Lectures on
Forms of Higher Degree

By

J.I. Igusa

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S. Raghavan

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Foreward

THE FOLLOWING lecture notes are based on my lectures at the Tata Institute of Fundamental Research from the middle of January to early March of 1978. My objective was not to give a systematic exposition of the theory of forms of higher degree but only to introduce young mathematicians to a new approach to this area. It is my view that although this theory is still at an early stage, it has all the features of becoming an important branch of Mathematics. If the reader finds these lecture notes intelligible, then it is largely due to the painstaking efforts of Professor Raghavan. I would like to thank him for his excellent collaboration to bring the notes into this final form.

9 March 1978 **Jun-Ichi Igusa**

Contents

Introduction

ONE OF THE principal objectives of modern number theory must be to develop the theory of forms of degree more than two,to the same satisfactory level in which the theory of quadratic forms is found today as the cumulative work of several eminent mathematicians and especially of C.L. Siegel. The importance of forms of higher degree as a serious number-theoretic object was pointed out already by Gauss([11], § 266) as follows: "...il suffira d'avoir recommendéce champ vaste á l'attention des gèométres, oú ils pourront trouver un trés-beau sujet d' exercer leurs forces, et les moyens de donner á l'Arithmetique transcendent de trés-beaux développements". It is unbelievable and highly remarkable how remarkable how young Gauss was able to predict the existence of a full-fledged theory of quadratic forms as it obtains today. 1

With a view to give some idea as to what may constitute such a theory of forms of higher degree, we shall describe two problems. Before we state the first one, let as fix the notation. Let $f(x) = f(x_1, x_2, \ldots, x_n)$ be a form i.e, a homogeneous polynomial degree $m \geq 2$ in n variables $x_1, \ldots, x_n$ with coefficients in the ring $\mathbb{Z}$ of integers. Let p be a fixed prime, e a non-negative integer and u an integer such that the g.c.d.(u, p^e) is l. Let us define the generalized Gaussian sum

$$F^*(u/p^e) = \frac{1}{p^{ne}} \sum_{\xi \bmod p^e} ((u/p^e)f(\xi))$$

where $\xi \in \mathbb{Z}^n$ and $\mathfrak{e}(\quad) = \exp(2\pi\sqrt{-1})$. The first problem is to develop a theory of such exponential sums associated with f and p. In this direction, we have the following *general theorem*:

2 *there exists an integer* $e_o \geq 0$ *such that for every* $e \geq e_o$, $F^{\star}(u/p^e)$ = *a fixed linear combination of expression of the form*

$$\mathcal{X}(u)(p^e)^{-\lambda}(\log p^e)^j$$

with $\mathcal{X}$ *equal to a Dirichlet character having a power of* p *as its conductor,* $0 \leq j < n$ *and* $\mathrm{Re}(\lambda)$ *being a positive rational number.*

If now we arrange $\mathrm{Re}(\lambda)$ for *lambda* occurring above in ascending order as

$$0 < \lambda_1 < \lambda_2 < \cdots$$

(where $\lambda_1, \lambda_2, \ldots$ are invariants of the affine hypersurface determined by the zeros of f), then for any $\epsilon > 0$, we have, for every $e \geq 0$,

$$F^{\star}(u/p^e) \leq c(p^e)^{-\lambda_1+\epsilon}$$

where c is a constant independent of e and u.

A *conjecture* in this connection states that except for a finite number of primes p depending on f and ϵ, we can replace the constant c above by 1, if $\lambda_1 > 1$.

In all the cases where $F^{\star}(u/p^e)$ can be closely examined or calculated, this conjecture has been verified. Indeed, in the particular case of interest when the projective hypersurface S defined by $f(x) = 0$ is non-singular, the conjecture is valid with $\lambda_1 = n/m$ (and further 0 instead of ϵ). The proof of the conjecture in this case depends on Deligne's solution of the Riemann-Weil hypothesis for zeta functions associated with non-singular projective varieties defined over finite fields.

In order to deal with the conjecture when S may not be non-singular, one approach may be to isolate the relevant part of Deligne's work used
3 to prove the conjecture and examine whether the assumption of S being non-singular may be dropped.

We now state the second problem we referred to earlier in the context of a theory of forms of higher degree. It is the following well-known conjecture:
if $m = 3$, *then* $f(x) = 0$ *has a non-trivial integral solution provided that* $n > 9$.

Davenport ([8]) has proved a similar statement with $n > 15$ in place of the condition $n > 9$. To bring down 15 to 9 by improving Davenport's method seems to be almost impossible. We have recently found a new approach to this conjecture in the non-singular case; this is a direct generalization of the method used by Weil to establish the Minkowski-Hasse theorem for quadratic forms connecting the global representability with local representability. Weil's method depends on a certain poisson formula and the use of the "metaplectic group" (which is a covering of SL_2). We have indeed a generalization of such a poisson formula to forms of higher degree. If our approach does really settle the conjecture for forms of higher degree, then we shall be rewarded with a generalization of the beautiful relation between quadratic forms and modular functions. We believe in the existence of a new satisfactory theory of forms of higher degree and hope that the solution of the conjectures above will occupy important positions in such a theory.

Chapter 1

A Theory of Mellin Transformations

IN THIS CHAPTER, we develop a theory of Mellin transformation for **4**
the multiplicative group of an arbitrary local field, Although that is the only case in which we shall be interested, we shall start by recalling the definition of the Mellin transformation in the general case.

We shall assume results in the theory of locally compact abelian groups and Fourier transforms.

1 Generalities

1.1

Let G be a locally compact abelian group and G^*, the Pontrjagin dual of G consisting of the group $\mathrm{Hom}(G, \mathbb{C}_1^\times)$ of all continuous homomorphisms of G into the group $\mathbb{C}_1^\times$) of complex numbers of absolute value 1. For $g \in G$ and $g^* \in G^*$, we write $\langle g, g^* \rangle = g^*(g)$.

Let dg be the Haar measure on G (unique upto a positive scalar factor) and $L^1(G)$ the space of complex-valued functions which are integrable on G. For F in $L^1(G)$, the Fourier transform F^* is defined by

$$F^*(g^*) = \int_G F(g) \langle g, g^* \rangle \, dg \quad \text{for} \quad g^* \in G^*;$$

then F^* belongs to the space $L^\infty(G^*)$, which is the completion relative to the uniform norm of the space of complex-valued continuous functions on G^* with compact support.

Let $\Lambda(G)$ be the space of continuous functions F in $L^1(G)$ whose Fourier transform F^* is in $L^1(G^*)$; then $F \mapsto F^*$ is a bijection of $\Lambda(G)$ on $\Lambda(G^*)$. The Haar measure dg^* on G^* can be normalized so that $(F^*)^*(g) = F(-g)$ for every $F \in \Lambda(G)$ and $g \in G$. This is just the Fourier inversion theorem and the measure dg^* is said to be the *dual* of dg.

1.2

Let $\Omega(G) = \mathrm{Hom}(G, \mathbb{C}^\times)$, the group of continuous homomorphisms of G into the group $\mathbb{C}^\times$ of non-zero complex numbers. An element ω of $\Omega(G)$ is called a *quasicharacter* of G. The fact that $\mathbb{C}^\times = \mathbb{R}_+^\times \times \mathbb{C}_1^\times$ (direct product) induces a corresponding decomposition for any ω in $\Omega(G)$. Indeed, let $r(g) = |\omega(g)|$ for any $g \in G$ and then $g \mapsto \omega(g)/r(g)$ is in G^* so that for every g in G we have

$$\omega(g) = r(g)\langle g, g^*\rangle \tag{1}$$

for a unique g^* in G^*.

For any complex-valued function F on G and any ω in $\Omega(G)$, let us define

$$Z(\omega) = \int_G F(g)\,\omega(g)\,dg \tag{2}$$

where, for the moment, we say nothing about the existence of the integral and merely remark that we have a function Z defined on a certain subset of $\Omega(G)$. We call the map $M : F \to Z$ the *Mellin transformation*.

Substituting (1) in (2), we have

$$Z(\omega) = \int_G F(g)\,r(g)\,\langle g, g^*\rangle\,dg$$

and the Fourier inversion theorem yields the following

Theorem 1.2. *If dg^* is the dual of the measure dg on a locally compact abelian group G, $\omega \in \Omega(G)$ and $F : G \to \mathbb{C}$ are given, then under the assumption that the function $g \mapsto F(g)r(g)$ is in $\Omega(G)$, the Mellin transform is defined and further*

$$F(g)r(g) = \int_{G^*} Z(\omega) \langle g, -g^* \rangle \, dg,$$

i.e. 6

$$F(g) = \int_{G^*} Z(\omega)\, \omega(g)^{-1} \, dg.$$

1.3

If we take $G = \mathbb{R}_+^\times$, then $\Omega(G)$ consists of $\omega(x) = x^s$ with $s = \sigma + it$ ($i = \sqrt{-1}$), s and t being real. Now $r(x) = x^\sigma$ and $\langle x, t \rangle = x^{it} = e^{ity}$ with $y = \log x$ (or equivalently $x = e^y$). The map $x \mapsto y = \log x$ gives a bijection $\mathbb{R}_+^\times \to \mathbb{R}$. Now

$$Z(s) = \int_0^\infty F(x)\, x^s \, d\log x = \int_{-\infty}^\infty F(e^y)\, e^{\sigma y} \, \mathfrak{e}(\frac{1}{2\pi} yt) \, dy.$$

The Lebesgue measure dy on $\mathbb{R}$ is its own dual, relative to $(y, y') \mapsto \mathfrak{e}(yy')$. Thus Fourier's inversion theorem gives, under the assumption that $F(x)x^\sigma \in \Omega(\mathbb{R}_+^\times)$ (or equivalently that $F(e^y)e^{\sigma y} \in \Omega(\mathbb{R})$)

$$F(y)e^{\sigma y} = \int_{-\infty}^\infty Z(s)\, \mathfrak{e}(-\frac{1}{2\pi} yt) \frac{1}{2\pi} \, dt$$

i.e.

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^\infty Z(s)\, x^{-\sigma - it} \, dt = \frac{1}{2\pi i} \int_{\sigma - \infty i}^{\sigma + \infty i} Z(s)\, x^{-s} \, ds$$

which is just the assertion of the Theorem above for $G = \mathbb{R}_+^\times$.

1.4

In the special case $G = \mathbb{R}_+^\times$ discussed above, the fact that $\Omega(\mathbb{R}_+^\times) \simeq \mathbb{C}$ plays an important role; we are thereby in a position to use the theory of functions of one complex variable. We would like to know if we have, in a general situation, $G^* \subset \Omega(G)$ with G^* being a real Lie group (possibly without separability) and $\Omega(G)$, its complexification. This is, in fact, true if and only if G is compactly generated (i.e. generated by a compact neighbourhood of 0). From the structure of locally compact abelian groups, we know that G is compactly generated if and only if there exists a unique maximal compact subgroup C with $G \simeq C \times \mathbb{R}^m \times \mathbb{Z}^n$. But then $G^* \simeq C^* \times \mathbb{R}^m \times (\mathbb{C}_1^\times)^n$ where C^* is discrete and $\Omega(G) \simeq C^* \times \mathbb{C}^m \times (C^\times)^n$. In this situation, one can expect to find a good theory of Mellin transforms. As a matter of fact, we shall take G to be the group $K^\times$ of
7 non-zero elements of a local field K, where, by a *local field*, we mean $\mathbb{R}$ or $\mathbb{C}$ or a finite extension of Hensel's field $\mathbb{Q}_p$ of p-adic numbers or the field of quotients of the ring of formal power-series with a finite coefficient field. In the case of $\mathbb{R}$-fields K (i.e. $K = \mathbb{R}$ or $\mathbb{C}$), we have $K^\times \simeq K_1^\times \times \mathbb{R}$ where $K_1^\times$ is the compact group of elements of K of absolute value 1. In the last two cases which we refer to as p-fields, $K^\times \simeq K_1^\times \times \mathbb{Z}$ where $K_1^\times$ is a similarly defined compact group in K; here K is not always compactly generated while $K^\times$ is compactly generated.

2 Asymptotic Expansions

Keeping in view some applications, later on, like the determination of the asymptotic behaviour of certain integrals naturally associated over fields with forms of higher degree and of the nature of the so-called "local singular series" for such forms, we now introduce a general notion of asymptotic expansions.

Let Y be a topological space and x_∞, a point of the closure $\overline{X}$ of a sub-space X of Y such that X is separable at x_∞ (i.e. the system of neighbourhoods at x_∞ has a countable base). Let $\phi_0, \phi_1, \phi_2, \ldots$ be a given sequence of complex-valued functions on X such that for every $k \geq 0$, we have

(i) $\phi_k(x) \neq 0$ for all x in X which are different from x_∞ and sufficiently close to x_∞ and

(ii) $\phi_{k+1}(x) = O(\phi_k(x))$ as x tends to x_∞ i.e., for given $\epsilon > 0$, $|\phi_{k+1}(x)| \leq \epsilon|\phi_k(x)|$ for x in X close enough to x_∞. We then say that a complex valued function f on x has an *asymptotic expansion as* x (in X) *tends to* x_∞, if there exists a sequence $\{a_n\}_{n\geq 0}$ of complex numbers such that for every $k \geq 0$ and all x in X close to x_∞,

$$f(x) = \sum_{i=0}^{k} a_i\phi_i(x) + O(\phi_{k+1}(x)) \tag{3}$$

i.e: $|f(x) - \sum_{i=0}^{k} a_i\phi_i(x)| \leq C|\phi_{k+1}(x)|$ for a constant $C > 0$ independent **8**
of x. In symbols we denote this by

$$f(x) \approx \sum_{k=0}^{\infty} a_k\phi_k(x) \text{ as } x \to x_\infty \tag{4}$$

The symbol O and o are in the sense of Hardy-Littlewood. For the given sequence $\{\phi_n\}$, condition (i) clearly implies the uniqueness of $a_0, a_1, a_2, \ldots$ and therefore an asymptotic expansion for f is unique, if it exists. Further, it is sufficient to assume that (3) holds for a cofinal set of natural numbers $k_1 < k_2 < \ldots$

Let $Y = \mathbb{R}^n$ and X, an open set in $\mathbb{R}^n$. Taking $x_1, \ldots, x_n$ as coordinates in $\mathbb{R}^n$, let for $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-negative integers $\alpha_1, \ldots, \alpha_n$, D^α denote the differential operator $\dfrac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}$. If, in the foregoing, $f, \phi_0, \phi_1, \phi_2, \ldots$ are infinitely differentiable and if, further, for every α,

$$(D^\alpha f)(x) \approx \sum_{k=k_\alpha}^{\infty} a_k(D^\alpha\phi_k)(x) \quad \text{as} \quad x \to x_\infty$$

where we assume that $D^\alpha\phi_k$ vanish for $0 \leq k \leq k_\alpha$ and for x close enough to x_∞, then we say that the *asymptotic expansion* of f as x tends to x_∞ is termwise differentiable.

Remarks. It may happen that the $a_k's$ in (4) depend on a parameter. Then we can talk of the *uniformity of the asymptotic expansion* relative to such a parameter and similarly of the *termwise differentiability* including the *differentiation with regard to the parameter*.

Let us illustrate the notion of asymptotic expansion for f, with some examples.

9 **Example 1.** Take $X = Y = \mathbb{R}$, $x_\infty = 0$, $f \in C^\infty(\mathbb{R})$ and $\phi_k(x) = x^k$ for $k = 0, 1, 2, , \ldots$ Then the Maclaurin expansion for f at $x = 0$ gives just an asymptotic expansion for f as x tends to 0, namely

$$f(x) \approx \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k \quad as \quad x \to 0$$

(Note that $\approx$ does not, in general, imply equality). Further, the asymptotic expansion is termwise differentiable, as is obvious.

This example is quite simple but all the important features of an asymptotic expansion are incorporated here. We now give a slightly more complicated example.

Example 2. Let $0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ be a monotonic increasing sequence of non-negative real numbers with no finite accumulation point and let $m_0, m_1, m_2, \ldots$ be a sequence of natural numbers. Take $Y = \mathbb{R}$, $X = \mathbb{R}_+^\times$, $x_\infty = 0$ and

$$x^{\lambda_0}(\log x)^{m_0-1}, \ldots, x^{\lambda_0}, x^{\lambda_1}(\log x)^{m_1-1}, \ldots, x^{\lambda_1}, \ldots$$

as $\phi_0(x), \phi_1(x), \ldots$ It is easy to check that condition (i) and (ii) are fulfilled. We can thus talk of the following asymptotic expansion for a function f as x tends to 0 , namely

$$f(x) \approx \sum_{k=0}^{m_k} \quad \sum_{m=1}^{m_k} a_{k,m} x^{\lambda_k} (\log x)^{m-1} \quad as \quad x \to 0$$

If we set

$$R_k(x) = f(x) - \sum_{i=0}^{k} \quad \sum_{m=1}^{m_i} a_{i,m} x^{\lambda_i} (\log x)^{m-1}$$

then (3) can be replaced for *this* sequence $\{\phi_n\}$ by

$$R_k(x) = o(x^\rho) \quad as \quad x \to 0 \text{ for every } \quad \rho < \lambda_{k+1} \tag{3$'$}$$

10 and for every $k \geq 0$. This is quite easy to verify. In fact, (3) $\Rightarrow$ (3)$'$ since $x^{-\rho}R_k(x) = x^{\lambda_{k+1}-\rho}O((\log x)^{m_{k+1}-1}) = o(1)$ in view of $\lambda_{k+1} - \rho$ being strictly positive. On the other hand, (3)$'$ $\Rightarrow$ (3) since $R_k(x) - R_{k+1}(x) = \sum\limits_{m=1}^{m_{k+1}} a_{k+1,m}x^{\lambda_{k+1}}(\log x)^{m-1} = O(x^{\lambda_{k+1}}(\log x)^{m_{k+1}-1})$ as $x \to 0$ while $R_{k+1}(x) = o(x^\rho)$ for $\lambda_{k+1} < \rho < \lambda_{k+2}$ implying that $R_k(x) = O(x^{\lambda_{k+1}}(\log x)^{m_{k+1}-1}) + o(x^\rho) = O(x^{\lambda_{k+1}}(\log x)^{m_{k+1}-1})$.

3 The Classical Case

3.1 The Statement of a Theorem

We are interested in explicitly defined function spaces on which the Mellin transformation is an isomorphism. We begin with the classical case where G is the multiplication group $\mathbb{R}_+^\times$ of positive real numbers.

Let $0 \leq \lambda_0 < \lambda_1 < \ldots$ be a strictly increasing sequence of non-negative real numbers with no finite limit point and let $m_0, m_1, m_2, \ldots$ be a sequence of natural numbers. We introduce two function spaces $\mathcal{F}$ and $\mathcal{Z}$ as follows. The space $\mathcal{F}$ is defined as the set of all complex-valued functions F on $\mathbb{R}_+^\times$ such that (i) $F \in C^\infty(\mathbb{R}_+^\times)$, (ii) $F(x)$ behaves like a Schwartz function as x tends to ∞ i.e., $F^{(n)} = o(x^{-\rho})$ as $x \to \infty$ for every ρ and every $n \geq 0$ and (iii)$F(x) \approx \sum\limits_{k=0}^{\infty} \sum\limits_{m=1}^{m_k} a_{k,m}x^{\lambda_k}(\log x)^{m-1}$ as $x \to 0$ (with constants $a_{k,m}$) and this asymptotic expansion is termwise differentiable. $\mathcal{F}$ is clearly nonempty, since the function $F = 0$ is in $\mathcal{F}$. If $\lambda_n = n$ for every $n \geq 0$, then $F(x) = e^{-x}$ (for $x \geq 0$) is in the corresponding $\mathcal{F}$. The space $\mathcal{Z}$ consists of all meromorphic functions $Z(s)$ of the complex variable s for which

1. $Z(s)$ has poles at most at the points $-\lambda_0, -\lambda_1, -\lambda_2, \ldots,$

2. $Z(s) - \sum\limits_{m=1}^{m_k} \frac{b_{k,m}}{(s+\lambda_k)^m}$ for suitable constants $b_{k,m}$ is holomorphic in a **11** neighbourhood of the point $s = -\lambda_k$ and

3. for every polynomial $P(s)$ in s, the function $P(s)Z(s)$ is bounded in every vertical strip $B_{\sigma_1,\sigma_2} = \{s = \sigma + ti \in \mathbb{C} \mid \sigma_1 \leq \sigma \leq \sigma_2\}$ for $-\infty < \sigma_1 < \sigma_2 < \infty$ after deleting small neighbourhoods of $-\lambda_k$ for every k from B_{σ_1,σ_2}. Under the correspondence M between $\mathcal{F}$ and $\mathcal{Z}$ in Theorem 3.1 below, Euler's gamma function $\Gamma(s)$ corresponds in $\mathcal{Z}$ to the function $F(x) = e^{-x}(x \geq 0)$ in $\mathcal{F}$ when $\lambda_n = n$ for every $n \geq 0$.

We shall now state and prove a theorem which has many good applications.

Theorem 3.1. *We have $M : \mathcal{F} \xrightarrow{\sim} \mathcal{Z}$. More precisely, for any F in $\mathcal{F}$,*

$$(MF)(S) = \int_0^\infty F(x)x^s \, d\log x$$

defines a holomorphic function on $\mathbb{C}_+ = \{s = \sigma + ti \in \mathbb{C} \mid \sigma > 0\}$ and its meromorphic continuation belongs to $\mathcal{Z}$. Conversely, if Z is in $\mathcal{Z}$, then

$$(M^{-1}Z)(x) = \frac{1}{2\pi i} \int_{\sigma-\infty i}^{\sigma+\infty i} Z(s)x^{-s} \, ds$$

gives rise to a function in $\mathcal{F}$ independently of σ for $\sigma > 0$. Moreover

$$b_{k,m} = (-1)^{m-1}(m-1)! \, a_{k,m} \tag{5}$$

for every k and m.

3.2

Before we commence the proof of Theorem 3.1, let us make a comment on its statement but, what is more important, namely, also a few remarks to simplify later arguments for the proof of the theorem.

For a moment, denote $M^{-1}Z$ in the statement of Theorem 3.1 by NZ instead. As soon as we know that $M(\mathcal{F}) \subset \mathcal{Z}$ and $N(\mathcal{Z}) \subset \mathcal{F}$, then NM is identity on $\mathcal{F}$ and MN is identity on $\mathcal{Z}$. This follows from Fourier's
12 inversion theorem. Therefore we can write $N = M^{-1}$.

Let $D = \sum\limits_{k=0}^{t} p_k(x)\frac{d^k}{dx^k}$ be a differential operator with coefficients $p_k(x)$ which are polynomials in x having complex coefficients. We say that D is *homothety-invariant* if, for every $f \in C^\infty(\mathbb{R}_+^\times)$ and $\lambda > 0$, we have $D(f(\lambda x)) = (Df)(\lambda x)$ for all x. Clearly D is such an operator, if and only if $p_k(x) = c_k x^k$ for every k. The space of such homothety-invariant differential operators is spanned over $\mathbb{C}$ by $x^k \frac{d^k}{dx^k}$ $(k = 0, 1, 2, \ldots)$. On the other hand,we have

$$\left(x\frac{d}{dx}\right)^2 \overset{\text{defn}}{=} \left(x\frac{d}{dx}\left(x\frac{d}{dx}\right)\right) = x\frac{d}{dx} + x^2\frac{d^2}{dx^2}$$

and generally, if

$$\left(x\frac{d}{dx}\right)^n = \sum_{i=1}^{n} c_{n,i}\ x^i \frac{d^i}{dx^i},$$

then

$$\left(x\frac{d}{dx}\right)^{n+1} = \sum_{i=1}^{n+1} c_{n+1,i}\ x^i \frac{d^i}{dx^i}$$

where $c_{n+1,i} = ic_{n,i} + c_{n,i-1}$. Thus the space of the homothety-invariant differential operators above is just the $\mathbb{C}$-span of the differential operators $1, x\frac{d}{dx}, (x\frac{d}{dx})^2, \ldots$

We now remark that $\mathcal{F}$ is stable under homothety-invariant differential operators (with polynomial coefficients). For this, it suffices to prove, in view of the foregoing, that $\mathcal{F}$ is invariant under $x\frac{x}{dx}$ and in turn to verify condition (iii) for xF' whenever $F \in \mathcal{F}$. Now

$$F(x) \approx \sum_{k=0}^{\infty} \sum_{i=1}^{m_k} a_{k,j} x^{\lambda_k} (\log x)^{j-1}$$

for $F \in \mathcal{F}$ as $x \to 0$ implies that

$$F'(x) \approx \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} a'_{k,j} x^{\lambda} k^{-1} (\log x)^{j-1}$$

where $a'_{k,j} = \lambda_k a_{k,j} + j a_{k,j+1}$ for $1 \leq j \leq m_k - 1$ and $a'_{k,m_k} = \lambda_k a_{k,m_k}$. Thus, for every $F \in \mathcal{F}$,

$$xF'(x) \approx \sum_{k=0}^{\infty} \sum_{j=1}^{m_K} a'_{k,j} x^{\lambda_k} (\log x)^{j-1} \text{ as } x \to 0$$

and so $xF' \in \mathcal{F}$.

13 For $\mathcal{Z}$, we have a corresponding property, namely that $\mathcal{Z}$ is stable under multiplication by polynomials in s. This is quite obvious, since for $z \in \mathcal{Z}$, multiplication by polynomials only serves to improve the situation with regard to conditions (1) and (2), while there is nothing new to be verified regarding condition (3).

We assert further that for $F \in \mathcal{F}$,

$$M(x^k F^{(k)}(x))(s) = (-1)^k (s + k - 1) \ldots s(MF)(s) \tag{6}$$

for every $s \in \mathbb{C}_+$and $k \geq 1$. For $k = 1$, the formula is clear since

$$\begin{aligned}
M(xF'(x))(s) &= \int_0^\infty xF'(x) x^s \, d \log x \\
&= F(x) x^s]_0^\infty - s \int_0^\infty F(x) x^s \, d \log x \\
&= -s(MF)(S), \text{ since the first term is 0.}
\end{aligned}$$

Assuming (6) with $k - 1$ in place of k (≥ 2), we have again, from integration by parts,

$$\begin{aligned}
M(x^K F^{(k)}(x))(s) &= F^{(k-1)}(x) x^{s+k-1}]_0^\infty - (s + k - 1) M(x^{k-1} F^{(k-1)}(x))(s) \\
&= 0 + (-1)^k (s + k - 1) \ldots s(MF)(s).
\end{aligned}$$

For the sake of completeness, we include the following well-known lemma on the interchange of differentiation and integration.

Lemma 3.2. *Let X be a locally compact measure space with dx, a Borel measure on X and let, for an interval $I = (a, b) \subset \mathbb{R}$, $f(x, t)$ be a contin-*
14 *uous function on $X \times I$ satisfying the conditions:*

$$f(x,t) \in L^1(X, dx) \text{ for every } t \in I,$$

$$\frac{\partial}{\partial t} f(x,t) \text{ is continuous and}$$
$$\left|\frac{\partial}{\partial t} f(x,t)\right| \leq \phi(x) \text{ for a } \phi \in L^1(X, dx) \text{ for every } t \in I.$$

Then

$$\frac{d}{dt}\int_X f(x,t)dx = \int_X \frac{\partial}{\partial t} f(x,t)dx.$$

Proof. The hypotheses imply the existence of the integrals $(\phi(t) \stackrel{\text{def}}{=}) \int_X f(x,t)dx$ and $(\psi(t) \stackrel{\text{def}}{=}) \int_X \frac{\partial}{\partial t} f(x,t)dx$. On the other hand, for every t, t' in I, we have $f(x,t') = f(x,t) + (t'-t)\frac{\partial}{\partial t} f(x,\tau)$ for some τ between t and t'. Therefore

$$\left|(\phi(t') - \phi(t))/(t'-t) - \psi(t)\right| \leq \int_X \left|\frac{\partial}{\partial t} f(x,\tau) - \frac{\partial}{\partial t} f(x,t)\right| dx$$

and the lemma follows from Lebesgue's theorem. □

3.3 Proof of Theorem 3.1

(i) First we show that for every $F \in \mathcal{F}, Z = MF$ is in $\mathcal{Z}$.

Let $s = \sigma + ti \in \mathbb{C}_+$ and $0 < \sigma_1 \leq \sigma \leq \sigma_2 < \infty$. Taking ϵ, n with $0 < \epsilon < \sigma_1$ and $n > \sigma_2$, define

$$\phi(x) = \begin{cases} x^{\sigma_1 - \epsilon} \max\limits_{0<x\leq 1}(x^{\epsilon}|F(x)|), & 0 < x \leq 1 \\ x^{\sigma_2 - n} \max\limits_{x\leq 1}(x^{n}|F(x)|), & x > 1 \end{cases}$$

Then it is easy to verify that $\phi \in L^1(\mathbb{R}_+^\times, d\log x)$ and further ϕ dominates $F(x)x^s$. Therefore the integral defining $Z(s)$ converges absolutely and the integrand being holomorphic, it follows, in view of the arbitrary nature of σ_1 and σ_2, that $Z(s)$ is holomorphic in $\mathbb{C}_+$. Since for every σ, σ_2 with $\sigma \leq \sigma_2 < \infty$, $F(x)x^s$ is dominated by a function in

$L^1((1,\infty), d\log x)$, say the restriction of the function ϕ above to $(1,\infty)$, it follows similarly that $\int\limits_1^\infty F(x)x^s\, d(\log x)$ is indeed an entire function of s. Now for every k and $\rho < \lambda_{k+1}$,

$$\begin{aligned} R_k &= F(x) - \sum_{i=0}^{k} \sum_{j=1}^{m_i} a_{i,j}\, x^{\lambda_i} (\log x)^{j-1} \\ &= o(x^\rho) \quad \text{by condition}(3)' \end{aligned} \tag{7}$$

15 Therefore in $(0,1]$, $R_k(x)x^s$ is dominated by the function

$$\psi(x) \overset{def}{=} x^{\sigma_1+\rho} \max_{0<x\leq 1} x^{-\rho} |R_k(x)|$$

which clearly belongs to $L^1((0,1], d(\log x))$ for every σ_1 with $-\lambda_{k+1} < \sigma_1 \leq \sigma$ and $-\sigma_1 < \rho < \lambda_{k+1}$. Thus, for similar reasons as above, $\int\limits_o^1 R_k(x)x^s\, d(\log x)$ is holomorphic in $s(=\sigma+ti)$ for $\sigma > -\lambda_{k+1}$. Finally, for any $\lambda \geq 0$ and $\sigma > 0$,we have

$$\int\limits_0^1 x^{s+\lambda} d(\log x) = \frac{1}{s+\lambda}.$$

Differentiating the integral $j-1$ times with respect to s for $j \geq 1$, we get (in view of Lemma 3.2) that

$$\int\limits_0^1 x^{s+1} (\log x)^{j-1} d(\log x) = \frac{(-1)^{j-1}(j-1)!}{(s+\lambda)^j} \tag{8}$$

Putting together all the facts above and using (7) and (8), we have

$$\begin{aligned} Z(s) &= \left(\int\limits_0^1 + \int\limits_1^\infty F(x)x^s d(\log x) \right) \\ &= \sum_{i=0}^{k} \sum_{j=1}^{m_i} \frac{b_{i,j}}{(s+\lambda_i)^j} + \int\limits_0^1 R_k(x)x^s d(\log x) + \int\limits_0^\infty F(x)x^s d(\log x) \end{aligned} \tag{9}$$

with $b_{ij} = (-1)^{j-1}(j-1)!a_{ij}$ where the second term in (9) represents a holomorphic function of $s(=\sigma + ti)$ for $\sigma > -\lambda_{k+1}$ and the third term is an entire function of s. Since k is arbitrary, we have thus verified conditions (1) and (2) for Z to belong to $\mathcal{Z}$. The existence of a function in $L^1([1,\infty))$ dominating the integrand shows that the third term in (9) is bounded for s in any vertical strip B_{σ_1,σ_2}; for a similar reason, the 16
second term is also bounded in B_{σ_1,σ_2} (with k chosen correspondingly) while the first term is bounded, if we delete therefrom neighbourhoods of the points $-\lambda_0, -\lambda_1, -\lambda_2, \ldots$. Thus condition (3) for Z to be in $\mathcal{Z}$ is fulfilled with $P(s) \equiv 1$. To verify condition (3) for arbitrary $P(s)$ we may assume, without loss of generality, that

$$P(s) = \sum_{i=0}^{n}(-1)^i a_i s(s+1)\ldots(s+i-1)$$

with a_i in $\mathbb{C}$. Then working with $\sum\limits_{i=0}^{n} a_i x^i F^{(i)}(x)$ in place of F above, as is legitimate, we may, in view of the remarks in §3.2, conclude that $P(s)Z(s)$ is bounded in every vertical strip B_{σ_1,σ_2}.

(ii) Let us now prove the converse that for every $Z \in \mathcal{Z}$, $F = M^{-1}Z \in \mathcal{F}$.

Let, for $0 < \sigma_1 < \sigma_2$, L denote the boundary (traversed anticlockwise) of the rectangle in the s-plane with vertices at $\sigma_2 + t_0 i$, $\sigma_1 + t_0 i$, $\sigma_1 - t_0 i$, $\sigma_2 - t_0 i$ (for $t_0 > 0$). Let $x > 0$; then by Cauchy's theorem, $\int\limits_L Z(s)x^{-s}\,ds = 0$, since the integrand is holomorphic in $\mathbb{C}_+$. But, in view of the growth condition satisfied by Z in vertical strips, the integral taken over the horizontal sides of L tends to 0 as t_0 tends to infinity. Thus

$$\int_{\sigma_1-\infty i}^{\sigma_1+\infty i} Z(s)x^{-s}\,ds = \int_{\sigma_2-\infty i}^{\sigma_2+\infty i} Z(s)x^{-s}\,ds$$

enabling us to conclude that, for $x > 0$,

$$F(x) = \frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i} Z(s)x^{-s}\,ds \tag{10}$$

is defined independently of $\sigma(>0)$. The growth condition (3) satisfied by Z ensures the absolute convergence of the integrals involved above. Using Lemma 3.2, we obtain for $k \geq 1$, that

$$x^k F^{(k)}(x) = \frac{1}{2\pi i} \int_{\sigma-\infty i}^{\sigma+\infty i} (-1)^k s(s+1)\dots(s+k-1)Z(s)x^{-s}\, ds \tag{11}$$

17 Condition (3) for Z again guarantees the absolute convergence of the integral and we conclude that $F \in C^\infty(\mathbb{R}_+^\times)$ and further F behaves at infinity like a Schwartz function, since 11 implies that

$$\left|x^{\sigma+k}F^{(k)}(x)\right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|s(s+1)\dots(s+k-1)Z(s)\right| dt < \infty.$$

We are left with proving condition (iii) for F to be in $\mathcal{F}$. Choose ρ such that $\lambda_k < \rho < \lambda_{k+1}$ and let, for $t_0 > 0$, the contours L_1 and L_2 be defined respectively as the boundary of the rectangle with vertices at $\sigma + t_0 i$, $-\rho + t_0 i$, $-\rho - t_0 i$, $\sigma - t_0 i$ and the union of the line segments $\{\sigma - ti \mid t_0 \leq t < \infty\}$, $\{u - t_0 i \mid -\rho \leq u \leq \rho\}$, $\{-\rho + ti \mid -t_0 \leq t \leq t_0\}$, $\{u + t_0 i \mid -\rho \leq u \leq \sigma\}$ and $\{\sigma + ti \mid t_0 \leq t < \infty\}$. While the contour L_1 is traversed in the counter-clock-wise direction, the contour L_2 is covered clockwise from $\sigma - i\infty$. From (10), we have

$$F(x) = \frac{1}{2\pi i}\left(\int_{L_1} + \int_{L_2}\right) Z(s)x^{-s}\, ds.$$

The integration over L_1 gives, by Cauchy's theorem, the sum of the residues of the integrand at the poles inside L_1. Condition (3) for Z implies that the contribution to the integral over L_2 from the horizontal segments tends to 0 as t_0 tends to ∞. Making t_0 tend to ∞, we obtain for $F(x)$ the expression

$$\frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i} Z(s)x^{-s} ds = \frac{1}{2\pi i}\int_{-\rho-\infty i}^{-\rho+\infty i} Z(s)x^{-s} ds + S \tag{12}$$

where S is the sum of the residues of $Z(s)x^{-s}$ at the points $s = -\lambda_0, -\lambda_1, -\lambda_2, \ldots$ The residue at $s = -\lambda_i$ is just the coefficient of $(s + \lambda_i)^{-1}$in the power-series expansion at $-\lambda_i$ of the function

$$x^{\lambda_i} e^{-(s+\lambda_i)\log x} \sum_{m=1}^{m_i} b_{i,m}(s + \lambda_i)^{-m}$$

and is seen to be equal to $\sum\limits_{m=1}^{m_i} x^{\lambda_i}(\log x)^{m-1} a_{i,m}$ with $a_{i,m}$ precisely as **18**
given by (5). In view of this, the integral on the right hand side of (12) is just $R_k(x)$ introduced earlier. However, the integral can be directly majorised by $\frac{1}{2\pi} x^{\rho} \int\limits_{-\infty}^{\infty} |Z(-\rho + ti)|dt$; we thus see for every $k \geq 1$, that $R_k(x) = O(x^{\rho})$ where ρ is arbitrary but subject to the condition that $\lambda_k < \rho < \lambda_{k+1}$. Consequently, $R_k(x) = o(x^{\rho})$ for every $\rho < \lambda_{k+1}$ and for every $k \geq 1$.

We have finally to prove the termwise differentiability of the asymptotic expansion of F as $x \to 0$. By (11) with $k = 1$ and our remarks in §3.2, $-sZ(s)$ is given in $\mathbb{Z}$ and condition (2) is fulfilled for it with

$$b'_{k,m} = \begin{cases} \lambda_k b_{k,m} - b_{k,m+1} & \text{for} \quad 1 \leq m \leq m_k - 1 \\ \lambda_k b_{k,m} & \text{for} \quad m = m_k \end{cases}$$

in place of $b_{k,m}$. This follows from the fact that

$$\begin{aligned} -sZ(s) &= (\lambda_k - (s + \lambda_k))Z(s) \\ &= (\lambda_k - (s + \lambda_k)) \sum_{m=1}^{m_k} \frac{b_k, m}{(s + \lambda_k)}^{m} + \text{ a function holomorphic at } -\lambda_K. \end{aligned}$$

Applying the arguments above to $-sZ(s)$ in place of $Z(s)$, we obtain, as indicated, the function $xF'(x)$ in place of $F(x)$ and further

$$xF'(x) \approx \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} \widetilde{a}_{k,m} x^{\lambda_k} (\log x)^{m-1}$$

where $\widetilde{a}_{k,m} = (-1)^{m-1} b'_{k,m}/(m-1)!$. But using (5), we see that $\widetilde{a}_{k,m}$ is the same as $a'_{k,m}$ defined in §3.2. From our remarks in §3.2, this means precisely that the asymptotic expansion of $F(x)$ as $x \to 0$ can be differentiated termwise once. Iteration of this procedure (using (6)), gives us that termwise differentiation is valid any number of times and Theorem 3.1 is completely proved.

4 The Case of $\mathbb{R}$-fields

4.1

19 In this section, we shall prove the analogues of Theorem 3.1 for the cases when G is multiplicative group $K^*(= K \setminus \{0\})$ of an $\mathbb{R}$-field K, i.e, for $K = \mathbb{R}$ or $\mathbb{C}$.

We first fix some notation applicable to any local field K. For $a \in K$, we define the modulus $|a|_K$ of a by

$$|a|_K = \begin{cases} \text{the rate of change of the measure in } K \text{ under } x \to ax \\ \text{for } x \in K \text{ and } a \neq 0 \\ 0, \quad \text{for } \ a = 0. \end{cases}$$

It is well-known that $|a|_{\mathbb{R}} = |a|$ and $|a|_{\mathbb{C}} = |a|^2$ where $|\ |$ denotes the usual absolute value in $\mathbb{R}$ or $\mathbb{C}$.

We had introduced in §1.2 the group $\Omega(K^\times)$ of quasicharacters of K as the group (with the compact open topology) of continuous homomorphisms of $K^\times$ into $\mathbb{C}^\times$. Then $\Omega(K^\times)^0$, the connected component of the identity, consists of ω_s for $s(= \sigma + ti) \in \mathbb{C}$ defined by $\omega_s(x) = |x|_K^s$ for every $x \in K^\times$. For $\omega \in \Omega(K^*)$, the associated quasicharacter $|\omega(x)|$ is given by $|x|_K^{\sigma(\omega)} = \omega_{\sigma(\omega)}(x)$, $\sigma(\omega)$ being in $\mathbb{R}$. We also set $\Omega_+(K^\times) = \{\omega \in \Omega(K^*) \mid \sigma(\omega) > 0\}$ and for an $\mathbb{R}$-field K, introduce the angular component $\mathrm{ac}(x)$ for $x \in K^\times$ by $\mathrm{ac}(x) = x/|x|$. For $x \in \mathbb{R}^\times$, we have $\mathrm{ac}(x) = \mathrm{sgn}(x)$, the usual sign-function on $\mathbb{R}$.

4.2 The Case $K = \mathbb{R}$

Let us assume, as before, that $0 \leq \lambda_0 < \lambda_1 < \ldots$ is a given strictly increasing sequence of non-negative real numbers with no finite accumulation point and $\{m_k\}_{k \geq 0}$ a given sequence of natural numbers. Correspondingly we define for $K = \mathbb{R}$ (or $G = \mathbb{R}^\times$) the function-spaces $\mathscr{F}$

and $\mathcal{Z}$ as follows. The space $\mathcal{F}(=\mathcal{F}(\mathbb{R}^\times))$ is defined as the set of all complex-valued functions F such that

(i) $F \in C^\infty(\mathbb{R}^\times)$, **20**

(ii) F behaves like a Schwartz function of x as $|x|$ tends to infinity i.e. $F^{(n)} = o(|x|^{-\rho})$ as $|x| \to \infty$ for every ρ and every $n \geq 0$, and

(iii) $F(x) \approx \sum\limits_{k=0}^{\infty} \sum\limits_{m=1}^{m_k} a_{k,m}(\operatorname{sgn} x)|x|^{\lambda_k}(\log|x|)^{m-1}$ as $|x| \to 0$ and this asymptotic expansion is termwise differentiable. We write

$$a_{k,m}(u) = a_{k,m,o} + u a_{k,m,1} \tag{13}$$

for $u = +1$ or -1; this may be regarded as the (trivial) Fourier expansion of functions on the group $\{1, -1\}$.

The group $\Omega(\mathbb{R}^\times) = \{\omega_s(\operatorname{sgn})^p; s \in \mathbb{C}, p = 0, 1\}$ consists of two copies of $\mathbb{C}$ indexed by $p = 0$ or 1.

The space $\mathcal{Z}$ $(= \mathcal{Z}(\Omega(\mathbb{R}^\times)))$ is defined as the set of all complex-valued functions Z on $\Omega(\mathbb{R}^\times)$ such that

(1) $Z(\omega_s(\operatorname{sgn})^p)$ is meromorphic on $\Omega(\mathbb{R}^\times)$ with poles at most for $s = -\lambda_0, -\lambda_1, -\lambda_2, \ldots$,

(2) $Z(\omega_s(\operatorname{sgn})^p) - \sum\limits_{m=1}^{m_k} \frac{b_{k,m,p}}{(s+\lambda_k)^m}$ is holomorphic for s close to $-\lambda_k$, for every $k \geq 0$ and

(3) for every polynomial $P \in \mathbb{C}[s]$ and every σ_1, σ_2, the function $p(s)Z(\omega_s(\operatorname{sgn})^p)$ is bounded for s in a vertical strip B_{σ_1,σ_2}, with neighbourhoods of the points $-\lambda_0, -\lambda_1, \ldots$ removed therefrom.

The following theorem is an immediate consequence of Theorem 3.1 and the definitions given above.

Theorem 4.2. *We have a bijective correspondence $M_{\mathbb{R}}$ (abbreviated as M) between $\mathcal{F} = \mathcal{F}(\mathbb{R}^\times)$ and $\mathcal{Z} = \mathcal{Z}(\Omega(\mathbb{R}^\times))$. More precisely, if $F \in \mathcal{Z}$, then*

$$(MF)(\omega) = \int\limits_{\mathbb{R}^\times} F(x)\omega(x)\, d^\times x \quad \textit{with} \quad d^\times x = \frac{dx}{2|x|_{\mathbb{R}}}$$

defines a holomorphic function on $\Omega_+(\mathbb{R}^\times)$ *which has a meromorphic continuation belonging to* $\mathcal{Z}$*. Conversely, for* $z \in \mathcal{Z}$*, the integral* 21

$$(M^{-1}Z)(x) = \sum_{p=0,1} \Big(\frac{1}{2\pi i} \int_{\sigma-\infty i}^{\sigma+\infty i} Z(\omega_s(\mathrm{sgn})^P)|x|_{\mathbb{R}}^{-s} ds\Big) (\mathrm{sgn}\, x)^{-p}$$

defines a function F *in* $\mathcal{F}$ *independently of* σ*, for* $\sigma > 0$*. Furthermore*

$$b_{k,m,p} = (-1)^{m-1}(m-1)!\, a_{k,m,p}$$

for every k, m *and* p*.*

Proof. By introducing suitable definitions, we shall reduce ourselves to the situation in §3. For the given F and for $x \in \mathbb{R}_+^\times$, let us define

$$F_p(x) = \frac{1}{2}(F(x) + (-1)^p F(-x)), \quad p = 0, 1. \tag{14}$$

Then $F(ux) = F_0(x) + uF_1(x)$ for $u = \pm 1$ and therefore, for every $x \in \mathbb{R}^\times$, we have

$$F(x) = F_0(|x|) + (\mathrm{sgn}\, x)F_1(|x|). \tag{15}$$

Similarly, for Z defined on $\Omega(\mathbb{R}^\times)$, let us introduce a function Z_p on $\mathbb{C}$ for $p = 0, 1$ by the prescription

$$Z_p(s) = Z(\omega_s(\mathrm{sgn})^p) \quad \text{for } s \in \mathbb{C} \tag{16}$$

Then, in a purely formal fashion, we have

$$M_{\mathbb{R}}F = Z \Leftrightarrow MF_p = Z_p \text{ for } p = 0 \text{ and } 1.$$

In fact, let $M_{\mathbb{R}}F = Z$. Then, by the definition of Z_p,

$$\begin{aligned} Z_p(s) &= \int_{\mathbb{R}^\times} F(x)|x|_{\mathbb{R}}^s (\mathrm{sgn}\, x)^p \frac{dx}{2|x|_{\mathbb{R}}} \\ &= \int_0^\infty \frac{1}{2}(F(x) + (-1)^p F(-x)) x^s d(\log x) \end{aligned}$$

$$= (MF_p)(s)$$

22 for $p = 0, 1$. Conversely, let $MF_p = Z_p$ for $p = 1, 2$, Then, any ω in $\Omega(\mathbb{R}^\times)$ being of the form $\omega_s(\mathrm{sgn})^p$ for $p = 0$ or 1 and s in $\mathbb{C}$, $Z(\omega) = Z_p(s)$ for a unique $p = 0$ or 1. But $Z_p(s) = (MF_p)(s)$ and we have only to retrace our steps in the foregoing to conclude that $Z(\omega) = (M_{\mathbb{R}}F)(\omega)$.

Finally, looking at the relations (14) - (16) above between F and F_0, F_1 and between Z and Z_0, Z_1, we conclude from the definitions of the various function spaces that

$$\begin{aligned} F \in \mathcal{F}(\mathbb{R}^\times) &\Rightarrow F_0, F_1 \in \mathcal{F} \\ &\Rightarrow Z_0, Z_1 \in \mathcal{Z} \qquad \text{by Theorem 3.1} \\ &\Rightarrow Z \in \mathcal{Z}(\Omega(\mathbb{R}^\times)) \end{aligned}$$

Similarly, we have

$$\begin{aligned} Z \in \mathcal{Z}(\Omega(\mathbb{R}^\times)) &\Rightarrow Z_0, Z_1 \in \mathcal{Z} \\ &\Rightarrow F_0, F_1 \in \mathcal{F} \qquad \text{by Theorem 3.1} \\ &\Rightarrow F \in \mathcal{F}(\mathbb{R}^\times). \end{aligned}$$

From the uniqueness of the coefficients in asymptotic expansions and the definitions (13) and (14), it follows that for $p = 0, 1$, the coefficients $a_{k,m,p}$ correspond to $F_p(x)$ in $\mathcal{F}$ (for $\mathbb{R}_+^\times$) in the same way as $a_{k,m}$ to $F \in \mathcal{F}(\mathbb{R}^\times)$ for every k, m. Similarly from (16), the coefficients $b_{k,m,p}$ feature in the expansion of $Z_p(s)$ at $s = -\lambda_k$ for $p = 0, 1$. The last 23
assertion of the theorem now follows from Theorem 3.1 together with the fact that, for $p = 0, 1$, the function F_p and Z_p correspond to each other (under the correspondence M in §3.1.) □

4.3 The Case $K = \mathbb{C}$

We shall now prove an analogue of Theorem 3.1 for the complex case.

Let, as before, $0 \le \lambda_0 \le \lambda_1 < \lambda_2 < \ldots$ be a strictly increasing sequence of non-negative real numbers with no finite accumulation point and $\{m_k\}_{k\ge 0}$ a sequence of natural numbers. Let $\mathcal{F}$ be the associated space of complex-valued functions F on $\mathbb{C}^\times$ such that

(i) $F \in \mathbb{C}^{\infty}(\mathbb{C}^{*})$, (ii) $F(x)$ behaves like a Schwartz function as $|x|_{\mathbb{C}}$ tends to infinity, namely,

$$\frac{\partial^{a+b}F}{\partial x^{a}\partial\overline{x}^{b}}(x) = o(|x|_{\mathbb{C}}^{-\rho})$$

as $|x|_{\mathbb{C}} \to \infty$, for every ρ and for every $a, b, \geq 0$ and (iii) $F(x) \approx \sum\limits_{k=0}^{\infty}\sum\limits_{m=1}^{m_k} a_{k,m}(\mathrm{ac}(x)).|x|_{\mathbb{C}}^{\lambda_k}(\log|x|_{\mathbb{C}})^{(m-1)}$ as $|x| \to 0$ with $a_{km} \in C^{\infty}(\mathbb{C}_1^{\times})$ for every k, m, is an asymptotic expansion which is termwise differentiable and uniformly in $\mathrm{ac}(x)$ even after termwise differentiation (in the sense that for every $k \geq 0$ and for every $\rho < \lambda_{k+1}$ and for given $\epsilon > 0$, there exists δ independent of $\mathrm{ac}(x)$ such that for $|x|_{\mathbb{C}} < \delta$, we have

$$|R_k(x)| = \left|\sum_{i=0}^{k}\sum_{m=1}^{m_i} a_{i,m}(\mathrm{ac}(x))|x|_{\mathbb{C}}^{\lambda_i}(\log|x|_{\mathbb{C}})^{m-1}\right| \leq \epsilon|x|_{\mathbb{C}}^{\rho}$$

and further, similar inequalities hold even after the termwise differentiation of the asymptotic expansion).

Since for every k, m we have $a_{k,m} \in C^{\infty}(\mathbb{C}_1^{\times})$, we have the Fourier expansion $a_{k,m}(u) = \sum\limits_{p\in\mathbb{Z}} a_{k,m,p}u^{p}$.

Putting $|x|_{\mathbb{C}} = r$ and $u = \mathrm{ac}(x) = \mathfrak{e}(\theta)$ for $x \in \mathbb{C}^{\times}$, we have $x = r^{\frac{1}{2}}.\mathrm{u}$
24 and

$$d^{\times}x \overset{\text{def}}{=} \frac{dx}{2\pi|x|_{\mathbb{C}}} = d(\log r)d\theta.$$

It is easy to check that

$$r\frac{\partial}{\partial r} = \frac{1}{2}(x\frac{\partial}{\partial x} + \overline{x}\frac{\partial}{\partial\overline{x}}) \quad \text{and} \quad \frac{\partial}{\partial\theta} = 2\pi i(x\frac{\partial}{\partial x} - \overline{x}\frac{\partial}{\partial\overline{x}}) \tag{17}$$

Suppose D is a differential operator of the form

$$\sum_{0\leq a,b\leq n} p_{a,b}(x,\overline{x})\frac{\partial^{a+b}}{\partial x^{a}\partial\overline{x}^{b}}$$

with polynomials $P_{a,b}$ in $x, \overline{x}$ having coefficients in $\mathbb{C}$. We call D homothety-invariant, if for every $F \in C^{\infty}(\mathbb{C}^{\times})$ and $t \in \mathbb{C}^{\times}$, we have $(DF)(tx) = D(F(tx))$ identically in x. The space of such homothety-invariant differential operators is easily seen to be spanned over $\mathbb{C}$ by differential operators of the form $(x\frac{\partial}{\partial x})^a(\overline{x}\frac{\partial}{\partial \overline{x}})^b$ for $a, b \geq 0$ in $\mathbb{Z}$ and hence, in view of (17), the differential operators $D_{a,b} = \frac{1}{(2\pi i)^b} r^a \frac{\partial^{a+b}}{\partial r^a \partial \theta^b}$ for $a, b, \geq 0$ in $\mathbb{Z}$ also generate the above space over $\mathbb{C}$.

For $F \in C^\infty(\mathbb{C}^\times)$ and for fixed $r = |x|_{\mathbb{C}}$, we have the (absolutely convergent) Fourier expansion

$$F(x) = \sum_{p \epsilon \mathbb{Z}} F_p(r) u^p. \tag{18}$$

We shall characterise F being in $\mathcal{F}(\mathbb{C}^\times)$ in terms of an alternative set of conditions involving Fourier coefficients F_p in (18) as follows:

(I) $F \in C^\infty(\mathbb{C}^\times) \Leftrightarrow F_p \in C^\infty(\mathbb{R}_+^\times)$ for every $p \in \mathbb{Z}$ and further, for every $a, b, \geq 0$ in $\mathbb{Z}$ and r_1, r_2 with $0 < r_1 < r_2 < \infty$, we have

$$\sup_{p \in \mathbb{Z}, r_1 \leq r \leq r_2} \left| p^b F_p^{(a)}(r) \right| < \infty \tag{19}$$

Moreover, when these equivalent conditions hold, the Fourier expansion (18) is termwise differentiable, i.e.

$$r^{-a} D_{a,b} F(x) = \frac{1}{(2\pi i)^b} \frac{\partial^{a+b}}{\partial r^a \partial \theta^b} F(x) = \sum_{p \epsilon \mathbb{Z}} p^b F_p^{(a)}(r) u^p \quad \text{for every } a, b \geq 0 \text{ in } \mathbb{Z} \tag{20}$$

(II) $F \in C^\infty(\mathbb{C}^\times)$ and $F(x)$ behaves like a Schwartz function as $r \to$ **25**
$\infty \Leftrightarrow \{F_p\}_{p \epsilon \mathbb{Z}}$ is as in I above and further, for every $a, b, \geq 0$ and every σ, we have

$$\sup_{p \epsilon \mathbb{Z}, r \geq 1} |r^\sigma p^b F_p^{(a)}(r)| < \infty \tag{21}$$

(III) $F \in C^\infty(\mathbb{C}^\times)$ and the asymptotic expansion $F(x) \approx \sum\limits_{k=0}^{\infty} \sum\limits_{m=1}^{\infty} a_{k,m}(u)$ $r^{\lambda_k} (\log r)^{m-1}$ as $r \to 0$ is termwise differentiable uniformly for u in $\mathbb{C}_1^\times \Leftrightarrow \{F_p\}_p \in \mathbb{Z}$ is as in I above; for every p, there exists an asymptotic expansion $F_p(x) \approx \sum\limits_{k=0}^{\infty} \sum\limits_{m=1}^{m_k} a_{k,m,p} r^{\lambda_k} (\log r)^{m-1}$ as $r \to$

0 which is termwise differentiable and further, for every $k \geq 0$, $a, b \geq 0$ in $\mathbb{Z}$ and $\sigma > -\lambda_{k+1}$, we have

$$\sup_{p\in\mathbb{Z}, 0<r\leq 1} \left| r^{\sigma+a} p^b R_{k,p}^{(a)}(r) \right| < \infty \tag{22}$$

where, for $x \in \mathbb{C}^\times$,

$$R_k(x) \overset{def}{=} F(x) - \sum_{i=0}^{k} \sum_{m=1}^{m_i} a_{i,m}(u) r^{\lambda_i} (\log r)^{m-1} = \sum_{p\in\mathbb{Z}} R_{k,p}(r) u^P$$

with

$$R_{k,p}(r) = F_p(r) - \sum_{i=0}^{k} \sum_{m=1}^{m_i} a_{i,m,p} r^{\lambda_i} (\log r)^{m-1}.$$

In (21), we may replace $r^\sigma p^b F_p^{(a)}(r)$by $r^\sigma (D_{a,b}F)_p(r)$ and similarly, in (22), $r^{\sigma+a} p^b R_{k,p}^{(a)}(r)$ by $r^\sigma (D_{a,b}R_k)_p(r)$.

We now give the proof of assertions, I, II, III above.

Proof of I. We shall use Lemma 3.2 in the following special form: namely if, for every $p \in \mathbb{Z}$, $f_p(t)$, $f_p^{'}(t)$ are continuous on $I = (a, b) \subset \mathbb{R}$
26 such that

$$\sum_{p\in\mathbb{Z}} |f_p(t)| < \infty, |f_p^{'}(t)| \leq c_p, \sum_{p\in\mathbb{Z}} c_P < \infty \text{ for every } t \in I, \text{ then}$$

$$\frac{d}{dt}\left(\sum_{p\in\mathbb{Z}} f_p(t)\right) = \sum_{p\in\mathbb{Z}} f_p'(t) \tag{23}$$

If $F \in C^\infty(\mathbb{C}^\times)$, then, for $r^{-a} D_{a,b}F$, we have corresponding to (18), the Fourier expansion

$$(r^{-a} D_{a,b}F)(x) \left(= \frac{1}{(2\pi i)^b} \frac{\partial^{a+b} F}{\partial r^a \partial \theta^b}(x)\right) = \sum_{p\in\mathbb{Z}} \widetilde{F}_p(r) u^P \tag{24}$$

converging absolutely (and hence uniformly for u in $\mathbb{C}_1^\times$), for every $a, b, \geq 0$ in $\mathbb{Z}$. Therefore, for every $p \in \mathbb{Z}$ we have

$$\widetilde{F}_p(r) = \int_0^1 r^{-a} D_{a,b} F(x) \mathfrak{e}(-p\theta) d\theta \text{ by definition of } \widetilde{F}_p$$

$$= p^b \int_0^1 \frac{\partial^a}{\partial r^a} F(x)\mathfrak{e}(-p\theta)d\theta, \quad \text{on integrating by parts } b \text{ times,}$$

$$= p^b \frac{d^a}{dr^a}\left(\int_0^1 F(x)\mathfrak{e}(-p\theta)d\theta\right), \quad \text{using (23)}$$

$$= p^b F_p^{(a)}(r)$$

Thus, for the expression on the left hand side of (19), we have the bound $\sup\limits_{r_1 \le |x|_{\mathbb{C}} \le r_2} |r^{-a} D_{a,b} F(x)|$ which is certainly finite. We have therefore proved (19) and (20).

To prove the reverse implication in I, we use (19) with $b+2$ in place of b and the fact that $\sup\limits_{r_1 \le r \le r_2} |F_0^{(a)}(r)| \le c < \infty$ for every $a, b \ge 0$ in $\mathbb{Z}$. Then, for a constant c' which may be taken to satisfy $c' \ge c$, we have

$$\sum_{p \in \mathbb{Z}} \left| p^b F^{(a)}(r) \right| \le c'(1 + 2\sum_{p=1}^{\infty} p^{-2}) < \infty \quad \text{for} \quad r_1 \le r \le r_2.$$

Applying (23) a sufficient number of times, we get $(r^{-a} D_{a,b} F)(x) = \sum\limits_{p \in \mathbb{Z}} p^b F_p^{(a)}(r) u^p$.

Proof of II. It is easy to check that any F in $C^\infty(\mathbb{C}^\times)$ behaves like a 27
Schwartz function as $|x|_{\mathbb{C}} \to \infty$ if and only if, for every σ and every $a, b, \ge 0$ in $\mathbb{Z}$, we have

$$\sup_{r \ge 1} \left| r^\sigma \frac{\partial^{a+b} F}{\partial r^a \partial \theta^b}(x) \right| < \infty$$

However, the last condition implies (21) for every σ and every $a, b \ge 0$ in $\mathbb{Z}$ in view of (20). Conversely, in can be deduced by assuming (21) with $(b+2)$ in place of b, just as above.

Proof of III. A given F in $C^\infty(\mathbb{C}^\times)$ has an asymptotic expansion as $r \to 0$ which is termwise differentiable uniformly in u, if and only if, for every $a, b, k \geq 0$ in $\mathbb{Z}$ and every $\sigma > -\lambda_{k+1}$,

$$\sup_{0<|x|_{\mathbb{C}}\leq 1} \left| r^{\sigma} \frac{1}{(2\pi i)^{b}} r^{a} \frac{\partial^{a+b} R_k}{\partial r^{a} \partial \theta^{b}}(x) \right| < \infty.$$

But proceeding exactly as above (in the proof of I) the last condition is seen to be equivalent to (22) being valid for every $a, b, k \geq 0$ in $\mathbb{Z}$ and every $\sigma > -\lambda_{k+1}$.

Finally, let us remark that $\mathcal{F}(\mathbb{C}^{\times})$ is stable under homothety-invariant differential operators, as is clear from the foregoing.

We may now proceed to define the function-space $\mathcal{Z}(\Omega(\mathbb{C}^{\times}))$. First, we observe that $\Omega(\mathbb{C}^{\times}) = \{\omega_s(\mathrm{ac})^p; s \in \mathbb{C}, p \in \mathbb{Z}\}$ consists of a countable number of copies of $\mathbb{C}$ indexed by p in $\mathbb{Z}$. The space $\mathcal{Z}(\Omega(\mathbb{C}^{\times}))$ is defined as the set of all complex-valued functions Z such that

1. Z is meromorphic on $\Omega(\mathbb{C}^{\times})$ with poles at most when $s = -\lambda_0$, $-\lambda_1, -\lambda_2, \ldots$,

2. $Z(\omega_s(\mathrm{ac})^p) - \sum\limits_{m=1}^{m_k} \frac{b_{k,m,p}}{(s+\lambda_k)^m}$ is holomorphic for s close enough to $-\lambda_k$, where $b_{k,m,p}$ are constants and $k \geq 0$ is arbitrary in $\mathbb{Z}$; and

28 3. for any $p \in \mathbb{Z}$ and s belonging to any vertical strip B_{σ_1,σ_2} with neighbourhoods of the points $-\lambda_0, -\lambda_1, -\lambda_2, \ldots$. removed therefrom and for every polynomial $P(s, p)$ in s and p with coefficients in $\mathbb{C}$ we have
$$|P(s, p) Z(\omega_s(\mathrm{ac})^p)| \leq c'' \tag{25}$$
for a constant c'' depending on P, Z, σ_1, σ_2 and the neighbourhoods of $-\lambda_0, -\lambda_1, \lambda_2, \ldots$ removed but neither on s nor on p.

For $Z \in \mathcal{Z}(\Omega(\mathbb{C}^{\times}))$, let us define the functions Z_p by $Z_p(s) = Z(\omega_s(\mathrm{ac})^p)$ for $s \in \mathbb{C}$ and $p \in \mathbb{Z}$. We may thus look upon Z as a collection of functions Z_p for $p \in \mathbb{Z}$. It is clear that $\mathbb{Z}(\Omega(\mathbb{C}^{\times}))$ is stable under $Z_p \mapsto QZ_p$ where Q is an arbitrary polynomial in p and s, for every $p \in \mathbb{Z}$. We are now in a position to state and prove

Theorem 4.3. *We have a bijective correspondence $M_{\mathbb{C}}$ between $\mathcal{F}(\mathbb{C}^{\times})$ and $\mathbb{Z}(\Omega(\mathbb{C}^{\times}))$. More precisely, for any F in $\mathcal{F}(\mathbb{C}^{\times})$,*

$$(M_{\mathbb{C}}F)(\omega) = \int\limits_{\mathbb{C}^{\times}} F(x)\omega(x)d^{\times}x \quad \textit{with} \quad d^{\times}x = \frac{dx}{2\pi|x|_{\mathbb{C}}} \tag{26}$$

defines a holomorphic function on $\Omega_{+}(\mathbb{C}^{\times})$ and its meromorphic continuation is in $\mathcal{Z}(\Omega(\mathbb{C}^{\times}))$. Conversely, for any $Z \in \mathcal{Z}(\Omega(\mathbb{C}^{\times}))$ and $x \in \mathbb{C}^{\times}$

$$(M_{\mathbb{C}}^{-1}Z)(X) = \sum_{p\in\mathbb{Z}} \left(\frac{1}{2\pi i} \int\limits_{\sigma-\infty i}^{\sigma-\infty i} Z(\omega_s(\mathrm{ac})^p)|x|_{\mathbb{C}^{-s}ds} \right) \mathrm{ac}(x)^{-p}$$

defines a function in $\mathcal{F}(\mathbb{C}^{\times})$ independently of σ for $\sigma > 0$. Moreover, for every $k \leq 0, m \leq 1$ and p in $\mathbb{Z}$, we have

$$b_{k,m,p} = (-1)^{m-1}(m-1)!a_{k,m-p}.$$

Proof. As in the case of $\mathbb{R}^{\times}$, we do fall back on Theorem 3.1 for the proof of this theorem; however, an additional complication is caused here by the maximal compact subgroup $\mathbb{C}_1^{\times}$ of $\mathbb{C}^{\times}$ being infinite, unlike the case of $\mathbb{R}^{\times}$ and forces us to seek, every time, estimates which are **29**
uniform with regard to p.

From I, II, III above, we know, for $F \in \mathcal{F}(\mathbb{C}^{\times})$, that $F_p \in \mathcal{F}$ (for $\mathbb{R}_{+}^{\times}$) for every $p \in \mathbb{Z}$ and further, $D_{a,b}F \in \mathcal{F}(\mathbb{C}^{\times})$ for every $a, b \geq 0$ in $\mathbb{Z}$. Substituting (18) into (26). with $d^{\times}x = d(\log r)d\theta)$, we have (on integrating first with respect to θ), $(M_{\mathbb{C}}F)_p = M(F_{-p})$. On the other hand, let us define, for Z_P, the function $Z_P^{\sharp}$ by

$$Z_p^{\sharp}(s) = (-1)^{a+b}(s+a-1)\ldots s.p^b Z_p(s) \tag{27}$$

for $a, b \geq 0$ in $\mathbb{Z}$ and let $D = D_{a,b}$. Again, substituting now (24) into (26) with $DF \in \mathcal{F}(\mathbb{C}^{\times})$ in place of F, we have similarly,

$$\begin{aligned} ((M_{\mathbb{C}}(DF))_P)(s) &= (M(r^a\widetilde{F}_{-p}))(s) \\ &= (M((-p)^b r^a F_p^{(a)}(r)))(s) \qquad \text{by (20)} \\ &= (-p)^b(-1)^a(s+a-1)\ldots s(M(F_p))(s) \qquad \text{by (6)} \end{aligned}$$

$$= (-1)^{a+b} p^b (s+a-1)\dots s(M(F_{-p}))(s).$$

Thus, if $Z = M_{\mathbb{C}}F$, then

$$Z_p^\sharp = (M_{\mathbb{C}}(DF))_p. \tag{28}$$

Let us assume now that $F \in \mathcal{F}(\mathbb{C}^\times)$ and show that $M_{\mathbb{C}}F$ gives a function in $\mathcal{Z}(\Omega(\mathbb{C}^\times))$. From above, $F_p \in \mathcal{F}(\text{for } \mathbb{R}_+^\times)$ for every $p \in \mathbb{Z}$ and therefore $(M_{\mathbb{C}}F)_p = M(F_{-p}) \in \mathcal{Z}$ for $\mathbb{R}_+^\times$ by Theorem 3.1. On the other hand, for $\sigma > -\lambda_{k+1}$, we have from (9),

$$(MF_p)(s) = \sum_{i=0}^{k}\sum_{m=1}^{m_i} \frac{(-1)^{m-1}(m-1)!a_{i,m,p}}{(s+\lambda_i)^m} + \int_0^1 R_{k,p}(r) r^s d(\log r) + \int_1^\infty F_P(r) r^s d(\log r).$$

Now $a_{i,m}(u) = \sum\limits_{p\in\mathbb{Z}} a_{i,m,p}u^p$ for $u \in \mathbb{C}_1^\times$ and therefore, $|a_{i,m,p}| \leq \|a_{i,m}\|_\infty$
30 (the supremum norm), for every p. Thus

$$\left|\sum_{i=0}^{k}\sum_{m=1}^{m_i} \frac{(-1)^{m-1}(m-1)!a_{a,m,p}}{(s+\lambda_i)^m|}\right| \leq \sum_{i=0}^{k}\sum_{m=1}^{m_i}(m-1)!\|a_{i,m}\|_\infty \epsilon^{-m} \text{ for } s+\lambda_i \geq \epsilon > 0.$$

Further, by (22), $|R_{k,p}(r)r^s| \leq c_1 r^{\sigma_1-\sigma_0}$ with $c_1 = \sup\limits_{0<r\leq 1, p\in\mathbb{Z}} |r^{\sigma_0}R_{k,p}(r)|$ for $-\lambda_{k+1} < \sigma_o < \sigma_1 \leq \sigma$ and we have consequently

$$\left|\int_0^1 R_{k,p}(r)r^s d(\log r)\right| \leq \frac{c_1}{\sigma_1-\sigma_0}$$

Again, using (21), we have

$$\left|F_p(r)r^s\right| \leq \sup_{r\geq 1, p\in\mathbb{Z}} \left|r^n.F_p(r)\right|.r^{\sigma_2-n} = c_2.r^{\sigma_2^{-n}}, \quad \text{say}$$

leading to

$$\left|\int_1^\infty f_p(r)r^s d(\log r)\right| \leq \frac{c_2}{n-\sigma_2}, \text{ for } \sigma \leq \sigma_2 < n.$$

Putting these together, we conclude that for every p in $\mathbb{Z}$, $((M_\mathbb{C}F)_p))(s)$ is bounded for s in vertical strips B_{σ_1,σ_2} with neighbourhoods of $-\lambda_0, -\lambda_1, -\lambda_2, \ldots$ removed therefrom and for $-\lambda_{k+1} < \sigma_1 < \sigma < \infty$ and all $k \geq 0$. To prove the corresponding assertion for $P(s,p)((M_\mathbb{C}F)_p)(s)$ with arbitrary polynomials $P(s,p)$, we may assume, without loss of generality, that $P(s,p) = (-1)^{a+b}p^b(s+a-b)\ldots s$ for some $a, b \geq 0$ in $\mathbb{Z}$. Now we can work with $D_{a,b}F$ instead of F and the required assertion follows in view of (28). The verification for $M_\mathbb{C}F$ to belong to $\mathcal{Z}$ is nothing but checking the conditions 1), 2), 3) for $(M_\mathbb{C}F)_p$ which are all satisfied as seen above.

Conversely, let us assume that Z is in $\mathcal{Z}$. Then, for every p in $\mathbb{Z}$, we have $Z_p \in \mathcal{Z}$ (for $\mathbb{R}_+^\times$) and by Theorem 3.1, $F_p \overset{\text{def}}{=} M^{-1}(\mathcal{Z}_p)$ is in $\mathcal{F}$ (for $\mathbb{R}_+^\times$). Therefore, for $\sigma > 0$, **31**

$$2\pi r^\sigma F_p(r) = \int_{-\infty}^\infty Z_{-p}(\sigma+ti)r^{-ti}dt.$$

On the other hand, there exists a constant $c_3 > 0$ depending on σ such that, for every $t \in \mathbb{R}$ and $p \in \mathbb{Z}$,

$$\max(1, |t|^2)|Z_p(\sigma+ti)| \leq c_3,$$

and hence we have

$$2\pi r^\sigma |F_p(r)| \leq \int_{-\infty}^\infty |Z_p(\sigma+ti)|dt \leq 4c_3.$$

Replacing Z by $Z^\sharp$ for which $(Z^\sharp)_p = Z_p^\sharp$ given by (27) from every $p \in \mathbb{Z}$, we obtain for $\sigma > 0$ and every $D_{a,b}$that

$$\sup_{p\in\mathbb{Z}, r>0} |r^\sigma (D_{a,b}F)_p(r)| < \infty.$$

But if we choose σ with $-\lambda_{k+1} < \sigma < -\lambda_k$ then we have, by our arguments concerning (12) in §3.3, that

$$2\pi r^{\sigma} R_{k,p}(r) = \int_{-\infty}^{\infty} Z_{-p}(\sigma + ti) dt.$$

On the other hand, there exists a constant c_4 depending on σ such that

$$\max(1, |t|^2)\ |Z_{-p}(\sigma + ti)| \leq c_4$$

for every $t \in \mathbb{R}$ and every $p \in \mathbb{Z}$. This gives first

$$2\pi r^{\sigma} |R_{k,p}(r)| \leq 4c_4,$$

and, as before, on replacing Z by $Z^{\sharp}$, we get

$$\sup_{p \in \mathbb{Z}, r \geq 0} |r^{\sigma} (D_{a,b} R_k)_p(r)| < \infty$$

for every $D_{a,b}$ with $a, b \geq 0$ in $\mathbb{Z}$ and every σ with $-\lambda_{k+1} < \sigma < -\lambda_k$. This implies (22) and putting all these together, we conclude from I, II, III that $M_{\mathbb{C}}^{-1} Z \in \mathcal{F}(\mathbb{C}^{\times})$.

32 The last assertion of Theorem 4.3 follows from the fact that $(M_{\mathbb{C}} F)_p = M(F_{-p})$ for every $p \in \mathbb{Z}$ and from the corresponding assertion of Theorem 3.1.

Our theorem is now completely proved. □

REMARK. The above theory is essentially one-dimensional, since $\Omega(K^{\times})$ for $K = \mathbb{R}, \mathbb{C}$ (or even a p-field) is a one-dimensional complex Lie group and we have used function theory only in the case of one complex variable. To generalise our results to general compactly generated abelian groups, there seems to be another problem, namely, that of guessing the right kind of asymptotic expansions to be employed, for which perhaps one needs the powerful intuition of a good applied mathematician!

5 The Case of p-fields

5.1

We deal now with a bijective correspondence between function spaces $\mathcal{F}$ and $\mathcal{Z}$ associated, in a similar manner, with p-fields which, as we recall, are just finite algebraic extensions of Hensel's field $\mathbb{Q}_p$ of p-adic numbers or fields $\mathbb{F}((t))$ of Laurent series in one variable t with coefficients in a finite field $\mathbb{F}$.

We first recall certain well-known facts about p-fields.

Let K be a p-field, $R = \{x \in K; |x|_k \leq 1\}$, the maximal compact subring of K and $P = \{x \in K; |x|_k < 1\}$ the (unique) maximal ideal of R. Then the residue field R/P is a finite field $\mathbb{F}_q$ consisting of q elements (with q equal to some power of p in the former case and equal to the cardinality of $\mathbb{F}$ in latter case). In fact, if $K = \mathbb{F}((t))$, then $R = \mathbb{F}[[t]]$, the ring of power-series in t over $\mathbb{F}, P = t\mathbb{F}[[t]]$, and further $\mathbb{F}$ and R/P are isomorphic and have the same cardinality, say q, so that we can write $\mathbb{F}_q$ **33**
instead of $\mathbb{F}$ in this case.

We take dx to be a Haar measure on K which is so normalised as to make the measure $m(R)$ of R equal to 1. Now $R = \coprod_a (P + a)$, the disjoint union of finitely many cosets $P + a$ of R modulo P and $m(P + a) = m(P)$ for every a. Therefore we have

$$1 = m(R) = \sum_{a \mod P} m(P + a) = [R : P]m(P)$$
$$i.e, \qquad m(P) = q^{-1}.$$

More generally, for any $e \in \mathbb{Z}$, we have

$$m(P^e) = \begin{cases} 1/[R : p^e] = 1/([R : P][P : P^2] \ldots [P^{e-1} : P^e]) & \text{for } e \geq 0 \\ [P^e : R] = [P^e : P^{e+1}] \ldots [P^{-1} : R] & \text{for } e < 0 \end{cases}$$
$$= q^{-e}$$

For $a \in K^\times$, we define the *order* $\operatorname{ord}(a)$ of a by $aR = P^{\operatorname{ord}(a)}$. Then $|a|_K$ introduced in §4 as the rate of change of measure in K under $x \mapsto ax$, is just $\frac{m(aR)}{m(R)} = m(P^{\operatorname{ord}(a)})$, i.e, $|a|_k = q^{-\operatorname{ord}(a)}$.

The group $R^\times$ of units in R is precisely $R\backslash P = \{x \in K; |x|_K = 1\}$ and $m(R^\times) = m(R) - m(P) = 1 - q^{-1}$.

We define $d^\times x$ as the Haar measure on K, with the obvious normalisation, namely, the one that makes the measure of $R^\times$ to be 1. It is now easy to see that

$$d^\times x = \frac{1}{1 - q^{-1}} \frac{dx}{|x|_K}$$

From the definition of ord(a) for $a \in K^\times$ above, it is clear that we have an exact sequence

$$\{1\} \to R^\times \xrightarrow{j} K^\times \xrightarrow{\text{“ord”}} \mathbb{Z} \to \{0\}$$

34 where j is the inclusion map and "ord" is the map $a \mapsto \text{ord}(a)$. For any e in $\mathbb{Z}$ the complete inverse image in K^* under the "ord" map i.e., $\{a \in K; |a|_k = q^{-e}\} = p^e \smallsetminus p^{e+1}$ is a compact open subset of K and called the e^{th} wall (around 0).

We choose a cross-section for the map "ord":$K^\times \to \mathbb{Z}$ above, by selecting an element π in $K^\times$ with ord $\pi = 1$, It is to be noted that π is not unique but once chosen, it is fixed once for all.

For every $x \in K^\times$, we define the angular component ac(x) as $x\pi^{-\text{ord}(x)}$ in $R^\times$ and again the definition is not intrinsic in view of the ambiguity about the choice of π.

The map $x \mapsto (\text{ord}(x), \text{ac}(x))$ gives, from above, an isomorphism $K^\times \simeq \mathbb{Z} \times \mathbb{R}^\times$. Thus $\Omega(K^\times) \simeq \mathbb{C}^\times \times (R^\times)^*$ under the isomorphism $\omega \Leftrightarrow (z, \mathcal{X})$ where, for $e \in \mathbb{Z}$ and $u \in R^\times$, we have $\omega(\pi^e u) = z^e \mathcal{X}(u)$. We could have written instead, for any $x = \pi^e u \in K^\times, \omega(x) = |x|_k^s \mathcal{X}(\text{ac}(x))(= q^{-es}$ $\mathcal{X}(u))$. But $|x|_K$ for $x \in K^\times$ always belongs to the set $\{q^e; e \in \mathbb{Z}\}$ and we have therefore to take s only modulo $2\pi i/\log q$. As a result, the right parameter to be taken is actually $z = q^{-s}$ and *not* s. As stated above, $\Omega(K^\times)$ consists of countably many copies of $\mathbb{C}^\times$ indexed by $(R^\times)^*$, the discrete dual of $R^\times$ (which now corresponds to the indexing set $\mathbb{Z}$ in the case of $K = \mathbb{C}$). For any function Z on $\Omega(K^\times)$ and for $\omega \in \Omega(K^\times)$ with $\omega(\pi^e u) = z^e \mathcal{X}(u)$ as mentioned above, we write $Z_{\mathcal{X}}(\omega)$ for $Z(\omega)$. Let $\Omega_+(K^\times) = \{\omega \in \Omega(K^\times); \sigma(\omega) > 0\}$.

We now proceed to define the spaces $\mathcal{F}(K^\times)$ and $\mathcal{Z}(\Omega(K^\times))$ for p-fields K. While doing so, we guided by the existence of accepted ana-

logues, for p-fields, of C^∞ functions behaving like Schwartz functions **35**
on $\mathbb{R}$-fields and also by the subsequent applications we have in mind.

5.2

Let $\Lambda = \{\lambda \in \mathbb{C}; \lambda \text{ modulo } 2\pi i/\log q; Re\lambda \geq 0\}$ be a given finite set and $\{m_k; k \in \wedge\}$ be a given set of natural numbers. Let $\mathcal{F}(K^\times)$ be the space of complex-valued functions F on $K^\times$ such that

(i) F is *locally constant* i.e, for any $x \in K^\times$, there exists a natural number n depending on x such that $F(x) = F(y)$ for all y with $y - x \in xP^n$,

(ii) $F(x) = 0$ for $|x|_k \gg 1$ (i.e, for all sufficiently large $|x|_K$), and

(iii) for $|x|_K \ll 1$ (i.e, for all sufficiently small $|x|_K$), we have

$$F(x) = \sum_{\lambda\in\Lambda} \sum_{m=1}^{m_\lambda} a_{\lambda,m}(\mathrm{ac}(x))|x|_K^\lambda (\log |x|_k)^{m-1} \tag{29}$$

with locally constant functions $a_{\lambda,m}$ on $\mathbb{R}^\times$.

Since $a_{\lambda,m}(u)$ are locally constant on $R^\times$, we may write

$$a_{\lambda,m}(u) = \sum_{\mathcal{X}\in(R^\times)^*} a_{\lambda,m,\mathcal{X}}\mathcal{X}(u) \tag{30}$$

as a finite Fourier series, with constant coefficients $a_{\lambda,m,\mathcal{X}}$.

For any $F \in \mathcal{F}(K^\times)$, we know that its restriction to the e^{th} wall $p^e \setminus p^{e+1}$ is locally constant and therefore we have, for $u \in R^\times$, the finite Fourier series

$$F(\pi^e u) = \sum_{\mathcal{X}\in(R^\times)^*} c_{e,\mathcal{X}}\mathcal{X}(u) \tag{31}$$

with constant coefficients $c_{e,\mathcal{X}}$. The set $\{\mathcal{X} \in (R^\times)^*; c_{e,\mathcal{X}} \neq 0$ for some e in $\mathbb{Z}\}$ is finite. Indeed, for all $e < 0$in $\mathbb{Z}$ with $|e|$ large, we have $F(\pi^e u) = 0$ for all u and therefore $c_{e,\mathcal{X}} = 0$ for all such e. From (29) and (30) we
have, for all sufficiently large e, **36**

$$c_{e,x} = \sum_{\lambda\in\wedge}\sum_{m=1}^{M_\lambda} a_{\lambda,m,\mathcal{X}}\, q^{-\lambda e}(-e\log q)^{m-1} \tag{32}$$

and the number of $a_{\lambda,m}$ involved and hence from (30), the number of $\mathcal{X}$ with $a_{\lambda,m,\mathcal{X}} \neq 0$ is finite. For the remaining finitely many e again, the number of $\mathcal{X}$with $c_{e,\mathcal{X}} \neq 0$ is finite, since (31) is a finite sum. The space $\mathcal{Z}(\Omega(K^\times))$ is defined as the set of complex-valued functions Z on $\Omega(K^\times)$ for which

(1) for every $\mathcal{X} \in (R^\times)^*$, there exist constants $b_{\lambda,m,\mathcal{X}}$ such that

$$Z_{\mathcal{X}}(z) - \sum_{\lambda\in\wedge}\sum_{m=1}^{m_\lambda}\frac{b_{\lambda,m,\mathcal{X}}}{(1-q^{-\lambda_z})^m}$$

is a polynomial in z and $z^{-1}$with complex co-efficients, and

(2) for almost all $\mathcal{X}$, $Z_{\mathcal{X}}(z)$ is identically zero.

Let, for every $m \geq 1$, the numbers $e_{m,1}, \ldots, e_{m,m}$ be defined by the following identity in t:

$$t^{m-1} = \sum_{j=1}^{m} e_{m,j}\frac{(t+j-1)(t+j-2)\ldots(t+1)}{(j-1)!} \tag{33}$$

Actually $e_{m,j}$ are integers and further, for $m = 1, 2\ldots$

$$e_{m,m} = (m-1)!; \quad e_{i,j} = 0 \quad \text{for} \quad i < j, \tag{34}$$

5.3

We may now state and prove

Theorem 5.3. *We have an isomorphism M_k (or briefly, M) between the spaces $\mathcal{F}(K^\times)$ and $\mathcal{Z}(\Omega(K^\times))$ for any p-field K. More precisely, for any $F \in \mathcal{F}(K^\times)$.*

$$(M_K F)(\omega) = \int_{K^\times} F(x)\omega(x)d^\times x$$

defines a function on $\Omega_+(K^\times)$ and its meromorphic continuation is in $\mathcal{Z}(\Omega(K^\times))$. Conversely, if Z is in $\mathcal{Z}(\Omega(K^\times))$, then $(M_K^{-1}Z)(x)$ is given by

$$\sum_{\mathcal{X}\in(R^\times)^*} (\mathrm{Residue}_{z=0}(Z_{\mathcal{X}}(z)z^{-\mathrm{ord}(x)-1}))\mathcal{X}(\mathrm{ac}(x))^{-1} \tag{35}$$

and defines a function in $\mathcal{F}(K^\times)$. Moreover, we have **37**

$$b_{\lambda,m,\mathcal{X}} = \sum_{j=m}^{m_\lambda} e_{j,m}(-\log q)^{j-1} a_{\lambda,j,\mathcal{X}^{-1}} \tag{36}$$

for every λ, m and $\mathcal{X}$, where $e_{j,m}$ are given by (33).

Proof. First, let F be given in $\mathcal{F}(K^\times)$ and $\omega \in \Omega_+(K^\times)$, so that $\omega(\pi^e u) = z^e\mathcal{X}(u)$ and $|z| = q^{-\sigma(\omega)} < 1$. Now

$$\begin{aligned}(M_K F)_{\mathcal{X}}(z) &= \int\limits_{K^\times} F(x)\omega(x)d^\times x\\ &= \sum_{e\in\mathbb{Z}} \int\limits_{R^\times} F(\pi^e u)z^e\mathcal{X}(u)d^\times u \qquad (37)\\ &= \sum_{e\in\mathbb{Z}} c_{e,\mathcal{X}^{-1}}z^e \qquad (38)\end{aligned}$$

on substituting (31) and using the orthogonality relations $\int\limits_{R^\times} \mathcal{X}'(u)\mathcal{X}(u)\, d^\times u$ is 1 or 0 according as $\mathcal{X}' = \mathcal{X}^{-1}$ or not. All the steps used above are justified, recalling that $|z| < 1$.

As remarked earlier, the number of $\mathcal{X}$ in $(R^\times)^*$ with the property that $c_{e,\mathcal{X}} \neq 0$ for some e (depending on $\mathcal{X}$) in $\mathbb{Z}$ finite. Hence, expect for finitely many $\mathcal{X}$, we have $(M_K F)_{\mathcal{X}} = 0$. In view of (37) (38), (31) and (32), we have

$$(M_k F)_{\mathcal{X}}(z) = \sum_e{}' c_{e,\mathcal{X}^{-1}}z^e + \sum_{e=e_0}^{\infty}\left\{\sum_{\lambda\in\wedge}\sum_{m=1}^{m_\lambda} a_{\lambda,m,\mathcal{X}^{-1}}q^{-\lambda e}(-e\log q)^{m-1}\right\}z^e$$

where $\sum_e{}'$ is a finite summation and $e_0 \geq 0$ in $\mathbb{Z}$ depends on F. Therefore, writing $\equiv$ to imply equality modulo $\mathbb{C}[z, z^{-1}]$, we have **38**

$$(M_kF)_{\mathcal{X}}(z) \equiv \sum_{e=0}^{\infty}\left(\sum_{\lambda\in\Lambda}^{\infty}\sum_{m=1}^{m_\lambda} a_{\lambda,m,\mathcal{X}^{-1}} q^{-\lambda e}(-e\log q)^{m-1}\right)z^e$$
$$= \sum_{e=0}^{\infty}\left(\sum_{\lambda\in\Lambda}\sum_{m=1}^{m_\lambda} a_{\lambda,m,\mathcal{X}^{-1}} q^{-\lambda e}(-\log q)^{m-1}\sum_{j=1}^{m} e_{m,j}\binom{e+j-1}{j-1}\right)z^e,$$

using 33,

$$= \sum_{\lambda\in\Lambda}\sum_{j=1}^{m_\lambda}\left(\sum_{m=j}^{m_\lambda} e_{m,j}(-\log q)^{m-1} a_{\lambda,m,\mathcal{X}^{-1}}\right)\sum_{e=0}^{\infty}\binom{e+j-1}{j-1}(q^{-\lambda}z)^e,$$

on reversing the order of the summations over j and m in the preceding line and then interchanging them with the summation over e. We remark that this is justified $|z| < 1$ and the summations over m, j and λ are al finite, using (36) to define $b_{\lambda,m\mathcal{X}}$ and the fact that $|q^{\lambda}z| \leq |z| < 1$,we have, finally,

$$(M_KF)_{\mathcal{X}}(z) \equiv \sum_{\lambda\in\Lambda}\sum_{j=1}^{m_\lambda}\frac{b_{\lambda,j,\,\mathcal{X}}}{(1-q^{-\lambda}z)^j}$$

and therefore M_kF is indeed in $\mathcal{Z}(\Omega(K^{\times}))$.

Conversely, let Z be given in $\mathcal{Z}(\Omega(K^{\times}))$, so that, for every $\mathcal{X} \in (R^{\times})^*$, we have $Z_{\mathcal{X}}(z) = \sum_{e\in\mathbb{Z}} d_{\mathcal{X},e}z^e$. where $0 < |z| < 1$ and further, in view of conditions (1) and (2) satisfied by Z, we have the following:

> $\{\mathcal{X} \in (R^{\times})^*; d_{\mathcal{X},e} \neq 0$ for some e in $\mathbb{Z}\}$ is a finite set, $d_{\mathcal{X},e} = 0$ for $e < 0$ in $\mathbb{Z}$ with $|e|$ large and $d_{\mathcal{X},e} = \sum_{\lambda\in\Lambda}\sum_{m=1}^{m_\lambda} b_{\lambda,m,\mathcal{X}}\binom{e+m-1}{m-1}q^{-\lambda e}$ for all large $e > 0$ in $\mathbb{Z}$. (39)

From (31) and (38), by reversing the process, we get, for $e \in \mathbb{Z}$ and $u \in R^{\times}$ that

$$(M_K^{-1}Z)(\pi^e u) = \sum_{\mathcal{X}} d_{\mathcal{X},e}\mathcal{X}(u)^{-1}$$
$$= \sum_{\mathcal{X}\in(R^{\times})^*}(\text{Residue}_{z=0}(Z_{\mathcal{X}}(z)z^{-e-1}))\mathcal{X}(u)^{-1}. \qquad (40)$$

39 It is now clear that $M_K^{-1}Z$ satisfies conditions (i) and (ii) defining $\mathcal{F}(K^{\times})$.

Also, for any given $\mathcal{X}$ and λ, equations (36) for $1 \leq m \leq m_\lambda$ considered as a set of equations in the m_λ unknowns $a_{\lambda,j,\mathcal{X}^{-1}}$ $(1 \leq j \leq m_\lambda)$ are solvable, in view of (34). From (40) and (38), we have, for all large $e > 0$ in $\mathbb{Z}$.

$$\begin{aligned}(M_K^{-1}Z)(\pi^e u) &= \sum_{\mathcal{X}} \left\{ \sum_{\lambda\in\Lambda} \sum_{m=1}^{m_\lambda} b_{\lambda,m,\mathcal{X}} \binom{e+m-1}{m-1} q^{-\lambda,e} \right\} \mathcal{X}(u)^{-1} \\ &= \sum_{\mathcal{X}} \left[\sum_{\lambda\in\Lambda} \sum_{m=1}^{m_\lambda} \left\{ \sum_{j=m}^{m_\lambda} e_{j,m}(-\log q)^{j-1} a_{\lambda,j,\mathcal{X}^{-1}} \right\} \right. \\ &\qquad \left. \binom{e+m-1}{m-1} q^{-\lambda e} \right] \mathcal{X}(u)^{-1},\end{aligned}$$

using the solvability of equations (36) with $1 \leq m \leq m_\lambda$ for $a_{\lambda,j,\mathcal{X}^{-1}}$. Again, reversing the order of the summations over j and m and then using (30) and (33), we have, for all large $e > 0$ in $\mathbb{Z}$,

$$\begin{aligned}(M_K^{-1}Z)(\pi^e u) &= \sum_{\lambda\in\Lambda} \sum_{j=1}^{m_\lambda} a_{\lambda,j}(u)^{-\lambda e} \left\{ \sum_{m=1}^{j} e_{j,m} \binom{e+m-1}{m-1} \right\} (-\log q)^{j-1} \\ &= \sum_{\lambda\in\Lambda} \sum_{m=1}^{m_\lambda} a_{\lambda,m}(\mathrm{ac}(x))|x|_k^\lambda (\log |x|_k)^{m-1}, \text{ with } x = \pi^e u.\end{aligned}$$

Thus, condition (29) is proved for $M_k^{-1}Z$ with $|x|_K \ll 1$, implying that $M_K^{-1}Z$ is in $\mathcal{F}(K^\times)$. □

Remark 1. In (29), we have *assumed* (while defining $\mathcal{F}(K^\times)$) that the functions $a_{\lambda,m}$ on $R^\times$ are locally constant. However, we shall now show that this is actually a *consequence* of F being locally constant on $K^\times$. For this, is suffices to prove that the functions

$$|x|_k^\lambda (\log |x|_K)^{m-1} \text{ for } \lambda \in \Lambda \text{ and } 1 \leq m \leq m_\lambda$$

on $\{|x|_k = q^{-e}$; e sufficiently large in $\mathbb{Z}\}$ are linearly independent over $\mathbb{C}$, If possible, let $a_{\lambda,m,1}$ be complex constants such that **40**

$$\sum_{\lambda\in\Lambda} \sum_{m=1}^{m_\lambda} a_{\lambda,m,1} |x|_K^\lambda (\log |x|_K)^{m-1} = 0 \tag{41}$$

for all $|x|_k \ll 1$. Then, proceeding exactly as in the proof of the first half of Theorem 5.3, we obtain $b_{\lambda,m,1}$ related to $a_{\lambda,m,1}$ as in (36) with $\mathcal{X} = 1$ such that

$$\sum_{\lambda\in\Lambda}\sum_{m=1}^{m_\lambda} \frac{b_{\lambda,m,1}}{(1-q^{-\lambda}z)^m} \equiv 0 \pmod{\mathbb{C}[z,z^{-1}]}. \tag{42}$$

Since, on the left side, $(1-q^{-\lambda}z)^{-m}$ has a pole q^λ and the points q^λ for $\lambda \in \Lambda$ are all distinct, (42) necessarily implies that $b_{\lambda,m,1}$ are all 0. Therefore, in view of (36) with $\mathcal{X} = 1$, the constants $a_{\lambda,m,1}$ in (41) are all 0.

Remark 2. Formula (35) for the inverse transform M_K^{-1} looks, on the face of it, quite different from its counterpart in the case of $\mathbb{R}$-fields. But the analogy will be obvious, if we rewrite it, for $\sigma > 0$, as

$$(M_K^{-1}Z)(x) = \sum_{\mathcal{X}\in(R^\times)^*}\left(\frac{\log q}{2\pi i}\int_{\sigma-\frac{\pi i}{\log q}}^{\sigma+\frac{\pi i}{\log q}} Z_{\mathcal{X}}(q^{-s})|x|_k^{-s}\,ds\right)\mathcal{X}(\mathrm{ac}(x))^{-1}$$

where now $\pi = 3.14\ldots$ (the length of the circumference of a circle of unit diameter!). In fact, this is immediate from (35), since $Z_{\mathcal{X}}(z)^{-e-1}$ is holomorphic in $0 < |z| < 1$ and for $\sigma > 0$, we have

$$\begin{aligned}\text{Residue}_{\,z=0}(Z_{\mathcal{X}}(z)z^{-e-1}) &= \frac{1}{2\pi i}\oint_{|z|=q^{-\sigma}} Z_{\mathcal{X}}(z)z^{-e}d(\log z)\\ &= \frac{\log q}{2\pi i}\int_{\sigma-\frac{\pi i}{\log q}}^{\sigma+\frac{\pi i}{\log q}} Z_{\mathcal{X}}(q^{-s})|x|_k^{-s}ds\end{aligned}$$

(on setting $z = q^{-s}$).

APPENDIX

POISSON FORMULA OF HECKE TYPE

This appendix is based on a preprint entitled "On a generalization of **41**
the Fourier transformation" by T. Yamazaki [55], where the special case corresponding to $\varkappa = 1 - \frac{1}{m}$ for $m = 2, 4, 6, \ldots$ below has been discussed. The Poisson formula that we shall derive here goes back, in its simplest form, to Hecke [14]. Our generalisation, besides being of interest on its own, may be expected to provide, together with a suitable enlargement of this theory so as to cover all the local fields, the 'metaplectic group' to be associated with diagonal forms of $m \geq 2$. In this connection, we should also refer to a series of papers by T. Kubota where he has generalised the Fourier transformation and proved a poisson formula for his transformation (See [30]).

1 The Unitary Operator W

We recall Theorem 3.1 on the bijective correspondence between the spaces $\mathcal{F}$ and $\mathcal{Z}$ associated with $\mathbb{R}_+^\times$.

Choosing $\lambda_k = k, m = 1$ for $k = 0, 1, 2 \ldots$, let us remark the space for $\mathbb{R}_+^\times$ can be characterised also as the space of $F \in C^\infty(\mathbb{R}_+^\times)$ such that $F(x)$ behaves like a Schwartz function as $x \to \infty$ and further $F \in C^\infty([0, \infty))$ by defining $F(0) = \lim_{x \to 0} F(x)$. The last condition is seen to be equivalent to F in $C^\infty(\mathbb{R}_+^\times)$ having a termwise differentiable asymptotic expansion $F(x) \approx \sum_{n=0}^\infty a_n x^n$ as $x \to 0$; one has merely to define $F(0) = a_o$, in order to show that conditions (iii) for F in $\mathcal{F}$ implies "$FC^\infty([0, \infty))$". The spaces $\mathcal{F}$ can also be described as the space of complex-valued functions obtained by restricting to $R_+^\times$, Schwartz functions on $\mathbb{R}$ (i.e, C^∞ functions $G(x)$ on $\mathbb{R}$ which behave like Schwartz functions as $|x| \to \infty$); **42**
this can be seen by using, for example, a theorem of H. Whitney [54] that C^∞ functions on closed subsets of $\mathbb{R}^n$ admit C^∞ extensions to the whole of $\mathbb{R}^n$.

The space $\mathcal{Z}$ for $\mathbb{R}_+^\times$ can be described alternatively as the family of complex-valued functions $Z(s)$ on $\mathbb{C}$ such that $Z(s)(\Gamma(s))^{-1}$ is an entire function of s and further, for every polynomial $P \in \mathbb{C}[s]$, the function $P(s)Z(s)$ is bounded in any vertical strip B_{σ_1,σ_2} with neighbourhoods of the points $0, -1, -2, \ldots$ removed therefrom. This is clear, since $1/\Gamma(s)$ is an entire function of s with simple zeros at $0, -1, -2, \ldots$

For any $a > 0$, the function $F(x) = e^{-ax}$ is clearly in $\mathcal{F}$ and further,

$$M(e^{-ax})(s) = \int_0^\infty e^{-ax} x^s d(\log x) = a^{-s}\Gamma(s) \tag{43}$$

almost by the definition of $\Gamma(s)$. More generally, for any τ in $\mathbb{C}$ with $Im(\tau) > 0$, we get, by the principle of analytic continuation, from (43), that

$$M(e^{i\tau x}(s) = (-i\tau)^{-s}\Gamma(s)$$

where $(-i\tau)^{-s} = e^{-sLog(-i\tau)}$ with Log denoting the principal branch of the logarithm. We also recall the following asymptotic formula for the gamma-function: namely,

$$|\mathrm{r}(s)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}} \exp\left(-\frac{\pi}{2}|t|\right)(1 + o(1)) \tag{44}$$

for $s = \sigma + ti$ in any vertical strip B_{σ_1,σ_2}, as $|s| \to \infty$.

Let us fix, once for all, a real number $\mathfrak{x} > 0$. Then, from (44), it is immediate that

$$\left|\frac{\Gamma(s)}{\gamma(\mathfrak{x} - s)}\right| = |t|^{2\sigma-\mathfrak{x}}(1 + o(1)) \tag{45}$$

43 for $s = \sigma + ti$ in a vertical strip B_{σ_1,σ_2}.

For any Z in $\mathcal{F}$, we define $Z^\times$ by

$$Z^\times(s) = \Gamma(s)\frac{Z(\mathfrak{x} - s)}{\Gamma(\mathfrak{x} - s)}$$

Then $Z(s)/\Gamma(s) = z(\mathfrak{x} - s)/\Gamma(\mathfrak{x} - s)$ therefore, an entire function. For any polynomial P, we have $P(s)Z(s)$ bounded in vertical strips with neighbourhoods of $0, -1, -2, \ldots$ removed therefrom and (45) then implies the same property for $Z^\times$ instead of Z. Thus $Z^\times$ is again in $\mathcal{Z}$ and moreover it is trivial to check that $(Z^\times)^\times) = Z$.

For any Z in $\mathcal{Z}$, if we define

$$\|Z\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|z(\frac{1}{2}\mathfrak{x} + ti)\right|^2 dt$$

then the growth condition on Z implies $\|Z\| < \infty$ and it is clear that $\| \quad \|$ is a norm in $\mathcal{Z}$. Since the left hand side of (45) is 1 for $s = \frac{1}{2}\mathfrak{x} + ti$, it is easily seen that $\|Z^{\times}\| = \|Z\|$ for every Z in $\mathcal{Z}$. We can define, similarly, for F in $\mathcal{F}$,

$$\|F\|^2 = \int_0^{\infty} |F(x)|^2 x^{\mathfrak{x}} d\log x.$$

The integral is finite since $\mathfrak{x} > 0$ and F behaves like a Schwartz function at infinity. We have again a norm in $\mathcal{F}$ given by $\| \quad \|$ but, as we shall see, it is the same as one obtained by transporting to $\mathcal{F}$ the norm from $\mathcal{Z}$ under $M^{-1}\colon \mathcal{Z} \xrightarrow{\sim} \mathcal{F}$.

We assert that $\|F\|^2 = \|MF\|^2$. In fact, if $Z = MF$, then

$$Z\left(\frac{1}{2}\mathfrak{x} + ti\right) = \int_0^{\infty} F(x)x^{\frac{\mathfrak{x}}{2}}, x^{ti} d\log x$$

is just the Fourier transform of $F(x)x^{\mathfrak{x}/2}$. Since $\frac{1}{2\pi}dt$ and $d\log x$ are dual **44**
measures, the Plancherel theorem gives

$$(\|MF\|^2 =)\|Z\|^2 = \|F\|^2.$$

We now define an operator $W : \mathcal{F} \to \mathcal{F}$ as the composite of the operators $M, Z \mapsto Z^{\times}$ and M^{-1}; more explicitly, we have, for any F in $\mathcal{F}$,

$$WF = M^{-1}((MF)^{\times}). \tag{46}$$

From $(Z^{\times})^{\times}$, it is immediate that $W^2 F = F$. Further

$$\|WF\| = \|M^{-1}((MF)^{\times})\| = \|(MF)^{\times}\| = \|MF\| = \|F\|.$$

Thus W is a unitary operator of order 2 on the pre-Hilbert space $\mathcal{F}$.

We shall make use of the following formula, later on: namely, for any $F \in \mathcal{F}$,

$$(WF)(0) = \frac{(MF)(\mathfrak{X})}{\Gamma(\mathfrak{X})} \tag{47}$$

This is easy to prove; in fact, if $(WF)(x) \approx a'_0 + a'_1 x + \ldots$ as $x \to 0$, then, by Theorem 3.1, $(M(WF))(S) - \frac{a'_0}{s}$ is holomorphic at $s = 0$ which implies that $\lim\limits_{s\to 0} \frac{(M((WF))(S)}{\Gamma(s)} = a'_o = WF(0)$. But $\frac{(M(WF))(s)}{\Gamma(s)} = \frac{(MF)(\mathfrak{X}-s)}{\Gamma(\mathfrak{X}-s)}$ and taking limits on both sides as $s \to 0$, we see that (47) is immediate.

Remarks. One can give another proof of (47) by using the relation

$$(WF)(x) = \int\limits_0^\infty k(xy)F(y)y^{\mathfrak{X}} d\log y$$

where $k(x) = \sum\limits_{n=0}^{\infty} \frac{(-x^n)^n}{n!\Gamma(n+\mathfrak{X})} = x^{-\frac{1}{2}(\mathfrak{X}-1)} J_{\mathfrak{X}-1}(2x^{\frac{1}{2}})$ and $J_{\mathfrak{X}-1}$ is the usual Bessel function of order $\mathfrak{X} - 1$.

It should be stressed that $\mathfrak{X}$ is involved in the definition of W.

2 A Poisson Formula

45 Consider a triple $\{\lambda, \mathfrak{X}, \gamma\}$, where λ and $\mathfrak{X}$ are positive real numbers and γ is a complex number. Following Hecke, we say that a complex-valued function ϕ on $\mathbb{C}$ is a function of signature $\{\lambda, \mathfrak{X}, \gamma\}$ if

1) $(s - \mathfrak{X})\phi(s)$ is an entire function of s and further ϕ is (at most) of polynomial growth in any vertical strip B_{σ_1,σ_2} for every $\sigma_1 < \sigma_2$.

2) $R(s) = (\frac{\lambda}{2\pi})^s \Gamma(s)\phi(s)$ satisfies the function equation $R(s) = \gamma R(\mathfrak{X} - s)$, and

3) for all sufficiently large σ, $\phi(s)$ is represented by an absolutely convergent Dirichlet series $\sum\limits_{n=1}^{\infty} \frac{a_n}{n^s}$.

Our condition (1) above is a variant of Hecke's assumption [14]): $1)'$ $(s-\mathfrak{X})\phi(s)$ is an entire function genus. Conditions 1), 2), 3) above are together equivalent to conditions $1)'$, 2), 3). With regard to 3), we should mention that there exist Dirichlet series e.g, $\sum(-1)^n/(\sqrt{n}(\log n)^s)$ convergent for all values of s, but never absolutely convergent. If there exists a function $\phi \neq 0$, of signature $\{\lambda, \mathfrak{X}, \gamma\}$, then condition (2) above implies $\gamma^2 = 1$, as is obvious on applying $s \mapsto \mathfrak{X} - s$ to the functional equation.

Theorem. *Suppose that $\phi \neq 0$, is a functional of signature $\{\lambda, \mathfrak{X}, \gamma\}$ and further, let*

$$a_0 = \gamma \left(\frac{\lambda}{2\pi}\right)^{\mathfrak{X}} \Gamma(\mathfrak{X}) Residue_{s=\mathfrak{X}} \phi(s).$$

Then we have, for every F, in $\mathcal{F}$, the Poisson formula

$$\sum_{n=0}^{\infty} a_n (WF)\left(\frac{2\pi n}{\lambda}\right) = \gamma \sum_{n=0}^{\infty} a_n F\left(\frac{2\pi n}{\lambda}\right)$$

Proof. From (46) and the functional equation for $R(s)$, we get

$$\frac{(M(WF))(s)}{\Gamma(s)} R(s) = \frac{(MF)(\mathfrak{X}-s)}{\Gamma(\mathfrak{X}-s)} \gamma R(\mathfrak{X}-s)$$

i.e. **46**

$$(M(WF))(s)\left(\frac{\lambda}{2\pi}\right)^{s} \phi(s) = \gamma (MF)(\mathfrak{X}-s)\left(\frac{\lambda}{2\pi}\right)^{\mathfrak{X}-s} \phi(\mathfrak{X}-s) \qquad (48)$$

Since ϕ has got at most a simple pole at $s\ (= \sigma + ti) = \mathfrak{X}$ and holomorphic everywhere else, we see, in view of Theorem 3.1 applied to WF (with $\lambda_k = k, m_k = 1$ for all $k \geq 0$) that the left side of (48) is holomorphic if $s \neq \mathfrak{X}, 0, -1, -2, \ldots$ For similar reasons, the right side of (48) is holomorphic for $\mathfrak{X} - s \neq, 0, -1, -2, \ldots$ i.e, for $s \neq 0, \mathfrak{X}, \mathfrak{X}+1, \mathfrak{X}+2, \ldots$. Thus the function of s represented by (48) is holomorphic except for $s = 0, \mathfrak{X}$ at most. Moreover, from condition (1) satisfied by ϕ and an application of Theorem 3.1 to F and WF again, we see that, even after multiplication by a polynomial $P(s)$, both sides of (48) remain bounded in any given vertical strip B_{σ_1,σ_2} with neighbourhoods of the poles above removed therefrom. Thus, we have, in particular, for any σ with $0 < \sigma < \mathfrak{x}$, that

$$\frac{1}{2\pi i}\int\limits_{\sigma-\infty i}^{\sigma-\infty i}(M(WF))(s)\left(\frac{\lambda}{2\pi}\right)^{s}\phi(s)\,ds = \frac{\gamma}{2\pi i}\int\limits_{\sigma-\infty i}^{\sigma+\infty i}(MF)(\mathfrak{x}-s)\left(\frac{\lambda}{2\pi}\right)^{\mathfrak{x}-s}\phi(\mathfrak{x}-s)\,ds, \quad (49)$$

both integrals converging absolutely. On the left side of (49), we can shift the line integration from $Re(s) = \sigma$ to $Re(s) = \sigma_0$ for any sufficiently large $\sigma_0 > 0$, provided that we take into account the residue of the integrand at $s = \mathfrak{x}$. The nice growth conditions satisfied by the integrands (in vertical strips) make, as before, the integrals on horizontal segments of fixed length tend to zero, as the segments recede to infinity on either side of the real axis. In this manner, we obtain that the left hand side of (49) is equal to

$$\frac{1}{2\pi i}\int\limits_{\sigma_0-\infty i}^{\sigma_0+\infty i}(M(WF))(s)\sum_{n=1}^{\infty}a_n\left(\frac{\lambda}{2\pi n}\right)^{s}ds - (M(WF))(\mathfrak{x})\left(\frac{\lambda}{2\pi}\right)^{\mathfrak{x}}\text{Residue}_{s=\mathfrak{x}}\phi(s)$$

47 which is the same as $\sum\limits_{n=1}^{\infty}a_n(WF)\left(\frac{2\pi n}{\lambda}\right) - \gamma a_o F(0)$. We have used here, the relation (47) with WF in place of F, the definition of a_o and the fact that $\gamma = \pm 1$; further, we have interchanged the integration on the line $Res = \sigma_o$ and the summation over n as justifiable without difficulty and also appealed to Theorem 3.1. By applying entirely similar arguments to the right hand side of (49), we see that it is equal to $\gamma\left(\sum\limits_{n=1}^{\infty}a_nF\left(\frac{2\pi n}{\lambda}\right) - \gamma a_o(WF)(0)\right)$ and our theorem is completely proved. □

3 Relation with Hecke's Theory

As we have remarked earlier, the theorem proved above, in a special but quite typical case, goes back to Hecke whose theory of the correspondence between Dirichlet series with functional equations and modular forms is well-known. We now make the connection of our theorem with Hecke's explicit.

For a complex number τ with $Im(\tau) > 0$, let us set $F(x) = e^{i\tau x}$ for $x \in \mathbb{R}_+^\times$. Then we know, from, above, that $(MF)(s) = \Gamma(s)(-i\tau)^{-s}$ and this gives us **48**

$$\begin{aligned}(W(e^{i\tau x}))(x) &\overset{defn}{=} (M^{-1}(\Gamma(s)\frac{(M(e^{i\tau x}))(\mathfrak{X}-s)}{\Gamma(\mathfrak{X}-s)}))(x)\\ &= (M^{-1}(\Gamma(s)(-i\tau)^{-(\mathfrak{X}-s)}))(x)\\ &= (M^{-1}(\Gamma(s)(-i\tau)^{-\mathfrak{X}}\left(\frac{i}{\tau}\right)^{-s}))(x)\\ &= (i\tau)^{-\mathfrak{X}}e^{-ix/\tau}.\end{aligned}$$

Thus, taking $F(x) = e^{i\tau x}$ in the theorem above, we get the relation

$$\sum_{n=0}^{\infty} a_n \mathfrak{e}\left(-\frac{n}{\lambda\tau}\right) = \gamma(-i\tau)^{\mathfrak{X}} \sum_{n=0}^{\infty} a_n \mathfrak{e}\left(\frac{n\tau}{\lambda}\right)$$

which becomes more transparent, on being restated as follows. Namely, under Hecke's correspondence

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \mapsto f(\tau) = \sum_{n=0}^{\infty} a_n \mathfrak{e}\left(\frac{n\tau}{\lambda}\right)$$

with $a_o = \gamma\left(\frac{\lambda}{2\pi}\right)^{\mathfrak{X}}\Gamma(\mathfrak{X})\text{Residue}_{s=\mathfrak{X}}\phi(s)$, the functional equation $R(s) = \gamma R(\mathfrak{X}-s)$ of $R(s) = \left(\frac{\lambda}{2\pi}\right)^s\Gamma(s)\phi(s)$ corresponds to the transformation-law $f\left(-\frac{1}{\tau}\right) = \gamma(-i\tau)^{\mathfrak{X}}f(\tau)$ which merely says that the function $f(\tau)$ obviously holomorphic in τ for $\text{Im}(\tau) > 0$ behaves like a modular form of weight $\mathfrak{X}$ under the transformations from the subgroup of $(PSL_2)(\mathbb{R})$ generated by $\tau \mapsto \tau + \lambda$ and $\tau \mapsto -1/\tau$. This was indeed the starting point of Hecke theory ([15]).

We go back now, to the general case starting from an arbitrary F in $\mathcal{F}$ and proceed to give an interpretation of the set-up above as follows. For any u in $\mathbb{R}$, let us denote by $\xi(u)$, the multiplication by e^{iux} of any function in $\mathcal{F}$ and further define $\eta(u)$ on u on $\mathcal{F}$ by **49**

$$\eta(u)F = W\xi(-u)W\,F$$

for any F. Let $M_p(\mathbb{R})$ (respectively $M_p(Z)$) denote the subgroup of the group of unitary automorphisms of $\mathcal{F}$ generated by $\xi(u)$ and $\eta(u)$ as u varies over $\mathbb{R}$ (respectively over $\mathbb{Z}.\lambda$). Finally, let us fix a function $\phi \neq 0$ of signature $\{\lambda, \mathfrak{X}, \gamma\}$ and define for any F in $\mathcal{F}$, a function θ_F on $M_p(\mathbb{R})$ by

$$\theta_F(g) = \sum_{n=0}^{\infty} a_n(gF)\left(\frac{2\pi n}{\lambda}\right) \quad \text{for} \quad g \in M_p(\mathbb{R})$$

Then , for any g_0 in $M_p(\mathbb{Z})$,

$$\theta_F(g_o g) = \theta_F(g) \text{ for every } g \in M_p(\mathbb{R}).$$

Also, it can be proved that $w = \mathfrak{e}\left(\frac{\mathfrak{X}}{4}\right)W$ belongs to $M_p(\mathbb{R})$. Moreover, if $(M_p)'(\mathbb{Z})$ denotes the subgroup of $M_p(\mathbb{R})$ generated by $\xi(u)$ for all u is $\mathbb{R}$ and w, then

$$\theta(g'_o g) = \mathcal{X}(g'_o)\theta_F(g) \quad (g \in M_p(\mathbb{R}))$$

for every g'_o in $(M'_p)(\mathbb{Z})$, where $\mathcal{X}$ is a character of $M_p(\mathbb{Z})$. All the assertions above except the one about w being in $M_p(\mathbb{R})$, are consequences of the Poisson formula.

50 The group $M_p(\mathbb{R})$ is known as the Metaplectic group and our interpretation above becomes significant in view of the following fact that can be proved: namely lat μ denote the cyclic subgroup of $\mathbb{C}_1^{\times}$ generated by $\mathfrak{e}(\mathfrak{X})$; then μ is contained in $M_p(\mathbb{R})$ and under the correspondence

$$\xi(u) \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \eta(u) \mapsto \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$$

We have

$$M_p(\mathbb{R})/\mu \xrightarrow{\sim} PSL_2(\mathbb{R}) \text{or} SL_2(\mathbb{R})$$

according as $\mathfrak{X}$ is or is not the quotient of an integer by an integer.

Remark. By using the same function $k(x)$ as in the remarks at the end of §1 above, we have

$$(\eta(t)F)(x) = (it)^{-\mathfrak{x}} e^{ix/t} \int_0^\infty k\left(\frac{xy}{t^2}\right) e^{iy/t} F(y) y^{\mathfrak{x}} d\log y$$

for every $t \neq 0$ in $\mathbb{R}$ and every F in $\mathcal{F}$; this integral representation of $(\eta(t)F)(x)$ can be used to determine the structure of $M_p(\mathbb{R})$ stated above.

Chapter 2

Dual Asymptotic Expansions

THIS CHAPTER IS essentially concerned with asymptotic expansions which are dual to the asymptotic expansions mentioned in chapter I, in a certain sense. We shall deal with the following two questions about the spaces $\mathcal{F}$ and $\mathcal{Z}$ with the additional assumption that $\lambda_o > 0$ in the case of $\mathbb{R}$-fields K and $Re(\lambda) > 0$ for every $\lambda \in \Lambda$ for p-fields K; namely, **51**

(i) characterising the space $(\omega^{-1}\mathcal{F})^* = \{$the Fourier transform $(\omega^{-1}F)^*F$ for $F \in \mathcal{F}\}$ and

(ii) giving necessary and sufficient conditions for the function $\omega^{-1}F$ to have its Fourier transform as well in L^1, in terms of $\omega^{-1}F$ and $Z = MF$.

The reasons for our careful discussion of asymptotic expansions in chapters I and II will become evident when we deal, later on, with the local arithmetic theory of forms of higher degree and a poisson formula of the Siegel-Weil type for such forms. At this stage, by way of pointing out just one reason, we shall merely highlight the following important problem. For locally compact abelian groups X and G with a continuous map $f : X \to G$, Weil has defined, for any ϕ in the space $\mathscr{S}(X)$ of Schwartz-Bruhat functions on X and any g^* in the dual G^* of G,

$$F^*_\phi(g^*) = \int\limits_X \phi(x)\langle f(x), g^*\rangle\, dx$$

where dx is the Haar measure on X and $\langle f(x), g^*\rangle = g^*(f(x))$. It is still **52** a fundamental problem to determine precisely when, for a given f, the function F^*_ϕ (clearly uniformly continuous on G^*) actually belongs to $L^1(G^*)$ for every ϕ in $\mathscr{S}(X)$. In his well-known paper [52], Weil calls this the "condition A" and clarifies its meaning for quadratic forms f, in the case of local fields and adele-spaces. For forms of higher degree, one can similarly give necessary and sufficient conditions involving an invariant associated with the surface determined by f, at least in the case when $G = K$.

1 Fourier Transforms of Quasi-characters

We shall quickly recall well-known results concerning Fourier transforms of quasi-characters.

1.1 Notation

Let X be a finite-dimensional vector space over a local field K with ψ, a nontrivial character of K. Let $[x, y]$ be a symmetric non-degenerate K-bilinear form $X \times X$, $|dx|$ denote a Haar measure on X and let X be identified with its dual X^*.

For every ϕ in $L^1(X)$, we define

$$\phi^*(x) = \int\limits_X \phi(y)\psi([x, y]) \quad |dy|.$$

If $\Lambda(X)$ is the space of continuous functions ϕ in $L^1(X)$ such that ϕ^* is also in $L^1(X)$, then the measure $|dx|$ can be normalised uniquely in such a manner that $(\phi^*)^*(x) = \phi(-x)$ for every ϕ belonging to $\Lambda(X)$. We shall
53 then call $|dx|$ the self-dual measure relative to $\psi([x, y])$.

If $\mathfrak{e}(\cdot) = exp(2\pi\sqrt{-1}\cdot)$, then for ψ, we have the standard choice

$$\phi(x) = \begin{cases} \mathfrak{e}(x) \text{ for } x \in K = \mathbb{R} \\ \mathfrak{e}(2Re(x)) \text{ for } x \in K = \mathbb{C} \end{cases}$$

In the case of p-fields K, we take a ψ which is 1 on R and non-trivial on P^{-1}. Further, if $[x, y] = xy$ for $x, y \in K$, then $|dx|$ is the same as the measure dx in chapter I, namely

$$dx = \begin{cases} \text{usual measure for } K = \mathbb{R} \\ \text{twice the usual measure for } K = \mathbb{C} \\ \text{the measure with } M(R) = 1 \text{ for } p\text{-fields} K. \end{cases}$$

1.2 The Space $\mathscr{S}(X)$ of Schwartz-bruhat Functions on X and the Space $\mathscr{S}(X)'$ of Tempered Distributions on X

The space $\mathscr{S}(\mathbb{R}^n)$ of Schwartz functions on $\mathbb{R}^n$ consists of all C^∞ functions from $\mathbb{R}^n$ to $\mathbb{C}$ such that for every linear differential operator $D = \sum a_{\alpha\beta} x^\alpha D^\beta$ with $x^\alpha = x_1^{\alpha_1}, \ldots x_n^{\alpha_n}, D^\beta = \frac{\partial_1^{\beta_1+\ldots\beta_n}}{\partial x_1^{\beta_1}\ldots\partial x_n^{\beta_n}}, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots \beta_n \geq 0$ in $\mathbb{Z}$, $a_{\alpha\beta} \in \mathbb{C}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we have $\|D\phi\|_\infty = \sup\limits_{x\in\mathbb{R}^n} |D\phi(x)| < \infty$. We topologize $E = \mathscr{S}(\mathbb{R}^n)$ with respect to the family of seminorms $\phi \mapsto \|D\phi\|_\infty$, as D varies over all such linear differential operators with polynomial coefficients. Thus, a sequence $\{\phi_n\}_{n\geq 0}$ in E tends to 0 if (and only if), for every D as above, the sequence $\{\|D\phi_n\|_\infty\}_n$ of real numbers tends to 0 as $n \to \infty$. The same topology can **54**
also be obtained as follows. Namely, we rearrange $\{\|(x^\alpha D^\beta)(\phi)\|_\infty; \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n), \alpha_i, \beta_j \geq 0$ in $\mathbb{Z}$ for all $i, j\}$ as $c_1(\phi)$, $c_2(\phi)$, $\ldots$, and define

$$\|\phi\| = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{c_m(\phi)}{1 + c_m(\phi)}.$$

Then we have

(i) $\|\phi\| = \|-\phi\| \geq 0$ with equality only when $\phi = 0$;

(ii) $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$ for every ϕ, ψ in E; and

(iii) the scalar multiplication $(\lambda, \phi) \mapsto \lambda\phi$ from $\mathbb{C} \times E$ to E is continuous.

While (i) and (ii) make E a metric space with $\|\phi - \psi\|$ as the distance between ϕ and ψ and $\| \quad \|$ being fairly close to a norm, (iii) in addition, makes E a topological vector space. It is direct verification to check that

E is complete. The dual E' consisting of all complex-valued $\mathbb{C}$-linear continuous maps of E is called the space of tempered distributions on X; since E satisfies (i), (ii), (iii) and is complete, by the Banach-Steinhaus theorem.

We now take up the definition of $\mathscr{S}(X)$, $\mathscr{S}(X)'$ for the case of p-fields K. Let G be a locally compact abelian group. We say that G has arbitrarily large (respectively arbitrarily small) compact open subgroups H, if for every compact subset C of G (respectively neighbourhood V of 0) there exists such an H containing C (respectively contained in V).
55 Whenever G has arbitrarily large and arbitrarily small compact open subgroups, we define the Schwartz-Bruhat space $\mathscr{S}(G)$ as the space of all complex-valued functions ϕ on G which are locally constant (i. e, constant in some neighbourhood of every point) and which have, in addition, compact support. If, for any subset S of G, we denote by φ_S the characteristic function of S, then $\mathscr{S}(G)$ is just the $\mathbb{C}$-span of the functions φ_{a+H} with a varying over G and H over subgroups described above. We may take $\mathscr{S}(G)'$ as just the algebraic dual of $\mathscr{S}(G)$, which consists of all $\mathbb{C}$-linear function on $\mathscr{S}(G)$; it is not necessary to topologies $\mathscr{S}(G)$, in order to define the dual space $\mathscr{S}(G)'$.

Both in the case of $\mathbb{R}$-fields and p-fields K, the Fourier transformation $\phi \mapsto \phi^*$ gives an automorphism of $\mathscr{S}(X)$ and hence an automorphism $T \mapsto T^*$ of the dual $\mathscr{S}(X)'$, if we set $T^*(\phi) = T(\phi^*)$ for every $\phi \in \mathscr{S}(X)$.

The following two definitions are needed for our purposes, later on.

Let $\varphi \in L^1_{\text{loc}}(X)$ i. e, φ is a complex-valued locally integrable function on X which has, in addition, at most polynomial growth (at infinity) whenever K is an $\mathbb{R}$-field. We can then define a tempered distribution $T_\varphi \in \mathscr{S}(X)'$ by simply setting, for every ϕ in $\mathscr{S}(X)$,

$$T_\varphi(\phi) = \int_X \phi(x)\varphi(x)\,|dx| \tag{50}$$

For the case of p-fields, only the convergence of the integral needs to be verified and this is taken care of by ϕ having compact support and φ being locally integrable. For $\mathbb{R}$-fields, the convergence of the integral at infinity is made possible by ϕ being Schwartz function and φ being

at most of polynomial growth there; as for the continuity, it is easy to **56**
check that $T_{\varphi}(\phi) \to 0$ as $\phi \to 0$ in $\mathscr{S}(X)$.

We remark that under the imbedding $\phi \to T_{\phi}, \mathscr{S}(X)$ is dense in $\mathscr{S}(X)'$.

Given T in $\mathscr{S}(X)'$, U open in X and $f \in L^1_{\text{loc}}(U)$, we say that $T = f$ in U, if

$$T(\phi) = \int_U \phi(x) f(x)\, |dx| \tag{51}$$

for every ϕ in $\mathscr{S}(X)$ with support of ϕ contained in U.

1.3

In the case when K is a p-filed, we shall have occasion to use the following facts frequently.

We shall assume ψ to be standard i. e, ψ is 1 on R and ψ is non-constant on P^{-1}. For any $\mathcal{X}(R^{\times})^*$, we denote by $e_{\mathcal{X}}$ the smallest natural number e such that $\mathcal{X}$ is 1 on the set $1 + P^e$; thus, by definition, we have always $e_{\mathcal{X}} \geq 1$. We assert that the following formulae are valid:

$$\int_{R^{\times}} \mathcal{X}(u)(\pi^{-e}u) du = \begin{cases} 1 - 1/q & \text{for } e \leq 0 \text{ and } \mathcal{X} = 1 \\ -1/q & \text{for } e \leq 1 \text{ and } \mathcal{X} = 1 \\ 0 & \text{for } e > 1 \text{ and } \mathcal{X} = 1 \\ 0 & \text{for } e \neq e_{\mathcal{X}} \text{ and } \mathcal{X} \neq 1 \end{cases} \tag{52}$$

Formulae (52) for $\mathcal{X} = 1$ are straightforward. We shall therefore prove only that integral is 0 for $\mathcal{X} \neq 1$ and $e \neq e_{\mathcal{X}}$. First, if $e > e_{\mathcal{X}}$, then the integral is the same as

$$\begin{aligned} &\sum_{a \in R^{\times}, \quad \text{mod } p^{e-1}} \int_{p^{e-1}} \mathcal{X}(a(1+x))\psi(\pi^{-e}a(1+x))\, dx \\ &= \sum_{a \in R^{\times}, \quad \text{mod } P^{e-1}} \mathcal{X}(a)\psi(\pi^{-e}a) \int_{P^{e-1}} \psi(\pi^{-e}x)\, dx \end{aligned}$$

The last integral over P^{e-1} is 0, since $P^{e-1} \ni x \to \psi(\pi^{-e}x)$ is a non- **57** trivial character. Similarly, if $e < e_\mathcal{X}$ and $e_\mathcal{X} \geq 2$, then the integral on the left side of (52) is equal to

$$= \sum_{a \in R^\times, \mod P^{e_\mathcal{X}-1}} \int_{P^{e_\mathcal{X}-1}} \mathcal{X}(a(1+x))\psi(\pi^{-e}a(1+x))\, dx$$
$$= \sum_{a \in R^\times, \mod P^{e_\mathcal{X}-1}} \mathcal{X}(a)\psi(\pi^{-e}a) \int_{p^{e_\mathcal{X}-1}} \mathcal{X}(1+x)\, dx$$

and the integral over $P^{e_{\mathcal{X}-1}}$ is 0 since $P^{e_{\mathcal{X}-1}} \ni x \to \mathcal{X}(1+x)$ is a non-trivial character of $P^{e_{\mathcal{X}-1}}$. Finally, if $e < e_\mathcal{X} = 1$, the left hand side of (52) is just $\int\limits_{R^\times} \mathcal{X}(u)du = 0$, since $\mathcal{X} \neq 1$.

We have the following supplement to (52). For any $\mathcal{X}$(possibly equal to 1) in $(R^\times)^*$, let us define

$$g_\mathcal{X} = \int_{R^\times} \mathcal{X}(u)\psi(\pi^{-e\mathcal{X}}u)du \tag{53}$$

Then, trivially, we have

$$\overline{g}_\mathcal{X} = \mathcal{X}(-1)g_{\mathcal{X}^{-1}} \tag{54}$$

Moreover, it is clear from (52) that

$$g_1 = -q^{-1} \tag{55}$$

If we write $u = a + \pi^{e_\mathcal{X}}v$ and let a run over representatives of $R^\times$ modulo $P^{e_\mathcal{X}}$ and v over R, then we can rewrite (53) as

$$g_\mathcal{X} = q^{-e_\mathcal{X}} \sum_{a \in R^\times \mod p^{e_\mathcal{X}}} \mathcal{X}(a)\psi(\pi^{-e_\mathcal{X}}a) \tag{56}$$

Thus $q^{e_\mathcal{X}}g_\mathcal{X}$ is a standard Gaussian sum and for $\mathcal{X} \neq 1$, we know indeed that

$$|g_\mathcal{X}|^2 = q^{-e_\mathcal{X}} \tag{57}$$

1.4

We are now in a position to recall the following well-known table of Fourier transforms of quasicharacters; for $K = \mathbb{R}$, one may refer to M. J. Lighthill [32] or I. M. Gel'fand and G. E. Shilov ([12]).

For any quasicharacter ω in $\Omega_+(K^\times)$, i. e, with $\sigma(\omega) > 0$, the function $\omega\omega_1^{-1}$ is in $L^1_{loc}(k)$ as may be verified at once from the convergence of the integral $\int |x|_k^{\sigma-1}\, dx$ taken over a compact neighbourhood of 0. Thus $T_{\omega\omega_1^{-1}}$ in the sense of (50) is defined and therefore, also $(T_{\omega\omega_1^{-1}})^\times$, although the latter is not, in general, of the form T_φ for some φ. But if we restrict it to $K^\times$, then 58

$$(T_{\omega\omega_1^{-1}})^\times \quad = b(\omega)\omega^{-1} \text{on } K^\times \tag{58}$$

for a constant $b(\omega)$ explicitly described below: namely, let ψ be standard and let $d = 1/2$ or 1 according as $K = \mathbb{R}$ or $\mathbb{C}$-field K and for $\omega = \omega_s(ac)^p$, we have

$$b(\omega) \overset{defn}{=} b_p(s) = i^{|p|}(2d\pi)^{d(1-2s)} \frac{\Gamma(ds + |p|/2)}{\Gamma(d(1-s) + |p|/2)}$$

and for a p-field K, we have, for $\omega = \omega_s \mathcal{X}$,

$$b(\omega) \overset{defn}{=} .b_{\mathcal{X}}(s) = \begin{cases} \frac{1-q^{-(1-s)}}{1-q^{-s}} & \text{for } \mathcal{X} = 1 \\ g_{\mathcal{X}} q^e \mathcal{X}^s & \text{for } \mathcal{X} \neq 1 \end{cases} \tag{59}$$

A transparent proof for the complete table (59) can be found in Weil's exposé ([53]) and we shall give a brief outline of his method.

The natural action of $K^\times$ on K induces the action of $K^\times$ on $\mathscr{S}(K)$ given by $\phi \mapsto g\phi$ with $(g^\phi)(x) = \phi(g^{-1}x)$ and hence, by duality, the action on $\mathscr{S}(K)'$ given by $T \mapsto gT$ with

$$(gT)(\phi) = T(g^{-1}\phi), i, e.(gT)(\phi(x)) = T(\phi(gx))$$

for ϕ in $\mathscr{S}(K)$. It can be immediately verified that for $T = T_{\omega\omega_1^{-1}}$ (in the sense of (50)) with $\omega \in \Omega_+(K^\times)$, we have 59

$$(gT)(\phi) = \int_k \phi(gx)(\omega\omega_1^{-1})(x)|dx|$$

$$= \omega(g)^{-1} \int_k \phi(x)(\omega\omega_1^{-1})(x)|dx| \quad using x \mapsto g^{-1}x \tag{60}$$
$$= \omega(g)^{-1} T(\phi)$$

for every $g \in K^\times$ and $\phi \in \mathscr{S}(K)$.

For a given $\rho \in \Omega(K^\times)$, suppose $\triangle \in \mathscr{S}(K)'$ satisfies the condition $g\triangle = \rho(g)^{-1}\triangle$ for every g in $K^\times$. Then we shall write $\triangle \in P(K, K^\times; \rho)$.

Taking the restriction of $\triangle$ to $K^\times$ i. e, by requiring ϕ in $\triangle(\phi)$ to have support contained in $K^\times$, we get a distribution on $K^\times$, which is relatively invariant under the action of $K^\times$ (i. e, picking up a factor $\rho(g)$ under the multiplication by g in $K^\times$). Then, by a method similar to the proof of the uniqueness of a relatively invariant measure (on $K^\times$), we can show that

$$\triangle \text{ on } k^\times \quad = c(\rho\omega_1^{-1})(x)\, dx \tag{61}$$

for a complex constant c which may be 0. For instance, this proof goes as follows in the case of a p-field K: if we put

$$\triangle'(\phi) = \triangle(\rho^{-1}\phi)$$

for every ϕ in $\mathscr{S}(K)$ with support of ϕ contained in $K^\times$, then $\triangle'$ becomes even invariant under the action of $K^\times$. Since, for every $e \geq 1$, we have

$$\varphi_{1+p^e}(x) = \sum_{g \in 1+p^e, \mod p^{e+1}} \varphi_{1+p^{e+1}}(g^{-1}x)$$

60 we obtain, by the invariance of $\triangle'$, that

$$\begin{aligned} q^e.\triangle'(\varphi_{1+p^e}) &= q^{e+1}\triangle'(\varphi_{1+p}e+1) \\ &= \text{a constant } c', \text{ say;} \end{aligned}$$

then, for every a in $K^\times$ and $e \geq 1$, we have

$$\triangle'(\varphi_{a(1+p^e)}) = c' q^{-e} = c' \int_{K^\times} \varphi_{a(1+p^e)}(x) \frac{dx}{|x|}_k$$

Then implies

$$\triangle' = c' \frac{dx}{|x|_k} i.e, \triangle = c'(\rho\omega_1^{-1})(x)\, dx \text{ in } K^\times.$$

Coming back to $\triangle \in P(K, K^\times; \rho)$, we see, from the simple relation $(g^{-1}\phi)^* = \omega_1(g)^{-1} g^{\phi^*}$ for every ϕ in $\mathscr{S}(K)$, that

$$\triangle \in P(K, K^\times; \rho) \Rightarrow \triangle^* \in (K, K^\times, \rho^{-1}\omega_1)$$

Taking $\triangle = T_{\omega\omega_1^{-1}}$ for $\omega \in \lambda_+(K^\times)$ and ω instead of ρ, we can see, in view of (60) and (61), that the relation (58) is valid. The function $b(\omega)$ can be determined by evaluating both sides of (58) at a suitable ϕ. For example, in the case of a p-field K, we can take for ϕ, the characteristic function of $1 + p^e \mathcal{X}$ and then we get the expression for $b_{\mathcal{X}}(s)$ in (59). We shall give some details in the case of a p-field K: if φ_E denotes the characteristic function of a compact open subset E of K, then, for any a, in K, and e in $\mathbb{Z}$, we have

$$\varphi^*_{a+P^e}(x) = \int\limits_{a+p^e} \psi(xy)dy = \psi(ax) \int\limits_{P^e} \psi(xy)dy$$

$$= \psi(ax) \begin{Bmatrix} m(P^e) \text{ if } x \in P^e \\ 0 \text{ otherwise,} \end{Bmatrix}$$

i. e. $$\varphi^*_{a+P^e}(x) = q^{-e}\psi(ax)\varphi_{P^{-e}}(x) \tag{62}$$

If $\sigma(\omega) > 0$ and $\mathcal{X} = 1$, then for $\phi = \varphi_{1+P}$, we obtain, using (52), that **61**

$$b(\omega) = \int\limits_{P^{-1}} \psi(x)|x|_k^{s-1}\, dx = \sum_{e=-1}^{\infty} q^{-es} \int\limits_{R^\times} \psi(\pi^e u)du$$

$$= -q^{s-1} + (1 - q^{-1}) \sum_{e=0}^{\infty} q^{-es}$$

$$= -q^{s-1} + \frac{1 - q^{-1}}{1 - q^{-s}}$$

$$= \frac{1 - q^{s-1}}{1 - q^{-s}}$$

If, on the other hand, $\mathcal{X} \neq 1$, then for $\phi = \varphi_{1+P^{e_{\mathcal{X}}}}$, we get again, in view of (52), that

$$b(\omega) = \int\limits_{P_{\mathcal{X}}^{-e}} \psi(x)|x|_K^{s-1}\mathcal{X}(ac(x))\, dx$$

$$= \sum_{e=-e_{\mathcal{X}}}^{\infty} q^{-es} \int_{R^{\times}} \psi(\pi^e u)\mathcal{X}(u)du$$
$$= q^{e_{\mathcal{X}} s} g_{\mathcal{X}}$$

1.5

For a moment, we shall write $\mathcal{X}$ for $(ac)^p$ as well and correspondingly denote $b_p(s)$ as $b_{\mathcal{X}}(s)$. If $Re(s) > 0$, then $\varphi(x) = \mathcal{X}(ac(x))|x|_K^{s-1}(\log |x|_K)^{m-1}$ is locally integrable on K (and at most of polynomial growth at infinity for $\mathbb{R}$ – fields K). Further, for the Fourier transform of φ (which we shall identify with the distribution T_φ), we have the relation

$$(\mathcal{X}(ac(x))|x|_K^{s-1}(\log |x|_K)^{m-1})^{\times} \quad \text{on } K^{\times} = \tag{63}$$
$$= \sum_{j=1}^{m} (-1)^{j-1} \binom{m-1}{j-1} \frac{d^{m-j} b_{\mathcal{X}}(s)}{ds^{m-j}} \mathcal{X}(ac(x))^{-1} |x|_K^{-s} (\log |x|_K)^{j-1}$$

62 (in the sense of the definition at the end of 1.2). The proof of (63) is as follows. If $\sigma(\omega) > 0$, then

$$\left(\frac{d^{m-1}}{ds^{m-1}}(\omega\omega_1^{-1})\right)^{*} = \frac{d^{m-1}}{ds^{m-1}}(\omega\omega_1^{-1})^{*} \tag{64}$$

and moreover, the left hand side of (63) is the same as that of (64); this is an identity in $\mathscr{S}(K)'$. If now we restrict the right hand side of (64) to $K^{\times}$, then in view of (58), it becomes equal to

$$\frac{d^{m-1}}{ds^{m-1}}(b(\omega)\omega^{-1})(x) = \frac{d^{m-1}}{ds^{m-1}}(b_{\mathcal{X}}(s)\chi(ac(x))^{-1}|x|_K^{-s});$$

on applying the Leibnitz formula, it coincides with the right hand side of (52).

If we take $\mathcal{X} = 1$, $s = 1$, $m = 1$ in (63), the left hand side becomes just $1^{\times}$ or the Dirac distribution δ_0 with $\{0\}$ as support while the right hand side is 0 since $b_1(1) = 0$, by (59); thus $\delta_0 = 1$ on $K^{\times}$, which is indeed true!

2 The Space $(\omega_1^{-1}\mathcal{F})^*$

2.1 Statement of a Theorem

(A) Given a strictly increasing sequence $\{\lambda_k\}_{k\geq 0}$ of positive real numbers with no finite accumulation point and a sequence $\{m_k\}_{k\geq 0}$ of natural numbers, the space $(\omega_1^{-1}\mathcal{F})^* = \{(\omega_1^{-1}G)^*; G \in \mathcal{F}\}$ in the case of an $\mathbb{R}$-field K, consists of all complex-valued C^∞ function **63**
$F^\#$ on K, with a termwise differentiable uniform asymptotic expansion of the form

$$F^\#(x) \approx \sum_{k=0}^{\infty}\sum_{m=1}^{m_k} a^\#_{k,m}(ac(x))|x|_K^{-\lambda_k}(\log|x|_K)^{m-1} \qquad (65)$$

as $|x|_K \to \infty$ with $a^\#_{k,m}$ denoting a C^∞ function on $K_1^\times$. Furthermore, if $F^\#$ is the same as the Fourier transform F^* of $F \in \omega_1^{-1}\mathcal{F}$ which has the termwise differentiable asymptotic expansion

$$F(x) \approx \sum_{k=0}^{\infty}\sum_{m=1}^{m_k} a_{k,m}(ac(x))|x|_K^{\lambda_k-1}(\log|x|_K)^{m-1} \qquad (66)$$

as $|x|_K \to 0$, then (65) is the termwise Fourier transform of the expansion (66).

(B) Given a finite set $\Lambda = \{\lambda \mod 2\pi i/\log q; Re(\lambda) > 0\}$ and natural numbers m_λ for every λ in Λ, the space $(\omega_1^{-1}\mathcal{F})^* = \{(\omega_1^{-1}G)^*; G \in \mathcal{F}\}$ in the case of a p-field K, consists of all complex-valued locally constant functions $F^\#$ on K such that

$$F^\#(x) = \sum_{\lambda\in\Lambda}\sum_{m=1}^{m_\lambda} a^\#_{\lambda,m}ac(x))|x|_k^{\lambda}(\log|x|_K)^{m-1} \qquad (67)$$

for all sufficiently large $|x|_K$. Further, if $F^\#$ is the Fourier transform F^* of an F in $\omega_1^{-1}\mathcal{F}$ for which

$$F(x) = \sum_{\lambda\in\Lambda}\sum_{m=1}^{m_\lambda} a_{\lambda,m}(ac(x))|x|_K^{\lambda-1}(\log|x|_K)^{m-1} \qquad (68)$$

for all sufficient small $|x|_K$, then (67) is obtained from (68) by taking the Fourier transform termwise.

(C) In both the cases (A) and (B), the inverse of the map $F \mapsto F^*$ from $\omega_1^{-1}\mathcal{F}$ to $(\omega_1^{-1}\mathcal{F})^*$ is given by

$$F(x) = \lim_{r\to\infty} \int\limits_{|y|_K \leq r} F^*(y)\psi(-xy)dy, \text{ for } x \in K^\times \tag{69}$$

64 **REMARKS.** For the proof of part (A) of Theorem 2.1 in the case $k = \mathbb{R}$, the reader may refer to [32], Chapter 4, §4.2, §4.3, A partial form of (A) for $K = \mathbb{R}$ is the principal result proved by Lighthill who has shown that for F in $\omega_1^{-1}\mathcal{F}$ the Fourier transform F^* has an asymptotic expansion of the form (65) as $|x| \to \infty$, without discussing its termwise differentiability, however. In this context, we merely remark that the differentiability of F^* is immediate and the termwise differentiability of the asymptotic expansion for $F^*(x)$ as $|x|_K \to \infty$ follows from the stability of $\omega_1^{-1}\mathcal{F}$ and $(\omega_1^{-1}\mathcal{F})^*$ under any homothety-invariant differential operator. In fact, for $K = \mathbb{R}$, we get $(DF)^* = -F^* - DF^*$ with $D = x\frac{d}{dx}$ and therefore, the asymptotic expansions of $-(DF^*)(x)$ as $|x| \to \infty$ is obtained simply as the sum, taken termwise, of the asymptotic expansion of $F^*(x)$ and $(DF)^*(x)$ as $|x| \to \infty$; we can easily verify that this is the same as the asymptotic expansion of $F^*(x)$ as $|x| \to \infty$ being termwise differentiable once and by repeated applications of D, the asymptotic expansion of $F^*(x)$ is seen to be termwise differentiable (i. e, any number of times). The proof of part (A) for the case$K = \mathbb{C}$ is left as an exercise. We therefore give only the proof for (B) and our proof given below for (B) is an adaptation to the case of p-fields, of Lighthill's proof mentioned above; actually, in the case of p-fields, we can also give a proof entirely avoiding the concept of distributions and using the properties of the integral considered in (52).

Before we proceed to consider assertions (B) and (C) of Theorem 2.1, let us prove the following lemma.

65 **Lemma 2.1.** *For arbitrary s in $\mathbb{C}$ and e_o in $\mathbb{Z}$, let*

$$\varphi(x)=\begin{cases}\mathcal{X}(ac(x))|x|_K^{s-1}(\log|x|_K)^{m-1} & \textit{for ord } (x)<e_0\\ 0 & \textit{for ord}(x)\geq e_o\end{cases}$$

where $\mathcal{X}\in(R^{\times})^*$. *Then* $\varphi^{\times}=0$, *in* $K\backslash p^{-e}\mathcal{X}^{e_o+1}$

Proof. If $(a+P^e)\cap P^{-e_{\mathcal{X}}-e_0+1}\neq\phi$, then ord$(a)<-e_{\mathcal{X}}-e_0+1$; for otherwise, ord(a)$\geq -e_{\mathcal{X}}-e_0+1=t$, say and $(a+P^e)\cap P^t=(a+P^e)\cap(a+P^t)=a+P^{max(e,t)}\neq\phi$.

Every ϕ in $\mathscr{S}(K)$ with support contained in $K\backslash P^{-e_{\mathcal{X}}-e_0+1}$ is a linear combination of functions of the form φ_{a+P^e} with $(a+P^e)\cap P^{-e_{\mathcal{X}}-e_0+1}=\phi$. Therefore, in order to prove the lemma, it suffices to show that $T^*_{\varphi}(\varphi_{a+P^e})=0$ for $a+P^e$ disjoint with $P^{-e_{\mathcal{X}}-e_0+1}$. And, indeed we have then

$$\begin{aligned}T^*_{\varphi}(\varphi_{a+P^e})&\overset{defn}{=}T_{\varphi}(\varphi^*_{a+Pe})\\ &=q^{-e}\int_{P^{-e}}\varphi(x)\psi(ax)\,dx, \qquad \text{by 62}\\ &=q^{-e}\sum_{-e\leq j<e_0}q^{-js}(-j\log q)^{m-1}\int_{R^*}\mathcal{X}(u)\psi(a\pi^j u)du.\end{aligned}$$

$\square$

Since ord(a)$<-e_{\mathcal{X}}-e_0+1$ from above, we have ord $(a\pi^j)=ord(a)+j<-e_{\mathcal{X}}-e_0+1+e_0-1=-e_{\mathcal{X}}$. Therefore, by (52), each one of the last mentioned integrals vanishes and the lemma is proved.

Proof of assertions (B) and (C) of Theorem 2.1.

We do not prove assertion (B) in its entirety. We shall only start from any F in $\omega_1^{-1}\mathcal{F}$, with the expansion (68) for sufficiently small $|x|_K$, say, for $ord(x)\geq e_0$, and show that (67) holds for the Fourier transform F^* of **66**
F is in $L^1(K)$ and further F has compact support. For any function f in $L^1(K)$ with compact support, we see trivially that f^* is locally constant; in fact, for some $t\geq 1$, we know that f vanishes outside P^{-t} and

$$f^*(x+z)=\int_{P^{-t}}f(y)\psi((x+z)y)dy=\int_{P^{-t}}f(y)\psi(xy)\psi(zy)dy=f^*(x)$$

for $z \in P^t$ and every $x \in K$. Thus F^* is locally constant on K.

Let us define $\varphi_0 : K \to \mathbb{C}$ by

$$-\varphi_0(x) = \begin{cases} \text{Right hand side of (68)} & \text{for ord } (x) < e_0 \\ 0 & \text{for ord } (x) \geq e_0 \end{cases}$$

Then $\varphi_0 \in L^1_{\text{Loc}}(K)$ and by Lemma 2.1, φ_0^* vanishes for all sufficiently large $|x|_K$. If, now, we define the function ϕ by setting $\phi(0) = 0$ and for $x \in K^*, \phi(x)$ by

$$F(x) = (\text{right hand side of (68)}) \ + \varphi_0(x) + \phi(x), \tag{70}$$

then clearly $\phi(x)$ is again locally constant and furthermore, since $\phi(x) = F(x)$ for $\text{ord}(x) < e_0$, ϕ has compact support, os that $\phi \in \mathscr{S}(K)$. Applying the Fourier transform to (70), we get, on using (63), that

$$\begin{aligned} F^* &= \left(\sum_{\lambda \in \Lambda} \sum_{m=1}^{m_\lambda} a_{\lambda,m}(ac(x))|x|_k^{\lambda-1}(\log|x|_K)^{m-1}\right)^* + \varphi_0^* + \phi^* \\ &= \sum_{\lambda \in \Lambda} \sum_{m=1}^{m_\lambda} a^\#_{\lambda,m}(ac(x))|x|_k^{-\lambda}(\log|x|_K)^{m-1} \end{aligned}$$

for all sufficiently large $|x|_K$, where

$$a^\#_{\lambda,m}(u) = \sum_{\mathcal{X} \in (R^*)^*} \left(\sum_{j=m}^{m_\lambda} (-1)^{m-1} \binom{j-1}{m-1} a_{\lambda,j,\mathcal{X}} \frac{d^{j-m} b_{\mathcal{X}}(\lambda)}{ds^{j-m}}\right) \mathcal{X}^{-1}(u) \tag{71}$$

67 Thus F^* has the expansion (67), for all large $|x|_K$, that is obtained from (68) by termwise application of Fourier transform. This also proves that any $F^\#$ in $(\omega_1^{-1}\mathcal{F})^*$ has an expression of the type (67) for all sufficiently large $|x|_K$.

It now remains for us to prove assertion (C) of Theorem 2.1 again only partly. We first remark that the integral in (69) exists since F^* is C^∞ or locally constant, we shall prove (69) only for a p-field K. The right hand side of (69) is just

$$\lim_{e \to \infty} \int_{P^{-e}} \psi(-xy) \left(\int_K \psi(yz) F(z) dz\right) dy = \lim_{e \to \infty} \int_K \left(\int_{P^{-e}} \psi((z-x)y) dy\right) F(z) dz$$

$$= \lim_{e\to\infty} q^e \int\limits_{x+P^e} F(z)dz,$$

since the inner integral over P^{-e} is q^e or 0 according as $z - x \in P^e$ or otherwise. But, for $x \neq 0$, F is constant on $x + P^e$ for sufficiently large e and therefore the limit above is just $F(x)$.

REMARKS. (1) For any F in $\omega_1^{-1}\mathcal{F}$ as above with the asymptotic expansion (66) or (68) as $|x|_K \to 0$, we have, for the Fourier transform F^* a corresponding asymptotic expansion (65) or (67) as $|x|_K \to \infty$. As remarked in (71) for the case of p-fields, the Fourier coefficients $a_{\lambda,m,\mathcal{X}}, a^{\#}_{\lambda,m,\mathcal{X}}$ respectively of the coefficients $a_{\lambda,m}(u), a^{\#}_{\lambda,m}(u)$ in the asymptotic expansion of F and F^*, are related as follows: namely, for every λ and $i \leq m \leq m_\lambda$,

$$a^{\#}_{\lambda,m,\mathcal{X}^{-1}} = (-1)^{m-1} \sum_{j=m}^{m_\lambda} \binom{j-1}{m-1} \frac{d^{j-m}b_{\mathcal{X}}}{ds^{j-m}}(\lambda) a_{\lambda,j,\mathcal{X}}. \qquad (71)'$$

If we fix λ and $\mathcal{X}$ in $(K_1^*)^*$ and arrange $a_{\lambda,j,\mathcal{X}}(1 \leq j \leq m_\lambda)$ and $(-1)^{m-1}$ $a^{\#}_{\lambda,m,\mathcal{X}^{-1}}(1 \leq m \leq m_\lambda$ as M_λ-rowed column vectors, then the coefficient-matrix corresponding to the relations $(71)'$ becomes an upper triangular matrix of m_λ rows and columns, with all the diagonal entries to **68** $b_{\mathcal{X}}(\lambda)$ and the entries just above the diagonal equal to $b'_{\mathcal{X}}(\lambda)$, $2b'_{\mathcal{X}}(\lambda)$, $\ldots(m_\lambda - 1)b'_{\mathcal{X}}(\lambda)$. Therefore, if $b_{\mathcal{X}}(\lambda) = 0$, then certainly $a^{\#}_{\lambda,m_\lambda,\mathcal{X}^{-1}}$ is 0 and moreover, $a_{\lambda,1,\mathcal{X}}$ disappears from the expression for $a^{\#}_{\lambda,m,\lambda^{-1}}$ for $1 \leq m < m_\lambda$. Thus the asymptotic expansion of $F^*(x)$ as $|x|_K \to 0$ although F is determined by F^*, on K^* in view of (69). For $F(x) = e^{-\pi x^2}$, the asymptotic expansion of $F^*(= F)^*$ as $|x| \to \infty$ is just $F^*(x) \approx 0$ while $F(x) \approx \sum\limits_{n=0}^{\infty} (-\pi)^n x^{2n}/n!$ as $|x| \to$ tends to zero!

For an $\mathbb{R}$-field K and $\mathcal{X} = (ac)^p$, we know from (59), that $b_{\mathcal{X}}(s) = 0$ if and only if $s = 1 + \frac{|p|}{2d} + \frac{n}{d}$ for $n = 0, 1, 2\ldots$; for such $s, b'_{\mathcal{X}}(s) \neq 0$. Thus the rank of the m_λ-rowed matrix above is $m_\lambda - 1$. If K is a p-field, then again from (59), $b_{\mathcal{X}}(s) = 0$, if and only if $\mathcal{X} = 1$, $q^s = q$; in this case, $b'_{\mathcal{X}}(s) \neq 0$. Hence, at least for p-fields of characteristic 0, the rank of the above-mentioned matrix is $m_\lambda - 1$.

(2) Let us define m_K to be 2, 2π or $1 - q^{-1}$ according as $K = \mathbb{R}, \mathbb{C}$ or a p-field so that, in all cases, we get

$$d^* x \text{ (the normalised Haar measure on } K) = \frac{dx}{m_K |x|_K}$$

For $\lambda_0 > 0$ in the case of $\mathbb{R}$-fields and $Re(\lambda) > 0$ for all λ in the case of p-fields, let for any $F \in \omega_1^{-1}\mathcal{F}$, F^* be its Fourier transform and $Z = M(m_K \omega_1 F)$. If we accept Theorem 2.1, then F^* determines F on K^* by (69) and F in turn, determines Z. Similarly, if we start from Z, then F, F^* are determined, Thus any one of F, F^*, Z can be chosen arbitrarily
69 in $\omega_1^{-1}\mathcal{F}$, $(\omega_1^{-1}\mathcal{F})^*$ and $\mathcal{Z}$ respectively and the other two are determined. But, we shall seldom start from F^*, so that the lack of a complete proof of Theorem 2.1 in these lectures will not present any discontinuity.

3 The Space $(\omega_1^{-1}\mathcal{F})^* \cap L^1(K)$

3.1

Let $0 < \lambda_0 < \lambda_1 <, \ldots, Re\lambda > 0$ for $\lambda \in \Lambda$ and M_k be as in §2.1, and let $F \in \omega_1^{-1}\mathcal{F}$ and F^*, the Fourier transform of F. Then although F is in $L^1(K)$, F^* may not be in $L^1(K)$. We proceed to characterise the subspace $(\omega_1^{-1}\mathcal{F})^* \cap L^1(K)$ by the following

Theorem 3.1. *For $F \in \omega_1^{-1}\mathcal{F}$, the following assertions are equivalent:*

(a) $F^* \in L^1(K)$:

(b) $F(0) \overset{defn}{=} \lim\limits_{|x|_K \to 0} F(x)$ *exists;*

(c) $\left.\begin{array}{l} Z_p(s)(p \neq 0), (s+1)Z_0(s) \\ Z_\chi(q^{-s})(\chi \neq 1), (1 - q^{-(s+1)})Z_1(q^{-s}) \end{array}\right\}$ are holomorphic for $Re(s) \geq -1$;

(d) $F(0) = \frac{1}{m_K} \begin{cases} \lim\limits_{s \to -1} (s+1) Z_0(s) \\ \lim\limits_{s \to -1} \left(1 - q^{-(s+1)}\right) Z_1(q^{-s}) \end{cases}$

and $F^* \in L^{-1}(K)$.

To prove the theorem, we will show that $a) \Rightarrow b) \Rightarrow c) \Rightarrow d)$. We first need to prove a lemma, to take care of the proof in the case of p-fields,

Lemma 3.1. *Let $f(z)$ be meromorphic in $|z| \leq r$ for $r > 0$, and let $f(z) = \sum_{n\in\mathbb{Z}} s_n z^n$ be its Laurent expansion around $z = 0$, Then $a = \lim_{n\to\infty} r^n s_n$ exists* 70 *and is finite if and only if $f(z) - \frac{a'}{1-z/r}$ is holomorphic in $0 < |z| \leq r$ for a constant a', in which case $a' = a$.*

Proof. We merely sketch a proof of this lemma which is rather elementary. If $g(z) = \sum_{n\in\mathbb{Z}} t_n z^n$ is a Laurent expansion of g meromorphic in $0 < |z| \leq r$, then g is holomorphic in $0 < |z| \leq r$ if and only if $\overline{\lim_{n\to\infty}} |t_n|^{1/n} < r^{-1}$; if $\lim_{n\to\infty} |t_n|^{i/n} < r^{-1}$ then clearly $\lim_{n\to\infty} r^n t_n = 0$. Thus $f(z) - \frac{a}{1-z/r}$ is holomorphic in $0 < |z| \leq r$, then $a < \infty$ and $a' = a$.

Now, if $a < \infty$, then f is holomorphic in $0 < |z| < r$. Let us refer to "$a = \lim_{n\to\infty} r^n s_n$ exists and is finite" as the property P of f. If f has property P, then so does Rf, for any polynomial R in $\mathbb{C}[z]$. We now assert that f cannot have a pole at any $\alpha \neq r$ on $|z| = r$; for otherwise, $(1 - \alpha^{-1}z)^{-1} = \sum_{n=0}^{\infty} \alpha^{-n} z^n$ will have the property P, which gives a contradiction. Moreover, f can have at most a simple pole at $z = r$; for, otherwise, $(1 - z/r)^{-2} = \sum_{n=0}^{\infty} (n + 1) r^{-n} z^n$ will also have the property P, leading to a contradiction, This proves the lemma. □

Proof of Theorem 3. 1

a) $\Rightarrow$ b). This is straightforward. In fact, if G, G^* are dual groups with dual measures dg, dg^*, then, for F_0 in $L^{-1}(G; dg)$ with its Fourier transform F_0^* in $L^{-1}(G^*; dg^*)$, we have $F_0(g) = (F_0^*)^*(-g)$ at every g where F_0 is continuous. Thus , in our situation, $F(x) = (F^*)^*(-g)$ at every g where F_0 is continuous. Thus, in our situation, $F(x) = (F^*)^*(-x)$ for $x \in K^*$ and $F(0) = \lim_{|x|_K\to 0} F(x)$ exists, since $(F^*)^*$ is continuous.

b)$\Rightarrow$ c) $\Rightarrow$ d)(for an $\mathbb{R}$-field K). We know that F has an asymptotic expansion as in (66) as $|x|_K \to 0$ with $\lambda_0 > 0$; it follows that $F(0) = \lim_{|x|_K\to 0} F(x)$ exists if and only if $\lambda_k \geq 1$ for every k and, in addition, when 71

$\lambda_k = 1$ for some k, then $m_k = 1$, $a_{k,1} = a_{k,1}(1)$. This is equivalent to saying that $F(x) \approx F(0) + \sum\limits_{\lambda_k > 1} \sum\limits_{m=1}^{m_k} a_{k,m}(ac(x))|x|_K^{\lambda_k - 1}(\log |x|_K)^{m-1}$. In view of Ch. I §4, this is equivalent to $Z_P(s)$ for every $p \neq 0$ and $Z_0(s) - \frac{m_K f(0)}{s+1}$ being holomorphic for $Re(s) \geq -1$, where now $Z = M_K(m_K \omega_1 F)$, since $(m_K \omega_1 F)(x) \approx m_K F(0)|x|_K + m_K \sum\limits_{\lambda_k > 1} \sum\limits_{m=1}^{m_k} a_{k,m}(ac(x))|x|_K^{\lambda_k}(\log |x|_K)^{m-1}$ as $|x|_K \to 0$ and further $b_{k,m,p} = (-1)^{m-1}(m-1)! a_{k,m,-p}$. This, in turn, implies that $F^*(x) \approx \sum\limits_{\lambda_k > 1} \sum\limits_{m=1}^{m_k} a^\sharp_{k,m}(ac(x))|x|_K^{-\lambda_k}(\log |x|_K)^{m-1}$ as $|x|_K \to \infty$, in view of Theorem 2.1 above, (71)′ and the fact that $b_0(1) = 0$. Such an asymptotic expansion clearly implies that F^* is in $L^1(K)$.

b)⇒ c) ⇒ d) (for a p-field K) If we write $F(\pi^e u) = \sum\limits_{\mathcal{X}} c_{e,\mathcal{X}} \mathcal{X}(u)$ for $e \in \mathbb{Z}$ and $u \in R^*$, then $(m_K \omega_1 F)(\pi^e u) = m_K \sum\limits_{\mathcal{X}} (q^{-e} c_{e,\mathcal{X}}) \mathcal{X}(u)$ and in view of (38), we have $Z_{\mathcal{X}}(z) = m_K \sum\limits_{e \in \mathbb{Z}} c_{e,\mathcal{X}} - 1(q^{-1} z)^e$. Now b) is equivalent to "$\lim\limits_{e \to \infty} c_{e,\mathcal{X}}$ is equal to $F(0)$ for $\mathcal{X} = 1$and to 0 for $\mathcal{X} = 1$ and to 0 for $\mathcal{X} \neq 1$". This, in turn, is equivalent to $Z_{\mathcal{X}}(z)$ for every $\mathcal{X} \neq 1$ and $Z_1(z) - \frac{m_K F(0)}{1 - q^{-1} z}$ being holomorphic in $0 < |z| \leq q$, because of Lemma 3.1 This, again, is equivalent to the fact that

$$F(x) = F(0) + \sum_{\substack{\lambda \in \Lambda \\ Re(\lambda) > 1}} \sum_{M=1}^{m_\lambda} a_{\lambda,m}(ac(x))|x|_K^{\lambda - 1}(\log |x|_K)^{m-1}$$

for all sufficiently small $|x|_K$. Just as above, this new implies

$$F^*(x) = \sum_{\substack{\lambda \in \Lambda \\ Re(\lambda) > 1}} \sum_{m=1}^{m_\lambda} a^\sharp_{\lambda,m}(ac(x))|x|_K^{-\lambda}(\log |x|_K)^{m-1}$$

72 for all large enough $|x|_K$, so that finally F^* is in $L^1(K)$. This completes the proof Theorem 3.1.

Chapter 3

Local Arithmetic Theory of Forms of Higher Degree

AS INDICATED BY the title, we shall apply the results of the first two 73
chapters to study the arithmetic of forms f of higher degree considered over local fields. Broadly speaking, many facts that we shall discover do represent generalizations to the case when f has degree greater than 2, of earlier results; however, several striking new aspects are thrown up, which, as is not surprising, could not have been expected for quadratic forms. These will be recognised clearly, as we proceed further.

1 Three Functions: F_ϕ, F^*_ϕ, Z_ϕ

We shall use the notation in Chapter II, §1, with the modification that $X = K^n$ for a local field K, $[x, y] = x_1y_1 + \cdots + x_ny_n$ for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in X. We shall use the standard ψ, so that the self-dual measure $|dx|$ becomes the product of the usual measure on K. We choose a non-constant polynomial $f(x) = f(x_1, \ldots, x_n)$ in $K[x_1, \ldots, x_n]$, since the case when f is a constant, is trivial.

The following well-known lemma will be useful to establish the continuity of a function that we shall define by means of an integral.

Lemma. *Let (X, dx) be a measure space, T a metric space (or just separable at every point) and $f(x, t) : X \times T \to \mathbb{C}$ be such that f is continuous*

in t for any fixed x and locally integrable in x for any fixed t and, further,
74 *$|f(x,t)| \leq \varphi(x)$ for a φ in $L^1(X, dx)$, for every t. Then $\int\limits_X f(x,t)dx$ defines a continuous function of t.*

Proof. is immediate, in view of Lebesgue's theorem. □

1.1 Definition of Z_ϕ

For $\omega \in \Omega_+(K^\times)$ extended by continuity to K and $\phi \in \mathscr{S}(K)$, the integral

$$Z_\phi(\omega) = \int\limits_X \phi(x)\omega(f(x))|dx|$$

defines a holomorphic function on $\Omega_+(K^\times)$; in fact, for

$$0 \leq \sigma(\omega) \leq \sigma_2 < \infty, \quad |\omega(f(x))| = |f(x)|_K^{\sigma(\omega)} < \max(1, |f(x)|_K)^{\sigma_2}$$

and $\phi(x)\max(1, |f(x)|_K)^{\sigma_2}$ is in $L^1(X)$.

1.2 Definition of F^*_ϕ

With f in $K[x_1, \ldots, x_n]$ chosen above, we define, for any ϕ in $\varphi(X)$, the function F^*_ϕ on K by

$$F^*_\phi(i^*) = \int\limits_X \phi(x)\psi(i^* f(x))|dx| \quad \text{for} \quad i^* \in K.$$

Applying the lemma above to $f(x,t) = \phi(x)\psi(tf(x))$ and $\varphi(x) = |\phi(x)|$ it follows that F^*_ϕ is continuous on K (and indeed, bounded and uniformly continuous). The definition of F^*_ϕ is deceptively simple but it is a rather difficult function to investigate.

1.3 The Measure $|\theta_i|$

The *critical set* C_f for f is the set of points a in X at which the gradient grad f of f is 0. If, therefore, we define $U(i) = f^{-1}(i) \backslash C_f$ for $i \in K$, then a is in $U(i)$ if and only if $a \in X$ satisfies $f(a) = i$ and further at least one partial derivative of f, say $\dfrac{\partial f}{\partial x_k}$, does not vanish at a. Since $f(x) = i$ on $U(i)$, we have the relation

$$\frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n = 0.$$

Taking the exterior product of both sides with the $(n-2)$ - form $dx_1 \wedge \ldots \wedge \hat{dx}_j \wedge \ldots \wedge \hat{dx}_k \wedge \ldots \wedge dx_n$ (where ˆ indicates that the differentials $dx_j, dx_k (j < k)$ have been omitted), we obtain 75

$$(-1)^{j-1}\frac{\partial f}{\partial x_j}dx_1 \wedge \ldots \wedge \hat{dx}_k \wedge \ldots \wedge dx_n + (-1)^{k-2}\frac{\partial f}{\partial x_k}dx_1 \wedge \ldots \wedge \hat{dx}_j \wedge \ldots \wedge dx_n = 0$$

Thus $\theta_i(x) = (-1)^{k-1}\left(\dfrac{\partial f}{\partial x_k}\right)^{-1} dx_1 \wedge \ldots \wedge \hat{dx}_k \wedge \ldots \wedge dx_n|_{U(i)}$ is a well-defined non-vanishing regular $(n-1)$ - form around $a \in U(i)$ and thereby giving rise to a global regular and non-vanishing $(n-1)$ - form on $U(i)$; this is just "the residue of $(f(x) - i)^{-1}dx$ along $U(i)$" and we denote it still by θ_i. Now θ_i induces on $U(i)$ a Borel measure $|\theta_i|$ such that for every continuous function φ on X with compact support disjoint with C_f, we have

$$\int\limits_X \varphi(x)|dx| = \int\limits_K \left(\int\limits_{U(i)} \varphi|\theta_i|\right)|di|. \tag{72}$$

Furthermore, for every such φ, the function $i \mapsto \int\limits_{U(i)} \varphi|\theta_i|$ is continuous on K. The proof depends on the implicit function theorem for X. Let us assume, after re-ordering the subscripts if necessary, that the partial derivative $\dfrac{\partial f}{\partial x_n}$ does not vanish at a given point $a = (a_1, \ldots, a_n)$ in $U(i)$. Since

$$\frac{\partial(x_1, \ldots, x_{n-1}, f(x))}{\partial(x_1, \ldots, x_{n-1}, x_n)}(a) = \frac{\partial f}{\partial x_n}(a) \neq 0, \quad \text{we may use } \; x_1, \ldots, x_{n-1}, f(x)$$

as local coordinates at a. If $x' = (x_1, \ldots, x_{n-1})$ and $a' = (a_1, \ldots, a_{n-1})$, then, by the implicit function theorem, there exists a convergent power-series $g(x', t)$ in $x' - a'$, $t - i$ with coefficients in K, such that, around a, $f(x) = t$ if and only if $x_n = g(x', t)$. Applying a partition of unity, if necessary, we may suppose that φ has its support "near a". Then for t
76 close to i, we have, by the very definition of $|\theta_i|$, that

$$\int\limits_{U(t)} \varphi|\theta_t| = \int\limits_{K^{n-1}} \varphi(x', g(x', t)) \left|\frac{\partial f}{\partial x_n}(x', g(x', t))\right|_K^{-1} |dx'|. \tag{73}$$

The lemma stated at the beginning of §1 ensures that the right hand side of (73) is continuous in t. Also, our assertions are independent of the manner in which the partition of unity is chosen. Now integration on K gives

$$\begin{aligned}\int\limits_K \left(\int\limits_{U(t)} \varphi|\theta_t|\right)|dt| &= \int\limits_{K^n} \varphi(x', g(x', t)) \left|\frac{\partial(x', x_n)}{\partial(x', t)}\right|_K |dx'|\,|dt| \\ &= \int\limits_X \varphi(x)|dx|\end{aligned}$$

implying (72).

1.4 Definition of F_ϕ

For the sake of simplicity, let us *assume*, in the sequel, that $C_f \subset f^{-1}(0)$ (equivalently, $U(i) = f^{-1}(i)$ for every $i \in K^\times$). If $f(x)$ is homogeneous and if the characteristic of K does not divide m, the degree of f, then this assumption is fulfilled; in fact, for $a = (a_1, \ldots, a_n) \in C_f$, we have

$$0 = a_1 \frac{\partial f}{\partial x_1}(a) + \cdots + a_n \frac{\partial f}{\partial x_n}(a) = mf(a)$$

which is impossible unless $f(a) = 0$. Our assumption is not serious when the characteristic of K is 0; for, then, by a theorem of Bertini, C_f is contained in finitely many fibres $f^{-1}(i)$ which can be reduced, after applying a suitable partition of unity to the situation where there is only one fibre and therefore (by translation), we may assume without loss of generality that $C_f \subset f^{-1}(0)$. For $i \in K^\times$, we define

$$F_\phi(i) = \int\limits_{U(i)} \phi|\theta_i|. \tag{73}$$

Then if ϕ has compact support disjoint with C_f, then we know from 77
above that F_ϕ is continuous on $K^\times$; if, however, ϕ has compact support which is not disjoint with C_f, we may apply a partition of unity to isolate points on our fibre and conclude that F_ϕ is continuous. Thus, at least for the case of p-fields, the continuity of F_ϕ is clear. However, if K is an $\mathbb{R}$-field and if ϕ in $\mathscr{S}(X)$ does not have compact support, even the convergence of the integral above is not clear, not to speak of the continuity of F_ϕ. We therefore take up the typical case $K = \mathbb{R}$ and show that the integral defining $F_\phi(i)$ is absolutely convergent for every ϕ in $\mathscr{S}(\mathbb{R}^n)$ and further, that F_ϕ is continuous on $\mathbb{R}^\times$. We may assume after a translation, that $C_f \subset f^{-1}(a)$ for some $a \neq 0$ and that $i = 0$ in (73). In order to prove the continuity of F_ϕ at $i(= 0)$, it is enough to show that for any $\epsilon > 0$, there exists $r = r(\epsilon) > 0$ ensuring that

$$\int\limits_{f^{-1}(b), ||x|| \geq r} |\phi|\ |\theta_b| < \epsilon \tag{74}$$

for every b with $|b| \leq \rho < |a|$, where, for $x = (x_1, \ldots, x_n| \in X$, $||x|| = (x_1^2 + \cdots + x_n^2)^{1/2}$; the rest is clear from the case of compact support. We now appeal to the following lemma due to Hörmander ([17]): for any polynomial $P(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$, there exist constants $c, \alpha > 0$, $\beta \geq 0$, such that

$$|P(x)| \geq c \cdot \frac{\mathrm{dis}(x, V(P))^\alpha}{\max(1, ||x||)^\beta} \quad \text{for every} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

where $V = V(P)$ is the variety of the real zeros of P and $\mathrm{dis}(x, V)$ is the
distance of x from the (closed) set V. The critical set C_f is precisely the 78
set of common zeros of $\dfrac{\partial f}{\partial x_i}(1 \leq i \leq n)$ and hence the same as $V(P_0)$ for

$P_0(x) = \sum\limits_{i=1}^{n} \left(\dfrac{\partial f}{\partial x_i}\right)^2$. We now assert that

$$\eta = \inf_{\substack{x \\ |f(x)| \leq \rho}} \{\max(1, \|x\|)^{\gamma} P_0(x)^{1/2}\} > 0 \tag{75}$$

for a constant $\gamma \geq 0$. By Hörmander's inequality, we have

$$P_0(x) \geq c \frac{\operatorname{dis}(x, C_f)^{\alpha}}{\max(1, \|x\|)^{\beta}} \quad \text{for every} \quad x \in X. \tag{76}$$

We claim that (75) holds for $\gamma = (1/2)((m-1)\alpha + \beta)$, m being the degree of f. Otherwise, let S be a sequence of points x with $|f(x)| \leq \rho$ such that

$$\lim_{x \in S} \max(1, \|x\|)^{2\gamma} P_0(x) = 0$$

implying incidentally that $\|x\| \to \infty$. This means, in view of (76), that

$$\lim_{x \in S} \|x\|^{(m-1)\alpha} (\operatorname{dis}(x, C_f))^{\alpha} < c^{-1} \cdot \lim_{x \in S} \|x\|^{2\gamma} P_0(x) = 0 \tag{77}$$

We may, therefore, assume that $\|x\| \geq 1$ and $\|x\|^{m-1} \operatorname{dis}(x, C_f) \leq 1$ for every x in C_f. On the other hand, for every x, there exists $y = (y_1, \ldots, y_n) \in C_f$ such that $\|x - y\| = \operatorname{dis}(x, C_f)$ and we have

$$\begin{aligned} |f(y) - f(x)| &\leq \sum_{k=1}^{m} \sum_{i_1, \ldots, i_k} \frac{1}{k!} \left| \frac{\partial^k f(x)}{\partial x_{i_1} \ldots \partial x_{i_k}} (y_{i_1} - x_{i_1}) \ldots (y_{i_k} - x_{i_k}) \right| \\ &\leq \sum_{k=1}^{m} c_k \|x\|^{m-k} \|y - x\|^k \quad \text{with constants} \quad c_k \\ &\leq c' \|x\|^{m-1} \|x - y\| \quad \text{for a constant} \quad c' \\ &\to 0 \qquad \text{by 77.} \end{aligned}$$

79 But this contradicts

$$|f(y) - f(x)| \geq |f(y)| - |f(x)| \geq |a| - \rho (> 0),$$

for every x in S. Thus 75 is valid with the chosen γ and

$$f^{-1}([-\rho, \rho]) \subset \bigcup_{k=1}^{n} W_k$$

where $W_k = \{x \in X; \max(1, \|x\|)^\gamma \left|\dfrac{\partial f}{\partial x_k}(x)\right| \geq \eta/\sqrt{n}\}$; in other words, on $f^{-1}([-\rho, \rho])$ at least one of the partial derivatives goes to 0 at most like $\|x\|^{-\gamma}$ and not too rapidly. On the other hand, since $\phi \in \mathscr{S}(\mathbb{R}^n)$, there exists a constant $c'' > 0$ such that for every $x = (x_1, \ldots, x_n) \in X$ and correspondingly $x' = (x_1, \ldots, \hat{x}_k, \ldots, x_n)$ in $\mathbb{R}^{n-1}$ with $\|x'\|^2 = \|x\|^2 - x_k^2$, we have

$$\max(1, \|x'\|)^n \max(1, \|x\|)^{\gamma+1} |\phi(x)| < c''$$

for $1 \leq k \leq n$. Therefore, for the left hand side of (74), we have the upper estimate

$$\sum_{k=1}^{n} \int\limits_{\substack{f^{-1}(b)\cap W_k \\ \|x\|\geq r}} |\phi(x)|\; |1/\frac{\partial f(x)}{\partial x_k}|\; |dx'|$$

$$\leq \frac{\sqrt{n}}{\eta} \sum_{k=1}^{n} \int\limits_{\substack{f^{-1}(b)\cap W_k \\ \|x\|\geq r}} \max(1, \|x\|)^\gamma |\phi(x)|\; |dx'|$$

$$\leq \frac{\sqrt{n}}{\eta} mnc'' r^{-1} \int\limits_{\mathbb{R}^{n-1}} \max(1, \|x'\|)^{-n} |dx'| = O(r^{-1}), \text{ as } r \to \infty.$$

This is sufficient to take care of the continuity of F_ϕ.

1.5 The Relations Between F_ϕ, F^*_ϕ and Z_ϕ

If $\sigma(\omega) > 0$, then, for $\phi \in \mathscr{S}(K)$, we have, by definition,

$$Z_\phi(\omega) = \int\limits_{X\setminus f^{-1}(0)} \phi(x)\omega(f(x))|dx| \tag{78}$$

$$= \int\limits_{K^\times} \left(\int\limits_{U(i)} \phi|\theta_i| \right) \omega(i)|di|$$

$$= M_K(m_K \omega_1 F_\phi)(\omega), \quad \text{in a formal sense.}$$

For every i^* in K, we have, on the other hand, **80**

$$\begin{aligned}
F_\phi^*(i^*) &= \int\limits_{X\setminus f^{-1}(0)} \phi(x)\psi(i^* f(x))|dx|, \quad \text{since} \quad f^{-1}(0) \text{ has measure } 0,\\
&= \int\limits_{K^\times}\left(\int\limits_{U(i)} \phi|\theta_i|\right)\psi(ii^*)|di|\\
&= \int\limits_{K^\times} F_\phi(i)\psi(ii^*)|di|\\
&= \int\limits_{K} F_\phi(i)\psi(ii^*)|di|, \quad \text{extending } F_\phi \text{ to } K.
\end{aligned}$$

i.e. F_ϕ^* is just the Fourier transform $(F_\phi)^*$ of F_ϕ.

We note that both the integrals considered above are absolutely convergent:

$$\int\limits_{K^\times} |F_\phi(i)\omega(i)|\ |di| \ \le \int\limits_X |\phi(x)|\ |f(x)|_K^{\sigma(\omega)}|dx| < \infty,$$

$$\int\limits_{K^\times} |F_\phi(i)\psi(ii^*)|\ |di| \le \int\limits_X |\phi(x)|\ |dx| < \infty$$

1.6 Statement of a Theorem

We now state the first substantial theorem, in our theory, for forms of higher degree.

81 **Theorem 1.6.** *Suppose that the characteristic of K is 0 and that $C_f \subset f^{-1}(0)$. Then there exists a set of λ's with Re $\lambda > 0$ such that Z_ϕ is in $\mathcal{Z}$, F_ϕ is in $\omega_1^{-1}\mathcal{F}$ and F_ϕ^* is in $(\omega_1^{-1}\mathcal{F})^*$. In particular, $Z_\phi(\omega)$ defined by (78) has a meromorphic continuation to the whole of $\Omega(K^\times)$ and further, $F_\phi(i)$ (respectively $F_\phi^*(i^*)$) possesses an asymptotic expansion as $|i|_K \to 0$ (respectively $|i^*|_K \to \infty$).*

The major part of the proof of this theorem which will be given in the next two sections, is to show that Z_ϕ is in $\mathcal{Z}$; once that is proved, it will follow that F_ϕ is in $\omega_1^{-1}\mathcal{F}$ and hence F_ϕ^* in $(\omega_1^{-1}\mathcal{F})^*$. Indeed, together with the absolute convergence of the integral for $Z_\phi(\omega)$ mentioned above and the continuity of F_ϕ, the fact that Z_ϕ is in $\mathcal{Z}$ will enable us to apply the Fourier inversion theorem and conclude that

$$F_\phi = m_K^{-1}\omega_1^{-1}M^{-1}(Z_\phi)$$

The rest of the assertions of Theorem 1.6 now follows from the theorems proved in Chapter I.

REMARKS. 1) The assumption that $C_f \subset f^{-1}(0)$ in Theorem 1.6 cannot be dropped. Consider, for example, the case when $n = 2$, $f(x) = 1+x_1x_2$ and ϕ is the characteristic function of $V = P^{e_1}\times P^{e_2}$ with e_1, e_2 in $\mathbb{Z}$ satisfying $e_1 + e_2 \geq 1$. Then $C_f = \{0\} \subset f^{-1}(1)$ and with our former notation,

$$\begin{aligned} Z_\chi(z) &= \int\limits_V \chi(1 + x_1x_2)|dx| \\ &= \sum_{j\geq e_1}(1 - q^{-1}) \int\limits_{1+P^{j+e_2}} \chi(u)du \\ &= q^{-\max(e_1+e_2,e_\chi)} \\ &\neq 0 \end{aligned}$$

for every χ. This violates condition (2) defining $\mathcal{Z}(\Omega(K^\times))$.

2) If K has characteristic $p > 0$, then the condition "p does not divide the degree of f" is necessary (although not sufficient) to ensure that Z_ϕ is in $\mathcal{Z}$. Consider now, for example, the case when $n = 1$, $f(x) = x^p$ and ϕ is the characteristic function of R. Then, for $|z| < q^{1/p}$, we can show that **82**

$$Z_\chi(z) = \begin{cases} \frac{1-q^{-1}}{1-q^{-1}z^p} & \text{if } \chi^p = 1 \\ 0 & \text{if } \chi^p \neq 1. \end{cases}$$

But the number of χ in $(R^*)^*$ with $\chi^p = 1$ is infinite; in fact, for the finite group $R/(1+P^{pj})$ with $j \geq 1$, the cardinality of the cokernel of the p^{th} power map is $q^{(p-1)j}$ and therefore, the number of χ in $(R^*)^*$ with $\chi^p = 1$ and $e_\chi \leq pj$ is $q^{(p-1)j}$. Again condition (2) in the definition of $\mathcal{Z}$ is violated by Z_ϕ.

1.7 Gaussian Sums and Singular Series

For any p-field K and $e \in \mathbb{Z}$, let us denote the n-fold product $P^e \times \ldots \times P^e$ of P^e by $(P^e)^{(n)}$ and for $e = 0$, $R^{(n)}$ by X^0. If $f(x) \in R[x_1, \ldots, x_n]$ and ϕ is the characteristic function of X^0, we simply write F, F^*, Z instead of F_ϕ, F^*_ϕ, Z_ϕ respectively. Then, without any further assumption on f, we have, for $i^* = \pi^{-e}u$ with $e \geq 0$ in $\mathbb{Z}$ and u in R^*, the following relation:

$$F^*(i^*) = q^{-ne} \sum_{\xi \bmod P^e} \psi(i^* f(\xi))$$

which is immediate on writing any $x \in X^0$ as $\xi + \pi^e y$ with ξ running over $X^0 \bmod (P^e)^{(n)}$ and $y \in X^0$. The sum over ξ modulo $(P^e)^{(n)}$ is usually called a *generalized Gaussian sum*. Although this relation is quite straightforward, the interesting fact is that, once we accept Theorem 1.6, then we get

$$q^{-ne} \sum_{\xi \bmod P^e} \psi(i^* f(\xi)) = \sum_{\lambda, m} a^\sharp_{\lambda,m}(ac(i^*))|i^*|_K^{-\lambda}(\log |i^*|_K)^{m-1}$$

83 = a fixed linear combination of functions of the form

$$\chi(ac(i^*))|i^*|_K^{-\lambda}(\log |i^*|_K)^{m-1}$$

for all sufficiently large $e = -\operatorname{ord}(i^*)$. Such a "stable behaviour" of a generalized Gaussian sum is quite remarkable.

We also recall that for i in R,

$$N_e(i) = \text{ the number of } \xi \bmod P^e \text{ with } f(\xi) \equiv i \pmod{P^e},$$

also plays an important role in the classical theory. If f is homogeneous of degree m, i is in $R\backslash\{0\}$ and the characteristic of K does not divide m, then we will see that

$$q^{-(n-1)e} N_e(i) \text{ is independent of } e, \text{ for } e \geq 2\operatorname{ord}(mi) + 1. \tag{79}$$

Further, for $i \neq 0$ in R, we have the relation

$$F(i) = \text{ the above "stable quotient" } q^{-(n-1)e} N_e(i) \tag{80}$$

$$= \lim_{e\to\infty} q^{-(n-1)e} N_e(i)$$

which is known as the *singular series* associated with f and i. The proof of (79) is given in the appendix to this chapter. However, if we assume that

$$K \text{ has characteristic } 0 \text{ and } C_f \subset f^{-1}(0)$$

then (even for f not necessarily homogeneous) we can use Theorem 1.6 to derive, at one stroke, the relations (79) in perhaps a less precise form and (80). The proof is as follows. In fact, the theorem implies that F_ϕ and, in particular, F (= F_ϕ for $\phi = \varphi_{X^0}$) is in $\omega_1^{-1}\mathcal{F}$ and hence, locally constant on $K^\times$. Thus there exists $e > \operatorname{ord}(i)$ such that $F(t) = F(i)$ for all $t \in i + P^e$. If now, we take φ to be the characteristic function of the 84
compact open set $X^0 \cap f^{-1}(i + P^e)$, the right hand side of (72) is just

$$\begin{aligned}
\int_K \left(\int_{U(t)} \varphi|\theta_t| \right) |dt| &= \int_{i+P^e} \left(\int_{U(t)\cap X^0} |\theta_t| \right) |dt| \\
&= \int_{i+P^e} F(t)|dt| \text{ by definition 73} \\
&= F(i)q^{-e}, \text{ since } F \text{ is constant on } i + P^e.
\end{aligned}$$

But the left hand side of (72) is merely

$$\begin{aligned}
m(X^0 \cap f^{-1}(i + P^e)) &= m(\{\xi \in X^0; f(\xi) \equiv i \pmod{P^e}\}) \\
&= N_e(i) \cdot m(\pi^e X^0) \\
&= q^{-ne} N_e(i).
\end{aligned}$$

Hence there exists $e > \operatorname{ord}(i)$ such that

$$F(i) = N_e(i)/q^{(n-1)e}$$

and this relation persists even if we replace e by a larger integer.

2 Preparations for the Proof of the “Main Theorem”

2.1

We shall prove the “main theorem” i.e. Theorem 1.6, only for p-fields K and merely give an outline of the proof when K is an $\mathbb{R}$-field. First we explain the concept of a resolution of (the singularities of) the surface defined by $f(x) = 0$ in a general set-up.

Let k be an arbitrary field, X an (absolutely) irreducible non-singular algebraic variety defined over k and D, a positive k-rational divisor of
85 X. Then a k-resolution

$$h : Y \to X$$

of D is defined by the following three conditions:

(i) h is an everywhere regular birational map of an irreducible non-singular algebraic variety Y to X, with both Y and h defined over k, such that h can be written as a product of monoidal transformations each with an irreducible non-singular centre;

(ii) h^{-1} is regular and hence biregular, at every simple point a of D; and

(iii) for every b in Y, all the irreducible components of $h^*(D)$ passing through b are defined over the field $k(b)$ and further are mutually transversal at b.

Although this may appear to be much too general, we shall deal in our applications only with the cases when X is either an affine or a projective space and D, the divisor defined by a polynomial. Also, that part of condition (i) which requires h to be a product of monoidal transformations may be replaced by the weaker condition that h is “projective”, if our immediate objective is only to prove Theorem 1.6. However, the full force of condition (i) will be used only later on. It should also be remarked that the monoidal transformations cannot, in general, be required to be defined over k itself.

We also consider the following situation. Suppose D' is the sum of irreducible subvarieties of codimension 1 in X which are defined over k and further mutually transversal at every point of X; in our applications, D' will be either 0 or irreducible (and non-singular). Moreover, let condition (iii) above be satisfied with $h^*(D + D')$ in place of $h^*(D)$. Then we say that h is a *k-resolution of* (D, D'). If h is a k-resolution of (D, D'), then h is also a K-resolution of (D, D') for any extension field 86
K of k. Moreover, for any k-open subset U of X, if $Y_U = h^{-1}(U)$ and D_U (respectively D'_U) are the restrictions of D (respectively D') to U, then h gives rise to a k-resolution $Y_U \to U$ of (D_U, D'_U). These assertions are quite obvious. But what is important is that *a k-resolution of* (D, D') *always exists, if k has characteristic* 0*;* and this is indeed a consequence of Hironaka's fundamental theorem ([16]).

2.2

If h is a k-resolution of (D, D'), then every irreducible component of $h^*(D + D')$ is non-singular. Let E be an irreducible component of $h^*(D)$; then we have

$$h^*(D) = \sum_E N_E \cdot E$$

where the integer N_E is defined as follows. If b is any point of E and if D is defined locally by the equation $g = 0$ around $a = h(b)$, then the multiplicity of E in $g \circ h = 0$ is denoted by N_E; this does not depend on the choice of b in E. We say that the k-resolution h is *tame* if the characteristic of k does not divide any N_E.

We need to define another component for our datum. Let us choose local coordinates $x_1, \ldots, x_n$ around a in X and local coordinates $y_1, \ldots,$ y_n around b in Y. Then the multiplicity of E in the local divisor defined by $\dfrac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)} = 0$ depends only on E and is denoted by $\nu_E - 1$. We call the pair (N_E, ν_E) the *numerical datum of h along E.*

We observe that $\nu_E \geq 1$ and further, that $\nu_E = 1$ if and only if h is biregular at every point of E not contained in any other component of $h^*(D)$. If $E_1, E_2, \ldots$ are the irreducible components of $h^*(D)$ passing through any point b of Y, then the cardinality of the set $\{E_i\}$ is at most

87 equal to the dimension n of X. If we rewrite the numerical datum of h along E_i as (N_i, ν_i), then we call $\{(N_i, \nu_i)\}_i$ the *numerical data of h at b.*

Suppose that K is an extension field of k and $h : Y \to X$ is a k-resolution of (D, D') as above. For any K-rational point b of Y (i.e. for $b \in Y_K$), let $x_1, \dots, x_n$ be local coordinates around $a = h(b)$ on X which are defined over k and let D be defined locally around a by the equation $g = 0$. Further, let us write $dx = dx_1 \wedge \dots \wedge dx_n$. We can then find local coordinates $y_1, \dots, y_n$ around b on Y which are defined over K and satisfy $y_1(b) = \dots = y_n(b) = 0$, such that we have

$$
\begin{aligned}
g \cdot h &= \epsilon \prod_i y_i^{N_i} \\
h^*(dx) &= \eta \prod_i y_i^{\nu_i - 1} \cdot dy
\end{aligned}
\tag{81}
$$

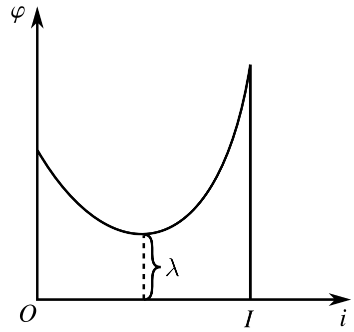
around b, with functions ϵ, η on Y that are defined over K and invertible in the local ring of Y at b. If K is a local field, we shall write X, $Y, \dots$ etc. for X_K, $Y_K, \dots$, consistent with our previous notation.

2.3 An Example

Suppose that k has characteristic 0 and D is an absolutely irreducible curve containing the origin 0, in a k-open subset X of the affine plane; let 0 be the only singular point of D and further, D absolutely analytically irreducible at 0. It is well-known that there exists then a unique minimal k-resolution of D. Let $E_i, i = 1, 2, \dots, I$ be the exceptional curve "created" at the stage of the i^{th} quadratic transformation. (We remark that such an ordering of the exceptional curves is used only here).
88 Then the graph of the function

$$\varphi(i) = \nu_i / N_i \ \text{ for } \ 1 \leq i \leq 1$$

has been examined in our paper [25]. In the first approximation, the graph looks as in the following figure.



The (strict) minimum value of φ, say λ, has a remarkably simple meaning. If $f(x) = 0$ is a local equation for D around 0 and $f_m(x)$ the leading form in $f(x)$, then after replacing x_1 by $x_1 + cx_2$ for some c in k, if necessary, we may suppose that $f_m(0, 1) \neq 0$. Then we can solve the equation $f(x) = 0$ by a power-series $x_2 = g(x_1)$ in $x_1^{1/m}$. Let x_1^r be the first non-integral power of x appearing in $g(x_1)$ with non-zero coefficient. Then the so-called “first characteristic exponent” r satisfies $r > 1$ and further,

$$\lambda = \frac{1}{m}\left(1 + \frac{1}{r}\right).$$

In particular, for $f(x) = x_1^n - x_2^m$ with m, n coprime and $1 < m < n$, we have $r = n/m$ and hence, $\lambda = \dfrac{1}{m} + \dfrac{1}{n}$.

2.4

The following lemma will prove useful later, for example, in settling the rationality of $Z_\phi(\omega)$.

Lemma 2.4. *For a p-filed K, let ω be in $\Omega(K^\times)$ with $\omega(\pi^i u) = z^i \chi(u)$ as before and further, let $\nu + N\sigma(\omega) > 0$ (or, equivalently $|q^{-\nu} z^N| < 1$) for given N, ν in $\mathbb{Z}$. Then, for any c in K and e in $\mathbb{Z}$, we have* 89

$$\int_{c+P^e} \omega(t)^N |t|_K^{\nu-1} |dt| = \begin{cases} (1-q^{-1})q^{-\nu e} z^{eN}/(1-q^{-\nu}z^N), & \text{if } \chi^N = 1 \text{ and } c \in P^e \\ 0, & \text{if } \chi^N \neq 1 \text{ and } c \in P^e \\ \chi^N(c\pi^{-\operatorname{ord}(c)}) q^{-(\nu-1)\operatorname{ord}(c)-e} z^{\operatorname{ord}(c)N}, & \text{if } \chi^N = 1 \\ & \text{on } 1 + c^{-1}P^e \text{ and } c \notin P^e \\ 0, & \text{if } \chi^N \neq 1 \text{ on } 1 + c^{-1}P^e \text{ and } c \notin P^e \end{cases} \tag{82}$$

Proof. Let $e_0 = \operatorname{ord}(c)$ and $u_0 = c\pi^{-e}$. Then, for $c \in P^e$, the left hand side of (82) is clearly the same as

$$\sum_{i \geq e} (q^{-\nu} z^N)^i \int_{R^\times} \chi^N(u) du = \begin{cases} \frac{(1-q^{-1})(q^{-\nu}z^N)^e}{(1-q^{-\nu}z^N)}, & \text{for } \chi^N = 1 \\ 0, & \text{if } \chi^N \text{ is non-trivial.} \end{cases}$$

If, however, $c \notin P^e$, then the left hand side of (82) is obviously equal to $\omega(c)^N |c|_K^\nu \int_{1+c^{-1}P^e} \chi^N(u) du$ on applying $t \mapsto ct$ and this is then seen to coincide with the expression on the right hand side. □

2.5

Let m denote a positive integer and let us denote the map $x \mapsto x^m$ in K by $[m]$. Thus, for any subset S of K, we have $S^{[m]} = \{s^m; s \in S\}$. The following lemma gives a very precise description of the images under $[m]$ of certain subsets of $R^\times$ in a p-field K.

Lemma 2.5. *If m is a natural number not divisible by the characteristic of a p-field K, then, for every integer $i > \operatorname{ord}(m)$, we have*

$$(1 + P^i)^{[m]} = 1 + mP^i.$$

90 *Proof.* Our proof depends on the following simple version of Hensel's lemma. For any $g(x)$ in $R[x_1, \ldots, x_n]$ and a in $X = K^n$, let $\mathscr{X} = \mathscr{X}(a)$ be defined by

$$\mathscr{X} = \mathscr{X}(a) = \min\left\{\operatorname{ord}\left(\frac{\partial g}{\partial x_1}(a)\right), \ldots, \operatorname{ord}\left(\frac{\partial g}{\partial x_n}(a)\right)\right\}$$

so that $\mathscr{X}(a) = \infty$ if and only if a is in C_f. *If for a in $X^0 \backslash C_f$, we have $g(a) \equiv 0 (\bmod P^{2\mathscr{X}+1})$, then there exists b in X^0 such that*

$$g(b) = 0 \text{ and } b \equiv a \pmod{g(a)P^{-\mathscr{X}}} \tag{83}$$

In particular, every simple point a mod P *on* $g(x) \equiv 0 \pmod P$ *can be "lifted" to a point b of the variety given by $g(x) = 0$, in X^0.* For the sake of completeness, we give a quick proof of (83). If $e_0 = \operatorname{ord}(g(a))$, then $e_0 \geq 2\mathscr{X} + 1$; let us exclude the trivial case when $e_0 = \infty$. We can construct, as follows, a sequence

$$a^{(0)} = a, \quad a^{(1)}, \quad a^{(2)}, \ldots$$

in X^0 such that

$$g(a^{(i)}) \equiv 0 \pmod{P^{e_0+i}} \text{ and } a^{(i)} \equiv a^{(i-1)} \pmod{P^{e_0+i-1-\mathscr{X}}} \tag{84}$$

for $i \geq 1$. For $i = 0$, the first condition in (84) is already true while the second condition is non-existent. Assume (84) to have been proved for some $i \geq 0$ and set

$$a^{(i+1)} = a^{(i)} + \pi^{e_0+i-\mathscr{X}} t$$

with an unknown $t = (t_1, \ldots, t_n)$ in X^0. Then we have

$$\begin{aligned} g(a^{(i+1)}) \equiv 0 \pmod{P^{e_0+i+1}} \Leftrightarrow \sum_{k=1}^{n} \pi^{-\mathscr{X}} \frac{\partial g}{\partial x_k}(a) t_k \\ \equiv -\pi^{-(e_0+i)} g(a^{(i)}) \pmod P \end{aligned}$$

in view of $e_0 \geq 2\mathscr{X} + 1$, on using the Taylor expansion of f. Now the definition of $\mathscr{X}$ enables us to establish 84 for $i + 1$ instead of i and 83 follows from the completeness of K. □

We proceed to derive Lemma 2.5 from our version of Hensel's **91**
lemma. It is clear that, for $i \geq \operatorname{ord}(m)$, we have $(1 + P^i)^{[m]} \subset 1 + mP^i$. In order to prove the reverse inclusion for $i > \operatorname{ord}(m)$, we have only to start from any element $1 + m\pi^i a$ of $1 + mP^i$ with $a \in R$ and set

$$g(x) = (m\pi^i)^{-1}((1 + \pi^i x)^m - (1 + m\pi^i a))$$

Then $g(x) \equiv x - a (\bmod P)$ and a mod P is a simple point on $g(x) \equiv 0 (\bmod P)$. Therefore, we have, from above, an element b of R with $g(b) = 0$, proving the desired inclusion and the lemma.

3 Proof of Theorem 1. 6 for p-Fields

3.1 Reformulation of the Theorem for p-Fields

Let K be a p-field and then, by our identification of X, Y etc. with X_K, Y_K etc. , X is the same as K^n. Instead of assuming that $C_f \subset f^{-1}(0)$, let us make the weaker assumption that, for the given ϕ in $\mathscr{S}(X)$, its support Supp(ϕ) satisfies the condition

$$C_f \cap \operatorname{Supp}\phi \subset f^{-1}(0). \tag{85}$$

Let us recall that $Z_\phi(\omega) = \int\limits_X \phi(x)\omega(f(x))|dx|$ where $\omega \in \Omega_+(K^\times)$ with $\omega(x) = z^{\operatorname{ord}}(x)\chi(x\pi^{-\operatorname{ord}}(x))$ for $x \in K^\times$. We shall prove that, if a tame resolution $h : Y \to X$ for the hypersurface D defined by $f(x) = 0$ exists, then

$$Z_\phi(\omega) \text{ is a rational function of } z, \text{ holomorphic in } 0 < |z| \leq 1 \tag{86}$$

and further, that

$$Z_\phi(\omega) \text{ vanishes identically in } z, \text{ for almost all } \chi \in (R^\times)^* \tag{87}$$

92 Moreover, it will turn out that $\Lambda \subset \{\lambda \in \mathbb{C}; \lambda \bmod 2\pi i/\log q, \lambda N_E \equiv \nu_E (\operatorname{mod} 2\pi i/\log q)$ i.e. $q^{\lambda N_E} = q^{\nu_E}\}$ and $m_\lambda \leq n$ for every $\lambda \in \Lambda$. Both the assumption 85 and the tameness of h are essential for the validity of 87. (See Remarks 1, 2 Ch. III, §1.6). Moreover, for our proof, we need to resolve only the K-rational singularities of D; no further singularities come into the picture or rather, need to be resolved. As remarked already in §2.1, the latter part of condition (i) for h is invoked only to ensure that h is a proper map.

3.2 Reduction of the Proof

For any b in $Y = Y_K$, we get, in view of 81, local coordinates $y_1, \ldots, y_n$ on Y defined over K such that $y_1(b) = \ldots\ldots = y_n(b) = 0$ and

$$f \circ h = \epsilon \prod_i y_i^{N_i}$$

$$h^*(dx) = \eta \prod_i y_i^{\nu_i - 1} dy \tag{88}$$

with invertible K-analytic functions ϵ, η around b and the characteristic of K dividing no N_i. We can modify any y_i in 88 by throwing in a factor such as an invertible K-analytic function around b and a fortiori, a factor from $K^\times$. In 88, if, for example, the product $\prod_i y_i^{N_i}$ is empty, it is to be interpreted as 1, as when no component passes through b. Let

$$m_0 = \max_b \max_i \{\mathrm{ord}(N_i)\} \tag{89}$$

where we first take the maximum over the finitely many natural numbers $N_1, N_2, \ldots$ and then take the maximum over all b in $h^{-1}(\mathrm{Supp}(\phi))$; it is clear that m_0 exists.

If $f(a) \neq 0$ for $a = h(b)$, then the assumption 85 enables us to **93**
include $f(x) - f(a)$ among the coordinate functions $y_1, \ldots, y_n$ so that, in this case, we may take

$$\begin{aligned} f \circ h (= \epsilon) &= \epsilon(b)(1 + y_1),\ f(a) = \epsilon(b) \\ h^*(dx) &= \eta dy \end{aligned} \tag{88$'$}$$

instead of 88. In the general case, we have still,

$$\begin{aligned} f \circ h &= \epsilon \prod_i y_i^{N_i}, \quad \epsilon = \epsilon(b)\left(1 + \sum_{|\alpha| \geq 1} A_\alpha y^\alpha\right), \quad \epsilon(b) \neq 0 \\ h^*(dx) &= \eta \prod_i y_i^{\nu_i - 1} dy, \quad \eta = \eta(b)\left(1 + \sum_{|\alpha| \geq 1} B_\alpha y^\alpha\right), \quad \eta(b) \neq 0 \end{aligned} \tag{88$''$}$$

where, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \geq 0$ in $\mathbb{Z}$, we have written y^α for $y_1^{\alpha_1} \ldots y_n^{\alpha_n}$, A_α for $A_{\alpha_1, \ldots, \alpha_n}$, B_α for $B_{\alpha_1, \ldots, \alpha_n}$ and $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$. Both the power-series above converge in a neighbourhood $(P^{\nu_0})^{(n)}$ of the origin for some $\nu_0 \geq 0$ and therefore $\lim A_\alpha \pi^{\nu_0|\alpha|} = \lim B_\alpha \pi^{\nu_0|\alpha|} = 0$ as $|\alpha| \to \infty$. Replacing, if necessary, each y_i by $\pi^\nu y_i$ for a suitably large integer ν, we may suppose, without loss of generality, that all the coefficients A_α, B_α are already in R. Thus, for all c close to b (i.e. with all $y_i(c)$ in P for example), we have, in (88)$''$,

$$|\eta(c)|_K = |\eta(b)|_K. \tag{90}$$

We now assert that it is possible to choose an open neighbourhood U of b sufficiently small so as to satisfy the two conditions:

(i) $Y \supset U \hookrightarrow (P^{m_0+1})^{(n)} \subset K^{(n)}$, via $c \mapsto (y_1(c), \ldots, y_n(c))$ from U to $K^{(n)}$

(ii) $\phi \circ h$ is constant on U.

94 This is possible since ϕ is locally constant, $\mathrm{Supp}(\phi)$ is compact and h is a proper map. In fact, we can express ϕ as a finite linear combination of characteristic functions of disjoint compact open subsets W of X which are of the form $a' + (P^e)^{(n)}$ and $\phi \circ h$ is constant on each compact open set $h^{-1}(W)$. We can now cover the compact open set $h^{-1}(\mathrm{Supp}\,\phi)$ by a finite number of U's, say $U_1, U_2, \ldots$ and by the standard process of taking $U_1, U_2 \backslash U_1, U_3 \backslash (U_1 \cup U_2)$, etc. , we may assume that $U_1, U_2, \ldots$ are already disjoint and non-empty. After imbedding each of these in K^n and decomposing them into disjoint cosets modulo P^{e_0} for a fixed $e_0 \geq m_0 + 1$, we call the resulting finitely many compact open sets $V_1, V_2, \ldots$ where each V_i is of the form $b' + (P^{e_0})^{(n)} \subset (P^{m_0+1})^{(n)}$, (for m_0 defined by (89)). In view of (88)$''$, (90), we now see that $Z_\phi(\omega)$ is a finite linear combination of integrals $Z(\omega)$ extended over $V = V_1$, $V_2, \ldots$, where

$$\begin{aligned} Z(\omega) &= \int\limits_V (\phi \circ h)\omega(f \circ h)|h^*(dx)| \\ &= \phi(h(b))\omega(\epsilon(b))|\eta(b)|_K \int\limits_V \chi\left(1 + \sum_{|\alpha|\geq 1} A_\alpha y^\alpha\right) \prod_i \omega(y_i)^{N_i} |y_i|_K^{\nu_i - 1} |dy| \end{aligned} \tag{91}$$

and the factor outside the integral in 91 is evidently a constant multiple of $z^{\mathrm{ord}(\epsilon(b))}$.

3.3

Before we proceed further, we first isolate the case when $f(a) = (f \circ h)(b) \neq 0$. We have then, by (88)$''$,

$$\begin{aligned}\int_V (\phi \circ h)\omega(f \circ h)|h^*(dx)| &= c' q^{-(n-1)e_0} \int_{b_1'+P^{e_0}} \chi(1+y_1)dy_1 \\ &= c'' \int_{1+P^{e_0}} \chi(t)dt \quad \text{since} \quad 1+b_1' \in R^\times \\ &= \begin{cases} c''' & \text{if } e_\chi \leq e_0 \\ 0 & \text{if } e_\chi > e_0 \end{cases}\end{aligned}$$

where c', c'', c''' are constants. Thus, *upto a constant factor.* **95**

$$Z(\omega) = \begin{cases} z^{\operatorname{ord}(f(a))} & \text{if } e_\chi \leq e_0 \\ 0 & \text{if } e_\chi > e_0 \end{cases} \tag{92}$$

in view of 91, since $f(a) = \epsilon(b)$.

3.4

Suppose now b is such that $f(a) = (f \circ h)(b) = 0$ (i.e. there exists i such that E_i passes through b). We try to pull out $\chi\left(1 + \sum\limits_{|\alpha|\geq 1} A_\alpha y^\alpha\right)$ from inside the integral in 91 over $V \simeq b' + (P^{e_0})^{(n)}$. We set $e = \max\{e_0, e_\chi\}$ and decompose $(P^{e_0})^{(n)}$ into cosets modulo P^e; namely, let

$$(P^{e_0})^{(n)} = \coprod_{b''} b'' + (P^e)^{(n)}$$

so that

$$V \simeq \coprod_c c + (P^e)^{(n)} \tag{93}$$

with $c = b' + b''$. Now

$$y \in c + (P^e)^{(n)} \subset (P^{m_0+1})^{(n)} \Rightarrow y \equiv c \pmod{P^e}$$

$$\begin{aligned}
&\Rightarrow y \equiv c \pmod{P^e\chi}\\
&\Rightarrow 1 + \sum_{|\alpha|\geq 1} A_\alpha y^\alpha \equiv 1 + \sum_{|\alpha|\geq 1} A_\alpha c^\alpha \pmod{P^{e_\chi}}\\
&\Rightarrow \chi\left(1 + \sum_{|\alpha|\geq 1} A_\alpha y^\alpha\right) = \chi\left(1 + \sum_{|\alpha|\geq 1} A_\alpha c^\alpha\right)
\end{aligned}$$

since both the series are $\equiv 1 \pmod{P^{m_0+1}}$ and, a fortiori, $\equiv 1 (\mathrm{mod} P)$. The integral in 91 is thus a sum of finitely many terms (indexed by $c = (c_1, \ldots, c_n)$ in 93) which, upto a factor q^{-er} (with r equal to the number of missing coordinate functions y_i in (88)$''$) are of the form

$$\chi\left(1 + \sum_{|\alpha|\geq 1} A_\alpha c^\alpha\right) \prod_i \int\limits_{c_i+P^e} \omega(t)^{N_i} |t|_K^{\nu_i - 1} |dt|.$$

96 Therefore, by Lemma 2.4, it is the quotient of a polynomial in z and z^{-1} by $\prod\limits_i (1 - q^{-\nu_i} z^{N_i})$ for $|z| < 1 < \min q^{\nu/N_i}$. As a result, $Z(\omega)$ is a rational function of z, holomorphic in $0 < |z| \leq 1$. Putting this together with 92, our assertion 86 is proved as well as the remark on the composition of A.

3.5 Proof of Assertion 87

It is sufficient to show that

$$Z_\phi(\omega) = 0 \text{ for } e_\chi > m_0 + e_0 \tag{94}$$

since the number of $\chi \in (R^\times)^*$ with $e_\chi \leq m_0 + e_0$ is finite. We shall assume 94 to be false and arrive at a contradiction; note that $f(a) = 0$ in view of 92. If $e = \max(e_0, e_\chi)$, then clearly, $e = e_\chi$. Now since $Z_\phi(\omega) \neq 0$, we must have

$$\int\limits_{c_i+P^e} \omega(t)^{N_i} |t|_K^{\nu_i - 1} |dt| \neq 0$$

for some c and some i, at least. By Lemma 2.4 above, we have

$$\chi^{N_i} = 1 \text{ on } R^\times \text{ for } c_i \in P^e,$$

$$\chi^{N_i} = 1 \text{ on } 1 + c_i^{-1}P^e \text{ for } c_i \notin P^e.$$

Since $c_i \in P^{m_0+1}$, $c_i^{-1}P^e \supset P^{-m_0-1}P^e$, so that $1 + P^{e-m_0-1} \subset 1 + c_i^{-1}P^e$ and both are clearly in $R^\times$. Thus $\chi^{N_i} = 1$ on $1 + P^{e-m_0-1}$, implying that $\chi = 1$ on $(1 + P^{e-m_0-1})^{[N_i]}$ which is the same as $1 + N_i P^{e-m_0-1}$ by Lemma 2.5, provided that $e - m_0 - 1 > \operatorname{ord}(N_i)$. But the last condition is indeed fulfilled, since $e - m_0 - 1 = e_\chi - m_0 - 1 \geq 1 + m_0 + e_0 - m_0 - 1 = e_0 \geq m_0 + 1 > \operatorname{ord}(N_i)$ for every i by 94 and 89. Since $\operatorname{ord}(N_i) + e - m_0 - 1 \leq m_0 + e - m_0 - 1 = e_\chi = 1$, we have $\chi = 1$ 97
on $1 + P^{e_\chi - 1}$ while $e_\chi > m_0 + e_0 \geq 2m_0 + 1 \geq 1$, giving the desired contradiction from the definition of e_χ. We have therefore proved 94 and indeed, as a result, that Z_ϕ is in $\mathcal{Z}$. Theorem 1.6 is completely proved for the case of p-fields, in view of the remarks immediately following the statement of the theorem.

REMARK. On a close examination of the proof of the rationality of $Z_\phi(\omega)$ above, it may be seen that neither the tameness of the resolution nor condition 85 is required to be assumed.

It may be of interest to exhibit the rationality of a related function. Suppose that $f(x)$ is an arbitrary polynomial in $R[x_1, \ldots, x_n]$. Consider the power-series

$$P_0(z) = \sum_{e=0}^{\infty} N_e \cdot (q^{-n}z)^e = 1 + N_1 q^{-n} z + N_2 q^{-2n} z^2 + \cdots$$

where $N_e = N_e(0)$ is the number of ξ in X^0, modulo P^e such that $f(\xi) \equiv 0(\operatorname{mod} P^e)$. (See Chapter III, § 1.7). Since, trivially, we have $N_e \leq q^{ne}$ for every $e \geq 0$, it follows that $P_0(z)$ is holomorphic in the unit disc $|z| < 1$. It has been conjectured in [6] (Page 47, Problem 9), that $P_0(z)$ represents a rational function of z. We shall now establish the validity of this conjecture. Let us decompose X^0 as $X^0 = \left(\coprod_{e \geq 0} E_e\right) \coprod E_\infty$, the disjoint union of $E_e = X^0 \cap f^{-1}(\pi^e R^\times)$ for $e \geq 0$ and of $E_\infty = X^0 \cap f^{-1}(0)$. Then, on E_e, $|f|_K^s$ is always equal to z^e and further

$$m(E_e) = m(X^0 \cap f^{-1}(P^e)) - m(X^0 \cap f^{-1}(P^{e+1})) = N_e \cdot q^{-ne} - N_{e+1} \cdot q^{-n(e+1)}.$$

Therefore we get

$$Z_1(z) = \int\limits_{X^0\backslash E_\infty} |f(x)|_K^s |dx| = \sum_{e=0}^{\infty}\{N_e(q^{-n}z)^e - N_{e+1}(q^{-n}z)^{e+1}z^{-1}\}$$
$$= P_0(z) - (P_0(z)-1)z^{-1},$$

98 implying that

$$P_0(z) = \frac{1 - zZ_1(z)}{1-z}$$

which is indeed a rational function of z from the considerations above.

4 Proof of Theorem 1.6 for $\mathbb{R}$-Fields

4.1 Reformulation of the Theorem for $\mathbb{R}$-Fields

Let $X = K^n$ for an $\mathbb{R}$-field K and for a given ϕ in $\mathscr{S}(X)$ and the chosen f, let us assume, as before, that

$$\mathrm{Supp}(\phi) \cap C_f \subset f^{-1}(0).$$

For $\omega = \omega_s(ac)^p \in \Omega(K^\times)$ with $\mathrm{Re}\,(s) > 0$, we recall the definition $Z_\phi(\omega) = \int\limits_X \phi(x)\omega(f(x))|dx|$. Using the same kind of resolution as in the case of p-fields, we shall prove that

$$\begin{array}{l} Z_\phi(\omega) \text{ has a meromorphic continuation to the entire } s\text{-plane,} \\ \text{having poles at most at the points } -\lambda = -(\nu_E/N_E + r/(2dN_E)) \\ \text{for } r = 0,1,2,\ldots \text{ and of order at most } n(=\dim X)\text{, notation} \\ \text{being the same as in §2 above.} \end{array} \tag{95}$$

(cf. [2]. [3]). Moreover, we shall prove that, for every polynomial $P_0(s,p)$ and every vertical strip B_{σ_1,σ_2}.

$$P_0(s,p)Z_\phi(\omega) \text{ is uniformly bounded for } s \text{ belonging to } B_{\sigma_1,\sigma_2} \tag{96}$$

with neighbourhoods of the poles of $Z_\phi(\omega)$ removed therefrom, for p in $\mathbb{Z}$.

99 Both 95 and 96 will together ensure that $Z_\phi \in \mathcal{Z}(\Omega(K^\times))$ and our theorem for $\mathbb{R}$-fields will follow, once again, in view of the remarks immediately following its statement in § 1.6.

4.2 Reduction of the Proof

We shall give an outline of the proof of 95 and 96 only for the case when ϕ has *compact* support: the general case will be dealt with later.

Using the same kind of K-resolution $h : Y \to X$ for the hypersurface defined by $f(x) = 0$ as in the case of p-fields, we have, for any b in Y, local coordinates $y_1, \ldots, y_n$ centered at b such that

$$f \circ h = \in \prod_i y_i^{N_i}, \quad h^*(dx) = \eta \prod_i y_i^{\nu_i - 1} dy \tag{97}$$

for invertible ϵ, η K-analytic around b. If $a = h(b)$ and $f(a) \neq 0$, then we have, as in $(88)'$,

$$f \circ h = f(a)(1 + y_1), \quad h^*(dx) = \eta dy. \tag{97}$$

If $f(a) = 0$, we have, on replacing y_1 by $(\pm\epsilon)^{1/N_1} y_1$ for example,

$$f \circ h = \delta \prod_{1 \le i \le n} y_i^{N_i}, \quad \delta = \begin{cases} \pm 1 & \text{for } K = \mathbb{R} \\ 1 & \text{for } K = \mathbb{C} \end{cases}$$

instead of the first relation in 97. The numerical data of the resolution at b are given by $\{N_i, \nu_i\}$.

Again $h^{-1}(\operatorname{Supp} \phi)$ is compact and we choose a finite covering by subsets which are homeomorphic (under the mapping by the n coordinate functions) to the polydisc C in K^n defined by $|y_i| < 1$ for $1 \le i \le n$. We choose a smooth partition of unity subordinate to the finite covering and proceeding exactly as in § 3, we see that $Z_\phi(\omega)$ becomes a finite sum **100**
of expressions $Z(\omega)$ where

$$Z(\omega) = \begin{cases} \omega(f(a)) \displaystyle\int_{K^n} \psi(y) \omega(1 + y_1) dy_1 & (98)' \\[2ex] \displaystyle\int_{K^n} \psi(y) \prod_i \omega(y_i)^{N_i} |y_i|_K^{\nu_i - 1} |dy| & (98)'' \end{cases}$$

depending on the two cases above and further ψ is a C^∞-function with support contained in the polydisc C. In $(98)''$ we may assume that $N_i \geq 1$ for $1 \le i \le n$, after reducing n if necessary.

4.3

The integral in $(98)'$ may be rewritten as $\int\limits_K \varphi(t)\omega(t)dt$ where now φ is a C^∞ function with support contained in the disc $|t-1| < 1$ and hence not containing 0. By the special case of our theory discussed in Chapter I, §§3-5, we may conclude that, in this case, $Z(\omega)$ is actually an entire function function of s and further satisfies the growth condition 96.

4.4

We shall prove that the integral in $(98)''$ represents a meromorphic function with poles of order at most n and all the poles of the form

$$-\frac{1}{N_i}\left(\nu_i + \frac{r}{2d}\right), \quad r = 0, 1, 2, \ldots : 1 \le i \le n \tag{99}$$

with $d = 1/2$ for $K = \mathbb{R}$ and 1 for $K = \mathbb{C}$ and further, that $Z(\omega)$ is bounded for s in B_{σ_1,σ_2} from where neighbourhoods of the poles have been removed. The last assertion will imply (96), in view of the familiar argument employed in Chapter I, §§ 4-5, in such a context: we could have well worked with $D\psi$ instead of ψ, where D is a suitable homothety-invariant differential operator, say $D = y_1^k \dfrac{\partial^k}{\partial y_1^k}$ or $r_1^k \dfrac{\partial^{k+\ell}}{\partial r_1^k \partial\theta^\ell}$ for $y_1 = \sqrt{r}_1 \oplus (\theta_1)$ according as $K = \mathbb{R}$or $\mathbb{C}$.

4.5

101 We first prove a simple lemma that will enable us to invoke an induction argument. By a C^∞ function $g(z_1, \ldots, z_n)$ on $\mathbb{C}^n$, we mean, as usual, a C^∞ function of the $2n$ variables $z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n$ (i.e. of Re (z_1), Im(z_1), etc.). For $K = \mathbb{R}$, we write $m_j(z)$ for z^j and for $K = \mathbb{C}$, $j = (j', j'')$ with $j', j'' \ge 0$ in $\mathbb{Z}$, we write $m_j(z)$ instead of $z^{j'}\overline{z}^{j''}$ and $|j|$ for $j' + j''$.

Lemma 4.5. *For an $\mathbb{R}$-field, let $\psi(z) = \psi(z_1, \ldots, z_n) \in C^\infty(K^n)$ and $t = (t_1, \ldots, t_n) \in \mathbb{Z}^n$ with all $t_i \ge 0$. Then there exist C^∞ functions*

$$A_{k,j}(z) = A_{k,j}(z_1, \ldots, \hat{z}_k, \ldots, z_n), B_{j_1,\ldots,j_n}(z)$$

for $|j_k| = t_k,\ (1 \le k \le n)$ *such that*

$$\psi(z) = \sum_{k=1}^{n} \sum_{|j|<t_k} A_{k,j}(z_1, \ldots, \hat{z}_k, \ldots, z_n) m_j(z_k) + \sum_{\substack{|j_k|=t_k\\ 1\le k\le n}} B_{j_1,\ldots,j_n}(z) \prod_{r=1}^{n} m_{j_r}(z_r) \tag{100}$$

Proof. If we fix $z' = (z_1, \ldots, z_{n-1})$ in K^{n-1} as a parameter and apply Taylor's theorem to $\psi(z)$, then we get

$$\psi(z) = \sum_{|j|<t_n} A_{n,j}(z_1, \ldots, z_{n-1}, \hat{z}_n) m_j(z_n) + \sum_{|j|=t_n} \psi_{1,j}(z) m_j(z_n)$$

where $A_{n,j} \in C^\infty(K^{n-1})$ and $\psi_{1,j} \in C^\infty(K^n)$. Next, we ragard $z'' = (z_1, \ldots, z_{n-2}, z_n) \in K^{n-1}$ as a parameter and apply Taylor's theorem to each $\psi_{1,j}$, obtaining

$$\psi_{1,j}(z) = \sum_{|j^*|<t_{n-1}} A^*_{n,j,j^*}(z'') m_{j^*}(z_{n-1}) + \sum_{|j^*|=t_{n-1}} \psi_{2,j,j^*}(z) m_{j^*}(z_{n-1})$$

with C^∞ functions A^*_{n,j,j^*}, ψ_{2,j,j^*}. We now denote $\sum\limits_{|j|=t_n} A^*_{n,j,j^*}(z'') m_j(z_n)$ **102**
as $A_{n-1,j^*}(z'')$. Iteration of this procedure yields the lemma. □

We take the case when $K = \mathbb{R}$ and substitute, in the integral $(98)''$, the expression given by 100 for $\psi(y)$ with $t_1 = \ldots = t_n = t$ large enough to satisfy the conditions

$$N_i\sigma_1 + \nu_i + t > 0 \ \text{ for } \ 1 \le i \le n. \tag{101}$$

Then we get

$$\begin{aligned} Z(\omega) = \sum_{k=1}^{n} \sum_{j=0}^{t-1} \int A_{k,j}(y_1, \ldots, \hat{y}_k, \ldots y_n) y_k^j \prod_{i=1}^{n} \omega(y_i)^{N_i} |y_i|^{\nu_i - 1} |dy| + \\ + \int B_{t,\ldots,t}(y) \prod_{i=1}^{n} y_i^t \omega(y_i)^{N_i} |y_i|^{\nu_i-1} |dy| \end{aligned} \tag{102}$$

the integrations being performed only over the compact support of ψ in the polydisc C. Since the integrand in the second integral in 102 is majorised by $(\sup_{y\in C}|B_{t,\ldots,t}(y)|)\prod_{i=1}^{n}|y_i|^{N_i\sigma_1+t+\nu-1}$ which is in $L^1(C)$ in view of 101, the second integral in 102 represents a holomorphic function of s bounded in B_{σ_1,σ_2}. On the other hand, an easy computation gives

$$\int_{-1}^{1}|y_k|^{N_k s+\nu_k+j-1}(\operatorname{sgn} y_k)^{N_k p+j}dy_k = \frac{1+(-1)^{N_k p+j}}{N_k s+\nu_k+j} \tag{103}$$

and using induction on n, it turns out the first integral in 102 is a meromorphic function of s with a pole of order at most n, possibly at points
103 of the form 99 with $r = 0, 1, \ldots, t$ and further bounded in B_{σ_1,σ_2} from which the neighbourhoods of the poles have been removed. Thus our assertions (95), (96) are verified for $Z(\omega)$ instead of $Z_\phi(\omega)$ and therefore for $Z_\phi(\omega)$ as well.

4.6

In the case $K = \mathbb{C}$, we use entirely similar arguments choosing $t_1 = t_2 = \ldots = t_n = t$ in Lemma 4.5 (for $K = \mathbb{C}$) large enough so that

$$N_i\sigma_1 + \nu_i + t/2 > 0 \text{ for } 1 \le i \le n.$$

Now the presence of poles of order at most n for $Z(\omega)$ at the points of the form (99) is made possible by the contribution from (102) which is a linear combination of the "principal parts", namely

$$\prod_{1\le j\le n}(N_j s + \nu_j + (k'_j + k''_j)/2)^{-1}$$

where k'_j, k''_j are ≥ 0 in $\mathbb{Z}$, such that

$$k'_j + k''_j < t, \quad k'_j - k''_j + N_j p = 0, \quad 1 \le j \le n. \tag{104}$$

Incidentally (104) implies that $|p| \le N_j|p| < t$ and therefore $Z_\phi(\omega)$ is holomorphic in the right half-plane $\sigma(\omega) \geq \sigma_1$, *for almost all p.* This remarkable fact seems to be the counterpart in the case of $\mathbb{R}$-fields of the assertion (87) in §3.1.

4.7

We have so far assumed that ϕ in $\mathscr{S}(X)$ has compact support. In the general case we compactify X by imbedding it in a projective space $X^\sharp = X \cup H_\infty$ with H_∞, a hyperplane at infinity. If $(f^\sharp)_0$ is the divisor of zeros of the extension $f^\sharp$ of f to $X^\sharp$, we use a K-resolution of $((f^\sharp)_0, H_\infty)$. We no longer have the guarantee that N_i are all positive but we have the **104**
compensating factor that $\mathscr{S}(X_K)$ can be identified with the subspace of C^∞ functions on $X^\sharp$ which are "infinitely divisible by the local equation of H_∞" (i.e. locally representable as a multiple of any given large power of $|g|_K$ where $g = 0$ is a local equation for H_∞); this makes everything work perfectly well here and we leave the verification of the details as an exercise. The proof of Theorem 1. 6 for $\mathbb{R}$-fields K is now complete.

5 Strongly Non-Degenerate Forms

We shall now study the implications of the preceding sections in a less general but perhaps the most important and typical situation to be considered.

5.1

Let k be an arbitrary field. We call a homogeneous polynomial f of degree $m \geq 2$ in $k[x_1, \ldots, x_n]$ *strongly non-degenerate* if $C_f = \{0\}$ or equivalently, if the projective hypersurface defined by $f(x) = 0$ is non-singular. If, for some extension K of k, the set of K-rational points of the critical set C_f for (homogeneous) f in $k[x_1, \ldots, x_n]$ consists only of $\{0\}$, then we call f *strongly non-degenerate over K*.

Suppose that $n > 2$ and $f(x)$ is strongly non-degenerate: then $f(x)$ is absolutely irreducible, as is not hard to see. In this situation, the quadratic transformation with centre at the origin 0 gives a k-resolution for the hypersurface defined by $f(x) = 0$. We now present some details concerning this well-known fact.

Let $x_1, \ldots, x_n$ be coordinates in an affine n-space X over k and let $w_1, \ldots, w_n$ be homogeneous coordinates in a projective $(n-1)$-space

$W = \mathbb{P}^{n-1}$. Let us consider, in the product $X \times W$, the subvariety Y defined by the equations

$$x_i w_j - x_j w_i = 0 \text{ for } 1 \leq i \leq j \leq n.$$

105 Then the quadratic transformation $h : Y \to X$ with centre 0 is just the projection $X \times W \to X$ restricted to Y. If W_1 denotes the affine $(n-1)$-space in W, defined by $w_1 \neq 0$, then $y_i = w_i/w_1$ for $1 < i \leq n$ form coordinates in W_1. Let Y_1 denote the intersection of Y with the affine $(2n-1)$-space $X \times W_1$ with coordinates $x_1, \ldots, x_n, y_2, \ldots, y_n$; then Y_1 is defined by

$$x_i - x_1 y_i = 0 \text{ for } 1 < i \leq n.$$

Therefore, if we put $y_1 = x_1$, then Y_1 becomes an affine n-space with coordinates $y_1, y_2, \ldots, y_n$ and further Y is covered by Y_1 together with $Y_2, \ldots, Y_n$ which are defined in a similar manner. In particular, Y is non-singular and if we put

$$E = h^{-1}(0) = \{0\} \times W \subset Y,$$

then h gives a k-isomorphism i.e. a biregular map of $Y \backslash E$ to $X \backslash \{0\}$, defined over k. We observe that $h^*(dx)$ is given in Y_1 by

$$h^*(dx) = dy_1 \wedge d(y_1 y_2) \wedge \ldots \wedge d(y_1 y_n) = y_1^{n-1} dy.$$

Moreover, $f \circ h$ is given in Y_1 by

$$f \circ h = y_1^m f(1, y_2, \ldots, y_n)$$

and $f(1, y_2, \ldots, y_n) = 0$ gives a local equation in Y_1, of an irreducible non-singular hypersurface in Y, say E' (this being just the "proper transform" of the hypersurface given by $f(x) = 0$). We see that E' and E are transversal to each other at every point of Y. Therefore h gives a k-resolution of the hypersurface defined by $f(x) = 0$ and is tame, if the
106 characteristic of k does not divide m. The numerical data of h along E' and E are $(1, 1)$ and (m, n) respectively.

5.2

Let us apply our theory to the case when k is a local field K, identifying X with X_K etc. , as usual. Then, for a p-field K and every ϕ in $\varphi(X)$, we have

$$Z_\phi(\omega) = \frac{P_{\phi,\chi}(z)}{(1 - z/q)(1 - z^m/q^n)}$$

with $P_{\phi,\chi}(z)$ in $\mathbb{C}[z, z^{-1}]$. Therefore, if the characteristic of k does not divide m and if $n > m$, then we get

$$F^*_\phi(i^*) = \sum_\lambda a^\sharp_\lambda(ac(i^*))|i^*|_K^{-\lambda} \tag{105}$$

for all sufficiently large $|i^*|_K$; here, λ satisfies $q^{m\lambda} = q^n$ (We steer clear of the case $n = m$, just to avoid multiple poles at $z = q$. See Ch. II, Theorem 3.1). Now (105) implies, for every i^* in K, that

$$|F^*_\phi(i^*)| \leq c \max(1, |i^*|_K)^{-n/m} \tag{106}$$

with a constant $c > 0$. (This follows just from the numerical data and involves no computation). The same estimate can be seen to be valid for the case of $\mathbb{R}$-fields as well, when ϕ has compact support; actually, even for arbitrary ϕ in $\mathscr{S}(X)$ in this case, we can draw the same conclusion, by using the same quadratic transformation as in §5.1 but extended to a projective space in which the affine space X is imbedded. All this enables us to assert that F^*_ϕ is an L^1-function on K and hence $F_\phi(0)$ **107**
exists for $n > m$ and the characteristic of K not dividing m. We shall see later that $F_\phi(0)$ is given by

$$F_\phi(0) = \int\limits_{U(0)} \phi|\theta_0|. \tag{107}$$

We have thus clarified "Condition (A)" of Weil ([52]) referred to earlier (see Ch. II), in the case of strongly non-degenerate forms over local fields. However, if we wish to examine the same relative to a global field, we need, at the present moment, to appeal to the theorem of Deligne ([9]). We shall now explain this point in some detail.

5.3

Let us start from a strongly non-degenerate form $f(x_1, \ldots, x_n)$ of degree $m(\geq 2)$ with coefficients in a global field k, assuming, as above, that $n > m$ and that m is not divisible by the characteristic of k. Let K denote a completion of k with respect to a non-archimedean valuation v on k. If we exclude a finite number of v's depending on $f(x)$, then, for every i^* in K, we have

$$|F^*(i^*)| \leq \max(1, |i^*|_K)^{-n/m} \tag{108}$$

In (108), we have written F^* for $F^*_{\varphi_0}$ corresponding to the characteristic function φ_0 of $X^0(= R^{(n)})$; further the constant c in 106) is replaced by 1 in (108). We give a sketch of the proof of (108) as follows.

Let $f(x)$ be in $R[x_1, \ldots, x_n]$, without loss of generality, and let $mR = R$; moreover, let the reduction $\overline{f}(x)$ of $f(x)$ modulo P be strongly non-degenerate. Writing i^* in $K^\times$ as $\pi^{-e}u$ with $e \geq 0$ in $\mathbb{Z}$ and u in $R^\times$, an elementary argument gives us

$$F^*(i^*) = \begin{cases} q^{n[-e/m]}, & \text{for } e \not\equiv 1 \bmod m; \\ q^{-n(1+(e-1)/m)} \sum\limits_{t \bmod P} \psi(\pi^{-1}ut)N_1(t), & \text{for } e \equiv 1 \bmod m. \end{cases} \tag{109}$$

108 For the first case in 109), we have immediately

$$|F^*(i^*)| \leq (q^e)^{-n/m} = |i^*|_K^{-n/m}.$$

But, in the second case, such an inequality holds if and only if

$$|q^{-n} \sum_{t \bmod P} \psi(\pi^{-1}ut)N_1(t)| \leq q^{-n/m} \tag{110}$$

On the left side of 110), the expression inside the sign for absolute value is merely the (generalized) Gaussian sum

$$q^{-n} \sum_{\xi \bmod P} \psi(\pi^{-1}uf(\xi))$$

and both sides of 110 are equal for $m = 2$. In the general case, the problem is to estimate the number of $\mathbb{F}_q$-rational points on $\overline{f}(x) = 0$ or on

$\overline{f}(x) = \bar{t}x_0^m$ with $\bar{t} \in \mathbb{F}_q^\times$, both the hypersurfaces being non-singular. We can indeed apply Deligne's theorem ([9]) to carry out such an estimation; for the details, we may refer the reader to our paper [21].

APPENDIX

Generalized Gaussian Sums and Singular Series (Local Case)

109 For any p-field K, we had introduced in §1.7, the functions $F = F_\phi$, $F^* = F^*_\phi$ corresponding to the characteristic function ϕ of $R^{(n)}$, $f(x) \in R[x_1, \ldots, x_n]$ and the standard character ψ on K. If

$$i^* = \pi^{-e}u \text{ with } e \geq 0 \text{ in } \mathbb{Z} \text{ and } u \in R^\times$$

we had the relation

$$F^*(i^*) = q^{-ne} \sum_{\xi \bmod P^e} \psi(i^* f(\xi))$$

where the right hand side is essentially a generalized Gaussian sum and on the basis of Theorem 1.6, we had referred to its "stable behaviour" for all sufficiently large $e = -\operatorname{ord}(i^*)$, namely, its being a fixed linear combination of expressions of the form $\chi(ac(i^*))|i^*|_K^{-\lambda}(\log|i^*|_K)^j$.

In the simplest non-trivial case where $f(x) = x^2$ and $2 \in R^\times$, we have

$$F^*(i^*) = \begin{cases} |i^*|_K^{-1/2} & \text{for even } e = -\operatorname{ord}(i^*) \geq 0 \\ q^{1/2} g_\chi \chi^{-1}(ac(i^*))|i^*|_K^{-1/2} & \text{for odd } e = -\operatorname{ord}(i^*) > 0 \end{cases} \tag{111}$$

where χ is the non-trivial character of order 2 on $R^\times$. Moreover, $(q^{1/2}g_\chi)^2 = \chi(-1) = \pm 1$ (cf. Relation [57]).

We shall give a quick sketch of the proof of 111. First $e_\chi = 1$, in view of the relation $(1+P)^{[2]} = 1 + P$ from Lemma 2.5. Now it is easy to see that

$$F^*(i^*) = \sum_{r \geq 0} q^{-r} \int_{R^\times} \psi(\pi^{-(e-2r)} ac(i^*) u^2)|du|.$$

110 But, for any continuous function φ on $R^\times$, we have

$$\int_R \varphi(x^2)|dx| = \int_{R^\times} \varphi(x)(1 + \chi(x))|dx|.$$

Taking $\varphi(u) = \psi(\pi^{-(e-2r)}ac(i^*)u)$, we now get

$$F^*(i^*) = \begin{cases} (1-q^{-1})\sum\limits_{r>e/2} q^{-r} - q^{-(e+1)/2+\chi(ac(i^*))}q^{-(e-1)/2}g_\chi & \text{for } e \text{ odd} \\ (1-q^{-1})\sum\limits_{r\geq e/2} q^{-r} & \text{for } e \text{ even} \end{cases}$$

on using 52) and this proves 111).

We now give the proof of assertion 79) which is well-known; namely, for homogeneous $f(x)$ of degree m in $R[x_1, \ldots, x_n]$ and $i \neq 0$ in R,

$$q^{-e(n-1)}N_e(i) \text{ is independent of } e, \text{ for } e \geq 1 + 2\,\mathrm{ord}(mi).$$

For any a in X, let us define $\mathscr{X} = \mathscr{X}(a)$ as $\min\limits_{1\leq i\leq n}\left\{ord\left(\dfrac{\partial f}{\partial x_i}(a)\right)\right\}$. Then $\mathscr{X}$ is a $\mathbb{Z}$-valued locally constant function on $X\backslash C_f$. For a in $X^0 = R^{(n)}$ and b in X with $b \equiv a(\mathrm{mod} P^{\mathscr{X}(a)+1})$, we have $\dfrac{\partial f}{\partial x_i}(b) \equiv \dfrac{\partial f}{\partial x_i}(a)(\mathrm{mod}\ P^{\mathscr{X}(a)+1})$ for every i and therefore $\mathscr{X}(a) = \mathscr{X}(b)$. Further, it is immediate, on using the Taylor expansion of f, that

$$a \equiv b \pmod{P^{e-\mathscr{X}(a)}}, \quad e \geq 2\mathscr{X}(a) \Rightarrow f(a) \equiv f(b) \pmod{P^e}.$$

Let us observe that, in view of the foregoing.

$$E(e, \mathscr{X}) = \{a \bmod P^e;\ f(a) \equiv 0(\mathrm{mod} P^e), \mathscr{X}(a) = \mathscr{X}\},$$
$$E'(e, \mathscr{X}) = \{a \bmod P^{e-\mathscr{X}};\ f(a) \equiv 0(\mathrm{mod} P^e), \mathscr{X}(a) = \mathscr{X}\}$$

are well-defined for $e \geq \mathscr{X}+1$ and $e \geq 2\mathscr{X}+1$ respectively. Therefore, if $e \geq 2\mathscr{X} + 1$, both are well-defined and the canonical map **111**

$$X^0/\pi^e X^0 \to X^0/\pi^{e-\mathscr{X}}X^0$$

gives rise to a surjective map from $E(e, \mathscr{X})$ to $E'(e, \mathscr{X})$ such that each fibre has the cardinality $[\pi^{e-\mathscr{X}}X^0; \pi^e X^0] = q^{n\mathscr{X}}$ independent of e. Similarly, the natural map

$$X^0/\pi^{e-\mathscr{X}+1}X^0 \to X^0/\pi^{e-\mathscr{X}}X^0$$

induces a surjective map $E'(e+1,\mathscr{X}) \to E'(e,\mathscr{X})$; actually $f(a+\pi^{e-\mathscr{X}}x) \equiv 0(\bmod P^{e+1})$ if and only if $\sum_{1\le i\le n} \pi^{-\mathscr{X}} \dfrac{\partial f}{\partial x_i}(a)x_i \equiv -\pi^{-e}f(a)(\bmod P)$ and the linear congruence has exactly q^{n-1} solutions x modulo P, since one of the elements $\pi^{-\mathscr{X}}\dfrac{\partial f}{\partial x_i}(a)$ is in $R^{\times}$. Thus, in this case, each fibre has q^{n-1} elements. As a consequence, we have, for $e \ge 2\mathscr{X}+1$,

$$\begin{aligned}\operatorname{card} E(e+1,\mathscr{X}) &= q^{n\mathscr{X}}\operatorname{card} E'(e+1,\mathscr{X})\\ &= q^{n\mathscr{X}}\operatorname{card} E'(e,\mathscr{X})\cdot q^{n-1}\\ &= \operatorname{card} E(e,\mathscr{X})q^{n-1}\end{aligned}$$

where "card" stands for the cardinality. We may therefore conclude that *for* $e \ge 2\mathscr{X}+1$,

$$q^{-e(n-1)}\operatorname{card} E(e,\mathscr{X}) \text{ is independent of } e. \tag{112}$$

We should perhaps remark that the canonical map $E(e+1,\mathscr{X}) \to E(e,\mathscr{X})$ is not always surjective.

We now start from a homogeneous $f(x)$ of degree m in $R[x_1,\ldots,x_n]$ and for a fixed i in $R\backslash\{0\}$, use $f(x)-i$ in place of the polynomial
112 $f(x)$ referred to in the preceding paragraph. Then we see that, for $e \ge \operatorname{ord}(mi)+1$, the set $[\xi \bmod P^e;\ f(\xi) \equiv i \pmod{P^e}\}$ is the disjoint union of $E(e,\mathscr{X})$ for $0 \le \mathscr{X} \le \operatorname{ord}(mi)$; in fact, for $e \ge \operatorname{ord}(i)+1$ and ξ in X^0 with $f(\xi) \equiv i(\bmod P^e)$, we have $\mathscr{X}(\xi) \le \operatorname{ord}(mi)$, in view of the relation

$$\sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i}(\xi) = mf(\xi) \equiv mi(\bmod mP^e).$$

We may now conclude from above that

$$q^{-e(n-1)}N_e(i) = \sum_{0\le\mathscr{X}\le\operatorname{ord}(mi)} q^{-e(n-1)}\operatorname{card}(E(e,\mathscr{X}))$$

is independent of $e \ge 2\operatorname{ord}(mi)+1$, by 112).

Next we proceed to prove assertion 80) namely, that the above stable quotient $q^{-e(n-1)}N_e(i)$ for $e \ge 2\operatorname{ord}(mi)+1$ which is, therefore, the same as $\lim\limits_{e\to\infty} q^{-e(n-1)}N_e(i)$, coincides with $\int\limits_{X^0\cap U(i)} |\theta_i|$. We first observe that if $\{a_{\nu,r}\}_r$ is a complete set of representatives $a_{\nu,r}$ of $E'(e,\nu)$ with $f(a_{\nu,r}) = i$, then $U(i)^0 = X^0 \cap U(i)$ is the disjoint union of $V_{\nu,r} = \{a \in U(i)^0; a \equiv a_{\nu,r}(\text{mod} P^{e-\nu})\}$ for varying r and for ν with $0 \leq \nu \leq \nu_0 = \text{ord}(mi)$.

From the implicit function theorem for analytic functions on complete fields, we have the following: namely, for a p-field K and any convergent power-series $g(x)$ in $R[[x_1,\ldots,x_n]]$ with $g(0) = 0$ and $\dfrac{\partial g}{\partial x_n}(0) \not\equiv 0(\text{mod}P)$, there exists a unique convergent power-series $\varphi(x')$ in $x' = (x_1,\ldots,x_{n-1})$ with coefficients in R such that $\varphi(0) = 0$ and further.

(i) $g(x',\varphi(x')) = 0$ for all x' sufficiently close to 0; and **113**

(ii) whenever $g(a) = 0$ for $a = (a_1,\ldots,a_n)$ close to 0, then $a_n = \varphi(a')$.

We shall, however, use a refinement of the same, which is not hard to prove: namely, if, in addition, we have $g(x) = \sum\limits_{\alpha} c_\alpha x^\alpha$ (with the usual notation) and $c_\alpha \equiv 0(\text{mod}P^{|\alpha|-1})$, then $\varphi(x')$ converges on the whole of $R^{(n-1)}$ and further satisfies

(i)′ $g(x',\varphi(x')) = 0$ for all $x' \in R^{(n-1)}$; and

(ii)′ for any $a = (a_1,\ldots,a_n) \in R^{(n)}$ with $g(a) = 0$, we have $a_n = \varphi(a')$.

Going back to the proof of 80), we shall examine closely the set

$$V = \{x \in X^0; f(x) = i, x \equiv \xi \pmod{P^{e-\nu}}\}$$

where ξ is one of the $a_{\nu,r}$ above. Putting $x = \xi + \pi^{e-\nu}y$, we see that V is defined in X^0 by the condition

$$0 = \pi^{-e}(f(x) - i) = \sum_{i=1}^{n} \pi^{-\nu}\frac{\partial f}{\partial x_i}(a)y_i + \pi^{-e}Q \tag{113}$$

where Q is a polynomial in $\pi^{e-\nu}y_1,\ldots,\pi^{e-\nu}y_n$ with coefficients in R and having no terms of degree ≤ 1. Now (113) gives an equation of the form

$$\sum_{i=1}^{n} c_i y_i + \sum_{|\alpha|\geq 2} c_\alpha y^\alpha = 0$$

where c_i, c_α are all in R and further, $c_\alpha \in P^{|\alpha|-1}$ (since $|\alpha|(e-\nu-1)-e+1 \geq e-2\nu-1 \geq 0$), while at least one of $c_1, \ldots, c_n$, say c_n, is not in P. By the above-mentioned refinement of the implicit function theorem (applied to K and $\pi^{-e}(f(x)-i)$ instead of $g(x)$), we see that V is the same as
114 $\{(y', \varphi(y')); y' \in R^{(n-1)}\}$. Now, on V,

$$\theta_i = (-1)^{n-1}\left(\frac{\partial f}{\partial x_n}\right)^{-1} dx_1 \wedge \ldots \wedge dx_{n-1}|_V$$

and therefore

$$|\theta_i| \text{ on } V = q^{-e(n-1)+n\nu}|dy'|$$

implying that

$$\int\limits_V |\theta_i| = q^{-e(n-1)+n\nu}$$

We have, finally from above,

$$\begin{aligned}
\int\limits_{U(i)^0} |\theta_i| &= \sum_{0\leq\nu\leq\nu_0}\sum_r \int\limits_{V_{\nu,r}} |\theta_i| \\
&= \sum_{0\leq\nu\leq\nu_0} q^{-e(n-1)}\,\mathrm{card}(E'(e,\nu))q^{n\nu} \\
&= \sum_{0\leq\nu\leq\nu_0} q^{-e(n-1)}\,\mathrm{card}(E(e,\nu)) \\
&= q^{-e(n-1)}N_e(i), \ \text{ for } e \geq 2\,\mathrm{ord}(mi)+1.
\end{aligned}$$

Chapter 4

Poisson Formula of Siegel-Weil Type

1 Formulation of a Poisson Formula

In this Chapter, we formulate a Poisson formula of general type which 115
corresponds to forms of higher degree in the same manner as the Siegel-Weil formula ([52]) is related to quadratic forms; the classical Poisson summation formula is again a special case of such a Poisson formula and it is well-known ([47]) that in this classical formula lies embedded a substantial part of the validity of the functional equation of Hecke L-series. Our Poisson formula, as in the case of Weil, is to be regarded as an identity between two tempered distributions on adelized vector spaces arising from global fields. We shall formulate sufficient conditions for the validity of the general Poisson formula. Moreover, we shall use the so-called "adelic language" ([50]), essentially to simplify the presentation.

1.1 Standard Notation

Let k be a *global field* i.e. an algebraic number field of finite degree over the rational number field or an algebraic function field of one variable with finite constant field. For any non-trivial absolute value $|\ |_v$ on k, denote by k_v the completion of k with respect to the metric $d(a, b) = |a-b|_v$

on k. Let us assume that $|\ |_v$ is normalized, as already described. We call v *archimedean* or *non-archimedean* according as k_v is an $\mathbb{R}$-field or a p-
116 field; archimedean valuations arise only in the case of algebraic number fields. We choose exactly one representative (and indeed normalised as well) in each "equivalence class" of valuations v. By S, we shall mean, in this article, a finite set of valuations v which always includes the archimedean valuations. For non-archimedean v, we denote the valuation ring by R_v or R and its maximal ideal by P_v or simply P.

1.2 Adelization

A subset U of an affine space X of dimension n is called *locally k-closed* (or *quasi-k-affine*) if $U = V \backslash W$ for k-closed subsets V, W in the affine space. If V is defined by the equations $f_1(x) = \ldots = f_r(x) = 0$ and W by $g_1(x) = \ldots = g_s(x) = 0$ for $f_1, \ldots, f_r, g_1, \ldots, g_s$ in $k[x_1, \ldots, x_n]$, then a belongs to U if and only if $f_1(a) = \ldots = f_r(a) = 0$ and further $g_j(a) \neq 0$ for at least one j with $1 \leq j \leq s$. If we define $U_v = U_{k_v} = \{x \in k_v^n; f_1(a) = \ldots = f_r(a) = 0$ and $g_j(a) \neq 0$ for at least one j with $1 \leq j \leq s\}$, then U_v is clearly locally compact. Further, for non-archimedean v, we see that $U_v^0 = \{a \in R^{(n)}; f_1(a) = \ldots = f_r(a) = 0$ and at least one $g_j(a)$ is a unit of R_v for $1 \leq j \leq s\}$ is compact. For any S described in §1.1 above, $\prod\limits_{v \in S} U_v$ is locally compact and $\prod\limits_{v \notin S} U_v^0$ is compact, so that

$$U_S \overset{\text{defn.}}{=} \prod_{v \notin S} U_v^0 \times \prod_{v \in S} U_v$$

is locally compact. For two such S, S' with $S \subset S'$, clearly U_S is open in $U_{S'}$.

For any locally k-closed U, we define the *adelization* U_A of U by $U_A = \underset{S}{\overset{\lim}{\longrightarrow}} U_S$ and U_S is open in U_A for every S. Then U_A is a locally
117 compact space and it is not hard to see that U_A depends only on U (although the requirement that for $a \in U_v^0$, $g_j(a)$ is a unit for at least one j, may look disturbing!). We can show that adelization is a "functor" and, in particular, for any k-morphism $j : U \to U'$ of locally k-closed subsets U, U' in affine spaces, we have a unique continuous map j_A :

$U_A \to U'_A$. The set U_k of k-rational points on U can be imbedded in U_A through the diagonal imbedding and it is, in fact, discrete in U_A.

1.3 Examples

We shall illustrate the foregoing with some examples.

(i) Let X be the affine n-space, so that $X_k \simeq k^n$. We denote X_A, for $n = 1$, by k_A, the ring of k-adeles; for any n, $X_A \simeq (k_A)^n$ and X_A/X_k is compact. If $n = 1$, $k = \mathbb{Q} \hookrightarrow \mathbb{Q}_A$ and any t in $\mathbb{Q}_A$ is of the form $(t_\infty, t_p, \ldots)$ with $t_\infty \in \mathbb{R}$ and t_p is in the ring $\mathbb{Z}_p$ of p-adic integers for almost all (non-archimedean) p. If $< t_p >$ denotes the "fractional part" of t_p, then $< t_p >= 0$ for almost all p so that $t' = t - \sum\limits_p < t_p >$ in $\mathbb{Q}_A$ has its non-archimedean components t'_p in $\mathbb{Z}_p$ for every p. It is now immediate that $\mathbb{Q}_A/\mathbb{Q} \simeq \mathbb{R}/\mathbb{Z} \times \prod\limits_{p \text{ prime}} \mathbb{Z}_p$ and hence, it is compact. The imbedding of $\mathbb{Q}$ as a discrete subgroup of $\mathbb{Q}_A$ corresponds precisely to the imbedding $\mathbb{Z} \hookrightarrow \mathbb{R}$.

(ii) For $f(x) \in k[x_1, \ldots, x_n]$ and i in k, we have introduced $U(i)$ as $f^{-1}(i)\backslash C_f$; thus, a belongs to $U(i)$ if and only if $f(a) = i$ and further, $\dfrac{\partial f}{\partial x_j}(a) \neq 0$ for at least one j. Clearly $U(i)$ is a locally k-closed subset and hence $U(i)_A$ is defined.

1.4 Tamagawa Measure

Suppose U is a non-singular locally k-closed subset of affine n-space **118**
and ω, an everywhere regular differential form of the highest degree on U vanishing nowhere and moreover, defined over k. We shall be dealing only with situations where such a form ω exists; for example if $U = X$, $\omega(x) = dx_1 \wedge \ldots \wedge dx_n$ in Example (i) above and $\omega(x) = \theta_i(x)$ for $U = U(i)$ in Example (ii).

By a procedure (which may not, however, work always), we can associate to ω, the so-called *Tamagawa measure* $|\omega|_A$ on U_A as follows. Let us start from the Borel measure $|\omega|_v$ on U_v associated with ω; in particular, on the open set U_v^0, we have a measure, say m_v. Let

us assume that the (infinite product) measure $\bigotimes_{v \notin S} |\omega|_v$ exists on $\prod_{v \notin S} U_v^0$; this, incidentally, happens to exist if and only if the infinite product $\prod_{v \notin S} m_v(U_v^0)$ is absolutely convergent. On the other hand, we always have the product measure $\bigotimes_{v \in S} |\omega|_v$ on the (finite) product $\prod_{v \in S} U_v$. Thus, we can define, for every S, the measure $|\omega|_A$ on $U_S = \prod_{v \notin S} U_v^0 \times \prod_{v \in S} U_v$ by $|\omega|_A = (\bigotimes_{v \notin S} |\omega|_v) \otimes (\bigotimes_{v \in S} |\omega|_v)$. However, we have, still, to remove the ambiguity that arises in our definition of $|\omega|_v$ above, due to our not having fixed a Haar measure on k_v^n, for $n = 1, 2, \ldots$. For this purpose, let us proceed as follows. Let us choose a non-trivial character $\psi : k_A/k \to \mathbb{C}_1^\times$. Then k_A may be identified with its dual by setting
119 $< a, b >= \psi(ab)$ for a, $b \in k_A$ and further, $k \simeq (k_A/k)^*$ via the map $c \mapsto \psi(ct)$. Let $j_v : k_v \to k_A$ be the imbedding which sends any t in k_v to the adele with t as the v-th component and 0 elsewhere; we set $\psi_v = \psi \circ j_v$. On $X_v = k_v^n$, we take as our measure, the n-fold product $|dx|_v$ of the measure on k_v which is self-dual relative to $(t, t') \mapsto \psi_v(tt')$; then, this is the same as the measure on $X_v = k_v^n$ self-dual relative to $((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto \psi_v(x_1 y_1 + \cdots + x_n y_n)$. Moreover, $\psi_v = 1$ on R_v and nontrivial on P_v^{-1}, for all but finitely many v; further, $m(X_v^0) = 1$ for such v. The measure $|dx|_A$ always exist on X_A and has the characteristic property that X_A/X_k has measure 1; on each open subgroup X_S defined above, $|dx|_A$ is just the product of the measures $|dx|_v$, in the usual sense.

1.5 The Schwartz-Bruhat Space $\mathscr{S}(X_A)$

If k is an algebraic function field of one variable with finite constant field, then X_A is a locally compact abelian group with arbitrarily large and small compact open subgroups in the sense of Chapter II, § 1.2. The Schwartz-Bruhat space $\mathscr{S}(X_A)$ is thus already familiar to us and so is the dual, $\mathscr{S}(X_A)'$ of tempered distributions.

Let, on the other hand, k be an algebraic number field and the degree $[k : \mathbb{Q}]$ of k over $\mathbb{Q}$ be finite. If S_∞ denotes the set of all archimedean valuations on k, we write, for the affine space X of dimension n,

$$X_\infty = \prod_{v \in S_\infty} X_v \simeq \mathbb{R}^{n[k:\mathbb{Q}]}$$
$$X_0 (= X_{\text{finite}}) = \varinjlim_S \prod_{v \in S \setminus S_\infty} X_v^0.$$

Let us remark that X_0 is a locally compact abelian group with arbitrarily large and small compact open subgroups and therefore the Schwartz-Bruhat space $\mathscr{S}(X_0)$ is already known to us. Moreover, we are also quite familiar with the Schwartz-Bruhat space $\mathscr{S}(X_\infty)$. We now define the *Schwartz-Bruhat space associated with* X_A by $\mathscr{S}(X_A) = \mathscr{S}(X_0) \otimes_{\mathbb{C}} \mathscr{S}(X_\infty)$; every element of this space is a finite linear combination of $\phi_0 \otimes \phi_\infty$ with $\phi_0 \in \mathscr{S}(X_0)$ and $\phi_\infty \in \mathscr{S}(X_\infty)$. We define the "topological dual" $\mathscr{S}(X_A)'$ of $\mathscr{S}(X_A)$ as the space of all $\mathbb{C}$-linear functionals on $\mathscr{S}(X_A)$ such that $T(\phi_0 \otimes \phi_\infty)$ depends continuously on ϕ_∞ in $\mathscr{S}(X_\infty)$ for every fixed ϕ_0 in $\mathscr{S}(X_0)$. Let $\{T_n\}_n$ be a sequence contained in $\mathscr{S}(X_A)'$ such that, for every $\phi \in \mathscr{S}(X_A)$, $\lim_n T_n(\phi)$ exists; denoting this limit by $T(\phi)$, we see trivially that T is $\mathbb{C}$-linear. Moreover, for every fixed ϕ_0 in $\mathscr{S}(X_0)$ and any $\phi_\infty \in \mathscr{S}(X_\infty)$, we have $T(\phi_0 \otimes \phi_\infty) = \lim_{n\to\infty} T_n(\phi_0 \otimes \phi_\infty)$, by the definition of T. Now since $\phi_\infty \mapsto T_n(\phi_0 \otimes \phi_\infty)$ belongs to $\mathscr{S}(X_\infty)'$ for every fixed $\phi_0 \in \mathscr{S}(X_0)$, the well-known completeness of $\mathscr{S}(X_\infty)'$ entails that $\phi_\infty \mapsto T(\phi_0 \otimes \phi_\infty)$ belongs to $\mathscr{S}(X_\infty)'$. Thus $\mathscr{S}(X_A)'$ is complete. It is called the *space of tempered distributions* on X_A, as usual. **120**

1.6 Poisson Formula

We are now in a position to formulate the general Poisson formula associated with $f(x) \in k[x_1, \ldots, x_n]$ and with $\mathscr{S}(X_A)$. We first make the simple remark that for every $i^* \in k$, $\psi(i^* f(x)) \in \mathscr{S}(X_A)'$, if we simply define $\psi(i^* f(x))(\phi) = \int_{X_A} \phi(x)\psi(i^* f(x))|dx|_A$, for every ϕ in $\mathscr{S}(X_A)$ and note that the integral converges absolutely.

Let us formulate a series of assumptions. **121**

(PF-1) The infinite sum $\sum_{i^* \in k} \psi(i^* f(x))$ belongs to $\mathscr{S}(X_A)'$, i.e. (equi valently) *the Eisenstein-Siegel* series $\sum\limits_{i^*\in k}\int\limits_{X_A}\phi(x)\psi(i^* f(x))|dx|_A$ associated with $f(x)$ converges absolutely for every ϕ in $\mathscr{S}(X_A)$.

(PF-2)′ For every i in k, the measure $|\theta_i|_A$ exists on $U(i)_A$.

(PF-2)″ If $j : U(i)_A \hookrightarrow X_A$ is induced by $U(i) \hookrightarrow X$, then for every i in k, the *global singular series* $j_*(|\theta_i|_A)$ or simply $|\theta_i|_A$ *associated with* $f(x)$ *and* i exists in $\mathscr{S}(X_A)'$ i.e. (equivalently) $\int\limits_{U(i)_A}\phi|\theta_i|_A$ is absolutely convergent for every ϕ in $\mathscr{S}(X_A)$.

(PF-3) The infinite sum $\sum\limits_{i\in k}|\theta_i|_A$ belongs to $\mathscr{S}(X_A)'$.

We say that the *Poisson formula holds for* $f(x)$ (relative to k) if *all the assumptions* (PF-1), (PF-2)′, (PF-2)″, (PF-3) *are valid* and *further*

(PF-4)
$$\sum_{i\in k}|\theta_i|_A = \sum_{i^*\in k}\psi(i^* f(x)) \tag{114}$$

The simplest example of a Poisson formula is furnished by considering the case when $n = 1$, $f(x) = x$ so that $\phi^*(x) = \int\limits_{k_A}\phi(y)\psi(xy)|dy|_A$. Then 114 reads
$$\sum_{i\in k}\delta(x-i) = \sum_{i^*\in k}\psi(i^* x)$$
since $|\theta_i|_A$ is just the Dirac measure supported at i. But the relation above is the same as saying
$$\sum_{i\in k}\phi(i) = \sum_{i^*\in k}\phi^*(i^*) \tag{115}$$
for every $\phi \in \mathscr{S}(k_A)$. As remarked already, there remains, built into the simple-looking classical Poisson formula (115), quite a substantial part
122 of the proof of the functional equation of Hecke's L-series rephrased in adelic language.

2 Criteria for the Validity of the Poisson Formula and Applications

2.1

Sufficient conditions for the validity of the Poisson formula stated in the last section are provided by the following theorem which is the second substantial theorem in our theory for forms of higher degree.

Theorem 2.1. *Let $f(x)$ denote a homogeneous polynomial of degree $m \geq 2$ in n variables $x_1, \ldots, x_n$ with coefficients in a global field k of characteristic not dividing m and let us assume that a tame k-resolution of the projective hypersurface defined by $f(x) = 0$ exists. Then the Poisson formula holds for $f(x)$ relative to k if the following two conditions are satisfied:*

(C1) *the codimension of C_f in $f^{-1}(0) \geq 2$ (i.e. equivalently, codim $(C_f) \geq 3$);*

(C2) *there exist $\sigma > 2$ and a finite set S of valuations v of k, such that, for every i^* in $k_v \backslash R_v$ and $v \notin S$,*

$$|F_v^*(i^*)| \leq |i^*|_v^{-\sigma}.$$

where $F_v^*(i^*) = \int_{X_v^0} \psi_v(i^* f(x))|dx|_v$.

The proof of this theorem will be given later. We merely remark that there exists a conjecture to replace condition (C2) by a geometric condition; this will be explained later. Condition (C1) is easy to verify and simply means that the hypersurface defined by $f(x) = 0$ is irreducible and normal.

2.2 Applications

We enumerate a series of applications of Theorem (2.1), assuming that k has characteristic 0, just for the sake of simplicity.

A.1) Let $f(x)$ be strongly non-degenerate i.e. let $f(x)$ be homogeneous with the critical set $C_f = \{0\}$. Then (C_1) is equivalent to the condition $n \geq 3$, since $\operatorname{codim}_{f^{-1}(0)}(C_f) = n-1$. Further, from 108), we know that, for all but finitely many v and for every $i^* \in k_v \backslash R_v$. 123

$$|F_v^*(i^*)| \leq |i^*|_v^{-n/m} \tag{108$'$}$$

as a consequence of Deligne's theorem ([9]); for $\operatorname{ord}(i^*)$ divisible by m, actually equality holds in (108)$'$. Thus (C2) is equivalent to $n > 2m$ and the Poisson formula holds for strongly non-degenerate $f(x)$ if $n > 2m$. In any case, the existence of a tame resolution is obvious here and even, in the case of function fields k, the Poisson formula holds, provided that m is not divisible by the characteristic of k.

A.1)$'$ If $f(x)$ is a non-degenerate quadratic form, we are just in the situation A.1) above with $m = 2$. Thus, the Poisson formula holds for $n > 4$. Incidentally, condition C2) is easy to verify, since

$$\begin{aligned} |F_v^*(i^*)|^2 &= \int\limits_{X_v^0 \times X_v^0} \psi_v(i^*(f(x) - f(y))|dx \wedge dy|_v \\ &= \left(\int\limits_{R_v^{(2)}} \psi_v(i^* x_1 y_1)|dx_1 \wedge dy_1| \right)^n \\ &= |i^*|_v^{-n} \end{aligned}$$

for every i^* in $k_v \backslash R_v$ and for all but finitely many v; we may replace $f(x)$ by $\sum\limits_i u_i x_i^2$ with $u_i \in R_v^\times$, in the computation above.

124 A.2) If $f(x)$ is just a homogeneous polynomial of degree m in n variables with coefficient in k, then by applying the techniques of Birch and Davenport for forms in many variables ([4]), it can be shown that, for any $\epsilon > 0$,

$$|F_v^*(i^*)| \leq |i^*|_v^{-(\operatorname{codim}(C_f)/2^{m-1}(m-1)[k:\mathbb{Q}])+\epsilon}$$

for every i^* in $k_v \backslash R_v$ and all but finitely many v. The condition (C1) is already fulfilled if we require that

$$\operatorname{codim}(C_f) > 2^m(m-1)[k:\mathbb{Q}](\geq 2)$$

and consequently the Poisson formula holds, subject to this requirement being satisfied. For further details, we refer to our paper [26].

2.3 Further Applications of the Poisson Formula

A.3) Let G be a "k-form" of SL_{2m} (i.e. an algebraic group defined over k and isomorphic to SL_{2m} over an algebraic closure $\overline{k}$ of k) and let $\rho : G \to GL(X)$ be a "k-form" of the $m(2m-1)$ - dimensional fundamental representation of G, where X is a certain affine space. If G is k-isomorphic to SL_{2m}, then we may identify X with the vector space of $2m$-rowed skew-symmetric matrices x on which the action of G via ρ is described by $x \to \rho(g)\cdot x = gx^t g$ for g in G. Upto a factor from $K^\times$, there exists a unique invariant of degree m, with coefficients in k; this invariant is precisely the *Pfaffian* $\mathrm{Pf}(x)$ with its well-known property that $\mathrm{Pf}(gx^t g) = \det(g)\,\mathrm{Pf}(x)$ for any g in GL_{2m}. In this case, it is known that $F_v^*(i^*) = \sum_{i=1}^{m-1} c_i |i^*|_v^{-(2i+1)}$, where $c_i = \sum_{\substack{1\leq j\leq m \\ j\neq i}} (1-q^{-(2j+1)})/(1-q^{-2(j-i)})$. In the general case, 125
there exists a unique invariant $f(x)$ which, for all but finitely many v, coincides with $\mathrm{Pf}(x)$ after a non-singular linear transformation with coefficients in R_v. As a result, the above formula for $F_v^*(i^*)$ is valid even in the general case and consequently, we have

$$|F_v^*(i^*)| \leq |i^*|_v^{-3+\epsilon}$$

for every i^* in $k_v \backslash R_v$ and for all but finitely many v. Further

$$\operatorname{codim}_{f^{-1}(0)}(C_f) = 5$$

and thus the Poisson formula is valid in this case. Complete details may be found in our paper [19].

A.4) Let G be a k-form of the connected, simply connected, simple group of type E_6 and let $\rho : G \to GL(X)$ be a k-form of the 27-dimensional fundamental representation of G. Then there exists a cubic invariant $f(x)$ known as "det" (in the theory of Jordan algebras) and unique upto a factor from k^*, as before ([28]). In this case, it can be shown that

$$\operatorname{codim}_{f^{-1}(0)}(C_f) = 9 \quad \text{and} \quad |F_v^*(i^*)| \leq |i^*|_v^{-5+\epsilon}$$

and hence the Poisson formula holds. The inequality above is a consequence of the more precise assertion:

$$F_v^*(i^*) = \frac{1-q^{-9}}{1-q^{-4}}|i^*|_v^{-5} - q^{-4}\frac{1-q^{-5}}{1-q^{-4}}|i^*|_v^{-9} \tag{116}$$

126 Formula 116) may be found in [36], § 14.

A.5) Consider the complete polarization $\det(a, b, c)$ of the cubic form "det" in example A.4) above; the form $\det(a, b, c)$ is symmetric trilinear in a, b, c and $\det(a, a, a) = 3! \det(a)$. Taking a non-degenerate symmetric bilinear form $Q(a, b)$ with coefficients in k, on the same space, define $a \times b$, $a^\sharp$ by

$$Q(a \times b, c) = \det(a, b, c), \quad a^\sharp = (1/2)(a \times a).$$

Then, for a suitable choice of $Q(a, b)$, we have

$$(a^\sharp)^\sharp = \det(a) \cdot a.$$

Let X be the $(2(27 + 1) =)56$ - dimensional space consisting of $x = (a, b, \alpha, \beta)$ where a, b are elements of the 27-dimensional space mentioned in A.4) and α, β are in $\mathbb{C}$. After Freudenthal, we take the quartic

$$f(x) = Q(a^\sharp, b^\sharp) + \alpha \det(b) + \beta \det(a) - (1/4)(Q(a, b) - \alpha\beta)^2.$$

The group of automorphisms of $f(x)$ has two connected components and the connected component G of the identity is a k-form of the connected, simply connected simple group of type E_7 and corresponds to its "first fundamental representation". Conversely, for a k-form of type E_7 with its 56-dimensional fundamental representation defined over k, there exists only one invariant, namely $f(x)$, upto a factor from $K^\times$.

For the quartic $f(x)$ above, we have

$$\operatorname{codim}_{f^{-1}(0)}(C_f) = 10 \quad \text{and} \quad |F_v^*(i^*)| \leq |i^*|_v^{-5-1/2+\epsilon}$$

and therefore, the Poisson formula holds in this case. The estimate for $F_v^*(i^*)$ follows from the more precise relation: 127

$$\begin{aligned} F_v^*(i^*) = {} & \frac{(1-q^{-14})(1-q^{-18})}{(1-q^{-4})(1-q^{-17})}\gamma(i^*)|i^*|_v^{-5-1/2} \\ & - q^{-4}\frac{(1-q^{-10})(1-q^{-14})}{(1-q^{-4})(1-q^{-9})}\gamma(i^*)|i^*|_v^{-9-1/2} \\ & + (q^{-14}+c(i^*))|i^*|_v^{-14}, \end{aligned}$$

where

$$\gamma(i^*) = \begin{cases} 1, & \text{for } \operatorname{ord}(i^*) \text{ even} \\ \chi(-\pi^{-\operatorname{ord}(i^*)}i^*)q^{1/2}g_\chi, & \text{for } \operatorname{ord}(i^*) \text{ odd} \end{cases}$$

with $\chi \neq 1, \chi^2 = 1$ and

$$c(i^*) = \begin{cases} q^{-13}\dfrac{(1-q^{-1})(1+q^{-13})(1-q^{-14})}{(1-q^{-9})(1-q^{-17})}, & \text{for } \operatorname{ord}(i^*) \text{ even} \\ q^{-17-1/2}\dfrac{(1-q^{-1})(1+q^{-4})(1-q^{-14})}{(1-q^{-9})(1-q^{-17})}\gamma(i^*), & \text{for } \operatorname{ord}(i^*) \text{ odd.} \end{cases}$$

For further details, one may refer to [27].

3 The Siegel Formula

In this section, we shall explain the significance of the Poisson formula in the simplest case and then derive the Siegel formula due to Weil, as a consequence of the Poisson formula for quadratic forms.

REMARK. Except for the case A.1) and hence also A.1)′, we have assumed that k has characteristic 0. But actually, G.R. Kempf has explicitly constructed, in the fall of 1974, k-resolutions for the hypersurface defined by $f(x) = 0$, in the cases A.3), A.4) and A.5), for an *arbitrary* field k. Since conditions C1), C2) are valid in the case of function fields as well, we see, by incorporating the results of Kempf, that the Poisson formula holds, if the characteristic of k does not divide $m!$ in the case A.3) and if k has characteristic different from 2, 3 in the cases A.4) and A.5). However, we might mention that, without any restriction on the characteristic of k, the Poisson formula is valid in the case A.3). **128**

3.1 Statement of the Siegel Formula

Let k be a global field of characteristic (different from 2), X the affine n-space and let $f(x)$ be a non-degenerate quadratic form in $n \geq 3$ variables $x_1, \ldots, x_n$ with coefficients in k. Let G be the special orthogonal group $\mathrm{SO}(f)$ of f, consisting of all linear transformations on $x_1, \ldots, x_n$ which leave $f(x)$ invariant and have determinant 1. Let dg be an invariant differential form on G of (maximal) degree $\frac{1}{2}n(n-1)$, defined over k. Then, for all but finitely many valuations v of k, we have

$$m(G_v^0) = \int\limits_{G_v^0} |dg|_v = \begin{cases} \prod\limits_{1 \leq i \leq (n-1)/2} (1 - q^{-2i}), & \text{for } n \text{ odd} \\ \prod\limits_{1 \leq i < n/2} (1 - q^{-2i}) \cdot (1 - \chi(d) q^{-n/2}), & \text{for } n \text{ even} \end{cases}$$

where $\chi^2 = 1$, $\chi \neq 1$ and the *discriminant* d of $f(x)$ is defined to be $(-1)^{n(n-1)/2} \det(t_{ij})$, if $f(x, y) = f(x + y) - f(x) - f(y)$ is written as
129 $\sum\limits_{i,j=1}^{n} t_{ij} x_i x_j$ with $t_{ij} = t_{ji}$. In particular, (for $n \geq 3$) we see, from above, that $\prod\limits_{v \notin S} m(G_v^0)$ is absolutely convergent, where S is the finite set of exceptional v mentioned above; thus the (adelic) measure $|dg|_A = \bigotimes'_v |dg|_v$ exists. The *Siegel formula (for the special orthogonal group* G) *due to Weil* ([52]) states that

$$(1/2) \int\limits_{G_A/G_k} \left(\sum_{\xi \in X_k} \phi(g\xi) \right) |dg|_A = \phi(0) + \sum_{i^* \in k} \int\limits_{X_A} \phi(x) \psi(i^* f(x)) |dx|_A \quad (117)$$

for every ϕ in $\mathscr{S}(X_A)$ and $n > 4$ (with $\psi \in (k_A/k)^*$ chosen already).

3.2

In the course of deriving the Siegel formula, we shall freely use the following theorems.

(i) *Minkowski-Hasse theorem:* $U(i)_A \neq \phi \Rightarrow U(i)_k \neq \phi$ *for every* $i \in k$ (for every $n \geq 1$).

(ii) *Witt's theorem:* $\xi_i \in U(i)_k (\neq \phi) \Rightarrow G_k \cdot \xi_i = U(i)_k$ *for every* $i \in k$ *and further* $G_A \cdot \xi_i = U(i)_A$ (if $n \geq 3$).

(iii) *For* $n = 3, 4$, $\tau_k(G) \overset{\text{def}}{=} \int\limits_{G_A/G_k} |dg|_A = 2.$

(iv) *For the twisted Fourier transformation* $\hat{}$ *on* X_A *defined by* $\hat{\phi}(x) = \int\limits_{X_A} \phi(y)\psi(f(x,y))|dy|_A$, *we have*

$$\psi(i^* f(x))\hat{} = \psi(-(i^*)^{-1} f(x))$$

for $n \geq 1$ *and every* $i^* \in k^\times$, *where* $f(x,y) = f(x+y) - f(x) - f(y)$.

Among the theorems mentioned above, (i) and (ii) are well-known. For the proof of (iii), we refer to Theorem 3.7.1 in the lectures [50] of Weil. For the proof of (iv), we refer to Théorème 2 and Théorème 5 in the paper [51] of Weil and Theorem 3 in our paper on "Harmonic **130** analysis and theta functions", Acta Math. 120(1968), 187-222.

3.3 Tamagawa Number

We recall that, on any algebraic group G defined over an arbitrary field k, there exists a left G-invariant differential form $dg \neq 0$, which is of degree equal to the dimension of G and further, defined over k; we shall simply call such a form a *gauge form on G*. A gauge form is automatically everywhere regular and non-vanishing on G. More generally, suppose that U is a non-singular algebraic variety defined over k with a k-rational point ξ, such that G acts transitively on U through $(g, x) \mapsto g \cdot x$ and further, the action of G is defined over k; then the stabiliser H at ξ is an algebraic subgroup of G defined over k. In such a case, we simply write $G/H = U$ and call U an algebraic homogeneous space defined over k. A G-invariant differential form $\theta \neq 0$ on U which is of degree equal to the dimension of U and defined over k, is called a *gauge form on U*; such a form does not exist, in general. But if it exists, then any other gauge form on U differs from θ only by a scalar factor from $k^\times$. We observe that if G is the special orthogonal group $\mathrm{SO}(f)$ of a quadratic form $f(x)$ over k, then θ_i is a gauge form on $U(i)$ for every i in k.

Let now k be a global field and let k_A, $k_A^\times$ be its adele and idele groups respectively, as before. For any idele $t = (\ldots, t_v, \ldots) \in k_A^\times$, define its modulus $|t|_A$ as the rate of measure change in k_A under the multiplication by t; then $|t|_A = \prod_v |t_v|_v$. For any c in $k^\times \hookrightarrow k_A^\times$, multiplication by c gives an automorphism of the compact group k_A/k; hence, for $c \in k^\times$, we have $|c|_A = 1$. This fact (well-known as the product formula for global
131 fields) has a far-reaching implication: namely, if a gauge form θ exists on $U = G/H$ and if, further, the Tamagawa measure $|\theta|_A$ also exists on U_A, then, for any other gauge form ω on U, $|\omega|_A$ (exists and) is the same as $|\theta|_A$, since $\omega = c\theta$ for $c \in k^\times$ and $|\omega|_A = |c\theta|_A = |c|_A|\theta|_A = |\theta|_A$. Thus $|\theta|_A$ depends only on U and k. Furthermore, if dg, dh are the gauge forms on G, H and if $G_A \cdot \xi = U_A$, then we will get $G_A/H_A = U_A$ and $|dg|_A$, $|dh|_A$, $|\theta|_A$ automatically "match together" in the sense that

$$\int_{G_A} \varphi(g)|dg|_A = \int_{U_A} \left(\int_{H_A} \varphi(gh)|dh|_A \right) |\theta(g \cdot \xi)|_A \tag{118}$$

for any continuous function φ on G_A with compact support. These are the excellent properties of the Tamagawa measures.

On the other hand, for any γ in G_k, we have $d(g\gamma) = c \cdot dg$ with $c \in k^\times$ and hence $|dg|_A$ is right G_k-invariant. Therefore, an intrinsic measure $\tau_k(G)$ can be introduced as

$$\tau_k(G) = m(G_A/G_k) = \int_{G_A/G_k} |dg|_A$$

and $\tau_k(G)$ is called the *Tamagawa number* of G relative to k (usually under the assumption that $\tau_k(G) < \infty$). Theorem (iii) in § 3.2 (that we have assumed) merely states that, for $n = 3$ or 4, we have $\tau_k(\mathrm{SO}(f)) = 2$. Furthermore, if $\mathbb{G}_a$ denotes the affine line considered as an additive algebraic group (defined over the prime field), then we have $\tau_k(\mathbb{G}_a) = m(k_A/k) = 1$. In order to proceed further, we shall prove the following elementary lemma.

Lemma 3.2. *Suppose that a locally compact group G splits as a semi-direct product of H by G' : $G = G' \cdot H$. Let dg, dg', dh denote left-invariant measures on G, G', H respectively such that $dg = dg' \otimes dh$.* **132**
Let Γ, Γ', Δ denote discrete subgroups of G, G', H respectively such that $\Gamma = \Gamma' \cdot \Delta$. Assume that dg, dg', dh are also right-invariant under Γ, Γ', Δ respectively and further that, for every γ in Γ', $d(\gamma h \gamma^{-1}) = dh$. Then we have

$$\int\limits_{G/\Gamma} \varphi(g) dg = \int\limits_{G'/\Gamma'} \left(\int\limits_{H/\Delta} \varphi(g'h) dh \right) dg' \tag{119}$$

for every continuous function φ on G/Γ with compact support.

Proof. We can find a continuous function φ_0 on G with compact support such that

$$\varphi(g) = \sum_{\gamma \in \Gamma} \varphi_0(g\gamma)$$

for every g in G. Similarly, if we define $\varphi_1(g)$ as $\sum\limits_{\delta \in \Delta} \varphi_0(g\delta)$ then we get

$$\begin{aligned} \int\limits_{G/\Gamma} \varphi(g) dg = \int\limits_{G} \varphi_0(g) dg &= \int\limits_{G'} \left(\int\limits_{H} \varphi_0(gh) dh \right) dg' \\ &= \int\limits_{G'/\Gamma'} \left(\sum_{\gamma' \in \Gamma'} \int\limits_{H/\Delta} \varphi_1(g'\gamma' h) dh \right) dg'. \end{aligned}$$

If we apply to the inner integral the measure-preserving automorphism $h \to \gamma'^{-1} h \gamma'$ of H/Δ, then (119) is immediate. $\square$

We now go back to our previous notation and denote by G, G', H algebraic groups defined over a global field such that G is the semi-direct product of H by G'. Let dg, dg', dh denote gauge forms on G, G', H. Then we can apply Lemma 3.2 to G_A, G'_A, H_A, G_k, G'_k, H_k, $|dg|_A$, $|dg'|_A$, $|dh|_A$ instead of G, G', H, Γ, Γ', Δ, Δ', Δ, dg, dg', dh respectively. In fact, all the conditions in the lemma are satisfied. Since we can replace φ in (119) by any non-negative measurable function on G/Γ, we get, on putting $\varphi = 1$, the formula

$$\tau_k(G) = \tau_k(G')\tau_k(H) \tag{120}$$

133 Since $\tau_k(\mathbb{G}_a) = 1$, formula (120) implies that $\tau_k(G) = \tau_k(G')$, whenever H is k-isomorphic to a direct product of $\mathbb{G}_a$.

3.4 Proof of the Siegel Formula

We proceed to prove the Siegel formula given by 117.

Applying induction on n, let us assume that the Tamagawa number of the special orthogonal group of any non-degenerate quadratic form over k in n' variables with $3 \leq n' < n$ is 2; in spite of the fact that $n > 4$ (as stated after formula (117)), our induction hypothesis is quite legitimate by virtue of Theorem (iii) assumed in § 3.2.

With the same notation as in Theorem (ii) assumed in § 3.2, we have clearly

$$X_k \backslash \{0\} = \coprod_{i \in k} U(i)_k = \coprod_{i \in k}{}' G_k \cdot \xi_i \tag{121}$$

where the accent (over the symbol for disjoint union) indicates that i runs only over the subset of k which ensures $U(i)_k \neq \phi$. For any such i, we set

$$H_i = \text{ the stabiliser of } G \text{ at } \xi_i.$$

We then assert that $\tau_k(H_i) = 2$ for every such i. The proof runs as follows. If $i \neq 0$, let $X' = \{x \in X; f(\xi_i, x) = 0\}$ and f' the restriction of f to X'. Then we have $\mathrm{SO}(f') = H_i$ and since X' has dimension $n - 1 \geq 4$, we see that $\tau_k(H_i) = 2$, by the induction hypothesis. Let now $i = 0$. Then choosing η from X_k with $f(\xi_0, \eta) = 1$ and putting $\eta_0 = -f(\eta)\xi_0 + \eta$, we obtain $\eta_0 \in X_k$, $f(\eta_0) = 0$ and $f(\xi_0, \eta_0) = 1$. Let us

now take X' to be the subspace of X defined by $f(\xi_0, x) = f(\eta_0, x) = 0$ and f' to be the restriction of f to X'. Then, as in the previous case, f' is non-degenerate and H_0 can be verified to be the semidirect product by 134
$\mathrm{SO}(f')$ of the group H of matrices of the form

$$\begin{pmatrix} 1 & -f'(c) & -{}^t cT' \\ 0 & 1 & 0 \\ 0 & c & 1_{n-2} \end{pmatrix}$$

where c is an arbitrary $(n-2)$-rowed column, $f'(x') = 1/2\, {}^t x' T' x'$ for $x' \in X'$ and 1_{n-2} is the $(n-2)$-rowed identity matrix. Since H is isomorphic to the $(n-2)$-fold product of $\mathbb{G}_a$ and since $n-2 \geq 3$, it follows by applying (120), that $\tau_k(H_0) = \tau_k(\mathrm{SO}(f')) = 2$, in view of the induction hypothesis once again.

For $\xi_i \in U(i)_k$, the map $g \mapsto \xi_i$ of G to $U(i)$ induces the continuous injection $G_A/(H_i)_A \to U(i)_A$ and Witt's theorem assumed in § 3.2 guarantees the surjectivity of the latter map. Thus we have a homeomorphism

$$G_A/(H_i)_A \simeq U(i)_A$$

(in view of a well-known theorem in the theory of topological groups).

We are now ready to start the proof of the Siegel formula. The left hand side of (117) is the same as

$$\tau_k(G)\phi(0) + \int\limits_{G_A/G_k} \left(\sum_{\xi \in X_k \setminus \{0\}} \phi(g\xi) \right) |dg|_A,$$

on pulling out the term corresponding to $\xi = 0$. By (121), we have

$$\begin{aligned} \sum_{\xi \in X_k \setminus \{0\}} \phi(g\xi) = \sum_{i \in k} \sum_{\xi \in U(i)_k} \phi(g\xi) &= \sum_{i \in k}{}' \sum_{\xi \in G_k \cdot \xi_i} \phi(g\xi) \\ &= \sum_{i \in k}{}' \sum_{\gamma \in G_k \bmod (H_i)_k} \phi(g\gamma\xi_i). \end{aligned}$$

Therefore, integrating over G_A/G_k, we get

$$\int\limits_{G_A/G_k} \left(\sum_{\xi \in X_k \setminus \{0\}} \phi(g\xi) \right) |dg|_A = \sum_{i \in k}{}' \int\limits_{G_A/(H_i)_k} \phi(g\xi_i) |dg|_A$$

$$= \sum_{i \in k}{}' \tau_k(H_i) \int\limits_{U(i)_A} \phi |\theta_i|_A; \tag{122}$$

135 we have used here the fact that integrating over $G_A/(H_i)_k$ with respect to $|dg|_A$ is the same as integrating over $(H_i)_A/(H_i)_k$ relative to $|dh|_A$ coming from a gauge form dh on H_i and then over $G_A/(H_i)_A = U(i)_A$ relative to $|\theta_i|_A$. We recall now that $\tau_k(H_i) = 2$ for every i in k such that $U(i)_k \neq \phi$.

In view of the Minkowski-Hasse theorem, $U(i)_k = \phi$ implies $U(i)_A = \phi$ and hence the integral over $U(i)_A$ is 0; hence we may legitimately remove the accent over the symbol of summation in (122). In this way, we get, for $n > 4$ and ϕ in $\mathscr{S}(X_A)$, that

$$\begin{aligned}\int\limits_{G_A/G_k} \left(\sum_{\xi \in X_k} \phi(g\xi)\right) |dg|_A &= \tau_k(G)\phi(0) + 2\sum_{i \in k} \int\limits_{U(i)_A} \phi |\theta_i|_A \\ &= \tau_k(G)\phi(0) + 2\sum_{i^* \in k} \int\limits_{X_A} \phi(x)\psi(i^* f(x))|dx|_A.\end{aligned} \tag{123}$$

The second step in (123) is a consequence of the Poisson formula 114. On the other hand, for $\Psi(x) = \phi(gx)$, we have

$$\begin{aligned}\hat{\Psi}(x) &= \int\limits_{X_A} \phi(gy)\psi(f(x,y))|dy|_A \\ &= \int\limits_{X_A} \phi(gy)\psi(f(gx,gy))|dy|_A \quad (y \to g^{-1}y) \\ &= \int\limits_{X_A} \phi(y)\psi(f(gx,y))|dy|_A \\ &= \hat{\phi}(gx).\end{aligned}$$

136 The usual Poisson formula implies that

$$\sum_{\xi \in X_k} \phi(g\xi) = \sum_{\xi \in X_k} \Psi(\xi) = \sum_{\xi \in X_k} \hat{\Psi}(\xi) = \sum_{\xi \in X_k} \hat{\phi}(g\xi). \tag{124}$$

Therefore, the left hand side of (123) is invariant if ϕ is replaces by $\hat{\phi}$. By Theorem (iv) assumed in § 3.2, we have

$$\int\limits_{X_A} \hat{\phi}(x)\psi(i^* f(x))|dx|_A = \int\limits_{X_A} \phi(x)\psi(-i^{*^{-1}} f(x))|dx|_A \tag{125}$$

for every i^* in $k^\times$. The invariance under $\phi \to \hat{\phi}$ of the right hand side of (123) implies, in view of (125), the invariance under $\phi \to \hat{\phi}$ of

$$\tau_k(G)\phi(0) + 2\int\limits_{X_A} \phi(x)|dx|_A = \tau_k(G)\phi(0) + 2\hat{\phi}(0).$$

Hence $\tau_k(G)$ is necessarily equal to 2 and the Siegel formula is proved.

REMARK. In the course of proving the Siegel formula, we have tacitly used the fact that $\tau_k(G) < \infty$. However, following H. Ariturk ([1]), the finiteness of $m(G_A/G_k)$ can be established inductively as follows. In view of Theorem (iii) assumed in § 3.2, the induction hypothesis entails that the Tamagawa number of the special orthogonal group of a non-degenerate quadratic form over k in n' variables with $3 \leq n' < n$ is not only finite but, in fact, equal to 2. Now we can certainly find $\phi \neq 0$ in $\mathscr{S}(X_A)$ such that $\phi \geq 0$ and $\phi(0) = 0$; it is clear then that $\hat{\phi}(0) > 0$. Moreover, as is well-known, we can also choose φ in $\mathscr{S}(X_A)$ such that $\varphi \geq 0$ and, further, $|\hat{\phi}(x)| \leq \varphi(x)$ for every x in X_A. By going through the previous argument for such ϕ, we have **137**

$$\begin{aligned}\int\limits_{G_A/G_k} \left(\sum_{\xi\in X_k} \phi(g\xi)\right)|dg|_A &= \int\limits_{G_A/G_k} \left(\sum_{\xi\in X_k\setminus\{0\}} \phi(g\xi)\right)|dg|_A \\ &= \sum_{i\in k} \tau_k(H_i) \int\limits_{U(i)_A} \phi|\theta_i|_A \\ &= 2\sum_{i\in k} \int\limits_{U(i)_A} \phi|\theta_i|_A \quad \text{(by induction)} \\ &< \infty,\end{aligned}$$

since the Poisson formula holds for $f(x)$ and hence (PF-3) is valid.
On the other hand,

$$\begin{aligned}\int\limits_{G_A/G_k}\left|\sum_{\xi\in X_k\setminus\{0\}}\hat{\phi}(g\xi)\right||dg|_A &\leq \int\limits_{G_A/G_k}\left(\sum_{\xi\in X_k\setminus\{0\}}\varphi(g\xi)\right)|dg|_A\\ &= 2\sum_{i\in k}\int\limits_{U(i)_A}\varphi|\theta_i|_A\\ &< \infty,\end{aligned}$$

in a similar manner. In other words, $\sum\limits_{\xi\in X_k}\phi(g\xi)$ and $\sum\limits_{\xi\in X_k\setminus\{0\}}\hat{\phi}(g\xi)$ for the chosen ϕ are both in $L^1(G_A/G_k)$ and hence the same is true of their difference as well. Thus the constant function $\hat{\phi}(0)>0$ is in $L^1(G_A/G_k)$ which implies that $m(G_A/G_k)<\infty$.

4 Other Siegel Formulas

4.1 General Comments

In order that we may be able to carry over the method of proof for the Siegel formula for orthogonal groups to other cases, we need to have,
138 besides the Poisson formula 114), suitable generalizations of Theorems (i), (ii), (iii) and (iv) assumed in § 3.2. In the situations covered by A.3), A.4), and A.5) in § 2.3, we have, indeed, $U(i)_k\neq\phi$ for every i in k; therefore, there is no need to look for an analogue of Theorem (i) assumed in § 3.2. Theorem (ii) assumed in § 3.2 can be replaces by the following general theorem (stated without proof).

Theorem 4.1. *Let $U=G/H$ denote an algebraic homogeneous space defined over a global field k and assume that both G and H are connected and simply connected. Then U_k decomposes into finitely many disjoint G_k-orbits $G_k\cdot\xi_\alpha$ and U_A becomes the disjoint union of G_A-orbits $G_A\cdot\xi_\alpha$ for the same ξ_α, with each $G_A\cdot\xi_\alpha$ open in U_A.*

We wish to emphasise that in the cases covered by A.3), A.4) and A.5) in § 2.3, every G-orbit in X defined over k satisfies the conditions in the above theorem. Even in the case of quadratic forms corresponding to A.1)′ of § 2.2, the stabilisers turn out to be connected and simply connected, if we use Spin(f) and G instead of SO(f). A systematic exposition (with references) of this subject can be found in [20]. We recall, in this context, a *conjecture of Weil* which states that $\tau_k(G) = 1$ for any connected simply connected algebraic group G defined over a global field k. Since the stabilisers (other than G) have smaller dimension, we can hope to verify this conjecture inductively at least in some cases.

We have already explained the nature of the analogue in the general case of Theorem (iii) of § 3.2. As for Theorem (iv), we remark that a generalization of the formula $\psi(i^* f(x))\hat{} = \psi(-(i^*)^{-1} f(x))$ is not known and may even not exist. (In this connection, we refer to our paper [24]). **139**
Therefore, following Mars ([36]), we proceed as follows. We choose a symmetric (or skew-symmetric) non-degenerate bilinear form $Q(x, y)$ on $X \times X$ with coefficients in k, such that G becomes invariant under $g \mapsto {}^t g$, where, for any g in $GL(X)$, we define the adjoint ${}^t g$ by the relation $Q(gx, y) = Q(x, {}^t gy)$. Such a choice of $Q(x, y)$ is always possible; for example, we may take $Q(x, y) = f(x, y)$ in A.1)′ of § 2.2. As in that case, we define the (twisted) Fourier transformation $\phi \to \hat{\phi}$ by

$$\hat{\phi}(x) = \int_{X_A} \phi(x)\psi(Q(x, y))|dy|_A \quad \text{for} \quad x \in X_A.$$

Then, for every g in $GL(X)_A$, we have

$$\sum_{\xi \in X_k} \phi(g\xi) = |\det(g)|_A^{-1} \sum_{\xi \in X_k} \hat{\phi}({}^t g^{-1} \xi);$$

this is just the usual Poisson formula. Since the automorphism $g \mapsto {}^t g^{-1}$ of G gives rise to measure-preserving homeomorphism of G_A/G_k with itself, we see that

$$I(\phi) = \int_{G_A/G_k} \left(\sum_{\xi \in X_k} \phi(g\xi) \right) |dg|_A$$

satisfies the relation

$$I(\phi_t) = |t|_A^{-n} I(\hat{\phi}_{t^{-1}}) \tag{126}$$

where $t \in k_A^\times$ and $\phi_t(x) = \phi(tx)$. For $t = 1$, this relation reduces to the invariance of $I(\phi)$ under the Fourier transformation $\phi \to \hat{\phi}$. The idea of Mars is that, without looking for an analogue of Theorem (iv) or rather of the invariance under $\phi \to \hat{\phi}$ of

$$\sum_{i^* \in k^\times} \int_{X_A} \phi(x)\psi(i^* f(x))|dx|_A,$$

140 one may consider, instead, the relation

$$\lim_{|t|_A \to \infty} I(\phi_t) = \lim_{|t|_A \to \infty} |t|_A^{-n} I(\hat{\phi}_{t^{-1}}) \tag{127}$$

after expressing both sides of (126) by "orbital integrals" and incorporating the Poisson formula. For instance, if we now go back to the case of a quadratic form $f(x)$ in $n > 4$ variables, then (127) will yield either the relation $\tau_k(G)\phi(0) = 2\phi(0)$ or the relation $\tau_k(G)\phi(0) = \phi(0)$ for every ϕ in $\mathscr{S}(X_A)$, according as $G = \mathrm{SO}(f)$ or $\mathrm{Spin}(f)$. (In the latter case, we need, of course, to assume correspondingly, that $\tau_k(\mathrm{Spin}(f)) = 1$ for $n = 3, 4$).

4.2 Other Siegel Formulas

We now indicate, in some detail, how one may obtain a Siegel formula in the general case; actually in the cases covered by A.3), A.4) of § 2.3 such a generalisation exists. Using Theorem 4.1, we have, for every ϕ in $\mathscr{S}(X_A)$,

$$\int_{G_A/G_k} \left(\sum_{\xi \in U_k} \phi(g\xi) \right) |dg|_A = \int_{U_A} \in \phi |\theta_U|_A \tag{128}$$

where θ_U is a gauge form on the homogeneous space U and further, $\epsilon(x) = \tau_k(H_\alpha)$ if x is in $G_A \cdot \xi_\alpha$. In all cases, we know that $U(i)$ is a G-orbit for every i in k and further, $f^{-1}(0)$ is a union of finitely many G-orbits. From (128), we see that

$$I(\phi) = \tau_k(G)\phi(0) + \sum_V \int_{V_A} \in \phi|\theta_V|_A + \sum_{i \in k} \int_{U(i)_A} \in \phi|\theta_i|_A \tag{129}$$

letting V run over all G-orbits in $f^{-1}(0)$ other than $U(0)$ and $\{0\}$. Further, we also have the Poisson formula

$$\sum_{i\in k}\int\limits_{U(i)} \phi|\theta_i|_A = \sum_{i^*\in k}\int\limits_{X_A} \phi(x)\psi(i^* f(x))|dx|_A.$$

Let us define the "completed Eisenstein-Siegel series" $E(\phi)$ for ϕ in $\mathscr{S}(X_A)$ by **141**

$$E(\phi) = \phi(0) + \sum_{V}\int\limits_{V_A} \phi|\theta_V|_A + \sum_{i^*\in k}\int\limits_{X_A} \phi(x)\psi(i^* f(x))|dx|_A$$

where V runs over the same G-orbits as in (129). Then a (conjectural) *Siegel formula* in the general case, is the simple (- looking) relation that, for every ϕ in $\mathscr{S}(X_A)$,

$$I(\phi) = E(\phi). \tag{130}$$

In the case covered by A.3) of § 2.3, it is known that $\tau_k(G) = \epsilon(x) = 1$ and therefore, the Siegel formula (130) is indeed valid. As for A.4) and A.5), we apply the method of Mars explained in § 4.1. Then we do get $\tau_k(G) = \epsilon(x) = 1$ as far as A.4) is concerned and therefore the Siegel formula (130) is valid here. Similarly, we can uphold the Siegel formula also for A.5), provided, however, that $\epsilon(x) = 1$ i.e. if the above-mentioned conjecture of Weil is verified for the stabilisers.

REMARK. For any $u \in k_A$, $\phi(x) \to \psi(uf(x))\phi(x)$ represents a unitary operator of $\mathscr{S}(X_A)$. The unitary operators of $\mathscr{S}(X_A)$ defined by

$$\phi(x) \to \Psi(x) = \psi(uf(x))\hat{\phi}(x) \to \hat{\phi}(-x)$$

which are obtained by conjugating unitary operators above with respect to $\phi \to \hat{\phi}$, generate, together, with them, what might be called a *metaplectic group* associated with $f(x)$. Let us denote this group by Mp_A **142**
and its subgroup obtained by restricting u to k, by Mp_k. Then $I(\phi)$ is invariant under the action of Mp_k. Therefore, if the Siegel formula (130) is valid, then the correspondence

$$Mp_A \ni \mathbb{S} \to E(\mathbb{S}\phi) \in \mathbb{C}$$

defines a continuous function on Mp_A which is left invariant or "automorphic" under Mp_k. In the case when $f(x)$ is a quadratic form and $Q(x, y) = f(x, y)$, Mp_A is just the *metaplectic group of Weil*; the invariance of $E(\mathbb{S}\phi)$ under $Mp_k = SL_2(k)$ gives the adelic version of the automorphic behaviour of the classical Eisenstein series.

4.3 A Final Comment

In the case covered by A.1) of § 2.3 i.e. when $f(x)$ is a strongly non-degenerate form of degree $m \geq 2$ in $n > 2m$ variables with m not divisible by the characteristic of k, there is no Siegel formula in the strict sense. In this case, we still take

$$E(\phi) = \phi(0) + \sum_{i^* \in k} \int_{X_A} \phi(x)\psi(i^* f(x))|dx|_A$$

as the completed Eisenstein-Siegel series but there is no group over which we can take the average of

$$I_0(\phi) = \sum_{\xi \in X_k} \phi(\xi).$$

However, at least a subgroup of what might be called a metaplectic group is readily available. In fact, we convert the product $P = \mathbb{G}_a \times \mathbb{G}_m$ into an algebraic group by defining

$$(u, t)(u', t') = (u + t^m u', tt').$$

143 (Here $\mathbb{G}_m$ is the multiplicative group $\mathbb{G}_a \backslash \{0\}$). For $(u, t) \in P_A = k_A \times k_A^\times$, and ϕ in $\mathscr{S}(X_A)$, we define

$$((u, t) \cdot \phi)(x) = |t|_A^{n/2} \psi(uf(x))\phi(tx).$$

Then $(I_0 - E)$ $((u, t) \cdot \phi)$ becomes a P_k-invariant continuous function on P_A and we can show that it vanishes at every "k-rational boundary point" of P_A in the following sense:

$$(I_0 - E)((u, t) \cdot \phi) = O(|t|_A^{m-n/2}) \text{ as } |t|_A \to \infty$$

and moreover,

$$(I_0 - E)((u,t)\cdot\phi) = O(|t|_A^{n/2-m}) \text{ as } |t|_A \to 0$$

but subject to the restriction that $(u+i^*)t^{-m}$ remains in a compact subset of k_A for some i^* in k. If we specialise f to be a quadratic form and k to be $\mathbb{Q}$, the theorem above gives the well-known behaviour of theta series as the variable in the complex upper half-plane approaches the rational points on the real axis. In the general case, the theorem above suggests that $(I_0 - E)((u,t)\cdot\phi)$ remains bounded as $|t|_A$ tends to 0. We have observed that this conjecture will have as its consequence a generalisation of the Minkowski-Hasse theorem (namely $U(i)_A \neq \phi$ implying that $U(i)_k \neq \phi$). For the details, we refer to our paper [23].

5 Siegel's Main Theorem for Quadratic Forms

As it is well-known, the main theorem of Siegel in the analytic theory of quadratic forms is equivalent to the existence of an identity between a certain weighted average of theta series and an Eisenstein series, when one considers non-degenerate integral quadratic forms in more than four **144**
variables. We shall see in this article how this identity can be derived from the Siegel formula for quadratic forms due to Weil, quite explicitly in the positive-definite case.

5.1 Siegel's Main Theorem

For any matrix M, we denote its transpose by ${}^tM$ and its determinant (whenever it makes sense) by $\det M$. If M is a real n-rowed symmetric non-singular matrix, we say that M is of *signature* (p,q) if exactly p eigenvalues of M are positive (and consequently $q = n-p$ eigenvalues are negative); M is called *positive-definite* or *indefinite* according as $q = 0$ or $pq > 0$.

Two n-rowed integral symmetric matrices T, T' are said to be *in the same* (equivalence) *class* (respectively *narrow class*) if $T = {}^tAT'A$ for an integral matrix A of determinant ±1 (respectively 1). We say that two non-singular n-rowed integral symmetric matrices T, T' of the same

signature *belong to the same genus*, if, for every prime number p, there exists a matrix A_p with entries in the ring $\mathbb{Z}_p$ of p-adic integers and of determinant 1, such that $T = {}^tApT'Ap$; we remark that this notion of a genus is the same as that of Siegel, in view of the fact every symmetric matrix M over $\mathbb{Z}_p$ admits an A over $\mathbb{Z}_p$ such that ${}^tAMA = M$ and $\det A = -1$ (See [40], I, Hilfssatz 19]. Clearly any two T, T' as above which are in the same class, belong to the same genus as well; thus the genus of T splits into classes. From the reduction theory of quadratic forms due to Minkowski, Hermite and Siegel, it is known that the genus of any non-singular integral symmetric matrix T consists of only a finite number of classes and therefore, only finitely many narrow classes. For the narrow classes, we choose a complete set of representatives say
145 $T_1(= T), T_2, \ldots, T_r$. We may further assume that if, for one of these T_i, there exists no integral matrix A of determinant -1 with ${}^tAT_iA = T_i$, then, along with T_i, the matrix $T_i^* = {}^tDT_iD$ also occurs among the r representatives for a fixed integral D of determinant -1; we may take, for example, D to be a diagonal matrix with -1 as its first diagonal element and 1 as the remaining diagonal elements. To each T_i, $1 \leq i \leq r$, we associate the quadratic form $f_i(x) = 1/2^t xT_ix$, where x now denotes the n-rowed column with elements $x_1, \ldots, x_n$; further, we write $f(x)$ instead of $f_1(x)$.

The special orthogonal group $\mathrm{SO}(f)$ of $f(x) = 1/2^t xTx$, consists of all matrices A with ${}^tATA = T$ and $\det A = 1$. It acts, in a natural fashion, on the affine n-space X. Denoting $\mathrm{SO}(f)$, for the present, by G, it is known that G is a semi-simple linear algebraic group (for $n > 2$) defined over $\mathbb{Q}$; the adele group G_A has the property that $G_A/G_{\mathbb{Q}}$ has finite measure. We assume that the measure μ on G_A is normalized so that

$$\mu(G_A/G_{\mathbb{Q}}) = 1. \tag{131}$$

The Siegel formula for orthogonal groups due to Weil may now be stated once again: namely, for $n > 4$ and for any ϕ in $\mathscr{S}(X_A)$, we have

$$\int\limits_{G_A/G_{\mathbb{Q}}} \left(\sum_{\xi \in X_{\mathbb{Q}}} \phi(g\xi) \right) d\mu(g) = \phi(0) + \sum_{i^* \in \mathbb{Q}} \int\limits_{X_A} \phi(x)\psi(i^* f(x))|dx|_A. \tag{132}$$

Let T be positive-definite, unless otherwise stated and let $T_1, \ldots, T_r$ be the above-mentioned representatives of the narrow classes in the genus of T. Corresponding to each $f_i(x) = 1/2\, {}^t x T_i x$ and a complex variable τ with $\mathrm{Im}(\tau) > 0$, let us define the theta series v_i by

$$v_i(\tau) = \sum_{\xi \in \mathbb{Z}^n} \mathfrak{e}(\tau f_i(\xi)).$$

It is not hard to show that this series converges absolutely, uniformly when $\mathrm{Im}\,\tau \geq \epsilon > 0$. Thus $v_i(\tau)$ is a holomorphic function of τ for $\mathrm{Im}(\tau) > 0$ and furthermore, $v_i(\tau)$ depends only on the class of T_i. For T_i and T_i^*, in particular, we have the same $v_i(\tau)$. Let e_i (respectively e_i^+) denote the number of integral matrices A of determinant ± 1 (respectively 1) such that ${}^t A T_i A = T_i$; then e_i is obviously finite and at least equal to 2. Moreover, $e_i = \delta_i e_i^+$ where $\delta_i = 2$ or 1 according as T_i admits an integral A with ${}^t A T_i A = T_i$, $\det A = -1$ or otherwise. With our understanding above, only one of T_i or $T_i^* = T_i[D]$ occurs among the r representatives if $\delta_i = 2$, while both of them occur for $\delta_i = 1$. 146

The analytic formulation of the main theorem of Siegel for representation of integers by T for $n > 4$ may now be given: namely,

$$\left(\sum_{i=1}^{r} v_i(\tau)/e_i^+\right) \Big/ \left(\sum_{i=1}^{r} 1/e_i^+\right)$$
$$= 1 + \frac{\mathfrak{e}(n/8)}{\sqrt{\det T}} \sum_{c,d} c^{-n/2} \left(\sum_{\xi \bmod c} \mathfrak{e}\left(-\frac{d}{c} f(\xi)\right)\right) (c\tau + d)^{-n/2} \quad (133)$$

where, on the right hand side, the summation is over all pairs of coprime integers c, d with $c \geq 1$ and further, with $cd\,f(\xi) \in \mathbb{Z}$ for all $\xi \in \mathbb{Z}^n$. In view of our remarks above, it is not difficult to check that the left hand side of (133) is the same as the "analytic genus-invariant" of T in the sense of Siegel; formula (133) is thus the same as formula (83) in the above-mentioned fundamental paper ([40], I).

5.2

We wish to prove that the Siegel formula (132) for positive-definite integral quadratic forms yields the analytic identity (133), if we merely

147 specialise ϕ in (132) suitably. But first, let us refer to another familiar notion of "classes in a genus" associated with an arbitrary linear algebraic group over a global field.

Let G be a linear algebraic group defined over a global field k of characteristic 0. The adele group G_A contains only finitely many distinct double cosets modulo G_Ω, G_k where G_Ω is just the open subgroup denoted earlier by G_{S_∞}:

$$G_A = \coprod_{i=1}^{r} G_\Omega g_i G_k \tag{134}$$

The number $r = r_k(G)$ is known sometimes as the class number of G. We shall prove, in § 5.6, that

$$r_{\mathbb{Q}}(SL_n) = 1 \tag{135}$$

Actually, more generally, even $r_k(SL_n) = 1$. If $G \subset GL(V)$ for a vector space V of dimension n, then G_A acts in a natural fashion, on the "lattices" in V_k; the orbit of a "lattice" under G_A (respectively G_k) constitutes the *genus* (respectively *class*) of the "lattice" and the number of classes in a genus is finite.

Going back to the n-rowed integral positive-definite matrix T once again, we take $G = \mathrm{SO}(f)$, for the associated $f(x) = 1/2\,{}^t xTx$. From (134) and (135), we have

$$(SL_n)_A = (SL_n)_{S_\infty} \cdot (SL_n)_{\mathbb{Q}}. \tag{135$'$}$$

Thus any g in $\mathrm{SO}(f)_A \subset (SL_n)_A$ can be written as

$$g = A \cdot C^{-1}, \quad A \in (SL_n)_{S_\infty}, \quad C \in (SL_n)_{\mathbb{Q}}. \tag{136}$$

148 Since ${}^tCTC = {}^t(g^{-1}A)Tg^{-1}A = {}^tATA$, it follows that ${}^tATA$ is in the genus of T; its class depends only on the double coset $G_\Omega g G_{\mathbb{Q}}$. Conversely, if T' is in the genus of T, then, in $(GL_n)_A$, $T' = {}^tATA$ for some A in $(SL_n)_{S_\infty}$; by the Minkowski-Hasse theorem, we know then that $T' = {}^tCTC$ for some C in $(SL_n)_{\mathbb{Q}}$ and thus T' corresponds to the double coset $G_\Omega(AC^{-1})G_{\mathbb{Q}}$. Therefore, for our r (narrow) class-representatives

$T_1, \ldots, T_r$ above, we may take $T_i = {}^tA_iTA_i = {}^tC_iTC_i$, $1 \leq i \leq r$ corresponding to $g_i = A_iC_i^{-1}$ as in (136) and the double coset decomposition 134 for $\mathrm{SO}(f)_A$. From (131), (134) and the invariance of μ, we obtain

$$\begin{aligned}
1 &= \sum_{i=1}^{r} \mu(G_\Omega\, g_iG_{\mathbb{Q}}/G_{\mathbb{Q}}) \\
&= \sum_{i=1}^{r} \mu(g_i^{-1}G_\Omega\, g_iG_{\mathbb{Q}}/G_{\mathbb{Q}}) \\
&= \sum_{i=1}^{r} \mu(g_i^{-1}G_\Omega\, g_i/(G_{\mathbb{Q}} \cap g_i^{-1}G_\Omega\, g_i)) \\
&= \sum_{i=1}^{r} \mu(G_\Omega/(g_iG_{\mathbb{Q}}g_i^{-1} \cap G_\Omega)) \\
&= \mu(G_\Omega) \sum_{i=1}^{r} 1/e_i^+
\end{aligned}$$

in view of the fact that $g_iG_{\mathbb{Q}}g_i^{-1} \cap G_\Omega \simeq A_i^{-1}G_\Omega A_i \cap C_i^{-1}G_{\mathbb{Q}}C_i = \mathrm{SO}(f_i)_{\mathbb{Z}}$ is of order e_i^+. Thus

$$\mu(G_\Omega) = 1\Big/\left(\sum_{i=1}^{r} 1/e_i^+\right). \tag{137}$$

5.3

For $\phi \in \mathscr{S}(X_A)$ in (132), we take $\phi = \phi_0 \otimes \phi_\infty$ in $\mathscr{S}(X_0) \bigotimes_{\mathbb{C}} \mathscr{S}(X_\infty)$, following the notation of Chapter IV, § 1.51.5, with

$$\phi_0 = \text{ the characteristic function of } \prod_p X_p^0 \text{ and}$$

$$\phi_\infty(x_\infty) = \mathfrak{e}(f(x_\infty)\tau).$$

The left hand side of (132) becomes just 149

$$\sum_{i=1}^{r} \int\limits_{G_\Omega/(g_iG_{\mathbb{Q}}g_i^{-1} \cap G_\Omega)} \left(\sum_{\xi \in X_{\mathbb{Q}}} \phi(gg_i\xi)\right) d\mu(g) \tag{138}$$

using the same arguments that led us to the proof of (137). The innermost series in (138) is the same as $\sum\limits_{\xi\in X_{\mathbb{Q}}} \phi(g' A_i \xi)$ with $g' = (g'_0, g'_\infty)$ in G_Ω. If now we use the definition of ϕ, then the last-mentioned series is seen to be precisely

$$\sum_{\xi\in\mathbb{Z}^n} \mathfrak{e}(f(g'_\infty A_{i,\infty}\xi)\tau) = \sum_{\xi\in\mathbb{Z}^n} \mathfrak{e}(f_i(\xi)\tau) = \nu_i(\tau).$$

It is now immediate that (138) is exactly $\mu(G_\Omega) \sum\limits_{i=1} \nu_i(\tau)/e_i^+$ and therefore, by (137), the left hand side of (132) is the same as that of (133).

5.4

What remains to be proved then is only that (132) has the same right hand side as (133), for ϕ as chosen as above. Since $\phi(0) = 1$, we are reduced to showing that the series on the right hand side of (132) is identical with the series over (c, d) in (133).

Since $(k_A/k)^* \simeq k$, the right hand side is independent of the choice of ψ and therefore we may assume that

$$\psi_p : Q_p \to Q_p/\mathbb{Z}_p \hookrightarrow \mathbb{R}/\mathbb{Z} \simeq \mathbb{C}_1^\times \quad \text{and}$$
$$\psi_\infty(x_\infty) = \mathfrak{e}(-x_\infty).$$

150 Any i^* in $\mathbb{Q}$ may be written as $-d/c$ with coprime c, d in $\mathbb{Z}$ and further $c \geq 1$. The integral over X_A in 132 becomes

$$\left(\prod_p \int\limits_{X_p^0} \psi_p\left(-\frac{d}{c} f(x)\right) |dx|_p\right) \cdot \int\limits_{X_\infty} \mathfrak{e}(f(x)(\tau + d/c))|dx|_\infty. \tag{139}$$

The integral over X_p^0 above may be rewritten as

$$\sum_{\xi \mod c\mathbb{Z}p} \int\limits_{c\mathbb{Z}_p^n} \psi_p\left(-\frac{d}{c} f(\xi + x)\right) |dx|_p. \tag{140}$$

Since $\psi_p\left(-\dfrac{d}{c}f(\xi+x)\right) = \psi_p\left(-\dfrac{d}{c}f(\xi)\right)\psi_p\left(-\dfrac{d}{c}f(x)\right)$, the integral over $c\mathbb{Z}_p^n$ in (140) is (upto the factor $\psi_p\left(-\dfrac{d}{c}f(\xi)\right)$ equal to

$$|c|_p^n \int\limits_{\mathbb{Z}_p^n} \psi_p(-cdf(x))|dx|_p = \begin{cases} |c|_p^n \text{ if } cdf(x) \text{ is } \mathbb{Z}_p - \text{valued on } \mathbb{Z}_p^n \\ 0, \text{ otherwise} \end{cases}$$

on noting that $\psi_p(-cdf(x))$ is a character of $\mathbb{Z}_p^n$. Hence the infinite product over primes p in (139) is precisely

$$c^{-n} \sum_{\xi \mod c} \mathfrak{e}\left(-\frac{d}{c}f(\xi)\right) \tag{141}$$

assuming, of course, the $cdf(x)$ is $\mathbb{Z}$-valued on $\mathbb{Z}^n$. On the other hand, the integral over X_∞ in (139) is clearly equal to

$$\begin{aligned} &\int\limits_{\mathbb{R}^n} \mathfrak{e}(1/2\,{}^t(Mx)(Mx)(\tau + d/c))|dx|_\infty \text{ with } {}^tMM = T \\ &= |\det M|^{-1}\left(\int\limits_{\mathbb{R}} \mathfrak{e}(1/2(\tau+d/c)t^2)dt\right)^n \\ &= (\det T)^{-1/2}\left(\sqrt{i/(\tau+d/c)}\right)^n \\ &= (\det T)^{-1/2}\mathfrak{e}(n/8)c^{n/2}(c\tau+d)^{-n/2} \end{aligned} \tag{142}$$

Taking into account (139), (140), (141) and (142), it is clear that our assertion at the beginning of § 5.45.4 is established. With this, the proof of the fact that the Siegel formula (132) implies (133) is also complete. **151**

5.5

If we consider an n-rowed integral symmetric matrix T of signature (p, q) with $0 < q = n - p < n$, then Siegel's main theorem takes the following form. For the special orthogonal group $G = SO(f)$ with $f(x) = 1/2\,{}^txTx$, G_∞ is no longer compact; $G_{\mathbb{Z}}$ is no more finite. One now considers the space $\mathscr{H}$ of all the *majorants* M ot T, which are just n-rowed symmetric positive-definite matrices such that $M^{-1}SM^{-1}S$ is equal to the n-rowed identity matrix. The group G_∞ acts on $\mathscr{H}$ via the maps $M \mapsto {}^t g_\infty M g_\infty$ for $g_\infty \in G_\infty$ and indeed transitively; the stabiliser of any point in $\mathscr{H}$ is a maximal compact subgroup of G_∞. Let μ be a measure on $\mathscr{H}$ such that $\mu(\mathscr{H}/G_{\mathbb{Z}}) = 1$. The main theorem of Siegel ([42]) now reads: for $n > 4$,

$$\int\limits_{\mathscr{H}/G_{\mathbb{Z}}} \left(\sum_{\xi \in X_{\mathbb{Z}}} \mathfrak{e}(f(\xi)\mathrm{Re}\,(\tau) + i^t\xi M\xi\,\mathrm{Im}(\tau))\right) d\mu(M)$$

$$= 1 + |\det T|^{-1/2} \oplus \left(\frac{p-q}{8}\right) \sum_{c,d} c^{-n/2}$$

$$\left(\sum_{\xi \mod c} \oplus \left(-\frac{d}{c} f(\xi)\right) (c\tau + d)^{-p/2} (c\overline{\tau} + d)^{-q/2}\right)$$

where c, d run over all pairs of coprime integers with $c \geq 1$ and $cdf(\xi) \in \mathbb{Z}$ for all $\xi \in \mathbb{Z}^n$. This formula can again be derived from (132), with a proper choice of ϕ; actually we may take $\phi = \phi_0 \otimes \phi_\infty$ where ϕ_0 is, as before, the characteristic function of $\prod\limits_p X_p^0$ but $\phi_\infty(x_\infty) = \exp(-2\pi^t x_\infty M x_\infty)$.

5.6

We now give an indirect but hopefully instructive proof of the assertion (135)′ used in the course of our arguments in § 5.2.

152 First, we give an outline of the proof of the fact that the Tamagawa number

$$\tau_k(SL_n) = \int\limits_{(SL_n)_A/(SL_n)_k} |dg|_A = 1 \tag{143}$$

for any $n \geq 1$ and any global field k. Since the assertion (143) is clear for $n = 1$, assume that it has been proved with $r \leq n-1$ in place of n, for $n > 1$. Let us now take G to be SL_n and X to be the same as the space $M_{n,1}$ of n-rowed vectors. For any field K, we have

$$X_K \backslash \{0\} = G_K \cdot e_1$$

where now e_1 is the unit vector with 1 as its first element and 0 elsewhere. The stabiliser H in G of the vector e_1 is a semi-direct product of SL_{n-1} and the $(n-1)$-fold product of the additive group $\mathbb{G}_a$. Hence the induction hypothesis yields $\tau_k(H) = 1$. For any ϕ in $\mathscr{S}(X_A)$, it can be shown that

$$\int\limits_{G_A/G_k} \left(\sum_{\xi \in X_k} \phi(g\xi) \right) |dg|_A = \tau_k(G)\phi(0) + \tau_k(H)\phi^*(0) \tag{144}$$

where ϕ^* is the Fourier transform of ϕ. The left hand side of (144) can be seen to be invariant under $\phi \mapsto \phi^*$, in view of the (classical) Poisson formula (115). But then the invariance of the right hand side under $\phi \mapsto \phi^*$ implies that $\tau_k(G) = \tau_k(H) = 1$, which establishes (143).

Coming back to the proof of (135)′, we need, in view of (133), only to show that the measure of

$$(SL_n)_{S_\infty}(SL_n)_{\mathbb{Q}}/(SL_n)_{\mathbb{Q}} = \prod_p SL_n(\mathbb{Z}_p) \times SL_n(\mathbb{R})/SL_n(\mathbb{Z})$$

is precisely 1. It is known that **153**

$$\mu(SL_n(\mathbb{Z}_p)) = (1 - p^{-n}) \ldots (1 - p^{-2})$$

and therefore

$$\mu\left(\prod_p SL_n(\mathbb{Z}_p)\right) = 1/\prod_{i=2}^{n} \zeta(i)$$

where ζ is the Riemann zeta function. In order to complete the proof of (135)′, we have only to show that

$$\mu(G_{\mathbb{R}}/G_{\mathbb{Z}}) = \mu(H_{\mathbb{R}}/H_{\mathbb{Z}}) \cdot \zeta(n). \tag{145}$$

Now, analogous to (144), we also have the relation

$$\int\limits_{G_{\mathbb{R}}/G_{\mathbb{Z}}} \left(\sum_{\xi} \phi_\infty(g\xi) \right) |dg|_\infty = \mu(H_{\mathbb{R}}/H_{\mathbb{Z}})\phi_\infty^*(0) \tag{146}$$

for ϕ_∞ in $\mathscr{S}(X_\infty)$, where ξ runs over all "primitive vectors" i.e. over all the elements of the orbit $SL_n(\mathbb{Z})e_1$ of the vector e_1 above. We may assume that $\phi_\infty \geq 0$, $\phi_\infty \neq 0$, so that $\phi^*_\infty(0) > 0$. If we replace $\phi_\infty(x)$ in (146) by $\phi_\infty(tx)$ for $t > 0$, then we obtain, instead,

$$\int\limits_{G_{\mathbb{R}}/G_{\mathbb{Z}}} \left(\sum_{\xi} \phi_\infty(gt\xi)\right) |dg|_\infty = \mu(H_{\mathbb{R}}/H_{\mathbb{Z}}) t^{-n} \phi^*_\infty(0). \tag{147}$$

Taking $t = 1, 2, 3, \ldots$ and summing up both sides of (147) over t, we get

$$\int\limits_{G_{\mathbb{R}}/G_{\mathbb{Z}}} \left(\sum_{\xi \in X_{\mathbb{Z}} \setminus \{0\}} \phi_\infty(g\xi)\right) |dg|_\infty = \mu(H_{\mathbb{R}}/H_{\mathbb{Z}}) \zeta(n) \phi^*_\infty(0). \tag{148}$$

Replacing $\phi_\infty(x)$ in (148) by $\phi_\infty(t'x)$ for $t' > 0$ and letting t' tend to 0, we have

$$\int\limits_{G_{\mathbb{R}}/G_{\mathbb{Z}}} \left(\int\limits_{X_{\mathbb{R}}} \phi_\infty(gx) |dx|_\infty\right) |dg|_\infty = \mu(H_{\mathbb{R}}/H_{\mathbb{Z}}) \zeta(n) \phi^*_\infty(0).$$

154 But now the left hand side is the same as $\mu(G_{\mathbb{R}}/G_{\mathbb{Z}})\phi^*_\infty(0)$ and 145 is proved.

6 Proof of the Criterion for the Validity of the Poisson Formula

We devote this section to giving an outline of the proof of Theorem 2.1.

Let us recall that k is a global field, ψ, a non-trivial character of k_A/k, $f(x) \in k[x_1, \ldots, x_n]$ a homogeneous polynomial of degree $m \geq 2$. We had further assumed that m is not divisible by the characteristic of k and that a tame k-resolution of the projective hypersurface defined by $f(x) = 0$ exists. Further, the two criteria stated are:

(C1) $r = \operatorname{codim}_{f^{-1}(0)} C_f \geq 2$; and

(C2) there exist $\sigma > 2$ and a finite set S of valuations v of k, such that, for every i^* in $k_v \backslash R_v$ and $v \notin S$, we have

$$|F_v^*(i^*)| = \left| \int_{X_v^0} \psi_v(i^* f(x)) |dx|_v \right| \leq |i^*|_v^{-\sigma}$$

We have to show that (C1) and (C2) imply the validity of (PF-1),...,(PF-3) and (PF-4).

6.1 Geometric Part of the Proof

Let D be a positive divisor on an irreducible non-singular algebraic variety X and $h : Y \to X$ be a K-resolution of D for any arbitrary field K. Let $\{(N_i, \nu_i)\}_i$ denote the numerical date of h at a point b in Y, in the sense of Chapter III, § 2.2. We say that h has property (P_0), at b, if $\nu_i \geq N_i$ for every i and if, further, the equality holds for at most one i. If, moreover, 155
in case such equality holds, say $\nu_{i_0} = N_{i_0}$, we have $\nu_{i_0} = N_{i_0} = 1$, then h is said to *have property* (P) *at* b. It may happen that h has property (P_0) at every point of Y_K but does not have property (P) at some point of Y_K. We have, however, the following

Lemma 6.1. *If h (is tame and) has property (P_0) everywhere, then h has necessarily property (P) everywhere.*

Proof. By restricting h to a Zariski open neighbourhood of $h^{-1}(h(b))$ for a point b in Y, we may assume that X is affine and hence that $D = (g)$ for some function g regular everywhere on X; moreover, we may suppose that there exists a "gauge form" dx on X, namely an everywhere regular differential form of degree $n = \dim(X)$ on X, vanishing nowhere. □

If the lemma is false, then, for the numerical data $\{(N_i, \nu_i)\}_i$ at some point in Y, we have $\nu_i > N_i$ for $i \neq i_0$ and $\nu_{i_0} = N_{i_0} \geq 2$; without loss of generality, let $i_0 = 1$. Thus there exists an irreducible component E_1 of $h^*(D)$ with $\nu_1 = N_1 \geq 2$ and $N_1 (= N_{E_1})$ not divisible by the characteristic of K, since h is tame. We shall construct a regular $(n-1)$-form $\theta \neq 0$ on E_1, as follows. At an arbitrary point b on E_1, there exist local coordinates $y_1, \ldots, y_n$ of Y centred at b such that

$$g \circ h = \epsilon \prod_i y_i^{N_i}, \quad h^*(dx) = \eta \prod_i y_i^{\nu_i - 1} dy_1 \wedge \ldots \wedge dy_n$$

with ϵ, η regular and non-vanishing around b and $y_1 = 0$ being the local equation for E_1. Then, for the $(n-1)$-form

$$\Theta = \left(N_1 \epsilon + y_1 \frac{\partial \epsilon}{\partial y_1}\right)^{-1} \eta \prod_{i>1} y_i^{\nu_i - N_i - 1} dy_2 \wedge \ldots \wedge dy_n$$

on Y regular along E_1, we have

$$h^*(dx) = d(g \circ h) \wedge \Theta.$$

156 Also, locally around b, the restriction θ of Θ to E_1, i.e.

$$\theta = \text{ the restriction to } E_1 \text{ of } (N_1 \epsilon)^{-1} \eta \prod_{i>1} y_i^{\nu_i - N_i - 1} dy_2 \wedge \ldots \wedge dy_n$$

is well-defined (i.e. independent of the choice of local coordinates), different from 0 and regular on E_1.

We recall that h can be factored into monoidal transformations each having irreducible non-singular centre and E_1 is "created" at one such stage, say $h' : Y' \to X'$; we have tacitly used the fact that $\nu_1 \geq 2$ and hence h is not biregular along E_1. Let Z denote the centre of h' and let $E' = (h')^{-1}(Z)$; then E' is irreducible and non-singular and the restriction of $Y \to Y'$ to E_1 gives a birational morphism $g' : E_1 \to E'$. Let θ' denote the image of θ under g'. Then θ' is different from 0 and everywhere regular on E'; if θ' is not regular on E', then $g^{-1}(C')$, for any component C' of its polar divisor, becomes a component of the polar divisor of θ, contradicting the regularity of θ. If $\pi = h'|_{E'}$, then $\pi : E' \to Z$ makes E' into a fibre bundle with the projective space $\mathbb{P}_{t-1}$ of dimension $t-1$ as fibre; here $t = (\text{codimension of } Z) \geq 2$. Choosing b' in E' at which θ' does not vanish and putting $a' = \pi(b')$, we choose a local gauge form dz on Z around a' and write $\theta' = \pi^*(dz) \wedge \rho$ with a $(t-1)$ form ρ on E' regular along $F = \pi^{-1}(a')$. This is possible and further, the restriction ρ_F of ρ to F is well-defined, different from 0 and regular on F. But F being isomorphic to $\mathbb{P}_{t-1}$, there is no regular form other than 0. This contradiction proves the lemma.

The following lemma is quite elementary.

Lemma 6.2. *Let $f(x)$ be a homogeneous polynomial of degree m in n variables $x_1, \ldots, x_n$ with coefficients in (an arbitrary field) k. Let X_0 (respectively X) denote the projective (respectively affine) spaces with $(x_1, \ldots, x_n)$ as its homogeneous (respectively affine) coordinates and $X^\sharp$, the projective space with $(1, x_1, \ldots, x_n)$ as its homogeneous coordinates. Let $H_\infty = X^\sharp \backslash X$ and $f^\sharp$ denote the rational function on $X^\sharp$ defined by $f(x)$. Let us assume that a k-resolution $h_0 : Y_0 \to X_0$ of the projective hypersurface defined by $f(x) = 0$ exists. Then h_0 gives rise to a k-resolution $h^\sharp : Y^\sharp \to X^\sharp$ of $((f^\sharp)_0, H_\infty)$ such that the numerical data of $h^\sharp$ are the numerical data of h_0 at some point of Y_0 possibly augmented by (m, n); here $(f^\sharp) = (f^\sharp)_0 - mH_\infty$ is the divisor of $f^\sharp$ on $X^\sharp$.* 157

Proof. Clearly $(1, x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$ gives a rational map $\mathscr{S}$: $X^\sharp \to X_0$ over k, regular except at $(1, 0, \ldots, 0)$. If $g : Z \to X^\sharp$ is the quadratic transformation centred at $(1, 0, \ldots, 0)$, then, $h_1 = \varphi \circ g : Z \to X_0$ is a k-morphism. If $Y^\sharp$ is the fibre product $Y_0 \times_{X_0} Z = \{(y, z) \in Y_0 \times Z; h_0(y) = h_1(z)\}$, define $h^\sharp : Y^\sharp \to X^\sharp$ as the composite of $h_0 \times 1 : Y^\sharp \to Z$ and $g : Z \to X^\sharp$. For X_0 (respectively Y_0), we have k-open coverings by $X_1, \ldots, X_n$ (respectively $Y_1, \ldots, Y_n$) where X_1 is the affine space obtained from X_0 by letting $x_i = 0$ define the hyperplane at infinity and $Y_i = h_0^{-1}(X_i)$ for $1 \leq i \leq n$. Let A_i (respectively $\mathbb{P}_{1,i}$) be the affine line (respectively projective line) with coordinate ring $k[1/x_i]$ (respectively homogeneous coordinate ring $k[t, tx_i]$ with a new variable t) for $1 \leq i \leq n$. Then X, $A_i \times X_i (1 \leq i \leq n)$ form a k-open covering of $X^\sharp$ and $\mathbb{P}_{1,i} \times X_i (1 \leq i \leq n)$ form a k-open covering of Z. For $Y^\sharp$, we have a k-open covering by $\mathbb{P}_{1,i} \times Y_i$. □

We show that $h^\sharp$ has the required property. Let $a^\sharp = h^\sharp(b^\sharp)$ for a point $b^\sharp$ of $Y^\sharp$. We have only to show that the irreducible components of $(h^\sharp)^*$ $((f^\sharp)_0 + H_\infty)$ are defined over $k(b^\sharp)$ and mutually transversal at $b^\sharp$ and 158
that the numerical data of $h^\sharp$ at $b^\sharp$ are the same as those of h_0 at a point

of Y_0 except possibly for the addition of (m, n). By changing the indices, we may assume that $b^\sharp \in \mathbb{P}_{1,1} \times Y_1$; thus we may write $b^\sharp = (a_1, b)$ with a_1 in the universal field or $a_1 = \infty$ and b in Y_1. Let

$$u_2 = x_2/x_1, \ldots, u_n = x_n/x_1.$$

Then there exist local coordinates $(v_2, \ldots, v_n)$ of Y_0 centred at b and defined over $k(b)$ such that

$$\begin{aligned} f(1, u_2, \ldots, u_n) &= \epsilon \prod_{i>1} v_i^{N_i}, \\ du_2 \wedge \ldots \wedge du_n &= \eta \prod_{i>1} v_i^{\nu_i - 1} dv_2 \wedge \ldots \wedge dv_n \end{aligned}$$

where ϵ, η are regular and non-vanishing around b. First, let $a^\sharp \in H_\infty$ i.e. $a_1 \neq \infty$. Then we put $y_1 = x_1 - a_1$, $y_2 = v_2, \ldots, y_n = v_n$ and $(y_1, \ldots, y_n)$ form local coordinates of $Y^\sharp$ centred at $b^\sharp$ and clearly defined over $k(b^\sharp)$ such that

$$\begin{aligned} f(x_1, \ldots, x_n) &= \epsilon (y_1 + a_1)^m \prod_{i>1} y_i^{N_i} \\ dx &= \eta (y_1 + a_1)^{n-1} \prod_{i>1} y_i^{\nu_i - 1} \cdot dy. \end{aligned}$$

Since $(x_1, \ldots, x_n)$ form local coordinates on $X^\sharp$ around $a^\sharp$ and $f(x_1, \ldots, x_n) = 0$ gives a local equation for $(f^\sharp)_0$, $h^\sharp$ has the required property at $b^\sharp$. On the other hand, let $a_1 = \infty$, then setting $y_1 = 1/x_1$, $y_2 = v_2, \ldots, y_n = v_n$, we see that $(y_1, u_2, \ldots, u_n)$ (respectively $(y_1, \ldots, y_n)$) form local coordinates on $X^\sharp$ (respectively $Y^\sharp$) around $a^\sharp$ (respectively centred at $b^\sharp$) and defined over $k(b^\sharp)$. Moreover, $f(1, u_2, \ldots, u_n) = 0$
159 (respectively $y_1 = 0$) gives local equations for $(f^\sharp)_0$ (respectively H_∞); also we have

$$\begin{aligned} f(1, u_2, \ldots, u_n) &= \epsilon \prod_{i>1} y_i^{N_i}, \\ dy_1 \wedge du_2 \wedge \ldots \wedge du_n &= \eta \prod_{i>1} y_i^{\nu_i - 1} \cdot dy \end{aligned}$$

This finally establishes the required property of $h^\sharp$ at $b^\sharp$.

We remark that the addition of (m, n) to the numerical data results if and only if $a_1 = 0$ i.e. if and only if $a^\sharp = h^\sharp(b^\sharp)$ has $(1, 0, \ldots, 0)$ as its homogeneous coordinates.

6.2 Key Lemmas

Let $f(x)$ be a polynomial in $x_1, \ldots, x_n$ with coefficients in a global field k, such that $C_f \subset f^{-1}(0)$. Defining X, $X^\sharp = X \cup H_\infty$ and $f^\sharp$ (the rational function extending f to $X^\sharp$) as in Lemma lem6.2, let us assume that a tame k-resolution $h^\sharp : Y^\sharp \to X^\sharp$ of $((f^\sharp)_0, H_\infty)$ exists. Then the restriction of $h^\sharp$ to $Y = f^{-1}(X)$ gives a tame k-resolution $h : Y \to X$ of the hypersurface $f(x) = 0$. In the following lemma, $\mathscr{D}(X_v)$ stands for the space of Schwartz-Bruhat functions on X_v with *compact support*; thus $\mathscr{D}(X_v) = \mathscr{S}(X_v)$ for non-archimedean v.

Lemma 6.3. *For given b in Y_v, choose $\phi_v \geq 0$ in $\mathscr{D}(X_v)$ with $\phi_v(h(b)) > 0$ and* (i) *assume that existence of $F_{\phi_v}(0)$ or* (ii) *make the stronger assumption that*

$$F_{\phi_v}(0) = \int\limits_{U(0)_v} \phi_v |\theta_0|_v. \tag{149}$$

Then h has property (P_0) or property (P) at b depending on (i) *or* (ii) *above. Conversely, if h has property (P) at every point of Y_v, then (149) holds for every ϕ_v in $\mathscr{D}(X_v)$; further, if $h^\sharp$ has property (P) at every point of $Y_v^\sharp$, then (149) holds for every ϕ_v in $\mathscr{S}(X_v)$.*

Proof. Using the notation of Chapter III, § 3.2, there exist, for the k_v-resolution $h : Y_v \to X_v$, local coordinates $y_1, \ldots, y_n$ of Y centred at b and defined over k_v such that **160**

$$f \circ h = \epsilon \prod_i y_i^{N_i}, \quad h^*(dx) = \eta \prod_i y_i^{\nu_i - 1} \cdot dy$$

with ϵ, η regular and non-vanishing around b. Now, $\epsilon, \eta, y_1, \ldots, y_n$ give rise to k_v-analytic functions on an open neighbourhood, say V, of b in Y_v.

By choosing V smaller still, if necessary, we may assume that already $\epsilon\eta$ is non-vanishing on V and also that, for the given ϕ_v.

$$|\epsilon(y)|_v^s|\eta(y)|_v\phi_v(h(y)) \geq c > 0$$

for a suitable constant c independent of y in V and s with $-1 \leq s \leq 0$. Then, for all such s,

$$Z_{\phi_v}(\omega_s) \geq c \int\limits_V \prod_i |y_i|_v^{N_i s+\nu_i-1}|dy|_v \tag{150}$$

On the other hand, from the existence of $F_{\phi_v}(0)$, we can conclude, in view of Chapter II, § 3.1, that the function $(s+1)Z_{\phi_v}(\omega_s)$, which is positive for $-1 < s \leq 0$, is also continuous and bounded in this range. On being multiplied by $s+1$, the right hand side of (150) is asymptotic to a constant multiple of $(s+1)/\prod\limits_i(N_i s+\nu_i)$ as $s \to -1$ from the right. Therefore, $\nu_i \geq N_i$ for every i and the equality $\nu_i = N_i$ can hold for at most one i. In other words, the existence of $F_{\phi_v}(0)$ entails that h has property (P_0) at b. □

We shall show next that (149) implies $\nu_{i_0} = N_{i_0} = 1$. For that purpose, we shall first obtain an expression for $F_{\phi_v}(0)$ without assuming 149. We take an arbitrary b' from Y_v at which $\phi_v(b') > 0$; then what we
161 have shown for b is applicable to b'. For the sake of simplicity, we shall write b again for b' and use the same notation as above. By changing indices, we may suppose that $i_0 = 1$. Let $\epsilon_1\left(N_1\epsilon + y_1\dfrac{\partial\epsilon}{\partial y_1}\right)^{-1}\eta$, locally around b in Y_v. Then, for every $i \neq 0$ in k, we have, $h^*(\theta_i)$ = the restriction of $\epsilon_1\prod_{j>1}y_j^{\nu_j-N_j-1}dy_2\wedge\ldots\wedge dy_n$ to $(f\circ h)^{-1}(i)$. Furthermore, if we denote E_1 by E, then the everywhere regular differential form θ on E (obtained by restriction as in Lemma (6.1) exists. If $E_v \neq \phi$, then E and θ are both defined over k_v. Moreover, θ gives rise to a positive Borel measure $|\theta|_v$ on E_v, with E_v as its exact support.

We may take the neighbourhood V of b above small enough so that $\epsilon_1 \neq 0$ on V. Then for every locally constant or C^∞ function φ_0 on V

with compact support, we have

$$\lim_{i\to 0}\int\limits_{(f\circ h)^{-1}(i)} \varphi_0|h^*(\theta_i)|_v = \int\limits_{E\cap V}\varphi_0|\theta|_v.$$

We know that h is biregular at every point of $h^{-1}(U(0))$; if $\nu_1 = N_1 = 1$ as long as only points of V are considered, we have

$$h^{-1}(U(0)) = E\setminus\cup \text{ Hyperplanes defined by } y_j = 0 \text{ for } N_j \geq 1.$$

Hence we get

$$\int\limits_{E\cap V}\varphi_0|\theta|_v = \int\limits_{h^{-1}(U(0))_v}\varphi_0|h^*(\theta_0)|_v.$$

We take a finite covering of the preimage, under h, of the support of ϕ_v, by means of open sets V as above. Choosing a partition of unity $(p_V)_V$ subordinate to this covering, we apply the observation above to each $\varphi_0 = (\phi_v\circ h)p_V$ and obtain

$$F_{\phi_v}(0) = \int\limits_{U(0)_v}\phi_v|\theta_0|_v + \sum_{\nu_E\geq 2}\int\limits_{E_v}(\phi_v\circ h)|\theta|_v.$$

But $|\theta|_v$ has E_v as its exact support and therefore, the validity of (149) **162**
now implies that no E with $\nu_E = N_E \geq 2$ can pass through a point b' where $\phi_v(b') > 0$. Therefore (149) implies that h has property (P) at b.

Conversely, suppose that h has property (P) at every point of Y_v. Then certainly (149) holds for every ϕ_v in $\mathscr{D}(X_v)$. If, further, h has property (P) at every $b^\sharp$ in $Y_v^\sharp$, then we choose local coordinates $y_1,\ldots,y_n$ on $Y^\sharp$ centred at $b^\sharp$ and defined over k_v such that, upto a regular and non-vanishing function around $b^\sharp$, $f^\sharp\circ h^\sharp$ becomes a product of (possibly negative) powers of $y_1,\ldots,y_n$ and further, $(h^\sharp)^*(dx)$ is another such power-product in $y_1,\ldots,y_n$ multiplied by a local gauge form around $b^\sharp$. Suppose, for example, the exponent of y_i in $f^\sharp\circ h^\sharp$ is negative. Then the "infinite divisibility" of ϕ_v by a local equation for H_∞ implies the infinite divisibility of $\phi_v\circ h^\sharp$ by y_i. Thus the component with $y_i = 0$ as a local equation becomes negligible and (149) holds for every ϕ_v in $\mathscr{S}(X_v)$.

In the sequel, we shall assume as in the beginning of § (6), that k is a global field, $f(x)$ in $k[x_1, \ldots, x_n]$ is homogeneous of degree $m(\geq 2$ and) not divisible by the characteristic of k and further, a tame k-resolution h_0 of the projective hypersurface defined by $f(x) = 0$ exists. We shall also use the notation of Lemma (6.2).

Lemma 6.4. *If F_v^* is in $L^1(k_v)$ for all but finitely many valuations v of k, then $h^\sharp$ has property (P) everywhere.*

Proof. Since h_0 is tame by assumption, all irreducible components of the total transform on Y_0 of the projective hypersurface defined by $f(x) = 0$ are defined over the separable closure k_s of k. In view of the quasi-compactness of the Zariski topology and the Zariski density
163 of the set of k_s-rational points on any irreducible variety defined over k_s, we can choose a finite number of k_s-rational points of Y_0 such that the set of the numerical data at these points contains the numerical data at every point of Y_0. Let k_1 denote the extension field of k obtained by adjoining (the coordinates of) all the finitely many points mentioned above; then k_1 is a finite separable extension of k and as a result, there exist infinitely many non-archimedean valuations on k_1 with degree 1 relative to k (This is a consequence of the face that the zeta function $\zeta_k(s)$ of a global field is holomorphic for Re $(s) > 1$ and has a pole at $s = 1$). We can choose one such valuation so that its restriction, say w, to k is not one of the finitely many valuations excluded in the hypothesis; then the field k_1 is contained in k_w and F_w^* is in $L^1(k_w)$. We have thus achieved a situation where the numerical data of $h^\sharp$ at any point of $Y^\sharp$ turn out to be also the numerical data of h at some point of $Y_v \cap h^{-1}(X_w^0)$. Since F_w^* is in $L^1(k_w)$, $F_v(0)$ exists and by Lemma 6.3, h has property (P_0) at every point of $Y_w \cap h^{-1}(X_w^0)$. But, then, by the construction, $h^\sharp$ has property (P_0) everywhere and hence, by Lemma 6.1, has property (P) everywhere and the proof is complete. □

As a consequence, by the last assertion in Lemma 6.3, (149) holds good, implying that $F_{\phi_v}^*$ is in $L^1(k_v)$ for every ϕ_v in $\mathscr{S}(X_v)$ and for every valuation v of k.

6.3 Standard Lemmas

For a non-archimedean valuation v of k and the characteristic function ϕ_v of X_v^0, we shall denote F_{ϕ_v} simply by F_v; then $F^*_{\phi_v} = F^*_v$ is the Fourier transform of F_v. Further, if F^*_v is in $L^1(k_v)$, then

$$F_v(i) = \int\limits_{k_v} F^*_v(i^*)\psi_v(-ii^*)|di^*|_v,$$

for every i in k_v. In the following two lemmas, we shall assume that v is non-archimedean, $\psi_v = 1$ on R_v, ψ_v is non-constant on P_v^{-1} and further **164**
$f(x) \in R_v[x_1, \ldots, x_n]$. Anyway, these conditions hold good for all but finitely many v.

Lemma 6.5. *If there exists $\sigma > 1$ such that $|F^*_v(i^*)| \leq |i^*|_v^{-\sigma}$ for every i^* in $k_v \backslash R_v$, then, for every i in R_v, we have*

$$|F_v(i) - 1| \leq c, q_v^{-(\sigma-1)}$$

for a constant c depending only on σ.

Proof. By our assumptions on v above, $F^*_v = 1$ on R_v. Since X_v^0 has measure 1 and since F^*_v is in $L^1(k_v)$ in view of the hypothesis, we have

$$F_v(i) - 1 = \int\limits_{k_v \backslash R_v} F^*_v(i^*)\psi_v(-ii^*)|di^*|_v$$

for every i in R_v. Thus

$$\begin{aligned}
|F_v(i) - 1| &\leq \int\limits_{k_v \backslash R_v} |i^*|_v^{-\sigma}|di^*|_v \\
&= (1 - q_v^{-1})(1 - q^{-(\sigma-1)})^{-1-(\sigma-1)}q_v \\
&\leq (1 - 2^{-(\sigma-1)})^{-1}q_v^{-(\sigma-1)}
\end{aligned}$$

and the lemma is proved. □

For i in R_v, we recall that $U(i)_v^0$ is a compact subset of $U(i)_v$ defined as follows. Namely, if $i \neq 0$, then $U(i)_v^0$ is just $U(i)_v \cap X_v^0$ and for $i = 0$, it is the subset of $U(i)_v \cap X_v^0$, where, in addition, $\mathrm{grad}_x f \not\equiv 0 (\mod P_v)$. We also recall that r is the codimension of C_f in $f^{-1}(0)$.

165 **Lemma 6.6.** *Under the same assumptions as in LEMMA 6.5 , we have, for every i in R_v,*

$$\left| \int\limits_{U(i)_v^0} |\theta_i|_v - 1 \right| \leq c'(q_v^{-(\sigma-1)} + q_v^{-r}) \tag{151}$$

with a constant c' independent of i and v.

Proof. Since the left hand side of (151) is at most equal to $|F_v(i)| + 1 \leq c+2$ (in view of Lemma 6.5), we may exclude some more finitely many valuations while proving this lemma. We may, in particular, assume that $m \not\equiv 0(\mod P_v)$. If $N_e(i)$ (respectively $N_e^0(i)$) is the number of ξ in X_v^0 modulo P_v^e such that $f(\xi) \equiv i(\mod P_v^e)$ (respectively $f(\xi) \equiv i(\mod P_v^e)$ and $\mathrm{grad}_\xi f \not\equiv 0 \pmod{P_v}$), for $e = 0, 1, 2, \ldots$ then

$$q_v^{-(n-1)e} N_e(i) = \int\limits_{P_v^{-e}} F_v^*(i^*)\psi_v(-ii^*)|di^*|_v.$$

As in the proof of Lemma 6.5, we have

$$|q_v^{-(n-1)} N_1(i) - 1| \leq c q_v^{-(\sigma-1)}$$

for a constant c independent of i and v. But it is elementary to prove that $N_1(i) = N_1^0(i)$ for $i \notin P_v$ and further that $q_v^{-(n-1-r)}(N_1(i) - N_1^0(i))$ is bounded uniformly for i in R_v and all v ([31]). Also, we have $q_v^{-(n-1)} N_1^0(i) = \int\limits_{U(i)_v^0} |\theta_i|_v$ for all but finitely many v. Putting these together, the estimate (151) follows. ◻

Lemma 6.7. *Suppose that, for every v in S_∞, $\sigma_v > 0$ and further that, for every i^* in k_v and every ϕ_v in $\mathscr{S}(X_v)$, we have*

$$|F_{\phi_v}^*(i^*)| \leq c(\phi_v) \max(1, |i^*|_v)^{-\sigma_v}$$

166 *with a constant $c(\phi_v) > 0$. Let us write*

$$X_\infty = \prod_{v\in S_\infty} X_v, \quad |dx|_\infty = \bigotimes_{v\in S_\infty} |dx|_v, \quad k_\infty = \prod_{v\in S_\infty} k_v$$

and let C_∞ denote a compact subset of $\mathscr{S}(X_\infty)$. Then there exists a constant $c \geq 0$ depending on C_∞ such that

$$\left| \int\limits_{X_\infty} \left(\prod_{v\in S_\infty} \psi_v(i_v^* f(x)) \right) \phi_\infty(x) |dx|_\infty \right| \leq c \prod_{v\in S_\infty} \max(1, |i^*|_v)^{-\sigma_v} \tag{152}$$

for every $i^* = (i_v^*)_v$ in k_∞ and every ϕ_∞ in C_∞.

Proof. We first remark that, on the left hand side of 152, ϕ_∞ is not necessarily of the form $\bigotimes_{v\in S_\infty} \phi_v$ (with ϕ_v in $\mathscr{S}(X_v)$); this is just the "raison d'etre" for this lemma.

If we put

$$T_{i^*}(\phi_v) = \max(1, |i^*|_v)^{\sigma_v} \cdot F^*_{\phi_v}(i^*),$$

then we get a subset $\{T_{i^*}\}_{i^*}$ of $\mathscr{S}(X_v)'$ parametrized by k_v. The hypothesis means precisely that this subset is "bounded" i.e. $|T_{i^*}(\phi_v)| \leq c(\phi_v)$ for every i^* in k_v. The proof of the lemma now follows from the two well-known facts given below:

(i) For any bounded subset B of $\mathscr{S}(\mathbb{R}^p)'$ and any compact subset C of $\mathscr{S}(\mathbb{R}^p)$, we have

$$\sup_{T\in B, \phi\in C} |T(\phi)| < \infty;$$

(ii) For bounded subsets B, B' respectively of $\mathscr{S}(\mathbb{R}^p)'$, $\mathscr{S}(\mathbb{R}^q)'$, the 167
set $\{T \otimes T'; T \in B, T' \in B'\}$ is bounded in $\mathscr{S}(\mathbb{R}^{p+q})'$

□

Lemma 6.8. *Let $\sigma_v > 1$ for all v and further, for all but finitely many v, let $\sigma_v \geq \sigma > 2$. Then*

$$\sum_{i^*\in k} \prod_v \max(1, |i^* + j^*|_v)^{-\sigma_v}$$

is bounded as j^* varies over k_A.

Proof. Since k_A/k is compact, we may restrict j^* to a compact subset of k_A. If v is non-archimedean and if j^*_v is in R_v, then

$$\max(1, |i^* + j^*_v|_v) = \max(1, |i^*|_v)$$

for every i^* in k_v. We may therefore assume $j^* = 0$ in the foregoing. In the special case, when $k = \mathbb{Q}$ and $\sigma_v = \sigma(> 2)$ for all v, we have

$$\prod_v \max(1|i^*|_v) = \max(c, |d|)$$

for $i^* = d/c$ with $c \geq 1$ and $(c, d) = 1$. By a simple calculation, it can be shown that

$$\sum_{i^* \in \mathbb{Q}} \prod_v \max(1, |i^*|_v)^{-\sigma} = \zeta(\sigma)(4\zeta(\sigma - 1) - \zeta(\sigma)) < \infty$$

where $\zeta(s)$ is the Riemann zeta function. For the case of any global field k, the proof can be found in [19], page 187 and [21], page 226. ◻

6.4 Proof of Theorem 2.1

We are now ready to prove that the criteria (C1) and (C2) imply the validity of the Poisson formula.

168 From (C1), (C2) and Lemma 6.6, we see that (PF-2)′ is valid i.e. the restricted product measure $|\theta_i|_A$ exists on $U(i)_A$, for every i in k. Further, by Lemma 6.5, (PF-2)′′ is valid i.e. the image measure under $U(i)_A \to X_A$ is tempered. On the other hand, for every ϕ in $\mathscr{S}(X_A)$ and every i^* in k_A, we set

$$F^*_\phi(i^*) = \int_{X_A} \phi(x)\psi(i^* f(x))|dx|_A.$$

We recall that ϕ is a finite sum of functions of the form $\phi_0 \otimes \phi_\infty$, in our earlier notation. We restrict ϕ_∞ to compact subset C_∞ of $\mathscr{S}(X_\infty)$; then there exists a constant $c \geq 0$ depending on ϕ_0 and C_∞ such that

$$\sum_{i^* \in k} |F^*_\phi(i^* + j^*)| \leq c \sum_{i^* \in k} \prod_v \max(1, |i^* + j^*|_v)^{-\sigma} < \infty$$

for every j^* in k_A, as a consequence of (C2) and Lemmas 6.3, 6.4, 6.7, 6.8. This establishes the validity of (PF-1) i.e. the absolute convergence of the Eisenstein-Siegel series. Since $\sum_{i^* \in k} F^*_\phi(i^* + j^*)$ represents a continuous periodic function of j^*, we get the Fourier expansion

$$\sum_{i^* \in k} F^*_\phi(i^* + j^*) = \sum_{i \in k} c_\phi(i)\psi(ij^*) \tag{153}$$

where

$$c_\phi(i) = \int_{k_A} F^*_\phi(j^*)\psi(-ij^*)|dj^*|_A,$$

provided that the series on the right hand side of (153) converges absolutely. This is certainly the case if the series is just a finite sum.

If ϕ is of the form $\bigotimes_v \phi_v$ with ϕ_v in $\mathscr{S}(X_v)$ for every v and further ϕ_v equal to the characteristic function of X^0_v for all but finitely many v, then (C2) together with Lemmas 6.3, 6.4, 6.5 implies that **169**

$$\begin{aligned} c_\phi(i) &= \int_{k_A} F^*_\phi(i^*)\psi(-ii^*)|di^*|_A \\ &= \prod_v \int_{k_v} F^*_{\phi_v}(i^*)\psi_v(-ii^*)|di^*|_v \\ &= \prod_v F_{\phi_v}(i) \\ &= \int_{U(i)_A} \Psi|\theta_i|_A. \end{aligned}$$

Therefore, this expression $c_\phi(i)$ is valid for every ϕ in $\mathscr{S}(X_A)$, by continuity and linearity. Now putting $j^* = 0$ in 153, we deduce that

$$\sum_{i^* \in k} F^*_\phi(i^*) = \sum_{i \in k} \int_{U(i)_A} \phi|\theta_i|_A \tag{154}$$

provided that (PF-3) holds i.e. the right hand side of (154) is absolutely convergent. As remarked earlier, this is certainly true, if the series is a

finite sum, as, for example, in the case of ϕ having compact support. Let us now consider the general case of ϕ in $\mathscr{S}(X_A)$, not necessarily having compact support. For any such ϕ, we know that there exists $\mathscr{S} \geq 0$ in $\mathscr{S}(X_A)$ such that $|\phi(x)| \leq \varphi(x)$ for every x in X_A. It is sufficient therefore to show that the right hand side of (152) converges for every $\phi \geq 0$ in $\mathscr{S}(X_A)$. Let us therefore assume that $\phi \geq 0$ and moreover, take ϕ to be of the form $\phi_0 \otimes \phi_\infty$ where ϕ_0 is the characteristic function of a large compact open subset of X_0 and $\phi_\infty \geq 0$ in $\mathscr{S}(X_\infty)$. We can easily find a monotone increasing sequence $\{\phi_{\infty,n}\}_n$ of non-negative C^∞ functions on X_∞ with compact support which converges to ϕ_∞ in $\mathscr{S}(X_\infty)$; then
170 $\{\phi_{\infty,n}\}_n \cup \phi_\infty$ is a compact subset of $\mathscr{S}(X_\infty)$. We put $\phi_n = \phi_0 \otimes \phi_{\infty,n}$ for $n = 1, 2, 3, \ldots$,; then every ϕ_n is in $\mathscr{S}(X_A)$ and has compact support. Further the monotone increasing sequence $0 \leq \phi_1 \leq \phi_2 \leq \ldots$ converges to ϕ. Therefore, for a constant $c \geq 0$, we have, in view of Lemma 6.7,

$$\sum_{i \in k} \int_{U(i)_A} \phi_n |\theta_i|_A \leq \sum_{i^* \in k} |F^*_{\phi_n}(i^*)| \leq c < \infty$$

for $n = 1, 2, 3, \ldots$. Since the left hand side is the integral of ϕ_n over the (disjoint) union of $U(i)_A$ for all i in k, we obtain, by Lebesgue's theorem on a monotone increasing sequence of non-negative functions, that

$$\sum_{i \in k} \int_{U(i)_A} \phi |\theta_i|_A = \lim_{n \to \infty} \sum_{i \in k} \int_{U(i)_A} \phi_n |\theta_i|_A \leq c.$$

The proof of the validity of the Poisson formula is now complete.

REMARK. We observe that the proof of our theorem becomes much simpler in the function-field case, since the consideration of X_∞ disappears. We might, moreover, mention that we could have applied to the left hand side of (153), the result of Weil on page 7 of [52] and thereby avoided the use of the monotone sequence $\{\phi_n\}_n$.

6.5 A Conjecture

We conclude this series of lectures by stating a conjecture: namely, if, with the notation of Lemma 6.2, we assume that the resolution

$h^\sharp : Y^\sharp \to X^\sharp$ of $((f^\sharp)_0, H_\infty)$ is tame and further that $\lambda_1 = \min_{\nu_E \geq 2}(\nu_E/N_E) > 1$, then for any $\sigma < \lambda_1$, we have

$$|F^*_{\phi_v}(i^*)| \leq c(\phi_v) \max(1, |i^*|_v)^{-\sigma}$$

for every ϕ_v in $\mathscr{S}(X_v)$, all i^* in k_v and all v. It is very likely that *we can take* $c(\phi_v) = 1$ *for the characteristic function* ϕ_v *of* X_v^0*, for all but finitely* **171**
many v. If this conjecture is true, then condition (C2) can be replaced by the simple geometric condition $\lambda_1 > 2$.

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