar{B}'}.\overline{\nabla_\gamma\varphi^{b\bar{A}\gamma B'}}\\ &= h_{\bar{b}a}\triangle_{\bar{\gamma}}\triangle_\beta\varphi^a_A\beta_{\bar{B}'}.\overline{\varphi^{b\bar{A}\gamma B'}} + p!(q-1)!A(\tilde{\vartheta}\varphi,\tilde{\vartheta}\varphi), \qquad \text{by (3.10).}\end{aligned}$$

92 Thus

$$\begin{aligned}\operatorname{div}\xi - \operatorname{div}\eta = p!(q-1)!\Big\{&A(\tilde{K}\varphi,\varphi) + A(\nabla''\varphi,\nabla''\varphi)\\ &-A(\tilde{\partial}\varphi,\tilde{\partial}\varphi) - A(\tilde{\vartheta}\varphi,\tilde{\vartheta}\varphi)\Big\} - 2\left(S_{\alpha}\beta_\beta\xi^\alpha - \overline{S^\beta_{\gamma\beta}}\eta^{\bar{\gamma}}\right).\end{aligned}$$

Let $\varphi\epsilon\mathscr{D}^{pq}(X,E)$. Then by Stokes' theorem we have

$$\|\nabla''\varphi\|^2 + (\tilde{K}\varphi,\varphi) = \|\tilde{\partial}\varphi\|^2 + \|\tilde{\vartheta}\varphi\|^2 + \frac{2}{p!(q-1)!}\int_X\left(S^\beta_{\alpha\beta}\xi^\alpha - \overline{S^\beta_{\gamma\beta}}\eta^{\bar{\gamma}}\right)dX. \tag{3.16}$$

Let $|\ S\ |$ be the length of the torsion form. Applying lemma A of Chapter 1 and the Schwartz inequality we get from (3.4), (3.5) and from (3.8), (3.9) the following estimates

$$\|\tilde{\partial}\varphi\|^2 \le 2(\overline{\partial}\varphi\|^2 + \|S\varphi\|^2) \le 2\|\overline{\partial}\varphi\|^2 + c\int_X |\ S\ |^2 A(\varphi,\varphi)dX$$

$$
\begin{aligned}
&= 2\|\overline{\partial}\varphi\|^2 + +c(|\ S\ |^2\ \varphi,\varphi),\\
\|\tilde{\vartheta}\varphi\|^2 \leq 2(\|\vartheta\varphi\|^2 + \|T\varphi\|^2) &\leq 2\|\vartheta\varphi\|^2 + c\int_X |\ S\ |^2\ A(\varphi,\varphi)dX\\
&= 2\|\vartheta\varphi\|^2 + c(|\ S\ |^2\ \varphi,\varphi),
\end{aligned}
$$

c being a universal positive constant (depending only on the dimension of X).

Furthermore, again by the Schwartz inequality, we have **93**

$$
\begin{aligned}
\frac{1}{p!(q-1)!}\ |\sum S_{\alpha}\beta_{\beta}\xi^{\beta}\ | &\leq \frac{1}{p!(q-1)!}\ |\ S\,||\xi\ |\leq q\ |\ S\ |\ A(\nabla''\varphi,\nabla''\varphi)^{\frac{1}{2}}A(\varphi,\varphi)^{\frac{1}{2}}\\
&\leq \frac{q\varepsilon}{2}\ |\ S\ |^2\ A(\varphi,\varphi) + \frac{q}{2\varepsilon}A(\nabla''\varphi,\nabla''\varphi),\\
\frac{1}{p!(q-1)!}\ |\sum \overline{S\beta}_{\gamma_\beta}\eta^{\overline{\gamma}}\ | &\leq \frac{1}{p!(q-1)!}\ |\ S\,||\eta\ |\leq q\ |\ S\ |\ A(\tilde{\vartheta}\varphi,\tilde{\vartheta}\varphi)^{\frac{1}{2}}A(\varphi,\varphi)^{\frac{1}{2}}\\
&\leq q\frac{\varepsilon}{2}\ |\ S\ |^2\ A(\varphi,\varphi) + \frac{q}{2\varepsilon}A(\tilde{\vartheta}\varphi,\tilde{\vartheta}\varphi),
\end{aligned}
$$

for any $\varepsilon > 0$.

Substituting these estimates in (3.16) we obtain for any $\varepsilon > 0$

$$
\begin{aligned}
\left(1-\frac{q}{\varepsilon}\right)\|\ \nabla''\ \varphi\|^2 + \left(\tilde{K}\varphi - \left(2c + 2q\varepsilon + \frac{qc}{\varepsilon}\right)|\ S\ |^2\ \varphi,\varphi\right)\\
\leq 2\|\overline{\partial}\varphi\|^2 + +2\left(1+\frac{q}{\varepsilon}\right)\|\vartheta\varphi\|^2.
\end{aligned}
$$

Setting, for instance $\varepsilon = 2q$, and

$$
\mathcal{K} = \tilde{\mathcal{K}} - \left(\frac{5}{2}c + 4n^2\right)|\ S\ |^2\ .Id : C^{pq}(X,E)\to C^{pq}(X,E) \tag{3.17}
$$

we obtain the following

Proposition 3.3. *For any $\varphi\epsilon\mathscr{D}^{pq}(X,E)$ $(q>0)$ we have*

$$
\frac{1}{2}\|\nabla''\varphi\|^2 + (\mathcal{K}\varphi,\varphi) \leq 3\left(\|\overline{\partial}\varphi\|^2 + \|\vartheta\varphi\|^2\right).
$$

If the hermitian metric on X is a Kähler metric, the above proposition can be considerably sharpened. In fact, in this case, $S \equiv 0$, $\overline{\partial} = \overline{0}$, 94
$\tilde{\vartheta} = \vartheta$. *It follows from (3.16) that for every $\varphi \epsilon \mathscr{D}^{pq}(X, E)$ $(q > 0)$ satisfies the identity*

$$||\nabla''\varphi||^2 + (\mathcal{K}\varphi, \varphi) = ||\overline{\partial}\varphi||^2 + ||\vartheta\varphi||^2 \quad \textit{(Kähler)} \tag{3.18}$$

where we have set $\mathcal{K} = \tilde{\mathcal{K}}$. Also in the hermitian case the choice of the numerical constants can be improved.

As a corollary to proposition 3.1 we have the following

Theorem 3.1. *If there exists a positive constant C_2 such that*

$$A(\mathcal{K}\varphi, \varphi) \geq C_2 A(\varphi, \varphi)$$

for every $\varphi \epsilon C^{pq}(X, E)$ $(q > 0)$ and at each point of X, then E is $W^{p,q}$-elliptic.

Remark. Effect on $\mathcal{K}$ of a change of the hermitian metric on E.

Let $\varphi : X\mathbb{R}$ be any C^∞- function. We wish to examine how the operator $\mathcal{K}$ above changes when we replace the hermitian metric h on E by $h' = e^\varphi h$. (The hermitian metric on the base X remains undisturbed). First, the curvature form s' of h' is given by

$$s' = s + \sum_{\alpha,\beta} \frac{\partial^2\varphi}{\partial\overline{z}^\alpha \partial z^\beta} \overline{dz^\alpha} \wedge dz^\beta . I_m,$$

where s, as above, is the curvature form of h; $(z^\alpha)_{1\leq\alpha\leq n}$ is a local coordinate system in an open set $U \subset X$ on which a local trivialization of E is assumed given; the symbol I_m stands for the $(m \times m)$ identity matrix. It is immediate from (3.13) that

$$(\mathcal{K}'\varphi)^a_{A\overline{B}} = (\mathcal{K}\varphi^a)_{A\overline{B}} + \sum_i (-1)^i \frac{\partial^2\varphi}{\partial z^\alpha \partial\overline{z}^{\beta_i}} \varphi^{a\alpha}_{A\overline{B'_i}}.$$

95 where $\mathcal{K}'$ is the analogue of $\mathcal{K}$, defined now with respect to the new hermitian metric h'.

Chapter 4

Vanishing Theorems

12 q-complete manifolds

Let X be a complex manifold and $\phi : X \to \mathbb{R}$ a C^∞-function. The Levi form $\mathscr{L}(\phi)$ of ϕ is the hermitian quadratic differential form on X defined as follows: let $z^1, \ldots, z^n$ be a coordinate system on an open set $U \subset X$; then **96**

$$\mathscr{L}(\phi)\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z}^\beta}\right) = \frac{\partial^2 \varphi}{\partial z^\alpha \partial \overline{z}^\beta}.$$

($\mathscr{L}(\varphi)$ is thus a hermitian form on the holomorphic tangent space).

Definition 4.1. *A C^∞ function $\phi : X \to \mathbb{R}$ is strongly q-pseudo-convex if the Levi form $\mathscr{L}(\phi)$ has at least $(n - q)$ positive eigen-values at every point of X.*

Definition 4.2. *A complex manifold X is q-complete if there exists a C^∞ function $\phi : X \to \mathbb{R}$ such that*

(i) ϕ is strongly q-pseudo-convex

(ii) for $c \in \mathbb{R}$, the set $\{x \mid x \in X, \phi(x) < c\}$ is relatively compact in X.

Lemma 4.1 (E. Calabi)**.** *Let H be a hermitian quadratic differential form on X and G a hermitian metric on X. Assume that H has at least p*

positive eigen values. Let $\varepsilon_1(x), ..., \varepsilon_n(x)$ be the eigen values of H (w.r.t. G) at x in decreasing order: $\varepsilon_r(x) \geq \varepsilon_{r+1}(x)$

97 *Then given $c_1, c_2 > 0$, G can be so chosen that*

$$l_H(x) = c_1 \varepsilon_p(x) + c_2 \operatorname{Inf}(0, \varepsilon_n(x)) > 0 \quad \textit{for all} \quad x \in X.$$

Proof. Let G be any complete hermitian metric whatever. Let $\sigma_1(x), \ldots, \sigma_n(x)$ be the eigen-values of H with reference to G arranged in decreasing order. We construct now a metric G on X whose eigen values are functions of $\{\sigma_i(x)\}_{1 \leq i \leq n}$ as follows: let $\lambda : X \to \mathbb{R}$ be a $C^\infty-$function (we will impose conditions on λ letter); let U be a coordinate open set in X with holomorphic coordinates $(z^1, ..., z^n)$; in U, we have $G = G_{U\alpha\overline{\beta}} dz^\alpha d\overline{z}^\beta$ so that $(G_{U\alpha\overline{\beta}})_{\alpha\beta}$ is a function whose values are positive definite hermitian matrices; then the matrix valued function $\widehat{G}_U = (\widehat{G}_{U\alpha\overline{\beta}})_{\alpha\beta}$ where $\widehat{G} = \sum \widehat{G}_{U\alpha\overline{\beta}} dz^\alpha d\overline{z}^\beta$ in U is defined by

$$\widehat{G}_U^{-1} = G_U^{-1} \sum_{\underset{r=0}{=}}^{\infty} \frac{\lambda(x)^r}{(r+1)!} (H_U G_U^{-1})^r$$

where H_U is the matrix valued function $(H_{U\alpha\beta})$ defined by

$$H = \sum H_{U\alpha\overline{\beta}} dz^\alpha d\overline{z}^\beta$$

in U. □

We now assert that $\widehat{G}_U$ define a global hermitian defferential form on X and under a suitable choice of λ, it is positive definite. To see that $\widehat{G}_U$ defines a global hermitian differential form on X, we need only prove the following. Let V be another coordinate open set with coordinates

98 complex $(w^1, ..., w^n)$. Let $J = \dfrac{\partial(z^1, ..., z^n)}{\partial(w^1, ..., w^n)}$ be the Jacobian matrix. As before let $G_V = (G_{V\alpha\beta})$ be defined by $G = \sum G_{V\alpha\overline{\beta}} dw^\alpha d\overline{w}^\beta$ in V. Then if $\widehat{G}_V$ is defined starting from G_V as $\widehat{G}_U$ from G_U, we have

$$J \widehat{G}_U^t \overline{J} = \widehat{G_V}.$$

We have in fact, writing J^* for ${}^t\bar{J}^{-1}$,

$$J^*\widehat{G}_U^{-1}J^{-1} = G_V^{-1}J\left(\sum_{r=0}^{\infty}\frac{\lambda(x)^1}{(r+1)!}(H_U G_U^{-1})^r\right)J^{-1}, \text{ since } JG_U{}^t\bar{J} = G_V.$$

It follows from the above that

$$\begin{aligned} J^*\widehat{G}_U^{-1}J^{-1} &= G_V^{-1}\sum_{r=0}^{\infty}\frac{\lambda(x)^r}{(r+1)!}(JH_U G_U^{-1}J^{-1})^r \\ &= G_V^{-1}\sum_{r=0}^{\infty}\frac{\lambda(x)^r}{(r=1)!}(H_V J^* G_U^{-1}J^{-1})^r \end{aligned}$$

since $JH_U^t\bar{J}H_V$. Hence we obtain

$$\begin{aligned} J^*\widehat{G}_U^{-1}J^{-1} &= G_V^{-1}\sum_{r=0}\frac{(\lambda(x))^r}{(r+1)!}(H_V G_V^{-1})^r \\ &= \widehat{G}_V^{-1} \end{aligned}$$

This proves that $\widehat{G}_U$ defines on X a global hermitian differential. We next show that $\widehat{G}$ is positive definite. For this we look for the eigenvalues of $\widehat{G}$ with reference to G. TO compute these, we may assume, in the above formula for $\widehat{G}_U$, that G_U is the identity matrix

Then we have **99**

$$\widehat{G}_U = \sum_{r=0}^{+\infty}\frac{\lambda(x)^r}{(r+1)!}H_U^r$$

It follows that the eigen values of $\widehat{G}_U$ are

$$\left\{\sum_{r=0}^{+\infty}\frac{\lambda(x)^r}{(r+1)!}\sigma_q(x)^r\right\}_{1\le q\le n}.$$

It is easily seen that these are all strictly greater than zero: this assertion simply means this: $f(t) = \frac{e^t-1}{t} = \sum_{r=0}^{\infty}\frac{t^r}{(r+1)!}$ for $t \neq 0$, $f(0) = 1$ (which is continuous in t) is everywhere greater than 0.

We will now look for conditions on λ such that $\widehat{G}$ satisfies our requirements. From the formula for $\widehat{G}$, we have

$$H_U \widehat{G}_U^{-1} = \sum_{r=0}^{+\infty} \frac{\lambda(x)^r}{(r+1)} (H_U G_U^{-1})^{r+1}.$$

Now the eigen values of H with respect to $\widehat{G}$ (resp. G) are simply those of the matrix $H_U \widehat{G}_U^{-1}$ (resp. $H_U G_U^{-1}$), Hence these eigen-values $\varepsilon_q(X)$ of H with reference to $\widehat{G}$ are

$$f(\lambda(x),\ \sigma_q(x))$$

where $f(s,t)$ is the function on $\mathbb{R}^2$ defined by

$$f(s,t) = \sum_{r=0}^{t\infty} \frac{S^r}{(r+1)} t^{r+1}$$

Since $\frac{\partial f(s,r)}{\partial t} = e^{st} > 0$ for any, $f(s,t)$ is monotone increasing in t.

100 Hence we have

$$\varepsilon_r(x) \geqslant \varepsilon_{r+1}(x) \text{ for } 1 \leqslant r \leqslant n-1.$$

Moreover $f(s,t) \geqslant t$ for $s \geqslant 0$. Thus , if we choose $\lambda(x) \geqslant 0$ for every $x \epsilon X$, then $\varepsilon_q(x) \geqslant \sigma_q(x) > 0$.

The choice of $\lambda(x)$ is now made as follows . Let , for every integer $\nu > 0$. $B_\nu = \left\{ x \mid d(x, x_0) \leqslant \gamma \right\}$ for some $x_0 \epsilon X$, the distance being i the metric G. The B_ν are then compact. Let $b_\nu = \operatorname*{Inf}_{x \in B_\gamma} (\sigma_p(x))$.

Then $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{\nu+1} \geqslant \ldots$.

Let $b(x)$ be a C^∞ function on X such that $b(x) > 0$ for $x \in X$ and $b(x) < b_\nu$ in $B_\nu - B_{\nu-1}$. Then clearly $b(x) \leq \sigma_p(x)$.

Finally let $\rho(x)$ be a C^∞ function on X such that $\rho(x) \geqslant d(x, x_0)$, and $k > \sqrt{\frac{C_2}{C_1}} b_1$ be a real constant. Set $\lambda(x) = \frac{2ke^{\rho(x)}}{b^2(x)}$. We have then

$$\varepsilon_q(x) = f(\lambda(x), \sigma_p(x)) = \sigma_p(x) + \frac{\lambda(x)}{2!} \sigma_p(x)^2 + \cdots$$

so that $$\varepsilon_p(x) \geqslant \frac{ke^{\rho(x)}}{b^2(x)}\sigma_p(x)^2 \geqslant ke^{\rho(x)} \geqslant k.$$

On the other hand,

$$\varepsilon_n(x) = f(\lambda(x), \sigma_n(x)) = \frac{1}{\lambda(x)}\left\{e^{\lambda(x)\sigma_n(x)-1}\right\} \geqslant \frac{-1}{\lambda(x)} = -\frac{b^2(x)}{2ke^{\rho(x)}} \geqslant -\frac{b_1^2}{k},$$
$$C_1\varepsilon_p(x) + C_2 \operatorname{Inf}(0, \varepsilon_n(x)) \geqslant C_1 k - C_2\frac{b_1^2}{k} \geqslant \frac{1}{k}(C_1k^2 - C_2b_1^2) > 0.$$

This proves the lemma.

Combining this with the Remark at the end of §2, we can assert moreover that G can be chosen to be complete and satisfy the condition of Lemma 4.1. 101

Let $\phi : X \to \mathbb{R}$ be a strongly q-pseudoconvex function on X. Assuming as a form H in the above lemma the Levi form $\mathscr{L}(\phi)$, and taking $p = n - q$, $c_1 = 1$, $c_2 = n$, we obtain

Lemma 4.2. *Let $\phi : X \to \mathbb{R}$ be a strongly q-pseudoconvex function on X. Then there exists a complete hermitian metric on X such that at each point of X.*

$$l_{\mathscr{L}(\phi)}(x) = \varepsilon_{n-q}(x) + n \;\; Inf \;\; (0, \varepsilon_n(x)) > 0.$$

Let $u \in C^{rs}(X, E)$ and let

$$\mathscr{L}(\phi)\{u, u\} = \frac{1}{p!q!} h_{\bar{b}a} \frac{\partial^2\phi}{\partial z^\alpha \partial \overline{z}^\beta} u^{a\ \alpha}_{A\ \ \overline{B}'} u^{\overline{bA\beta B'}}$$

Lemma 4.3. *If $\phi : X \to \mathbb{R}$ is strongly q-pseudoconvex then for any $u \in C^{rs}(X, E)$, with $s \geq q + 1$ the following inequality holds at any point of X*

$$\mathscr{L}(\phi)\{u, u\} \geq l^{A(u,u)}_{\mathscr{L}(\phi)}.$$

Proof. At any point $x \in X$, we have

$$\mathscr{L}(\phi)\{u, u\} = h_{\bar{b}a}\sum_\beta \varepsilon_\beta(x) u_{\overline{A\beta B'}} u^{\overline{A\beta B'}}$$

Since 102

$$s \geq q+1, \qquad \text{i.e } s+n-q \geq n+1,$$

then every s-tuple of indices contains at least one the indices $1, 2 \ldots, n-q$ of the positive eigenvalues $\varepsilon_1(x), \ldots, \varepsilon_{n-q}(x)$. Hence

$$\frac{1}{p!q!} h_{\overline{b}a} \sum_{\beta=1}^{n-q} u_{A\beta\overline{B}'} u^{\overline{A\beta B'}} \geq A(u,u),$$

and therefore

$$\begin{aligned}
\mathscr{L}(\phi)\{u,u\} \geq \varepsilon_{n-q}(x) A(u,u) + \frac{1}{p!q!} \\
\left\{ \underset{n-q+1}{\varepsilon}(x) u_{\overline{An-q+1B'}} u^{\overline{An-q+1B^1}} \right\} + \varepsilon_n(x) u_{An\overline{B}'}\ u^{\overline{AnB'}} \Big\} \\
\geq \varepsilon_{n-q}(x) A(u,u) + \operatorname{Inf}(o, \varepsilon_n(x)) A(u,u).
\end{aligned}$$

This proves our lemma. □

Lemma 4.4. *ϕ be a strongly q-pseudo convex function on X and μ a positive real valued function on $\mathbb{R}$ such that $\mu'(t) \geq 0$, $\mu''(t) > 0$.*

Then

$$\mathscr{L}(\mu(\phi))(u,u) = \mu'(\phi)\mathscr{L}(\phi)(u,u) + (\mu'')(\phi) \left| \frac{\partial\phi}{\partial z^\alpha} u^\alpha \right|^2$$

Hence

$$\mathscr{L}(\mu(\phi))(u,u) \geq \mu'(\phi)\mathscr{L}(\varphi)(u,u),$$

and $$l_{\mathscr{L}(\mu(\phi))} \geq \mu'(\phi) l_{\mathscr{L}(\phi).}$$

The Lemma follows from a direct computation of the Levi form $\mathscr{L}(\mu(\phi))$ and from the fact since $\mu''(\phi) \left| \frac{\partial\phi}{\partial z^\alpha} u^\alpha \right|^2$ is positive semi definite,

103 *the eigenvalues of $\mathscr{L}(\mu(\phi))$ are not than the corresponding eigenvalues of $\mu'(\varphi)\mathscr{L}(\varphi)$.*

Lemma 4.5. *Let X be a hermitian manifold and ϕ a strongly q-pseudo convex proper function such that at each $x \in X$*

$$l_{\mathscr{L}(\phi)}(x) = \varepsilon_{n-q}(X) + n\,\mathrm{Inf}(0, \varepsilon_n(x)) > 0,$$

where $\{\varepsilon_p\}$ are the eigen values of $\mathscr{L}(\varphi)$ with respect to the metric on X arranged in decreasing order. Then given any continuous function $g : X \to \mathbb{R}$ there exists a sequence $\{a_\nu\}_{1\leq\nu<\infty}$ of real constants such that for any C^∞ function $\mu : \mathbb{R} \to \mathbb{R}$ satisfying (i) $\mu'(t) > 0$ (ii) $\mu''(t) \geq 0$ and (iii) $\mu'(t) \geq a_\nu$ for $\nu \leq t < \nu + 1$, we have

$$l_{\mathscr{L}(\mu(\phi))}(x) > g(x).$$

Proof. Since the sets $K_\nu = \{x | \nu \leq \phi(x) \leq \nu + 1\}$ are compact and $l_{\mathscr{L}(\phi)}(x) > 0$ for all $x \in X$, we can find a_ν such that

$$a_\nu l_{\mathscr{L}(\phi)}(x) > g(x) \text{ for } x \in K_\nu$$

the lemma now follows from Lemma 4.4 □

13 Holomorphic bundles over q-complete manifolds

Lemma 4.6. *Let E be a holomorphic vector bundle on a q-complete complex manifold. Let $\varphi : X \to \mathbb{R}$ be a strongly q-pseudo-convex C^∞ function on X such that $\{x|\phi(x) < c\} \subset\subset X$ for every $c \in \mathbb{R}$. Let ds^2 be a hermitian metric on X such that $l_{\mathscr{L}(\phi)}(x) > 0$ with respect to this metric. Let h be any hermitian metric on X. Then there is a sequence $(a_\nu)_{1\leq\gamma<\infty}$ of real constants such that the following property holds.*

Let μ be any C^∞ functions on $\mathbb{R}$ such that $\mu'(t) \geq a_\nu$ for $\nu \leq t \leq \nu + 1$ 104
and $\mu'(t) > 0$, $\mu''(t) \geq 0$ for all t. Let $\mathcal{K}$ and (resp. $\mathcal{K}_{-\mu'})$ be the operators (linear over C^∞ functions) from $C^{rs}(X, E) \longrightarrow C^{rs}(X, E)$ defined in chapter 3. (See (3.12) and (3.17)) with respect to the hermitian metrics ds^2 and h (resp. $h = e^{-\mu(\phi)h}$. Then for $\psi \in C^{r,s}(X, E)$ with $s > q$, we have

$$A_{-\mu}(\mathcal{K}_{-\mu}\psi, \psi) \geq e^{-\mu(\phi)} A(\psi, \psi).$$

Proof. From the Remark at the end of Chapter 3,

$$\mathcal{K}_{-\mu}\psi = \mathcal{K}\psi - \sum(-1)^i \frac{\partial^2\mu(\phi)}{\partial_z^\beta \partial\overline{z}^{\beta_i}} \varphi_A^a \frac{\beta}{B_i'}.$$

It follows that

$$\begin{aligned} A_{-\mu}(\mathcal{K}_{-\mu}\psi, \psi) &= e^{-\mu(\phi)}\{\mathscr{L}(\mu(\phi))(\psi,\psi)\} + e^{-\mu(\phi)}A(\mathcal{K}\psi,\psi) \\ &\geq e^{-\mu(\phi)}\{l_{\mathscr{L}(\mu(\phi)} + f\}A(\psi,\psi), \end{aligned}$$

Where $f : X \longrightarrow \mathbb{R}$ is a continuous function depending on $\mathcal{K}$. By Lemma 4.4. $(a_\nu)_{1\leq\nu<\infty}$ can be so chosen that for any convex μ satisfying

(i) $\mu'(t) > 0$, (ii) $\mu''(t) \geq 0$ (iii) $\mu'(t) > a_\nu$ for $\nu \leq t < \nu+1$, we have

$$c_1 l_{\mathscr{L}(\mu(\varphi)} + f \geq 1.$$

Hence

$$A_{-\mu(\varphi)}(\mathcal{K}_{-\mu}\psi, \psi) \geq e^{-\mu(\varphi)}A(\psi,\psi).$$

□

105 **Theorem 4.1.** *Let X be a q-complete manifold so that there exists $\varphi : X \longrightarrow \mathbb{R}$ satisfying : (i) φ is strongly q-pseudoconvex and (ii), for $c \in \mathbb{R}, \{x \mid \varphi(x) \leq c\}$ is compact X. Let E be a holomorphic vector bundle on X. Then there exists a complete hermitian metric ds^2 on X and a hermitian metric h along the fibres of E and real constants $\{a_\nu\}_{1\leq\nu<\infty}$ and $c > 0$ such that for every C^∞ function $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying*

(i) $\mu'(t) > 0$, (ii) $\mu''(t) \geq 0$ *and* (iii) $\mu'(t) \geq a_\nu$ *for* $\nu \leq t < \nu+1$ *we have*

$$3(\|\overline{\partial}\psi\|_{-\mu} + \|\vartheta_{-\mu}\psi\|^2_{-\mu}) \geq \|\psi\|^2_{-\mu} \text{ for every } \psi \in \mathscr{D}^{r,s}(X,E), s > q.$$

Hence E is W^{rs}-elliptic for $s > q$, with respect to the metric $e^{-\mu(\varphi)}h$.

Proof. From proposition 3.1., we have

$$(\mathcal{K}_{-\mu}\psi, \psi) \leq 3\left\{\|\overline{\partial}\psi\|^2_-\mu + \|\vartheta_{-\mu}\psi\|^2\right\}.$$

(The subscript $-\mu$ means that the operators scalar product etc. are defined with reference to the metric ds^2 on the base and $e^{-\mu(\varphi)}h$ along the fibres of E). The theorems then follow from Lemma 4.2, Lemma 4.6 and the above inequality. □

Remark . If we choose a μ satisfying the conditions of Theorem 4.1, then for any function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\lambda'(t) \geq 0, \lambda''(t) \geq 0$ for $t \in \mathbb{R}, \mu + \lambda$ again satisfies the conditions of the theorems.

Consequently if we denote the metric on the base by ds^2 and the hermitian metric $e^{-\mu(\varphi)}h$ by h', with respect to (ds^2, h') we have the following properties for $s > q$: **106**

1) There is a C^∞ function $\phi : X \to \mathbb{R}$ such that $\phi(x) \geq 0$

2) For every C^∞ non decreasing convex function $\lambda : \mathbb{R} \to \mathbb{R}E$ is $W^{rs}-$ elliptic with respect to $(ds^2, e^{-\lambda(\phi)}h')$

3) The $W^{rs}-$ ellipticity constant is independent of ν.

4) The set $B_c = \{x \mid x \in X, \varphi(x) \leq c\}$ is compact in X for every $c \in \mathbb{R}$. These conditions imply conditions $C_1^*, C_2^*, C_3^*, C_4^*$ of §4.

From Theorem 4.1 and from Theorem 1.3 we have the following corollary.

Theorem 4.2. *Let X be a q-complete complex manifold and E a holomorphic vector bundle on X. Then for*

$$s \geq q + 1, r \geq 0, H^s(X, \Omega^r(E)) = 0.$$

Proof. In fact, for $\varphi \in C^{rs}(X, E)(s > q)$, there is a suitable μ as in Theorem 4.1 above (i.e $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^∞ function such that $\mu'(t) > 0, \mu''(t) \geq 0$ and $\mu'(t) > a_\nu$ for $\nu \leq t < \nu + 1, \{a_\nu\}_{1 \leq \nu < \infty}$ being chosen so that Theorem 4.1 holds for this sequence of real constants) such that

$$||\varphi||_{-\mu}^2 = \int_X e^{-\mu(\phi)} A(\varphi, \varphi) dX < \infty.$$

On the other hand, Theorem 4.1. ensures us that with respect to the metric $e^{-\mu(\varphi)}h$ (h as in Theorem 4.1.) and the complete hermitian metric ds^2 (as in Theorem 4.1.) on X, we have $W^{rs}-$ ellipticity for $s > q$. We deduce Theorem 4.2. from Theorem 1.3 of Chapter 1. $\square$

Theorem 4.3. *The cohomology groups $H^s_k(X, \Omega^r(E))$ (cohomology with compact supports) vanish for $s \leq n - q - 1$ for any vector bundle E on the q-complete complex manifold X. Moreover, $H^{p-q}_k(X, \Omega^r(E))$ has a structure of a separated topological vector space.* 107

Proof. The theorem can be proved by applying Serre's duality to Theorem 4.2. It can also be given the following direct proof, which is based on the results of §4.

Let us choose a metric h' along the fibres of the dual bundle E^* in such a way that conditions 1), 2), 3), 4) of the Remark after Theorem 4.1. be satisfied (by E^*). Corresponding to the metric $e^{-\lambda(\phi)}h'$ on E^*, we choose the "dual" metric ${}^t(e^{-\lambda(\phi)}h')^{-1} = e^{\lambda(\phi)^t}h'^{-1}$ on E. Let $\varphi \in C^{rs}(X, E)$, with $s \leq n - q - 1$. Then $*\#\varphi \in C^{n-rn-s}(X, E^*)$. Since $n - s \geq q + 1, *\#\varphi$ satisfies the inequality

$$A_{E^*,-\lambda}(\mathcal{K}_{E^*,-\lambda} * \#\varphi, *\#\varphi) \geq A_{E^*,-\lambda}(*\#\varphi, *\#\varphi),$$

i.e.

$$A_{E^*,-\lambda}(\mathcal{K}_{E^*,-\lambda} * \#\varphi, *\#\varphi) \geq A_{E,\lambda}(\varphi, \varphi).$$

□

Hence, for any $\varphi \in \mathscr{D}^{rs}(X, E)$, with $s \leq n - q - 1$ we have

$$||\varphi||^2_{E,\lambda} \leq 3(||\overline{\partial} * \#\varphi||^2_{E^*,-\lambda} + ||\vartheta_{E^*,-\lambda} * \#\varphi||^2_{E^*,-\lambda})$$

i.e. by (1.4), (1.5), (1.6),

$$||\varphi||^2_{E,\lambda} \leq 3(||\overline{\partial}\varphi||^2_{E,\lambda} + ||\vartheta_{E,\lambda}\varphi||^2)$$

108 This inequality holds for any $\varphi \in \mathscr{D}^{rs}(X, E)$, with $s \leq n - q - 1$, and for any C^∞, nondecreasing, convex function $\lambda : \mathbb{R} \to \mathbb{R}$.

Our theorem follows from Theorem 1.4. of §4 and from Theorem 1.8.

Let now X be a q-complete manifold and $\phi : X \to \mathbb{R}^+$ a strongly q-pseudo convex function on X such that for

$c \in \mathbb{R}, B_c = \{x | x \in X, \phi(x) < c\} \subset\subset X$. We set $B_c = Y$ for some $c \in \mathbb{R}$.

Lemma 4.7. *B_c is q-complete.*

Proof. Let μ be a C^∞, nondecreasing convex function on $(-\infty, c)$ such that $\{t \mid t \in \mathbb{R}, \mu(t) < t_\circ\}$ is relatively compact in $(-\infty, c)$ for every $t_\circ \in \mathbb{R}$. Then the two conditions of Definitions 4.2 are satisfied by the function $\mu(\phi)$. This proves the lemma

We adopt the following notation

$$F^{rs}(X, E) = \left\{\psi \in \mathscr{L}^{rs}_{\text{loc}}(X, E); \overline{\partial}\psi = 0\right\}.$$
$$Q^{rs}(X, E) = \left\{\psi \in \mathscr{L}^{rs}_{-\nu\phi}(X, E) \text{ for positive integer } \nu, \overline{\partial}\psi = 0\right\}.$$

Finally $\rho : F^{rs}(X, E) \to F^{rs}(Y, E)$ is the restriction map. Clearly $Q^{rs}(X, E) \subset CF^{rs}(X, E)$ and we have □

Theorem 4.4. *If $s \geq q$, $\rho(Q^{rs}(X, E))$ is dense in $F^{rs}(Y, E)$. (The metric on X and that along the fibres are chosen as ds^2 and h respectively in the Remark after Theorem 4.1.)*

Proof. We denote $Q^{rs}(X, E)$ simply by Q. Let μ be a continuous linear functional on $F^{rs}(Y, E)$ which vanishes on $r(Q)$. It is sufficient to prove that μ vanishes on all of $F^{rs}(Y, E)$.

First, we extend μ to a continuous linear form on $\mathscr{L}^{rs}_{\text{loc}}(Y, E)$ (Hahn- **109**
Banach extension theorem). By the representation theorem it follows that there is a $\psi \in \mathscr{L}^{rs}(Y, E)$ with compact support in Y such that $\mu(u) = (\psi, u)$ for every $u \in \mathscr{L}^{rs}_{\text{loc}}(Y, E)$.

Let $c_\circ = \sup\{\varphi(x) \mid x \in$. Support $\psi\}$. Since Support ψ is compact in $Y = B_c$, then

$$c_\circ < c.$$

Now, let $\lambda : \mathbb{R} \to \mathbb{R}$ be a real C^∞ nondecreasing, convex function such that

$$\lambda(t) = 0 \text{ for } t \leq< c_\circ$$
$$\lambda(t) = 0 \text{ for } t < c_\circ,$$
$$0 \leq \lambda' \text{ (t) } \leq 1.$$

Then, for $t > c_\circ$,

$$\lambda(t) \leq t - c_\circ \leq t.$$

Let
$$u \in \mathscr{L}^{rs}_{-\nu\lambda(\phi)}(X, E) \text{ for some } \nu > 0, \bar{\partial} u = 0.$$
Then $u \in Q$, because
$$||u||^2_{-\nu\phi} = \int_X e^{-\nu\phi} A(u,u)dX \leq \int e^{-\nu\lambda(\phi)} A(u,u)dX = ||u||^2_{-\nu\lambda(\phi)}$$
Thus we have
$$(\psi, u) = \mu(\rho(u)) = 0$$

110 Now under these conditions, it follows from Theorem 1.6, taking into account the remark after Theorem 4.1 that there exists $\mathcal{X} \in \mathscr{L}^{r,s+1}$ such that

(i) $\psi = \vartheta\mathcal{X}$ and (ii) support $\mathcal{X} \subset \{x | \varphi(x) \leqslant c_\circ\}$.

Let $c_\circ < c_1 < c$ and introduce a complete hermitian metric on Y which coincides with the given one B_1. With respect to the new metric,
$$\psi \in \mathscr{L}^{rs}(Y, E), \mathcal{X} \in \mathscr{L}^{r,s+1}(Y, E)$$

Now let $u \in \mathrm{F}^{\mathrm{rs}}(Y, E)$ be an arbitrary form such that $\bar{\partial}\mathrm{u} = 0$. Then there is a C^∞, nondecreasing function $\sigma : (-\infty, c) \to \mathbb{R}$ such that $\sigma(t) = 0$ for $t \leqslant c_1$, and
$$u \in \mathscr{L}^{rs}_{-\sigma(\phi)}(Y, E).$$

We have then
$$(\psi, u) = (\vartheta\mathcal{X}, u) = (\vartheta_{-\sigma(\phi)}\mathcal{X}, u)_{-\sigma(\phi)} = (\mathcal{X}, \bar{\partial}u)_{-\sigma(\phi)} = 0.$$

Hence the linear form μ vanishes on $F^{rs}(Y, E)$. This proves the theorem. □

Corollary 1. *$Q^{rs}(X, E)$ is dense in $F^{rs}(X, E)$ (in the topology of L^2- convergence on compact sets).*

Proof. This is simply the case $X = Y$ (note that in the proof of Theorem 4.4, the case $c = \infty$ is covered). □

111 **Corollary 2.** *$\rho(F^{rs}(X, E))$ is dense in $(F^{rs}(Y, E)$.*

Proof. In fact $\Omega \subset F^{rs}(X, E)$. □

Suppose now that X is a stein manifold. Then X is is 0-complete. Hence we can find a C^∞ function $\phi : X \to \mathbb{R}$ such that $\mathscr{L}(\phi)$ is positive definite and such that for $c \in \mathbb{R}$, $\{x | x \in \mathcal{X}, \phi\mathcal{X} < c\} \subset \alpha X$. If then E is a holomorphic vector bundle on X, we obtain from Corollary 1, the following result.

Corollary 3. *Let $Q(X, E)$ denote the space of holomorphic sections of E over the Stein manifold X. There exists a complete hermitian metric on X and a hermitian metric along the fibres of E such that the set $\{f | f \in Q(X, E); \int_X e^{-\nu\phi} |f|^2 < \infty$ for some integer $\nu\}$ is dense in $Q(X, E)$ in the topology of uniform convergence on compact sets.*

In view of Corollary 1, to prove Corollary 3, we need only prove the following

Lemma 4.8. *Let D be a domain in $\mathbb{C}^n$ and $\{f_m\}$ a sequence of holomorphic functions on D such that for every compact $K \subset D$, $\{f_m\}$ converges in $L^2(K)$ (= space of square summable functions on K). Then $\{f_m\}$ converges uniformly on every compact set K.*

Proof. Let K be a compact set contained in D. Then there is a constant $c = c(K) > 0$ such that for every point $z_\circ \in K$. The poly-disc $P_{z_\circ} = \{z \big| |z^i - z^i_o| \leq c\}$ of radius c is contained in D and $\bigcup\limits_{z_\circ \in K} P_{z_o} \subset\subset D$. We have then from the Cauchy-integral formula,

$f(z_\circ) = \frac{1}{(\pi c^2)^n} \int_{P_{z_o}} f(z) dv$. ($dv$ = Lebesque measure in $\mathbb{C}^n$)

It follows on applying this to f^2, that 112

$$|f(z_\circ)|^2 \leqslant \frac{1}{\mathrm{vol} P_{z_\circ}} \int_{P_{z_o}} |f|^2 \, dv \leqslant \frac{1}{\mathrm{vol} P_{z_\circ}} \int_{P_{z_o}} \|f\|^2_{K'}$$

where K' is the (compact) closure of $\bigcup\limits_{z_\circ \in K} P_{z_o}$. The lemma is immediate from the above inequality. □

In the special case $q = 0$, Corollary 2 together with the lemma above yields the following result.

Theorem 4.5. *Let X be a Stein manifold and $\varphi : X \to \mathbb{R}$ a strongly 0-pseudo convex C^∞ function on X such that for*

$c \in \mathbb{R}, \{\mathcal{X} \mid \phi(\mathcal{X}) < c\} \subset\subset X$. Then the pair (X, B_c), $c \in \mathbb{R}$ is a Runge pair. In other words, the set of holomorphic functions on B_c which are restrictions of global holomorphic functions on X is dense in the space of all holomorphic functions on B_c in the topology of uniform convergence on compact subsets (of B_c).

14 Examples of q-complete manifolds

Let X be an m-complete manifold and $f_1, \ldots, f_{q+1}$ be any $q+1$ holomorphic functions on V. Let $Z = \{x | f_i(x) = 0 \text{ for } 1 \leqslant i \leqslant q+1\}$.

Assertion $Y = V - Z$ is $(m+q)$-complete.

Proof. Let $\varphi : V \to \mathbb{R}$ be a strongly m-pseudo-convex function V such that for every $c \in \mathbb{R}$,

$$B_c = \{x \mid x \in V, \varphi(x) < c\} \subset\subset X.$$

113 Because of the second condition, we can choose a C^∞ function

$$\lambda : \mathbb{R} \to \mathbb{R} \text{ such that } \lambda'(t) > 0, \lambda''(t) \geq 0,$$

$\lambda(t) > 0$ for $t \in \mathbb{R}$, $\lambda(t) \to \infty$ as $t \to \infty$ and further

$$\lambda(\varphi) > \sum_{i=1}^{q+1} f_i \overline{f}_i \text{ on } Y.$$

Set $\psi = \lambda(\varphi) - \log \sum_{i=1}^{q+1} f_i \overline{f_i}$. Since $\mathscr{L}(\log \sum_{i=1}^{q+1} f_i \overline{f_i})$ is positive semi-definite with at most q positive eigen-values, ψ strongly $(m+q)$-pseudo convex. It remains to prove that

$$B'_c = \{X \mid x \in Y, \psi(x) < c\} \subset\subset y$$

for every $c \in \mathbb{R}$. Now $\psi(x) \leq c$ implies that

$$\lambda(\varphi) - \sum_{i=1}^{q+1} \log\ f_i\ \overline{f_i} \leq c$$

or again that

$$\frac{e^{\lambda(\phi)}}{\sum\limits_{i=1}^{q+1} f_i \overline{f_i}} \leq e^c,$$

i,e.,

$$\sum_{i=1}^{q+1} f_i \overline{f}_i \geq e^{\lambda(\varphi)-c} > \delta > 0.$$

□

Hence $B_c \subset X - U$, where U is a neighborhood of Z. On the other hand; since $\lambda(\varphi) > \sum\limits_{i=1}^{q+1} f_i \overline{f}_i, e^{\lambda(\varphi)}/\lambda(\varphi) < e^c$, hence $\lambda(\varphi) < e^c = C$, on B'_c. **114**

Hence $B'_c \subset B_c$. On the other hand since $\overline{B'_c}$ is closed in $X - Z$ in $X - U$, it is closed in X. Hence $\overline{B'_c}$ is compact. This concludes the proof of our assertion.

In particular, if X is a Stein manifold, then Y is q-complete.

If Z is a complete intersection, then $q+1$ is the complex codimension of the submanifold Z of X. Let now Z be any analytic subset of the Stein manifold X, of complex codimension $q + 1$ at each point. To $Y = X - Z$ q-complete?

If $q = 0$, and if Z is a divisor, then the answer to this question is positive i.e. $Y = X - Z$ is a Stein manifold as it was shown by Serre ([6] p. 50) R.R. Simha [31] has proved that, if $q = 0$, if X is a complex codimension 2 and if Z is an analytic set in X of complex codimension 1 at each point, then $Y = X - Z$ is a Stein space. Examples ([23] Satz 12, 17; [14]) show that this result is false when $\dim_{\mathbb{C}} X > Z$.

An example, due to G. Sorani and V. Villani [33] , shows that if $q > 0$ then Y is not necessarily q-complete. Before discerning that example, we shall establish the following

Theorem 4.6 ([32]). *Let X be q-complete of complex dimension n. Let Ω^n denote the sheaf of germs of holomorphic n-forms on X. The natural map*

$$H^q(X, \Omega^n) \to H^{n+q}(X, \mathbb{C})$$

is surjective.

115 *Proof.* For any complex manifold, we have spectral sequence $(E_r^{st})_o \leq r \leq \infty$ such that E_∞ is the associated graded group of $H^*(X, \mathbb{C})$, and $E_1^{st} = H^t(X, \Omega^s)$ [11]. In the case when X is q-complete, $H^t(X, \Omega^s) = 0$ for $t > q$. In particular for $t+s = n+q$, $E_t^{st} = 0$ for all (s, t) different from (n, q). Hence $E_r^{s,t} = 0$ for $(s, t) \neq (n, q)$, $s + t = n + q$. Moreover, all of $E_1^{n,q}$ are cycles for d_1 since $\Omega^{n+1} = 0$. Hence $E_1^{n,q} \to E_2^{n,q}$ is surjective. For $r \geq 2$, $E_r^{n-r,q+r-1} = 0$ since $q + r - 1 > q$ and $E^{n+r,q+r+1} = 0$. Hence $E_2^{n,q} \backsimeq E_\infty^{n,q}$. Hence $H^{n+q}(X, \mathbb{C}) \simeq E_\infty^{n,q} = E_2^{n,q}$. (We note that $E_\infty^{s,t} = 0$ for $s + t = n + q$, $(s, t) \neq (n, q)$). This proves the theorem. □

We consider now the subsets of $\mathbb{C}^{2n}$

$$Z_1 = \{z \in \mathbb{C}^{2n} \big| z_i = 0 \text{ for i } > n\}, \qquad X_1 = \mathbb{C}^{2n} - Z_1,$$
$$Z_2 = \{z \in \mathbb{C}^{2n} \big| z_i = 0 \text{ for i } \leq n\}, \qquad X_2 = \mathbb{C}^{2n} - Z_2,$$
$$Z = Z_1 \cup Z_2, \quad Y = \mathbb{C}^{2n} - Z = X_1 \cap X_2.$$

The analytic set $Z \subset \mathbb{C}^{2n}$ has complex codimension n at each point. We shall prove that Y is not $(n-1)$-complete, when $n \geq 2$.

Since the point $\{0\}$ is a complete intersection of codimension $2n$ in $\mathbb{C}^{2n}$, then $U = \mathbb{C}^{2n} - \{0\}$ is $(2n-1)$-complete. Let Ω^{2n} be the sheaf of group of holomorphic 2n-forms. Then by Theorem 4.6.

$$H^{2n-1}(U, \Omega^{2n}) \to H^{2n-1}(U, \mathbb{C})$$

116 is surjective. On the other hand, since U is contractible on the unit sphere of $\mathbb{C}^{2n}$, then $H^{2n-1}(U, \mathbb{C}) \approx \mathbb{C}$. Hence

$$H^{2n-1}(U, \Omega^{2n}) \neq 0.$$

The exact cohomology sequence of Mayer-Victoris yields

$$\to H^r(U,\Omega^{2n}) \to H^r(X_1,\Omega^{2n}) \oplus H^r(X_z,\Omega^{2n})$$
$$\to H^r(X_1 \cap X_2,\Omega^{2n}) \to H^{r+1}(U,\Omega^{2n}) \to \quad (4.1)$$

Since $Z_i (i = 1,2)$ is a complete intersection of codimension n, X_i is $(n-1)$-complete.

Hence
$$H^r(X_i,\Omega^{2n}) = 0 \quad for \quad r \geq n (i = 1,2).$$

It follows from (4.1) that

$$H^r(X_1 \cap X_2,\Omega^{2n}) \approx H^{r+1}(U,\Omega^{2n}) \quad \text{for} \quad r \geq n.$$

Then
$$H^{2n-2}(X_1 \cap X_2,\Omega^{2n}) \neq 0.$$

If $n \geq 2$, then $2n-2 \geq n$. This shows that $X_1 \cap X_2$ is not $(n-1)$-complete.

15 A theorem on the supports of analytic functionals

We shall apply the methods developed above to obtain-following the ideas of [16], §2.5 with minor technical changes - a generalization, due to Martinean [23] of a theorem originating with Pólya [27]. We first prove two propositions which are of independent interest.

Proposition 4.1. *Let ϕ be a C^∞ plurisubharmonic function on $\mathbb{C}^n$. Let* 117
$\|\omega\|_{-\varphi}$ denote the norm of a C^∞ form ω of type (p,q) with respect to the Euclidean metric on the base and the metric $e^{-\phi}$ on the trivial line bundle. Let $\psi = \phi + 2\log(1+|z|^2)$. Then there is a constant $C > 0$ such that for any C^∞ form ω of type (p,q), $q \geq 1$, with $\overline{\partial}\omega = 0$, $\|\omega\|_\phi < \infty$, there is a C^∞ form ω' of type $(p,q-1)$ with $\overline{\partial}\omega' = \omega$, $\|\omega'\|_{-\psi} \leq C\|\omega\|_{-\phi}$.

Proof. Consider the operator $\mathcal{K}_{-\psi}$ with respect to the metric $e^{-\psi}$. We have, since the Euclidean metric is flat,

$$A_\psi(\mathcal{K}_{-\psi}u,u) = e^{-\psi}\mathscr{L}(\psi)\{u,u\}$$

$$\geq 2e^{-\psi}(1+|\bar{z}|^2)^{-2}A(u,u) = 2(1+|z|^2)^{-2}AA_{-\psi}(u,u),$$

since $\mathscr{L}(\psi)\{u,u\} \geq 2\mathscr{L}(log(1+|z|^2))\{u,u\} \geq 2(q+|z|^2)^{-2}A(u,u)$. Hence $\mathcal{K}_{-\psi}$ is positive definite with eigenvalues $\geq 2(1+|z|^2)^{-2}$ at the point z; $\mathcal{K}_{-\psi}^{-1}$ is also positive definite, and its eigenvalues are $\leq \frac{1}{2}(1+|z|^2)^2$. By consequence we have

$$\begin{aligned}|A_{-\psi}(u,v)| &\leq A_{-\psi}(\mathcal{K}_{-\psi}^{-1}u,u)A_{-\psi}(\mathcal{K}_{-\psi}v,v)\\ &\leq \frac{1}{2}A_{-\psi}((1+|z|^2)^2u,u)A_{-\psi}(\mathcal{K}_{-\psi}v,v).\end{aligned}$$

Hence, for forms u, v of type (p,q), we have

$$|(u,v)_{-\psi}|^2 \geq \frac{1}{2}||(1+|z|^2)u||^2_{-\psi}(\mathcal{K}_{-\psi}v,v)_{-\psi}, \tag{4.2}$$

whenever the forms u, v are such that the norms on the right are finite. □

118 On $\mathscr{D}^{p,q}(\mathbb{C}^n)$, we introduce the norm M by

$$M(u)^2 = (\mathcal{K}_{-\psi}u,u)_{-\psi} + ||\bar{\partial}u||^2_{-\psi} + ||\vartheta_{-\psi}u||^2_{-\psi},$$

and let $V^{p,q}$ be the completing of $\mathscr{D}^{p,q}$ with respect to this norm. Obviously $V^{pq} \subset \mathscr{L}^{pq}_{\text{loc}}$. For $u \in \mathscr{D}^{p,q}$, we have, by (3.18),

$$(\mathcal{K}_{-\psi}u,u) \geq ||\bar{\partial}u||^2_{-\psi} + ||\vartheta_{-\psi}u||^2_{-\psi}.$$

Thus the norm M on $\mathscr{D}^{p,q}$ is equivalent to that defined by

$$M'(u)^2 = ||\bar{\partial}u||^2_{-\psi} + ||\vartheta_{-\psi}u||^2_{-\psi}.$$

Let $\omega \in \mathscr{L}^{p,q}_{\text{loc}}$, and let $||\omega||_{-\varphi} < \infty$. Then, for $u \in \mathscr{D}^{p,q}$, we have, by (4.2) (dropping the factor $\frac{1}{2}$ as we may)

$$\begin{aligned}|(u,\omega)_{-\psi}|^2 &\leq ||(1+|z|^2)\omega||^2_{-\psi}M'(u)^2\\ &= ||\omega||^2_{-\phi}M'(u)^2\end{aligned} \tag{4.3}$$

Thus, the linear form $u \rightsquigarrow (u,\omega)_{-\psi}$ is continuous on $V^{p,q}$ so that there exists a unique $x \in V^{p,q}$ with

$$\begin{aligned}(u,\omega)_{-\psi} &= (\overline{\partial} u, \overline{\partial} x)_{-\psi} + (\vartheta_{-\psi} u,\ \vartheta_{-\psi}\ x)_{-\psi}\\ &= (\square_{-\psi} u, x)_{-\psi}, u \in \mathscr{D}^{p,q}.\end{aligned}$$

Thus we have

$$\omega = \square_{-\psi} x$$

and, moreover, by (4.3) **119**

$$M'(x)^2 = (x,\omega)_{-\psi} \leqslant \| \omega \|_{-\phi}\ M'(x),$$

so that

$$\| \overline{\partial} x \|^2_{-\psi} + \| \vartheta_{-\psi} x \|^2_{-\psi} \leqslant \| \omega \|^2_{-\phi}\ .$$

Let us assume now that ω *is* C^∞ and that $\overline{\partial}\omega = o$. Then by Corollary 1 to Theorem 1.2, we have, for any $\sigma > 0$,

$$\begin{aligned}\| \vartheta_{-\psi} \overline{\partial} x \|_{\psi} &\leqslant \frac{1}{\sigma} \| \overline{\partial}\omega \|^2_{\psi} + \sigma \| \overline{\partial} x \|^2_{-\psi}\\ &= \sigma \| \overline{\partial} x \|_{-\psi},\end{aligned}$$

and letting $\sigma \to 0$, we obtain

$$\vartheta_{-\psi} \overline{\partial} x = 0,$$

so that

$$\square_{-\psi} x = \overline{\partial} \vartheta_{-\psi}\ x;$$

since, as proved above, $\| \vartheta_{-\psi} x \|_{-\psi} \leq \| \omega \|_{-\phi}$, we obtain the proposition on setting $\omega' = \vartheta_{-\psi} x$.

We point out that the above proposition depends on a weaker condition than $W^{p,q}$-ellipticity, in fact, proposition 4.1 is a particular case of the following general statement, which can be established by a similar argument.

Proposition 4.2. *Let X be a complex manifold endowed with a complete hermitian metric. Let $\pi : E \to X$ be a holomorphic vector bundle on X and let h be a hermitian metric along the fibres of E. Assume that* **120**

$$A(\mathcal{K}\varphi,\varphi) > 0$$

at each point of X and for every $\varphi \in C^{pq}(X, E)$. Let $\varepsilon(x)$ be the least eigenvalue of the (positive definite) hermitian from $A(\mathcal{K}\varphi, \varphi)_x (x \in X)$ acting on the space of the (p, q)-form with values in E. Then $\varepsilon(x) > 0$.

If $\varphi \in C^{pq}(X, E)$, $\overline{\partial}\varphi = 0$ is such that

$$\int_X \frac{1}{\varepsilon(x)} A(\varphi, \varphi) dX < \infty,$$

there exists a from $\psi \in C^{p,q-1}(X, E)$ such that

$$\overline{\partial}\psi = \varphi,$$
$$\int_X A(\psi, \psi) dX \leq 3 \int_X \frac{1}{\varepsilon(x)} A(\varphi, \varphi) dX.$$

Proposition 4.3. *Let ϕ be a plurisubharmonic function on $\mathbb{C}^n$ such that there exists a constant $C > 0$ for which $|\phi(z) - \phi(z')| \leq C$ whenever $|z - z'| \leq 1$. Let V be a (complex) subspace of $\mathbb{C}^n$ of codimension k, and f a holomorphic function on V such that $\int |f|^2 e^{-\varphi} d_V v < \infty$ ($d_V v$ = Lebesgue measure on V). Then, there exits a holomorphic function F on $\mathbb{C}^n$ with $F|V = f$ and*

$$\int_{\mathbb{C}^n} |F|^2 e^{-\varphi} (1 + |z|^2)^{3k} dv \leqslant M \int_V |f|^2 e^{-\varphi} d_V v,$$

121 *where M is a constant independent of f; here $dv = d_{\mathbb{C}^n} v$ is the Lebesgue measure in $\mathbb{C}^n$*

Proof. We may clearly suppose that V has codimension 1 in $\mathbb{C}^n$. We assume that V is the subspace $z^n = 0$.

Then f is a holomorphic function in $\mathbb{C}^n$, depending only on $z^1, \ldots,$ z^{n-1}. Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a C^∞ function on $\mathbb{R}$, such that

$$0 \leqslant \lambda(t) \leqslant 1,$$
$$\lambda(t) = 1 \text{ for } t \leqslant \frac{1}{4}$$
$$\lambda(t) = 0 \text{ for } t \geqslant 1.$$

Let $C_1 = \operatorname{Sup} \left| \dfrac{d\lambda}{dt} \right|$.

We will construct a function μ in such a way that

$$F(z_1, \ldots, z_n) = \lambda(|z^n|^2)\ f(z_1, \ldots, z_{n-1}) - z^n \mu(z_1, \ldots, z_n)$$

satisfies all the requirements of the lemma.

Let

$$\sigma = \overline{\partial}\lambda(|z^n|^2)\ f,$$

σ is a C^∞, $\overline{\partial}$-closed form, such that $|z^n| \leq \frac{1}{2}$. Thus the form $\omega = \frac{1}{z^n}\sigma$ is a $(0,1)$- from which is C^∞ and $\overline{\partial}$-closed on $\mathbb{C}^n$, i.e. $\omega \in C^{01}(\mathbb{C}^n, \mathbb{C})$. Moreover

$$\text{Supp } \omega \subset \left\{ z = (z^1, \ldots, z^n) | \leqslant |z^n| \frac{1}{2} \leqslant 1 \right\}.$$

Since **122**

$$| \phi(z, ..., z^{n-1}, z^n) - \phi(z, ..., z^{n-1}, 0) | \leq C \text{ for } | z^n | \leqslant 1,$$

then $$\int\limits_{|z^n|<1} e^{-\varphi'} | f |^2 \ dv \leqslant \pi e^C \int\limits_V e^{-\rho} | f |^2 \ dv_V$$

On the other hand

$$||\omega||^2_{-\phi} = \int\limits_{\frac{1}{2}<|z^n|<1} e^{-\phi} A(\omega, \omega) dv \leqslant (2)^2 \int\limits_{\frac{1}{2}<|z^n|<1} e^{-\phi} | f |^2 d \leqslant C_2 \int\limits_V e^{-\phi} | f |^2 dv_V$$

$(C_2 = 4C_1^2 \pi e^C)$. In particular, $||\omega||^2_{-\phi} < \infty$

Assume now that ϕ is C^∞. By Proposition 4.1, there exists a C^∞ function $\mu : \mathbb{C}^n \to \mathbb{C}$ such that

$$= \overline{\partial}\mu$$

$$\int\limits_{\mathbb{C}^n} \frac{e^{-\phi}}{(1 + | z |^2)^n} | \mu |^2 dv \leqslant ||\omega||^2_{-\phi} \leqslant c_2 \int\limits_V e^{-\phi} | f |^2 dv.$$

If ϕ is not C^∞, we introduce a regularising function, i.e. a C^∞ function $\alpha(z) = \alpha(| z |) \geqslant 0$, on $\mathbb{C}^n$, with support $\subset \{| z | \leqslant 1\}$, and such that $\int\limits_{\mathbb{C}^n} \alpha dv = 1$. Given $0 < \varepsilon < 1$, the function

$$\varphi_\varepsilon(z) = \int\limits_{\mathbb{C}^n} \phi(z - \varepsilon z') \alpha(z') dv(z')$$

is a C^∞ plurisubharmonic function on $\mathbb{C}^n$, such that

$$\phi_\varepsilon(z) \to \phi(z) \text{ as } \varepsilon \to 0.$$

123 Moreover

$$| \phi_\varepsilon(z) - \phi(z) | \leqslant \int_{\mathbb{C}^n} \alpha(z') \mid \phi(z - \varepsilon z') - \phi(z) \mid d(z') \leqslant C,$$

and therefore $\quad | \phi_\varepsilon(z) - \phi_\varepsilon(z') | \leqslant 3C \text{ for } | z - z' | \leqslant .$

Hence there exists a C^∞ function $\mu_\varepsilon : \mathbb{C}^n \to \mathbb{C}$, such that

$$\bar{\partial}\mu_\varepsilon = \omega$$

$$\int_{\varepsilon^n} \frac{e^{-\phi}}{(1+ | z |^2)^2} | \mu_\varepsilon |^2 dv \leqslant e^c \int_{\mathbb{C}^n} \frac{e_\varepsilon^{-\phi}}{(1+ | z |^2)^2} | \mu_\varepsilon |^2 dv \leqslant C_3 \int_V e^{-\phi} | f |^2 \; dv_V.(C_3 = C_1^2 \pi e^{5c})$$

Letting $\varepsilon \to 0$, we can find a distribution μ such that

$$\bar{\partial}\mu = \omega_1$$

$$\int_{\mathbb{C}^n} \frac{e^{-\phi}}{(1+ | z |^2)^2} | \mu |^2 dv \leqslant C_3 \int_V e^{-\phi} | f |^2 dv_V.$$

Consider now the distribution

$$F = \lambda(| z^n |^2) f - z^n \mu.$$

We have

$$\bar{\partial}F = \bar{\partial}\lambda \cdot f - z^n \bar{\partial}\mu = 0.$$

Hence F is a holomorphic function $\mathbb{C}^n$.

124 Furthermore

$$\int_{\mathbb{C}^n} \frac{e^{-\phi}}{(1+ | z |^2)^3} - | F |^2 dv$$

$$\leqslant 2 \int\limits_{|z^n|<|} \frac{e^{-\phi}}{(1+\mid z\mid^2)^3} - \mid f\mid^2 dv + 2\int\limits_{\mathbb{C}} \frac{\mid z^n\mid^2 e^{-\phi}}{(1+\mid z\mid^2)^3} \mid \mu \mid^2 dv$$

$$\leqslant 2\pi e^C \int\limits_V e^{-\phi} \mid f\mid^2 dv_V + 2\int\limits_{\mathbb{C}^n} \frac{e^{-\phi}}{(1+\mid z\mid^2)^2} \mid \mu\mid^2 dv$$

$$\leqslant C_4 \int\limits_V e^{-\phi} \mid f\mid^2 dv_V (C_4 = 2\pi e^C + 2C_3).$$

This proves our proposition. □

Lemma 4.9. *Let ϕ be as in Proposition 4.2. Let $F : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function such that*

$$\int\limits_{\mathbb{C}} \frac{e^{-2\phi}}{(1+\mid z\mid^2)^N} \mid F\mid^2 dv = M^2 < \infty,$$

for some $N \geqslant 0$. Then there exists a positive constant C_5 depending only on C and on N such that

$$\mid F(z)\mid \leqslant C_5 M e^{\phi(z)} (1+\mid z\mid)^N e^{\phi(z)}.$$

Proof. Let $z_0 \in C^n$ and let $P(z_0)$ be the poly-disc with centre z_0 and radius 1. Then by Cauchy's formula we have

$$F(z_0)^2 = \frac{1}{\pi^n} \int_{P(z_0)} F(z)^2 dv,$$

whence 125

$$\mid F(z_o)^2 \mid \leqslant \frac{1}{\pi^n} \int\limits_{P(z_o)} \mid F\mid^2 d.$$

Furthermore

$$e^{-\phi(z_o)} \leqslant e^{\varepsilon} e^{-\phi}(z) \text{ for } z \in P_1(z_o)$$

and
$$\frac{1}{(1+\mid z_o\mid)^2} \leqslant \left(\frac{1+2n}{1+\mid z\mid}\right)^2 \leqslant \frac{(1+2n)^2}{1+\mid z\mid^2} \quad for \ z \in P_1(z_o)$$

In conclusion

$$\frac{e^{-2\phi(z_o)}}{(1+\mid z_o)^{2N}} \mid F(z_o)\mid^2 \leqslant \frac{e^{2\varepsilon}(1+2n)^{2N}}{\pi^n} \int\limits_{P(z_o)} \frac{e^{-2\phi}}{(1+\mid z\mid^2)} \mid F^2 \mid dv$$
$$\leqslant \frac{e^{2\varepsilon}(1+2n)^{2N}}{\pi^n} M^2.$$

This concludes the proof of our lemma. □

Before going on to state the theorem of Martinean-pólya, we need some definitions.

Let $\mathscr{H} = \mathscr{H}(\mathbb{C}^n)$ denote the vector space of holomorphic function on $\mathbb{C}^n$ with the topology of compact convergence. An *analytic functional* μ is a continuous linear functional on $\mathscr{H} \cdot \mu$ is said to be *supported by a compact set* K if for any open set $U \supset K$ there is a constant M_U such that sort any $f \in \mathscr{H}$ we have

$$\mid \mu(f)\mid \leq M_U \sup_{z\in U} \mid f(z)\mid .$$

126 For $z = (z_1, \ldots, z_n)$, $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$, let $\langle z, \zeta\rangle = \sum z_i\zeta_i$. For $\zeta \in \mathbb{C}^n$, let $e_\zeta \in \mathscr{H}$ be defined by $e_\zeta(z) = e^{\langle z,\zeta\rangle}$. The *Laplace* transform of μ is the holomorphic function $\breve{\mu}$ on $\mathbb{C}^n$ defined by

$$\breve{\mu}(\zeta) = \mu(e_\zeta).$$

Let K be compact and H the function defined by $H_K(\zeta) = \sup\limits_{z\in K}$ Re $\langle z, \zeta\rangle$. Clearly H_K is continuous, and hence, being the supremum of plurisubharmonic functions, is itself plurisubharmonic. If K_ε denotes the set $\{z \in \mathbb{C}^n \mid d(z, K) \leq \varepsilon\}$, we have $H_{K_\varepsilon}(\zeta) = H_K(\zeta) + \varepsilon \mid \zeta \mid$

Theorem 4.7 (Martineao-Polya)**.** *In order that the analytic functional μ be supported by the convex compact set K, it is necessary and sufficient that for every $\varepsilon > 0$, there exist a constant $C_\varepsilon > 0$ such that*

$$\mid \breve{\mu}(\zeta)\mid \leqslant C_\varepsilon e^{H_K(\zeta)+\in|\zeta|}.$$

Proof. Suppose that μ is supported by K. Then, by definition

$$|\tilde{\mu}(\zeta)| = |\mu(e_\zeta)| \leq C_\varepsilon \sup_{z\in K_\varepsilon} |e_\zeta(z)| = C_\varepsilon e^{\underset{K_\varepsilon}{\{\sup\langle z,\zeta\rangle\}}} \leq C_\varepsilon e^{H_K(\zeta)+\varepsilon|\zeta|}$$

To prove the converse, we proceed as follows. $\mathscr{H}$ is a closed subspace of the space $\mathscr{C}$ of continuous function on $\mathbb{C}^n$; hence (Hahn - Banach) μ extends to a linear functional $\mu : \mathscr{C} \to \mathbb{C}$, hence defines a measure. It is clearly sufficient to prove that any $\varepsilon > 0$, there exists distribution ν with support in K_ε such that

$$\mu(f) = \nu \text{ for all } f \in \mathscr{H}.$$

Let ξ' be the space of distributions with compact support in $\mathbb{C}^n$. For **127**
$\nu \in \xi'$, let $\hat{\nu}$ denote the Fourier transform of ν considered as a function of $2n$ real variables $u_1, \ldots, u_{2n}$:

$$\hat{\nu}(u_1, \ldots, u_{2n}) = \nu(e'_u),$$

where $$e'_u(z) = \exp\left(-i(u_1 \operatorname{Re} z_1 + u_2 \operatorname{Im} z_1 + \ldots + u_{2n} \operatorname{Im} z_n)\right).$$

□

Clearly $\widehat{\nu}$ has an extension to a holomorphic function on $\mathbb{C}^{2n}$; further since linear combination of the function e_ζ are dense in $\mathscr{H}$, we have $\nu(g) = \mu(g)$ for all $g \in \mathscr{H}$ if and only if $\mu(e_\zeta) = \nu(e_\zeta)$ for all ζ, i.e. if and only if

$$\breve{\mu}(\zeta) = \widehat{\nu}(i\zeta_1, -\zeta_i, ..., i\zeta_n, -\zeta_n). \tag{4.4}$$

Therefore, because of the Paley-Wiener theorem, it is sufficient to construct an entire function $\widehat{\nu}$ on $\mathbb{C}^{2n}$ satisfying (4.4), for which, we have further,

$$|\widehat{\nu}(u)| \leqslant C_\varepsilon(1+|u|)^{N\phi(u)}. \tag{4.5}$$

for some $N > 0$; here $\phi(u) = \sup\limits_{x\in K_\varepsilon}(x_1 \operatorname{Im} u_1 + \ldots + x_{2n} \operatorname{Im} u_{2n})$ and $z_j = x_{2j-1} + ix_{2j}$.

Consider the subspace V of $\mathbb{C}^{2n}$ consisting of points $(i\zeta_i, -\zeta_1, ..., i\zeta_n, -\zeta_n)$, where $\zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{C}^n$. If $u = (i\zeta_1, ..., -\zeta_n)$, we have $\phi(u) = H_{K_\varepsilon}(\zeta) = H_K(\zeta) + \varepsilon|\zeta|$.

Further, $\phi(u)$ is plurisubharmonic in $\mathbb{C}^{2n}$ and there exists a constant **128**
$C > 0$ such that $|\phi(u) - \phi(u')| \leqslant C$ for $|u - u'| \leqslant 1$. Clearly $\breve{\mu}$ defines a holomorphic function f on V if we set $f(i\zeta_1, \ldots, -\zeta_n) = \breve{\mu}(\zeta_1, \ldots, \zeta_n)$. Since for any $\delta > 0$, $|\breve{\mu}(\zeta)| \leqslant C_\delta e^{H_K(\zeta)+\delta|\zeta|}$ by hypothesis and $\phi(u) = H_K(\zeta) + \varepsilon|\zeta|$ on V, we have

$$|f(u)| \leqslant C_\delta e^{\phi(u)+(\delta-\varepsilon)|u|} \quad \text{on} \quad V.$$

If $\delta < \varepsilon$, we clearly have therefore

$$\int_V |f(u)|^2 e^{-2\phi(u)} d_V v < \infty.$$

By Proposition 4.2 there exists F, holomorphic in $\mathbb{C}^{2n}$ such that $F|V = f$, and

$$\int_{\mathbb{C}^{2n}} |F(u)|^2 e^{-2\phi(u)} (1 + |u|^2)^{-3n} dv < \infty.$$

By Lemma 4.9 this implies that there exists a constant $C > 0$ such that

$$|F(u)| \leqslant M(1 + |u|)^{3n} e^{\phi(u)},$$

so that if we set $\widehat{\nu}(u) = F(u)$, the conditions (4.4) and (4.5) are satisfied. This completes the proof of the theorem.

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