C. P. Ramanujam

A Tribute

C.P. RAMANUJAM

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Preface

This collection of the publications of Chidambaram Padmanabhan Ramanujam and of the several papers, in his memory, by his friends and colleagues is intended to be a tribute to a distinguished colleague.

C.P. Ramanujam joined the School of Mathematics of the Tata Institute of Fundamental Research as a Research Student in 1957. In his meteoric career he had done brilliant work in Number Theory and Algebraic Geometry. He was, without doubt, one of the outstanding young Indian mathematicians of the last twenty years. Gifted with a remarkably deep and wide mathematical culture, he was at home as much with abstract Algebraic Geometry as with analytic methods of Number Theory. A conversation with him was a rewarding experience for students and colleagues alike. He passed away, in Bangalore, when he was hardly 37 years old.

I should like to thank all those who contributed papers to this volume, and in addition Jal R. Choksi, K. Gowrisankaran, Pavaman Murthy, Raghavan Narasimhan and K. Srinivasacharyulu for their assistance. I am also grateful to C.P. Rajagopal, brother of Ramanujam, for some biographical details concerning Ramanujam.

K.G. Ramanathan

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(No. 2)

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1. Cubic forms over algebraic number fields : *Proceedings of the Cambridge Philosophical Society,* 59 (1963) 683-705 21
2. Sums of *m*-th powers in *p*-adic rings : *Mathematika*, 10 (1963), 137-146 59
3. A note on automorphism groups of algebraic varieties *Mathematische Annalen,* 156 (1964), 25-33 73
4. On a certain purity theorem : *Journal of the Indian Mathematical Society,* 34 (1970), 1-10 87
5. A topological characterisation of the affine plane as an algebraic variety : *Annals of Mathematics*, 94 (1971), 69-88 97
6. Remarks on the Kodaira Vanishing Theorem : *Journal of the Indian Mathematical Society,* 36 (1972), 41-51 123
7. The invariance of Milnor's number implies the invariance of the topological type (jointly with Le Dung Trang) *American Journal of Mathematics*, 98 (1976), 67-78 135

8. On a geometric interpretation of multiplicity : *Inventiones Mathematicae*, 22 (1973), 63-67 151
9. Supplement to the article "Remarks on the Kodaira Vanishing Theorem" : *Journal of the Indian Mathematical Society:* 38 (1974), 121-124 159
10. Appendix to the paper of C.S. Seshadri entitled "Quotient space by an abelian variety" : *Mathematische Annalen*, 152 (1963), 192-194 165
11. The theorem of Tate, Appendix I to the book entitled "Abelian Varieties" by D. Mumford Tata Institute of Fundamental Research Studies in Mathematics, (1974), 240-260 169

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C.P. Ramanujam

By K.G. Ramanathan

Chidambaram Padmanabhan Ramanujam was born in Madras on January 9, 1938, as the eldest son of Shri C. Padmanabhan, an advocate in the High Court, Madras. Ramanujam had his primary and secondary education at Ewart's School, Madras and later joined the Sir M. Ct. Muthiah Chetty High School at Vepery, Madras. He passed his High School examination at the early age of 14 in 1952 and joined, as many talented young students in Madras did, the Loyola College, Madras, for his Intermediate and then the Mathematics Honours. He passed out of the Loyola College with the B.A. (Hons.) degree in 1957 with a second class which was unusual for a gifted student like him. During his High School and College days he was interested in Chemistry and Tennis. He had set up, in his house, a small laboratory in Chemistry and would perform experiments along with his friend Raghavan Narasimhan, now at Chicago University. The Laboratory was closed down after the 'inevitable', namely, a small accident. 1

During his Honours years he came under the guidance of the late Reverend Fr. C. Racine of the Loyola College with whom he kept up regular correspondence almost until his (Ramanujam's) death. He had, as many students of Fr. Racine's had, a great regard and respect, bordering on affection for Fr. Racine. On Fr. Racine's suggestion he applied in 1957 for admission to the School of Mathematics of the Tata Institute of Fundamental Research at Bombay. In his letter to the Institute recommending Ramanujam, Fr. Racine wrote, "He has certainly originality of mind and the type of curiosity which is likely to suggest that he will develop into a good research worker if given sufficient opportunity". Ramanujam amply justified Fr. Racine's estimate of him. Ramanujam,

before he entered the Tata Institute, also came, briefly, under the influence of the late Professor T. Vijayaraghavan, the eminent mathematician who was, until 1954, the Director of the Ramanujan Institute of Math-
2 ematics at Madras. Ramanujam (and his friend Raghavan Narasimhan who also joined the Tata Institute in 1957) displayed a deep knowledge of Analytic Number Theory even at the time of joining the Tata Institute.

At the Tata Institute he learnt mathematics with an avidity and speed that was often frightening. He had given expert Colloquium talks and participated in seminars and displayed within two years of stay, versatility and depth in mathematics which are rare. Unfortunately however, he soon found himself in possession of a vast amount of mathematical expertise which he could not 'cash in', by solving problems of importance. The success of a few others at the Institute led to great frustration and Ramanujam decided to leave the Institute. He felt, wrongly of course, that he was inadequate in solving problems in mathematics and that he would be, perhaps, a better teacher in a university or a college. He began applying to various colleges and universities in India and it was lucky for the Tata Institute and Ramanujam that he was not selected in any of the places he had applied to.

It was then that he came to work with me and he started with a problem relating to Lie groups and Differential geometry connected with the work of C.L. Siegel. Early in 1961 he took up the problem regarding Diophantine equations especially related to those over algebraic number fields. The important problem was that raised by C.L. Siegel regarding the generalisation of Waring's problem to algebraic number fields, namely, to find a $g = g(m)$ *independent of the degree of the algebraic number field* K, so that every totally positive integer v with sufficiently large norm, which is in the ring $J_m(K)$ generated by mth powers of integers of K can be written

$$v = x_1^m + \cdots + x_g^m$$

as the sum of mth powers of totally positive integers x_i. At that time many significant results were being obtained by Davenport and his co-
3 workers D.J. Lewis and B.J. Birch. Davenport had proved that every cubic form with rational coefficients in at least $g = 32$ variables had

a non-trivial rational zero. Ramanujam set about the task of trying to generalise Davenport's method to cubic forms over algebraic number fields by using Siegel's generalisation of the major and minor arcs of the Hardy-Littlewood-Ramanujan circle method. In this attempt he succeeded eminently by first simplifying Siegel's method and then proving that every cubic form in 54 variables over any algebraic number field K had a non-trivial zero over that field. Ramanujam himself knew that the number 54 could be considerably reduced, even to Davenport's 29; but he was not interested in doing this. Recently Ramanujam's idea of proof was adapted by Pleasants who showed that Ramanujam's result holds with $g = 16$.

However, Waring's problem in algebraic number fields was always at the back of his mind. In the meantime some very interesting results had been published. Rose Marie Stemmler had written a thesis on the easier Waring's problem in algebraic number fields, namely, of expressing $v > 0$ in the form $x_1^m \pm x_2^m \pm \cdots \pm x_g^m$. She showed that g can be bounded by an integer depending only on m and *not* on the degree of the field K provided m is, for instance, a prime number. Birch then used a generalisation of Hua's mean value theorem to show that indeed the method of Stemmler gives Siegel's conjecture in the case m is a prime p, and that every totally positive integer of sufficiently large norm, in $J_p(K)$ is indeed a sum of at most $2^p + 1$ totally positive pth powers. The extension of this result for arbitrary natural number m was tied up with obtaining a local to global theorem as was shown by Birch and Körner. Ramanujam attacked the general problem of representation of elements of $J_m(A)$, the ring generated by mth powers of elements of a complete discrete valuation ring A, as sums of a fixed numbers of mth powers. He showed that whatever A might be, $8m^5$ summands would be enough. Birch had shown, almost simultaneously, that if $s > 2^m + 1$, then every integer of sufficiently large norm which is sum of s, mth powers modulo all powers of prime ideals, is itself such a sum of mth powers of integers in K. The problem raised by Siegel was thus solved. Independently, at about the same time, Birch also solved this p-adic problem by an entirely different method. 4

Ramanujam was promoted as Associate Professor for his brilliant

work on Number Theory. He protested very strongly against this but was prevailed upon by friends and colleagues to accept the position. This promotion, besides being deserved in his case, also showed, as was the intention of the School of Mathematics, that promotion to any position in the Institute was possible on just brilliant accomplishments. He wrote his thesis in 1966 and early in 1967 he took his Doctoral examination at which Carl. L. Siegel was one of the examiners. Siegel very greatly appreciated not only Ramanujam's knowledge in Number Theory but also his general mathematical culture especially in Analysis.

Ramanujam had attended during the first few years of his stay at the Institute a large number of courses of lectures by visiting Professors. He wrote the notes of Lectures of Max Deuring during 1958–59 on "the theory of algebraic functions of one variable" and in 1964–65 of the lectures of I.R. Shafarevich on "minimal models and birational transformations of two dimensional schemes". Professor Shafarevich was very appreciative of Ramanujam's contributions to the Lecture notes. He wrote, "I want to thank him (*Ramanujam*) for the splendid job he has done. He not only corrected several mistakes but also complemented proofs of many results that were only stated in oral exposition. To mention some of them, he has written the proofs of the Castelnuovo theorem ... of the chain condition ..., the example of Nagata of a non-projective surface ... and the proof of Zariski's theorem ...". In the case of Mumford's lectures on Abelian Varieties which he gave in 1968 at the Tata Institute, Mumford wrote, "...these lectures were subsequently written up, and improved in many ways, by C.P. Ramanujam. The present text is the result of a joint effort", and further, "... C.P. Ramanujam continuing my lectures at the Tata Institute lectured on and wrote up notes on Tate's theorem on homomorphisms between abelian varieties over finite fields".

5 The contacts with Shafarevich and Mumford led him on to Algebraic Geometry and his progress and deep understanding in this field was phenomenal. Of his work in Algebraic Geometry, Mumford has written an excellent account which also is included in this volume. His influence on the members of the School of Mathematics was indeed very great. He was a source of inspiration and of ideas to the members of the School

of Mathematics. He used to hold seminars on Commutative Algebra and on aspects of Algebraic Geometry. To talk to him on mathematics, especially Algebraic Geometry, was to get one's ideas clarified and sometimes get one's problems solved. He was very generous in giving his ideas, especially to the younger members of the School.

In 1964 he participated in the International Colloquium on Differential Analysis at the Institute and in 1968, the International Colloquium on Algebraic Geometry at which Grothendieck and Mumford, among others, greatly admired Ramanujam's knowledge and virtuosity. Both of them became fast friends and admirers of his and invited him to Paris and Harvard.

It was in 1964 that he had the first of the many attacks of the cruel malady that was ultimately to take his life. The ailment was diagnosed by the medical officer to be "Schizophrenia with severe depression". This depression made him feel that he was "inadequate" for mathematical research and he therefore decided to take a position in a University. In July 1965 he was offered a Professorship, with tenure, at the Panjab University in Chandigarh. However, he went there only for a year. He was immensely liked by colleagues and students there, but the illness struck him again and amidst tragic circumstances he had to cut short his stay there after about 8 months. Shortly after he rejoined the Tata Institute in June 1965, he received an invitation from the Institut des Hautes Etudes Scientifiques, Paris, to spend six months there. The Tata Institute deputed him to Bures-Sur-Yvette with full salary but again he returned before the projected period of stay was over, since the cruel illness struck him. I still remember his telephoning to me after arrival at Bombay airport—sans his baggage—saying that he would write to me **6** about his 'whereabouts in a few days'. Such bouts with his unfortunate illness continued throughout his short life. In February 1970, he wrote to the Director of the Institute, "due to personal reasons, I have to leave the Institute immediately. Please accept this letter of resignation". The Director, rightly, refused to accept the resignation and wrote that "in his disturbed state of mind the above cannot be treated straightaway as meaningful". However, he later resigned from the Institute and went for some time to Warwick University, at the instance of Mumford during

the Algebraic Geometry year there. He also visited Italy for some time and made many friends there. When he came back, he was requested to accept a Professorship at the Tata Institute of Fundamental Research but stay at Bangalore where the Institute had just started the programme in applications of mathematics. He lectured there on Analysis regularly and won great respect from the students as well as from many members of the Indian Institute of Science.

During the years he stayed at the Institute he had the attacks of his cruel illness several times. At those times, he resolved to take his life and it appears that once he *did* make a serious effort to end his life but the attempt was discovered and he was treated promptly. Due to great frustration he again decided to leave Bangalore and sever his connexion with the Tata Institute and had written several letters to me about it. He was recommended to the Indian Institute for Advanced Studies at Simla for a permanent Professorship. He died on October 27, 1974, after taking an overdose of barbiturates, before the offer was made by that Institute.

During his Bangalore days he became increasingly interested in mysticism and especially in 'miracles' which were being talked about in Bangalore at that time. He had many discussions with me about these 'miracles' and he said that he would investigate the truth about these miracles. He wanted to do this, both because of his intense curiosity and also because he was interested in finding out whether his malady
7 could be cured through these methods. He was becoming somewhat 'religious-minded' and began also chanting sacred hymns.

Ramanujam was intensely human in his relations with others and honest to a fault both in mathematics and in his dealings with fellow mathematicians and students—qualities that are rare, more so now-a-days. He was modest and self-effacing. In mathematics he had very high standards, though he would not make caustic remarks about the work of others especially if it concerned a young man who was trying to come up. He would always, as he told me many a time, think about himself and his own work, which curiously enough, he felt, was incommensurate with his position at the Institute. However, when it came to promotions he applied very strict standards. He was an understanding

colleague, warm-hearted, and an evening with him could be rewarding mathematically and socially. He liked high class classical Indian music and loved books. Buying books, mostly on mathematics but sometimes on art, mysticism etc., especially first editions or rare books became an obsession with him. His enormous private library of mathematical books was bought by the Tata Institute of Fundamental Research.

Ramanujam was a very good lecturer and loved his students. His colleagues and friends loved him and got great benefit in mathematics from discussion with him. He was elected Fellow of the Indian Academy of Sciences in 1973. In his death the country in general, and the Tata Institute in particular, has lost an outstanding mathematician, a warm colleague and a great human being. He is mourned by one and all.

The work of C.P. Ramanujam in Algebraic Geometry

By D. Mumford

It was a stimulating experience to know and collaborate with C.P. Ramanujam. He loved mathematics and he was always ready to take up a new thread or to pursue an old one with infectious enthusiasm. He was equally ready to discuss a problem with a first year student or a colleague, to work through an elementary point or to puzzle over a deep problem. On the other hand, he had very high standards. He felt the spirit of mathematics demanded of him not merely routine developments but the *right* theorem on any given topic. He wanted mathematics to be beautiful and to be clear and simple. He was sometimes tormented by the difficulty of these high standards, but, in retrospect, it is clear to us how often he succeeded in adding to our knowledge, results both new, beautiful and with a genuinely original stamp. **8**

Our lives and researches intertwined considerably. I first met him in Bombay in 1967–68, when he took notes on my course in Abelian Varieties and we worked jointly on refining and understanding better many points related to this theory. Later, in 1970–71, we were together in Warwick where he ran seminars on étale cohomology and on classification of surfaces. His excitement and enthusiasm was one of the main factors that made that "Algebraic Geometry year" a success. We discussed many topics involving topology and algebraic geometry at that time, and especially Kodaira's Vanishing Theorem. My wife and I spent many evenings together with him, talking about life, religion and customs both in India and the West and we looked forward to a warm and continuing friendship. His premature death was a great shock to all who knew him. I will always miss his companionship and collaboration in

the enterprise of mathematics.

I will give a short survey of his contributions to algebraic geometry. Perhaps his most perfect piece of work is his proof that a smooth
9 affine complex surface X, which is contractable *and* simply connected at ∞, is isomorphic to the plane C^2. The proof of this is not simple and uses many techniques; in particular, it shows how well he knew his way about in the classical geometry of surfaces! What is equally astonishing is his very striking counter-example showing that the hypothesis "simply connected at ∞" cannot be dropped. The position of this striking example in a general theory of 4-manifolds and particularly in a general theory of the topology of algebraic surfaces is yet to be understood. As mentioned above, the Kodaira Vanishing Theorem was an enduring interest of his. Both of us were particularly fascinated by this "deus ex machina", an intrusion of analytic tools (i.e. harmonic forms) to prove a purely algebraic theorem. His two notes on this subject went a long way to clarifying this theorem:

(a) he proves it by merely topological, not analytic, techniques and

(b) he finds a really satisfactory definitive extension of the theorem to a large class of non-ample divisors on surfaces.

This second point is absolutely essential for many applications and was used immediately and effectively by Bombieri in his work on the pluricanonical system $|nK|$ for surfaces of general type. His result is that if D is a divisor on X, such that $(D^2) > 0$ and $(D.C) \geq 0$ for all effective curves C, then $H^1(X, \mathcal{O}(-D)) = (0)$.

His earliest paper, on automorphisms group of varieties, is a definitive analysis of the way this group inherits an algebraic structure from the variety itself. This work employs the techniques of functors, e.g., families of automorphisms developed by Gröthendieck at about the same time. His paper "On a certain purity theorem" addresses itself to a question of Lang that puzzled almost all algebraic geometers at that time: *given a proper surjective morphism $f : X \to Y$ between smooth varieties, is the set*

$$\left\{ y \in Y \mid f^{-1}(Y) \text{ singular} \right\}$$

of codimension 1 *in Y?* Here he provides a topologico-algebraic analysis of one good case where it is true, and describes a counter example to the general case worked out jointly with me. We again see his fascination with the interactions between purely topological techniques and algebro-geometric ones.

This interest comes out again in his joint paper with Le Dung-Trang, **10**
whose Main Result is described in the title: "The invariance of Milnor's number implies the invariance of the topological type". Here they are concerned with a family of hypersurfaces in C^{n+1} : $F_t(z_0, \ldots, z_n) = 0$, with isolated singularities at the origin, whose coefficients are C^∞ functions of $t \in [0, 1]$. They show that when Milnor's number μ_t, giving the number of vanishing cycles at the origin, is independent of t, then if $n \neq 2$, the germs of the maps $F_t : C^{n+1} \to C$ near 0 are independent of t, up to homeomorphism (if $n = 2$, they get a slightly weaker result). A beautiful and intriguing corollary is that the Artin local ring

$$C[[z_0, \ldots, z_n]]/\left(F, \frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n}\right)$$

already determines the topology of the map F near 0.

Finally, his paper "On a geometric interpretation of multiplicity" proves essentially the following elegant theorem: *If $Y \subset X$ is a closed subscheme defined by $I_Y \subset \mathcal{O}_X$, which blows up to a divisor $E \subset X'$, then*

$$\frac{(-1)^{n-1}(E^n)}{n!} = \left[\begin{array}{c}\textit{leading coefficient of the polynomial} \\ P(k) = X\left(\mathcal{O}_x/I_Y^k\right), k \gg 0\end{array}\right].$$

In addition to these published papers, Ramanujam made many contributions to my book "Abelian Varieties", while writing up notes from my lectures. Reprinted here is the Appendix by him on Tate's Theorem on abelian varieties over finite fields; and the following extraordinary theorem: It had been proven by Weil that if X is a projective variety and $m : X \times X \to X$ is a morphism, then if m makes X into a group, m must satisfy the commutative law too. Ramanujam proved that if m merely possessed a 2-sided identity ($m(x, e) = m(e, x) = x$), then m must

also have an inverse and satisfy the associative law, hence make X into a group!

A Note on C.P. Ramanujam

By S. Ramanan

For sheer elegance and economy, I have come across few mathematicians who were C.P. Ramanujam's equal. He made so many remarks which clarified and threw light on different branches of mathematics that personally I derived immense mathematical pleasure in his company. I list below three of his unpublished mathematical comments which exemplify his mathematical style. **11**

1. *Any two points of an irreducible variety can be connected by an irreducible curve*. To prove this one notices that by Chow's lemma, one may assume that the variety in question is projective. Then one could blow up the two points and in any projective imbedding of this blow-up, take a generic hyperplane section which is irreducible of lower dimension. Since this hyperplane meets the two exceptional divisors, the problem reduces to one of lower dimension and hence proves the assertion by induction.

2. In the early days of algebraic K-theory, M.P. Murthy asked me if I could think of any example to show that $SL(2,A) \to SL(2,A/p)$ need not be surjective. I suggested that as a universal example (for C-algebras), one should take $A = C[X,Y,Z,T]$, $p =$ the principal ideal generated by $XY - ZT$ and try to show that $\left(\frac{\bar{X}}{Z}\ \frac{\bar{T}}{Y}\right)$ in $SL(2,A/p)$ cannot be in the image of an element in $SL(2,A)$. Ramanujam, who was around, immediately came up with the following proof unbeatable for its beauty and simplicity.

   Let $\begin{pmatrix} f & k \\ h & g \end{pmatrix}$ be the unimodular matrix over A reducing to $\left(\frac{\bar{X}}{Z}\ \frac{\bar{T}}{Y}\right)$.

Then the map $M(2,C) \to SL(2,C)$ given by

$$\begin{pmatrix} x & t \\ z & y \end{pmatrix} \mapsto \begin{pmatrix} f(x,y,z,t) & k(x,y,z,t) \\ h(x,y,z,t) & g(x,y,z,t) \end{pmatrix}$$

12 is easily seen to be a retraction, which is impossible since $M(2,C)$ is contractible, and $SL(2,C)$ is homotopically equivalent to the three dimensional sphere.

3. In our study of the moduli of stable vector bundles with trivial determinant over a projective nonsingular curve of genus 2, M.S. Narasimhan and I came upon the following question: If X is a 3-dimensional projective nonsingular curve, and $Y \subset X$ an ample divisor isomorphic to P^2 with $X - Y$ simply connected, can one conclude that X is isomorphic to P^3? Ramanujam's solution of this problem follows.

Theorem. *Let X be a projective, nonsingular variety of dimension $n \geq 3$ and $Y \cong P^{n-1}$ a closed subvariety of X of codimension 1, with $X - Y$ affine and $H_1(X - Y, Z)$ torsion free. Then $X \cong P^n$.*

Proof. By Lefschetz, valid even under the assumption $X-Y$ affine, $H_i(Y) \to H_i(X)$ is an isomorphism for $i \leq n-2$ and surjective for $i = n-1$. Now if n is even, $H_{n-1}(Y) = 0$, while if n is odd, $H_{n-1}(Y) \cong Z$ and rank $H_{n-1}(X)$ is clearly ≥ 1, so that in any case, $H_i(Y) \to H_i(X)$ is an isomorphism for $i \leq n-1$. By the universal coefficient theorem, so is $H^i(X) \to H^i(Y), i \leq n-1$. In particular, $H^1(X) = 0, H^2(X) = Z$ and $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Thus we obtain $H^1(X, \mathcal{O}_X^*) \cong H^2(X,Z) \approx Z$ and that $H^1(X, \mathcal{O}_X^*) \to H^1(Y, \mathcal{O}_Y^*)$ is an isomorphism.

We wish to show that the line bundle $L = L_Y$ defined by the divisor Y generates $H^1(X, \mathcal{O}_X^*)$, or what is the same, the Poincaré dual of the class in $H_{2n-2}(X)$ defined by Y generates $H^2(X,Z)$. Since $H_{2n-2}(Y) \to H_{2n-2}(X)$ is clearly nontrivial and $H_{2n-2}(X) \cong H^2(X) \cong Z$, it is enough to show that $H_{2n-2}(X,Y)$ has no torsion. But this follows from the assumption that $H_1(X - Y) \cong H^{2n-1}(X,Y)$ has no torsion.

Now L/Y generates $H^1(Y, \mathcal{O}_Y^*)$. Let f be a nonconstant rational function, regular on $X - Y$ and having a pole of order r along Y. If σ is the canonical section of L, then $f\sigma^r|Y$ is a nonzero section of $L^r|Y$ and hence $L|Y$ cannot be negative. Thus $L|Y$ is ample.

For any $p \in Z$, consider the exact sequence **13**

$$0 \to L^{p-1} \xrightarrow{\otimes\sigma} L^p|Y \to 0.$$

One deduces, by induction from the cohomology exact sequence, that

(i) for $p \geq 0, H^1(X, L^p) = 0$ and hence

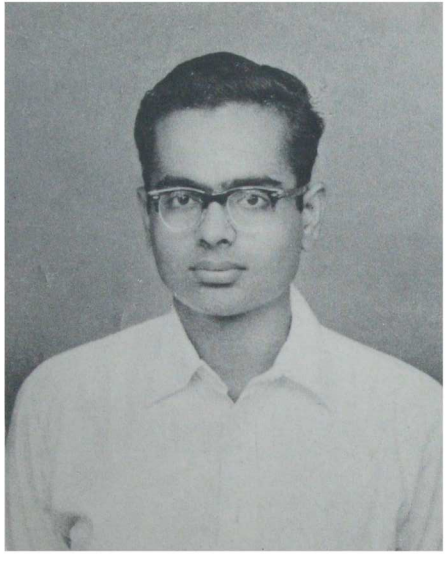
(ii) for $p \geq 1$,

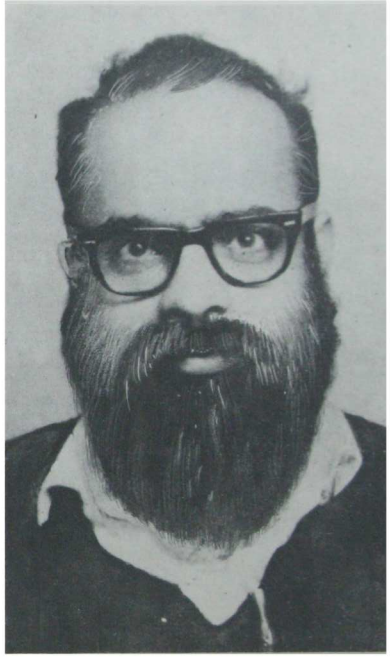
$$0 \to H^0(X, L^{p-1}) \to H^0(X, L^p) \to H^0(Y, L^p) \to 0$$

is exact. Thus in the graded algebra $A = \sum_{p \geq 0} H^0(X, L^p)$ there is an element σ of degree 1 which is not a zero divisor such that A/σ is a polynomial algebra in $(n-1)$ variables. From this it is easy to conclude that A itself is a polynomial algebra in n variables. Thus for some $d > 0, L^d$ is very ample and the homogeneous coordinate ring of X for the projective imbedding given by L^d is the same as that for P^n for the d-fold Segre imbedding. Hence $X \cong P^n$, is itself very ample and Y is a hyperplane.

□

JANUARY 9, 1938 --- OCTOBER 27, 1974





The dominated convergence theorem.

If $f_n \in L^1(X,V)$, $f_n \to f$ a.e and $\exists\, g \in L^1(X)$ such that $\|f_n\| \le g$, then $f \in L^1(X,V)$ and $\int f_n \to \int f$

Proof. In fact, by the dominated convergence theorem for real fns, $\int \|f_m - f_n\| d\mu \to 0$ as $m,n \to \infty$, so $\exists\, f' \in L^1(X,V)$ such that $f_n \to f'$ in $L^1(X,V)$. Replacing by a subsequence s.t. $\int \|f_n - f_{n+1}\| < 2^{-n}$, we may suppose $f_n \to f'$ a.e. Hence $f = f'$ a.e. Q.E.D.

Definition. A function $f: X \to V$ is measurable if it is the almost everywhere limit of a sequence of simple functions (i.e, V-valued linear combination of char. fns of measurable subsets, not necessarily of finite measure).

Theorem. Suppose S is σ-finite. For a function $f: X \to V$, the following are equivalent:

(1) f is measurable

(2) $f^{-1}(U)$ is measurable for every U open in V and $\exists\, E \subset X$, $\mu(E) = 0$, such that $f(X-E)$ is separable.

(3) For every $L \in V^*$, $L \circ f$ is measurable and $\exists\, E \subset X$, $\mu(E) = 0$, $f(X-E)$ is separable.

Facsimile of a page from Ramanujam's Notes on Analysis

Cubic forms over alǵebraic number fields

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Davenport has proved [3] that any cubic form in 32 or more variables with rational coefficients has a non-trivial rational zero. He has also announced that he has subsequently been able to reduce the number of variables to 29. Following the method of [3], we shall prove that any cubic form over any algebraic number field has a non-trivial zero in that field, provided that the number of variables is at least 54. The following is the precise form of our result. **21**

Theorem. *Let K be any algebraic number field and let $C(X) = C(X_1, \ldots, X_m)$ be any homogeneous cubic form with coefficients in K. If the number m of variables is greater than* 53*, there exist $X_1^0, \ldots, X_m^0$ in K, not all zero, such that $C(X^0) = C(X_1^0, \ldots, X_m^0) = 0$.*

Throughout this paper, we shall use the following notations:

Γ, R and C denote the fields of rational, real and complex numbers respectively.

K is a finite algebraic extension of Γ of degree n.

Z and $\mathfrak{o}$ are the rings of integers of Γ and K respectively, and $\mathfrak{d}$ the different ideal of K over Γ.

$\omega_1, \ldots, \omega_n$ is a basis of $\mathfrak{o}$ over Z, and $\rho_1, \ldots, \rho_n$ the dual basis of $\mathfrak{d}^{-1}$ over Z, determined uniquely by the conditions $S(\omega_i \rho_j) = \delta_{ij}$, where S denotes the trace over Γ.

K_R is the n-dimensional commutative algebra $R \otimes_\Gamma K$ over R; we identify an element x of K with the element $1 \otimes x$ of K_R; then $\omega_1, \ldots, \omega_n$ form a basis of K_R over R, and for any y in K_R, we write $y = \sum y_i \omega_i$. For any $y \in K_R, S(y)$ and $N(y)$ will denote the trace and norm respectively of y over R. By a box $B = B(a_1, b_1, \ldots, a_n, b_n)$ in K_R, where the a_i, b_i are real numbers with $0 < b_i - a_i \leqslant 1$, we shall mean the set of y in K_R with $a_i \leqslant y_i < b_i (i = 1, \ldots, n)$. B_0 will denote the set of y in K_R with $-1 \leqslant y_i < 1$.

$(K_R)^m$ will denote the m^{th} Cartesian power of K_R, considered as an mn-dimensional vector space over R. A box $\mathscr{B}$ in $(K_R)^m$ is the Carte-
22 sian product $\prod_{i=1}^{m} B_i$ of boxes B_i in K_R; similarly, $\mathscr{B}_0$ will denote $(B_0)^m$. Unless otherwise stated, all summations will be over 'integral' vectors $X = (X_1, \ldots, X_m)$ of $(K_R)^m, X_i \in \mathfrak{a}$, which are subject to the conditions stated either under the summation sign or immediately after.

For real values of $x, e(x)$ will denote $e^{2\pi i x}$ as usual and $||x||$ the distance of x from the nearest integer. We shall use the O, o, notation of Landau-Bachmann as well as the $\ll$ notation of Vinogradoff, whichever is more convenient. The parameters on which the constants involved in these notations depend will be explicitly mentioned, if not clear from the context.

1 General cubic exponential sums

Since the estimations of this and the next sections are straightforward extensions of those of the corresponding §§ 3 and 4 of [3], we shall only state the results or give brief sketches of the proofs, referring to [3] for details.

Let

$$\Gamma(X) = \sum_{i,j,k=1}^{m} \gamma_{ijk} X_i X_j X_k$$

be a cubic form with coefficients in K_R which are symmetric in all three indices. We associate to this cubic form the bilinear forms $B_j(X, Y) (j = 1, \ldots, m)$ on $(K_R)^m$ with values in K_R and the bilinear forms $\lambda_{jp}(X, Y)$

$(j = 1, \ldots, m;\ p = 1, \ldots, n)$ on $(K_R)^m$ with values in R by means of the equations

$$B_j(X, Y) = \sum_{i,k} \gamma_{ijk} X_i Y_k = \sum_{p=1}^{n} \lambda_{jp}(X, Y)_{\rho_p}.$$

For a box $\mathscr{B}$ in $(K_R)^m$ and a real $P > 1$, we set

$$S = S(P, \mathscr{B}) = \sum_{X \in P\mathscr{B}} e(S(\Gamma(X))).$$

Throughout this section, the constant implied by the sign $\ll$ will depend only on m and n, and will in particular be independent of the coefficients of $\Gamma(X)$.

LEMMA 1.1.

$$|S|^4 \ll P^{mn} \sum_{X,Y \in P\mathscr{B}_0} \prod_{j=1}^{m} \prod_{p=1}^{n} \min\left(P, ||6\lambda_{j,p}(X, Y)||^{-1}\right).$$

This is proved by following the first step of Weyl's method of estimating trigonometrical sums in one variable, exactly like Lemma 3.1 of [3].

The above lemma remains valid even if $\Gamma(X)$ contains quadratic and linear terms, as the proof of the lemma shows. This remark will be useful to us later.

We now find conditions under which the inequality

$$|S| > P^{(m-\kappa)n} \tag{1}$$

will be valid, where $0 < \kappa < m$.

LEMMA 1.2. *Suppose (1) holds and let N denote the number of distinct pairs of integral points X, Y satisfying*

$$X, Y \in P\mathscr{B}_0, \quad ||6\lambda_{jp}(X, Y)|| < P^{-1};$$

then

$$N \gg P^{(2m-4\kappa)n} (\log P)^{-mn}.$$

Proof. If $\{x\}$ denotes the fractional part of x for real x, and $r_{jp}(j = 1,\dots,m; p = 1,\dots,n)$ are integers satisfying $0 \leqslant r_{jp} \leqslant P$, the simultaneous inequalities

$$P^{-1}r_{jp} \leqslant \{6\lambda_{jp}(X,Y)\} < P^{-1}(r_{jp}+1)$$

23 cannot hold for more than $N(X)$ integer points, Y lying in any fixed box with edges not exceeding P in length, where $N(X)$ denotes for any $X \in P\mathscr{B}_0$ the number of Y satisfying the inequalities stated in the lemma. Since $P\mathscr{B}_0$ can be subdivided into 2^{mn} such boxes, we obtain

$$\begin{aligned}\sum_{Y\in P\mathscr{B}_0}\prod_{j=1}^{m}\prod_{p=1}^{n}&\min\left(P,||6\lambda_{jp}(X,Y)||^{-1}\right)\\ &\ll N(X)\prod_{j,p}\sum_{r_{j,p}=0}^{P-1}\min\left(P,\frac{P}{r_{jp}},\frac{P}{P-r_{jp}-1}\right)\\ &\ll N(X)(P\log P)^{mn};\end{aligned}$$

summing over all $X \in P\mathscr{B}_0$ and applying Lemma 1.1, we get the estimate for N. □

LEMMA 1.3. *Suppose (1) holds, and let* $0 < \theta < 1$. *Let* N_2 *denote the distinct pairs of integer points* X, Y *satisfying*

$$X, Y \in P^{\theta}\mathscr{B}_0, \quad ||6\lambda_{jp}(X,Y)|| < P^{-3+2\theta}.$$

Then
$$N_2 \gg \left(P^{2m\theta-4\kappa}(\log P)^{-m}\right)^n.$$

Proof. For fixed X and $Y_i = \sum_{q=1}^{n} Y_{iq}\omega_q$, we have

$$6\lambda_{jp}(X,Y) = 6S\left(B_j(X,Y)\omega_p\right) = 6\sum_{k,q} S\left(\sum_i \gamma_{ijk}X_i\omega_p\omega_q\right)Y_{k,q},$$

which shows that the coefficient of $Y_{k,q}$ in $6\lambda_{jp}$ is equal to the coefficient of Y_{jp} in $6\lambda_{k,q}$. Thus we may apply Lemma 3.3 of [3] to the forms in Y

obtained by substituting a fixed value of X in $6\lambda_{jp}(X,Y)$, and similarly also to the forms in X obtained by fixing Y.

Fixing X in $P\mathscr{B}_0$ and applying the aforementioned lemma to the forms $6\lambda_{jp}(X,Y)$ in Y with $A = P, Z_1 = P^{-1+\theta}, Z_2 = 1$, and summing over all integral X in $P\mathscr{B}_0$, we find that the number N_1 of distinct pairs of integral points X, Y satisfying

$$X \in P\mathscr{B}_0, \quad Y \in P^\theta \mathscr{B}_0, \quad ||6\lambda_{jp}(X,Y)|| < P^{-2+\theta}$$

is $\gg P^{(m+m\theta-4\kappa)n}(\log P)^{mn}$, by Lemma 1.2. Another application of the lemma to the forms in X obtained by fixing $Y \in P^\theta\mathscr{B}_0$ with $A = P^{\frac{1}{2}(3-\theta)}$,
$Z_1 = P^{-\frac{3}{2}(1-\theta)}, Z_2 = P^{-\frac{1}{2}(1-\theta)}$, and summation over Y, gives the result. □

LEMMA 1.4. *Let $\mathscr{Y}$ be a set of at least T distinct integer points Y in K^m, with $Y \in y\mathscr{B}_0$. Suppose there are at most r points of $\mathscr{Y}$ linearly independent over K. Then*

$$T \leqslant Ay^{rn},$$

where A depends only on m and n.

Further, if W is a positive integer with $1 < W \leqslant T$, there exist W distinct points $Y^{(1)}, \ldots, Y^{(W)}$ in $\mathscr{Y}$ such that

$$Y^{(1)} - Y^{(w)} \in ByW^{1/rn}T^{-1/rn}\mathscr{B}_0 \quad (w = 1, \ldots, W),$$

where B is again a constant depending only on m and n.

Proof. Lemma 3.5 of [3] is our lemma in the special case $K = \Gamma$. Our lemma clearly reduces to this special case if we identify K^m with Γ^{mn} by means of the basis $\omega_1, \ldots, \omega_n$, and notice that if there are at most r independent points in $\mathscr{Y}$ over K, there are at most rn independent points over Γ. □

Definition. *Let $m, n, r, \kappa, \theta (0 < \theta < 1), \delta (> 0)$, be given with $m\theta > 4\kappa$. For $P > 1$, define r to be the greatest positive integer with the* **24**

property that there exists a non-zero integer point X and r integer points $Y^{(1)}, \ldots, Y^{(r)}$ independent over K such that

$$X \in P^{\theta}\mathscr{B}_0, \quad Y^{(s)} \in P^{\theta}\mathscr{B}_0, \quad ||6\lambda_{jp}(X, Y^{(s)})|| < P^{-3+2\theta+\delta}$$

for $s = 1, \ldots, r$.

Note that under assumption (1) and for P large, $r \geqslant 1$ by Lemma 1.3.

LEMMA 1.5. *Suppose that*

$$0 < \xi \leqslant \theta, \quad 0 < \eta \leqslant \theta, \quad 0 < B \leqslant P^{-3+2\theta+\delta}$$

and that λ and ϵ are positive satisfying

$$\lambda > m\xi, \quad \lambda > r\eta.$$

Then there exists a $P_0 = P_0(m, n, \xi, \eta, \theta, \delta, \lambda, \epsilon)$, such that if $P > P_0$ and there are more than $P^{(\lambda+\epsilon)n}$ distinct pairs of integer points X, Y, neither zero, satisfying

$$X \in P^{\xi}\mathscr{B}_0, \quad Y \in P^{\eta}\mathscr{B}_0, \quad ||6\lambda_{jp}(X, Y)|| < B, \tag{2}$$

then

$$\lambda + \epsilon \leqslant m\xi + r\eta,$$

and there are more than $P^{mn\xi}$ distinct pairs of integer points X, Y neither zero, with

$$X \in P^{\xi}\mathscr{B}_0, \quad Y \in P^{\eta-(\lambda-m\xi)/r}\mathscr{B}_0, \quad ||6\lambda_{jp}(X, Y)|| < 2B.$$

Proof. For each $X \neq 0$, let $T(X)$ denote the number of $Y \neq 0$ satisfying (2). Because of the assumptions on ξ, η and B, we can apply the first part of Lemma 1.4 to deduce that

$$T(X) \leqslant AP^{r\eta};$$

since by assumption

$$\sum_{\substack{X \in P^{\xi}\mathscr{B}_0 \\ X \neq 0}} T(X) \geqslant P^{(\lambda+\epsilon)n}$$

We deduce that $P^{n(\lambda+\epsilon)-mn\xi-rn\eta} \leqslant A$, which shows that $\lambda+\epsilon \leqslant m\xi+r\eta$ unless P is small (the smallness as made precise in the statement of the lemma).

For $s = 0, 1, 2, \ldots$, let X_s denotes the number of integer points $X \neq 0, X \in P^{\xi}\mathscr{B}_0$, for which

$$AP^{r\eta\eta}2^{-s-1} \leqslant T(X) < AP^{r\eta\eta}2^{-s}.$$

It is clearly sufficient to take $\ll \log P$ values of s to include all X for which $T(X) \neq 0$. Since we have

$$AP^{r\eta\eta}\sum_s 2^{-s}X_s > P^{(\lambda+\epsilon)n},$$

there exists at least one s for which

$$P^{r\eta\eta}2^{-s}X_s \geqslant A'P^{(\lambda+\epsilon)n}(\log P)^{-1},$$

A' being a constant dependent only on m and n. For this s, and ρ defined by the equation $P^{\rho} = P^{r\eta\eta-\frac{1}{2}\epsilon n}2^{-s}$, we have

$$X_s \geqslant A''(m,n,\epsilon)P^{\lambda n-\rho+\frac{1}{3}\epsilon n},$$

and since $X_s \ll P^{mn}$ trivially, it follows that $\rho > \lambda n - mn\xi + \frac{1}{4}\epsilon n$ unless P is small.

Assuming P large enough, we now have X_s integer points $X \neq 0$ to **25**
each of which there correspond $T(X)$ integer points $Y \neq 0$ such that (2) is satisfied. We apply to the set of points Y corresponding to a particular X the second part of Lemma 1.4 with

$$y = P^{\eta}, \quad T = T(X) \geqslant \tfrac{1}{2}AP^{\rho+\frac{1}{2}\epsilon n}, \quad W = [P^{\rho-\lambda n+mn\epsilon}] + 1.$$

The condition $1 < W < T(X)$ is satisfied since $0 < n(\lambda - m\xi) < \rho$. We thus deduce that to each of the X_s points there correspond W distinct points $Y^{(1)}, \ldots, Y^{(W)}$ with

$$Y^{(1)} - Y^{(w)} \in B^1P^{\eta}P^{(\rho-\lambda n+mn\xi)/rn}P^{-(\rho+\frac{1}{2}\epsilon n)rn}\mathscr{B}_0 = B^1P^{\eta-(\lambda-m\xi)/r-\frac{1}{2}\epsilon r}\mathscr{B}_0,$$

where B^1 is a constant depending only on m and n. Denoting $Y^{(1)} - Y^{(w)}$ by $Y'^{(w)}$, we see that to each of the X_s points there correspond $W-1$ distinct integer points $Y \neq 0$ satisfying

$$Y \in P^{\eta-\{(\lambda-m\xi)/r\}}\mathscr{B}_0, \quad ||6\lambda_{jp}(X,Y)|| < 2B,$$

if P is large enough. All the resulting pairs satisfy the condition of our lemma, and their number is

$$(W-1)X_s \geqslant \tfrac{1}{2}A''P^{\rho-\lambda n+mn\xi}P^{\lambda n-\rho+\frac{1}{3}\epsilon n} \geqslant P^{mn\xi}$$

if P is large. □

The lemma is thus proved.

LEMMA 1.6. *Let $m, \kappa, \theta (0 < \theta < 1)$ and $\delta (> 0)$ be fixed independently of P and suppose that $m\theta > 4\kappa$ and that (1) holds. Then there exists a P_1 depending only on m, n, θ, κ and δ such that for $P \geqslant P_1$, we have*

$$r \geqslant m - 2\kappa/\theta.$$

Proof. The proof simply consists of repeated applications of the previous lemma, though the details are complicated.

We define several sequences of numbers depending on m, n, κ, θ and δ and on certain positive parameters $\epsilon_0, \epsilon_1, \ldots$, which will later be fixed in terms m, n, κ, θ and δ, as follows:

$$\left.\begin{aligned} &\xi_0 = \theta,\ \eta_0 = \theta,\ B_0 = P^{-3+2\theta},\ \lambda_0 = 2m\theta - 4\kappa - 2\epsilon_0, \\ &\xi_{q+1} = \eta_q - \frac{(\lambda_q - m\xi_q)}{r},\ \eta_{q+1} = \xi_q,\ B_{q+1} = 2B_q,\ \lambda_{q+1} = \\ &\quad = m\xi_q - \epsilon_{q+1}, \end{aligned}\right\} \quad (3)$$

for $q \geqslant 0$.

On substituting for η_q and λ_q in the expression for ξ_{q+1} when $q \geqslant 1$, we obtain

$$\xi_{q+1} = \xi_{q-1} - \frac{(m\xi_{q-1} - \epsilon_q - m\xi_q)}{r},$$

$$\xi_q - \xi_{q+1} = \omega(\xi_{q-1} - \xi_q) - \tfrac{1}{r}\epsilon_q,$$

where $\omega = m/r - 1$, and hence by induction,

$$\xi_q - \xi_{q+1} = \omega^q(\xi_0 - \xi_1) - \tfrac{1}{r}\sum_{t=1}^{q} \epsilon_t \omega^{q-t} \quad (q \geqslant 1).$$

Summation gives

$$\xi_0 - \xi_{q+1} = (\xi_0 - \xi_1)\sum_{s=0}^{q} \omega^q - \tfrac{1}{r}\sum_{s=0}^{q}\sum_{t=1}^{q} \epsilon_t \omega^{s-t}.$$

Now assume contrary to the required conclusion that $r < m - 2\kappa/\theta$. Since r is an integer, it follows that $m - 2\kappa/\theta - r$ must be greater than a positive number depending only on m, κ and θ. We shall show that there **26** exist $\phi(m, n, \theta, \kappa) > 0$ and an integer $Q = Q(m, n, \theta, \kappa)$ such that for $\epsilon_0 < \phi(m, n, \theta, \kappa)$ we have

$$\xi_0 - \xi_1 > \frac{m\theta - 4\kappa}{2m}$$

and

$$\xi_0 < (\xi_0 - \xi_1)\sum_{s=0}^{Q} \omega^s.$$

The first is clearly ensured if $\epsilon_0 < (m\theta - 4\kappa)/2m$. It follows that if $\omega \geqslant 1$, we can ensure the second by taking $Q + 1 > 2m\theta/(m\theta - 4\kappa)$. Hence assume $\omega < 1$, that is,

$$\omega \leqslant 1 - 1/m.$$

Since $\xi_0 - \xi_1$ is bounded above by 2, it is clearly enough to show that if ϵ_0 is small enough,

$$\xi_0 < (\xi_0 - \xi_1)\sum_{s=0}^{\infty} \omega^s = (1 - \omega)^{-1}(\xi_0 - \xi_1),$$

i.e.

$$\theta < \frac{r}{2r - m}\frac{(m\theta - 4\kappa - 2\epsilon_0)}{r} = \frac{m\theta - 4\kappa - 2\epsilon_0}{2r - m},$$

i.e. $$2r < 2m - \frac{3\kappa}{\theta} - \frac{2\epsilon_0}{\theta},$$

which holds if ϵ_0 is less than $\theta(m - 2\kappa/\theta - r)$, and hence if ϵ_0 is less than $\phi'(m,n,\kappa,\theta)$. It follows easily that for $\epsilon_0 < \phi'(m,n,\kappa,\theta)$, we have

$$\xi_0 < (\xi_0 - \xi_1)\sum_{s=0}^{Q}\omega^s + \mu(m,n,\kappa,\theta),$$

where $\mu(m,n,\kappa,\theta) > 0$. Hence it follows that there exists a

$$\psi(m,n,\kappa,\theta) > 0$$

such that if $\epsilon_i < \psi(m,n,\kappa,\theta)$ for $i = 0,\ldots,Q$ we have

$$\xi_0 < (\xi_0 - \xi_1)\sum_{s=0}^{Q}\omega^s - \tfrac{1}{r}\sum_{s=0}^{Q}\sum_{t=1}^{s}\epsilon_t\omega^{s-t} = \xi_0 - \xi_{Q+1},$$
$$\xi_{Q+1} < 0.$$

We shall now choose ϵ_i suitably depending only on m,n,θ,κ and δ to show that if the above inequality holds, P must be bounded above by a function of these parameters.

For every $q \leqslant Q$, we wish to choose $\epsilon_0,\ldots,\epsilon_q$ with $\epsilon_i < \psi(m,n,\kappa,\theta)$ and $P_q(m,n,\kappa,\theta,\delta)$ such that the following conditions hold:

$$\xi_s > 0,\ \xi_s - \xi_{s+1} > 0,\ \lambda_s > m\xi_s,\ \lambda_s > r\eta_s\ (s = 0,\ldots,q),$$

and if $P > P_q$, the inequalities

$$X \in P^{\xi_s}\mathscr{B}_0,\ Y \in P^{\eta_s}\mathscr{B}_0,\ ||6\lambda_{jp}(X,Y)|| < B_s$$

have at least $P^{n(\lambda_s+\epsilon_s)}$ pairs of solutions with $X, Y \neq 0$.

For $q = 0$, choose $\epsilon_0 = \frac{1}{2}\psi(m,n,\kappa,\theta)$. It follows from our previous considerations that the first set of conditions hold. Also by Lemma
27 1.3, the second set of inequalities has $\gg \left(P^{2m\theta-4\kappa}(\log P)^{-m}\right)^n$ solutions. Since the number of solutions with $X = 0$ or $Y = 0$ is $\ll P^{mn\theta}$ and

$mn\theta < 2mn\theta - 4\kappa n$, we deduce that if $P > P_0(m,n,\kappa,\theta)$, the number of solutions with $X \neq 0, Y \neq 0$ is $\geqslant (P^{2m\theta-4\kappa-\epsilon_0})^n = P^{n(\lambda_0+\epsilon_0)}$.

Suppose we have now chosen $\epsilon_0,\ldots,\epsilon_q$ and P_q depending only on m,n,κ and θ satisfying our conditions. Applying Lemma 1.5 to the second set of inequalities with $s = q$, we deduce that if

$$P > \max(P_q, P_0(m,n,\xi_q,\eta_q,\theta,\delta,\lambda_q,\epsilon_q)) = P_{q+1}(m,n,\theta,\delta,\kappa)$$

we have
$$\lambda_q + \epsilon_q \leqslant m\xi_q + r\eta_q,$$

and that the number of pairs X, Y with $X \neq 0, Y \neq 0$ satisfying

$$X \in P^{\xi_q}\mathscr{B}_0,\ Y \in P^{\eta_q-(\lambda_q-m\xi_q)/r},\ ||6\lambda_{jp}(X,Y)|| < B_{q+1}$$

is greater than $P^{mn\xi_q} = P^{n(\lambda_{q+1}+\epsilon_{q+1})}$. Since $\lambda_{jp}(X,Y) = \lambda_{jp}(Y,X)$ we may inter-change X and Y in the above set of inequalities to deduce that the second set of conditions are satisfied for $s = q+1$. As to the first set, we have

$$\xi_{q+1} = \eta_q - \frac{(\lambda_q - m\xi_q)}{r} > 0,$$

$$\xi_{q+1} - \xi_{q+2} = \omega(\xi_q - \xi_{q+1}) - \frac{1}{r}\epsilon_{q+1} > 0,$$

if $\epsilon_{q+1} < \omega(\xi_q - \xi_{q+1})$, and since the right-hand side depends only on m,n,κ,θ,δ, if $\epsilon_{q+1} < \psi_{q+1}(m,n,\kappa,\theta,\delta)$. Also

$$\lambda_{q+1} - m\xi_{q+1} = m(\xi_q - \xi_{q+1}) - \epsilon_{q+1} > 0$$

again if $\epsilon_{q+1} < \psi'_{q+1}(m,n,\kappa,\theta,\delta)$; and finally,

$$\lambda_{q+1} - r\eta_{q+1} = (m-r)\xi_q - \epsilon_{q+1} > 0,$$

if $\epsilon_{q+1} < \psi''_{q+1}(m,n,\kappa,\theta,\delta)$. Thus if we choose

$$\epsilon_{q+1} = \tfrac{1}{2}\min(\psi_{q+1}, \psi'_{q+1}, \psi''_{q+1}),$$

all our conditions are satisfied and our induction is complete.

Thus we obtain for $s = Q+1$ that $\xi_{Q+1} > 0$ unless $P < P_{Q+1}$ $(m,n,\kappa,\theta,\delta)$ and since we have already shown that $\xi_{Q+1} < 0$ (and since Q depends only on m,n,κ,θ,δ) we deduce that if $r < m - 2\kappa/\theta, P$ must necessarily be bounded by a function of m,n,κ,θ and δ. The proof of the lemma is complete. □

2 Two particular types of exponential sums

Let

$$C(X) = C(X_1, \ldots, X_m) = \sum_{i,j,k=1}^{m} c_{ijk} X_i X_j X_k$$

be a cubic form with coefficients which are integers of K, symmetric in all three indices. *We shall assume throughout this section that $C(X)$ does not represent zero non-trivially; that is, that $X = (0, \ldots, 0)$ is the only solution of $C(X) = 0$ in K.*

The cubic form $C(X)$ is said to *represent* the cubic form $C'(U) = C'(U_1, \ldots, U_s)$ in the s variables $U_1, \ldots, U_s$ if there exists a linear transformation

$$X_i = \sum_{t=1}^{s} p_{it} U_t, \ p_{it} \in \mathfrak{o} \ (i = 1, \ldots, m; t = 1, \ldots, s),$$

of rank s which transforms $C(X)$ into $C'(U)$. It follows that we must have $s \leqslant m$. We shall say that $C(X)$ *splits with remainder r* if it represents a form in $r + 1$ variables of the type

$$a_0 U_0^3 + C_1(U_1, \ldots, U_r).$$

28 We define the following bilinear forms (not to be confused with the B_j defined in § 1) associated to $C(X)$:

$$B_j(X, Y) = \sum_{i,k} c_{ijk} X_i Y_k.$$

LEMMA 2.1. *Suppose there is a non-zero integral point Z and r linearly independent integral points $Y^{(1)}, \ldots, Y^{(r)}$ such that*

$$B_j(Z, Y^{(s)}) = 0 \quad (j = 1, \ldots, m; s = 1, \ldots, r).$$

Then $r \leqslant m - 1$, and $C(X)$ splits with remainder r.

Proof. The linear transformation

$$X_i = Z_i U_0 + Y_i^{(1)} U_1 + \cdots + Y_i^{(r)} U_r$$

can be easily seen to be of rank precisely $r+1$, and transforms $C(X)$ into a form of the type $a_0U_0^3 + C^1(U_1,\dots,U_r)$, proving our assertion.

We will now apply Lemma 1.6 to two particular exponential sums which will arise in our later work.

Let $\mathscr{B}$ be a fixed box in $(K_R)^m$; for any $\alpha \in K_R$ and $P > 1$, we define

$$E(\alpha) = \sum_{X \in P\mathscr{B}} e(S(\alpha C(X))).$$

Let δ be any positive number. We then have □

LEMMA 2.2. *Let κ and θ be fixed numbers satisfying*

$$0 < \theta < 1, \quad 0 < 4\kappa < m\theta.$$

Let $C(X)$ be any cubic form over K with integral coefficients which does not represent zero non-trivially and which does not split with remainder r, where r is the least integer satisfying

$$r \geqslant m - \frac{2\kappa}{\theta}.$$

Then there exists a P_0 (depending on K, C, θ, κ and δ) such that for $P \geqslant P_0$, either

$$|E(\alpha)| \leqslant P^{(m-\kappa)n} \tag{4}$$

or there exists an integer $\mu \in \mathfrak{o}, \mu \neq 0$, and $\mu \in P^{2\theta+\delta}B_0$, and a $\lambda \in \mathfrak{d}^{-1}$ such that

$$\mu\alpha - \lambda = \sum_{i-1}^{n} \epsilon_i P_i, \quad \epsilon_i \in R, \quad |\epsilon_i| < P^{-3+2\theta+\delta}. \tag{5}$$

Proof. We apply Lemma 1.6 with $\Gamma(X) = \alpha C(X)$. It follows that for large P, either (4) holds or there exists an integral point X and $r \geqslant m - 2\kappa/\theta$ integral points $Y^{(1)},\dots,Y^{(r)}$, in $P^\theta\mathscr{B}_0$, linearly independent over K, such that the following conditions hold:

$$6\alpha B_j(X, Y^{(s)}) = \lambda_{j,s} + \sum_i \epsilon_i(j,s)\rho_i, \lambda_{j,s} \in \mathfrak{d}^{-1}, \quad \epsilon_i(j,s) \in R,$$

$$|\epsilon_i(j,s)| < P^{-3+2\theta+\delta} \quad (i,j=1,\ldots,n; s=1,\ldots,r).$$

All the $B_j(X,Y^{(s)})$ cannot be zero, since it would then follow from Lemma 2.1 that $C(X)$ splits with remainder r. Also since $X, Y^{(s)} \in P^\theta \mathscr{B}_0$, it follows that

$$6B_j(X,Y^{(s)}) \in AP^{2\theta}\mathscr{B}_0,$$

29 where A is a constant depending only on K and C. Our lemma therefore follows if we take $\mu = 6B_j(X,Y^{(s)})$ with j and s so chosen that $B_j(X,Y^{(s)}) \neq 0$.

We note that in view of the remark made at the end of Lemma 1.1, the above lemma holds without change even if we replace the sum $E(\alpha)$ by the sum

$$\sum_{X \in P\mathscr{B}} e(S(\alpha C(X)) + L(X)),$$

where $L(X)$ is a real linear form in the X_{ij}.

Now let γ be any non-zero element of K. We associate with γ the integral ideal $\mathfrak{a}_\gamma$ which is the denominator of $(\gamma)\mathfrak{d}$, that is to say $\mathfrak{a}_\gamma = (g.c.d(\mathfrak{o}, \gamma\mathfrak{d}))^{-1}$. Let $l = (l_{ij})(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$ be any system of rational integers. We define

$$S_1(l,\gamma) = \sum_{X \bmod N\mathfrak{a}_\gamma} e\left(S(\gamma C(X)) + \sum_{i,j} \frac{l_{ij}}{N\mathfrak{a}_\gamma} X_{ij}\right),$$

where $X = (X_1,\ldots,X_m), X_i = \sum_j X_{ij}\omega_j$, and each X_{ij} runs through a complete system of residues modulo $N\mathfrak{a}_\gamma$. We also put

$$S_1((0),\gamma) = (N\mathfrak{a}_\gamma)^{m(n-1)} \sum_{X \bmod N\mathfrak{a}_\gamma} e(S(\gamma C(X))) = (N\mathfrak{a}_\gamma)^{m(n-1)} S_\gamma.$$

□

LEMMA 2.3. *Let $C(X)$ be a fixed cubic form in m variables which does not represent zero non-trivially, and κ a fixed number satisfying*

$$0 < \kappa < \frac{1}{8}m.$$

Suppose $C(X)$ does not split with remainder r, where r is the least integer satisfying

$$r \geqslant m - 4\kappa.$$

Then, for any $\gamma \in K$ and any system of rational integers $l = (l_{ij})$, we have

$$|S_1(l,\gamma)| < (N\mathfrak{a}_\gamma)^{mn-\kappa} \tag{6}$$

unless $N\mathfrak{a}_\gamma$ is bounded above by a constant depending only on κ, the field K and the form C. In particular, if $N\mathfrak{a}_\gamma$ is large, we obtain

$$|S_\gamma| < (N\mathfrak{a}_\gamma)^{m-\kappa}. \tag{7}$$

Proof. We apply Lemma 2.2 (and the remark made at the end of that lemma) with the box $\mathscr{B} = \{X : 0 \leqslant X_{ij} < 1\}, \gamma$ instead of $\alpha, N\mathfrak{a}_\gamma$ instead of $P, (1/2n) - \delta$ instead of θ, and κ/n instead of κ. The conditions of that lemma are verified if δ is small enough. Hence if $N\mathfrak{a}_\gamma$ is large enough and (6) does not hold, there is a non-zero integer μ in $(N\mathfrak{a}_\gamma)^{1/n-\mathfrak{d}}B_0$ and $a\lambda \in \mathfrak{d}^{-1}$ such that

$$\mu\gamma - \lambda = \sum_{i=1}^{n} \epsilon_i \rho_i, \quad |\epsilon_i| < (N\mathfrak{a}_\gamma)^{-3-\delta+1/n}.$$

But since $\mathfrak{a}_\gamma(\mu\gamma - \lambda) \subset \mathfrak{d}^{-1}$, $N\mathfrak{a}_\gamma\epsilon_i$ must be a rational integer, and since

$$|N\mathfrak{a}_\gamma\epsilon_i| < (N\mathfrak{a}_\gamma)^{-2+1/n-\delta} < 1$$

for $N\mathfrak{a}_\gamma$ large, it follows that $\epsilon_i = 0$ for $1 \leqslant i \leqslant n$, and $\mu\gamma \in \mathfrak{d}^{-1}$. By the definition of $\mathfrak{a}_\gamma$, we must have $\mu \in \mathfrak{a}_\gamma$, and $N\mathfrak{a}_\gamma$ divides $N\mu$. But since

$$\mu \in (N\mathfrak{a}_\gamma)^{1/n-\delta}\mathscr{B}_0, \quad |N\mu| \leqslant A(N\mathfrak{a}_\gamma)^{1-n\delta},$$

where A depends only on K, which implies that for large $N\mathfrak{a}_\gamma$, we must **30**
have $|N\mu| < N\mathfrak{a}_\gamma$, and therefore $\mu = 0$, which is a contradiction. This proves the inequality (6). The inequality (7) follows from (6) since we have $S_1((0),\gamma) = (N\mathfrak{a}_\gamma)^{m(n-1)}S_\gamma$. □

3 The contribution of the major arcs

We shall continue to use the notations of the previous section. Further, we define $\mathscr{R}$ to be the parallelepiped in K_R defined by

$$\mathscr{R} = \left\{ x = \sum_{i=1}^{n} x_i \rho_i \mid 0 \leqslant x_i < 1 \right\}.$$

For any $\gamma \in K$, we define the 'major arc' B_γ to be the set of $\alpha \in K_R$ such that if

$$\beta = \alpha - \gamma = \sum_{i=1}^{n} \beta_i \rho_i,$$

we have $|\beta_i| < P^{-2-\delta}(N\mathfrak{a}_\gamma)^{-1}$. We assert that if $\gamma \neq \gamma'$ and $N\mathfrak{a}_\gamma \leqslant P^{1-2\delta}, N\mathfrak{a}_{\gamma'} \leqslant P^{1-2\delta}, B_\gamma$ and $B_{\gamma'}$ cannot intersect. In fact, if they did intersect, we must have

$$\gamma - \gamma' = \sum \gamma_i \rho_i, \quad |\gamma_i| < P^{-2-\delta}\left((N\mathfrak{a}_\gamma)^{-1} + (N\mathfrak{a}_{\gamma'})^{-1}\right).$$

But since $\mathfrak{a}_\gamma \mathfrak{a}_{\gamma'}(\gamma - \gamma') \subset \mathfrak{d}^{-1}, N\mathfrak{a}_\gamma N\mathfrak{a}_{\gamma'}\gamma_i$ must be a rational integer for all i. Since $\left|N\mathfrak{a}_\gamma N\mathfrak{a}_{\gamma'}\gamma_i\right| < P^{-2-\delta}(N\mathfrak{a}_\gamma + N\mathfrak{a}_{\gamma'}) < 2P^{1-2\delta}P^{-2-\delta} = 2P^{-1-3\delta} < 1$, we must have $\gamma_i = 0$ and $\gamma = \gamma'$.

Now let $C(X)$ be a cubic form in m variables with coefficients integers in K and symmetric in all three indices. Our ultimate goal is to find, under suitable conditions on C and for a suitable choice of the box $\mathscr{B}$, an asymptotic formula for the number $\mathscr{N}(P)$ of solutions of the equation $C(X) = 0$ with X integral and $X \in P\mathscr{B}$. If $\alpha = \sum \alpha_i \rho_i$ denotes a general point of $\mathscr{R}$ and $d\alpha$ denotes the measure $d\alpha_1, \ldots, d\alpha_n$, we have the following integral representation for $\mathscr{N}(P)$:

$$\mathscr{N}(P) = \int_{\mathscr{R}} E(\alpha) d\alpha. \tag{8}$$

In this section, we shall establish under certain conditions on C, and the box $\mathscr{B}$ being suitably chosen, an asymptotic formula for the contribution to the integral on the right of (8) of the major arcs B_γ with

$N\mathfrak{a}_\gamma \leqslant P^{1-2\delta}$. Because of the definition of $\mathscr{R}$ and the periodicity of $E(\alpha)$ in the variables α_i, the contribution of these major arcs is given by

$$\sum_{N\mathfrak{a}\leqslant P^{1-2\delta}} \sum_{\substack{\gamma \bmod \mathfrak{d}^{-1} \\ \mathfrak{a}\gamma=\mathfrak{a}}} \int_{B_\gamma} E(\alpha)d\alpha.$$

The main result of this section is the following:

LEMMA 3.1. *Let $C(X)$ be any cubic form in more than nine variables with integral coefficients in K which does not represent zero non-trivially. Assume that (in the notation of § 2) we have a uniform estimate of the form*

$$|S_1(l,\gamma)| \ll (N\mathfrak{a}_\gamma)^{mn-\omega}$$

with $\omega > 2$. Then for P tending to infinity suitably, and with a suitable choice of the box $\mathscr{B}$, we have the following asymptotic formula

$$\sum_{N\mathfrak{a}\leqslant P^{1-2\delta}} \sum_{\substack{\gamma \bmod \mathfrak{d}^{-1} \\ \mathfrak{a}\gamma=\mathfrak{a}}} \int_{B\gamma} E(\alpha)d\alpha = AP^{(m-3)n} + o(P^{(m-3)n}) \quad (0 < A < \infty).$$

The whole of this section is devoted to the proof of this lemma, **31** which will be carried out in several steps.

Let $g(t_1,\dots,t_p)$ be a function of p variables defined in a parallelepiped $A_i^1 \leqslant t_i \leqslant A_i^2$, and let Q be a subset $(r_1 < \cdots < r_q)$ of the set of integers $\{1,2,\dots,p\}$. We shall adopt the following conventions: h_Q will denote an element of Z^Q, that is, a set of integers $(h_{r_1},\dots,h_{r_q})$, and $|h_Q|$ the sum $\sum_{i=1}^{q} h_{r_i}$. If s is a fixed integer, we shall denote the set $(s,\dots,s)$ by $(s)_Q$. We shall write $\partial_Q^{h_Q} g$ in place of

$$\frac{\partial^{|h_Q|} g}{(\partial t_{r_1})^{h_{r_1}} \cdots (\partial t_{r_q})^{h_{rq}}}.$$

If a and b are two integers, the symbol $\sum_{h_Q=a}^{b}$ will imply summation over all the h_Q with $a \leqslant h_{r_i} \leqslant b (1 \leqslant i \leqslant q)$. An integral of the form

$\int_{A_Q^1}^{A_Q^2} g\, dt_Q$ will stand for the multiple integral

$$\int_{A_{r_1}^1}^{A_{r_1}^2} \cdots \int_{A_{r_q}^1}^{A_{r_q}^2} g\, dt_{r_1} \cdots dt_{r_q}.$$

Finally, we shall mean by the symbol $[g(t_1, \ldots, t_p)]\Delta_Q$ the multiple difference expression

$$\sum_{\lambda_i = 1 \text{ or } 2} (-1)^{\sum_i \lambda_i} g\left(t_1, \ldots, A_{r_1}^{\lambda_1}, \ldots, A_{r_2}^{\lambda_2}, \ldots, A_{r_q}^{q}, \ldots\right).$$

With these notations, we can state

LEMMA 3.2. *Let $f(x) = f(x_1, \ldots, x_k)$ be a function of k variables defined and continuous with all its partial derivatives up to order s in an open set containing the parallelepiped $A_i \leqslant x_i \leqslant B_i (i = 1, \ldots, k)$, where we assume the A_i and B_i to be non-integral. Let $z = (z_1, \ldots, z_k)$ be a set of k integers and let q be any integer $\geqslant 1$. Then we have*

$$\sum_{\substack{A_i < x < B_i \\ x_i \equiv z_i (q)}} f(x_1, \ldots, x_k) = \sum_{\Phi} (-1)^{\mu} q^{\nu(s-1)-\lambda}$$

$$\times \left[\sum_{h_J = 0}^{s-1} q^{|h_J|} \prod_{j \in J} \psi_{h_j}\left(\frac{\xi_j - z_j}{q}\right) \int_{A_1}^{B_1} \int_{A_K}^{B_K} \prod_{k \in K} \psi_{s-1}\left(\frac{\xi_k - z_k}{q}\right) \partial_J^h J \partial_K^{(s)} K f\, d\xi_I d\xi_K \right]_{\Delta_J},$$

where Φ runs through all partitions of $[1, k]$ into three disjoint sets of integers

$$I = \{i_1 < \ldots < i_\lambda\}, \quad J = \{j_1 < \ldots < j_\mu\} \quad \text{and} \quad K = \{k_1 < \ldots < k_\nu\},$$

and for any integer $h \geqslant 0$, we write

$$\psi_h(x) = (-1)^{h+1} \sum_{\substack{1 = -\infty \\ l \neq 0}}^{\infty} \frac{e(lx)}{(2\pi i l)^{h+1}}.$$

This lemma can be proved by induction on the number of variables, starting from Lemma 5.2 of [3]. We omit the details.

Note that ψ_h is bounded, being continuous and periodic.

LEMMA 3.3. *Let* $\xi_i = \sum_{j=1}^{n} \xi_{ij}\omega_j (i = 1, \ldots, m)$ *and* $\beta = \sum_{1}^{n} \beta_j \rho_j$ *be elements of* K_R*, and let* $h_{ij} (1 \leqslant i \leqslant m; 1 \leqslant j \leqslant n)$ *be non-negative* **32** *integers. Considering* $e(S(\beta C(\xi)))$ *as a function of the real variables* ξ_{ij} *and* β_k*, we have identically*

$$\frac{\partial^h}{\partial \xi_{11}^{h_{11}} \ldots \partial \xi_{m,n}^{h_{m,n}}} (e(S(\beta C(\xi))) = e(S(\beta C(\xi))) \sum_{\frac{1}{2}h \leqslant |\alpha| \leqslant h} \beta^{\alpha} \Phi_{\alpha}(\xi_{11}, \ldots, \xi_{mn})),$$

where α *runs through all multi-indices* $(\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geqslant 0$*, such that* $|\alpha| = \sum \alpha_i$ *lies between* $\frac{1}{3}h$ *and* h *(*h *stands for* $\sum_{i,j} h_{i,j}$*) and* Φ_{α} *is a homogeneous real polynomial in the* ξ_{ij} *of degree* $3|\alpha| - h$*.*

In particular, if $|\xi_{ij}| \leqslant BP$ *and* $|\beta_i| \leqslant \mu$*, with* $P^3\mu > 1$*, then the left side of the above identity is* $\leqslant B'(P^2\mu)^h$*, where* B' *depends only on* B*,* h_{ij}*, the coefficients of* C *and the field* K*.*

Proof. The first part can be proved easily by induction on $h = \sum_{i,j} h_{ij}$. It follows that if $|\xi_{ij}| \leqslant BP$ and $|\beta_i| \leqslant \mu$, and $P^3\mu > 1$, the left side is bounded in absolute value by a constant multiple of

$$\sum_{\frac{1}{3}h \leqslant \nu \leqslant h} \mu^{\nu} P^{3\nu - h} = P^{3\nu_0 - h} \mu^{\nu_0} \left\{ (P_{\mu}^3)^{h - \nu_0 + 1} - 1 \right\} / (P^3\mu - 1) \ll$$

$$P^{3\nu_0 - h} \mu^{\nu_0} (P^3\mu)^{h - \nu_0} = (P^2\mu)^h,$$

where ν_0 denotes the least integer $\geqslant h/3$.

The lemma is proved. □

We shall now start on the proof of Lemma 3.1. Let $I = I(P)$ denote the expression on the left in Lemma 3.1, and $I_{\gamma} = \int_{B\gamma} E(\alpha) d\alpha$, so that we have

$$I = \sum_{N\mathfrak{a} \leqslant P^{1-2\delta}} \sum_{\gamma \bmod \mathfrak{d}^{-1}} I_{\gamma}.$$

For α in B_{γ}, we write $\alpha = \beta + \gamma$, so that we have

$$\beta = \sum_i \beta_i \rho_i, \quad |\beta_i| < P^{-2-\delta} (N\mathfrak{a}_{\gamma})^{-1}.$$

Suppose the box $P\mathscr{B}$ is defined by the inequalities $A_{ij} < X_{ij} < B_{ij}$, where we assume that the A_{ij} and B_{ij} are non-integral. We then have for α in B_γ,

$$E(\alpha) = \sum_{Z_{ij}=1}^{N\mathfrak{a}_\gamma} e(S(\gamma C(Z))) \sum_{\substack{X_{ij}\equiv Z_{ij}(N\mathfrak{a}_\gamma) \\ A_{ij}<X_{ij}<B_{ij}}} e(S(\beta C(X))), \tag{9}$$

where Z stands for $(Z_1, \ldots, Z_m)$, $Z_i = \sum_{j=1}^{n} Z_{ij}\omega_j$. Let s be a fixed positive integer such that $s\delta > n+1$. We substitute for the inner sum in the above expression from Lemma 3.2.

Still retaining the notations of Lemma 3.2, let $\Phi = I \cup J \cup K$ be any partition of the set of double indices (i, j), $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, with K non-empty, that is $\nu > 0$, and let h_J be a fixed element of $[0, s-1]^J$. By Lemma 3.3, and using the fact that the A_{ij} and B_{ij} are $O(P)$, we deduce that the term arising from Lemma 3.2 corresponding to such a Φ and h_J is

$$\ll (N\mathfrak{a}_\gamma)^{\nu(s-1)-\lambda+|h_J|}(P^{-\delta}(N\mathfrak{a}_\gamma)^{-1})^{|h_J|+s\nu}P^{\lambda+\nu} = P^{-\delta(s\nu+|h_J|)}(PN\mathfrak{a}_\gamma^{-1})^{\lambda+\nu},$$

and since $PN\mathfrak{a}_\gamma^{-1} \geqslant P^{2\delta} > 1$, and $\nu \geqslant 1$, the above expression is $\ll P^{-\delta s}(P(N\mathfrak{a}_\gamma)^{-1})^{mn}$. Summation over the Z_{ij} and integration over
33 B_γ gives the contribution of the terms corresponding to a fixed Φ and h_J to I_γ to be $\ll P^{mn-\delta s}(P^{-2+\delta}N\mathfrak{a}_\gamma)^{-n}$, and hence the contribution to I is

$$P^{(m-2-\delta)n-s\delta} \sum_{N\mathfrak{a}\leqslant P^{1-2\delta}} (N\mathfrak{a})^{1-n} \ll P^{(m-2-\delta)n-(s+2)\delta+1} \ll P^{mn-3n-\delta}$$

since $s\delta > n+1$. This takes care of the terms arising from Lemma 3.2 corresponding to the partitions Φ for which K is non-empty. For the estimation of the remaining terms, we have to make a proper choice of the box $\mathscr{B}$. The possibility of this choice is given by the next

LEMMA 3.4. *Let* $C(\xi) = \sum_{i=1}^{n} C_i(\xi_{11}, \ldots, \xi_{mm})\omega_i$ *, where each* C_i *is a cubic form with rational integral coefficients in the variables* ξ_{ij}*. We*

assume that $m > 1$ *and that* $C(\xi)$ *does not represent zero non-trivially. Then there exists a box*

$$\mathscr{B} = \{\xi \in (K_R)^m | a_{ij} < \xi_{ij} < b_{ij}\}$$

in $(K_R)^m$ *such that the following conditions hold:*

(a) $a_{ij} = m_{ij}/2N$, $b_{ij} = n_{ij}/2N$, *where* m_{ij} *and* n_{ij} *are odd integers, and* N *a positive integer. We have* $b_{ij} - a_{ij} < 1$;

(b) there is a point $\xi^0 = (\xi^0_{ij})$ *in* $\mathscr{B}$ *and a set* R *of* n *couples* (p_j, q_j) $(j = 1, \ldots, n)$ *such that we have*

$$C_i(\xi^0) = 0 \quad (i = 1, \ldots, n),$$

$$\det \left| \frac{\partial C_i}{\partial \xi_{(p_j,q_j)}}(\xi^0) \right| \neq 0.$$

Let T *denote the set of couples* $(i, j)(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ *not belonging to* R;

(c) for any subset S *of* R, *there is a subset* S' *of* $\{1, 2, \ldots, n\}$ *such that for any fixed values of* ξ_{ij} *for* $(i, j) \in R - S$ *satisfying* $a_{ij} < \xi_{ij} < b_{ij}$, *the mapping* $(\xi_T, \xi_S) \to (\xi_T, \eta_{S'})$ *defined by* $\eta_i = C_i(\xi)$ *is an analytic isomorphism of the domain defined by* $a_T < \xi_T < b_T$, $a_S < \xi_S < b_S$ *onto a domain in the* $(\xi_T, \eta_{S'})$ *space containing the domain defined by* $a_T < \xi_T < b_T$, $|\eta_{S'}| < \epsilon$ (ξ_T, ξ_S, *etc., stand for the set of variables* ξ_{ij} *with* $(i, j) \in T$, S, *etc., respectively;* ϵ *is some fixed positive quantity).*

In order not to interrupt our discussion, we shall postpone the proof of this lemma to the end of this section.

Let us now turn to the estimation of those terms arising from the substitution from Lemma 3.2 in (9), corresponding to a fixed partition Φ of the double-indices into two sets I and J (in the notations of the same lemma) with J non-empty, and to a fixed h_J. Let $I = \{(i_1, j_1), \ldots, (i_p, j_p)\}$ and $J = \{(i'_1, j'_1), \ldots, (i'_q, j'_q)\}$, with $q > 0$, and

let $h_J = (h_{(i'_1,j'_1)}, \ldots, h_{(i'_q,j'_q)})$. Such a term when summed over the Z_{ij} and integrated over B_γ gives

$$\pm(N\mathfrak{a}_\gamma)^{-p+h}T \sum_{\frac{1}{3}h\leqslant|\alpha|\leqslant h} \int_{A_I}^{B_I} \Phi_\alpha(\xi) \left\{ \int_{|\beta|<\mu} \beta^\alpha e(S(\beta C(\xi)))d\beta \right\} d\xi_1, \tag{10}$$

where the Φ_α are as in Lemma 3.3, each one of the variables

$$\xi_{\{i'_1,j'_1\}}, \ldots, \xi_{\{i'_q,j'_q\}}$$

is fixed at a corresponding A_{ij} or B_{ij}, μ stands for the expression

$$P^{-2-\delta}(N\mathfrak{a}_\gamma)^{-1},$$

and T is defined by the equation

$$T = \sum_{Z_{ij}=1}^{N\mathfrak{a}_\gamma} e(S(\gamma C(Z)))\psi_{h(i'_1,j'_1)}\left(\frac{\xi_{(i'_1,j'_1)} - Z_{(i'_1,j'_1)}}{N\mathfrak{a}_\gamma}\right)\ldots\psi_{h(i'_q,j'_q)}\left(\frac{\xi_{(i'_q,j'_q)} - Z_{(i'_q,j'_q)}}{N\mathfrak{a}_\gamma}\right).$$

34 Let us put

$$S_r(x) = \int_{-x}^{x} t^r e(t)dt.$$

Then (10) becomes

$$\pm(N\mathfrak{a}_\gamma)^{-p+h}T \sum_{\frac{1}{3}h\leqslant|\alpha|\leqslant h} \int_{A_I}^{B_I} \Phi_\alpha(\xi) \prod_{r=1}^{n} \frac{S_{\alpha_r}(\mu C_r(\xi))}{(C_r(\xi))^{\alpha_r+1}} d\xi_I.$$

If we change ξ (including the fixed ξ_J) to $P\xi$, and use the fact that Φ_α is homogeneous of degree $3|\alpha| - h$, the above expression becomes

$$\pm(N\mathfrak{a}_\gamma)^{-p+h}T \sum_{\frac{1}{3}h\leqslant|\alpha|\leqslant h} P^{p-3n-h} \int_{a_I}^{b_I} \Phi_\alpha(\xi) \prod_{r=1}^{n} \frac{S_{\alpha_r}(P^3\mu C_r(\xi))}{(C_r(\xi))^{\alpha_r+1}} d\xi_I.$$

Now let $S = R \cap I$ and S' the subset of $\{1, 2, \ldots, n\}$ corresponding to S by Lemma 3.4(c); let $T' = I - R \cap I = T \cap I$. For any fixed values of

ξ_{R-I} in the respective intervals, the mapping $\xi_I = (\xi_S, \xi_{T'}) \to (\eta_{S'}, \xi_{T'})$ defined by $\eta_i = C_i(\xi)$ is an analytc isomorphism of the parallelepiped $a_I < \xi_I < b_I$ onto a domain in the $(\eta_{S'}, \xi_{T'})$ space, whose Jacobian we shall denote by $J(\xi_{T'}, \eta_{S'})$. Writing ϕ for $P^3\mu$, the above expression then transforms into

$$\pm (N\mathfrak{a}_\gamma)^{-p+h} T \sum_{\frac{1}{3}h \leqslant |\alpha| \leqslant h} P^{p-3n-h}$$
$$\times \int \prod_{r \in S'} \frac{S_{\alpha_r}(\phi\eta_r)}{\eta_r^{\alpha_r+1}} \left\{ \int_{\mathscr{D}(\eta_{S'})} \prod_{r \notin s'} \frac{S_{\alpha_r}(\phi C_r(\xi))}{(C_r(\xi))^{\alpha_r+1}} \Phi_\alpha(\xi) |J(\xi_{T'}, \eta_{S'})| d\xi_{T'} \right\} d\eta_{S'},$$

where the integration with respect to $\eta_{S'}$ is over a certain bounded domain and for every $\eta_{S'}$ in this domain, $\mathscr{D}(\eta_{S'})$ is a uniformly bounded domain in the $\xi_{T'}$-space. It is easy to see (by direct evaluation, or otherwise), that we have $S_p(x) \ll x^p$ for all x and $S_p(x) \ll x^{p+1}$ for $x \leqslant 1$. Since $\Phi_\alpha(\xi)$ and J are absolutely and uniformly bounded on any bounded domain, the above integral is

$$\ll \phi \sum_{r \notin S'} (\alpha_r + 1) \prod_{r \in S} \left\{ \int_{-A}^{A} |\frac{S_{\alpha_r}(\phi\eta_r)}{\eta_r^{\alpha_r+1}}| d\eta_r \right\} \ll \phi^{|\alpha|+n_0} \log^n \phi \ll \phi^{|\alpha|+n_0} \log^n P,$$

where n_0 denotes the number (n – number of elements in S). Hence (10) is bounded by

$$|T|(P(N\mathfrak{a}_\gamma)^{-1})^{p-h} P^{-3n} (P^3\mu)^{n_0+|\alpha|} \log^n P = |T|(P(N\mathfrak{a}_\gamma)^{-1})^{p-h+n_0+|\alpha|}$$
$$P^{-3n-\delta(n_0+|\alpha|)} \log^n P \ll |T|(P(N\mathfrak{a}_\gamma)^{-1})^{p+n_0} P^{-3n-\delta n_0}.$$

We shall now estimate $|T|$. We have for any integer $k \geqslant 0$ and any integer $Q \geqslant 0$,

$$\psi_k(x) = (-1)^{k+1} \sum_{l=-Q}^{Q}{}' \frac{e(lx)}{(2\pi i l)^{k+1} + R_Q},$$

where the $'$ over the summation sign means that there is no term corresponding to $l = 0$, and

$$|R_Q| \ll \frac{1}{Q^{k+1}||x||}.$$

We shall now assume that P varies through odd multiples of the integer denoted by N in Lemma 3.4(a). It follows that for Z_{ij} integral and $\xi_{ij} = Pa_{ij}$ or Pb_{ij},

$$\left|\left|\frac{\xi_{ij} - Z_{ij}}{N\mathfrak{a}_\gamma}\right|\right| \gg \frac{1}{N\mathfrak{a}_\gamma},$$

35 and on on substituting from the above expression for $\psi_k(x)$ into T, with $Q = (N\mathfrak{a}_\gamma)^{mn+1}$, we obtain

$$|T| \leqslant \left(\frac{1}{2\pi}\right)^{h+q} \sum_{l_1,\ldots,l_q=-(N\mathfrak{a}_\gamma)^{mn+1}}^{(N\mathfrak{a}_\gamma)^{mn+1}} |l_1|^{-h_{(i'_1,j'_1)^{-1}}} \cdots |l_q|^{-h_{(i'_q,j'_q)^{-1}}} \times \left| \sum_{Z_{ij}=1}^{N\mathfrak{a}_\gamma} e(S(\gamma C(Z))) + \frac{l_1 Z_{\{i'_1,j'_1\}} + \ldots + l_q Z_{\{i'_q,j'_q\}}}{N\mathfrak{a}_\gamma} \right|$$

and by our assumption in Lemma 3.1, we get

$$|T| \ll (N\mathfrak{a}_\gamma)^{mn-\omega} (\log N\mathfrak{a}_\gamma)^{mn} \ll (N\mathfrak{a}_\gamma)^{mn-\omega'}$$

for any fixed $\omega' < \omega$. Substituting in our earlier estimate for (10), summing over all $\gamma \bmod \mathfrak{d}^{-1}$ with $\mathfrak{a}_\gamma = \mathfrak{a}$ and finally summing over all $\mathfrak{a}$ with $N\mathfrak{a} \leqslant P^{1-2\delta}$ we deduce that the total contribution to I of the terms arising from a partition Φ for which J is non empty is

$$\begin{aligned} &\ll P^{p+n_0-3n-\delta n_0} \log^n P \sum_{N\mathfrak{a}<P^{1-2\delta}} (N\mathfrak{a})^{mn-p-n_0-\omega'+1} \\ &\ll P^{p+n_0-3n-\delta n_0} \log^{n+1} P (1 + P^{mn-p-n_0-\omega'+2}). \end{aligned}$$

Since by definition $n_0 \leqslant q$, we must have $p + n_0 \leqslant p + q = mn$, and if $p + n_0 = mn$, we must have $n_0 = q \geqslant 1$. Hence if we put $\rho = \min(\delta, \omega' - 2)$, the above expression is $O(P^{mn-3n-\rho} \log^{n+1} P)$ which is $o(P^{(m-3)n})$.

To complete the proof of Lemma 3.1, it only remains to estimate the main term which arises from the partition Φ for which J and K are empty, so that I is the set of all couples (i, j), $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant$

n. A calculation exactly similar to the one made above shows that the contribution of this term to I_γ is

$$(N\mathfrak{a}_\gamma)^{mn} S_1((0),\gamma) \int_{P\mathscr{B}} \left\{ \int_{-\mu}^{\mu} e(S(\beta C(\xi)))d\beta \right\} d\xi$$
$$= P^{(m-3)n}(N\mathfrak{a}_\gamma)^{-1} S_\gamma \int_{\mathscr{B}} \left(\prod_{r=1}^{n} \frac{\sin \phi C_r(\xi)}{C_r(\xi)} \right) d\xi$$

with the notations of the previous paragraph. Changing from (ξ_R, ξ_T) to (η, ξ_T), the above becomes

$$P^{(m-3)n}(N\mathfrak{a}_\gamma)^{-m} S_\gamma \int_{a_T}^{b_T} d\xi_T \left\{ \int_{\mathscr{D}(\xi_T)} \prod_{r=1}^{n} \frac{\sin \phi\eta_r}{\eta_r} |J(\xi_T,\eta)| d\eta \right\},$$

where $J(\xi_T,\eta)$ denotes the Jacobian of the above transformation and $\mathscr{D}(\xi_T)$ is a certain uniformly bounded domain in the η-space which always contains the domain $|\eta| < \epsilon$, by Lemma 3.4(c).

Now split the domain $\mathscr{D}(\xi_T)$ into 2^n parts in each of which a certain subset of the variables η remain $> \epsilon$, while the others remain $< \epsilon$. We shall show that on any of these subdomains on which at least one η_i remains $> \epsilon$, the integral over this subdomain of $\prod_{r=1}^{n} \eta_r^{-1} \sin \phi\eta_r J(\xi_T,\eta)$ tends to zero uniformly with respect to ξ_T as ϕ tends to ∞. Assume for instance that $\eta_n > \epsilon$ in such a subdomain. We rewrite the integral of the above function as a repeated integral

$$\int \frac{\sin \phi\eta_1}{\eta_1} d\eta_1 \int \frac{\sin \phi\eta_2}{\eta_2} d\eta_2 \quad \ldots \quad \int \frac{\sin \phi\eta_n}{\eta_n} d\eta_n.$$

Now, since the boundary of $\mathscr{D}(\xi_T)$ consists of algebraic hypersurfaces, **36**
it is easy to see that the line through any point $(\eta_1, \ldots, \eta_n)$ in $\mathscr{D}(\xi_T)$ parallel to the η_n-axis meets the boundary of $\mathscr{D}(\xi_T)$ in a uniformly bounded number of points. Hence the last integral with respect to η_n is over a uniformly bounded number of intervals in η_n on all of which intervals we have $|\eta_n| > \epsilon$. Since $|J(\xi_T,\eta)|/\eta_n$ is uniformly bounded and of uniformly bounded variation with respect to η_n in any bounded interval on

which $|\eta_n| > \epsilon$, we deduce that the above repeated integral is

$$\ll \frac{1}{\phi}\left\{\int_{-A}^{A}\left|\frac{\sin\phi\eta}{\eta}\right|d\eta\right\}^{n-1} \ll \frac{\log^{n-1}\phi}{\phi}$$

$\to 0$ uniformly as $\phi \to \infty$, and since $\phi = P^3\mu = P^{1-\delta}(N\mathfrak{a}_\gamma)^{-1} \geqslant P^\delta$, also as $P \to \infty$. Hence the 'main term' of I_γ is

$$P^{mn-3n}(N\mathfrak{a}_\gamma)^{-m}S_\gamma\left[\int_{a_T}^{b_T}d\xi_T\left\{\int_{|\eta|<\epsilon}\prod_r\frac{\sin\phi\eta_r}{\eta_r}|J(\xi_T,\eta)|d\eta\right\}+o(1)\right].$$

Now since $|J(\xi_T,\eta)|$ (being always equal to $J(\xi_T,\eta)$ or to $-J(\xi_T,\eta)$ in $\mathscr{D}(\xi_T)$) is analytic in η, we may write

$$|J(\xi_T,\eta)| = |J(\xi_T,0)| + \sum_i \eta_i\psi_i(\xi_T,\eta),$$

where the ψ_i are again analytic. It can be shown by an argument similar to the one used above that

$$\int_{|\eta|<\epsilon}\prod_r\frac{\sin\phi\eta_r}{\eta_r}\eta_i\psi_i(\xi_T,\eta)d\eta$$

$\to 0$ as $P \to \infty$ for all i uniformly. Hence the main term reduces to

$$P^{(m-3)n}(N\mathfrak{a}_\gamma)^{-m}S_\gamma\left\{\int_{a_T}^{b_T}|J(\xi_T,0)|d\xi_T\left(\int_{-\epsilon}^{\epsilon}\frac{\sin\phi x}{x}dx\right)^n+o(1)\right\}$$

$$= P^{(m-3)n}(N\mathfrak{a}_\gamma)^{-m}S_\gamma(A+o(1)),$$

where $$A = \pi^n\int_{a_T}^{b_T}|J(\xi_T,0)|d\xi_T > 0$$

and $o(1)$ tends to zero uniformly as $P \to \infty$.

On summing over the γ for which $\mathfrak{a}_\gamma = \mathfrak{a}$ and summing over all $\mathfrak{a}$ with $N\mathfrak{a} \leqslant P^{1-2\delta}$, we see that the main term in I is

$$AP^{(m-3)n}\sum_{\substack{\gamma \bmod \mathfrak{d}^{-1}\\ N\mathfrak{a}_\gamma\leqslant P^{1-2\delta}}}(N\mathfrak{a}_\gamma)^{-m}S_\gamma + o\left(P^{(m-3)n}\sum_{\substack{\gamma \bmod \mathfrak{d}^{-1}\\ N\mathfrak{a}_\gamma\leqslant P^{1-2\delta}}}(N\mathfrak{a}_\gamma)^{-m}|S_\gamma|\right).$$

Since $(N\mathfrak{a}_\gamma)^{m(n-1)}S_\gamma = S_1((0),\gamma)$, the assumption of Lemma 4.1 implies that

$$S_\gamma = O((N\mathfrak{a}_\gamma)^{m-\omega}),$$

with $\omega > 2$, and hence the series

$$\mathfrak{r} = \sum_{\gamma \bmod \mathfrak{d}^{-1}} (N\mathfrak{a}_\gamma)^{-m} S_\gamma$$

converges absolutely.

Hence we obtain finally that **37**

$$I = A\mathfrak{r}P^{(m-3)n} + o(P^{(m-3)n}),$$

and Lemma 3.1 will be proved completely if we can show that $\mathfrak{r} > 0$. This will be our next task.

LEMMA 3.5. *Under the assumptions of Lemma 3.1. we have*

$$\mathfrak{x} > 0.$$

Proof. It can be shown by standard arguments that if γ_1 and γ_2 are elements of K different from zero such that $\mathfrak{a}_{\gamma_1}$ and $\mathfrak{a}_{\gamma_2}$ are coprime, we have $S_{\gamma_1}S_{\gamma_2} = S_{\gamma_1+\gamma_2}$. For any prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, we define

$$\chi(\mathfrak{p}) = 1 + \sum_{p \geqslant 1} (N\mathfrak{p})^{-mp} \sum_{\substack{\gamma \bmod \mathfrak{d}^{-1} \\ \mathfrak{a}_\gamma = \mathfrak{p}^p}} S_\gamma;$$

this series is absolutely convergent, since it is majorized by the absolute values of the terms of the series for $\mathfrak{r}$, and we deduce easily that

$$\mathfrak{r} = \prod_{\mathfrak{p}} \chi(\mathfrak{p}),$$

the product being absolutely convergent. To prove the lemma, it is therefore sufficient to show that $\chi(\mathfrak{p}) > 0$ for all $\mathfrak{p}$.

Now, let $M(\mathfrak{p}^N)$ denote for any integer N the number of solutions of the congruence

$$C(X) \equiv O(\mathfrak{p}^N),$$

when the X_i run through a complete system of residues modulo $\mathfrak{p}^N$. Then it can be shown easily that

$$\sum_{p=0}^{N} (N\mathfrak{p})^{-mp} \sum_{\substack{\gamma \bmod \mathfrak{d}^{-1} \\ \mathfrak{a}_\gamma = \mathfrak{p}^p}} S_\gamma = (N\mathfrak{p})^{-(m-1)N} M(\mathfrak{p}^N),$$

and therefore

$$\chi(\mathfrak{p}) = \lim_{N\to\infty} (N\mathfrak{p})^{-N(m-1)} M(\mathfrak{p}^N).$$

Let $K_\mathfrak{p}$ denote the $\mathfrak{p}$-adic completion of the field K. Then it is well known (and is easily established) that if the form $C(X)$ has a non-trivial *non-singular zero* in $K_\mathfrak{p}$, then

$$\lim_{N\to\infty} (N\mathfrak{p})^{-N(m-1)} M(\mathfrak{p}^N) > 0.$$

From the fact that $C(X)$ does not represent zero non-trivially it follows that it is non-degenerate, i.e. we cannot reduce it to a form in a fewer number of variables by making a non-singular linear change of variable in $C(X)$. But for a non-degenerate cubic form over a $\mathfrak{p}$-adic field in more than 9 variables, the existence of a non-singular zero has been proved by several authors (see, for instance, (2)).

This finishes the proof of Lemma 3.5. □

Lemma 3.1 is therefore completely proved, except for the proof of Lemma 3.4, which we shall now give.

Proof of Lemma 3.4. It is enough to prove the existence of a point ξ^0 in $(K_R)^m$, $\xi^0 \neq 0$, such that

$$C_i(\xi^0) = 0 \quad (i = 1, \ldots, n), \qquad \text{Rank} \left\| \frac{\partial C_i}{\partial \xi_{(j,k)}} (\xi^0) \right\| = n.$$

38 For, once we have found such a ξ^0, we can find a box $\mathscr{B}$ satisfying the conditions *(a)*, *(b)* and *(c)* by the implicit function theorem (to get *(a)*, we use the fact that the rational numbers of the form $a/2N$, where a is an odd integer and N a positive integer, are dense in R). We are therefore

reduced to proving the existence of a real non-trivial, non-singular zero of the variety defined by the equations $C^i(\xi) = 0 (1 \leqslant i \leqslant n)$.

Let $\chi^{(1)}, \ldots, \chi^{(r_1)}$ denote the distinct isomorphisms of K into R, and $\chi^{(r_1+1)}, \ldots, \chi^{(n)}$ the distinct isomorphisms of K into C such that $\chi^{(r_1+r_2+p)} = \overline{\chi}^{(r_1+p)} (1 \leqslant p \leqslant r_2)$. For any $x \in K$, we shall put $x^{(i)} = \chi^{(i)}(x)$, and $C^{(i)}(X)$ will denote the form deduced from $C(X)$ by applying $\chi^{(i)}$ to its coefficients. For $1 \leqslant p \leqslant r_2$, we put

$$C^{(r_1+p)}(X + iY) = C_1^{(r_1+p)}(X, Y) + iC_2^{(r_1+p)}(X, Y),$$

where $C_1^{(r_1+p)}$ and $C_2^{(r_1+p)}$ are forms in the variables $X = (X_{ij})$ and $Y = (Y_{ij})$ with real coefficients.

If we now make the following real non-singular transformation of the variables ξ_{ij},

$$\left.\begin{aligned} \eta_i^{(p)} &= \sum_j \xi_{ij} \omega_j(p) && (1 \leqslant p \leqslant r_1), \\ \eta_i'^{(r_1+p)} &= \sum_j \xi_{ij} \omega_j'^{(r_1+p)} \\ \eta_i''^{(r_1+p)} &= \sum_j \xi_{ij} \omega_j''^{(r_1+p)} \end{aligned}\right\} \quad (1 \leqslant p \leqslant r_2),$$

where $\omega_j'^{(r_1+p)}$ and $\omega_j''^{(r_1+p)}$ stand for the real and imaginary parts of $\omega_j^{(r_1+p)}$, the variety in the ξ-space defined by $C_i(\xi) = 0 (i = 1, \ldots, n)$ gets transformed into the variety in the η-space defined by the equations

$$\left.\begin{aligned} C^{(p)}(\eta_1^{(p)}, \ldots, \eta_m^{(p)}) &= 0 \quad (1 \leqslant p \leqslant r_1), \\ C_1^{(r_1+p)}(\eta_1'^{(r_1+p)}, \ldots, \eta_m'^{(r_1+p)}, \eta_1''^{(r_1+p)}, \ldots, \eta_m''^{(r_1+p)}) &= 0 \\ C_2^{(r_1+p)}(\eta_1'^{(r_1+p)}, \ldots, \eta_m'^{(r_1+p)}, \eta_1''^{(r_1+p)}, \ldots, \eta_m''^{(r_1+p)}) &= 0 \end{aligned}\right\} (1 \leqslant p \leqslant r_2),$$

and is therefore the product of the varieties V_p defined by

$$C^{(p)}(X) = 0 (1 \leqslant p \leqslant r_1)$$

and the varieties W_q defined by $C_1^{(r_1+q)}(X,Y) = 0$, $C_2^{(r_1+q)}(X,Y) = 0 (1 \leqslant q \leqslant r_2)$. Hence it is enough to find a real non-trivial, non-singular point on each of the varieties V_p, W_q. It is shown in [3], Lemma 6.1 that such a point exists on each V_p. As for W_q, it is easy to see that if $Z = X + iY$ is a non-singular zero of $C^{(r_1+q)}(Z) = 0$, where X and Y are real, (X,Y) is a real non-singular zero of W_q. This concludes the proof of Lemma 3.4.

We make a final remark concerning the exponential sum $S_1(l,\gamma)$. Suppose $C(X)$ and $C'(Y)$ are two cubic forms in the disjoint sets of variables X and Y, and $S_1(l,\gamma)$ and $S_1'(l',\gamma)$ are the corresponding sums. Then if $S_1''((l,l'),\gamma)$ denotes the sum corresponding to the form $C(X) + C'(Y)$ in the variables X and Y, we have clearly

$$S_1''((l,l'),\gamma) = S_1'(l,\gamma)S_1'(l',\gamma)$$

and therefore the uniform estimates

$$|S_1(l,\gamma)| \ll (N\mathfrak{a}_\gamma)^{mn-\omega}, \quad |S_1'(l',\gamma)| \ll (N\mathfrak{a}_\gamma)^{m'n-\omega'}$$

imply the uniform estimate $|S_1''((l,l'),\gamma)| \ll (N\mathfrak{a}_\gamma)^{(m+m')n-(\omega+\omega')}$, where m and m' are the number of variables X, and Y respectively. We shall make use of this remark later.

4 Proof of the theorem

39 We preserve the notations of §§2, 3.

We shall denote by $\mathfrak{m}$ the set of points of $\mathscr{R}$ which do not belong to the major arcs, i.e. $\mathfrak{m} = \mathscr{R} - \bigcup_{N\mathfrak{a}_\gamma \leqslant P^{1-2\delta}} B_\gamma$. For any θ with $0 < \theta < 1$, we shall denote by $\mathfrak{m}_\theta$ the set of points $\alpha \in \mathfrak{m}$ for which there does not exist an integer $\lambda \in P^{2\theta+\delta}B_0$, $\lambda \neq 0$, and a $\mu \in \mathfrak{d}^{-1}$ such that we have

$$\lambda\alpha - \mu = \sum \epsilon_i\rho_i, \quad |\epsilon_i| < P^{-3+2\theta+\delta}.$$

We shall show that if $\theta < \{1 - \delta(n+2)\}/2n$ and P is large enough, $\mathfrak{m}_\theta = \mathfrak{m}$. Suppose on the contrary that there exists an integer λ and an

element $\mu \in \mathfrak{d}^{-1}$ satisfying the above conditions. Putting $\gamma = \mu/\gamma$, we have clearly $\lambda \in \mathfrak{a}_\gamma$ and therefore

$$N\mathfrak{a}_\gamma \leqslant N\lambda \ll P^{n(2\theta+\delta)},$$

and by our assumption on θ, it follows that if P is large enough, $N\mathfrak{a}_\gamma \leqslant P^{1-2\delta}$. Moreover, we have

$$\alpha - \gamma = \alpha - \mu/\lambda = \lambda^{-1}\sum \epsilon_i\rho_i = \sum_{i=1}^{n}\epsilon_i\left(\sum_{j=1}^{n}\alpha_{ij}\rho_j\right)$$
$$= \sum_{j=1}^{n}\left(\sum \alpha_{ij}\epsilon_i\right)\rho_j = \sum_{j=1}^{n}\beta_j\rho_j,$$

where (a_{ij}) is the regular representation matrix of λ^{-1} with respect to the basis $\rho_1, \ldots, \rho_n$ of K over Γ. Hence it follows that (a_{ij}) is the inverse of the regular representation matrix of λ with respect to the same basis. Hence for any $i, j (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n)$, we have

$$a_{ij} \ll P^{(n-1)(2\theta+\delta)}(N\lambda)^{-1},$$
$$|\beta_j| \ll P^{-3+2\theta+\delta+(n-1)(2\theta+\delta)}(N\lambda)^{-1} \ll P^{-3+n(2\theta+\delta)}(N\mathfrak{a}_\gamma)^{-1},$$

and therefore for P large, $|\beta_j| \ll P^{-2-2\delta}(N\mathfrak{a}_\gamma)^{-1}$, which shows that α lies on the 'major arc' B_γ, and hence cannot be in $\mathfrak{m}$. Our contention is proved.

For any θ with $0 < \theta < 1$, we shall denote by $\mathscr{E}(\theta)$ the set of points $\alpha \in \mathscr{R}$ for which there exists an integer $\lambda \in P^{2\theta+\delta}B_0$, $\lambda \neq 0$, and a $\mu \in \mathfrak{d}^{-1}$, such that

$$\lambda\alpha - \mu = \sum \epsilon_i\rho_i, \quad |\epsilon_i| < P^{-3+2\theta+\delta}.$$

We have $$\mathfrak{m} - \mathfrak{m}_\theta = \mathfrak{m} \cap \mathscr{E}(\theta) \subset \mathscr{E}(\theta).$$

We want to estimate the measure of $\epsilon(\theta)$. For a fixed λ and μ, the set of α having the above property is equal to $(P^{-3+2\theta+\delta})^n(N\lambda)^{-1}$, since multiplication by λ multiplies the n-dimensional measures of sets by $N\lambda$.

Since for any fixed integer λ, there correspond at most $O(N\lambda)$ elements $\mu \in \mathfrak{d}^{-1}$ for which there exists an $\alpha \in \mathscr{R}$ satisfying the above inequalities with respect to this λ and μ, we obtain

$$|\mathscr{E}(\theta)| \ll \sum_{\lambda \in P^{2\theta+\delta}B_0} (P^{-3+2\theta+\delta})^n (N\lambda)^{-1} N\lambda \ll (P^{-3+4\theta+2\delta})^n.$$

We need two lemmas which give estimates for exponential sums involving a single cube.

LEMMA 4.1. *For $\alpha \in \mathscr{R}$, B a fixed box in K_R, d an integer in K and $P > 1$, we define*

$$T_d(\alpha) = \sum_{X \in PB} e(S(\alpha d X^3)).$$

Then for $\alpha \in \mathfrak{m}_\theta$ and P large, we have

$$|T_d(\alpha)| < P^{(1-\frac{1}{4}\theta)n}.$$

40 *Proof.* We apply Lemma 1.3 of §1 to $\Gamma(X) = \alpha d X^3$, with $\theta + \frac{1}{3}\delta$ instead of θ and $\kappa = \frac{1}{4}\theta$; we have $B_1(X, Y) = 6\alpha d XY$. We conclude that if the above estimate for $|T_d(\alpha)|$ is not valid, and P large enough, there exist $X \in P^{\theta+\frac{1}{3}\delta}B_0$, $Y \in P^{\theta+\frac{1}{3}\delta}B_0$, neither zero, and $\mu \in \delta^{-1}\mathfrak{d}$ such that

$$6\alpha d XY - \mu = \sum_1^n \epsilon_i \rho_i,$$

$$|\epsilon_i| < P^{-3+2\theta+\frac{2}{3}\delta} < P^{-3+2\theta+\delta}.$$

For, the number of solutions of these inequalities is, by Lemma 1.3, $\gg \{P^{\theta+\frac{2}{3}\delta}(\log P)^{-1}\}^n$ and the number of solutions with X or Y zero is $\ll P^{n\theta}$.

But now, $6dXY$ is a non-zero integer and is in $P^{2\theta+\delta}B_0$ for P large. This contradicts the fact that $\alpha \in \mathfrak{m}_\theta$, proving the lemma.

□

LEMMA 4.2. *Let $\gamma \in K$ and let $l_1, \ldots, l_n$ be any set of rational integers. Then if*

$$X = \sum_1^n X_i \omega_i$$

runs through a complete system of residues mod $N\mathfrak{a}_\gamma$, *we have for any* $\epsilon > 0$,

$$\left| \sum_{X \bmod N\mathfrak{a}_\gamma} e\left(S(\gamma d X^3) + \frac{l_1 X_1 + \ldots + l_n X_n}{N\mathfrak{a}_\gamma} \right) \right| \ll (N\mathfrak{a}_\gamma)^{n-\frac{1}{3}+\epsilon}.$$

Proof. Let $\mathfrak{a} = \text{g.c.d}\left(\gamma, \dfrac{\sum l_i' \rho_i}{N\mathfrak{a}_\gamma}\right)$ and $\mathfrak{b} = (1, \mathfrak{a}\mathfrak{d})^{-1}$.

It is clear that $\mathfrak{a}_\gamma$ divides $\mathfrak{b}$, which in turn divides $N\mathfrak{a}_\gamma$. The above sum can be rewritten as

$$\sum_{X \bmod N\mathfrak{a}_\gamma} e\left(S\left(\gamma d X^3 + \frac{\sum l_i \rho_i}{N\mathfrak{a}_\gamma} X \right) \right) = \frac{(N\mathfrak{a}_\gamma)^n}{N\mathfrak{b}} \sum_{X \bmod \mathfrak{b}} e\left(S\left(\gamma d X^3 + \frac{\sum l_i \rho_i}{N\mathfrak{a}_\gamma} X \right) \right).$$

By Hua (4), the latter sum is $\ll (N\mathfrak{b})^{\frac{2}{3}+\epsilon}$, and therefore for the sum we started with, we obtain

$$|\sum| \ll (N\mathfrak{a}_\gamma)^n (N\mathfrak{b})^{\frac{1}{3}+\epsilon} \ll (N\mathfrak{a}_\gamma)^{n-\frac{1}{3}+\epsilon}$$

since $N\mathfrak{b} \geqslant N\mathfrak{a}_\gamma$. Lemma 4.2 is proved. □

LEMMA 4.3. *Let*

$$C(X, Y) = C^{(1)}(X_1, \ldots, X_{m_1}) + d_1 Y_1^3 + \ldots + d_t Y_t^3$$

be a cubic form with symmetric integral coefficients in K which does not represent zero. We assume moreover that

(i) $0 \leqslant t \leqslant 8$;

(ii) if r is the least integer $> 7 - \dfrac{1}{2}t$, *then* $m_1 \geqslant 2(r+1)$.

Then $C^{(1)}(X)$ *splits with remainder* $m_1 - r$.

Proof. Assuming that $C^{(1)}(X)$ does not split with remainder $m_1 - r$, we shall show that $C(X, Y)$ represents zero non-trivially.

For any $\gamma \in K$, and any sets of rational integers

$$l = (l_{ij})\ (1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant n)$$
$$\text{and} \quad l' = (l'_{kj})\ \ (1 \leqslant k \leqslant t, 1 \leqslant j \leqslant n),$$

let $S_1^{(1)}(l, \gamma)$ and $S_1(l, l', \gamma)$ be the exponential sums associated to the forms $C^{(1)}$ and C respectively. We may apply Lemma 2.3 to the form $C^{(1)}$ with $\kappa = \frac{1}{4}(r+1) - \delta$, since **41**

$$0 < 8\kappa = 2(r+1) - 8\delta < m_1$$

and $C^{(1)}(x)$ does not split with remainder $m_1 - r$ which is the least integer

$$\geqslant m_1 - 4\kappa = m_1 - r - 1 + 4\delta.$$

Hence we obtain $\quad |S_1^{(1)}(l, \gamma)| \ll (N\mathfrak{a}_\gamma)^{m_1 n - \frac{1}{4}(r+1)+\delta}$.

This, coupled with Lemma 4.2 and the remark made at the end §3, gives

$$|S_1(l, l', \gamma)| \ll (N\mathfrak{a}_\gamma)^{(m_1+t)n - \frac{1}{4}(r+1) - \frac{1}{3}t + 2\delta}$$

and

$$\frac{1}{4}(r+1) + \frac{1}{3}t - 2\delta \geqslant \frac{1}{4}(r+1) + \frac{1}{8}t - 2\delta = \frac{1}{4}(r+1+\frac{1}{2}t) - 2\delta > 2$$

if δ is small enough, by our assumption (ii). Hence by Lemma 3.1, there is a box $\mathscr{B}$ in $(K_R)^{m_1+t}$ such that if $E(\alpha)$ denotes the exponential sum attached to $C(X, Y)$, the contribution of the major arcs to the integral of $E(\alpha)$ is

$$AP^{(m_1+t-3)n} + o(P^{(m_1+t-3)n}) \quad (A > 0).$$

We shall now show that the contribution of the 'minor arcs' $\mathfrak{m}$ is $o(P^{(m_1+t-3)n})$. Let $\mathscr{B}^{(1)}$ denote the projection of the box $\mathscr{B}$ in $(K_R)^{m_1+t}$ onto the product of the first m_1 components and B_i the projection into the

$(m_1 + i)$th component for $i = 1, \ldots, t$. Let $E^{(1)}(\alpha)$ denote the exponential sum associated to $C^{(1)}$ and the box $\mathscr{B}^{(1)}$, and $T_{d_i}(\alpha)$ the exponential sum associated to $d_i Y_i^3$ and the box B_i, for $i = 1, \ldots, t$. We then have

$$E(\alpha) = E^{(1)}(\alpha) \prod_{i=1}^{t} T_{d_i}(\alpha).$$

The conditions of Lemma 2.2 are fulfilled by the form $C^{(1)}(X)$ with $\kappa = \frac{1}{2}(r+1)\theta(1-\delta)$, and we obtain the estimate

$$|E^{(1)}(\alpha)| < P^{(m_1 - \frac{1}{2}(r+1)\theta(1-\delta))n}$$

for P large and $\alpha \in \mathfrak{m}_\theta$. Hence by Lemma 4.1, we get for $\alpha \in \mathfrak{m}_\theta$

$$|E(\alpha)| < P^{(m_1 - \frac{1}{2}(r+1)\theta(1-\delta) + t - \frac{1}{4}t\theta)n}.$$

Applying this with $\theta = \frac{7}{8}$, we get

$$\int_{\mathfrak{m}_{\frac{7}{8}}} |E(\alpha)| d\alpha = o(P^{(m_1+t-3)n}),$$

since

$$\begin{aligned} m_1 - \frac{1}{2}(r+1)\frac{7}{8}(1-\delta) + t - \frac{7t}{32} &< m_1 + t - \frac{1}{2}(8 - \frac{1}{2}t)\frac{7}{8}(1-\delta) - \frac{7t}{32} \\ &= m_1 + t - \frac{7}{2} + \frac{7\delta}{32}(16 - t) < m_1 + t - 3 \end{aligned}$$

if δ is small.

Now let $\frac{7}{8} = \theta_0 > \theta_1 > \theta_2 > \ldots > 0$ be a decreasing sequence of **42** positive real numbers, to be chosen suitably later. We shall estimate the integral of $E(\alpha)$ on the set $\mathfrak{m}_{\theta_{g+1}} - \mathfrak{m}_{\theta_g}$. Since $\mathfrak{m}_{\theta_{g+1}} - \mathfrak{m}_{\theta_g} \subset \mathscr{E}(\theta_g)$, we obtain by the above estimate for $E(\alpha)$,

$$\int_{\mathfrak{m}_{\theta_{g+1}} - \mathfrak{m}_{\theta_g}} |E(\alpha)| d\alpha \ll |\mathscr{E}(\theta_g)| P^{U_n} \ll P^{V_n},$$

where

$$U = m_1 - \frac{1}{2}(r+1)\theta_{g+1}(1-\delta) + t - \frac{1}{4}t\theta_{g+1},$$

and

$$V = m_1 + t - 3 + 4\theta_g + 2\delta - \frac{1}{2}(r+1)\theta_{g+1}(1-\delta) - \frac{1}{4}t\theta_{g+1}.$$

Hence $$\int_{\mathfrak{m}_{\theta_{g+1}} - \mathfrak{m}_{\theta_g}} |E(\alpha)|d\alpha = o(P^{(m_1+t-3)n})$$

if $$4\theta_g - \frac{1}{2}(r+1)\theta_{g+1} - \frac{1}{4}t\theta_{g+1} + \delta(2 + \frac{7}{16}(r+1)) < \theta$$

i.e. $$4(\theta_g - \theta_{g+1}) + \theta_{g+1}\{4 - \frac{1}{2}(r+1) - \frac{1}{4}t\} < -\{2 + \frac{7}{16}(r+1)\}\delta.$$

This inequality clearly holds if θ_{g+1} remains greater than a fixed positive quantity (independent of δ), δ is small enough and the jumps $\theta_g - \theta_{g+1}$ are less than a positive quantity. Hence we can find a finite sequence

$$\frac{7}{8} = \theta_0 > \theta_1 > \ldots > \theta_G \text{ with } \frac{1}{4n} < \theta_G < \frac{1-\delta(n+2)}{2n},$$

such that

$$\int_{\mathfrak{m}_{\theta_{g+1}} - \mathfrak{m}_{\theta_g}} |E(\alpha)|d\alpha = o(P^{(m_1+t-3)n}).$$

Since $\mathfrak{m}_{\theta_G} = \mathfrak{m}$, we finally reach the conclusion that the number $\mathscr{N}(P)$ of integral points (X, Y) in $P\mathscr{B}$ for which $C(X, Y) = 0$ satisfies

$$\mathscr{N}(P) = \int_{\mathscr{R}} E(\alpha)d\alpha = AP^{(m_1+t-3)n} + o(P^{(m_1+t-3)n}) \quad (A > 0).$$

Since this is positive for P large, it follows that $C(x, Y)$ represents zero non-trivially, which is a contradiction, proving Lemma 4.3.

We have tabulated below for $0 \leqslant t \leqslant 8$ the associated values of r and the minimum permissible value of m_1 for the validity of Lemma 4.3.

t	r	**Minimum permissible value of m_1**
0	8	18
1	7	16
2	7	16
3	6	14
4	6	14
5	5	12
6	5	12
7	4	10

The main theorem stated at the beginning of this paper can be easily deduced from the above lemma.

Let $C(X)$ be a form with symmetric integral coefficients in K in **43**
at least 54 variables. If it does not represent zero, the above lemma says that it must represent a form of the type $C_1(X') + d_1Y_1^3$, where C_1 is a form in at least 46 variables. The form $C_1(X') + d^1Y_1^3$ cannot represent zero either, and another application of the lemma shows that C_1 represents a form of the type $C_2(X'') + d_2Y_2^3$, where C_2 is a form in at least 39 variables. Hence $C(X)$ represents the form $C_2(X'') + d_1Y_1^3 + d_2Y_2^3$. Continuing in this way, we get finally that $C(X)$ represents a form of the type

$$C_8(Z) + d_1Y_1^3 + \ldots + d_8Y_8^3,$$

which therefore cannot represent zero non-trivially. Let d_0 be any value taken by $C_8(Z)$ with $Z \neq (0)$; then $C(X)$ represents $d_0Y_0^3 + d_1Y_1^3 + \ldots + d_8Y_8^3$ and hence this form cannot have a non-trivial zero. But by a theorem of Birch ((1), p.458), we know that any diagonal cubic form in at least 9 variables represents zero non-trivially. Hence there exist $Y_0^0, \ldots, Y_8^0$, not all zero, such that $Y_i = Y_i^0$ is a zero of the form written above. This is a contradiction.

The theorem is therefore proved. □

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Sums of m-th Powers in p-Adic Rings

By C.P. Ramanujam

0 Introduction and notations

Let A be a complete discrete valuation ring of characteristic zero with finite residue field, and for any integer $m > 1$, let $J_m(A)$ be the subring of A generated by the m-th powers of elements of A. We will prove that any element of $J_m(A)$ is a sum of at most $8m^5$ m-th powers of elements of A. We will also prove a similar assertion when the residue field of A is only assumed to be perfect and of positive characteristic, with the number $\Gamma(m)$ of summands depending only on m and not on A.

It follows from this and Theorem 2 of Birch [1] (see also Körner [2]) that any totally positive integer of sufficiently large norm in *any* algebraic number field belonging to the order generated by the m-th powers of integers of this field is actually a sum of at most max $(2^m + 1,\ 8m^5)$ m-th powers of totally positive integers of this field. This answers a question raised by Siegel [3] in the affirmative.

The author has been informed that Dr. B. J. Birch has also solved substantially the same problem by a different method, and that Dr. Birch's work will be published in the Mordell issue of *Acta Arithmetica*.

Notations. A will denote a complete discrete valuation ring, $\mathfrak{p}$ its maximal ideal, and π a generator of $\mathfrak{p}$. The residue field $k = A/\mathfrak{p}$ is assumed to be perfect of characteristic $p > 0$, and $q = p^f$ will denote the number of elements in k when this is finite. Let e be the ramification index of A, and B the maximal unramified complete discrete valuation ring contained in A with the same residue field k.

For any integer $m > 1$, we write $m = n \cdot p^r$, with $r \geq 0$ and $(n, p) = 1$. $J_m(A) = J_m$ will denote the subring of A generated by the m-th powers of elements of A. When k is finite, we denote by f_1 the least positive

divisor of f such that $\dfrac{p^f - 1}{p^{f_1} - 1}$ divides n, and we write $q_1 = p^{f_1}$. We put $\tau = r + 1$ or $r + 2$ according as p is greater than or equal to 2.

$\gamma_1(m)$, $\gamma_2(m)$, $\Gamma(m)$ denote positive integers depending only on m and not on A or p.

1 The unramified case.

We start with a known lemma (Lemma 5 of Birch [1] or Theorem 7 of Stemmler [4]), whose proof we reproduce for completeness.

Lemma 1. *When k is infinite, $J_n = A$ and any element of A is the sum of at most $\gamma_1(n)$ n-th powers.*

When k is finite, let k_1 be its subfield with q_1 elements. Then J_n is
46 *the inverse image of k_1 under the natural homomorphism of A onto k, and any element of J_n is the sum of at most $(n + 1)$ n-th powers.*

[Mathematika 10 (1963), 137-146]

Proof. When k is finite, k_1 is evidently the set of elements of k which are sums of n-th powers of elements of k. By Theorem 1 of Tornheim [5], any element of k_1 is the sum of n n-th powers. Hence any unit α of A whose image in k lies in k_1 can be written as

$$\alpha \equiv \sum_{i=1}^{n} x_i^n (\mathrm{mod}\ \mathfrak{p}), \qquad x_i \in A.$$

Since at least one x_i is a unit and $(n, p) = 1$, it follows from Hensel's lemma that α is a sum of n n-th powers. If α were a non-unit, we write $\alpha = (\alpha - 1) + 1^n$, and the assertion follows.

If k is infinite, it follows from Theorem 2 and the proofs of Theorem 4 and its Corollary 1 of Tornheim [5] that any element of k is a sum of $\gamma_1'(n)$ n-th powers of k, and our assertion follows as before.

The next lemma says that a congruence modulo a high power of p can be refined to an equality, and is again well known (see Theorem 12 of Stemmler [4]). □

Lemma 2. *If*

$$\sum_{i=1}^{a} x_i^m \equiv y (\text{mod } p^\tau A)$$

and x_1 is a unit, there is an $x_1' \in A$, $x_1' \equiv x_1 (\text{mod } pA)$ with $y = (x_1')^m + \sum_2^a x_i^m$.

Proof. Suppose we have found a $z \in A$ with $z \equiv x_1 (\text{mod } pA)$ satisfying

$$z^m \equiv y - \sum_2^a x_i^m (\text{mod } p^s A)$$

for some $s \geq \tau$. If we put $z_1 = z + \lambda p^{s-r}$, it follows easily that

$$z_1^m \equiv z^m + z^{m-1} \cdot n\lambda p^s (\text{mod } p^{s+1}),$$

and since nz^{m-1} is a unit, we can find $\lambda \in A$ such that

$$z_1^m \equiv y - \sum_2^a x_i^m (\text{mod } p^{s+1} A).$$

Since A is complete, the lemma follows. □

Lemma 3. *Let C be a commutative ring, $\mathfrak{a}$ an ideal of C, and s an integer ≥ 0. We denote by $\mathfrak{a}^{(p^s)}$ the set of p^s-th powers of elements of $\mathfrak{a}$, and by $\mathfrak{a}_s$ the set of elements of C of the form $a_0 + pa_1 + \ldots + p^s a_s$ with $a_i \in \mathfrak{a}^{(p^{s-i})}$. Then C_s is a subring[1] of C, and $\mathfrak{a}_s$ an ideal of C_s. If $x, y \in C$, with $x \equiv y (\text{mod } \mathfrak{a})$, then $x^{p^s} \equiv y^{p^s} (\text{mod } \mathfrak{a}_s)$.*

Proof. This being trivial for $s = 0$, we may assume that $s \geq 1$, and that **47**
the lemma holds with $s - 1$ instead of s. Since $C_s = C^{(p^s)} + pC_{s-1}$, the inclusions $C_s \cdot C_s \subset C_s$ and $pC_{s-1} + pC_{s-1} \subset pC_{s-1}$ are trivial, and only the inclusion $C^{(p^s)} + C^{(p^s)} \subset C_s$ remains to be verified. For $x, y \in C$, we have

$$x^{p^s} + y^{p^s} = (x + y)^{p^s} - \sum_{0<i<p^s} \binom{p^s}{i} x^i y^{p^s - i}, \tag{1}$$

[1] C_S is derived from C as $\mathfrak{a}_S$ is from $\mathfrak{a}$,

and if $p^k \| i$, $p^{s-k} \| \binom{p^s}{i}$, and hence by induction hypothesis, $-\sum$ belongs to pC_{s-1}, and the right side belongs to C_s. Thus C_s is a ring, and $\mathfrak{a}_s$ a subring of C_s, and clearly $C_s \mathfrak{a}_s \subset \mathfrak{a}_s$. Finally, if $z = x + y$ with $y \in \mathfrak{a}$, it follows from (1) that $x^{p^s} \equiv z^{p^s} (\mathrm{mod}\ \mathfrak{a}_s)$, which completes the proof of the lemma. □

It follows that the set

$$J'_m = \left\{ x_0^{p^r} + p x_1^{p^{r-1}} + \ldots + p^r x_r | x_i \in J_n \right\}$$

is a subring of A, and we have evidently the inclusion

$$J'_m \subset J_m.$$

Proposition 1. *Assume that $r \geq 1$ and that A is unramified. If k is infinite, $J_m = J'_m = A$ and any element of A is a sum of at most $\gamma_2(m)$ m-th powers.*

If k is finite, any element of J_m is a sum of at most $\dfrac{p^\tau - 1}{p-1} n + \delta$ m-th powers, where $\delta = 1$ if $n = p - 1$ and $\delta = 0$ otherwise. Moreover, $J_m = J'_m$ unless $p = 2$ and $f_1 \neq f$.

Proof. First consider the case when k is finite and $p > 2$. We put $m_s = n \cdot p^s (0 \leq s \leq r)$, and we shall show that (with the notation of Lemma 3) any element of $(J_n)_s$ is the sum of at most $\dfrac{p^{s+1} - 1}{p-1} n$ m_s-th powers modulo $p^{s+1}A$. For $s = 0$, this is the theorem of Tornheim quoted earlier. Suppose the result is true for s, and let $x = x_0^{p^{s+1}} + y \in (J_n)_{s+1}$, with $y \in p(J_n)_s$. By the theorem of Tornheim, there exist $y_1, \ldots, y_n \in A$ such that $x_0 \equiv \sum_1^n y_i^n \ (\mathrm{mod}\ pA)$, hence

$$x_0^{p^{s+1}} \equiv \left(\sum_1^n y_i^n \right)^{p^{s+1}} (\mathrm{mod}\ p^{s+2}A) \quad \text{and} \quad \left(\sum_1^n y_i^n \right)^{p^{s+1}} \equiv \sum_1^n y_i^{m_{s+1}} (\mathrm{mod}\ p(J_n)_s),$$

as follows from (1) with $s+1$ instead of s. Thus we obtain by induction hypothesis,

$$x \equiv \sum_1^n y_i^{m_{s+1}} + p \sum_1^N z_j^m, \quad (\mathrm{mod}\ p^{s+2}A),$$

where $N = \dfrac{p^{s+1}-1}{p-1} \cdot n$. Since the residue field is perfect, we can find **48**
t_j such that $t_j^p \equiv z_j (\mathrm{mod}\ pA)$, and hence $t_j^{m_{s+1}} \equiv z_j^{m_s} (\mathrm{mod}\ p^{s+1}A)$, which gives

$$x \equiv \sum_1^n y_i^{m_{s+1}} + p \sum_1^N t_j^{m_{s+1}} \pmod{p^{s+2}A},$$

i.e., x is a sum of $n + pN = n \cdot \dfrac{p^{s+2}-1}{p-1}$ m_{s+1}-th powers modulo $p^{s+2}A$.

This completes the induction, and we obtain for $s = r$ that any $x \in J'_m$ is the sum of at most $\dfrac{p^r-1}{p-1} \cdot n$ m-th powers modulo $p^{r+1}A$, and it is easily deduced from Lemma 2 that x is actually the sum of this many m-th powers if $x \not\equiv 0(p^\tau)$. Further, if $n < p-1$ and ζ is a primitive $(p-1)$-th root of unity in A, $\sum_{i=0}^{p-2} \zeta^{mi} = 0$, and if $n \geq p-1$, $0 \equiv \underbrace{1 + \ldots + 1}_{p^\tau \text{ times}} (\mathrm{mod}\ p^\tau)$.

This proves the assertion when $x \equiv 0(p^\tau)$, in view of Lemma 2.

Next consider the case when $J_n = A$, and suppose that every element of A is a sum of λ n-th powers modulo pA. Since the residue field is perfect, it follows that sums of λ m-th powers form a system of representatives for the residue classes mod pA. Since any element $x \in A$ can be represented as $x = x_0 + x_1 p + \ldots + x_{\tau-1} p^{r-1} (\mathrm{mod}\ p^\tau A)$ with x_i chosen from this system of representatives, it follows from Lemma 2 that any element of A which is $\not\equiv 0 \pmod{p^\tau}$ is a sum of

$$(1 + p + \ldots + p^{\tau-1})\lambda = \frac{p^\tau - 1}{p-1} \cdot \lambda$$

m-th powers. Elements $x \equiv 0(p^\tau)$ can be dealt with as before. This settles the case of the infinite residue field as well as the case of $p = 2$ and $J_n = A$.

If $p = 2$ and $J_n \neq A$, it follows from Lemma 1 that $f_1 \neq f$ and hence $f_1 \leq f/2$, $n \geq \dfrac{2^f - 1}{2^{f_1} - 1} \geq 2^{f/2} + 1$, and $2^f \leq (n-1)^2$. We give an argument whose idea goes back to Siegel [3].

Let $\bar{x}_1^m, \ldots, \bar{x}_d^m$ be a minimal set of generators (as an additive group) of the image of $J_m(A)$ in $A/2^{r+2}A$, and q_i the index of the subgroup generated by $\{\bar{x}_1^m, \ldots, \bar{x}_{i-1}^m\}$ in the subgroup generated by $\{\bar{x}_1^m, \ldots, \bar{x}_i^m\}$. Then

$q_i \leq 2^{r+2}$, each q_i is a positive power of 2 and $q_1 \dots q_d \leq 2^{(r+2)f}$. Any element of $J_m(A)$ can therefore be written as a sum of at most $\sum_{i=1}^{d}(q_i - 1)$ m-th powers modulo $2^{r+2}A$. We shall find an upper bound for

$$\sum_{i=1}^{d}(q_i - 1) = \sum_{i=1}^{d} \min(q_i - 1, 2^{r+2} - 1).$$

49 Let N be the supremum of all sums $S = \sum_{i=1}^{l} \min(n_i - 1, 2^{r+2} - 1)$, where l, $n_1, \dots, n_l$ vary subject to the following conditions: each $n_i (1 \leq i \leq l)$ is a positive power of 2 and $n_1 \dots n_l \leq 2^{(r+2)f}$. Suppose $l = d'$, $n_i = q'_i$ $(1 \leq i \leq d')$ is a choice of the variables for which S attains its supremum, with l minimal. We assert that for all but one i, $q'_i \geq 2^{r+2}$. If not, suppose for instance that $q'_{d'-1} < 2^{r+2}$, $q'_{d'} < 2^{r+2}$. Since all the q'_i are powers of 2, it follows that $q'_{d'-1} + q'_{d'} \leq 2^{r+2}$. If we define $q''_i = q'_i$ for $1 \leq i \leq d' - 2$, $q''_{d'-1} = q'_{d'-1} \cdot q'_{d'}$, we have

$$\begin{aligned} &\min(q'_{d'-1} - 1, 2^{r+2} - 1) + \min(q'_{d'} - 1, 2^{r+2} - 1) \\ &= (q'_{d'-1} - 1) + (q'_{d'} - 1) \leq \min(q''_{d'-1} - 1, 2^{r+2} - 1), \\ N = \sum_{i=1}^{d'} \min(q'_i - 1, 2^{r+2} - 1) &\leq \sum_{i=1}^{d'-1} \min(q''_i - 1, 2^{r+2} - 1) \leq N, \end{aligned}$$

since $q''_1 \dots q''_{d'-1} = q'_1 \dots q'_{d'} \leq 2^{(r+2)f}$, which contradicts the minimality of d'. Hence for all but one i, $q'_i \geq 2^{r+2}$, and all the q'_i are ≥ 2, and therefore

$$2^{(r+2)(d'-1)+1} \leq q'_1 \dots q'_{d'} \leq 2^{(r+2)f}, d' \leq f.$$

It follows that

$$\begin{aligned} N = \sum_{1}^{d'} \min(q'_i - 1, 2^{r+2} - 1) &\leq f(2^{r+2} - 1) \\ &\leq \frac{2\log(n-1)}{\log 2}(2^{r+2} - 1) < n(2^{r+2} - 1). \end{aligned}$$

It follows from Lemma 2 that any element of $J_m(A)$ is a sum of at most $(2^{r+2} - 1)n$ m-th powers. This completes the proof of Proposition 1. □

2 The case when the residue field is large

Proposition 2. *When k is infinite, $J_m = J'_m = A$, and any element of A is a sum of at most $\gamma_3(m)$ m-th powers.*

When k is finite, and $f_1 > r \geq 1$, any element of J_m is a sum of at most $\{(r+1)(p^r+1)+1\}\left\{\dfrac{p^\tau - 1}{p-1}n + \delta\right\}$ m-th powers, where $\delta = 1$ if $n = p-1$ and $\delta = 0$ otherwise; and the equality $J_m = J'_m$ holds unless $p = 2$ and $f \neq f_1$.

Proof. When either k is infinite or $f_1 > r$, we can find elements $u_0, \ldots, u_{p^r} \in J_m(B)$ such that their residue field images are distinct, by Lemma 1. It follows that we can find $a_0, \ldots, a_{p^r} \in J_m(B)$ such that

$$p^l x^{p^{r-1}} = \sum_{i=0}^{p^r} a_i (u_i x + 1)^{p^r}. \tag{2}$$

In fact, writing $\binom{p^r}{p^{r-l}} = p^l \alpha_l$, where α_l is a unit in $J_m(B)$, we have only **50**
to solve the system of linear equations

$$\sum_{i=0}^{p^r} a_i u_i^\nu = \begin{cases} 0 \text{ if } \nu \neq p^{r-l}, & 0 \leq \nu \leq p^r \\ \alpha_l^{-1} \text{ if } \nu = p^{r-l} \end{cases}$$

for the a_i in $J_m(B)$. The determinant $\Delta = \prod_{i<j}(u_i - u_j)$ is an element of $J_m(B)$ invertible in B, hence in $J_m(B)$, since $\Delta(\Delta^{m-1}(\Delta^{-1})^m) = 1$, $\Delta^{m-1}(\Delta^{-1})^m \in J_m(B)$. Thus the a_i can be solved for in $J_m(B)$.

Now for $x \in \mathfrak{p}$, $(1+u_i x)^{p^r}$ is an m-th power by Hensel's lemma. It follows that if any element of $J_m(B)$ is a sum of λ m-th powers, any element of $\mathfrak{m}_r$ (in the notation of Lemma 3) is the sum of at most $\lambda(r+1)(p^r+1)$ m-th powers. But since any element of A is congruent to an element of B modulo $\mathfrak{p}$, it follows from Lemma 3 that any element of $J_m(A)$ or $J'_m(A)$ is congruent to an element of $J_m(B)$ or $J'_m(B)$ respectively modulo $\mathfrak{p}_r$.

Proposition 2 follows from this, if we substitute for λ from Proposition 1. □

Remarks. (1) It follows from the proof that the Proposition is true for any complete local ring with residue field finite and $f_1 > r$, or

perfect and infinite of characteristic $p > 0$, since such a ring contains a homomorphic image of an unramified complete discrete valuation ring of characteristic zero with the same residue field, by the structure theorems of Cohen.

(2) When $n = 1$, that is when $m = p^r$, the passage from $J_m(A)$ to $J_m(B)$ [or from $J'_m(A)$ to $J'_m(B)$] is not necessary, and we can apply (2) to any element $x \in A$ directly. Thus if every element of $J_m(B)$ is a sum of at most λ m-th powers, every element of $J_m(A)$ is a sum of at most $(r+1)(p^r+1)\lambda$ m-th powers.

Moreover when $n = 1$, the case $f = r$ can also be dealt with in the same manner. In fact if ζ is a primitive $(p^f - 1)$-th root of unity in B, we have the identities

$$(p^f - 1)\binom{p^r}{p^k} x^{p^k} = \sum_{i=0}^{p^f-2} \zeta^{-ip^k}(\zeta^i x + 1)^{p^r}, \quad 0 < k < r,$$

$$(p^f - 1)(p^r x + x^{p^r}) = \sum_{i=0}^{p^f-2} \zeta^{-i}(\zeta^i x + 1)^{p^r},$$

and ζ is a p^r-th power in B.

(3) Since $p^{f_1} - 1 \geq \dfrac{p^f - 1}{n}$, the condition $f_1 > r$ is certainly fulfilled if $p^f > m$. Since the case $p^f = m$ is also covered by Remark (2), we may assume in what follows that $p^f < m$.

3 The general case

51 We assume A to have finite residue field, and that $f_1 \leq r$, $p^f < m$. The method is similar to that of §2, but more complicated.

Let ζ be a primitive $(q_1 - 1)$-th root of unity in B. It is easily seen that $\zeta \in J_m(B)$. For any k with $0 \leq k \leq r$, we wish to establish an identity of the form

$$\sum_j \sum_{i=0}^{q_1-2} a_{ij}(\zeta^i \lambda_j x + 1)^{p^r} = p^{r-k}(\lambda x)^{p^k} \tag{3}$$

with a_{ij}, λ_j and λ suitably chosen. This identity is equivalent to the system of linear equations

$$\binom{p^r}{p^k}\sum_{i,j} a_{ij}\zeta^{iv}\lambda_j^v = \begin{cases} 0 \text{ if } v \neq p^k, & 0 \leq v \leq p^r, \\ p^{r-k}\lambda^{p^k} \text{ if } v = p^k. \end{cases}$$

If we put $\sum_i a_{ij}\zeta^{iv} = b_{jv}$, then $b_{jv} = b_{jv'}$ if $v \equiv v' (\mathrm{mod}\ (q_1 - 1))$. We naturally choose $b_{jv} = 0$ if $v \not\equiv p^k (\mathrm{mod}\ (q_1 - 1))$, and we will choose $b_{jv} = c_j$ for $v \equiv p^k (\mathrm{mod}\ (q_1 - 1))$ later. The a_{ij} are then determined by the c_j by the equations $a_{ij} = u_i \cdot c_j$ with $u_i \in J_m(B)$. Let us write $p^k = \sigma(q_1 - 1) + \rho_1$, $p^r - \rho_1 = \kappa(q_1 - 1) + \rho_2$, with $0 \leq \rho_1, \rho_2 < q_1 - 1$; if $q_1 > 2$ and l and s are the least non-negative residues of k and r modulo f_1, it is easily checked that

$$\sigma = \frac{p^k - p^l}{q_1 - 1}, \rho_1 = p^l, \text{ and } \kappa = \frac{p^r - p^s}{q_1 - 1} \text{ or } = \frac{p^r - p^s}{q_1 - 1} - 1$$

according as $s \geq 1$ or $s < l$. We let the index j vary through the range $0 \leq j \leq \kappa$. We write $\binom{p^r}{p^k} = p^{r-k}m_k$, where m_k is a unit in $J_m(B)$. Putting $\lambda_j^{q_1-1} = \mu_j$, $m_k c_j \lambda_j^{\rho_1} = d_j$, so that $a_{ij} = m_k^{-1}\lambda_j^{-\rho_1}u_i d_j$, the equation (3) becomes equivalent to the system of linear equations.

$$\sum_{j=0}^{\kappa} d_j \mu_j^t = \begin{cases} 0 \text{ if } t \neq \sigma, & 0 \leq t \leq \kappa \\ \lambda^{p^\kappa} \text{ if } t = \sigma. \end{cases}$$

We choose $\lambda_j = \pi^{jm} (0 \leq j \leq \kappa)$, so that $\mu_j = \pi^{jm(q_1-1)}$. If $S_\alpha(X_0, \ldots, X_\beta)$ denotes the elementary symmetric function of degree α in the variables $X_0, \ldots, X_\beta$, the solution of the above system of linear equations for the d_j is

$$d_j = \pm\lambda^{p^k} \frac{S_{\kappa-\sigma}(\mu_0, \ldots, \hat{\mu}_j, \ldots, \mu_\kappa)}{\prod_{\alpha \neq j}(\pi^{jm(q_1-1)} - \pi^{\alpha m(q_1-1)})}$$

and hence

$$a_{ij} = \pm\frac{m_k^{-1}u_i\lambda^{p^k}S_{\kappa-\sigma}(\mu_0, \ldots, \hat{\mu}_j, \ldots, \mu_\kappa)}{\pi^{jm\rho_1}\prod_{\alpha \neq j}(\pi^{jm(q_1-1)} - \pi^{\alpha m(q_1-1)})}$$

where the symbol '$\hat{}$' over a letter means that it is to be omitted. Now

$$S_{\kappa-\sigma}(\mu_0,\ldots,\hat{\mu}_j,\ldots,\mu_\kappa) = \pi^{m(q_1-1)(\kappa-\sigma-1)(\kappa-\sigma)/2} P_j(\pi^{m(q_1-1)}),$$

52 where P_j is a polynomial of degree $\leq (\sigma+1)(\kappa-\sigma)$ with rational integral coefficients. The above expression for the a_{ij} may be rewritten as

$$a_{ij} = \pm \frac{m_k^{-1} u_i \lambda^{p^k} \pi^{(\frac{1}{2}[(\kappa-\sigma-1)(\kappa-\sigma)]-\frac{1}{2}[(j-1)j]-(\kappa-j)j)m(q_1-1)-jm\rho_1} P_j(\pi^{m(q_1-1)})}{\prod_{\alpha=1}^{j}(1-\pi^{m\alpha(q_1-1)}) \prod_{\alpha=1}^{\kappa-j}(1-\pi^{m\alpha(q_1-1)})}$$

We wish to choose λ in such a way that it is divisible by as small a power of $\mathfrak{p}$ as possible, and such that the a_{ij} are sums of as few a number of m-th powers as possible.

Let $\mathfrak{p}^X$ be the power of $\mathfrak{p}$ dividing λ. In order that the a_{ij} belong to A at all, we must have

$$p^k X + \left(\frac{(\kappa-\sigma-1)(\kappa-\sigma)}{2} - \frac{(j-1)j}{2} - (\kappa-j)j\right) m(q_1-1) - jm\rho_1 \geq 0 \qquad (0 \leq j \leq \kappa).$$

The minimum of the left side of the above inequality as j varies in the range $0 \leq j \leq \kappa$ is attained for $j = \kappa$, so that we must have [on substituting $p^k = \sigma(q_1-1)\rho_1$]

$$p^k(X - \kappa m) + m(q_1-1)\frac{\sigma(\sigma+1)}{2} \geq 0.$$

We choose X in such a way that equality holds. We put

$$\lambda = \pi^X \prod_{j=1}^{\kappa}(1-\pi^{im(q_1-1)}).$$

It is easily checked that the polynomial $\prod_{j=1}^{\alpha}(1-Y^j)\prod_{j=1}^{\kappa-\alpha}(1-Y^j)$ divides $\prod_{j=1}^{\kappa}(1-Y^j)$, so that we obtain

$$a_{ij} = \pi^{um} Q_j(\pi^{m(q_1-1)}), \quad u = u(j) \geq 0$$

where Q_j is a polynomial with coefficients in $J_m(B)$ of degree

$$\leq (\sigma+1)(\kappa-\sigma) + p^k\frac{\kappa(\kappa+1)}{2} - \frac{j(j+1)}{2} - \frac{(\kappa-j)(\kappa-j+1)}{2}$$
$$= (\sigma+1)(\kappa-\sigma) + (p^k-1)\frac{\kappa(\kappa+1)}{2} + j(\kappa-j).$$

It follows from (3) that for any $x \in \mathfrak{p}^{X+1}$, $p^{r-k}x^{p^k}$ is a sum of at most

$$N(q_1-1)\left[(\kappa+1)\left\{(\sigma+1)(\kappa-\sigma) + (p^k-1)\frac{\kappa(\kappa+1)}{2}\right\} + \sum_{j=0}^{\kappa} j(\kappa-j)\right]$$

m-th powers, where N denotes the smallest integer such that any element of $J_m(B)$ is the sum of at most N m-th powers. Let us put $\kappa_0 = \left[\frac{p^r}{q_1-1}\right]$, so that $\kappa \leq \kappa_0$ for all k; the above integer is then $\leq N(q_1-1)p^k\kappa_0(\kappa_0+1)^2/2$.

Hence if we put $Y = 1 + \sup_{0\leq k\leq r} X$ and $\mathfrak{a} = \mathfrak{p}^Y$, any element of $\mathfrak{a}_r$ (in **53**
the notation of Lemma 3) is the sum of at most

$$\frac{N}{2}(q_1-1)\kappa_0(\kappa_0+1)^2\frac{p^{r+1}-1}{p-1}$$

m-th powers. Since

$$X = \kappa m + p^{r-k}n(q_1-1)\frac{\sigma(\sigma+1)}{2} = \kappa m + n\frac{(\sigma+1)}{2}(p^r - \rho_1 p^{r-k}),$$

and σ and $(p^r - \rho_1 p^{r-k})$ are both non-decreasing as k increases, we see that

$$\sup_{0\leq k\leq r} X = \kappa_0 m + n(q_1-1)\frac{\kappa_0(\kappa_0+1)}{2}.$$

Now, if $\bar{J}$ denotes the image of $J_m(A)$ in the ring $A_r/\mathfrak{a}_r + p^\tau A$, the order (as an abelian group) of $\bar{J}$ is of the form p^Q where

$$Q \leq (r+1)f\left(\kappa_0 m + n(q_1-1)\frac{\kappa_0(\kappa_0+1)}{2} + 1\right),$$

by Lemma 3. Choose a minimal set $\bar{x}_1^m, \ldots, \bar{x}_d^m$ of generators of $\bar{J}$ (as an abelian group), and denote by q_i the index of the subgroup $\{\bar{x}_1^m, \ldots, \bar{x}_{i-1}^m\}$ in the subgroup $\{\bar{x}_1^m, \ldots, \bar{x}_i^m\}$ for $1 \leq i \leq d$, so that $p^r \geq q_i > 1$, each q_i is a power of p and $q_1 \ldots q_d = p^Q$. Any element of $\bar{J}$ is then a sum of at most $\sum_1^d (q_i - 1)$ elements of the form $\bar{x}_i^m$. Arguing as in the last part of the proof of Proposition 2, we obtain that

$$\sum_{i=1}^{d} (q_i - 1) \leq \sum_{j=1}^{d'} \min(q'_j - 1, p^r - 1),$$

where each q'_j is again a positive power of p, $\prod q'_j = p^Q$, and all but one of the q'_j satisfy $q'_j \geq p^\tau$. Thus we obtain

$$p^{(d'-1)\tau+1} \leq p^Q,$$

$$d' - 1 < (r+1) f\left(\kappa_0 m + n(q_1 - 1)\frac{\kappa_0(\kappa_0+1)}{2} + 1\right)/\tau$$

$$\leq f\left(\kappa_0 m + n(q_1 - 1)\frac{\kappa_0(\kappa_0+1)}{2} + 1\right),$$

i.e.

$$d' \leq f\left(\kappa_0 m + n(q_1 - 1)\frac{\kappa_0(\kappa_0+1)}{2} + 1\right),$$

and hence

$$\sum_1^d (q_i - 1) \leq f(p^\tau - 1)\left(\kappa_0 m + n(q_1 - 1)\frac{\kappa_0(\kappa_0+1)}{2} + 1\right).$$

54 It follows that any element of $J_m(A)$ is a sum of at most

$$\frac{N}{2}(q_1 - 1)\kappa_0(\kappa_0+1)^2 \frac{p^{r+1} - 1}{p - 1} + f(p^\tau - 1)\left(\kappa_0 m + n(q_1 - 1)\frac{\kappa_0(\kappa_0+1)}{2} + 1\right)$$

m-th powers. Now if $f = f_1$, then $f \leq r$ by assumption, and if $f \neq f_1$, then $f_1 \leq f/2$, so that

$$n \geq \frac{p^f - 1}{p^{f_1} - 1} \geq p^{f/2} + 1, \quad f \leq \frac{2\log(n-1)}{\log p}.$$

Using this and the value for N as given by Proposition 1, one easily deduces after a little calculation that the above integer is $< 8m^5$. In view of Propositions 1 and 2, we thus have

Proposition 3. *Let A be any complete discrete valuation ring of characteristic zero with finite residue field, and m an integer > 1. Then any element of the subring $J_m(A)$ of A generated by m-th powers of elements of A is a sum of at most $8m^5$ m-th powers.*

Remark. As is needless to remark, the bound $8m^5$ is quite rough, and it is possible to obtain better estimates for particular values of m. In view of the fact that the "global to local" reduction of the number field case to that of a p-adic field works when the number of variables is at least $2^m + 1$ (Birch [1] and Körner [2]), and since $8m^5 < 2^m + 1$ for $m \geq 27$, it might not be without interest to check that $2^m + 1$ variables suffice for $m \leq 26$. For most values of this range, this is a consequence of our estimates, but the author has not checked this for all $m \leq 26$.

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A Note on Automorphism Groups of Algebraic Varieties *

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Matsusaka has proved [3] that the maximal connected group of automorphisms of a projective variety can be endowed with the structure of an algebraic group. Our aim in this note is to extend this result to arbitrary complete varieties. More generally, we shall show that a "connected" and "finite dimensional" group G of automorphisms (see below for precise definitions) of any algebraic variety X can be endowed with the structure of an algebraic group variety. The main line of argument is similar to the one used by Chevalley [2] and Seshadri [7] in the construction of the Picard variety, but somewhat simpler. We shall prove that the linear map of the Lie algebra of this algebraic group into the space of vector fields on X which associates to any tangent vector at the identity element of G the corresponding "infinitesimal motion" is an injection. It follows easily that G satisfies the universal property for connected algebraic families of automorphisms of X containing the identity, that is, that any algebraic family of automorphisms of an algebraic variety X parametrised by a variety T is induced by a morphism of T into G. As an application, we shall prove that the maximal connected group of automorphisms of a (locally isotrivial) principal fibre space over a complete variety has a structure of a group variety. We have been informed by the referee that this result has also been obtained by H. Matsumura. 55

All varieties will be assumed to be irreducible, and defined over an algebraically closed field K.

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We shall say that a family $\{\varphi_t\}_{t\in T}$ of automorphisms of a variety X, where the parametrising set T is also a variety, is an *algebraic family* if the map $T \times X \to X$ given by $(t, x) \xrightarrow{\varphi} \varphi_t(x)$ is a morphism. It is clear that if $\lambda : S \to T$ is a morphism, the family $\{\varphi_{\lambda(s)}\}_{s\in S}$ is again algebraic, and that if $\{\psi_s\}_{s\in S}$ is another algebraic family, $\{\psi_s \circ \varphi_t\}_{(s,t)\in S\times T}$ is also an algebraic family. Let $(\tilde{X}, p)$ be the normalisation of X. For every $t \in T$, φ_t lifts to a unique automorphism φ_t of $\tilde{X}$ such that $p \circ \tilde{\varphi}_t = \varphi_t \circ p$. We shall show that $\{\tilde{\varphi}_t\}_{t\in T}$ is an algebraic family of automorphisms of $\tilde{X}$. Let $(\tilde{T}, q)$ be the normalisation of T. Then $(\tilde{T} \times \tilde{X}, q \times p)$ is the normalisation of $T \times X$, and the morphism $T \times X \xrightarrow{\varphi} X$ lifts to a unique morphism $\bar{\varphi} : \tilde{T} \times \tilde{X} \to \tilde{X}$ such that $p \circ \bar{\varphi} = \varphi \circ (q \times p)$. It follows that for
56 any $\tilde{t} \in \tilde{T}$, the morphism of $\tilde{X}$ onto itself given by $\tilde{x} \to \bar{\varphi}(\tilde{t}, \tilde{x})$ coincides with $\tilde{\varphi}_q(\tilde{t})$. Let $\Gamma_{\bar{\varphi}}$ be the graph of $\bar{\varphi}$ in $\tilde{T} \times \tilde{X} \times \tilde{X}$, and $\Gamma_{\tilde{\varphi}}$ its image in $T \times \tilde{X} \times \tilde{X}$ by $q \times I_{\tilde{X}} \times I_{\tilde{X}}$ denoting the identity map of $\tilde{X}$. $\Gamma_{\tilde{\varphi}}$ is then the graph of the map $T \times \tilde{X} \xrightarrow{\tilde{\varphi}} \tilde{X}$ given by $(t, \tilde{x}) \to \tilde{\varphi}_t(x)$ and since q is proper, $\Gamma_{\bar{\varphi}}$ is closed. Since the projection of $\Gamma_{\tilde{\varphi}}$ onto the product $\tilde{T} \times \tilde{X}$ of the first two factors is an isomorphism and $q \times I_{\tilde{X}} : \tilde{T} \times \tilde{X} \to T \times \tilde{X}$ is proper, it follows that the projection of $\Gamma_{\tilde{\varphi}}$ onto $T \times \tilde{X}$ is a morphism which is proper and bijective.

Let p_1 and p_2 be the projections of $\Gamma_{\tilde{\varphi}}$ onto the second and third factors. It is easily checked that if τ is a tangent vector at a point $(t, \tilde{x}, \tilde{\varphi}_t(\tilde{x}))$ to $\Gamma_{\bar{\varphi}}$ whose image by the differential map of the projection onto T is zero, and if dp_i and $d\varphi_t$ are the differential maps of p_i and φ_t, we have $dp_2(\tau) = d\tilde{\varphi}_t(dp_1(\tau))$. It follows that the differential map of the projection of $\Gamma_{\tilde{\varphi}}$ onto $T \times \tilde{X}$ is everywhere injective. It follows from [5, Appendix, Exposé 5, p. 5-28] that $\Gamma_{\tilde{\varphi}} \to T \times \tilde{X}$ is an isomorphism everywhere, which shows that $\tilde{\varphi}$ is a morphism and $\{\tilde{\varphi}_t\}_{t\in T}$ is an algebraic family.

A similar argument proves that the family $\{\varphi_t^{-1}\}_{t\in T}$ of inverses of an algebraic family is again algebraic. Since the set of points of a parametrising variety for which the corresponding elements of an algebraic family become the identity automorphism is a closed set, it follows that if $\{\varphi_t\}_{t\in T}$ and $\{\psi_s\}_{s\in S}$ are two algebraic families, the set of $(s, t) \in S \times T$ for which $\varphi_t = \psi_s$ is closed.

We shall say that a group G of automorphisms of a variety X is a *connected group of automorphisms* if any automorphism belonging to G is a member of an algebraic family which also contains the identity automorphism of X. We shall say that G is *finite dimensional* if there exists an integer N such that if $\{\varphi_t\}_{t\in T}$ is any algebraic family of automorphisms contained in G (i.e. $\varphi_t \in G$ for every $t \in T$) and such that $\varphi_t \neq \varphi_{t'}$ if $t \neq t'$ *(an injective family)*, we have $\dim T \leq N$. The smallest integer N having this property is then called the *dimension* of G.

We need a final definition. Let $\{\varphi_t\}_{t\in T}$ be an algebraic family of automorphisms of X and $\varphi : T \times X \to X$ the defining morphism. Let τ be a tangent vector to T at a point t_0, and for any x, let τ_x be the tangent vector to $T \times X$ at (t_0, x) which is the image of τ by the differential mapping of the morphism $\theta_x : T \to T \times X$ given by $\theta_x(t) = (t, x)$. We can then define a vector filed $d\varphi(\tau)$ on X whose value $d\varphi(\tau)_x$ at the point x is given by $d\varphi(\tau)_x = d\varphi(\tau_{\varphi_{t_0}} - 1_x)$. The vector field $d\varphi(\tau)$ is immediately verified to be regular (i.e., maps regular functions on open subsets of X into regular functions). We shall say that the family $\{\varphi_t\}_{t\in T}$ is *infinitesimally injective* at a point $t_0 \in T$ if $d\varphi$ is an injection of the tangent space at t_0 into the vector space of regular vector fields on X.

We now state our result.

Theorem. *Let G be a connected finite dimensional group of automorphisms of an algebraic variety X. Then there exists a unique structure of an algebraic variety on G which makes of it an algebraic group (of dimension = dimension of G) such that the following condition holds:*

a. The automorphisms of X belonging to G, when considered as a family of automorphisms of X parametrised by the identity map of G onto G, is an algebraic family which is infinitesimally injective 57
on G. In other words, if $\chi : G \times X \to X$ is defined by $\chi(\varphi, x) = \varphi(x)$, χ is a morphism of algebraic varieties, and if $\tau \neq 0$ is a tangent vector at any point of G, $d\varphi(\tau) \neq 0$.

Further, G has the following universal property:

b. If $\{\varphi_t\}_{t\in T}$ is any algebraic family of automorphisms of X such that $\varphi_t \in G$ for every $t \in T$, there is a unique morphism $\overline{\varphi} : T \to G$

such that $\bar{\varphi}(t) = \varphi_t$.

We will first show that (a) implies (b), and this in turn trivially implies the uniqueness assertion of the theorem. By an earlier remark, the subset Γ of $T \times G$ consisting of those (t, φ) for which $\varphi_t = \varphi$ is a closed subset of $T \times G$. Since this is precisely the graph of $\bar{\varphi}$, we have only to verity that the projection is an isomorphism. We shall prove that the differential map of the projection of Γ onto T is injective at every point of Γ. Let p_1 and p_2 be the projections of Γ onto T and G respectively. Let $\{\psi_\gamma\}_{\gamma\in\Gamma}$ be the algebraic family on X defined by $\psi_\gamma = \varphi_{p_1(\gamma)} = p_2(\gamma)$. If τ is any tangent vector at a point γ_0 to Γ, we have evidently $d\psi(\tau) = d\psi(dp_1(\tau)) = d\chi(dp_2(\tau))$. It follows by (a) that if $dp_1(\tau) = 0$, then $dp_2(\tau) = 0$ and hence $\tau = 0$.

By Lemmas 1 and 2 below and Zariski's main theorem, we can find $y_1, \ldots, y_m \in X$ such that the morphism $\lambda : G \to X^m$ defined by $\lambda(g) = (gy_1, \ldots, gy_m)$ is a radicial covering of the non-singular locally closed subvariety $W = \lambda(G)$ of X^m by G. Since $\mu : T \to W$ given by $\mu(t) = (\varphi_t(y_1), \ldots, \varphi_t(y_m))$ is a morphism, its graph $\Gamma_\mu \subset T \times W$ is closed irreducible. Since $I_T \times \lambda : T \times G \to T \times W$ is a radicial covering and $(I_T \times \lambda)^{-1}(\Gamma_\mu) = \Gamma$, Γ is irreducible and $\Gamma \to T$ is proper. We may now apply the result of [5, Appendix to Exposé 5, p. 5-28] to conclude that $\Gamma \xrightarrow{p_1} T$ is an isomorphism, and $\bar{\varphi} = p_2 \circ p_1^{-1}$ is the required morphism.

Hence we have only to construct a structure of variety on G satisfying (a).

Lemma 1. *Let $\{\varphi_t\}_{t\in T}$ be an algebraic family of automorphisms of a variety X parametrised by a (not necessarily irreducible) algebraic space T. Then there exists a finite number of points $x_1, \ldots, x_n \in X$ such that if $\varphi_t(x_i) = \varphi_{t'}(x_i)$ $(i = 1, \ldots, n; t, t' \in T)$, then $\varphi_t = \varphi_{t'}$.*

Proof. For any $x \in X$, let S_x be the closed subset of $T \times T$ consisting of all (t, t') such that $\varphi_t(x) = \varphi_{t'}(x)$ and S the subset of (t, t') such that $\varphi_t = \varphi_{t'}$. Then we have $S = \bigcap_{x\in X} S_x$, and since $T \times T$ is a noetherian space, $S = \bigcap_{i=1}^{n} S_{x_i}$, where $x_1, \ldots, x_n$ are a finite number of points of X. This is the assertion of the lemma. □

Lemma 2. *Let G be a connected group of automorphisms of a variety X, and x any point of X. Then the orbit $Gx = \{\varphi(x) | \varphi \in G\}$ is a locally closed non-singular subvariety of X. Moreover if G is finite dimensional,* $\dim Gx \leqq \dim G$.

Proof. By a well known theorem of Chevalley, if $\{\varphi_t\}_{t\in T}$ is any algebraic family of automorphisms of X, the subset $\varphi_T(x) = \{\varphi_t(x) | t \in T\}$ is an irreducible constructible subset of X of dimension at most equal to the dimension of X. It follows that we can choose a family $\{\varphi_t\}_{t\in T}$ such that $\varphi_t \in G$ for all $t \in T$ and $\dim \varphi_T(x)$ is a maximal. By replacing this family by $\{\varphi_t \circ \varphi_{t_0}^{-1}\}_{t\in T}$, if necessary, we may assume that φ_{t_0} is the identity of X for some $t_0 \in T$. The closure $\overline{\varphi_T(x)}$ of $\varphi_T(x)$ is a closed irreducible **58**
subvariety of X and $\varphi_T(x)$ contains an open subset U of $\overline{\varphi_T(x)}$. We assert that $Gx \subset \overline{\varphi_T(x)}$. If not, there would exist a $\psi \in G$ such that $\psi(x) \notin \overline{\varphi_T(x)}$. Since G is connected, there exists a family $\{\psi_s\}_{s\in S}$ such that for two points $s_0, s_1 \in S$, we have $\psi_{s_0} = I_X$, $\psi_{s_1} = \psi$. The algebraic family of automorphisms $\{\psi_t \circ \psi_s\}_{(t,s)\in T\times S}$ is contained in G, and contains both the families $\{\varphi_t\}_{t\in T}$ and $\{\psi_s\}_{s\in S}$. Hence it follows that $\overline{\varphi_T(x)} \subsetneqq \overline{\varphi_T \circ \psi_S(x)}$ (since $\psi_{s1}(x) \notin \overline{\varphi_T(x)}$), $\dim \varphi_T \circ \psi_S(x) = \dim \overline{\varphi_T \circ \psi_S(x)} > \dim \overline{\varphi_T(x)} = \dim \varphi_T(x)$, which contradicts our choice of the family $\{\varphi_t\}$.

Thus, $Gx \subset \overline{\varphi_T(x)}$, and since $Gx = \bigcup_{\varphi\in G} \varphi(U)$ and each $\varphi(U)$ is open in $\overline{\varphi_T(x)}$, Gx is open in $\overline{\varphi_T(x)} = Gx$. Hence Gx is locally closed in X, and since G acts transitively on the variety Gx, it is non-singular.

Suppose now that G is of finite dimension N, and $\dim Gx > N$. It follows from the first part of the proof that there is an algebraic family $\{\varphi_t\}_{t\in T}$ contained in G such that $\dim \varphi_T(x) = \dim Gx > N$. Since the morphism $T \to Gx$ given by $t \to \varphi_t(x)$ is dominant, it is easy to see that there is a subvariety T_1 of T such that $\dim T_1 = \dim Gx$ and the morphism $T_1 \to Gx$ given by $t_1 \to \varphi_{t_1}(x)$ is again dominant. The function field $R(T_1)$ of T_1 is therefore algebraic over the function field $R(Gx)$ of Gx. Hence, by replacing T_1 by an open subset, we may further assume that the fibers of the morphism $t_1 \to \varphi_{t_1}(x)$ are finite, and in particular that for any $t_1 \in T_1$, there are only a finite number of $t_2 \in T_1$ such that $\varphi_{t_1} = \varphi_{t_2}$. If we can construct by a suitable "descent" an

injective family of automorphisms of the same dimension, we would have the required contradiction.

By Lemma 1, there exist $y_1, \ldots, y_n \in X$ such that $\varphi_{t_1}(y_i) = \varphi_{t_2}(y_i)$ $(i = 1, \ldots, n)$ implies that $\varphi_{t_1} = \varphi_{t_2}$. Let $y = (y_1, \ldots, y_n) \in X^n$, and let $\{\varphi^n_{t_1}\}_{t_1 \in T_1}$ be the algebraic family of automorphisms of X^n defined by $\varphi^n_{t_1}(x_1, \ldots, x_n) = (\varphi_{t_1}(x_1), \ldots, \varphi_{t_1}(x_n))$. Since $\varphi^n_{T_1}(y)$ is constructible we may assume that $\varphi^n_{T_1}(y)$ is actually a locally closed normal subvariety of X^n, by replacing T_1 by an open subset. Since the fibers of $t_1 \to \varphi^n_{t_1}(y)$ are finite, we may assume (by replacing T_1 by its normalisation in a normal algebraic extension of $R(\varphi^n_{T_1}(y))$ containing $R(T_1)$) that $R(T_1)$ is normal over $R(\varphi^n_{T_1}(y))$. Let T_1' be the normalisation of $\varphi^n_{T_1}(y)$ in the purely inseparable closure of $R(\varphi^n_{T_1}(y))$ in $R(T_1)$. By replacing by an open subset, we may assume that the morphism $T_1 \xrightarrow{\lambda} T_1'$ is a Galois covering, so that T_1 is the quotient of T_1' by a finite group $\prod$. If $t_1, t_2 \in T_1$ have the same image in T_1', then $\varphi^n_{t_1}(y) = \varphi^n_{t_2}(y)$ and hence $\varphi_{t_1} = \varphi_{t_2}$, and conversely, if $\varphi_{t_1} = \varphi_{t_2}$, $\varphi^n_{t_1}(y) = \varphi^n_{t_2}(y)$ and hence $\lambda(t_1) = \lambda(t_2)$ since T_1' is a purely inseparable covering of an open subset of $\varphi^n_{T_1}(y)$. Thus, the defining morphism $\varphi : T_1 \times X \to X$ commutes with the action of $\prod$ on $T_1 \times X$, and hence "passes down" to a morphism $\varphi' : T_1' \times X \to X$ such that $\varphi' \circ (\lambda \times I_X) = \varphi$. Then φ' defines an injective algebraic family of dimension = dimension of $T_1 > N$, which is a contradiction.

Lemma 2 is proved. □

We now proceed to the proof of the theorem. Let $\{\varphi_t\}_{t \in T}$ be an injective family contained in G and containing the identity such that
59 $\dim T = \dim G$ and T is normal. We assert that any element ψ of G can be written as $\varphi_{t_1} \circ \varphi_{t_2}^{-1}$ with $t_1, t_2 \in T$. In fact, by an application of Lemma 1 to the union of the two families $\{\varphi_t\}_{t \in T}$ and $\{\psi \circ \varphi_t\}_{t \in T}$, we deduce that there exist a finite number of points $x_1, \ldots, x_n$ such that $\varphi_t(x_i) = \varphi_{t'}(x_i)$ implies that $\varphi_t = \varphi_{t'}$ and also such that $\varphi_t(x_i) = \psi \circ \varphi_{t'}(x_i)$ implies that $\varphi_t = \psi \circ \varphi_{t'}$. Let $x = (x_1, \ldots, x_n) \in X^n$, and make G act on X^n componentwise. Then Gx is an irreducible locally closed subvariety of X^n whose dimension is the dimension N of G, by Lemma 2 and because the morphism $t \to \varphi^n_t(x)$ of T into Gx is injective. Also $\varphi^n_T(x)$ and $\psi \circ \varphi^n_T(x)$

contain open subsets of Gx, since both are of dimension N. Hence these two subsets of Gx have a non-void intersection, so that $\varphi_{t_1}^n(x) = \psi \circ \varphi_{t_2}^n(x)$ for some $t_1, t_2 \in T$, which implies (by our choice of x) that $\varphi_{t_1}^n = \psi \circ \varphi_{t_2}$, $\psi = \varphi_{t_1} \circ \varphi_{t_2}^{-1}$. Thus the algebraic family $\{\varphi_t \circ \varphi_{t'}^{-1}\}_{(t,t') \in T \times T}$ contains all the elements of G. Hence by Lemma 1, there exist a finite number of points $y_1, \ldots, y_m \in X$ such that $\varphi(y_i) = \varphi'(y_i)(i = 1, \ldots, m)$, $\varphi, \varphi' \in G$, implies that $\varphi = \varphi'$.

Let y be the point $(y_1, \ldots, y_m) \in X^m$, and Gy the orbit of y for the action of G componentwise on X^m. It follows from Lemma 2 that Gy is an irreducible locally closed non-singular subvariety of X^m of dimension N. Since the morphism $T \to Gy$ given by $t \to \varphi_t(y)$ is dominant and injective, $R(T)$ is purely inseparable over $R(Gy)$. If the characteristic is zero, it follows from Zariski's main theorem that T is isomorphic to an open subset of Gy, and we put $Z = Gy$. If the characteristic is $p > 0$, we can find an integer $n \geqq 0$ such that $R(T) \subset R(Gy)^{p^{-n}}$. In this case, let Z be the normalisation of Gy in $R(Gy)^{p^{-n}}$. By replacing T by its normalisation in $R(Gy)^{p^{-n}}$ we may assume that T is an open subset of Z. Let $\pi : Z \to Gy$ denote the projection of Z onto Gy (and the identity map if characteristic is zero). Since Z is the normalisation of Gy in $R(Gy)^{p^{-n}}$, and since G acts (as an abstract group) as a group of automorphisms of Gy, it can be made to act as a group of automorphisms of Z in such a way as to commute with the projection π. Since π is bijective, it follows from our choice of y that for any $z \in Z$, there is a unique element $\varphi_z \in G$ such that $\varphi_z(y) = \pi(z)$, and it follows that for any $\psi \in G$, we have $\varphi_{\psi z} = \psi \circ \varphi_z$. Also the map of Z onto G given by $z \to \varphi_z$ is a bijection. Since the family of automorphisms $\{\varphi_z\}_{z \in Z}$ is algebraic when restricted to the open subset T, and since G acts transitively (and simply) as a group of automorphisms of Z, it follows that $\{\varphi_z\}_{z \in Z}$ is an algebraic family.

We have thus constructed an algebraic family $\{\varphi_z\}_{z \in Z}$ parametrised by a non-singular variety Z, such that (a) $z \to \varphi_z$ is a bijection of Z onto G, and (b) G (as an abstract group) acts on Z in such a way that for $\psi \in G$, $z \in Z$, we have $\varphi_{\psi z} = \psi_z \circ \varphi$. If the family $\{\varphi_z\}_{z \in Z}$ is infinitesimally injective (this always holds when the characteristic of K is zero, as is well known) on the whole of Z, we transport the algebraic structure of Z

onto G by the above bijection, and we are through. Suppose that this is not so, so that the characteristic p of K is > 0. At any point $z \in Z$, let T_z be the tangent space of Z, and let T'_z be the kernel of the linear map $d\varphi$ of T_z into the space of vector fields on X. Because of (b), the dimension
60 of T'_z is the same for all $z \in Z$. Since locally on Z T'_z is defined by the vanishing of a finite number of regular differential forms, it follows that the family of vector spaces $\{T'_z\}_{z\in Z}$ defines s sub-bundle T' of the tangent bundle of Z. It is easy to verify that if X and Y are vector fields on an open set of Z such that their values at any point z of this open set belong to T'_z, the same is true of $[X, Y]$ and X^p. Thus, T' is an integrable sub-bundle of the tangent bundle of Z, in the sense of Cartier (see Exposé 6 of [5]). Hence there exists a non-singular variety Z' and a radicial covering $p : Z \to Z'$ of height one such that the kernel of dp at any $z \in Z$ is precisely T'_z. Further, there exists a morphism $\varphi' : Z' \times X \to X$ such that $\varphi' \circ (p \times I_X) = \varphi$ (Theorem 2 and Proposition 7, exposé 6, [5]). Thus, the family $\{\varphi'_{z'}\}_{z'\in Z'}$ parametrised by Z' and defined by $\varphi'_{p(z)} = \varphi_z$ for $z \in Z$ is again algebraic. Also the action of G on Z "goes down" to an action of G on Z' since it leaves the sub-bundle T' invariant. Finally we have a morphism of Z' onto Gy defined by $z' \to \varphi'_{z'}(y)$, which implies that $R(Z) \supset R(Z') \supset R(Gy)$, $[R(Z) : R(Z')] > 1$. If the family on Z' is not infinitesimally injective, we may repeat the above method of descent, to get a Z'' with $R(Z) \supset R(Z') \supset R(Z'') \supset R(Gy)$, $[R(Z') : R(Z'')] > 1$. Since $[R(Z) : R(Gy)] < \infty$, we must arrive at a bijective and infinitesimally injective algebraic family in a finite number of steps. Transporting the algebraic structure of the parametrising variety of this family to G, we arrive at a structure of an algebraic variety on G, such that $G \times X \xrightarrow{\chi} X$ defined by $\chi(\varphi, x) = \varphi(x)$ is a morphism and the family $\{\chi_\varphi\}_{\varphi\in G}(\chi_\varphi = \varphi)$ is infinitesimally injective.

Now, in the proof of the fact that part (a) of the theorem implies (b), we never used the fact that the group operations on G are algebraic. Thus we may apply (b) to the algebraic family $\{\chi_\varphi \circ \chi^{-1}_{\varphi'}\}_{(\varphi,\varphi')\in G\times G}$ to deduce that the map $G \times G \to G$ given by $(\varphi, \varphi') \to \varphi \circ \varphi'^{-1}$ is a morphism.

The theorem is completely proved.

We now apply the theorem to the group of all automorphisms of a

semi-complete variety which can be connected to the identity automorphism by an algebraic family parametrised by an irreducible variety. (From the preliminary remarks made at the beginning of this note, it follows that such automorphisms form a group under composition.) We shall say that a variety X is *semi-complete* if for any torsion free coherent algebraic sheaf $\mathscr{F}$ on X, the vector space $H^0(X, \mathscr{F})$ (over K) of sections is finite dimensional. By the theorem, we have only to show that there exists an integer N such that if $\{\varphi_t\}_{t\in T}$ is any injective algebraic family of automorphisms of X, $\dim T \leqq N$. The normalization $(\tilde{X}, p)$ of X is again semi-complete; for if $\mathscr{F}$ is coherent and torsion free on $\tilde{X}$, its direct image $p_*(\mathscr{F})$ on X is again coherent and torsion free, and $H^0(\tilde{X}, \mathscr{F}) \cong H^0(X, p_*(\mathscr{F}))$. Also the family $\{\varphi_t\}_{t\in T}$ lifts to a family $\{\tilde{\varphi}_t\}_{t\in T}$ of automorphisms of $\tilde{X}$, which is again injective. Let Y be the closed set of singular points of $\tilde{X}$. Then any automorphism of $\tilde{X}$ leaves $\tilde{X} - Y$ stable. Any coherent torsion free sheaf $\mathscr{F}_1$ on $\tilde{X} - Y$ admits of an extension $\mathscr{F}$ to $\tilde{X}$, which is again torsion free, and since $\tilde{X}$ is normal and codim $Y \geqq 2$, it follows that the map $H^0(\tilde{X}, \mathscr{F}) \cong H^0(\tilde{X} - Y, \mathscr{F}_1)$ is an isomorphism.

Thus, we are reduced to the case of a non-singular semi-complete variety X. In this case, it is well known that the group of coherent sheaves of principal ideals on X (the Cartier divisors) is canonically **61** isomorphic to the free group generated by the subvarieties of codimension one in X. Let $f_1, \ldots, f_p$ be non-constant rational functions on X generating the function field $R(X)$ over K, and let $D = \sum_{j=1}^{n} D_j$ be any positive divisor on X with D_j prime, such that $D + \operatorname{div}(f_i) \geqq 0$, and $D + \operatorname{div}(f_i - 1) \geqq 0$. If $\{\varphi_t\}_{t\in T}$ is any algebraic family of automorphisms of X such that for some $t_0 \in T$, φ_{t_0} = Identity and for any $t \in T$, the inverse image $\varphi_t^*(D)$ of D equals D, *I* assert that φ_t = Identity. In fact, since T is irreducible, it follows that we must have $\varphi_t(D_j) = D_j$ for any j, $1 \leqq j \leqq n$. If V_j is the discrete valuation on $R(X)$ defined by D_j, it follows that $V_j(f \odot \varphi_t) = V_j(f)$. Hence, we must have $\operatorname{div}(f_i) = \operatorname{div}(f_i \odot \varphi_t)$, $\operatorname{div}((f_i - 1) \odot \varphi_t) = \operatorname{div}(f_i - 1)$ for $1 \leqq i \leqq p$. Since X is semi-complete, the functions $\dfrac{f_i \odot \varphi_t}{f_i}$ and $\dfrac{(f_i - 1) \odot \varphi_t}{f_i - 1}$, being everywhere regular, must

be constants. Thus we obtain

$$f_i \odot \varphi_t = a_i f_i,$$
$$(f_i - 1) \odot \varphi_t = b_i(f_i - 1) = f_i \odot \varphi_t - 1 = a_i f_i - 1,$$

which shows that $a_i = b_i = 1$ and $f_i \odot \varphi_t = f_i$. Thus, φ_t induces the identity automorphism on $R(X)$, and hence must be the identity.

Now if $\{\varphi_t\}_{t\in T}$ is any irreducible algebraic family of automorphisms of X with φ_{t_0} = Identity, $\{\varphi_t^*(D)\}_{t\in T}$ is an algebraic family of divisors on X with $\varphi_{t_0}^*(D) = D$. If $\mathrm{Pic}(X)$ is the (connected) Picard variety of X, we thus get a morphism $\psi : T \to Pic(X)$ defined by $\psi(t) = Cl(\varphi_t^*(D) - D)$ ([5], Exposé 8, corollary to Theorem 3). Let T_1 be an irreducible component of $\psi^{-1}(\psi(t_0))$ containing t_0. We then have $\dim T \leqq \dim T_1 + \dim \mathrm{Pic}(X)$, by the dimension theorem ([1], Chapter III, Theorem 2). But now, for every $t \in T_1$, $\varphi_t^*(D) - D$ is linearly equivalent to zero, and thus we have an injective morphism $\xi : T_1 \to P^r$, where P^r is the projective space which parametrises the complete linear system containing D ([5], corollary to Prop. 7 and Theorem 2, Exposé 5]). Hence, we deduce that $\dim T \leqq \dim |D| + \dim \mathrm{Pic}(X)$.

We have thus proved

Corollary 1. *Let X be a semi-complete variety. Then the group G of all automorphisms of X which can be connected to the identity automorphism by an irreducible family can be given the structure of a group variety such that the map $G \times X \to X$ given by $(\varphi, x) \to \varphi(x)$ is a morphism. The induced linear map of the Lie algebra $\mathfrak{g}$ of G into the (finite dimensional) vector space of regular vector fields on X is an injection. G has the universal mapping property for all irreducible algebraic families of automorphisms containing the identity.*

Remark. By substituting G for T in the argument preceding the corollary, we see that G is an extension of a subgroup of $\mathrm{Pic}(X)$ by a linear group. In particular, when $\mathrm{Pic}(X)$ is trivial, G is a subgroup of the projective group of the projective space which defines the linear system $|D|$. Further, when X is itself projective, we may clearly assume (by adding to D a high multiple of an ample divisor) that D is a hyperplane section

in a projective imbedding of X. It follows that G is the restriction to X of a group of projective transformations of the ambient projective space (for this projective imbedding).

Now, let P be a locally isotrivial principal fiber space over a com- **62**
plete variety X with structure group G. ([6], Exposé 1, §2.2.) We will show that the group H of automorphisms of P which commute with the action of G on P and which can be connected to the identity automorphism of P is finite dimensional. Let $q : P \to X$ be the projection. If $\{\varphi_t\}_{t\in T}$ is any injective algebraic family of automorphisms of P with φ_{t_0} = Identity for some $t_0 \in T$, it is easy to check that it induces an algebraic family of automorphisms $\{\bar{\varphi_t}\}_{t\in T}$ of X such that $q \circ \varphi_t = \bar{\varphi}_t \circ q$. By Corollary 1 and the dimension theorem, it is sufficient to bound the dimension of any algebraic family $\{\varphi_t\}_{t\in T}$ such that φ_{t_0} = Identity and $q(\varphi_t(x)) = q(x)$, that is, a family which fixes the base space.

Let φ be any automorphism of P which is identity on X, so that for any $p \in P$, there is a unique $\psi(p) \in G$ such that $\varphi(p) = p\psi(p)$. Then ψ is a morphism of P into G satisfying $\psi(pg) = g^{-1}\psi(p)g$. Let $Ad(P)$ denote the bundle associated to P with fiber G for the action of G on the left of G by inner automorphisms and $\eta : P \times G \to Ad(P)$ the canonical map ([6], Exposé 1, §3.3). There is then a unique regular section $\sigma : X \to Ad(P)$ such that $\eta(p, \psi(p)) = \sigma(q(p))$. Suppose now that H is a closed normal subgroup of G, and let P' be the principal fiber space with structure group G/H deduced from G ([6], Exposé 1, §3.3). It is clear that φ induces an automorphism of P', which is the identity if and only if the morphism $\psi : P \to G$ defined above maps P into H, or equivalently, if the section σ of the bundle $Ad(P)$ has values in the sub-bundle with fiber H.

Assume first that the structure group G is linear, so that we may assume it to be a subgroup of a full linear group $Gl(n)$. Let G act on the vector space $M(n)$ of all (n, n) matrices on the left by inner automorphisms, and let V be the associated vector bundle. Then $Ad(P)$ is a sub-bundle of V. If $\{\varphi_t\}_{t\in T}$ is an injective family of automorphisms of P, we therefore get for each $t \in T$ a section $\sigma_t : X \to V$ of V, and it is easy to see that $\sigma : T \times X \to V$ defined by $\sigma(t, x) = \sigma_t(x)$ is a

morphism. Since X is complete, the vector space $\mathscr{L}$ of sections of V is finite dimensional, and if $\mathscr{L}$ is provided with the structure of an affine space, $t \in T \to \sigma_t \in \mathscr{L}$ is clearly a morphism which is injective. Hence $\dim T \leqq \dim_K \mathscr{L}$, and we are through in this case.

Next suppose G is any connected algebraic group, and C the centre G. Then G/C is a linear group ([4], §4, Lemma 3). Since we know the finite dimensionality of the group of automorphisms of the principal bundle with group G/C deduced from P, it is sufficient to prove the finite dimensionality of the group of automorphisms of P which induce the identity on the base and on the associated bundle with G/C as fiber. If $\{\varphi_t\}_{t\in T}$ is any injective family of automorphisms of this group with φ_{t_0} = Identity, we get a morphism $\sigma : T \times X \to C$ such that if $\sigma_t : X \to C$ is defined by $\sigma_t(x) = \sigma(t, x)$, $\sigma_t \neq \sigma_{t'}$ if $t \neq t'$, since the bundle associated to P with fiber C and the (trivial) action of G on C by inner automorphisms is trivial. Also we have $\sigma_{t_0}(X) = e$ in C. Let C' be the maximal linear subgroup of C, so that C/C', is an abelian variety, and $j : C \to C/C'$ the canonical homomorphism. Since $j \circ \sigma(t_0 \times X) = e$
63 in C/C', it follows from well known theorems on abelian varieties that $j \circ \sigma$ depends only on T, that is, there is a morphism $\xi : T \to C/C'$ such that $j \circ \sigma(t, x) = \xi(t)$. Hence if T_1 is any irreducible component of $\xi^{-1}(e)$ containing t_0, $\dim T \leqq T_1 + \dim C/C'$. But for any $t \in T_1$, $\sigma_t(X) \subset C'$, and since X is complete and C' is linear, $\sigma_t(X)$ is a single point $\sigma_t \in C'$, and the morphism $T_1 \to C'$ given by $t \to \sigma_t$ is injective. Hence, $\dim T \leqq \dim C' + \dim C/C'$.

Finally, suppose G is not connected, and let G_0 be the connected component of G containing e. Then P may be considered as a principal bundle with structure group G_0 over the Galois covering $Y = P \times^G G/G_0$ of X. Any connected family of automorphisms of P over X induces the identity on Y, and hence may be considered as a family of automorphisms of P over Y. Since Y is again complete we are reduced to the previous case.

Thus, we have proved

Corollary 2. *Let P be a locally isotrivial principal fiber space with base a complete variety X and structure group G. Let* $\mathrm{Aut}^0(P)$ *be the group*

of all automorphisms of P which can be connected to the identity by an irreducible algebraic family, and $\mathrm{Aut}^0_\chi(P)$ *the subgroup of* $\mathrm{Aut}^0(P)$ *consisting of those automorphisms which leave the base fixed. Then* $\mathrm{Aut}^0(P)$ *can be made into an algebraic group variety in such a way that the map* $\chi : \mathrm{Aut}^0(P) \times P \to P$ *defined by* $\chi(\varphi, p) = \varphi(p)$ *is a morphism.* $\mathrm{Aut}^0_\chi(P)$ *is a closed subgroup of* $\mathrm{Aut}^0(P)$*. The linear map* d_χ *maps the tangent space at e to* $\mathrm{Aut}^0(P)$ *(resp.* $\mathrm{Aut}^0_\chi(P)$*) injectively into the vector space of G-invariant vector fields on P (resp. G-invariant vector fields on P which are tangential to the fiber at any point of P).*

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On a Certain Purity Theorem

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1 Statement of result

Let X and Y be connected complex manifolds and $f : X \to Y$ a proper and flat[1] holomorphic map. Let $D \subset X$ be the analytic set where the differential df is not of maximal rank and $E = f(D)$ the set of critical values of f. It has been conjectured that E is pure of codimension one in Y. This has been proved by I. Dolgacev [1] and R. R. Simha [2] when the general fibre is a compact Riemann surface of genus $g \geq 1$. We prove the conjecture when the general fibre is the Riemann sphere, thereby establishing the conjecture when the relative dimension $\dim X - \dim Y$ is one. (When the general fibre is not connected, use Stein factorisation and purity of branch-locus). **65**

Specifically, we prove the following

Theorem . *Let X, Y be connected complex manifolds, $f : X \to Y$ a proper flat holomorphic map such that the general fibre of f is the Riemann sphere, D the set points in X where df is not of maximal rank and $E = f(D)$. Then E is pure of codimension one in Y.*

[1] Since X and Y are manifolds and f is proper, the assumption of flatness is equivalent to the assumptions that f is surjective and that all the fibres of f are of the same dimension everywhere on X.

2 Proof of Theorem

Let us assume to the contrary that the analytic set E is not pure of codimension one. We first achieve several reductions, which are valid even when the general fibre is only assumed of dimension one.

Firstly, since the set of $(n, n+1)$ matrices not of maximal rank is
66 an analytic subset pure of codimension 2 in the space of all $(n, n+1)$ matrices, it follows that any irreducible component of D is of codimension $\leqslant 2$ in X, hence any component of E is of codimension $\leqslant 2$ in Y. Thus if E is not pure of codimension one, by removing a certain analytic set from Y, we may assume that E is a connected sub-manifold of codimension two in Y.

Next, let $D' = \{x \in X \operatorname{rank}(df)_x \leqslant \dim Y - 3\}$. We assert that $\dim D' \leqslant \dim X - 3$. If not, D' has an irreducible component Z of codimension $\leqslant 2$ in X, and there would exist an open subset U of Z of smooth points x where $\dim(\ker(df)_x \cap T_x(U)) \geqslant 2$, so that there would exist a holomorphic vector field on a non-void open subset U' of U, not tangential to the fibres of f but mapped to zero by df. This is clearly impossible, since the restriction of f to any integral curve of this vector field must be constant. Also, we see that if Z is a component of D' of dimension equal to $\dim X - 3$, at any regular point of Z, $\ker d(f|Z) \neq 0$, so that dimension $f(Z) \leqslant \dim Y - 3$. Hence, by throwing out $f(D')$ from Y, we may assume that $\operatorname{rank}(df) \geqslant \dim Y - 2$ everywhere on X.

Let $D_i (1 \leqslant i \leqslant p)$ be the components of D, so that each D_i is of codimension 2 in X and $f(D_i) = E$. We can clearly choose a point $y \in E$ having the following properties: (i) every component of the fibre over y of the map $f|D_i : D_i \to E$ meets the set $(D_i)_{\text{reg}}$ of smooth points of D_i, and (ii) on every component of the fibre over y of the induced map $g_i : (D_i)_{\text{reg}} \to E$, g_i is of maximal rank.

Lemma. *For a dense subset Ω of the Grassmannian of two dimensional subspaces of the tangent space $T_y(Y)$ to Y at y, we have*

$$Im(df)_x + V = T_y(Y), \forall x \in f^{-1}(y), \forall V \in \Omega.$$

Proof. Let $C_1, \ldots, C_q$ be the irreducible components of $f^{-1}(y) \cap D$. If for some C_i, $\operatorname{rank}(df)_x = \dim Y - 2$ for all $x \in C_i$, by (i) and (ii) above,

we see that $Im(df)_x = T_y(E)$ for all $x \in C_i$. Then define Γ_i to be the set of 2-dimensional subspaces of $T_y(Y)$ meeting $T_y(E)$ in a space of dimension $\geqslant 1$. On the other hand, suppose C is a component such that $\text{rank}(df)$ is $\dim(Y) - 1$ generically on C_i. Then there is a finite set of points $x_1, \ldots, x_r$ on C_i such that $\text{rank}(df)_{x_j} = \dim Y - 2$ and $\text{rank}(df)_x = \dim Y - 1$ for $x \in C_i$, $x \neq x_j$ for any j. Let $\sum_j$ be the **67** set of 2-dimensional subspaces of $T_y(Y)$ meeting $Im(df)_{x_j}$ in a non-zero subspace. Let $\sum'$ be the set of 2-dimensional subspaces of $T_y(Y)$ contained in $Im(df)_x$ for some $x \in C_i - \{x_1, \ldots, x_r\}$. Put $\Gamma_i = \sum' \cup \bigcup_j \sum_j$.

A simple argument (counting constants) shows that each Γ_i and hence $\Gamma = \bigcup \Gamma_i$ is a countable union of closed nowhere dense subsets of the Grassmannian of 2-dimensional subspaces of $T_y(Y)$. Hence if Ω is the complement of Γ in this Grassmannian, Ω has the required property.

Now for the next reduction. Choose $y \in E$ as above, and a 2-dimensional subspace V of $T_y(Y)$ transversal to $T_y(E)$ with $V \in \Omega$. Choose a submanifold Y' of dimension two in a neighbourhood of y meeting E only at y and having V for its tangent space. Put $X' = f^{-1}(Y')$, so that X' is a submanifold of X proper and flat over Y'. Further, if $g : X' \to Y'$ is induced by f, g is of maximal rank outside $g^{-1}(y)$. Now for any flat holomorphic map ϕ of complex manifolds, ϕ is of maximal rank along a fibre if and only if the "correct" fibre (as an analytic space, possibly with nilpotent elements in its structure sheaf, whose defining ideal sheaf is generated by the maximal ideal of the local ring of the point in the image space) is reduced and smooth. Now, the fibres of f and g over y are the same, so that g cannot be of maximal rank all along $g^{-1}(y)$.

We are thus reduced to the case where Y is an open neighbourhood in C^2 of the closed ball $D_1 = \{(z_1, z_2) \in C^2 \| z_1|^2 + |z_2|^2 \leq 1\}$ and E consists of the single point $0 \in C^2$. We now make use of the hypothesis that the general fibre (hence every fibre $f^{-1}(z)$ for $z \neq 0$) is holomorphically isomorphic to the Riemann sphere.

Now, we assert that the inclusion $f^{-1}(0) \hookrightarrow f^{-1}(D_1)$ is a homotopy equivalence. To see this, note that since X can be triangulated such that $f^{-1}(0)$ is a sub-complex, there is a fundamental system of neighbour-

hoods $\{U_n\}_{n\geq 1}$ of $f^{-1}(0)$ such that $f^{-1}(0)$ is a strong deformation retract of each U_n. Further, if D_ϵ is the closed disc of radius $\epsilon < 1$ around 0 in C^2, $f^{-1}(D_\epsilon)$ is a strong deformation retract of $f^{-1}(D_1)$. In fact, one can
68 retract $f^{-1}(D_1)$ to $f^{-1}(D_\epsilon)$ along a projectable vector field on X whose projection to Y is an inward radial vector field around 0. Now, we have

$$D_1 \supset U_p \supset D_\epsilon \supset U_q$$

for some integers $p, q > 0$ and some ϵ with $0 < \epsilon < 1$. Since $U_p \supset U_q$ and $D_\epsilon \subset D_1$ are homotopy equivalences, the inclusion $U_q \hookrightarrow D_1$ induces isomorphisms of homotopy groups and is hence a homotopy equivalence. This proves the assertion.

Now, it is not difficult to show that $f^{-1}(D_1 - (0)) \to D_1 - (0)$ is a holomorphic fibre bundle with fibre the Riemann sphere and structure group the projective group $PGL(1, \mathbb{C})$. Since $D_1 - (0)$ is of the homotopy type of S^3 and $\pi_2(PGL(1, \mathbb{C})) \approx \pi_2(SL(2, \mathbb{C})) \approx \pi_2(SU(2)) = \pi_2(S^3) = (e)$, this fibration is topologically trivial. Let M be the compact manifold $f^{-1}(D_1)$ with boundary ∂M homeomorphic to $S^2 \times S^3$. Now, $f^{-1}(0)$ is the union of a number of irreducible one-dimensional compact analytic sets. Let ν be this number. Then by Lefschetz duality, we have (with $F = f^{-1}(0)$[2])

$$Z^\nu \approx H_2(F) \approx H_2(M) \approx H^4(M, \partial M) \approx H^3(\partial M) \approx Z \Rightarrow \nu = 1,$$

so that $f^{-1}(0)$ is irreducible. Also,

$$H_1(F) \approx H_1(M) \approx H^5(M, \partial M) \approx H^4(\partial M) = (0),$$

so that F is locally irreducible (i.e., without nodes) at every point and the normalisation of F is the Riemann sphere.

Let $\mathfrak{m}$ be the sheaf of ideals on Y defining the origin 0, and denote by $F = f^{-1}(0)$ the "correct" fibre with structure sheaf $\mathcal{O}_F = \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X$. Since $\mathfrak{m}$ is defined by two elements and F is of dimension one by the unmixedness theorem of Macaulay, the sheaf of ideals $\mathfrak{m}\mathcal{O}_X$ has

[2] All homologies, cohomologies are with integer coefficients. Further, H_c^* denotes cohomology with compact support.

no embedded components at any point, hence is primary with radical the sheaf of prime ideals $\mathfrak{p}$ defining the reduced fibre. Thus, if $\mathfrak{m}\mathscr{O}_X \neq \mathfrak{p}$, that is if F is not reduced, we can find a series of ideals $\mathscr{O}_F = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \ldots \supset \mathfrak{a}_m = (0)$ such that each quotient $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is an $\mathscr{O}_X/\mathfrak{p}$-module and

$$\sum_i \text{ Generic rank } \mathscr{O}_X/\mathfrak{p} \ \ (\mathfrak{a}_i, \mathfrak{a}_{i+1}) = r > 1.$$

We now recall some "well-known" definitions. Let X, Y be con- **69**
nected complex manifolds and $f : X \to Y$ a holomorphic map. Let Z be an irreducible closed analytic subset of Y of codimension p such that $f^{-1}(Z)$ is pure of codimension p. One then defines the *analytic cycle* (*i.e.*, formal linear combination of closed irreducible analytic subsets) $f^*(Z)$ in X to be $\sum n_i C_i$ where C_i are the components of $f^{-1}(Z)$ and n_i is defined as follows. Let $\mathfrak{q}$ be the defining ideal of Z. Then the sheaves $Tor_r^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{O}_Y/\mathfrak{q})$ are coherent sheaves on X with support in $f^{-1}(Z)$. On an open subset of C_i, we can find coherent sheaves F_{pj},

$$Tor_r^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{O}_Y/\mathfrak{q}) = F_{r0} \supset F_{r1} \supset \ldots \supset F_{rn_r} = (0)$$

such that F_{rj}/F_{rj+1} is an $\mathscr{O}_{C_i}$-module. Define the integer n_{ir} as

$$n_{ir} = \sum \quad \text{Generic rank on} \quad C_i \quad \text{of} \quad F_{rj}/F_{rj+1}$$

and put

$$n_i = \sum (-1)^r n_{ir}$$

Next, let X be a connected complex manifold and Z a *compact* irreducible analytic set on X of dimension r. Then there is a unique generator $\xi \in H_{2r}(Z) \approx Z$ determined by the orientation of Z, since Z can be triangulated and is an oriented compact connected pseudomanifold of dimension $2r$ (see Seifert-Threlfall [3]). Let ξ' be the image of ξ in $H_{2r}(X)$ and $Cl(Z) \in H_c^{2(n-r)}(X)$, the Poincaré dual of ξ'. We call $Cl(Z)$ the cohomology class (with compact support) associated to Z. If $z_c^p(X)$ denotes the free abelian group on the irreducible compact analytic subsets of codimension p, we obtain a homomorphism $Cl : z_c^p(X) \to H_c^{2p}(X)$ on extending by linearity.

We then have the following (acceptable, "well-known" but nowhere explicitly proved) result (See remark at the end of the paper).

If $f : X \to Y$ is a proper holomorphic map and Z, a compact analytic irreducible subset of Y of codimension p such that $f^*(Z)$ is defined, we have

$$H_c^{2p}(f)(Cl(Z)) = Cl(f^*(Z))$$

70 Now, let us return to our original situation. Let $\dot{D}_1$ be the interior of D_1 and P any point of $\dot{D}_1$. Then $Cl(P) \in H_c^4(\dot{D}_1)$ is the fundamental cohomology class with compact support of $\dot{D}_1$. Put $\dot{M} = M - \partial M$, so that $f|\dot{M} : \dot{M} \to \dot{D}_1$ is a proper holomorphic map. Since f is flat, $Tor_i^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{O}_Y/\mathfrak{m}) = (0)$ for $i > 0$. Thus, if $\mathfrak{m}\mathscr{O}_X \neq \mathfrak{p}$, we see that $f^*(0)$ is the cycle rF_0 where r is an integer > 1 and F_0 is the (reduced) fibre. By the result stated, $H_c^4(f)(\xi) = r.Cl(F_0)$ with $r > 1$. Now, consider the following commutative diagram

$$\begin{array}{ccccc}
Z \approx H^3(\partial M) & \xrightarrow{\beta_1}_{\sim} & H^4(M, \partial M) & \xleftarrow{\beta_2}_{\sim} & H_c^4(\dot{M}) \\
\wr\uparrow f_1^* & & \uparrow f_2^* & & \uparrow f_3^* \\
Z \approx H^3(S^3) & \xrightarrow{\alpha_1} & H^4(D_1, S^3) & \xleftarrow[\sim]{\alpha_2} & H_c^4(\dot{D}_1)
\end{array}$$

where f_1^* are the maps induced by f in cohomology. Here, α_1, α_2 and β_2 are well-known to be isomorphisms, and f_1^* is an isomorphism since $\partial M \to S^3$ is a trivial fibration with fibre S^2. Finally, β_1 is also an isomorphism since $H^3(M) = H^4(M) = (0)$. It follows that f_3^* is an isomorphism. Hence it cannot take a generator ξ of $H_c^4(\dot{D}_1)$ to an r-th multiple of an element of $H_c^4(M)$ where $r > 1$.

This contradiction shows hat $\mathfrak{m}\mathscr{O}_X = \mathfrak{p}$, so that F is a reduced (and irreducible) analytic set of dimension one. Since $\chi(\mathscr{O}_{f^{-1}(z)}) = 1$ for $z \in Y$, $z \neq 0$ and f is flat, we have $\chi(\mathscr{O}_F) = 1$ and $H^1(F, \mathscr{O}_F) = (0)$. This implies that F is smooth (since singularities increase the arithmetic genus). By what was said earlier, f is of maximal rank along F which is a contradiction to our assumption that 0 is a critical value of f. The theorem is proved. □

Remark . The result stated about the naturality of Cl with respect to proper maps is certainly reasonable, but nowhere explicitly proved to our knowledge. Borel-Haefliger [4] associate cohomology classes (with arbitrary support) to arbitrary (not necessarily compact) cycles, and prove naturality. Presumably their methods would also prove the above result.

However, we prefer to give an *ad hoc* proof in our situation. **71**

Let P be a point of $f^{-1}(0)$ where the reduced fibre F_{red} is smooth and choose a locally closed submanifold Z of dimension 2 through P transversal to the reduced fibre having P as the unique point of intersection with F. By shrinking D_1 if necessary, we may assume that $Z \cap M$ is proper over D_1 with finite fibres. Since Z and F have intersection number one in the topological sense (Seifert-Threlfall [3]), using the well-known relationship between intersection of cycles and cup-product, we see that if $\alpha \in H^4_c(M) \approx H^4(M, \partial M)$ is the class associated to the reduced fibre and $\zeta \in H_4(M, \partial M)$ the cycle defined by Z, we have $\langle \alpha, \zeta \rangle = 1$. Now, the generator ξ of $H^4(D_1, S^3) \approx H^4_c(\dot{D}_1)$ can be represented by a real 4-form ω with compact support in $\dot{D}_1$ and

$$\int_{\dot{D}_1} \omega = 1.$$

Since we have shown that $H^4(D_1, S^3) \to H^4(M, \partial M)$ is an isomorphism, $f^*(\omega)$ represents a generator of $H^4_c(\dot{M}) \approx H^4(M, \partial M)$. Thus, we see that α is the μ-th multiple of the class of $f^*(\omega)$ in $H^4(M, \partial M)$ with $\mu \in Z$. We thus obtain

$$1 = \langle \alpha, \zeta \rangle = \mu \int_Z f^*(\omega).$$

Now, if $\mathfrak{m}\mathcal{O}_X \neq \mathfrak{p}$, it is easy to see that $\mathfrak{m}$ together with the principal ideal in $\mathcal{O}_{P,X}$ defining Z cannot generate the maximal ideal of $\mathcal{O}_{P,X}$. Hence, $f|Z : Z \to Y$ is not an isomorphism at P. By local analytic geometry, there is a neighborhood of 0 in Y, which we may assume to be $\dot{D}_1$, such that outside of a proper analytic subset of this neighborhood, $f|Z$ is a

covering of degree $r > 1$. Hence,

$$1 = \mu \int_{Z \cap \dot{M}} f^*(\omega) = \mu r \int_{\dot{D}_1} \omega = \mu r$$

which shows that $\mu = r = 1$, a contradiction. This completes the proof.

72 **Addendum.** It was pointed out to us in a letter by R. R. Simha that the appeal to the theorem on the continuity of the Euler characteristic at the end of the proof of the theorem is unnecessary, since, once we know that the special fibre is reduced, the set of points where f is not of maximal rank is of codimension $\geqslant 3$, and hence is empty.

This enables us to extend the proof also to the case when the general fibre is a non-singular curve of arbitrary genus. In fact, in the notation of the above proof, we have only to show that $H^4_c(\dot{D}_1) \to H^4_c(\dot{M})$ is surjective (hence an isomorphism, since Rank $H^4_c(\dot{M})$ is $\geqslant 1$). The rest of the proof does not utilise the genus zero assumption. By the Wang sequence of the fibration $\partial M \to S^3$, we see that $H^3(S^3) \to H^3(\partial M)$ is surjective and since $H^4(M) = 0$, $H^1(\partial M) \to H^4(M, \partial M) = H^4_c(\dot{M})$ is also surjective. This proves the assertion, and we get a uniform proof for all genera.

Added in Proof (Nov. 16, 1970)

The author is informed that the result has also been proved in the case of algebraic varieties when the genus of the general fibre is zero, by I. Dolgacev and M. Raynaud.

The following very simple counter-example to the conjecture when the dimension of the fibre is $\geqslant 2$ was pointed out by Professor David Mumford.

Let X be a compact non-singular surface, $x_0 \in X$, $p_2 : X \times X \to X$ the second projection, and $\sum_1$ and $\sum_2$ the sections $\{x_0\} \times X$ and Δ the diagonal. Let I be the sheaf of ideals defining $\sum_1 \cup \sum_2$. Blow up $X \times X$ with respect to the sheaf of ideals I, to obtain Y. Denote the composite $Y \xrightarrow{\sigma} X \times X \xrightarrow{p_2} X$ by f. Then Y non-singular outside of $\sigma^{-1}(x_0, x_0)$ and f is of maximal rank outside $\sigma^{-1}(x_0, x_0)$. If we can show that (i) Y is

non-singular, and (ii) $\sigma^{-1}(x_0, x_0)$ is of dimension 2, it would follow that $f^{-1}(x_0)$ has at least two components, and hence f cannot be regular all along $f^{-1}(x_0)$.

To check these statements, we might as well blow up $\mathbb{C}^4$ with respect to the sheaf of ideals defined by $\{z_1 = z_2 = 0\} \cup \{z_3 = z_4 = 0\}$ i.e. with **73** respect to the sheaf of ideals generated by $\{z_1z_3, z_1z_4, z_2z_3, z_2z_4\}$. The resulting space is covered by four open subsets, all of which are isomorphic, a typical one being the affine algebraic variety with co-ordinate ring

$$\mathbb{C}\left[z_1, z_2, z_3, z_4, \frac{z_4}{z_3}, \frac{z_2}{z_1}, \frac{z_2z_4}{z_1z_3}\right] = \mathbb{C}\left[z_1, z_3, \frac{z_4}{z_3}, \frac{z_2}{z_1}\right]$$

which is isomorphic to $\mathbb{C}^4$. The intersection of this open set with the fibre over the origin is given by $z_1 = z_3 = 0$, hence is 2-dimensional.

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A topological characterisation of the affine plane as an algebraic variety

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1 Statement of results

This investigation arose from an attempt to answer the following question, raised by M. P. Murthy: If X is an affine algebraic variety over a field k such that $X \times A^1 \approx A^3$, where A^n denotes the affine n-space over k, is X isomorphic to A^2? It was hoped that when $k = C$, just the hypothesis that X is a non-singular, affine, rational and contractible surface would imply that $X \approx \mathbb{C}^2$. This, however, is not true as we show by a counter-example. We are unable to decide if the variety Y of this counter-example also satisfies the condition $Y \times \mathbb{C} \approx \mathbb{C}^3$.[1] 75

On the other hand, we do prove the following positive result.

Theorem. *Let X be a non-singular complex algebraic surface which is contractible AND simply connected at infinity. Then X is isomorphic to the affine two-space as an algebraic variety.*

The idea of the proof is to embed X as a Zariski open subset of a complete non-singular surface such that the complement is a union of non-singular curves with normal crossings, and to use the contractibility and simple-connectedness at infinity of X to work out the geometric

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[1] Examples of compact 4-manifolds with boundary which are contractible and whose boundary is a homology 3-sphere but not a homotopy sphere have been given by several authors (Mazur, Poinereau, ——). Our counter-example also provides another such.

configuration of these curves. We prove that after suitable transformations, we are reduced to the case where the complement consists of a single non-singular curve, from which it follows easily that the compactification is the projective plane and the curve a line.

We shall first prove the theorem, and then proceed to give the counter example. The methods of the proof of the theorem are analogous to those of Mumford [2].

2 Proof of the theorem

Let X be as in the statement of the theorem. By the Nagata compacti-
76 fication theorem [3] and the theorem of resolution of singularities of an algebraic surface [1], we may assume X embedded as a Zariski open subset of a complete non-singular (hence projective, see [4]) surface $\bar{X}$, such that if $\bar{X} - X = Y$ and $Y_i (1 \leqq i \leqq n)$ are the components of Y, each Y_i is a complete non-singular curve, and we have the following further conditions:

(i) For two distinct indices i, j, either $Y_i \cap Y_j = \oslash$ or $Y_i \cap Y_j$ consists of a single point where Y_i and Y_j meet transversally.

(ii) For three distinct indices i, j, k, $Y_i \cap Y_j \cap Y_k = \oslash$.

The contractibility of X implies some further conditions. In fact, by Poincaré duality on X, we get[2]

$$H^i(\bar{X}, Y) \approx H^i_c(X) \approx H_{4-i}(X) \approx \begin{cases} 0 & \text{if } i < 4 \\ Z & \text{if } i = 4, \end{cases}$$

so that by the cohomology exact sequence, we have isomorphisms

$$H^i(\bar{X}) \approx H^i(Y), \qquad 0 \leqq i \leqq 3.$$

Hence Y is connected, $H^2(\bar{X}) \to H^2(Y)$ is an isomorphism and $H^3(\bar{X}) = 0$. By duality, $H_1(\bar{X}) = 0$, hence also $H^1(Y) = 0$. From this, we deduce the following three conditions:

[2] All homologies and cohomologies have integral coefficients, unless otherwise indicated. Cohomology with compact supports is denoted by H^*_c.

(iii) Any two components Y_i, Y_j can be joined by a chain $Y_i = Y_{i_1}, Y_{i_2}, \ldots, Y_{i_p} = Y_j$ such that $Y_{i_k} \cap Y_{i_{k+1}} \neq \emptyset$, $1 \leqq k \leqq p-1$.

(iv) There is no chain of components $Y_{i_1}, \ldots, Y_{i_p} (p \leqq 3)$ such that $Y_{i_k} \cap Y_{i_{k+1}} \neq \emptyset$, $1 \leqq k \leqq p-1$, and $Y_{i_p} \cap Y_{i_1} \neq \emptyset$.

(v) Each Y_i is isomorphic to the complex projective line P^1.

In fact (iii) is just the connectedness of Y. As for (v), let Y' be the union of all the connected components of Y excepting Y_i. Since $Y_i \cap Y'$ is a finite set of points, the Mayer-Vietoris sequence gives us that $H^1(Y') \oplus H^1(Y_i) = 0$, which implies (v). Next suppose there is a chain of components satisfying the conditions of (iv). Let Y' be the union of the components of the chain, and Y'' the union of $Y_{i_1}, \ldots, Y_{i_{p-1}}$. Then Y'' is connected and $Y'' \cap Y_{i_p}$ consists of at least two points, and it follows from the Mayer-Vietoris sequence

$$\begin{aligned} 0 \to Z = H^0(Y') \to Z^2 \approx H^0(Y'') \oplus H^0(Y_{i_p}) \\ \to Z^\nu \approx H^0(Y'' \cap Y_{i_p}) \to H^1(Y') \end{aligned}$$

with $\nu \geqq 2$ that $H^1(Y') \neq 0$. If Y''' is the union of the components of Y not belonging to the chain, the exact sequence

$$0 = H^1(Y) \to H^1(Y') \oplus H^1(Y''') \to H^2(Y' \cap Y''') = 0$$

leads to a contradiction. Thus, (i)-(v) are established. We may 77
also assume the further condition:

(vi) If some Y_i has self-intersection number -1, it meets at least three other $Y_j (j \neq i)$.

In fact, suppose $(Y_i^2) = -1$ and Y_i meet at most two other Y_j. Then we can contract Y_i to obtain a non-singular complete surface $\bar{X}'$ such that X is isomorphic to an open subset X' of $\bar{X}'$, and $Y' = \bar{X}' - X'$ satisfies (i)-(v). If this does not suffice, we repeat the above process, till (vi) is satisfied.

Now $H^3(\bar{X}) = 0$ implies $H_2(\bar{X})$ is torsion-free, so that $H_2(Y) \to H_2(\bar{X})$ is an isomorphism. Since $H_2(Y)$ is freely generated by the

fundamental classes of its components $Y_i (1 \leqq i \leqq n)$, we see that $H^2(\bar{X})$ is freely generated by the cohomology classes associated to the divisors $Y_1, \ldots, Y_n$. Hence, with the usual notations,

$$H^2(\bar{X}, C) \approx H^1(\bar{X}, \Omega^1)$$

and $H^2(\bar{X}, \mathcal{O}) = H^0(\bar{X}, \Omega^2) = 0$. Since further $H^1(\bar{X}, C) = 0$, $H^1(\bar{X}, \mathcal{O}) = 0$ and $\chi(\mathcal{O}_{\bar{X}}) = 1$.

Now, denote by $L = L(Y)$ the vector space $\sum_1^n \mathbb{R} Y_i$ over $\mathbb{R}$ with basis Y_i, endowed with the quadratic form given by the intersection number extended by linearity. By the Hodge index theorem, we have

(vii) If W is any subspace of L on which the intersection form is positive semi-definite, dim $W \leqq 1$.

Now, consider any non-singular complete surface V and a Zariski-closed subset F of V of codimension one with irreducible components $F_1, \ldots, F_n$ satisfying conditions (i)-(vii) (with Y, Y_i replaced by F, F_i). To such a pair (V, F), we attach a graph $\Gamma = \Gamma(V, F)$ and weights which are integers to each of the vertices of Γ as follows. We take the vertices to be in one-one correspondence with the components F_i of F, and link two vertices if and only if the corresponding components meet. The self-intersection number of a component is the weight of the corresponding vertex. This graph is a tree, that is, is connected and contains no loops, by (ii) and (iv).

Let $\{P_a\} (\alpha = 1, \ldots, \lambda)$ be the various points of intersection of distinct components of F. We can choose disjoint closed neighborhoods $\bar{U}_\alpha$ of P_α and holomorphic co-ordinate systems $z_\alpha = (z_\alpha^1, z_\alpha^2)$ valid in a neighborhood of $\bar{U}_\alpha$ such that:

(a) z_α maps $\bar{U}_\alpha$ homeomorphically onto the polycylinder $\{z = (z^1, z^2) \in C^2 | |z^i| \leqq 1\}$.

78 (b) If F_i and F_j are the two components of F through P_α, no other

component of F meets $\bar{U}_\alpha$ and $F_i \cap \bar{U}_\alpha$ and $F_j \cap \bar{U}_\alpha$ respectively are defined by the equations $z_\alpha^1 = 0$, $z_\alpha^2 = 0$ in $\bar{U}_\alpha$.

Choose a Riemannian metric ds^2 on V such that in a neighborhood of $\bar{U}_\alpha$, $ds^2 = |dz_\alpha^1|^2 + |dz_\alpha^2|^2$. Then there is an $\epsilon > 0$ with $2\epsilon < 1$ such that: (a) the exponential map maps the open subset of the tangent bundle of V consisting of tangent vectors of length $< 2\epsilon$ diffeomorphically onto an open neighborhood of the diagonal in $V \times V$, (b) the image of the 2ϵ-ball in the tangent space at any point of V by the exponential map is a convex open neighborhood of that point, and (c) if $i \neq j$, $d(F_i - \bigcup_\alpha \bar{U}_\alpha, F_j - \bigcup_\alpha \bar{U}_\alpha) > 2\epsilon$. Denote by $\bar{V}_\delta = \bar{V}_\delta(F)$ the set of points in V whose distance from F is $\leqq \delta$, by $S_\delta = S_\delta(F)$ the set of points whose distance from F equals δ and set $V_\delta(F) = V_\delta = \bar{V}_\delta - S_\delta$. Then for $\delta < \epsilon$, S_δ is the boundary and V_δ the interior of $\bar{V}_\delta$ in V. From now on, δ always satisfies $0 < \delta < \epsilon$. We denote by $\bar{V}_{\delta i}, S_{\delta i}$, and $V_{\delta i}$ the constructs corresponding to $\bar{V}_\delta, S_\delta$, and V_δ for F_i instead of F. Then $\bar{V}_\delta = \bigcup_i \bar{V}_{\delta i}$ and $V_\delta = \bigcup_i V_{\delta i}$. The closed ball bundle $B_{\delta i}$ of radius δ in the normal bundle of F_i in V is homeomorphic to $\bar{V}_{\delta i}$ by the exponential map $\exp_i : B_{\delta i} \to \bar{V}_{\delta i}$. We denote the inverse of this homeomorphism by $\log_i$. Then $\log_i$ carries $S_{\delta i}$ onto the circle bundle (of radius δ) assosiated to the normal bundle of F_i.

Lemma 1. *There is a homeomorphism*

$$S_\delta \times (0, 1] \xrightarrow{\varphi} \bar{V}_\delta - F$$

such that $\varphi(x, 1) = x$ and $\varphi(S_\delta \times (0, \delta')) \cup F$ form a fundamental system of neighborhoods of F on V. Further, F is a strong deformation retract of $\bar{V}_\delta(F)$.

Proof. A point x of S at a distance $> \epsilon$ from any P_α lies on a unique $S_{\delta i}$. We then define for $0 < t \leqq 1$,

$$\varphi(x, t) = \exp_i(t \log_i x).$$

Define φ on each $(S_\delta \cap \bar{U}_\alpha) \times (0,1]$ in terms of the co-ordinates z_α by

$$\varphi(z_\alpha, t) = \begin{cases} (tz_\alpha^1, z_\alpha^2) & \text{if } |z_\alpha^2| \geqq \epsilon \\ (z_\alpha^1, tz_\alpha^2) & \text{if } |z_\alpha^1| \geqq \epsilon \\ \left(tz_\alpha^1, \left(t + (1-t)\dfrac{|z_\alpha^2| - \delta}{\epsilon - \delta}\right) z_\alpha^2\right) & \text{if } \epsilon > |z_\alpha^2| \geqq \delta \\ \left(\left(t + (1-t)\dfrac{|z_1^\alpha| - \delta}{\epsilon - \delta}\right) z_\alpha^1, tz_\alpha^2\right) & \text{if } \epsilon > |z_\alpha^1| \geqq \delta. \end{cases}$$

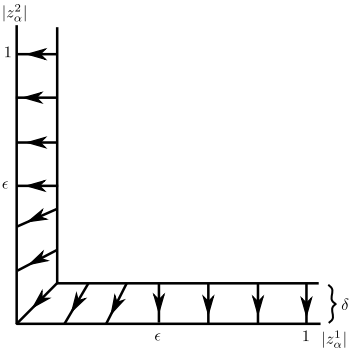

Fig. 1

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It is easily checked that φ is well-defined on $S_\delta \times (0,1]$ and satisfies the conditions of the lemma. □

Corollary. *The fundamental group at ∞ of $U = V - F$ is isomorphic to the fundamental group of S_δ for some (or any) $\delta < \epsilon$.*

We will now examine how S_δ is built up from the $S_{\delta_i} (1 \leqq i \leqq n)$ or, what is the same, from the circle bundles $\sum_i$ associated with the normal bundles, considered as complex line bundles, of F_i in V. For each i, we take the unique oriented circle bundle $\sum_i$ over $F_i \approx S^2$[3] of degree equal to the self-intersection number of F_i. If F_i and F_j intersect at a point

[3] We denote by S^n the n-sphere, by D^n the closed n-disc. For a subset Y of a topological space X, we denote by $\dot{Y}$, $\bar{Y}$ and ∂Y the interior, closure and boundary, respectively, of Y in X.

P_α, let D^2, D'^2 be closed discs of radius δ on F_i and F_j, respectively. Denote by π_i the projection $\sum_j \to F_i$. We have orientation-preserving trivialisations over the base $\pi_i^{-1}(D^2) \approx D^2 \times S^1$, $\pi_j^{-1}(D'^2) \approx D'^2 \times S^1$. We set $\sum_i' = \sum_i - (\pi_i^{-1}(D^2))^\circ$, and $\sum_j' = \sum_j - (\pi_j^{-1}(D'^2))^\circ$, so that these are manifolds with boundaries $D^2 \times S^1$, $D'^2 \times S^1$. We then identify $D^2 \times S^1$ and $D'^2 \times S^1$ by a homeomorphism carrying $\partial D^2 \times \{x\}$ onto $\{y\} \times S^1$ (any x, some y depending on x) and $\{x\} \times S^1$ onto $\partial D'^2 \times \{y\}$ (any x, some y), preserving orientations. If we perform this operation for all the points of intersection of different components of F, we end up with S_δ.

Thus, we see that S_δ is determined topologically by the following data: the number of components of F, the pairs among them which intersect, and their self-intersection numbers; in other words, the weighted graph associated with F.

Note that to any subsystem $G = \{G_1, G_2, \ldots, G_m\}$ of curves of $F = \{F_1, \ldots, F_n\}$, there corresponds a sub-graph Γ' of Γ such that two vertices of Γ' are linked in Γ' if and only if they are linked in Γ, and conversely, to any such sub-graph Γ', there corresponds a sub-system of curves of F. **80**

We introduce some definitions. Let Γ be any connected graph, and f a vertex of Γ. The connected components of the graph obtained by removing f from the vertices of Γ and deleting the links at f are called the branches of Γ at f. We say that f is a branch point if the number of branches at f is at least three. Now suppose Γ is a graph with weights $\{w_e\}$ which are real numbers attached to the vertices $\{e\}$ of Γ. On the real vector space $L(\Gamma)$ with the vertices of Γ as basis, we introduce a symmetric bilinear form $Q = Q(\Gamma)$ by defining

$$Q(e, e) = w_e \qquad \text{for any vertex } e \text{ of } \Gamma,$$

$$Q(e, f) = \begin{cases} 1 & \text{if } e \text{ and } f \text{ are distinct vertices linked in } \Gamma \\ 0 & \text{if } e \text{ and } f \text{ are distinct vertices not linked in } \Gamma. \end{cases}$$

Note that when Γ is the weighted graph arising from a pair (V, F) as above, Q is nothing but the intersection form.

Lemma 2. *Let Γ be a graph with real weights $\{w_e\}$ attached to the vertices $\{e\}$ of Γ. Let e, e' be two vertices of Γ such that there is a unique*

link $e\bar{e}'$ through e in Γ. Suppose $w_e \neq 0$. Let Γ' be the graph with the same vertices as Γ, but with the link $e\bar{e}'$ omitted from the links of Γ. Define weights $\{w'_f\}$ on the vertices $\{f\}$ of Γ' by

$$w'_f = w_f \quad \text{if} \quad f \neq e',$$
$$w'_{e'} = w_{e'} - w_e^{-1}.$$

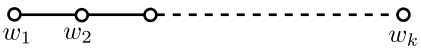
Then the quadratic forms $Q(\Gamma)$ and $Q(\Gamma')$ are equivalent by a unimodular linear transformation. In particular, they have the same discriminant (with respect to the basis consisting of the vertices of Γ).

Proof. Express $Q(\Gamma)$ in terms of the basis of $L(\Gamma)$ given by

$$e, e' - w_e^{-1}e, f(f \neq e, e').$$

□

Corollary. *The determinant of the following graph with real weights w_i*



$(k \geqq 1)$

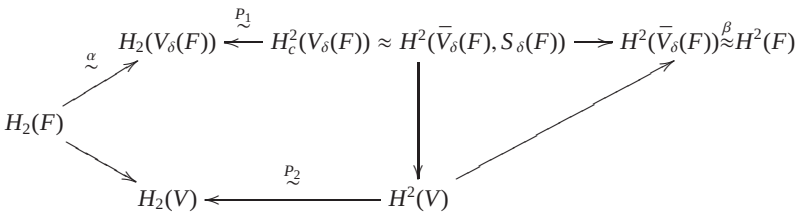
satisfying $w_1 < -1$, $w_i \leqq -2$ for $i > 1$, is greater than one in absolute value.

Proof. Successive applications of the lemma, starting from the left, show that the discriminant equals $\prod_1^k q_i$, where $q_1 = w_1$, $q_{i+1} = w_{i+1} - q_i^{-1}$, Since $q_i < -1$, the result follows.

We shall denote the discriminant of $Q(\Gamma)$ by $d(\Gamma)$ and we shall write $d(F)$ for $d(\Gamma(Y, F))$. □

81 **Lemma 3.** *Let (V, F) be a pair satisfying conditions (i)-(vii). Then $d(F) \neq 0$ if and only if $H_i(S_\delta(F))$ is finite for small δ, and in that case, $d(F)$ equals the order of $H_1(S_\delta(F))$.*

Proof. We have the following commutative diagram



Here, P_1, P_2 are the isomorphisms of Poincaré duality theory, obtained by forming cap products with the fundamental classes of $V_\delta(F)$ (in the homology group of locally finite chains) and $\bar{V}$, respectively, and α and β are isomorphisms since $F \hookrightarrow V_\delta(F) \hookrightarrow \bar{V}_\delta(F)$ are homotopy equivalences. It follows that the composite map $H_2(F) \to H^2(F)$ obtained by following the upper row is that given by the intersection product $H_2(F) \times H_2(F) \to Z$. Hence, $d(F) \neq 0$ if and only if this pairing is non-degenerate over Q, and in this case $d(F)$ equals the order of the cokernel of $H_2(F) \to H^2(F)$, i.e., the order of the cokernel of $H^2(\bar{V}_\delta(F), S_\delta(F)) \to H^2(\bar{V}_\delta(F))$. Now

$$H^3(\bar{V}_\delta(F), S_\delta(F)) \approx H_c^3(V_\delta(F)) \approx H_1(V_\delta(F)) \approx H_1(F) = 0,$$

so that the sequence

$$H^2(\bar{V}_\delta(F), S_\delta(F)) \to H^2(\bar{V}_\delta(F)) \to H^2(S_\delta(F)) \to 0$$

is exact. Further, by Poincaré duality on $S_\delta(F)$, $H^2(S_\delta(F)) \approx H_1(S_\delta(F))$. This proves the lemma. □

Now, let (V, F) satisfy conditions (i)-(vii) and let Γ be its graph. For any connected subgraph Γ' of Γ such that two vertices of Γ' are linked in Γ' if and only if they are linked in Γ, let F'' denote the corresponding subsystem of curves of F. We shall denote by $\pi(\Gamma')$ the fundamental group $\pi_1(S_\delta(F'))$ for small δ. Note that this is the fundamental group at infinity of $V - F'$. In particular, if (W, G) is another pair satisfying (i)-(vii) such that $V - F$ and $W - G$ are isomorphic varieties, $\pi_1(S_\delta(F)) \approx \pi_1(S_\delta(G))$.

Lemma 4 ([2]Mumford)**.** *Let (V, F) satisfy (i)-(vii), and let Γ be its graph. Let e be a vertex of Γ and $\Gamma_1, \ldots, \Gamma_p$ the branches at e. If $\pi(\Gamma) = (e)$, there are at most two branches Γ_i such that $\pi(\Gamma_i) \neq (e)$.*

Proof. Let G_i be the system of curves corresponding to Γ_i, and G the component of F corresponding to e. Then $S_\delta(F)$ is obtained as follows. From each $S_\delta(G_i)$, one removes an open solid torus $(D_i^2 \times S^1)^\circ$. Let **82**
$\pi : \sum(G) \to G$ be the oriented circle bundle over G of degree equal

to the self-intersection number of G. Then one removes from $\sum(G)$ certain open solid tori of the form $\pi^{-1}(D_i'^2)^\circ$, $D_i'^2$ being disjoint discs on G. Choose trivialisation of $\sum(G)$ over the $D_i'^2$ preserving orientation: $\pi^{-1}(D_i'^2) \overset{\lambda_i}{\approx} D_i'^2 \times S^1$. Now one identifies $\partial D_i^2 \times S^1$ and $\lambda_i^{-1}(\partial D_i'^2 \times S^1)$ by a homeomorphism φ_i such that $\lambda_i \circ \varphi_i$ takes circles of the form $\partial D_i^2 \times \{x\}$ onto circles of the form $\{y\} \times S^1$.

Choose an open subset U of G containing all the D_i' with a homeomorphism $U \approx R^2$ taking the D_i onto discs in $\mathbf{R}^2$ with centers on the unit circle and small radius, such that the centers of $D_1'^2, D_2'^2, \ldots$ occur in cyclic order on the unit circle (see Fig. 2). Join the origin O of $\mathbf{R}^2$ to points x_i of $\partial D_i'^2$ by straight lines l_i. Since the fibration π restricted to U is trivial, we can choose a base point

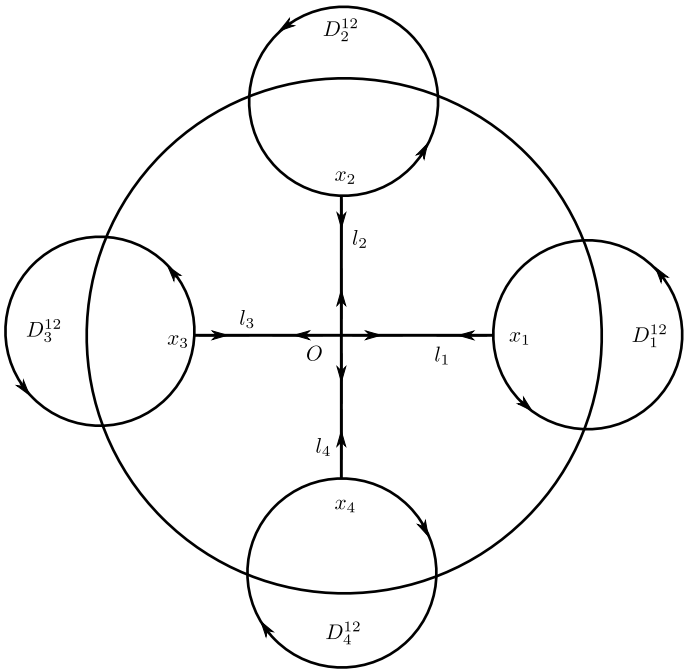


Fig. 2

83 O' in $\sum(G) - \cup_i \pi^{-1}(D_i'^2)$, curves l_i' lifting l_i leading from O' to a point x_i'

lying over x_i and loops γ_i' at x_i' whose projection $\pi(\gamma_i')$ is the loop $\partial D_i'^2$ oriented positively at x_i. Finally let δ_i be the loop at x_i' describing the fiber $\pi^{-1}(x_i) \approx S^1$ once positively.

The fundamental group H of $\sum(G) - \bigcup_i \pi^{-1}(D_i'^2)^\circ$ is generated by $c_i = l_i'\gamma_i' l_i'^{-1}$, $d_i = l_i'\delta_i l_i'^{-1}$ with the relations $d_1 = d_2 = \ldots = d_p (= d$, say), $c_i d_j c_i^{-1} d_j^{-1} = e$ and $c_1 c_2 \ldots c_p = d^r$ for some $r \in \mathbf{Z}$. Let G_i' be the fundamental group of $S_\delta(G_i) - (D_i^2 \times S^1)^\circ$ based at the point y_i corresponding to the point x_i' on $\pi^{-1}(\partial D_i'^2)$. By van-Kampen's theorem, $\pi_1(S_\delta(G_i)) = G_i''$ is the quotient of G_i' by the normal subgroup generated by the loop $\partial D_i^2 \times x_i$ (for some point $x_i \in S^1$) through y_i. Again by van-Kampen's theorem, the fundamental group $\pi(\Gamma)$ of $S_\delta(F)$ is the quotient of the free product

$$G_1' * G_2' * \ldots * G_p' * H$$

by the normal subgroup generated by $d_i(\partial D_i^2 \times x_i)^{-1} (1 \leqq i \leqq p)$ and $c_i \xi_i'^{-1}$ for some $\xi_i' \in G_i'$. On dividing each G_i by $\partial D_i^2 \times x_i$ and H by the (central) subgroup generated by d, we see that $\pi(\Gamma)$ has as quotient the group

$$G_1'' * G_2'' * \ldots * G_p''/\{\xi_1 \ldots \xi_p\}$$

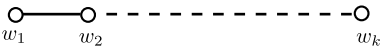
where ξ_i are the images of ξ_i' in G_i'', and for any η, we denote by $\{\eta\}$ the normal subgroup generated by η.

Now, if there are three indices i for which G_i'' is non-trivial, $\pi(\Gamma)$ has as quotient a group of the form

$$H_1 * H_2 * H_3/\{\eta_1\eta_2\eta_3\}$$

where H_i are non-trivial groups and $\eta_i \in H_i$. Since $\pi_1(\Gamma) = \{e\}$ by the assumption, the above group should also be trivial. But by Mumford's lemma [2] this group is non-trivial, which is a contradiction. □

Lemma 5. *Let* (V, F) *be a pair satisfying (i)-(vii) with graph* Γ. *Suppose there is a subgraph* Γ' *of the form*



$(k \geqq 1)$

(where the w_i denote the weights) with $\pi(\Gamma') = (e)$ such that no vertex of Γ' is a branch point of Γ. Then there is a pair (W, G) satisfying (i)-(v) and (vii) such that the following conditions hold:

(i) There is an isomorphism of varieties $V - F \approx W - G$;

(ii) The graph of (W, G) is obtained from the graph Γ of (V, F) by pinching Γ' to a single vertex in Γ and giving it a weight $+1$. (The weights at vertices not in Γ' may get changed.)

Proof. We proceed by induction on k. When $k = 1$, $w_1 = \pm 1$ since
84 $\pi(\Gamma') = (e)$, and $w_1 = -1$ is ruled out by assumption (vi). Assume that the result holds when the number of vertices in Γ' is $< k$. Now, all the weights w_i cannot be < 0, since they would then be $\leqq -2$ and $|d(\Gamma')| > 1$. If there are at least three non-negative weights, some two vertices which are not linked must have non-negative weights, and there is a two dimensional subspace of $L(\Gamma)$ on which $Q(\Gamma)$ is positive semi-definite. Hence the number of non-negative weights is either one or two, and if it is two, the corresponding vertices must be linked and one of the weights must be 0.

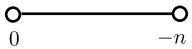
Suppose the number of non-negative weights is one, say $w_{i_0} \geqq 0$ and $w_i \leqq -2$ for $i \neq i_0$. By Lemma 2, the discriminant $d(\Gamma')$ equals

$$d(\Gamma') = q_1 \cdot q_2 \ldots q_{i_0-1} \cdot r_1 \cdot r_2 \ldots r_{k-i_0} \qquad (w_{i_0} - q_{i_0-1}^{-1} - r_{k-i_0}^{-1})$$

where $q_1 = w_1$, $q_{i+1} = w_{i+1} - q_i^{-1}$ for $i \geqq 1$, $r_1 = w_k$, $r_{i-1} = w_{k-i+2} - r_i^{-1}$ for $i \leqq k$. (Omit the q's if $i_0 = 0$ and the r's if $i_0 = k$). Thus, q_i, $r_i < -1$ and if $w_{i_0} \geqq 1$, $|d(\Gamma')| > 1$. Thus, $w_{i_0} = 0$, and

$$d(\Gamma') = -q_1 \cdots q_{i_0-2} \cdot r_1 \cdots r_{k-i_0} - q_1 \cdots q_{i_0-1} \cdot r_1 \cdots r_{k-i_0-1}$$

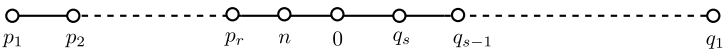
if $2 \leqq i_0 \leqq k - 1$. This again leads to $|d(\Gamma')| > 1$. Thus, i_0 must be 1 or k, and we may assume $i_0 = 1$ by symmetry. Again, one sees that necessarily $k = 2$, so that Γ' has to be the weighted graph



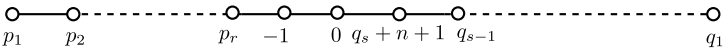
$n \in Z, \quad n \geqq 2.$

Let L be the line corresponding to the first vertex. If this vertex is linked to a vertex f of Γ not in Γ', let L' be the line corresponding to f. Choose an arbitrary point P on L if there is no such f, and choose P to be the point of intersection of L and L' if there is an f. Blow up P, and contract the proper transform of L. Repeat this procedure $(n-1)$ times, till we get a pair (W', G'), satisfying conditions (i)-(v) and (vii) such that $V - F \approx W' - G'$, and the graph of (W', G') is isomorphic to Γ. Further, the weights of Γ' are 0 and -1 respectively. Contract the line corresponding to the weight -1, to get (W, G) as required.

Next suppose there are two non-negative weights in Γ'. By condition (vii), these must be linked and one of them must be 0. Thus, Γ' looks like



with $r \geqq 0$, $s \geqq 0$, $n \geqq 0$, $r + s + 2 = k$. Blow up the point of intersection of the lines corresponding to the vertices with weights n and 0, and contract the proper transform of the line corresponding to the vertex of weight 0. Repeat this $n + 1$ times to get (W', G') satisfying conditions (i)-(v) and (vii) with a graph isomorphic to the original graph but the weights changed as follows:

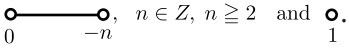


Now shrink the line corresponding to the vertex with weight -1. We get a pair (W'', G'') satisfying (i)-(v) and (vii). The graph of this pair is obtained from Γ by pinching two adjacent vertices to one and replacing the weights p_r and 0 by $p_r + 1$ and 1, respectively. Since k is now decreased, the assertion of the lemma follows by induction hypothesis. □

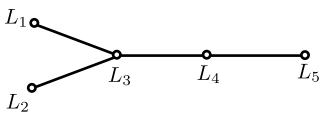
Remarks. (1) The pair (W, G) obtained from the lemma need not necessarily satisfy (vi), since certain weights at vertices of Γ not in Γ' are also altered. However, the only weights which are altered are those at vertices directly linked to the first or last vertex of

Γ'. Now, we can shrink suitable exceptional curves of the first kind successively to obtain a pair $(\bar{W}, \bar{G})$ which satisfies all the conditions (i)-(vii), with $\bar{W} - \bar{G} \approx V - F$. However, the graph of $(\bar{W}, \bar{G})$ is not describable in such a simple fashion as that of (W, G), though we can make the following assertion. *The number of branch points of the graph of* $(\bar{W}, \bar{G})$ *is the same as that of* Γ. In fact, if this were not so, we would necessarily have (a) the maximal linear subgraph Γ'' of Γ containing Γ' but not any branch point of Γ must be linked to exactly one vertex of Γ so that it is a branch at a branch point of Γ, and (b) the system of curves corresponding to Γ'' must be collapsible to a single point on a non-singular surface. But now (b) implies that the restriction of the intersection form to the space spanned by the vertices of Γ'' is negative definite. This is not so on Γ', as we have shown above.

(2) During the course of the proof, we have also established that there is no system of curves satisfying (i)-(vii) whose graph is linear, contains no non-negative weights, and has discriminant ± 1; and that the only systems satisfying (i)-(vii) with a linear graph of discriminant 1 in absolute value containing a unique vertex with non-negative weight are those with the graphs



Lemma 6. *On a complete non-singular surface* V *with* $H^1(V, \mathcal{O}_V) = 0$, *there cannot exist a system of five lines* $L_i (1 \leqq i \leqq 5)$ *satisfying conditions (i)-(v) whose weighted graph looks like*



with $(L_3^2) = -1$ *and* $(L_5^2) \geqq 0$.

Proof. Shrink L_3 on V to a point on a non-singular surface V' and denote by $L_i' (i \neq 3)$ the image of L_i in V'. Then L_1', L_2' and L_4' meet at a

point P, and L'_4 and L'_5 meet transversally at a point $Q \neq P$. Since **86**
$H^1(V', \mathcal{O}_{V'}) = 0$ we have the exact sequence

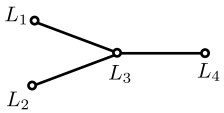
$$0 \to C = H^0(V', \mathcal{O}_{V'}) \to H^0(V', \mathcal{O}_{V'}(L'_5)) \to H^0(L'_5, \mathcal{O}_{V'}(L'_5)|L'_5) \to 0.$$

Thus the complete linear system $|L'_5|$ cuts out on L'_5 the complete linear system of divisors of degree $n = (L'^2_5) \geqq 0$. Hence $|L'_5|$ has no base points, and since $(L'_4 \cdot L'_5) = 1$, $|L'_5|$ cuts out the complete linear system of divisors of degree 1 on L'_4. Choose a divisors $D \in |L'_5|$ such that D cuts out the point P on L'_4. Since $(L'_5 \cdot L'_1) = (L'_5 \cdot L'_2) = 0$, D must have L'_1 and L'_2 for components but not L'_4. But then,

$$(L'_5 \cdot L'_4) = (D \cdot L'_4) \geqq ((L'_1 + L'_2) \cdot L'_4) \geqq 2,$$

a contradiction. □

Corollary. *With V as in Lemma 6, there cannot exist a system of four lines $L_i (1 \leqq i \leqq 4)$ satisfying (i)-(v) whose graph looks like*



and $\left(L_3^2\right) = 0, \left(L_4^2\right) > 0.$

Proof. If we blow up the point of intersection of L_3 and L_4 on V, we are landed with a system of five lines as in Lemma 6.

Let us call a subgraph Γ' of a graph Γ *simple* if Γ' is linear and no vertex of Γ' is a branch point of Γ. We shall say that a branch point of Γ is *extremal* if all but one of the branches at this point are simple. We say that a connected subgraph Γ' of Γ is *spherical*[4] if $\pi(\Gamma') = (e)$.

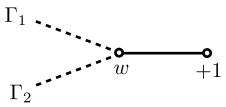
The proof of the main theorem follows immediately from the following □

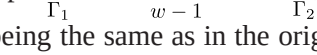
[4]This only means that if F' is the system of curves corresponding to Γ', $S_\delta(F)$ is a homotopy 3-sphere.

Proposition. *There is no pair* (V, F) *with* $H^1(V, \mathcal{O}_V) = 0$ *satisfying (i)-(vii) such that* $\pi_1(S_\delta(F)) = (e)$ *and the graph* Γ *not linear.*

Proof. Suppose in fact that there is such a pair. By Lemma 5 and the succeeding remark (1), we may assume that any simple branch at any branch point which is spherical consists of a single vertex with weight 1. Let $k \geqq 1$ be the number of branch points of Γ.

Suppose first that $k = 1$, and let e be the branch point. Then the branches at e are all simple; if these are $\geqq 4$ in number, at least two of them must be spherical by Lemma 4 and hence consist of single vertices with weight 1. But this clearly contradicts (vii). Thus there are exactly
87 three branches, and one of them is spherical, hence consists of a vertex of weight 1. The graph looks like



where Γ_i are simple branches. By (vii), the weights of the vertices on Γ_i must be $\leqq -2$, and the weight w at e is $\leqq 0$. If $w = 0$, we get a contradiction by the Corollary to Lemma 6. If $w \leqq -1$, use Lemma 2 to conclude that $d(\Gamma)$ = the discriminant of the graph 

, the weights in this graph at vertices on Γ_i being the same as in the original one. Since $w-1 \leqq -2$, we see that $|d(\Gamma)| > 1$ by the Corollary to Lemma 2 and we get a contradiction by Lemma 3.

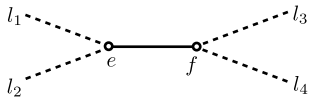
Hence suppose $k > 1$, and that we have ruled out the possibility of a pair as in the proposition with fewer branch points. Let S denote the set of extremal branch points of Γ. We assert that $\mathrm{Card}(S) \geqq 2$. In fact, start from an arbitrary branch point e of Γ. We assert that $\mathrm{Card}(S) \geqq 2$. In fact, start from an arbitrary branch point e of Γ. There is a branch at e containing a branch point of Γ, since $k \geqq 2$, and there are at least two such branches at e if e is not extremal. Travel along one of these branches (without retracting one's path) till one reaches a branch point f. If f is not extremal, we can choose a branch at f not containing e but containing a branch point. Travel along this branch. Continuing in this

manner, we see that on every branch at e which is not simple, there is an extremal branch point. This proves that $\mathrm{Card}(S) \geqq 2$.

Denote by S' the subset of S consisting of those vertices $e \in S$ such that every simple branch at e carries only weights < 0 (hence $\leqq -2$). Because of (vii), we have $\mathrm{Card}(S') \geq \mathrm{Card}(S) - 1 \geqq 1$.

Let e be an arbitrary element of S'. There must be exactly two simple branches at e, since otherwise there would exist a simple spherical branch at e by Lemma 4 and this branch would consist of a single vertex with weight 1 by assumption, contradicting the fact that $e \in S'$. Let Γ' be the non-simple spherical branch at e and f the point of this branch linked to e in Γ. If f is a branch point of Γ', the system of curves corresponding to Γ' would satisfy (i)-(vii) and Γ' is spherical, which is impossible by the induction hypothesis. For the same reason, it cannot happen that f is not a branch point of Γ. Thus f is a branch point of Γ with exactly three branches in Γ at f.

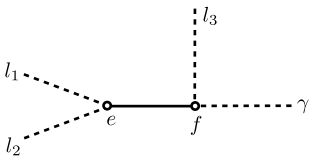
We deal separately with the case $k = 2$. The graph looks like



where e and f are linked branch points and $l_i (1 \leqq i \leqq 4)$ are simple branches such that the weights on l_1, l_2 and one of l_3, l_4, say l_3, are negative. Let δ_i be the discriminants of l_i and Δ_1 and Δ_2 the discriminants of $\overset{l_1}{\circ}\!\!-\!\!-\overset{e}{\circ}\!\!-\!\!-\overset{l_2}{\circ}$ and $\overset{l_3}{\circ}\!\!-\!\!-\overset{f}{\circ}\!\!-\!\!-\overset{l_4}{\circ}$ respectively. By taking the weights to be variables and applying Lemma 2 repeatedly to remove the links on the simple branches and the links connecting these branches to the branch points, one obtains the identity $\mathrm{disc}(\Gamma) = \Delta_1\Delta_2 - \delta_1\delta_2\delta_3\delta_4$. Since $\overset{l_3}{\circ}\!\!-\!\!-\overset{f}{\circ}\!\!-\!\!-\overset{l_4}{\circ}$ is spherical, $|\Delta_2| = 1$, and since the weights on $l_i (1 \leqq i \leqq 3)$ are $\leqq -2$, $|\delta_i| > 1$ and hence $\geqq 2$ by the Corollary to Lemma 2. If $\delta_4 = 0$, the quadratic form of l_4 is degenerate, and hence by (vii) should be negative semi-definite with null-space of dimension one. If this happens, there must be a non-negative weight in l_4 and this weight must then necessarily be 0. The corresponding vertex must belong to the null-space of the quadratic form. But then, this vertex cannot

be linked to any other vertex of l_4, so that l_4 should reduce to a single vertex with weight 0. But then, the graph $l_1 e l_2$ cannot be spherical, since we would then have $w_e = -1$ by remark (2) following Lemma 5, in contradiction to Lemma 6. Suppose then that $\delta_4 \neq 0$, so that $|\delta_4| \geqq 1$ and $|\delta_1\delta_2\delta_3\delta_4| \geqq 8$; since $1 = |d(\Gamma)| = |\Delta_1\Delta_2 - \delta_1\delta_2\delta_3\delta_4|$ and $|\Delta_2| = 1$, we must have $|\Delta_1| \neq 1$ and again $l_1 e l_2$ cannot be spherical. Applying Lemma 4 to the branch point f, we deduce that l_4 is spherical, hence reduces to a single vertex with weight 1. Because of (vii) we must have $w_f \leqq 0$, and by the Corollary to Lemma 6, $w_f \neq 0$, so that $w_f \leqq -1$. Using Lemma 2, the discriminant of $l_3 f l_4$ equals that of $l_3 f$ with w_f replaced by $w_f - 1$, so that all the weights of the latter graph are $\leqq -2$ and its discriminant exceeds 1 in absolute value, a contradiction.

Thus, we assume from now on that $k \geqq 3$. Reverting to our old notations, we have seen that if $e \in S'$, there are two simple branches at e, and on the third spherical branch Γ', the point f linked to e is a branch point of Γ with three branches. Since Γ' has at least one branch point, if $w_f \neq -1$, the system of curves corresponding to Γ' satisfies conditions (i)-(vii) and Γ' is spherical, which is impossible by the induction hypothesis. Furthermore, Γ' satisfies conditions (i)-(v) and (vii), and (vi) can be ensured by successively contracting exceptional curves corresponding to non-branch points. Since Γ' is spherical, this implies that the above process of shrinking suitable exceptional curves should eliminate all branchings. As is easily seen, this implies in paricular that one of the branches at f is simple with all weights less than, 0, hence $\leqq -2$. Thus, in the vicinity of a point $e \in S'$, Γ looks like



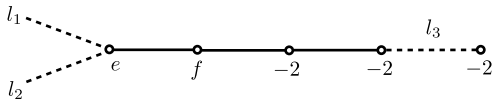
$(e \in S')$

89 where the l_i are simple branches, all having weights $\leqq -2$, $w_f = -1$, and γ is an arbitrary branch containing at least one branch point of Γ.

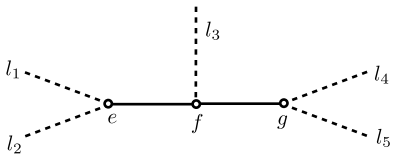
Define $S'' = \{e \in S' | w_e = -1\}$. Suppose $S'' \neq \varnothing$; then the intersection form is 0 on $e + f$. We assert that this implies that there is no

non-negative weight. In fact, if $w_g \geqq 0$ for some g, then by condition (vii), g must be linked to either e or f. But this is again impossible by Lemma 6. In particular, we have $S = S'$. Suppose on the other hand that $e \in S' - S''$. Then $l_1 e l_2$ cannot be spherical by the Corollary to Lemma 5 and the remarks following it. Thus, Lemma 4 applied to f gives us that γ is spherical. Since γ contains branch points of Γ, we deduce as before that the point on γ linked to f is a branch point of Γ with exactly three branches.

Let us dispose of the case $k = 3$. Suppose first that $S'' \neq \emptyset$ and $e \in S''$. Denote by g the third branch point besides e and f. We know that all weights are < 0. Suppose g is not linked to f directly. We can then apply Lemma 4 to Γ at g to get a contradiction to our induction hypothesis. By the same token, there are exactly three branches at g, two of them simple. Now, we know that Γ' is spherical. Apply Lemma 4 to Γ' at g to deduce that all the weights on l_3 equal -2. Apply Lemma 4 to Γ at g to deduce that

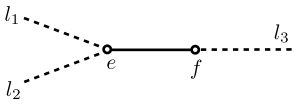


is spherical. Note that $w_e = w_f = -1$. Shrink the curves corresponding to vertices on the branch at e containing l_3, to get a linear spherical graph with all but one of the weights $\leqq -2$ and a weight at a unique vertex which is not an end vertex $\geqq 0$. This is impossible. Thus, $S'' = \emptyset$, and in this case, by what we have seen earlier, the graph looks like

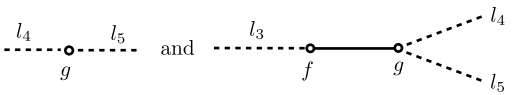


with all the weights on l_1, l_2 and l_3 and one of l_4 or l_5, say l_4, negative, and $w_f = -1$, $w_e \neq -1$. Now, l_5 cannot be spherical since it would then consist of a single vertex of weight 1 by assumption, contradicting

Lemma 6. Applying Lemma 4 at g, we see that



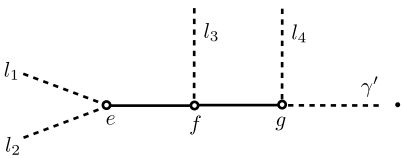
is spherical. Thus the horizontal branch of this graph at e should be collapsible, and all the weights on l_3 should be equal to -2. Now, both the graphs



90

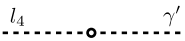
are spherical, hence have discriminants $= \pm 1$. But the discriminant of the second graph equals that of the first graph with w_g replaced by $w_g + r$ where r is an integer $\geqq 2$. But clearly the discriminant of the first graph is a linear function of w_g, the coefficient of w_g being $\delta_4\delta_5$, where δ_i is the discriminant of l_i. Now, δ_i are intergers, $|\delta_4| \geqq 2$, so that the above can happen only if $\delta_5 = 0$. But then the quadratic form of l_5 must be negative semi-definite with null-space of dimension one. Thus, at least one weight on l_5 is $\geqq 0$, hence equal to 0. This vertex must therefore belong to the null-space, and must be orthogonal to all the vertices for the quadratic form. Thus, l_5 consists of a single vertex of weight 0, contradicting Lemma 6.

Hence $k = 3$ is impossible, and we assume from now on that $k \geqq 4$. We look more closely at the branch γ when $e \in S' - S''$. Since γ is spherical and contains at least two branch points of Γ, one of which, say g, is linked to f, we deduce as before that $w_g = -1$ and one of the three branches at g is simple with weights $\leqq -2$. Thus, in the vicinity of e, Γ looks like

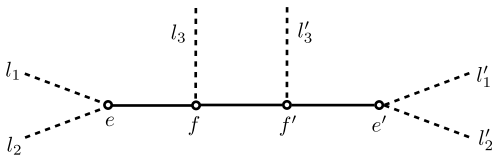


Further, Γ' spherical gives us that all the weight on l_3 should be -2. Again, a repetition of an earlier argument applied to the branch γ tells us that there is a branch point of Γ in γ' linked to g in Γ. But then, γ being spherical implies that all the weights on l_4 equal -2.

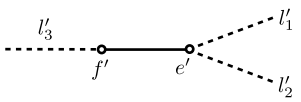
Now, collapse all exceptional curves at non-branch points of Γ', that is, collapse the branch $l_3 f$ at g. We are left with the graph γ:



except that the weight at g is replaced by $w_g + r = r - 1$, r an integer $\geqq 2$. But this graph fulfills all the conditions (i)-(vii), contains a branch point and is spherical, which is impossible. Thus for $k \geqq 4$, $S' - S'' = \emptyset$, $S' = S''$, and hence (since $\mathrm{Card}(S') \geqq 1$) $S = S' = S''$, $\mathrm{Card}(S) \geqq 2$. Choose two distinct vertices $e, e' \in S''$, and denote the objects corresponding to the objects l_i, f, γ associated to e' by the same symbols with a prime. Since $e + f$ and $e' + f'$ both have self-intersection 0, one of e, f must either coincide with or be linked to one of e', f'. But e is linked only to f besides non-branch points and e' only to f' besides non-branch points. Thus, either $f = f'$ or f and f' are linked. If $f = f'$, looking at the **91**
structure of Γ near e and e', we see that there can be only three branch points e, $f = f'$ and e'. Hence f and f' are linked, and Γ looks like



with all weights < 0, $w_e = w_{e'} = w_f = w_{f'} = -1$. Now, the fact that the non-simple branch at e is spherical gives us (by our induction hypothesis) that all the weights on l_3 are equal to -2. Collapsing $l_3 f$ on this branch by successively shrinking lines corresponding to non-branch vertices with weight -1 gives us that the graph



where the weight at f' is replaced by $w_{f'} + r = r - 1$, $r \geqq 2$, is spherical. But this new graph satisfies all the conditions (i)-(vii) and contains a branch point. This is impossible.

Thus the Proposition is proved. □

The theorem follows almost immediately. In fact, imbed X in $\bar{X}$ such that $\bar{X} - X = Y$ satisfies conditions (i)-(vii) as explained at the beginning of this section. By the Corollary to Lemma 1 and the Proposition $\Gamma(Y)$ does not contain branch points, and is hence linear. By Lemma 5, we may assume that $Y \approx \mathbf{P}^1$ and has self-intersection 1. The exact sequence

$$0 \longrightarrow \mathbf{C} = H^0(\bar{X}, \mathscr{O}_{\bar{X}}) \longrightarrow H^0(\bar{X}, \mathscr{O}_{\bar{X}}(Y)) \longrightarrow H^0(Y, \mathscr{O}_{\bar{X}}(Y)|Y) \longrightarrow H^1(\bar{X}, \mathscr{O}_{\bar{X}}) = 0$$

shows that the complete linear system $|Y|$ has no base points, and defines a morphism φ of degree 1 (since $(Y^2) = 1$) into $\mathbf{P}^2$. Since the Neron-Severi group of $\bar{X}$ is $\mathbf{Z}$, φ cannot contract any curve, so that φ is an isomorphism. Hence $X = \bar{X} - Y \approx \mathbf{P}^2 - \{\text{line}\} = \mathbf{C}^2$, proving the theorem.

3 The counter-example

Consider in the projective plane $\mathbf{P}^2$ a cubic curve C_1 with a cusp and a non-degenerate conic C_2 meeting C_1 at two distinct points P and Q, which are simple on C_1 and different from the inflexion point of C_1, with intersection numbers 5 and 1 respectively. (That such a configuration exists, and in fact, C_1 and either P or Q can be chosen arbitrarily, is
92 easily seen, for instance using the fact that C_1-{cusp} is an algebraic group isomorphic to the additive group of $\mathbf{C}$, and a conic meets C_1 at simple points $P_i (1 \leqq i \leqq 6)$, each point being repeated as many times as the intersection multiplicity, if and only if $\sum_1^6 P_i = 0$ for this group

law.) Blow up the point Q to obtain a variety F and let C_i' be the proper transform of C_i. The veriety $X = F - C_1' - C_2'$ is our counter-example.

Put $C' = C_1' \cup C_2'$, so that C' is topologically the wedge of two 2-spheres, so that $H_1(C') = 0$ and $H_2(C')$ is the free abelian group on the classes defined by C_1' and C_2'. Now, F is simply-connected, and $H_2(F)$ is generated by the class E of the exceptional curve and the class H defined by the total transform of a line in $\mathbf{P}^2$. But in $H_2(F)$, we have the relations $C_1' = 3H - E$, $C_2' = 2H - E$. Further, $H_3(F) \approx H^1(F) = 0$. Thus $H_i(C) \to H_i(F)$ is an isomorphism for $0 \leqq i \leqq 3$, and $H_i(F, C) = 0$ for $0 \leqq i \leqq 3$. Thus we also have

$$H_i(X) = H_c^{4-i}(X) = H^{4-i}(F, C) = 0, \quad 1 \leqq i \leqq 4.$$

To show that X is contractible, it suffices to show that $\pi_1(X)$ is abelian, since it would then follow that $\pi_1(X) = H_1(X) = (e)$.

Now F is a projective line bundle over $\mathbf{P}^1$ whose fibers are the proper transforms of the lines in $\mathbf{P}^2$ through the point Q in F. Since the intersection number of any fiber with C_2' is 1, C_2' is a section of this fibration $\pi : F \to \mathbf{P}^1$ and the restriction of π to $F - C_2'$ is an affine line bundle over $\mathbf{P}^1$. If P' is the inverse image of P, we may assume that $\pi(P)$ is the point at infinity. If $G = F - C_2 - \pi^{-1}(\infty)$, G is an affine line bundle over $\mathbf{P}^1 - (\infty) = \mathbf{C}$. Since affine line bundles over $\mathbf{C}$ are trivial, we have $G \approx \mathbf{C} \times \mathbf{C}$ such that the restriction of π to G identifies itself to the first projection of $\mathbf{C} \times \mathbf{C}$ onto $\mathbf{C}$. Now, the proper transform of a line through Q in $\mathbf{P}^2$ meets C_1' in one point if and only if either the line passes through the singularity of C_1 or the line is tangent to C_1 at a point which is simple and distinct from Q. Now, it is easily seen that there is exactly one line through Q tangent to C_1 at a simple point other than Q and this point has to be distinct from P or the inflexional point of C_1. Since C_1' and C_2' meet only at the point P', $G \cap C_1'$ is proper over $\mathbf{C}$, of degree 2, and exactly two fibers of the map $G \cap C_1 \to \mathbf{C}$ reduce to a single point. Denoting the coordinates in $\mathbf{C} \times \mathbf{C}$ by (z, w), we see that the equation of C_1' in $\mathbf{C} \times \mathbf{C}$ has the form $w^2 + a(z)w - p(z) = 0$, $a, p \in \mathbf{C}[z]$. Replacing w by $w + \varphi(z)$ for a suitable polynomial φ, we may assume that the equation is $w^2 = p(z)$. Since exactly two fibers in C_1' reduce to

a point, $p(z)$ has exactly two zeros and after a linear change of z and w we may assume the equation of C_1' in $\mathbf{C} \times \mathbf{C}$ has the form

$$w^2 = z^m(z-1)^n, \qquad m, n \geqq 1.$$

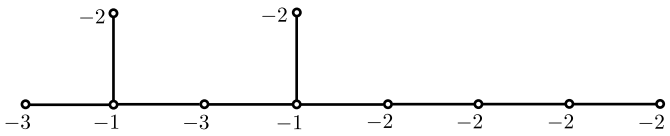
93 We may assume that the singularity of C_1' lies over 0, and the the point over 1 in C_1' is simple. This gives $n = 1$, and the fact that the singularity of C_1' is an ordinary cusp (i.e., on blowing up the singular point, the proper transform of C_1 is non-singular and touches the exceptional curve exactly one point) gives us that $m = 3$. Thus, $C_1' \cap \mathbf{C} \times \mathbf{C}$ has the equation

$$w^2 = z^3(z-1).$$

Now, $\mathbf{C} \times \mathbf{C} - \{z(z-1) = 0\} - C_1'$ is a locally trivial fiber space over $\mathbf{C} - \{0, 1\}$ with typical fiber the complex plane with two points removed. The homotopy exact sequence shows that $\pi_1(\mathbf{C} \times \mathbf{C} - \{z(z-1) = 0\} - C_1')$ is generated by the fundamental group of the fiber and loops at the base points whose images by π are loops around 0 and 1 in $\mathbf{C} - \{0, 1\}$. Choose the base point to be $(1/2, M) = A$ with M sufficiently large. Further note that

$$\pi_1(\mathbf{C} \times \mathbf{C} - \{z(z-1) = 0\} - C_1') \longrightarrow \pi_1(\mathbf{C} \times \mathbf{C} - C_1)$$

is surjective. We can choose loops at A lifting the loops at $1/2$ in $\mathbf{C} - \{0, 1\}$ around 0 and 1 with the w-co-ordinate constantly M, and these loops can be contracted in $\mathbf{C} \times \mathbf{C} - C_1'$. Thus we see that $\pi_1(\mathbf{C} \times \mathbf{C} - C_1')$ is generated by the fundamental group of any fiber of $\mathbf{C} \times \mathbf{C} - C_1' \to \mathbf{C}$ over a $\lambda \in \mathbf{C}$, $\lambda \neq 0, 1$. Since the w-co-ordinates of the two points of C_1' lying over a λ tend to 0 as λ tends to 1, if U is any connected neighborhood of $(1, 0)$ in $\mathbf{C} \times \mathbf{C}$, $\pi_1(U - C_1') \to \pi_1(\mathbf{C} \times \mathbf{C} - C_1')$ is surjective. Now choose U such that there are co-ordinates (z', w') valid in U and mapping it on to the polycylinder $\{(z', w') \| z'| < 1, |w'| < 1\}$ such that $C_1' \cap U$ is mapped onto $z' = 0$. Then $\pi_1(U - C_1')$ is infinite cyclic. We have thus proved that X is contractible. Now, after successive blowings up of points on $C_1' \cup C_2'$ and its inverse images, starting from F, we get an embedding $X \hookrightarrow \bar{X}$ where $\bar{X}$ is a non-singular complete surface and the pair $(\bar{X}, \bar{X} - X)$ satisfies conditions (i)-(vii) of § 2. The associated graph takes the form



This graph is not spherical, as is seen by applying Lemma 4 successively at the two branch points. On the other hand, this system of curves forms a basis for $H_2(\bar{X})$, and hence by Poincaré duality over $\mathbf{Z}$, the discriminant of this graph is one in absolute value. Denoting this system of curves on $\bar{X}$ by Y, with the notations of § 2, we see that for small δ, $\bar{X} - V_\delta(Y) = M$ is a contractible compact 4-manifold with boundary ∂M a homology sphere but not a homotopy sphere.

Consider the manifold $M' = M \times [0,1] \times [0,1]$ with boundary **94**

$$\partial M' = \partial M \times [0,1] \times [0,1] \cup M \times \{0,1\} \times [0,1] \cup M \times [0,1] \times \{0,1\}.$$

Then M' is contractible and $\partial M'$ is a homotopy sphere. By the h-cobordism theorem, $\partial M'$ is homeomorphic to a sphere and M' to a disc. It follows from well-known theorems that $X \times \mathbf{R}^2$ is homeomorphic to $\mathbf{R}^6$.

We may thus conclude that either X is an affine rational non-singular surface with $X \times \mathbf{C}$ isomorphic as an algebraic variety to $\mathbf{C}^3$ but X not isomorphic to $\mathbf{C}^2$; or, that there is a non-singular affine rational variety $Y(= X \times \mathbf{C})$ of dimension 3 homeomorphic to $\mathbf{C}^3$ but not isomorphic to $\mathbf{C}^3$. We are unable to decide which of these possibilities holds in fact, or in other words whether $X \times \mathbf{C} \approx \mathbf{C}^3$ or not.

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Remarks on the Kodaira Vanishing Theorem*

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1 Introduction

The object of this note is three-fold. In the first part, we give a de- **95**
duction of the Kodaira-Nakano-Akizuki vanishing theorem [1] from the Lefschetz hyperplane-section theorem [6], using a lemma of Mumford. In the second part, we prove a vanishing theorem for the first cohomology for varieties in all characteristics. The idea of this section is due to Franchetta [2], but Franchetta failed to prove the basic Lemma 3. In the third part, we return to the case of the complex base-field, and give what seems to be the most general form known of the Kodaira vanishing theorem for the cohomologies of a line bundle in this case. We were told of the existence of such a generalisation by D. Mumford.

2 The Kodaira-Nakano-Akizuki vanishing theorem

The statement runs as follows:

Theorem 1. *Let X be a complex non-singular algebraic variety, and L a line bundle on X such that L^{-1} is ample (in the sense of Grothendieck, that is, for some $n > 0$, L^{-n} is induced from the hyperplane bundle*

*This work was done while the author was at the Mathematics Institute, University of Warwick, with financial assistance from Harvard University.

$\mathcal{O}_{\mathbf{P}^N}(1)$ by an imbedding $X \hookrightarrow \mathbf{P}^N$). Then, for $p + q < \dim X$,

$$H^q(X, \Omega^p \otimes L) = (0).$$

For the proof, we need the following simple lemma, whose construction is due to Mumford [7].

96 **Lemma 1.** *Let X be a variety over the complex field, L a line bundle on X, $\sigma \in H^0(X, L^m) - (0)$ where m is positive integer and D the divisor of σ. Then there is an m-fold cyclic covering $f : X' \to X$ ramified precisely over the support of D such that $f(L)$ admits a section τ satisfying*

$$\tau^m = f^*(\sigma) \quad in \quad H^0(X', f^*(L^m)).$$

If further X and D are non-singular, so are X' and div. τ, *and in the neighbourhood of a point of the support of D, the covering f is isomorphic to the covering $\mathbf{C}^n \to \mathbf{C}$ given by*

$$(z_1, z_2, \ldots, z_n) \longmapsto (z_1, z_2, \ldots, z_n^m)$$

restricted to neighbourhoods of the origin.

Proof. Define X' to be the subvariety of the total space of L given by

$$X' = \{x \in L | x^m = \sigma(\pi(x)) \quad \text{in} \quad L^m\}$$

where $\pi : L \to X$ is the projection, $f : X' \to X$ the restriction of π, and on $f^*(L) = L \times X^{X'}$, define a section $\tau : X' \to f^*(L)$ by $\tau(x) = (x, x)$. All the conditions are trivially verified. □

Proof of Theorem 1. We proceed by induction on $n = \dim X$. The statement being trivial for $n = 1$, we assume $n > 1$ and that the result holds for varieties of smaller dimension.

By assumption and the theorem of Bertini, there is an integer $m > 0$ and $\sigma \in H^0(X, L^{-m})$ such that $D = \text{div.}\,\sigma$ is a non-singular hyperplane section of X for some projective embedding. By Lemma 1, $\exists f : X' \to X$ a cyclic m-fold covering with X' non-singular and $\tau \in H^0(x, f^*(L^{-1}))$ such that $\tau^m = f^*(\sigma)$ and $D' = \text{div.}\,\tau$ is non-singular. Also, D' is the

support of a hyperplane section of X' for some embedding, since $f^*(L^{-1})$ is ample on X' (E. G. A., Chap. II). By the Lefschetz hyperplane section theorem [6],

$$H^i(X', \mathbf{C}) \to H^i(\mathbf{C}', \mathbf{C})$$

is an isomorphism for $i < n-1$ and is injective for $i = n-1$. By the Hodge decomposition theorem and its naturality ([8], p. 129), it follows that

$$H^p(X', \Omega^q_{X'}) \to H^p(D', \Omega^q_{D'}) \tag{1}$$

is an isomorphism for $p+q < n-1$ and is injective for $p+q = n-1$. **97**
Let $\mathscr{I}$ be the sheaf of ideals defining D', so that $\mathscr{I}$ is isomorphic to the sheaf of germs of sections of $f^*(L)$. We have the exact sequence (E. G. A., Chap. IV, Pt. 1)

$$0 \to \mathscr{I} \otimes \mathscr{O}_{D'} \to \Omega^1_{X'} \otimes_{OX'} \mathscr{O}_{D'} \to \Omega^1_{D'} \to 0,$$

and hence for any $q > 0$, the exact sequence

$$0 \to \Omega^{q-1}_{D'} \otimes \mathscr{I} \otimes \mathscr{O}_{D'} \to \Omega^q_{X'} \otimes \mathscr{O}_{D'} \to \Omega^q_{D'} \to 0$$

and $\Omega^{q-1}_{D'} \otimes \mathscr{I} \otimes \mathscr{O}_{D'} \approx \Omega^{q-1}_{D'} \otimes f^*(\mathscr{L})$ where as usual we denote by a script letter the sheaf of germs of sections of the line bundle denoted by the same letter in bold type. By induction hypothesis, $H^p(D', \Omega^{q-1}_{D'} \otimes f^*(\mathscr{L})) = (0)$ for $p+q < n$, so that $H^p(X, \Omega^q_{X'} \otimes \mathscr{O}_{D'}) \to H^p(\Omega^q_{D'})$ is an isomorphism for $p+q < n-1$ and is injective for $p+q = n-1$. This together with (1) gives us that

$$H^p(X', \Omega^q_{X'}) \to H^p(D', \Omega^q_{X'} \otimes \mathscr{O}_{D'})$$

is an isomorphism for $p+q < n-1$ and is injective for $p+q = n-1$. It follows that

$$H^p(X', f^*(L) \otimes \Omega^q_{X'}) = H^p(X', \mathscr{I}\Omega^q_{X'}) = (0) \tag{2}$$

for $p+q < n$. Having proved the theorem for X', it would follow for X and the line bundle L on X, provided the inclusion of sheaves $\mathscr{L} \otimes \Omega^q_X \hookrightarrow f_*(f^*(\mathscr{L}) \otimes \Omega^q_{X'}) \approx \mathscr{L} \otimes f_*(\Omega^q_{X'})$ admits a splitting, or

equivalently, if the natural inclusion $i : \Omega^q_X \hookrightarrow f_*(\Omega^q_{X'})$ splits. (This is because the morphism f is finite, so that for any $\mathscr{F}$ coherent on X', $H^p(X', \mathscr{F}) \approx H^p(X, f_*(\mathscr{F}))$.)

To give a splitting of i, let U be any open set of X and ω a regular differential form on $f^{-1}(U)$. Define a form $\tilde{\omega}$ on $f^{-1}(U)$ by $\tilde{\omega} = \sum_{\phi \in G} \phi^*(\omega)$ where G denotes the Galois groups of X' over X. Then $\tilde{\omega}$ is regular and G-invariant on $f^{-1}(U)$. We assert that there is a unique form $Tr\omega$ regular on U such that $f^*(Tr\omega) = \tilde{\omega}$. This is in fact clear if U does not contain any branch points of f. At points of the branch-locus, this
98 follows from a simple calculation using the local description of f given in Lemma 1[†]. Now, $(1/m)\, Tr$ gives the required splitting.

3 The method of Franchetta

Let X be a non-singular variety of dimension $n \geq 2$ over an algebraically closed field of any characteristic and D an *effective* divisor on X. We define (following Franchetta) D to be *numerically connected* if there is an ample H on X such that for any decomposition $D = D_1 + D_2$, $D_i > 0$, we have $(H^{n-2}.\, D_1.\, D_2) > 0$. It is clear that the effective divisor without multiple components is numerically connected if and only if its support is connected, and that for any effective D, if nD is numerically connected, so is D. Further, if H' is a generic (hence non-singular) hyperplane section of X for an embedding given by any multiple of H, D is numerically connected if and only if $D.\, H'$ is numerically connected on H'.

Lemma 2. *If for some $N > 0$, ND moves in an algebraic system of effective divisors without fixed components, and $(H^{n-2}.\, D^2) > 0$ for some ample H, then D is numerically connected.*

[†]In fact, for any separable morphism of non-singular varieties $f : X' \to X$ one can define a homomorphism

$$Tr : f_*(\Omega^q_{X'}) \to \Omega^q_X.$$

Proof. By a remark above, we may replace D by ND and assume $N = 1$. By a second remark, we may successively take general hyperplane sections of X for an embedding given by a multiple of H, till we arrive at a surface. The other hypotheses are stable for general hyperplane sections.

Thus, assume X a non-singular surface and $D = D_1 + D_2$, $D_i > 0$. Since D moves in an algebraic system without fixed components we have that $(D.D_i) = (D'.D_i) \geqslant 0$ where D' is algebraically equivalent to D and effective, and has no common components with D. Since $(D^2) > 0$, at least one of these is a strict inequality. Hence - $(D_1.D_2) \leqslant (D_i^2)(i = 1, 2)$ with atleast one strict inequality. Suppose now that $(D_1.D_2) \leqslant 0$, so that $0 \leqslant -(D_1.D_2) \leqslant (D_i^2)(i = 1, 2)$.

Apply the Hodge index theorem for surfaces (valid in any characteristic, [4]) to the subgroup of the Neron-Severi group generated by **99**
D_1, D_2. The intersection form must have one positive and one negative eigenvalue on this subspace if the D_i are numerically independent. But if they are numerically dependent, we must have $aD_1 = bD_2$ with a, b positive integers, and since $(D^2) > 0$, $(D_1.D_2) > 0$. Hence they are independent, and we get

$$(D_1^2).(D_2^2) - (D_1.D_2)^2 = \det\begin{pmatrix} (D_1^2) & (D_1.D_2) \\ (D_1.D_2) & (D_2^2) \end{pmatrix} < 0,$$

contradicting our earlier inequalities $0 \leqslant -(D_1.D_2) \leqslant (D_1^2)$. Hence $(D_1.D_2) > 0$. □

Lemma 3. *If D is a numerically connected effective divisor on a non-singular variety X, $H^0(D, \mathcal{O}_D)$ consists of constants.*

Proof. We proceed by induction on $n = \dim X$. Suppose $n > 2$ and the result holds in smaller dimensions. Choose an embedding of X in $\mathbf{P}^N$ such that $H^0(D, \mathcal{O}_D(-1)) = (0)$. (This is possible since D does not have points as embedded components). Let H be a general hyperplane section. Then we have the exact sequence $0 \to H^0(D, \mathcal{O}_D) \to H^0(D \cap H, \mathcal{O}_{D\cap H})$, and the last group consists of constants, by induction

hypothesis applied to the pair $(H, D \cap H)$. Hence so does $H^0(D, \mathscr{O}_D)$, and we are through.

Thus, one has only to prove the lemma when $\dim X = 2$. Let us suppose there is a section σ of $H^0(D, \mathscr{O}_D)$ which is not a scalar. Since D_{red} is connected, $H^0(D, \mathscr{O}_{D_{\text{red}}})$ consists of constants, and the image of σ in $H^0(\mathscr{O}_{D_{\text{red}}})$ is a scalar λ. Replacing σ by $\sigma - \lambda$ we may assume that σ is a section of $\mathscr{R}$, the root ideal of $\mathscr{O}_D$. One can evidently find an effective divisor D_1 which is maximum, satisfying the conditions $0 < D_1 < D$, $\sigma \in \mathscr{I}_{D_1}\mathscr{O}_D$. Let $D = D_1 + D_2$. We have two exact sequences of coherent sheaves on X,

$$0 \longrightarrow \mathscr{O}_{D_2} \xrightarrow{\sigma} \mathscr{O}_D \longrightarrow \mathscr{O}_D/\sigma\mathscr{O}_D \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{O}_D/\sigma\mathscr{O}_D \longrightarrow \mathscr{O}_{D_1} \longrightarrow 0,$$

where $\mathscr{F}$ is a sheaf supported by finitely many points. Calculate Chern classes (in the ring of rational or algebraic equivalence classes of cycles, *see* [5]). We have

$$c(\mathscr{O}_D) = c(\mathscr{O}_{D_2}).\, c(\mathscr{O}_D/\sigma\mathscr{O}_D) = c(\mathscr{O}_{D_2}).\, c(\mathscr{O}_{D_1}).\, c(\mathscr{F}),$$
$$= \frac{1}{1-D} = \frac{1}{1-D_1}.\frac{1}{1-D_2}.\, c(\mathscr{F}).$$

100 If $\mathscr{F}$ has support at the points P_i and $l(\mathscr{F}_{P_i}) = n_i$, we have $c(\mathscr{F}) = 1 - \sum n_i P_i$. Thus,

$$(1 - D_1).\,(1 - D_2) = (1 - D).\,(1 - \sum n_i P_i),$$
$$(D_1.\, D_2) = -\sum n_i \leqslant 0.$$

□

Lemma 4. ‡ *Let $f : X' \to X$ be a proper birational morphism of a non-singular surface X' onto a normal surface X, and L a line bundle on X' such that for any irreducible curve C on X' contracted to a point by f, $(C.\, c_1(L)) \geqslant 0$. Then, $(R^1 f)(\Omega^2_{X'} \otimes L) = 0$.*

‡Analogous results have been obtained over the complex field by Grauert, also in the case of higher dimensions.

Proof. The sheaf $R^1 f(\Omega^2_{X'} \otimes L)$ is concentrated at the finitely many points P where $\dim f^{-1}(P) = 1$. Let P be one of these points. If we can show that for any effective divisor D with support $f^{-1}(P)$, we have $H^1(D, L \otimes \Omega^2_{X'}/\mathscr{I}_D \Omega^2_{X'}) = 0$, it would follow from the 'Théorème fondamentale des morphismes propres' (E. G. A., Chap.III) that $R^1 f(\Omega^2_{X'} \otimes L)_P = (0)$ and we would be through. Now, the sheaf $\omega_D = \mathscr{I}_D^{-1} \Omega^2_{X'}/\Omega^2_{X'}$ is dualising on D, so that we have to show that $H^0(D, L^{-1} \otimes \mathscr{I}_D^{-1}/\mathscr{O}_{X'}) = 0$. If not, let σ be an non-zero element of $H^0(D, L^{-1} \otimes \mathscr{I}_D^{-1}/\mathscr{O}_{X'})$. We can find a maximum divisor D_1 with $0 \leqslant D_1 \leqslant D$ such that $\sigma \in H^0(D, L^{-1} \otimes \mathscr{I}_{D_1} \mathscr{I}_D^{-1}/\mathscr{O}_{X'})$, and since $\sigma \neq 0$, $D_1 \neq D$. It then follows that σ generates $L^{-1} \otimes \mathscr{I}_{D_1} \mathscr{I}_D^{-1}/\mathscr{O}_{X'}$ generically on the components of the support of this sheaf. If we set $D_2 = D - D_1$, this sheaf is nothing but $L^{-1} \otimes \mathscr{I}_{D_2}^{-1}/\mathscr{O}_{X'}$. Further, the annihilator of σ is easily seen to be $\mathscr{I}_{D_2}$, so that we get an exact sequence

$$0 \longrightarrow \mathscr{O}_{D_2} \xrightarrow{\sigma} L^{-1} \otimes \mathscr{I}_{D_2}^{-1}/\mathscr{O}_{X'} \longrightarrow \mathscr{F} \longrightarrow 0$$

where $\mathscr{F}$ has zero dimensional support. Calculating Chern classes, we get

$$c(L^{-1} \otimes \mathscr{I}_{D_2}^{-1}/\mathscr{O}_{X'}) = c(\mathscr{O}_{D_2}).\, c(\mathscr{F}),$$

i.e., **101**

$$(1 - D_2).\,(1 - c_1(L) + D_2) = (1 - c_1(L)).\,(1 - \sum n_i P_i),$$

where P_i are the points supporting $\mathscr{F}$ and $n_i = l_{P_i}(\mathscr{F}_{P_i})$. This gives

$$(D_2^2) \geqslant (D_2^2) - (c_1(L).\, D_2) = \sum n_i \geqslant 0,$$

which is impossible, since by a well-known result [9], for any nonzero divisor E with support in $f^{-1}(P)$, $(E^2) < 0$. This is a contradiction. □

Lemma 5. *Let X be a non-singular complete surface and L a line bundle on X such that for some $n > 0$, the complete linear system determined by $H^0(X, L^n)$ has no fixed components and is not composite with a pencil. Then $H^1(X, L^{-N}) = 0$ for all large N.*

Proof. By ([10], Theorem 6.2), we may assume that the linear system determined by $H^0(X, L^n)$ has no base points at all. Hence, there is a morphism $\phi : X \to \mathbf{P}^m$ for some $m > 0$ such that $Y = \phi(X)$ is a surface and $L^n \simeq \phi^*(\mathcal{O}_{\mathbf{P}m}(1))$. Further, by (E.G.A., Chap. II), we may assume Y normal and ϕ birational. Determine N_0 so that for $N \geqslant N_0$ and $0 \leqslant v < n$, $H^1(R^0\phi(\Omega^2_{X'} \otimes L^v) \otimes \mathcal{O}_Y(N)) \approx H^1(R^0\phi(\Omega^2_{X'} \otimes L_{N_{n+v}})) = (0)$. For any curve C contracted to a point by ϕ, the restriction of L^n to C is trivial, hence $(c_1(L).C) = 0$. Thus by Lemma 4, $R^1\phi(\Omega^2_{X'} \otimes L^v) = 0$ for $v \geqslant 0$. It follows from the Leray spectral sequence that $H^1(X', \Omega^2_{X'} \otimes L_{N_{n+v}}) = (0)$ for $0 \leqslant v < n$, $N \geqslant N_0$; and the lemma follows by Serre duality.

□

Lemma 6. *Let X be a complete normal variety. Assume further in the case of positive characteristic that the Frobenius homomorphism on $H^1(X, \mathcal{O}_X)$ is injective. For any effective divisor D on X, define $\alpha(D)$ to be the dimension of the kernel of the homomorphism $H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$. Then, $\alpha(D) = \alpha(D_{\text{red}})$.*

Proof. It is clearly sufficient to show that if $D_1 \leqslant D_2 \leqslant 2D_1$, $\alpha(D_1) = \alpha(D_2)$.

First suppose that the characteristic is zero. Since the ideal $\mathcal{I}_{D_1}\mathcal{O}_{D_2}$ is of square zero, we have the exact sequence $0 \to \mathcal{I}_{D_1}\mathcal{O}_{D_2} \to \mathcal{O}^*_{D_2} \to \mathcal{O}^*_{D_1} \to 1$ from which the exact sequence

$$H^0(\mathcal{O}^*_{D_1}) \to H^1(\mathcal{I}_{D_1}\mathcal{O}_{D_2}) \to \operatorname{Pic} D_2 \to \operatorname{Pic} D_1.$$

102 Since $H^0(\mathcal{O}^*_{D_1})$ is divisible (use binomial series), it follows that the kernel of $\operatorname{Pic} D_2 \to \operatorname{Pic} D_1$ is torsion-free. If we set $K(D_i) = \operatorname{Ker}(\operatorname{Pic}^\circ X \to \operatorname{Pic} D_i)$, $K(D_i)$, are closed subgroups of the abelian variety $\operatorname{Pic}^\circ X$, and torsion elements are dense in $K(D_1)/K(D_2)$. Thus, $K(D_1) = K(D_2)$ and the assertion follows on passing to Lie algebras.

Suppose now that the characteristic is $p > 0$.

Since $\mathcal{I}_{D_1}\mathcal{O}_{D_2}$ is of square zero, it follows from the exactness of $H^1(\mathcal{I}_{D_1}\mathcal{O}_{D_2}) \to H^1(\mathcal{O}_{D_2}) \xrightarrow{\lambda} H^1(\mathcal{O}_{D_1})$ that the Frobenius kills the kernel of λ. Since the Frobenius is semi-simple on $H^1(\mathcal{O}_X)$ by assumption,

it is so on any quotient, and the image of $H^1(\mathscr{O}_X) \to H^1(\mathscr{O}_{D_2})$ cannot meet Ker λ. □

Remark. An 'evaluation' of $\alpha(D)$ has been given by Kodaira when X is a surface in characteristic zero and D is reduced. We give a rapid derivation of this, valid for any non-singular X in characteristic zero and any effective D. We have clearly that $\alpha(D) = \alpha(D_{\text{red}})$ is the dimension of the kernel of $\text{Pic}^\circ X \to \text{Pic}\, D_{\text{red}}$. But if $\eta : X \to A$ is the Albanese map, and B the abelian subvariety of A generated by the differences $\eta(x) - \eta(y)$ where x and y belong to the same connected component of D_{red}, since $\text{Pic}^\circ X = \text{Pic}^\circ A$, one sees easily (using the Jacobians of normalisations of generic curves on the various components of D_{red}) that the dimensions of the kernels of $\text{Pic}^\circ X \to \text{Pic}\, D_{\text{red}}$ and $\text{Pic}^\circ A \to \text{Pic}^\circ B$ are the same. But the dimension of this last group equals the dimension of the space of 1-forms on A which induce the 1-form 0 on B; and since D_{red} generates B (in the above sense), it also equals the dimension of the space of 1-forms on X which induce the 1-form 0 on D_{red}. Thus we have

$$\alpha(D) = \text{dim. ker.}\,(H^0(X, \Omega^1_X) \to H^0(D_{\text{red}}, \Omega^1 D_{\text{red}})).$$

Theorem 2. *Let X be a non-singular projective variety of dimension $\geqslant 2$ and D an effective divisor on X such that for some $n > 0$, $|nD|$ has no fixed components and is not composite with a pencil. Assume further* **103** *that either the characteristic is zero or that it is positive and that the Frobenius is injective on $H^1(\mathscr{O}_X)$. Then $H^1(\mathscr{O}_X(-D)) = (0)$.*

Proof. First suppose $\dim X = 2$. The exact sequence

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$$

leads to the cohomology sequence

$$H^0(X, \mathscr{O}_X) \to H^0(D, \mathscr{O}_D) \to H^1(X, \mathscr{O}_X(-D)) \to H^1(X, \mathscr{O}_X) \to H^1(D, \mathscr{O}_D)$$

and the result follows from Lemmas 2, 3, 5 and 6.

Next suppose $n = \dim X > 2$ and that the result holds in smaller dimensions. Replacing the given projective embedding by the one given

by hypersurfaces of sufficiently large degree, we may assume that $H^1(X, \mathcal{O}_X(-D)(-1)) = (0)$. Hence if H is a general hyperplane section for this embedding, the homomorphism $H^1(X, \mathcal{O}_X(-D)) \to H^1(H, \mathcal{O}_H(-D.H))$ is injective. On the other hand, the assumptions fulfilled by D on X are also fulfilled by $D.H$ on H. This completes the induction. □

4 A generalisation of the Kodaira vanishing theorem for algebraic varieties

We assume the following theorem:

Theorem [11]. *Let X be a projective variety over the complex field, L a line bundle on X such that there is an integer $n > 0$ and a birational morphism $\phi : X \to Y \subset \mathbf{P}^N$ such that $\phi^*(\mathcal{O}_Y(1)) \approx L$. Then $H^i(X, L^{-1}) = 0$ for $0 \leqslant i < \dim X$.*

We deduce from this the following

Theorem 3. *Let X be a projective non-singular variety of dimension n over the complex field, L a line bundle on X, m an integer with $1 \leqslant m \leqslant n$ such that for some $N > 0$, the complete linear system determined by $H^0(X, L^N)$ has the dimension of its base point set $\leqslant n - m$ and has a projective image of dimension $\geqslant m$. Then $H^i(X, L^{-1}) = (0)$ for $0 \leqslant i < m$.*

Proof. We fix m, and use induction on n starting from $n = m$. Assume first that $n > m$ and that the theorem is true for varieties of dimen-
104 sion $n - 1$. Using a suitable projective embedding, we may assume that $H^i(X, L^{-1}(-1)) = (0)$ for $0 \leqslant i < n$. If H is a general hyperplane section for this embedding, it follows that $H^i(X, L^{-1}) \to H^2(H, (L_H)^{-1})$ is injective for $0 \leqslant i < n$, where L_H denotes the restriction of L to H. Further, the hypotheses made on the pair (X, L) are evidently also satisfied by the pair (H, L_H) for H general. We are thus reduced to proving the theorem when $n = m$.

Thus, assume $n = m$. Since the base points of the complete linear system determined by $H^0(X, L^N)$ are isolated, by ([10], Theorem

6.2), replacing N by a larger multiple, we may assume there are no base points. Thus there is a morphism $\phi : X \to Y \subset \mathbf{P}^k$ with $\dim X = \dim Y = n$ such that $\phi^*(\mathscr{O}_Y(1)) \approx L^N$. By (E.G.A. Chap II), we may even assume ϕ birational. But now, the result follows from the theorem quoted above. □

Remarks. (1) It is not clear if in Theorem 3, the hypothesis on the set of base points of the complete linear system determined by $H^0(X, L^N)$ can be weakened to the assumption that the base point set be of dimension $\leqslant n-2$.

For $m = 2$, Theorem 3 specialises to Theorem 2 over $\mathbf{C}$, which however is proved under the further assumption that $H^0(X, L) \neq (0)$.

(2) A generalisation of the theorem of Nakano on the vanishing of $H^p(X, \Omega^q \otimes L^{-1})$ along the lines of the theorem quoted at the beginning of this section does not exist. In fact, let $\sigma : X \to \mathbf{P}^3$ be the morphism obtained by blowing up $\mathbf{P}^3$ at (1, 0, 0, 0), and $L = \sigma^*(\mathscr{O}_{\mathbf{P}^3}(1))$. Then one shows easily that $H^1(X, \Omega^1_X \otimes L^{-1}) \neq (0)$.

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The Invariance of Milnor's Number Implies The Invariance of the Topological Type

By Lê Dũng Tráng and C. P. Ramanujam.

Introduction. We are interested in analytic families of n-dimen- **107**
sional hypersurfaces having an isolated singularity at the origin. In this paper we only consider the case when the Milnorś number of the singularity at the origin does not change in this family. Under this hypothesis with $n = 1$, H. Hironaka conjectured that the topological type of the singularity does not change. We give a proof of this conjecture in the more general case of C^∞ family of n-dimensional hypersurfaces of dimension $n \neq 2$. The hypothesis $n \neq 2$ comes from the fact we are using h-cobordism theorem. Actually a more general conjecture should be the following:

0.1 let $F(z_0, \ldots, z_n, t)$ be analytic in the z_i and smooth (i.e. C^∞) in t. Suppose that for any $t_0 \in \mathbf{R}$ the complex hypersurface $F(z_0, \ldots, z_n, t_0) = 0$ of $\mathbf{C}^{n+1}$ has an isolated singularity at the origin. Suppose that the Milnor's number μ_{t_0} of this singularity, say the complex dimension of $\mathbf{C}\{z_0, \ldots, z_n\}/(\partial F_{t_0})$ is the ideal generated by the partial derivatives $(\partial \mathbf{F}/\partial z_i)(z_0, \ldots, z_n, t_0)$ $(i = 0, \ldots, n)$ in this algebra, does not depend on t_0. Then the pair composed of the smooth part of the hypersurface $F = 0$ in a neighborhood U of 0 in $\mathbf{C}^{n+1} \times \mathbf{R}$ and $(\{0\} \times \mathbf{R}) \cap U$ satisfies Whitney conditions at each point of $(\{0\} \times \mathbf{R}) \cap U$, (cf. [17] p. 540).

If such a conjecture is true we obtain easily our result by using Thom-Mather isotopy theorem (cf. [16] and [7]).

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Actually recent results of B. Teissier [14] give a numerical condition to get Whitney conditions. Precisely let (X, x) be a germ of hypersurface in $\mathbf{C}^{n+1}$ with an isolated singularity, let E be a k-dimensional affine subspace of $\mathbf{C}^{n+1}$ passing through x. If E is chosen sufficiently general the Milnor's number of $X \cap E$ at x does not depend on E, we denote this number by $\mu^{(k)}(X, x)$. Then B. Teissier proved that if in an analytic family of germs of hypersurfaces $(X_t, 0)$ in $\mathbf{C}^{n+1}$ defined by $F(z_0, \ldots, z_n, t) = 0$ and having an isolated singularity at the origin 0 the numbers $\mu^{(k)}(X_t, 0)$ for $1 \leqslant k \leqslant n+1$ do not depend on t, the smooth
108 part of the hypersurface $F(z_0, \ldots, z_n, t) = 0$ in a neighborhood U of $\mathbf{C}^{n+1} \times \mathbf{C}$ satisfies Whitney conditions along $(\{0\} \times \mathbf{C}) \cap U$.

When $n = 1$ our result shows that the multiplicity of the curves of our analytic family is constant, because in this case we know that the multiplicity of a plane curve singularity is a topological invariant.

Actually when $n = 1$, O. Zariski in [19] and [20] proved that if the sum of $\mu_t + m_t - 1$, where μ_t is the Milnor's number of the singularity and m_t its multiplicity, is independent of t then the smooth part of the surface $F(x, y, t) = 0$ in a neighborhood U of 0 in $\mathbf{C}^3$ satisfies Whitney conditions along $(\{0\} \times \mathbf{C}) \cap U$. In his terminology the preceding surface is equisingular along its singular locus at 0. Then the constancy of μ_t implying the one of m_t, Teissier's result is analogous to Zariski's result.

0.2 It is then natural to conjecture, following B. Teissier (cf. [14]) that the constancy of the $\mu_t^{(n+1)}$ implies the constancy of the $\mu_t^{(k)} (1 \leqslant k \leqslant n)$, which would imply (0.1). Recently J. Brianson and J. P. Speder disproved this conjecture.

1 Milnor's Results on the Topology of Hypersurfaces

In this paragraph we first recall Milnor's results on the topology of hypersurfaces (cf. [9]).

Let $f : U \subset \mathbf{C}^{n+1} \to \mathbf{C}$ be an analytic function on an open neigh-

borhood U of 0 in $\mathbf{C}^{n+1}$. We denote

$$B_\epsilon = \{z | z \in \mathbf{C}^{n+1} : ||z|| \leqslant \epsilon\}$$
$$S_\epsilon = \partial B_\epsilon = \{z | z \in \mathbf{C}^{n+1} : ||z|| = \epsilon\}.$$

Then:

Theorem 1.1. *For $\epsilon > 0$ small enough the mapping $\varphi_\epsilon : S_\epsilon - \{f = 0\} \to S^1$ defined by $\varphi_\epsilon(z) = f(z)/|f(z)|$ is a smooth fibration.*

Theorem 1.2. *For $\epsilon > 0$ small enough and $\epsilon \gg \eta > 0$ the mapping $\psi_{\epsilon,n} : \dot{B}_\epsilon \cap f^{-1}(\partial D_\eta) \to \mathbf{S}^1$ defined by $\psi_{\epsilon,\eta}(z) = f(z)/|f(z)|$, where $\partial D_\eta = \{z | z \in \mathbf{C} : |z| = \eta\}$, is a smooth fibration isomorphic to $\varphi_\epsilon$by an isomorphism which preserves the arguments. We call the fibrations of Theorems 1.1 and 1.2 the Milnor's fibrations of F at 0.*

Corollary 1.3. *The fibers of φ_ϵ have the homotopy type of a n-dimensional finite CW-complex.*

Theorem 1.4. *For $\epsilon > 0$ small enough, S_ϵ transversally cuts the smooth part of the algebraic set H_0 defined by $f = 0$. If 0 is an isolated critical point of f, then the pairs $(S_\epsilon, S_\epsilon \cap H_0)$ for any ϵ small enough are diffeomorphic and $(B_\epsilon, B_\epsilon \cap H_0)$ is homeomorphic to $(B_\epsilon, C(S_\epsilon \cap H_0))$,* **109** *where $C(S_\epsilon \cap H_0)$ is the real cone, union of segments with vertices at 0 and at a point of $S_\epsilon \cap H_0$.*

Theorem 1.4 says that, when 0 is an isolated critical point of f, for $\epsilon > 0$ small enough, the topology of the pair $(B_\epsilon, B_\epsilon \cap H_0)$ does not depend on ϵ. Then if g is another analytic function defined in a neighborhood of 0, having an isolated critical point at 0, we say that the hypersurfaces H_0 and H_0' defined by $f = 0$ and $g = 0$ have *the same topological type at 0* if for $\epsilon > 0$ small enough there is a homeomorphism $(B_\epsilon, B_\epsilon \cap H_0) \xrightarrow{\sim} (B_\epsilon, B_\epsilon \cap H_0')$.

From [9] and [11] we have:

Theorem 1.5. *If 0 is an isolated critical point of f, for $\epsilon > 0$ small enough, the fibers of φ_ϵ have the homotopy type of a bouquet of μ spheres of dimension n with*

$$\mu = \dim_{\mathbf{C}}(\mathbf{C}\{z_0, \ldots, z_n\}/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n)).$$

(A bouquet of spheres is the topological space union of spheres having a single point in common).

We call the number of spheres *the Milnor's number* of the critical point 0 of F or the *number of vanishing cycles* of F at 0.

Actually we have:

Proposition 1.6. *The germ of morphism $\psi : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ whose components are the partial derivatives of f is an analytic covering of degree μ. On another hand, for $\epsilon > 0$ small enough, the mapping $\psi_\epsilon : S_\epsilon \to \mathbf{S}^{2n+1}$ defined by*

$$\psi_\epsilon(z) = (\partial f/\partial z_0(z), \ldots, \partial f/\partial z_n(z)) \Big/ \sqrt{\sum_{i=0}^{n} |\partial f/\partial z_i(z)|^2}$$

has the degree μ.

Finally recall a well-known result of P. Samuel (cf. [12]):

Theorem 1.7. *Let $f : U \subset \mathbf{C}^{n+1} \to \mathbf{C}$ be an analytic function on a neighborhood U of 0 in $\mathbf{C}^{n+1}$. Suppose $f(0) = 0$ and 0 is an isolated critical point. Then there exists a polynomial $f_0 : \mathbf{C}^{n+1} \to \mathbf{C}$ with an isolated critical point at 0 and an analytic isomorphism of a neighborhood U_1 of 0 onto a neightborhood U_2 of 0 which sends the points of $f = 0$ on points of $f_0 = 0$.*

2 The Main Theorem

We prove the following theorem:

Theorem 2.1. *Let $F(t, z)$ be a polynomial in $z = (z_0, \ldots, z_n)$ with coefficients which are smooth complex valued functions of $t \in I = [0, 1]$*
110 *such that $F(t, 0) = 0$ and such that for each $t \in I$, the polynomials $(\partial F/\partial z_i)(t, z)$ in z have an isolated zero at 0. Assume moreover that the integer*

$$\mu_t = \dim_{\mathbf{C}} \mathbf{C}\{z\} / \left(\frac{\partial f}{\partial z_0}(t, z), \ldots, \frac{\partial f}{\partial z_n}(t, z) \right)$$

is independent of t. Then the monodromy fibrations of the singularities of $F(0,z) = 0$ *and* $F(1,z) = 0$ *at 0 are fo the same fiber homotopy. If further* $n \neq 2$, *these fibrations are even differentiably isomorphic and the topological types of the singularities are the same.*

A crucial step in the proof of the theorem is the following lemma:

Lemma 2.2. *Let* $f \in \mathbf{C}[z_0, \ldots, z_n]$, $f(0) = 0$, *and* $R > 0$, $\epsilon > 0$ *such that*

(i) $\dim_{\mathbf{C}} \mathbf{C}\{z\}/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n) = \mu < \infty$;

(ii) *for any z with* $0 < ||z|| < R$ *and* $|f(z)| \leqslant \epsilon$, *we have* $df \neq 0$;

(iii) *for any* $z \in S_R = \{z| \ ||z|| = R\}$ *with* $|f(z)| \leqslant \epsilon$, $d(f|S_R)$ *is of rank 2;*

(iv) *for some* w_0 *with* $0 < |w_0| \leqslant \epsilon$, $F_{w_0} = \{z| \ ||z|| \leqslant R, f(z) = w_0\}$ *is of the homotopy type of a bouquet of* μ *n-spheres.*

Then the map

$$\{z| \ ||z|| \leqslant R, |f(z)| = \epsilon\} \xrightarrow{f/|f|} \mathbf{S}^1 \tag{A}$$

is a fibration fiber homotopy equivalent to the monodromy fibration of the singularity of $f = 0$ *at 0. If further* $n \neq 2$, *it is even diffeomorphic to the monodromy fibration, and there is a homeomorphism of the set*

$$\{z| \ ||z|| \leqslant R, |f(z)| = \epsilon\} \cup \{z| \ ||z|| = R, |f(z)| \leqslant \epsilon\}$$

with the sphere $\{z| \ ||z|| = \delta\}$ *(*δ *small) which maps the set* $\{z| \ ||z|| = R, f(z) = 0\}$ *onto the set* $\{z| \ ||z|| = \delta, f(z) = 0\}$.

Proof. It is clear from (ii) and (iii) that the map

$$f : \{z| \ ||z|| \leqslant R, \ 0 < |f(z)| \leqslant \epsilon\} \to \{w|0 < |w| \leqslant \epsilon\} \tag{B}$$

is a locally trivial differentiable fibration, hence so is (A). Choose δ, η with $0 < \delta < R$, $0 < \eta < \epsilon$ such that

$$f/|f| : \{z| \ ||z|| \leqslant \delta, |f(z)| = \eta\} \to \mathbf{S}^1 \tag{C}$$

is the monodromy fibration of the singularity. This fibration is contained in the fibration 111

$$f/|f| : \{z| \; ||z|| \leqslant R, \; |f(z)| = \eta\} \to \mathbf{S}^1, \qquad \text{(D)}$$

and (A) and (D) are diffeomorphic fibrations since (B) has been shown to be a fibration. Thus it suffices to show that the inclusion of (C) in (D) is a fiber homotopy equivalence. By the theorem of Dold ([4] Theorem 6.3) and the homotopy sequence of a fibration, it suffices to show that the inclusion of the fiber G_1 over 1 of (C) in the fiber F_1 over 1 of (D) is a homotopy equivalence. Now, since the spheres S_R and S_δ of radii R and δ respectively are transversal to the manifold $f(z) = \eta$, for all z_0 close enough to the origin, the pair of manifolds with boundaries (F_1, G_1) is diffeomorphic to the pair

$$(\{z|f(z) = \eta, ||z - z_0|| \leqslant R\}, \; \{z/f(z) = \eta, ||z - z_0|| \leqslant \delta\}).$$

Choose such z_0 such that the distance function $||z - z_0||$ has only nondegenerate critical points on $f(z) = \eta$. Since the indices of this function at these critical points are $\leqslant n$, it follows that F_1 is obtained from G_1, up to homotopy type, by attaching cells of dimension $\leqslant n$. Now, it follows from (iv) that F_1 is of the homotopy type of the wedge μ n-spheres, and the same holds of G_1, by [9]. It clearly follows that for $n = 1$, the inclusion $G_1 \hookrightarrow F_1$ is a homotopy equivalence. For $n > 1$, both spaces are simply connected, and it suffices to show that $H_i(G_1, \mathbf{Z}) \to H_i(F_1, \mathbf{Z})$ is an isomorphism for all i. This is clearly so for $i \neq n$ and it also holds for $i = n$ since $H_{n+1}(F_1, G_1; \mathbf{Z}) = 0$, $H_n(G_1, \mathbf{Z})$ and $H_n(F_1, \mathbf{Z})$ are both free of rank μ and $H_n(F_1, G_1; \mathbf{Z})$ is torsion-free.

We have thus shown that the inclusion of (C) in (D) is a homotopy equivalence. We have yet to establish the stronger assertions of the lemma for $n \neq 2$. First notice that if $X = \{z|\delta \leqslant ||z|| \leqslant R, |f(z)| \leqslant \eta\}$, then $f : X \to \{w||w| \leqslant \eta\}$ is a differentiable fibration. This follows from (ii), (iii) and the fact that $d(f|S_\delta)$ is of rank 2 on $X \cap S_\delta$. It follows that X is diffeomorphic to a product of the disc $\{w||w| \leqslant \eta\}$ and the fiber X_w of X over any point w of this disc, in particular $X_\eta = F_1 - \operatorname{Int} G_1$. Now, $H_*(F_1, G_1; Z) = H_*(X_\eta, \partial G_1; \mathbf{Z}) = (0)$. So that

$H_*(\partial G_1, \mathbf{Z}) \to H_*(X_\eta; \mathbf{Z})$ is an isomorphism. For $n = 1$, it follows from this and the classification of surfaces with boundary that X_η is diffeomorphic to $I \times \partial G_1$. For $n > 2$, ∂G_1 is simply connected ([9]) and the inclusion $\partial G_1 \subset X_\eta$ is a homotopy equivalence. Further, by considering the Morse function $-||z - z_0||$ with z_0 close to the origin, and noticing that its indices at critical points on F_1 are $\geqslant n$, ([1]) we see that $\pi_1(\partial F_1) \to \pi_1(F_1)$ is injective, so that ∂F_1 is also simply connected. If we can show that $H_*(X_\eta, \partial F_1; \mathbf{Z}) = (0)$, it would follow that the inclusion $\partial F_1 \hookrightarrow X_\eta$ is a homotopy equivalence. From the homology exact sequence of the triple $(F_1, X_\eta, \partial F_1)$, it suffices to show that $H_*(F_1, \partial F_1) \to H_*(F_1, X_\eta)$ is an isomorphism, or passing to cohomology, that $H_c^*(\operatorname{Int} G_1) \to H_c^*(\operatorname{Int} F_1)$ is an isomorphism, where H_c^* denotes cohomology with compact support. By Poincare duality, this is the same as saying that $H_*(G_1, \mathbf{Z}) \to H_*(F_1, \mathbf{Z})$ is an isomorphism, which is certainly true. **112**

Thus, the inclusions $\partial G_1 \subset X_\eta$ and $\partial F_1 \subset X_\eta$ are homotopy equivalences, the spaces are simply connected and we are in real dimensions $\geqslant 6$. It follows from the h-cobordism theorem [8] that X_η is diffeomorphic to $I \times \partial G_1$. Thus, the fibration (D) is obtained from the fibration (C) (which restricted to the boundary of the total space of (C) is trivial) by attaching $S^1 \times$ collar. Hence the two fibrations are diffeomorphic by a diffeomorphism φ. Further, if we fix a diffeomorphism $X \overset{\lambda}{\leftarrow} D \times I \times \partial G_1$ over D, where D denotes the disc $\{w||w| \leqslant \eta\}$, we may assume that the points $\lambda(w, 0, x)$ and $\lambda(w, 1, x)$ correspond to each other under φ. But now, the canonical diffeomorphism of $D \times 0 \times \partial G_1$ and $D \times 1 \times \partial G_1$ goes over by λ into a diffeomorphism of $\{z|\ ||z|| = \delta, |f(z)| \leqslant \eta)\}$ onto $\{z|\ ||z|| = R, |f(z)| \leqslant \eta)\}$ such that φ and ψ are equal at points where both are defined and ψ carries the zero set of f (i.e., $\lambda(0 \times 0 \times \partial G_1)$) in its domain onto the zero set of f $(i.\epsilon.,, \lambda(0 \times 1 \times \partial G_1))$ in its range. Now, by [9] there is a homeomorphism of $\{z|\ ||z|| \leqslant \delta, |f(z)| = \eta\} \cup \{z|\ ||z|| = \delta, |f(z)| \leqslant \eta\}$ onto S_δ which is the identity on the second of these sets. This establishes the last assertion of the lemma, except for the fact that ϵ is replaced by η. But this clearly does not matter, in view of assumptions (ii) and (iii).

Q.E.D. □

We need another simple lemma:

Lemma 2.3. *Let K be a compact convex set in $\mathbf{C}^{n+1}$ and $g_1, \ldots, g_{n+1}$ holomorphic functions in a neighborhood of K such that on* $\operatorname{Bd} K$, $\sum |g_i|^2 > 0$. *Orient* $\operatorname{Bd} K$ *(which is homeomorphic to S^{2n+1}) such that a radial projection onto an interior sphere has degree $+1$. Then the g_i have only finitely many common zeros in K, and the degree μ of the map $\partial K \to S^{2n+1}$ given by $z \mapsto (\sum |g_i(z)|^2)^{-1/2}\,(g_1(z), \ldots, g_{n+1}(z))$ equals $\sum_{P \in K} \mu_p$, where*

$$\mu_p = \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^{n+1}_P}/(g_1, \ldots, g_{n+1}).$$

Proof. Since the g_i have no common zeros on $\operatorname{Bd} K$, their common zeros in K form an analytic set in $\mathbf{C}^{n+1}$, and K being compact, they must be a finite set. We proceed by induction on the number N of these common zeros. If $N = 1$, let $P \in \operatorname{Int} K$ be the unique common zero, B a small around P contained in $\operatorname{Int} K$ and S the boundary of B. The map $\operatorname{Bd} K \to S^{2n+1}$ is then homotopic to the composite $\operatorname{Bd} K \xrightarrow{\alpha} S = BdB \to S^{2n+1}$ where α is radial projection and the map $\operatorname{Bd} B \to S^{2n+1}$ is the map defined above with K replaced by S. The result then follows from [9].

113 If $N > 1$, choose a hyperplane H such that there are no common zeros of the g_i on H and such that if H^+ and H^- are the closed half-spaces defined on H, $H^+ \cap K$ and $H^- \cap K$ each contain at least one common zero of the g_i. Since we may assume by induction that the result holds for each of the compact convex sets $H^+ \cap K$ and $H^- \cap K$, we have only to show that the degree of $\operatorname{Bd} K \to S^{2n+1}$ equals the sum of the degrees of $\operatorname{Bd}(K \cap H^+) \to S^{2n+1}$ and $\operatorname{Bd}(K \cap H^-) \to S^{2n+1}$. Since the intersection $\operatorname{Bd}(K \cap H^+) \cap \operatorname{Bd}(K \cap H^-)$ is contractible on each of these boundaries, the assertion reduces to the standard and easy fact that if $f, g : S^{2n+1} \to S^{2n+1}$ are maps preserving some base point, $h : S^{2n+1} \to S^{2n+1} \bigvee S^{2n+1}$ the pinching map, the composite $S^{2n+1} \xrightarrow{h} S^{2n+1} \bigvee S^{2n+1} \xrightarrow{(f,g)} S^{2n+1}$ has degree equal to the sum of the degrees of f and g. Q.E.D. □

Proof of Theorem 2.1. We set $f_t(z) = F(t,z)$. It is clearly enough to prove the theorem with I replaced by some smaller interval $[0,\delta]$ with $\delta > 0$. Denote by μ the constant value of μ_t, $t \in I$. Choose R and ϵ such that the conditions (ii), (iii) and (iv) hold with f replaced by f_0, and also such that $\sum_1^{n+1} |\partial f_0/\partial z_i|^2 > 0$ on $||z|| = R$. By continuity, one sees that there is a $\delta > 0$[1] such that for any $t \in [0,\delta]$, (iii) holds and also $\sum_1^{n+1} |\partial f_t/\partial z_i|^2 > 0$ on $||z|| = R$. The maps $S_R \to S^{2n+1}$ are defined by $||\operatorname{grad} f_t||^{-1}$. $(\partial f_t/\partial z_i, \ldots, \partial f_t/\partial z_{n+1})$ for various t are all homotopic, hence all of the degree μ. On the other hand, we know that at the origin, $\dim_{\mathbf{C}} \mathbf{C}\{z\}/(\partial f_t/\partial z_1, \ldots, \partial f_t/\partial z_{n+1}) = \mu$. It follows from Lemma 2.3 that (ii) is also fulfilled by f_t, R and ϵ if $t \in [0,\delta] = I_1$.

Let $X = \{(t,z) \in I_1 \times \mathbf{C}^{n+1} |\ ||z|| \leqslant R,\ 0 < |F(t,z)| = \epsilon\}$, and define $\Phi : X \to I_1 \times \{w | 0 < |w| \leqslant \epsilon\}$, $\Phi(t,z) = (t, F(t,z))$. Then Φ has compact fibers, and is of maximal rank on the sets $\{(t, z \in X |\ ||z|| < R)\}$ as well as when restricted $X \cap I_1 \times S_R^{2n+1}$. Thus, Φ is a differentiable fibration. Similarly, if $Y : \{(t,z) \in I_1 \times S_R | |F(t,z)| \leqslant \epsilon\}$, the map $\psi : Y \to I_1 \times \{w | |w| \leqslant \epsilon\}$, $\psi(t,z) = (t, F(t,z))$ is a smooth fibration. It follows that ψ is differentiably a trivial fibration, and that $\Phi | X \cap (0 \times \mathbf{C}^{n+1}) : X \cap (0 \times \mathbf{C}^{n+1}) \to \{w | 0 < |w| \leqslant \epsilon\}$ and $\Phi | X \cap (\delta \times \mathbf{C}^{n+1}) : X \cap (\delta \times \mathbf{C}^{n+1}) \to \{w | 0 < |w| \leqslant \epsilon\}$ are smoothly isomorphic fibrations such that this isomorphism is compatible with the chosen trivialisation of ψ.

The theorem now follows from Lemma 1. Q.E.D.

We now deduce some corollaries. In order to shorten statements we introduce the following definition. We say that the isolated hypersurface singularities at 0 in $\mathbf{C}^{n+1}$ defined by the equations $f = 0$ and $g = 0$ are of the *same type* if (i) their monodromy fibrations are fiber homotopic, and (ii), for $n \neq 2$, these fibrations are differentiably isomorphic, and there is a homeomorphism of S^{2n+1} onto itself carrying the zero set of f on S^{2n+1} onto that of g, this implies in particular that the matrices of the monodromy transformations on the n-th homology of the fiber of the **114**
Milnor's fibrations of f and g are inner conjugate in $\mathbf{SL}(\mu, \mathbf{Z})$, μ being the number of vanishing cycles.

Corollary 2.4. *Let* $f \in \mathbf{C}[z_1, \ldots, z_{n+1}]$, *and* $p_i \in \mathbf{Z}$, $p_i > 0 (1 \leqslant i \leqslant n+$

1) *be weights for the variables* z_i, $f = f_N + f_{N+1} + \ldots$ *the decomposition of* f *into quasi-homogeneous polynomials* f_p *of weight* p *with* $f_N \neq 0$, *suppose 0 is an isolated critical point of* $f_N = 0$. *Then the same holds for* $f = 0$, *and the singularities* $f = 0$ *and* $f_N = 0$ *at* 0 *are of the same type.*

Proof. Put $F(t,z) = f_N + tf_{N+1} + t^2 f_{N+2} + \cdots$, so that for $t \neq 0$, $F(t,z) = t^{-N} f(t^{p_1} z_1, \ldots, t^{p_{n+1}} z_{n+1})$. Choose R so that $(d f_N)_z \neq 0$ for $0 < ||z|| \leqslant R$, and let $\mu = \dim_{\mathbf{C}} \mathbf{C}\{z\}/(\partial f_N/\partial z_1, \ldots, \partial f_N/\partial z_{n+1})$. Then for t close enough to 0, $d_z F(t,z) \neq 0$ on $||z|| = R$, so that $F(t,z)$ has only isolated critical points in $||z|| \leqslant R$. Since $F(t,z)$ derives from $f(z)$ by a trivial coordinate transformation, $f(z)$ has 0 for an isolated critical point. Now choose $R_1 > 0$ such that both f and f_N have 0 as their only critical point in $||z|| \leqslant R_1$. Then the same is true of $F(t,z)$ for $0 \leqslant t \leqslant 1$. Further for all $t \in [0,1]$, the maps $S_R \to S^{2n+1}$ given by

$$z \mapsto ||\operatorname{grad}_z F(t,z)||^{-1} \left(\frac{\partial f_t}{\partial z_1}, \ldots, \frac{\partial f_t}{\partial z_{n+1}} \right)$$

are homotopic, hence have the same degree. It follows from Lemma 2.3 that $\mu = \dim_{\mathbf{C}} \mathbf{C}\{z\}/(\partial f_t/\partial z_1, \ldots, \partial f_t/\partial z_{n+1})$ is independent of t. Now appeal to the theorem. Q.E.D. □

Remark 2.5. It is in fact possible to make a stronger statement. There is a diffeomorphism $\varphi : S_\delta \to S_\delta$ (δ small) isotopic to the identity which maps $K = \{z \in S_\delta | f(z) = 0\}$ onto $K_0 = \{z \in S_\delta | f_N(z) = 0\}$ and such that for $z \in S_\delta - K$, $f(z)/|f(z)| = f_N(\varphi(z))/|f_N(\varphi(z))|$, i.e., $\varphi | S_\delta - K$ is a fiber preserving diffeomorphism of the monodromy fibration of f onto that of f_N. This holds for all $n \geqslant 1$ without exception. This is a consequence of the fact that if we define $f_t(z) = F(t,z)$ as above, there is a $\delta > 0$ such that, for any with $0 < ||z|| = \delta$, and any $t \in [0,1]$,grad log $f_t = \lambda$. z with λ complex implies that $|arg\lambda| < \pi/4$ and $||z|| = \delta$, $f_t = 0$ imply that $d(f_t | S_\delta)_z$ is of rank 2.

To state the next corollary, we need a few definitions and facts from commutative algebra. Let A be a noetherian normal local domain of

dimension $n+1$ with maximal ideal $\mathfrak{M}$, and $\mathfrak{a}$ an $\mathfrak{M}$-primary ideal. An element $f \in A$ is said to be integral over $\mathfrak{a}$ if it satisfies an equation

$$f^m + a_1 f^{m-1} + \ldots + a_m = 0, \quad a_i \in \mathfrak{a}^i.$$

The set $\bar{\mathfrak{a}}$ of elements of A integral over $\mathfrak{a}$ form an ideal containing $\mathfrak{a}$, called the integral closure of $\mathfrak{a}$. (See [13], Appendix 4 and [10] for **115**
all properties made use of.) If $e(\mathfrak{a})$ is the multiplicity of $\mathfrak{a}$, we have $e(\mathfrak{a}) = e(\bar{\mathfrak{a}})$ ([10], Theorem 1). Further, if R runs through all the discrete valuation rings containing A and having the same quotient field as A, we have $\bar{\mathfrak{a}} = \cap R \cdot \mathfrak{a}$ ([13], Appendix 4, Theorem 3). Finally, let $\alpha_i (1 \leqslant i \leqslant k)$ be elements of A generating an $\mathfrak{M}$-primary ideal $\mathfrak{a}$. Then there exists a finite set of polynomials $P_\alpha(x_{ij})$ over the residue field in the variables $x_{ij} (1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n+1)$ such that if $v = (n_{ij}) \in A^{k(n+1)}$ and $\mathfrak{a}_n = (\sum_{i=1}^k n_{i1}\alpha_i, \ldots, \sum_1^k n_{in+1}\alpha_i)$; then $\mathfrak{a}$ is integral over $\mathfrak{a}_n$ (i.e., $\mathfrak{a} \subset \bar{\mathfrak{a}}_n$) if and only if $P_\alpha(\bar{n}_{ij}) \neq 0$ for some α, where $\bar{n}_{ij}$ denotes the image of n_{ij} in the residue field. (See [10], Sec. 5, Theorem 1.)

Lemma 2.6. *Let A be a regular local ring of dimension $n+1$ containing a field k of characteristic 0 which gets mapped onto the residue field of A. Suppose $D_i (1 \leqslant i \leqslant n+1)$ is a system of k-derivations of A such that the induced maps $D_i : \mathfrak{M}/\mathfrak{M}^2 \to A/\mathfrak{M}$ form a basis of the dual of $\mathfrak{M}/\mathfrak{M}^2$. Then if $f \in \mathfrak{M}$ such that the ideal $(D_1 f, \ldots, D_{n+1} f)$ is primary for the maximal ideal, f belongs to the integral closure of $(D_1 f, \ldots, D_{n+1} f)$.*

Proof. It suffices to show that if $A \subset R$, where R is a discrete valuation ring, $f \in \sum_1^{n+1} R \cdot D_i f$, we may further assume that the maximal ideal of A is contained in that of R, since otherwise $\sum_1^{n+1} R \cdot D_i f = R$ and the assertion is trivial. We may further suppose R complete. We can then find a subfield L of R such that $L \supset k$ and R is the formal power series ring $L[[T]]$ over L in some T generating the maximal ideal of R. Let D denote the L-derivation d/dt of R, so that clearly $Rf \subset R \cdot Df$. It suffices to show that $R \cdot Df \subset \sum_1^{n+1} R \cdot D_i f$. Let $\mathrm{Der}_k(A, R)$ be the R-module of k-derivations of A into R continuous for the $\mathfrak{M}$-adic topology, so that $D|A \in \mathrm{Der}_k(A, R)$. We will be done if we show that $\mathrm{Der}_k(A, R)$ is generated by the $D_i (1 \leqslant i \leqslant n+1)$ choose a basis $x_i (1 \leqslant i \leqslant n+1)$

of $\mathfrak{M}$ such that $D_i x_j = \delta_{ij} (\mathrm{mod}\ \mathfrak{M})$. The homomorphism $\mathrm{Der}_k(A, R) \to R^{n+1}$ which sends D to $(Dx_1, \ldots, Dx_{n+1})$ is clearly injective. It is also surjective and the images of the D_i generated R^{n+1} by Nakayama, since it is so when we pass to the quotient modulo the maximal ideal of R. This proves the lemma. Q.E.D.

□

We are now ready for the next corollary. For any $f \in \mathbf{C}[z_1, \ldots, z_{n-1}]$ with an isolated singular point at (0), define

$$\mathfrak{a}_1(f) = \sum_1^{n+1} \mathbf{C}\{z\} \cdot \frac{\partial f}{\partial z_i} = \{Df/D \quad \text{a continuous } \mathbf{C}\text{ - derivation of } \mathbf{C}\{z\}\},$$

$$\mathfrak{a}_2(f) = \mathbf{C}\{z\} f + \mathfrak{a}_1(f),$$

$$\mathfrak{a}_3(f) = \overline{\mathfrak{a}_1(f)} = \text{ integral closure of } \mathfrak{a}_1(f).$$

116 It follows from Lemma (2.6) that $\mathfrak{a}_1(f) \subset \mathfrak{a}_2(f) \subset \mathfrak{a}_3(f)$. Further, it is clear that for any $\mathbf{C}$-automorphism φ of $\mathbf{C}\{z\}$, $\mathfrak{a}_i(\varphi(f)) = \varphi(\mathfrak{a}_i(f))$. Define the artinian local rings $A_i(f)$ by $A_i(f) = \mathbf{C}\{z\}/\mathfrak{a}_i(f)$.

Corollary 2.7. *Suppose $f, g \in \mathbf{C}[z]$ and that for some $i (1 \leqslant i \leqslant 3)$ we are given an isomorphism over $\mathbf{C}$, $\lambda : A_i(f) \cong A_i(g)$. Then the singularities of $f = 0$ and $g = 0$ at 0 are of the same type.*

Proof. First we start with proving weaker statements. Suppose $\mathfrak{a}_1(f) \supset \mathfrak{M}^N$ and $g - f \in \mathfrak{M}^{N+2}$. Then $(\partial g/\partial z_i) - (\partial f/\partial z_i) \in \mathfrak{M}^{N+1} \subset \mathfrak{M}\mathfrak{a}_1(f)$, which shows that $\mathfrak{a}_1(f) = \mathfrak{a}_1(g)$. Hence for any t, if $h_t = tg + (1-t)f$, $\mathfrak{a}_1(f) = \mathfrak{a}_1(h_t)$, and it follows from the theorem that the singularities of f and g are of the same type. (Compare to [15].)

Next suppose φ is a $\mathbf{C}$-automorphism of $\mathbf{C}\{z\}$ and $\varphi(f) = g$. We want to say that the singularities of f and g are of the same type. Let $J(\varphi)$ be the Jacobian matrix of φ, $\gamma(t)$ a smooth path in $\mathbf{GL}(n+1, \mathbf{C})$ connecting E to $J(\varphi)^{-1}$, and ϕ_t the automorphism of $\mathbf{C}\{z\}$ induced by $\gamma(t)$. Applying the theorem to the family $h_t = \phi_t \circ \varphi(f)$, we see that we may assume φ has Jacobian matrix the identity, so that we have $\varphi(z_i) = z_i + \psi_i(z)$, where ψ_i begins with second degree terms. By what we said

at the beginning of the proof, if ψ_i' denotes the power series obtained from ψ_i by leaving out from ψ_i all terms of degree N (N large) and φ' the automorphism of $\mathbf{C}\{z\}$ defined by $\varphi'\{z_i\} = z_i + \psi_i'(z)$, $\varphi(f) \equiv \varphi'(f) \pmod{\mathfrak{M}^N}$ so that $\varphi\{f\}$ and $\varphi'(f)$ have the same monodromy. Thus we may assume that the ψ_i are polynomials. Now define an automorphism φ_t of $\mathbf{C}\{z\}$ by $\varphi_t(z_i) = z_i + t\psi_i(z)$ and apply the theorem to $h_i = \varphi_t(f)$. Our assertion follows.

Suppose now that we have a $\mathbf{C}$-isomorphism $\bar{\lambda} : A_i(f) \xrightarrow{\sim} A_i(g)$. We can then find a $\mathbf{C}$-automorphism λ of $\mathbf{C}\{z\}$ which makes the diagram

$$\begin{array}{ccc} \mathbf{C}\{z\} & \xrightarrow{\lambda} & \mathbf{C}\{z\} \\ \downarrow & & \downarrow \\ \mathbf{C}\{z\}/\mathfrak{a}_i(f) & \xrightarrow{\bar{\lambda}} & \mathbf{C}\{z\}/\mathfrak{a}_i(g) \end{array}$$

commutative, the vertical arrows being the natural maps. It follows that $\lambda(\mathfrak{a}_i(g)) = \mathfrak{a}_i(g)$. On the other hand, we have $\lambda(\mathfrak{a}_i(f)) = \mathfrak{a}_i(\lambda(f))$. Finally, since $\overline{\mathfrak{a}_i(f)} = \mathfrak{a}_3(f)$, we deduce that $\mathfrak{a}_3(\lambda(f)) = \mathfrak{a}_3(g)$. But now, since the singularities of f and $\lambda(f)$ are of the same type, we may assume that we have $\mathfrak{a}_3(f) = \mathfrak{a}_3(g)$. But now, there is a finite set $S \subset \mathbf{C}$, $0 \notin S$, $1 \notin S$, such that for $t \in \mathbf{C} - S$, the integral closure of the ideal $(t(\partial f/\partial z_i) + (1-t)\partial g/\partial z_i)$ also equals $\mathfrak{a}_3(f)$. Join 0 and 1 in $\mathbf{C}$ by a smooth path $\gamma(t)$ not containing any point of S, and apply **117**
the theorem to the family $h_t = \gamma(t) \cdot f + (1 - \gamma(t)) \cdot g$. Note that $\mu_t = \dim_{\mathbf{C}} \mathbf{C}\{z\}/(\partial h_t/\partial z_i)) = e(\mathfrak{a}_3(f))$ is constant. Q.E.D. □

3 Plane Curve Case.

Let us show that our main theorem implies the conjecture 0.1 of the introduction when $n = 1$. Because of Zariski's results in [19], it is sufficient to prove that the multiplicity m_t of the curve $F(z_0, z_1, t) = 0$ of $\mathbf{C}^2$ at 0 is independent of t.

But O. Zariski in [21] and M. Lejeune in [6] proved that:

Theorem 3.1. *The topological type of a plane curve singularity at a singular point is determined by the topological type of each analytically irreductible component of this curve at the singular point and the intersection numbers of any pairs of distinct branches.*

Then consider two plane curves defined by $f = 0$ and $g = 0$ and having an isolated singular point at 0. Suppose that these curves have the same topological type at 0. Then there is a one-to-one correspondence ϕ between their anayltically irreductible branches, such that the branch γ_i of $f = 0$ at 0 has the same topological type as the branch $\phi(\gamma_i)$ of $g = 0$ at 0. Thus, as the multiplicity at 0 is the sum of the multiplicities of each branch, it suffices to prove that the multiplicity is a topological invariant of an analytically irreductible plane curve at its singular point.

More generally using results of K. Brauner in [2], W. Burau in [3] and O. Zariski in [18] we have:

Theorem 3.2. *Puiseux pairs (cf. [5]) of ananalytically irreductible plane curve singularity depends only on the topology of the singularity.*

If $(m_1, n_1), \ldots, (m_g, n_g)$ are the Puiseux pairs, one knowns that $n_1 \ldots n_g$ is the multiplicity of the singularity (cf. [5]).

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On a Geometric Interpretation of Multiplicity

By C.P. Ramanujam (Madras)

The interpretation is as follows. Let Y be a projective variety of dimension n, y a point of Y, $\mathcal{Q}$ an ideal in the local ring $\mathcal{O}_{Y,y}$ primary for the maximal ideal, and $f : X \to Y$ a proper birational morphism of a non-singular variety X into Y such that $\mathcal{Q}\mathcal{O}_X$ is an invertible sheaf of ideals of X. Then the multiplicity of $\mathcal{Q}$ in $\mathcal{O}_{Y,y}$ equals $(-1)^{n-1}(D^n)$, where D is the effective divisor on X defined by $\mathcal{Q}\mathcal{O}_X$.

Following a suggestion by Bombieri, we present a generalisation of this result to sheaves of ideals defining closed subsets of arbitrary dimension. Also, we have given the result in the context of schemes.

All schemes considered will be noetherian and seperated, and all morphisms considered seperated of finite type.

First, we recall some basic facts on the blow-up of a scheme with respect to a coherent sheaf of ideals. (See [1].) Let Y be a scheme and $\mathcal{I}$ a coherent sheaf of ideals in $\mathcal{O}_Y$, defining a closed subscheme Z of Y. Let $\mathcal{R}(\mathcal{I})$ be the quasi-coherent sheaf of graded $\mathcal{O}_Y$-algebras defined by

$$\mathcal{R}(\mathcal{I}) = \mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \dots .$$

Then the blow-up of Y with respect to $\mathcal{I}$ or the blow-up of Z on Y is defined to be the Y-scheme $X = \operatorname{Proj} \mathcal{R}(\mathcal{I})$. If Y is affine with $\Gamma(Y, \mathcal{O}_Y) = A$ and $\Gamma(Y, \mathcal{I}) = I$, X is covered by affine open sets isomorphic to $\operatorname{Spec} I \cdot f^{-1}$ where f runs through the elements (or even a set of generators) of I and $I \cdot f^{-1}$ is the subring of the of quotients $A_f = A[f^{-1}]$ consisting of elements of the form g/f^n, $g \in I^n$. If $\varphi : X \to Y$ is the structural morphism, $\mathcal{I}\mathcal{O}_X$ is an invertible sheaf of ideals of $\mathcal{O}_X$ which is very ample for φ, and the restriction of φ to $\varphi^{-1}(Y - z)$ is an isomorphism of this open subscheme onto $Y - Z$. Further, if U is any non-void

open subset of X, it cannot happen that $\varphi(U) \subset Z$, since then $\mathscr{I} \cdot \mathscr{O}_U$ would be contained in the sheaf of nilpotents of $\mathscr{O}_U$ and hence cannot be invertible. Hence, if Z contains no components of Y, or equivalently if $Y - Z$ is dense in Y, φ maps an open dense subset of X isomorphically onto an open dense subset of Y, i.e., φ is birational.

Now, let $\psi : Y' \to Y$ be a morphism such that $\mathscr{I}\mathscr{O}_{Y'}$ is invertible. We assert that there is a unique morphism $\tilde{\psi} : Y' \to X$ such that $\varphi \circ$
120 $\tilde{\psi} = \psi$. To prove this, we may clearly assume that $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} B$ are affine and that if $I = \Gamma(Y, \mathscr{I})$, $IB = f \cdot B$ for some $f \in I$, and also that the image of f in B is a non-zero divisor. Then $B \to B_f$ is injective, and under the induced homomophism $A_f \to B_f$, the image of $I \cdot f^{-1}$ is contained in B. This proves the assertion. Finally, let ξ be the generic point of a component of the closed subset of Y' defined by $\mathscr{I}\mathscr{O}_{Y'}$, so that $\mathscr{O}_{\xi,Y'}$ is a local ring of dimension one. Since $(\mathscr{I}\mathscr{O}_{Y'})_\xi$ is a principal ideal generated by a non-zero divisor contained in the maximal ideal of $\mathscr{O}_{\xi,Y'}$, the maximal ideal of $\mathscr{O}_{\xi,Y'}$ is not associated to (0). Hence if $\eta_1, \eta_2, \ldots, \eta_p$ are generic points of components of Y' containing ξ, the natural homomorphism $\mathscr{O}_{\xi,Y'} \to \prod_1^p \mathscr{O}_{\eta_i,Y'}$ is injective.

Next we recall the definitions and some basic facts about the degree of an invertible sheaf. Let m be an integer $\geqq 0$, and X a proper scheme over Spec Λ, where Λ is an artinian ring, with $\dim X \leqq m$. Then, if $\mathscr{L}$ is an invertible sheaf on X, and $\mathscr{I}$ any coherent sheaf on X,

$$P(n) = \sum_{i=0}^{n} (-1)^i l_\Lambda(H^i(X, \mathscr{I} \otimes \mathscr{L}^n))$$

is a polynomial in n of degree $\leqq m$. This is well-known when Λ is a field (see for instance [2]), and the general case follows easily. The coefficient of n^m in $\chi(\mathscr{L}^n)$ is of the form $\dfrac{d_m(\mathscr{L})}{m!}$, where $d_m(\mathscr{L})$ is an integer called the degree of $\mathscr{L}$ (the integer m being fixed once for all). This has the following properties:

(i) If X_i are the irreducible components of X with reduced structure

and $\mathscr{L}_i = \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{O}_{X_i}$ and ξ_i is the generic point of x_i, we have

$$d_m(\mathscr{L}) = \sum_i l(\mathscr{O}_{X,\xi_i}) d_m(\mathscr{L}_i).$$

(ii) If $f : Y \to X$ is a proper morphism, X irreducible and reduced of dimension m and $\dim Y \leqq m$, and if ξ is the generic point of X,

$$d_m(f^*(\mathscr{L})) = \Big(\sum_{f(\eta)=\xi} \dim_{k(\xi)} \mathscr{O}_{Y,\eta}\Big) \cdot d_m(\mathscr{L}).$$

We can now state the main theorem.

Theorem. *Let X be a noetherian scheme of dimension n and Y a closed subscheme of X defined by the coherent sheaf of ideals $\mathscr{I}$. Suppose Y is a proper scheme over Spec Λ where Λ is an artinian ring. Suppose further that $f : X' \to X$ is a proper birational morphism such that $\mathscr{I}\mathscr{O}_{X'}$ is an invertible sheaf of ideals. Then we have:*

(i) there exists a polynomial $P(T)$ of degree $\leqq n-1$ such that for all large N, **121**

$$P(N) = \sum_{i=0}^{n} (-1)^i l_\Lambda(H^i(X, \mathscr{I}^N/\mathscr{I}^{N+1})),$$

(ii) the coefficient of T^{n-1} in $P(T)$ is $\dfrac{1}{(n-1)!} \cdot d_{n-1}(\mathscr{I}\mathscr{O}_{X'})$.

Proof. Let $g : \bar{X} \to X$ be the blow-up of X with respect to the sheaf of ideals $\mathscr{I}$. Then $\mathscr{I}\mathscr{O}_{\bar{X}}$ is an invertible sheaf relatively very ample for g, hence $\mathscr{I}\mathscr{O}_{\bar{X}}/\mathscr{I}^2\mathscr{O}_{\bar{X}}$ is relatively very ample for the morphism $g^{-1}(Y) = Y \times_X \bar{X} \to Y$. It follows that for N large $R^i g(\mathscr{I}^N\mathscr{O}_{\bar{X}}/\mathscr{I}^{N+1}\mathscr{O}_{\bar{X}}) = (0)$ for $i > 0$ and $g_*(\mathscr{I}^N\mathscr{O}_{\bar{X}}/\mathscr{I}^{N+1}\mathscr{O}_{\bar{X}}) \simeq \mathscr{I}^N/\mathscr{I}^{N+1}$. Hence from the Leray spectral sequence, for N large we have

$$\sum_{i=0}^{n} (-1^i) l_\Lambda(H^i(X, \mathscr{I}^N/\mathscr{I}^{N+1})) = \sum_{i=0}^{n} (-1)^i l_\Lambda(H^i(Y \times_X \bar{X}, \mathscr{I}^N\mathscr{O}_{\bar{X}}/\mathscr{I}^{N+1}\mathscr{O}_{\bar{X}}))$$

and the right side is a polynomial of degree $\leqq \dim Y \times_X \bar{X} < \dim \bar{X} \leqq n$ ([1]). This proves (i), and also (ii) for the special morphism g.

Now, f factorises as $g \circ h$ where $h : X' \to \bar{X}$ is birational proper. We have to show that $d_{n-1}(\mathscr{I}\mathscr{O}_{X'}) = d_{n-1}(\mathscr{I}\mathscr{O}_{\bar{X}})$. Since $\mathscr{I}\mathscr{O}_{\bar{X}}$ is an invertible sheaf of ideals, we have $h^*(\mathscr{I}\mathscr{O}_{\bar{X}}) \simeq \mathscr{I}\mathscr{O}_{X'}$. Put $\bar{Y} = Y \times_X \bar{X}$, $Y' = Y \times_X X'$; in view of the preceding remarks, it suffices to show that if η is the generic point of an irreducible component of $\bar{Y}$ of dimension $n-1$, we have

$$l(\mathscr{O}_{\bar{Y},\eta}) = \sum_{\substack{\eta' \in Y' \\ h(\eta')=\eta}} l_{\mathscr{O}_{\bar{Y},\eta}}(\mathscr{O}_{Y',\eta'}).$$

Set $A = \mathscr{O}_{\bar{X},\eta}$. Then the morphism $X' \times_X \operatorname{Spec} A \to \operatorname{Spec} A$ is proper with finite fibers, hence is a finite morphism. Thus, $X' \times_X \operatorname{Spec} A$ is isomorphic over $\operatorname{Spec} A$ to $\operatorname{Spec} B$ where B is an A-algebra which is a finite A-module. Let $f \in A$ be a generator of $(\mathscr{I}\mathscr{O}_{\bar{X}})_\eta$. Then $\mathscr{O}_{\bar{Y},\eta} \simeq A/fA$ and

$$\prod_{\substack{\eta' \in Y' \\ h(\eta')=\eta}} \mathscr{O}_{Y',\eta'} \simeq B/fB.$$

We have thus to show that $l_A(A/fA) = l_A(B/fB)$. Assume for the moment that $A \to B$ is injective and $l_A(B/A) < \infty$. Then we have

$$l_A(B/fA) = l_A(B/fB) + l_A(fB/fA) = l_A(B/A) + l_A(A/fA).$$

But since f is a non-zero divisor in B, multiplication by f induces an isomorphism $B/A \simeq fB/fA$ and $l_A(B/A) = l_A(fB/fA)$. Inserting this in the above equality, the desired result follows.

To prove that $A \to B$ is an injection and $l_A(B/A) < \infty$, let K be the product of the (artinian) local rings of generic points of those components of X which contain η and L the product of the (artinian) local
122 rings of generic points of those components of X which contain a point of $h^{-1}(\eta)$. The natural homomorphisms $A \to K$ and $B \to L$ are injection in view of the remarks made earlier. Further, $K \to L$ is an isomorphism since h is birational. Hence B can be identified with a finitely generated submodule of the total quotient ring K of A. Hence there is a non-zero

divisor $\lambda \in A$ such that $\lambda B \subset A$, and $B/A \simeq \lambda B/\lambda A \hookrightarrow A/\lambda A$ which is of finite length over A.

The proof of the theorem is complete. □

Remarks. (1) Suppose A is a local ring of dimension $n \geqq 1$ and $\mathcal{Q}$ an ideal primary for the maximal ideal. The theorem is then applicable to $\operatorname{Spec} A$, Y the closed subscheme defined by $\mathcal{Q}$ (and $\Lambda = A/\mathcal{Q}$). The polynomial $P(N)$ then equals $l_A(\mathcal{Q}^N/\mathcal{Q}^{N+1})$ for N large, and its leading coefficient is $\dfrac{1}{(n-1)!} e_{\mathcal{Q}}(A)$, where $e_{\mathcal{Q}}(A)$ is the multiplicity of $\mathcal{Q}$.

Suppose now that $\mathcal{Q}, \mathcal{Q}'$ are two ideals primary for the maximal ideal and $g : X_1 \to X$ a proper birational morphism such that $\mathcal{Q}\mathcal{O}_{X_1} = \mathcal{Q}'\mathcal{O}_{X_1}$. We can then find $h : X' \to X_1$ proper birational such that $\mathcal{Q}\mathcal{O}_{X'} = \mathcal{Q}'\mathcal{O}_{X'}$ is invertible. (Take for instance X' to be the blow-up of X_1 with respect to $\mathcal{Q}\mathcal{O}_{X_1}$.) It follows from the theorem applied to $g \circ h$ that $e_{\mathcal{Q}}(A) = e_{\mathcal{Q}'}(A)$. Now recall that if $\mathcal{Q}' \supset \mathcal{Q}$, $\mathcal{Q}'$ is said to be integral over $\mathcal{Q}$ if every x in $\mathcal{Q}'$ satisfies an equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with $a_i \in \mathcal{Q}^i$ (see [3]). We assert that if $\mathcal{Q}'$ is integral over $\mathcal{Q}$, there is a $g : X_1 \to X$ proper birational such that $\mathcal{I}\mathcal{O}_{X_1} = \mathcal{I}'\mathcal{O}_{X_2}$. Infact, choose X_1 so that $\mathcal{Q}\mathcal{O}_{X_1}$ and $\mathcal{Q}'\mathcal{O}_{X_1}$ are both invertible. For any $x_1 \in X_1$, we have

$$\mathcal{Q}\mathcal{O}_{X_1,x_1} = \lambda\mathcal{O}_{X_1,x_1} \subset \mathcal{Q}'\mathcal{O}_{X_1,x_1} = \mu\mathcal{Q}_{X_1,x_1}$$

with $\lambda, \mu \in \mathcal{Q}_{X_1,x_1}$, $\lambda = \mu\nu$, $\nu \in \mathcal{O}_{X_1,x_1}$. Now μ must be integral over $\lambda\mathcal{O}_{X_1,x_1}$, hence satisfies an equation

$$\mu^m + a_1\mu\nu \cdot \mu^{m-1} + a_2(\mu\nu)^2\mu^{m-2} + \cdots + a_m(\mu\nu)^m = 0, \quad a_i \in \mathcal{O}_{X_1,x_1},$$

i.e.,

$$\mu^m(1 + a_1\nu + a_2\nu^2 + \cdots + a_m\nu^m) = 0.$$

Since μ is not a zero divisor, it follows that $1 + \nu(a_1 + a_2\nu + \cdots + a_m\nu^{m-1}) = 0$, and ν is a unit.

Thus, if $\mathcal{Q}'$ is integral over $\mathcal{Q}$, $e_{\mathcal{Q}}(A) = e'_{\mathcal{Q}}(A)$. This is a result due to Northcott and Rees ([4]).

(2) Let the assumptions be as in the theorem. Suppose further that Y is a local complete intersection of codimension r in X, i.e., suppose for every $y \in Y$, $\mathscr{I}_y$ is generated by an $\mathscr{O}_{X,y}$-sequence of length r. Then $\mathscr{I}/\mathscr{I}^2$ is locally free of rank r on Y and $\oplus_{m\geqq 0}\mathscr{I}^m/\mathscr{I}^{m+1}$ is isomorphic to the symmetric algebra $S(\mathscr{I}/\mathscr{I}^2)$ over $\mathscr{O}_Y$ (see [5]). Then $\chi(\mathscr{I}^N/\mathscr{I}^{N+1}) = P(N)$ where P is a polynomial, for all $N \geqq 0$.

123 Now, suppose further that a theory of Chern classes is defined on Y with values in a graded ring $A(Y)$ with the usual properties. (For instance, $\Lambda = \mathbb{C}$, Y reduced and $A(Y)$ is the even dimensional integral cohomology of $Y(\mathbb{C})$; or Λ an algebraically closed field, Y non-singular projective over Λ and $A(Y)$ the Chow ring of rational equivalence; or any Λ, and $A(Y)$ is the associated graded ring of $K(Y)$ with respect to the λ-filtration, tensored with $\mathbb{Q}$.) Since $\dim Y \leqq n-r$, we have a homomorphism $A^{n-r}(Y) \to \mathbb{Q}$ which we denote by $\xi \mapsto \xi[Y]$. There is then a universal polynomial Q (depending only on n and r) in the Chern classes c_i of $\mathscr{I}/\mathscr{I}^2$ such that the coefficient of N^{n-1} in $P(N)$ equals $Q(c_i)[Y]$.

These are general assertions having to do with a locally free sheaf ξ ($(\mathscr{I}/\mathscr{I}^2)$ in our case) of rank r and its symmetric powers on a scheme Y of dimension m proper over $\operatorname{Spec}\Lambda$, Λ an artinian ring. To prove them, let $\mathbb{P}(\xi^*)$ be the projective bundle associated to the dual sheaf ξ^* on Y, and $\mathscr{O}(1)$ the canonical invertible sheaf on $\mathbb{P}(\xi^*)$. If $\pi : \mathbb{P}(\xi^*) \to Y$ is the projection, we have $R^i\pi(\mathscr{O}(N)) = (0)$, $i > 0$, $N \geqq 0$ and $\pi_*(\mathscr{O}(N)) \simeq S^N(\xi)$. Hence by the Leray spectral sequence, $\chi(Y, S^N(\xi)) = \chi(\mathbb{P}(\xi^*), \mathscr{O}(N))$, which shows that $\chi(Y, S^N(\xi))$ is a polynomial in N. Further, if ξ is the first Chern class of $\mathscr{O}(1)$ on $\mathbb{P}(\xi^*)$, the coefficient of N^{r+m-1} in this polynomial is $\dfrac{1}{(r+m-1)!}\xi^{r+m-1}[\mathbb{P}(\xi^*)]$. Now, ξ satisfies an equation $\xi^r - c_1\xi^{r-1} + \cdots \pm c_r = 0$, where c_i are the Chern classes of $\mathscr{E}$ and $A(\mathbb{P}(\mathscr{E}^*))$ is considered as an $A(Y)$-algebra via π. Dividing ξ^{r+m-1} by $\xi^r - c_1\xi^{r-1} + \cdots \pm c_r$ leaves a remainder

$$\alpha_0 + \alpha_1\xi + \cdots + \alpha_{r-1}\xi^{r-1}$$

where the α_i are universal polynomials in $c_1, \ldots, c_r$. Further, by the

projection formula, $\alpha_i \xi^i[\mathbb{P}(\mathscr{E}^*)] = 0$ for $i < r-1$ and $\alpha_{r-1}\xi^{r-1}[\mathbb{P}(\mathscr{E}^*)] = \alpha_{r-1}[Y]$. Thus, we may take $Q = \dfrac{1}{(r+m-1)}\alpha_{r-1}$.

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Supplement to the Article "Remarks on the Kodaira Vanishing Theorem"

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Throughout this Paper, we work over an algebraically closed field of characteristic zero, which we may assume is the field of complex numbers. We adhere to the notations of the paper referred to in the title. 125

Our first remark was pointed out to us by E. Bombieri, and is used by him in his investigations of pluri-canonical surfaces.

If X is a complete non-singular surface and D an effective divisor on X with $(D^2) > 0$, then

$$H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$$

is injective.

Proof. By the Riemann-Roch theorem, $\dim H^0(X, \mathcal{O}_X(nD))$ increases quadratically in n. Hence for some $n > 0$, if F is the divisor of base components of $|nD|$, $|nD| - F$ is a linear system without base components not composite with a pencil. We then have, in the terminology of Lemma 6 of (2),

$$\alpha(D) = \alpha(nD) \leqslant \alpha(nD - F) = 0$$

by Theorem 2 and Lemma 6 of (2).

Our next result is the following □

*C. P. Ramanujam suddenly passed away in Bangalore on October 27, 1974 (Ed.)

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Theorem. *Let X be a complete non-singular surface and $\mathcal{L}$ an invertible sheaf on X such that $(c_1(\mathcal{L})^2) > 0$ and $(c_1(\mathcal{L}) \cdot C) \geqslant 0$ for any curve C on X. Then $H^i(X, \mathcal{L}^{-1}) = 0$, $i = 0, 1$.*

126 *Conversely, if $(c_1(\mathcal{L})^2) > 0$ and $H^i(X, \mathcal{L}^{-n}) = 0$, $i = 1, 2$ and n large, for any curve C on X, $(c_1(\mathcal{L}) \cdot C) \geqslant 0$.*

Proof. We first prove the sufficiency of the condition.

Suppose first that $H^0(X, \mathcal{L}) \neq 0$, so that $\mathcal{L} \simeq \mathcal{O}_X(D)$, D an effective divisor. We show that D is numerically connected. Suppose on the contrary that $D = D_1 + D_2$, D_i effective, $(D_1 \cdot D_2) = -\lambda \leqslant 0$. Since $(D_1^2) \pm (D_1 \cdot D_2) = (D \cdot D_1) \geqslant 0$ by assumption, $(D_1^2) \geqslant \lambda$ and similarly $(D_2^2) \geqslant \lambda$. Now, if D_1 and D_2 are linearly dependent modulo numerical equivalence, we must have aD_1 numerically equivalent to bD_2 where a and b are positive integers, since D_i are effective. But then, since $(D^2) > 0$, $(D_1 \cdot D_2) > 0$. Thus, D_i are independent modulo numerical equivalence, and by the Hodge index theorem,

$$\det \begin{pmatrix} (D_1^2) & -\lambda \\ -\lambda & (D_2^2) \end{pmatrix} = (D_1^2)(D_2^2) - \lambda^2 < 0,$$

which is a contradiction. Thus, D is numerically connected. But now, the assertion follows from our earlier remark, Lemma 3 of (2) and the exact sequence

$$H^0(X, \mathcal{O}_X) \to H^0(D, \mathcal{O}_D) \to H^1(X, \mathcal{L}^{-1}) \to H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D).$$

Next, consider the case when $\mathcal{L}$ does not admit a non-zero section. Our hypotheses clearly imply that for H ample, $(c_1(\mathcal{L}) \cdot H) > 0$, and hence $H^0(X, \Omega_X^2 \otimes \mathcal{L}^{-n}) = 0$ for n large. Hence, by Riemann-Roch, there is an $n > 0$ such that $\mathcal{L}^n$ admits a non-zero section σ with $\mathrm{div}\sigma = D$. By Lemma 1 of (2) and the theorem of resolution of singularities, we can find a complete non-singular surface Y and a surjective morphism $f : Y \to X$ such that $f^*(\mathcal{L})$ admits a section τ with $\tau^n = f^*(\sigma)$ in $f^*(\mathcal{L}^n)$. For any curve C' on Y, we have

$$(c_1(f^*(\mathcal{L})) \cdot C') = (f^*(c_1(\mathcal{L})) \cdot C') = (c_1(\mathcal{L}) \cdot f^*(C')) \geqslant 0,$$

so that $f^*(\mathcal{L})$ on Y satisfies the hypothesis and also admits a non-zero section. Hence, by the first part and Serre duality,

$$H^i(Y, \Omega_Y^2 \otimes f^*(\mathcal{L})) = 0, \ i > 0.$$

Now, by Lemma 4 of (2) and Stein factorisation,

$$R^i f(\Omega_Y^2) = 0, \quad i > 0.$$

Thus, by the Leray spectral sequence, **127**

$$H^i(X, f_*(\Omega_Y^2) \otimes \mathcal{L}) = 0, \quad i > 0.$$

We have a splitting $1/m \quad Tr : f_*(\Omega_Y^2) \to \Omega_X^2 (m = \deg f)$ of the natural homomorphism $\Omega_X^2 \to f_*(\Omega_Y^2)$ (See footnote on p.44 of (2)), which gives

$$H^i(X, \Omega_X^2 \otimes \mathcal{L}) = 0, \quad i > 0,$$

and the result follows by Serre duality.

We next prove the converse part of the theorem. If H is ample, $(H \cdot c_1(\mathcal{L})) \neq 0$, by the Hodge index theorem. By our hypothesis and Riemann-Roch, $H^0(X, \Omega_X^2 \otimes \mathcal{L}^n) \neq 0$ for n large, so that $(H \cdot (K + nc_1(\mathcal{L}))) > 0$ for n large. Hence $(H \cdot c_1(\mathcal{L})) > 0$ and $H^0(K \otimes \mathcal{L}^{-n}) = 0$ for n large. Hence $(H \cdot c_1(\mathcal{L})) > 0$ and $H^0(K \otimes \mathcal{L}^{-n}) = 0$ for n large. Again by Riemann-Roch, $H^0(\mathcal{L}^n) \neq 0$ for some $n > 0$, and replacing $\mathcal{L}$ by $\mathcal{L}^n$, we may assume $H^0(\mathcal{L}) \neq 0$, $\mathcal{L} \simeq \mathcal{O}_X(D)$ for some effective divisor D.

From the cohomology exact sequence of the short exact sequence

$$0 \to \mathcal{O}_X(-(n+1)D) \xrightarrow{D} \mathcal{O}_X(-nD) \to \mathcal{O}_D \otimes \mathcal{O}_X(-nD) \to 0$$

we deduce that $H^0(\mathcal{O}_D \otimes \mathcal{O}_X(-nD)) = 0$ for n large. We can clearly find a morphism $f : \tilde{D} \to D$ such that (i) f is finite, (ii) every connected component of $\tilde{D}$ is irreducible, and (iii) there is a finite set S of points on D such that $f | f^{-1}(D - S)$ in an isomorphism onto $D - S$. We then have an exact sequence

$$0 \to \mathcal{O}_D \to f_*(\mathcal{O}_{\tilde{D}}) \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a sheaf supported at finitely many points. We deduce from this that

$$\dim H^0(\tilde{D}, f^*(\mathcal{O}_X(-nD))) = \dim H^0(D, f_*(\mathcal{O}_{\tilde{D}}) \otimes \mathcal{O}_X(-nD))$$

is bounded. Hence for any component C of D, we must have $(D \cdot C) \geqslant 0$. But for a curve C' which is not a component of D, the inequality $(D \cdot C') \geqslant 0$ is obviously satisfied. Q.E.D. □

Remarks. (1) Using the Lemma of Enriques-Severi-Zariski-Serre as in the proof of Theorem 2 of (2), it is easy to deduce from the above theorem the following generalisation to higher dimensional varieties:

If X non-singular projective of dimension n and $\mathcal{L}$ an invertible
128 *sheaf on X with $(c_1(\mathcal{L})^n) > 0$ and $(c_1(\mathcal{L}) \cdot C) \geqslant 0$ for any curve C on X, then $H^1(X, \mathcal{L}^{-1}) = 0$.*

(2) The assumptions that D effective, $(D^2) > 0$ and $(D \cdot C) \geqslant 0$ do not imply that $|nD|$ has no base components for some $n > 0$, as is shown by the following example:

Let C be an elliptic curve, and X the projective line bundle on C obtained by compactifying a line bundle $\mathcal{L}$ of degree-1 on C by adding points at infinity to the fibers. Let D_1 be the zero section and F the fiber over a point P of C. Then,

$$((D_1 + F)^2) = (D_1^2) + 2(D_1 \cdot F) = -1 + 2 = 1.$$

If C is a curve not a component of $D_1 + F$, we have evidently $(C \cdot (D_1 + F)) > 0$. On the other hand, $((D_1 + F) \cdot F) = 1$ and $((D_1 + F) \cdot D_1) = (D_1^2) + (D_1 \cdot F) = -1 + 1 = 0$. If some linear system $|n(D_1 + F)|$ does not have D_1 for a base curve, we deduce that $\mathcal{O}_{D_1} \otimes \mathcal{O}_X(n(D_1 + F))$ is trivial on D_1, i.e., $\mathcal{L}^n \otimes \mathcal{O}_C(nP)$ is trivial on C. But we can choose P such that $\mathcal{L} \otimes \mathcal{O}_C(P)$ is not of finite order. Then D_1 is a base component of each of the linear systems $|n(D_1 + F)|$.

Our theorem is therefore strictly stronger than Theorem 2 of (2).

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Appendix

By C.P. Ramanujam

The proof of Prop. 3 has been essentially global. The following result gives a "local proof" of Prop. 3. **130**

Proposition *. *Let A be a noetherian analytically normal local ring with maximal ideal $\mathfrak{m}$. Let B be a noetherian local ring with maximal ideal $\mathfrak{n}$ such that B contains A and has the same residue field as A. Let us suppose that $\mathfrak{n} \cap A = \mathfrak{m}$ and that $\mathfrak{n}$ is generated by $\mathfrak{m}$ together with elements $x_1, \ldots, x_r$ of B, where $r = \dim B - \dim A$.*

Then any prime ideal $\mathfrak{p}$ of height one in B which is not contained in $\mathfrak{m} \cdot B$ is principal.

Proof. Let A and B be the completions of A and B respectively. The hypothesis that $\mathfrak{n}$ is generated by $\mathfrak{m}$ together with $r = \dim B - \dim A$ elements on $\mathfrak{n}$ implies that B is isomorphic to the formal power series ring $A\left[[X_1, \ldots, X_r]\right]$ in r variables over A, as can be shown easily. Since $\mathfrak{p}$ is of height one in B, there exist elements $a, b \in B$ such that $(a : b) = \mathfrak{p}$, and hence by a well-known property of completions (B is B flat), we deduce that $(aB : bB) = \mathfrak{p}B = \mathfrak{p}$, which implies in turn (since $\hat{B}$ is normal) that all the associated prime ideals of $\mathfrak{p}B$ in B are of height one. Moreover, since $\mathfrak{p} \not\subset \mathfrak{m} \cdot B$, $\mathfrak{p} \not\subset \mathfrak{m} \cdot B = \mathfrak{m} \cdot B$, and the same holds for all the associated prime ideals of $\mathfrak{p}$. If we could prove that each of these associated prime ideals is principal, it would then follow that $\mathfrak{p}B$ is principal, and hence that $\mathfrak{p}$ is principal (since B is B-flat).

*This proposition could be used to prove straightaway that the Weil divisor C in Prop. 4 is a divisor; thereby one could avoid going to the variety Z for the proof of Prop. 4.

Thus, it is enough to prove the theorem when A is a complete normal local domain, and B a formal power series ring $A\left[[X_1, \ldots, X_r]\right]$ in r variables over A. □

We shall do this in a series of lemmas.

Lemma 1. *Let A be a local ring with maximal ideal $\mathfrak{m}$, and f an element of $A\left[[X_1, \ldots, X_p]\right]$ which does not lie in the ideal $\mathfrak{m}A\left[[X_1, \ldots, X_p]\right]$. Then there is an automorphism φ of $A\left[[X_1, \ldots, X_p]\right]$ which fixes A such that $\varphi(f) \notin (\mathfrak{m}, X_1, \ldots, X_{p-1})$.*

Proof. The proposition being trivial for $p = 1$, we may assume that $p \geqq 2$. We proceed by induction on p.

Let $p = 2$ and $f = \sum_{r=s} a_{r,s} X_1^r X_2^s$, and let a_{r_0,s_0} be a coefficient which is not in $\mathfrak{m}$. We may assume that r_0 is minimal with this property. Let N be any integer greater than S_0, and define

$$\varphi(X_1) = X_1 + X_2^N$$
$$\varphi(X_2) = X_2.$$

Then φ extends to a unique automorphism of $A[[X_1, X_2]]$, which fixes A, by continuity. The coefficient of $X_2^{r_0N+S_0}$ in $\varphi(f)$ is the sum of the $a_{r,s}$ for which $rN + s = r_0N + s_0$. This implies that N divides $s - s_0$, and since s and s_0 are non-negative and $N > s_0$, that $s \geqq s_0$, and $r \leqq r_0$. By the minimality of r_0, we see that those $a_{r,s}$ which occur in the sum with $(r, s) \neq (r_0, s_0)$ are in $\mathfrak{m}$, and since $a_{r_0.s_0} \notin \mathfrak{m}$, the coefficient of $X_2^{r_0N+s_0}$ in $\varphi(f)$ is not in $\mathfrak{m}$.

131 Now let p be any integer > 2 and assume that the lemma is valid for $p - 1$ variables. Writing $f = \sum_{q=0} f_q X_p^q$, with $f_q \in A\left[[x_1, \ldots, X_{p-1}]\right]$, we see that there is an $f_{q_0} \notin \mathfrak{m}A\left[[X_1, \ldots, X_{p-1}]\right]$, and by induction hypothesis, there is an automorphism φ' of $A\left[[X_1, \ldots, X_{p-1}]\right]$ such that $\varphi'(f_{q_0}) \notin (\mathfrak{m}, X_1, \ldots, X_{p-2})$. Extending φ' to $A\left[[X_1, \ldots, X_p]\right]$ by putting $\varphi'(X_p) = X_p$ (and continuity), we see that $\varphi'(f) \in A\left[[X_1, \ldots, X_{p-2}]\right]\left[[X_{p-1}, X_p]\right]$ and $\varphi'(f) \notin (\mathfrak{m}, X_1, \ldots, X_{p-2})$. Since we have proved the lemma for $p =$

2, there is an automorphism φ'' of $A\left[[X_1, \ldots, X_p]\right]$ such that $\varphi''(\varphi'(f)) \notin (\mathfrak{m}, X_1, \ldots, X_{p-2}, X_{p-1})$, and $\varphi = \varphi'' \circ \varphi'$ fulfills the requirements. □

Lemma 1 clearly reduces the proof of the proposition to the case when $B = A[[X]]$ and A are before. We shall assume this from now on.

Lemma 2. *Let A be a complete local ring with maximal ideal $\mathfrak{m}$ and $f \in A[[X]]$, $f \notin \mathfrak{m}[[X]]$. Let r be the least integer such that $f_r \notin \mathfrak{m}\left(f = \sum_0^\infty f_r X^r\right)$. If $g \in A[[X]]$ there exists a unique $h \in A[[X]]$ such that*

$$g = hf + R,$$
$$R = \sum_0^{r-1} r_i X^i,$$

In particular, f is the associate of a unique polynomial $X^r + \alpha_1 X^{r-1} + \ldots + \alpha_r$, $\alpha_i \in \mathfrak{m}$.

Proof. The proof of the Weierstrass preparation theorem as given in Zariski and Samuel (Vol. II, Commutative Algebra, p. 139) carries over almost word for word. □

Lemma 3. *A is as in Lemma 2, $\mathfrak{a}$ an ideal of $A[[X]]$, $\mathfrak{a} \not\subset \mathfrak{m}[[X]]$. Then $\mathfrak{a}$ is generated by $\mathfrak{a} \cap A[X]$.*

Proof. Let f be any element of $\mathfrak{a}$ which is not in $\mathfrak{m}[[X]]$, and r as in Lemma 1. If g is any element of $\mathfrak{a}$, it follows from Lemma 1 that $g = hf + R$, $R \in A[X]$, and f is the associate of an element of $A[X]$. Lemma 3 is proved.

We can now complete the proof of the proposition. By Lemma 3 it is enough to show that $\mathfrak{p} \cap A[X] = \mathfrak{p}_1$, is a principal ideal of $A[X]$. Let S be the set of non-zero elements of A; then $A[X]_S = Q[X]$, Q being the quotient field of A. We assert that $S \cap \mathfrak{p}_1 = \emptyset$. In fact, if $\alpha \in \mathfrak{p}_1 \cap S$, $\mathfrak{p}_1$ (being of height one) must be associated to $\alpha A[X]$. But since $\alpha \in A$, the associated prime ideals of α in $A[X]$ are all the extensions to $A[X]$ of the

prime ideals associated to α in A, and all these extensions are $\subset \mathfrak{m}[X]$. This contradicts the fact that $\mathfrak{p}_1 \not\subset \mathfrak{m}[X]$.

It follows that $\mathfrak{p}_1 Q[X]$ is a prime ideal of $Q[X]$. Let $f = X^n + a_1 X^{n-1} + \cdots + a_n, a_i \in Q$ be the monic polynomial which generates $\mathfrak{p}_1 Q[X]$. We assert that $a_i \in A$. In fact, let $\bar{X}$ be the image of X in $Q[X]/\mathfrak{p}_1 Q[X] = Q[\bar{X}]$. Since $\mathfrak{p} \not\subset \mathfrak{m}[[X]]$, it follows from Lemma 2 that there is a monic polynomial over $A[X]$ which is in $\mathfrak{p}$, and hence in $\mathfrak{p}_1$, this proves that $\bar{X}$ is integral over A. But since A is integerally
132 closed, the minimal polynomial of $\bar{X}$ over Q, i.e., f, must be in $A[X]$, and our assertion is proved. But now, since $\mathfrak{p}_1$ is prime, we must have $\mathfrak{p}_1 Q[X] \cap A[X] = \mathfrak{p}_1(A[X])_S \cap A[X] = \mathfrak{p}_1$, and $f \epsilon \mathfrak{p}_1$. Since f is monic, and divides any element of $\mathfrak{p}_1$ in $Q[X]$, it does so in $A[X]$. □

Our result is prove.

Added in proof. The proposition in the Appendix has been proved in a special case by P. Samuel (Sur une conjecture de Grothendieck, Comptes Rendus, Paris, Tome 255).

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Appendix I
The Theorem of Tate

By C.P. Ramanujam

We have seen that if X and Y are abelian varieties over a field k (algebraically closed) and l a prime different from the characteristic of k, the natural homomorphism **134**

$$\mathbf{Z}_l \otimes_{\mathbf{Z}} \mathrm{Hom}(X, Y) \xrightarrow{T_1} \mathrm{Hom}_{\mathbf{Z}_l}(T_l(X), T_l(Y)) \tag{1}$$

is injective. It is obviously of importance to know the image.

By a *field of definition* of X, we shall mean a subfield k_0 of k over which there is a *group scheme* X_0 and an isomorphism of *group schemes* $X_0 \otimes_{k_0} k \xrightarrow{\sim} X$. We also say that X is an abelian variety defined over k_0. Now choose a common field of definition k_0 for X and Y; we may and shall assume k_0 to be of finite type over the prime field. Let $\bar{k}_0$ be the algebraic closure of k_0 in k. Since n_X and $n_Y (n \in \mathbf{Z})$ are morphisms defined over k_0, their respective kernels X_n and Y_n consist entirely of $\bar{k}_0$-rational points, or in other words, the points of finite order in X and Y are $\bar{k}_0$-rational (i.e., they come from $\bar{k}_0$-rational points of $X_0 \otimes_{k_0} \bar{k}_0$ and $Y_0 \otimes_{k_0} \bar{k}_0$ resp.) Thus, if $G = G(\bar{k}_0/k_0)$ is the Galois group of k_0 over k_0, G acts on the groups X_n and Y_n compatible with the homomorphisms $X_{mn} \xrightarrow{m} X_n$ and $Y_{mn} \xrightarrow{m} Y_n$. Hence G acts continuously on the $\mathbf{Z}_l$-modules $T_l(X)$ and $T_l(Y)$, where we give G the Krull topology and $T_l(X)$ and $T_l(Y)$ their l-adic topologies.

Furthermore, let φ be any homomorphism of X into Y. Since the points of finite order in X are $\bar{k}_0$-rational and k-dense, and φ maps points of finite order into points of finite order, φ is defined over $\bar{k}_0$ and hence over a finite extension k_1 of k_0 in $\bar{k}_0$ (i.e., comes by base extension from

a homomorphism of group schemes $X_0 \otimes_{k_0} k_l \to Y_0 \otimes_{k_0} k_1$). It then follows that if $H = G(\bar{k}_0/k_1) \subset G$ is the subgroup of G fixing k_1, for any $\bar{k}_0$-rational point x of X, we have $h\varphi(x) = \varphi(hx)$ for all $h \in H$. Thus, if we make G act on $\mathrm{Hom}_{Z_l}(T_l(X), T_l(Y))$ by putting $(g\lambda)(x) = g(\lambda(g^{-1}x))$ for $g \in G$, $\lambda \in \mathrm{Hom}_{\mathbf{Z}}(T_l(X), T_l(Y))$ and $x \in T_l(X)$, we see that $hT_l(\varphi) = T_l(\varphi)$ for $h \in H$. In other words, for any $\varphi \in \mathrm{Hom}(X, Y)$, there is a neighborhood H of the identity in G fixing $T_l(\varphi)$. We see therefore that if for any G-module M we denote by $M^{(G)}$ the subgroup of elements fixed by some neighborhood of e, the image of (1) is contained in $\mathrm{Hom}_{\mathbf{Z}_l}(T_l(X), T_l(Y))^{(G)}$, and we obtain a homomorphism

$$\mathbf{Z}_l \otimes_{\mathbf{Z}} \mathrm{Hom}(X, Y) \to \mathrm{Hom}_{\mathbf{Z}_l}(T_l(X), T_l(Y))^{(G)}. \tag{2}$$

Tate *conjectures* that (2) is an isomorphism (or equivalently (2) is surjective) for any field of definition k_0 of finite type over the prime field. He proves this for abelian varieties defined over a finite field. It has also been proved when k_0 is an algebraic number field and $X = Y$ is of dimension one (Serre). We shall now reproduce Tate's proof in the case of finite fields.

Theorem 1. *Let X and Y be abelian varieties defined over a finite field k_0. Then the homomorphism ((2)) is an isomorphism.*

The rest of this section will be devoted to the proof, which we give in several steps.

Step I. *It suffices to show that the homomorphism*

$$\mathbf{Q}_l \otimes_{\mathbf{Z}} \mathrm{Hom}(X, Y) = \mathbf{Q}_l \otimes_{\mathbf{Q}} \mathrm{Hom}^0(X, Y) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_{\mathbf{Q}_l}(V_l(X), V_l(Y))^{(G)} \tag{3}$$

where $V_l(X) = \mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(X)$, is bijective.

In fact, if this were so, the image of the homomorphism (1) would be a $\mathbf{Z}_l$-submodule of maximal rank in $\mathrm{Hom}_{\mathbf{Z}_l}(T_l(X), T_l(X))^{(G)}$. On the other hand, if this image is M, $\mathrm{Hom}_{\mathbf{Z}_l}(T_l(X), T_l(X))/M$ has no torsion; for, suppose $\varphi \in \mathrm{Hom}_{\mathbf{Z}_l}(T_l(X), T_l(X))$ and $l_\varphi \in M$. We can find $\psi_n \in$

$\mathrm{Hom}(X, Y)$ such that $T_l(\psi_n) \to l_\varphi$, so that for all large n, $T_l(\psi_n) = l_{\varphi_n, \varphi_n} \in \mathrm{Hom}_{\mathbf{Z}_l}(T_l(X), T_l(Y))$. Thus, $T_l(\psi_n)(T_l(X)) \subset lT_l(Y)$ and ψ_n vanishes on X_l. Thus ψ_n admits a factorisation $\psi_n = \chi_n \circ l_X = l\chi_n$ and the χ_n converge to a certain χ in $\mathbf{Z}_l \otimes_{\mathbf{Z}} \mathrm{Hom}(X, Y)$ with $T_l(\chi) = \varphi$, so that $\varphi \in M$. This proves our assertion, and establishes that $M = \mathrm{Hom}_{\mathbf{Z}_l}(T_l(X), T_l(Y))^{(G)}$.

Step II. *It suffices to show that for any abelian variety X defined over a* **136**
finite field

$$\mathbf{Q}_l \otimes_{\mathbf{Q}} \mathrm{End}^0(X) \xrightarrow{\lambda} \mathrm{End}_{\mathbf{Q}_l}(V_l(X))^{(G)} \tag{4}$$

is an isomorphism.

In fact, if (4) is an isomorphism with $X \times Y$ instead of X, since we have the direct sum decompositions as G-modules

$$\begin{array}{ccccccccc}
\mathbf{Q}_l \otimes \mathrm{End}^\circ(X \times Y) & \xleftarrow{\sim} & (\mathbf{Q}_l \otimes_{\mathbf{Q}} \mathrm{End}^0 X) & \oplus & (\mathbf{Q}_l \otimes_{\mathbf{Q}} \mathrm{End}^0 Y) & \oplus (\mathbf{Q}_l \otimes_{\mathbf{Q}} \mathrm{Hom}^0(X, Y)) & \oplus & (\mathbf{Q}_l \otimes_{\mathbf{Q}} \mathrm{Hom}^0(Y, X)) \\
\downarrow & & \downarrow & & \downarrow & \downarrow & & \downarrow & \\
\mathrm{End}(V_t(X \times Y)) & \xleftarrow{\sim} & \mathrm{End}(V_t(X)) & \oplus & \mathrm{End}(V_l(Y)) & \oplus \mathrm{Hom}(V_l(X), V_l(Y)) & \oplus & \mathrm{Hom}(V_l(Y), V_l(X)), &
\end{array}$$

the above diagram is commutative and the first vertical arrow is an isomorphism, (3) is an isomorphism.

Thus, henceforward we shall restrict ourselves to a single abelian variety X defined over a finite field, and prove that (4) is an isomorphism.

Step III. *It suffices to show that λ is an isomorphism for one l and that* $\dim_{\mathbf{Q}_l} \mathrm{End}(V_l(X))^{(G)}$ *is independent of* $1 \neq \mathrm{char.}\ k$.

This follows from the fact that the dimension of the left member in (4) is independent of l and λ is always injective.

Step IV. *To establish* (4) *for an l, it suffices to show the following. Let E be the image of λ, and F the intersection over all neighborhoods of e in G of the subalgebras generated in* $\mathrm{End}_{\mathbf{Q}_l}(V_l)$ *by these neighborhoods. Then F is the commutant of E in* $\mathrm{End}(V_l)$.

In fact, since E is semi-simple, if the above were true, it would follow from von Neumann's density theorem that E is the commutant of F. Further, since $\mathrm{End}_{\mathbf{Q}_l}(V_l)$ is of finite dimension over $\mathbf{Q}_l$, F actually

equals the subalgebra generated by some neighborhood, and all smaller neighborhoods generate the same subalgebra. Hence the commutant of F is precisely $\mathrm{End}_{\mathbf{Q}_l}(V_l)^{(G)}$.

Now, Steps I-IV are valid for any field of definition k_0, and do not
137 make use of the fact that k_0 is finite. The next step makes use of a certain hypothesis which is easily seen to be true for finite fields, and probably holds for any field of finite type over its prime field. We therefore state it as a hypothesis, verify it for a finite field and deduce its consequences.

Let X be an abelian variety defined over a field k_0, l a prime. We then state the

Hypothesis (k_0, X, l): *Let d be any integer $\geqslant 1$. Then there exist upto isomorphism only a finite number of abelian varieties Y defined over k_0, such that:*

(a) there is an ample line bundle L on Y defined over k_0 with $\chi(L) = d$;

(b) there exists a k_0-isogeny $Y \to X$ of l-power degree.

(If $X \simeq X_0 \otimes_{k_0} k$ and $Y \simeq Y_0 \otimes_{k_0} k$, a line bundle L on Y is said to be defined over k_0 if there is a line bundle L_0 on Y_0 such that $L \simeq L_0 \otimes_{k_0} k$, and an isogeny $Y \to X$ is said to be defined over k_0 if it arises from a homomorphism $Y_0 \to X_0$ by base extension to k.)

For finite fields k_0, we have in fact the following stronger

Lemma 1. *If k_0 is a finite field, $d > 0$ and $g > 0$, there are upto isomorphism only finitely many abelian varieties Y defined over k_0 of dimension g and carrying an ample line bundle L defined over k_0 with $\chi(L) = d$.*

Proof. The line bundle L^3 on Y is defined over k_0 and gives a projective embedding of Y as a k_0-closed subvariety of projective space of dimension $3^g\chi(L) - 1 = 3^g d - 1$. The degree of this subvariety is the g-fold self intersection number of L^3, that is, $3^g.(L)^g.d = 3^g.g!d$. Thus, there corresponds to it a k_0-rational point of the Chow variety of cycles of dimension g and degree $3^g.g!d$ in $\mathbf{P}^{3^g d-1}$. Since k_0 is a finite field, there are only finitely many k_0-rational points on this Chow variety, which proves the lemma. □

Step V. *Suppose X is an abelian vareity defined over k_0 and L an ample line bundle on X defined over k_0. Suppose for a prime $l \neq$ char. k_0, $Hyp(k_0, X, l)$ holds. Let $W \subset V_l(X)$ be a subspace of $V_l(X)$* **138** *which is G-stable and is maximal isotropic for the skew-symmetric form E^L. Then there is an element $u \in E$ with $u(V_l(X)) = W$.*

Proof. Set $T = T_l(X)$, $V = V_l(X)$ and for each integer $n \geqslant 0$,

$$T_n = (T \cap W) + l^n T$$

If $\psi_n : T_n(X) \to X_{l^n}$ is the natural homomorphism, set $K_n = \psi_n(T_n)$, $Y_n = X/K_n$ and let $\pi_n : X \to Y_n$ be the natural homomorphism. Then $(l^n)_X$ factors as $X \xrightarrow{\pi_n} Y_n \xrightarrow{\lambda_n} X$. Since $\pi_n \circ \lambda_n \circ \pi_n = l^n \pi_n$, we obtain that $\pi_n \circ \lambda_n = (l^n)_{Y_n}$, so that

$$\lambda_n((Y_n)_{l^n}) = \ker \pi_n = K_n.$$

Furthermore, for any $m \geqslant n$, $\lambda_n((Y_n)_{l^m}) \supset (\lambda_n \circ \pi_n)(X_{l^m}) = X_{l^{m-n}}$, so that $T_l(\lambda_n)(T_l(Y_n)) \supset l^n T_l(X)$, $T_l(\lambda_n)(T_l(Y_n)) = T_n$.

We now verify that Y_n is an abelian variety defined over k_0 and that λ_n is a k_0-morphism. First note that if $X = X_0 \otimes_{k_0} k$ where X_0 is a group scheme over k_0, the points of X_{l^n}, are separably algebraic over k_0. In fact, since $(l^n)_X : X \to X$ is étale, so is $(l^n)_{X_0} : X_0 \to X_0$, so that $(l^n)_{X_0}^{-1}(0)$ consists of points whose residue fields are separably algebraic over k_0. Further, since W and hence $(T \cap W) + l^n T$ are G-stable by assumption, K_n is also G-stable. Thus the fact that Y_n as a variety is defined over k_0 and $\pi_n : X \to Y_n$ is defined over k_0 follow from the following general □

Lemma 2. *Let X be a quasi-projective variety defined over k_0 and Π a finite group of automorphisms of X such that*

(i) *there is a separably algebraic extension of k_0 over which the automorphisms of Π are defined;*

(ii) *for any automorphism σ of the algebraic closure $\bar{k}_0$ of k_0 over k_0, $\Pi^\sigma = \Pi$.*

Then X/Π is defined over k_0 and the natural map $X \to X/\Pi$ is defined over k_0.

139 *Proof.* Choose a Galois extension k_1/k_0 over which the automorphisms of Π are defined. If U is a k_0-affine open subset of X, $\bigcap_{\lambda\in\Pi} \lambda U$ is again k_0-affine open and Π-stable, by assumption (ii) of the lemma. Thus, we may assume X affine, $X = \operatorname{Spec} A$, $A = A_0 \otimes_{k_0} k$, A_0 being a k_0-algebra. If $A_1 = A_0 \otimes_{k_0} k_1$, Π operates on A_1, and $A_1^{\pi} \otimes_{k_1} k = A^{\Pi}$. Further, $G = \operatorname{Gal}(k_1/k_0)$ operates on A_1, and since by assumption $\lambda\Pi\lambda^{-1} = \Pi$ for any $\lambda \in G$, it follows that A_1^{Π} is G-stable. Let B be the k_0-algebra of G-invariants, and $\{\theta_i\}$ a basis of k_1/k_0. If $\sum a_i \otimes \theta_i \in A_1^{\Pi}$, $a_i \in A_0$, then $\lambda(\sum a_i \otimes \theta_i) = \sum a_i \otimes \lambda(\theta_i) \in A_1^{\pi}$, and since $\det(\lambda(\theta_i))_{\lambda\in G,i} \neq 0$, $a_i \otimes 1 \in A_1^{\Pi} \cap A_0 \subset B$, which proves that $A_1^{\Pi} = B \otimes_{k_0} k_1$. □

This proves the lemma.

Thus, as a variety, Y_n is defined over k_0 and $\pi_n : X \to Y_n$ is defined over k_0, so that $\pi_n(0)$ is k_0-rational. Since the addition map $m : Y_n \times Y_n \to Y_n$ and the inverse $I : Y_n \to Y_n$ are defined over a separably algebraic extension of k_0 and are invariant under the action of $\operatorname{Gal}(\bar{k}_0/k_0)$, they are again defined over k_0. Hence Y_n is defined over k_0 as an abelian variety. Now, $(l^n)_X = \lambda_n \circ \pi_n$ and π_n are defined over k_0, hence λ_n is defined over k_0, since for a k_0-regular function φ on an open subset of X, $\varphi \circ \lambda_n$ is k_1-regular for a Galois extension k_1 of k_0 and invariant under $\operatorname{Gal}(k_1/k_0)$ since $\varphi \circ \lambda_n \circ \pi_n = \varphi \circ (l^n)_X$ is.

Let d be the degree of the ample line bundle L on X. We shall produce on ample line bundle of degree $2^g . d$ defined over k_0 on each Y_n. We have

$$\begin{aligned} e_l(x, \varphi_{\lambda_n^*(L)}(y)) &= E^{\lambda_n^*(L)}(x, y) \\ &= E^L(\lambda_n(x), \lambda_n(y)) \in e^L(T_n, T_n) \\ &= e^L(l^n T, T \cap W + l^n T) \subset l^n M_l \end{aligned}$$

for any $x, y \in T_l(Y_n)$, since W is isotropic for e^L. Since $e_l : T_l(Y_n) \times T_l(\hat{Y}_n) \to M_l$ is non-degenerate, it follows that $\varphi_{\lambda_n^*(L)}(T_l(X)) \subset l^n T_l(X)$, $\varphi_{\lambda_n^*(L)} = l^n \psi$ for some $\psi : Y_n \to Y_n$.

140 It follows from the theorem of §23 that $\psi = \varphi_{L_n}$ for some line bundle L_n on Y_n defined over the algebraic closure, hence over some normal extension of k_0. We may assume that L_n is symmetric. Hence, if p denotes the characteristic of k_0 if this is positive and $p = 1$ if the characteristic is zero, for a suitable integer $N > 0$, $L_n^{p^N}$ is defined over a Galois extension k_1 of k_0. We now have

Lemma 3. *Let Y be an abelian variety defined over k_0 and $L \in \mathrm{Pic}^0 Y$ a line bundle defined over the algebraic closure $\bar{k}_0$ of k_0 in k. Let $\sigma \in \mathrm{Gal}(\bar{k}_0/k_0)$, and denote by $\sigma(L)$ the line bundle on Y defined over $\bar{k}_0$ obtained by pulling back L by the morphism $1_{Y_0} \times \mathrm{Spec}\,\sigma : Y_0 \otimes_{k_0} \bar{k}_0 \to Y_0 \otimes_{k_0} \bar{k}_0$. Then $\sigma(L) \in \mathrm{Pic}^0 Y$.*

Proof. Let M be an ample line bundle on Y defined over k_0, and consider the line bundle $N = m^*(M) \otimes p_1^*(M)^{-1} \otimes p_2^*(M)^{-1}$ on $Y \times Y$. We can find an algebraic point $y \in Y$ such that $N|\{y\} \times Y = L$. It is then easy to see that $N|\{\sigma y\} \times Y = \sigma(L)$, so that $\sigma(L) \in \mathrm{Pic}^0 Y$.

We shall make use of the notation $\sigma(L)$ introduced in the lemma in future. Further, if L_1 and L_2 are two line bundles on Y with $L_1 \otimes L_2^{-1} \in \mathrm{Pic}^0 Y$, we shall write $L_1 \equiv L_2$.

Resuming the earlier discussion, if $\sigma \in \mathrm{Gal}(k_1/k_0)$, we see that $\sigma(L_n^{p^N})$ is also symmetric, and $\sigma(L_n^{p^N})l^n \equiv \lambda_n^*(L^{p^N})$, so that since $NS(Y_n)$ is torsion-free $\sigma(L_n^{p^N}) \equiv L_n^{p^N}$ for every $\sigma \in \mathrm{Gal}(k_1/k_0)$. Hence, $\sigma(L_n^{p^N})$ and $L_n^{p^N}$ differ by an element of order 2 in $\mathrm{Pic}^0 Y$. Thus, if we put $M_n = L_n^{2p^N}$, we have $\sigma(M_n) \simeq M_n$ for every $\sigma \in \mathrm{Gal}(k_1/k_0)$ and $M_n^{l^n} \equiv \lambda_n^*(L^{2p^N}) \equiv L_n^{2p^N l^n}$. We now have □

Lemma 4. *Let Y be a complete variety defined over k_0 with a k_0-rational point, k_1 a Galois extension of k_0 and L a line bundle on Y defined over k_1 such that for every $\sigma \in \mathrm{Gal}(k_1/k_0)$, $\sigma(L) \simeq L$. Then L can be defined over k_0.*

Proof. Put $X = X_0 \otimes_{k_0} k$, $X_1 = X_0 \otimes_{k_0} k_1$ and let $\pi : X_1 \to X_0$ the natural morphism. Since projective modules of constant rank over semi-local rings are free, we can find an affine covering $\mathfrak{u} = (U_i)_{i \in I}$ of X_0 such that

$L/\pi^{-1}(U_i)$ is free. Let $\{\alpha_{ij}\}$ be the 1-cocycle with respect to the covering $\{\pi^{-1}(U_i)\}$ with values in $O^*_{X_1}$ defining L. Our assumption implies that for any $\sigma \in G = \mathrm{Gal}(k_1/k_0)$, we have $\beta_{i,\sigma} \in \Gamma(\pi^{-1}(U_i), O^*_{X_1})$ such that 141

$$\frac{\sigma(\alpha_{ij})}{\alpha_{ij}} = \frac{\beta_{i,\sigma}}{\beta_{j,\sigma}}.$$

If y_0 is a k_0-rational point of X_0 with $y_0 \in U_{i_0}$ and $y_1 = \pi^{-1}(y_0)$, by dividing the $\beta_{j,\sigma}$ by $\beta_{i_0,\sigma}(y_1)$, we may assume that $\beta_{i_0,\sigma}(y_1) = 1$. Now,

$$\frac{\sigma\tau(\alpha_{ij})}{\alpha_{ij}} = \frac{\beta_{i,\sigma\tau}}{\beta_{j,\sigma\tau}} = \frac{\sigma\tau(\alpha_{ij})}{\sigma(\alpha_{ij})}, \frac{\sigma(\alpha_{ij})}{\alpha_{ij}} = \frac{\sigma(\beta_{i,\tau})}{\sigma(\beta_{j,\tau})} \cdot \frac{\beta_{i,\sigma}}{\beta_{j,\sigma}},$$

so that for any $\sigma, \tau \in G$, $i, j \in I$,

$$\beta_{i,\sigma\tau}\sigma(\beta_{i,\tau})^{-1}\beta_{i,\sigma}^{-1} = \beta_{j,\sigma\tau}\sigma(\beta_{j,\tau})^{-1}\beta_{j,\sigma}^{-1} \text{ in } U_i \cap U_j.$$

Since X is complete, $\beta_{i,\sigma\tau}\sigma(\beta_{i,\tau})^{-1}\beta_{i,\sigma}^{-1} = C_{\sigma,\tau}$ for all i, and taking $i = i_0$ and evaluating at y_1, we see that

$$\sigma(\beta_{i,\tau})\,\beta_{i,\sigma\tau}^{-1}\,\beta_{i,\sigma} = 1.$$

Grant for the moment that if A is any local ring of X_0 and $B = A \otimes_{k_0} k_1$, and if B^* is the group of units of B, $H^1(G, B^*) = (1)$. We deduce that there is a covering $\{U_{i,\alpha}\}_{\alpha\in A_i}$ of U_i and $\gamma_{i,\alpha} \in \Gamma(\pi^{-1}(U_{i,\alpha}), O^*_{X_1})$ such that

$$\beta_{i,\sigma} = \frac{\sigma(\gamma_{i\alpha})}{\gamma_{i\alpha}} \quad \text{in} \quad \pi^{-1}(U_{i\alpha}).$$

The cocycle $\alpha_{ij}\dfrac{\gamma_{j\beta}}{\gamma_{i\alpha}}$ with respect to the covering $\{U_{i\alpha}\}_{\substack{\alpha\in A_i\\ i\in I}}$ is cohomologous to $\{\alpha_{ij}\}$, and we have

$$\sigma\left(\alpha_{ij}\frac{\gamma_{j\beta}}{\gamma_{i\alpha}}\right) = \frac{\beta_{i\sigma}}{\beta_{j\sigma}} \cdot \alpha_{ij}\frac{\beta_{j\sigma}}{\beta_{i\sigma}}\frac{\gamma_{j\beta}}{\gamma_{i\alpha}} = \frac{\gamma_{j\beta}}{\gamma_{i\alpha}}\alpha_{ij},$$

$$\alpha_{ij}\frac{\gamma_{j\beta}}{\gamma_{i\alpha}} \in \Gamma(U_{i\alpha} \cap U_{j\beta}, O^*_{X_0}).$$

It only remains to show that if A is the local ring of a point on X_0 and $B = k_1 \otimes_{k_0} A$, $H^1(G, B^*) = \{1\}$. Let R be the quotient field of B; we then have the exact sequence

$$H^0(G, R^*) \to H^0(G, R^*/B^*) \to H^1(G, B^*) \to H^1(G, R^*) = \{1\}.$$

An element of $H^0(G, R^*/B^*)$ is represented by an element $f \in R^*$ such **142**
that $\dfrac{\sigma f}{f} \in B^*$ for all $\sigma \in G$, and we shall show that we can write $f = gu$ where $g \in H^0(G, R^*)$ and $u \in B^*$. Writing $f = \dfrac{f_0}{F}$ with $f_0 \in B$, $F \in A$, we see that we may assume that $f \in B$. Now, since $\dfrac{\sigma f}{f} \in B^*$ for all $\sigma \in G$, the ideal Bf is G-invariant, hence of the form $B\mathfrak{U}$, $\mathfrak{U}$ being an ideal in A. But now, since $Bf \simeq k_1 \otimes_{k_0} \mathfrak{U}$, $\mathfrak{U}$ is A-projective, hence principal. Thus, $Bf = Bg$ for some $g \in A$ and $f = gu$, $g \in A$, $u \in B^*$. □

This completes the proof of Lemma 4.

It follows that the line bundle M_n on Y_n is defined over k_0. Now, the g.c.d. of $2p^N$ and l^n is 1 or 2 according as l is 2 or not. Hence, we can find integers a, b such that $2ap^N + bl^n = 2$. Define $N_n = M_n^a \otimes \lambda_n^*(L)^b$, so that N_n is defined over k_0. Further,

$$N_n \equiv L_n^{2ap^N + bl^n} = L_n^2,$$

so that N_n is ample with
$$\begin{aligned}\chi(N_n) = 2^g \chi(L_n) &= 2^g . l^{-ng} \chi(\lambda_n^*(L)) \\ &= 2^g . l^{-ng} \deg . \lambda_n \chi(L) \\ &= 2^g \chi(L) = 2^g d.\end{aligned}$$

An alternative way of producing an ample line bundle of degree $2^g d$ on Y_n is to observe that (i) if Y is an abelian variety defined over k_0, and we construct $\hat{Y}$ as $Y/K(L)$ for a line bundle L defined over k_0, $\hat{Y}$ is defined over k_0 and the Poincaré bundle P on $Y \times \hat{Y}$ is defined over k_0; (ii) if $\varphi_{\lambda_n^*(L)} = l^n \psi$, since $\lambda_n^*(L)$ is defined over k_0, so is $\varphi_{\lambda_n^*(L)}$ and hence also ψ; and finally; (iii) if P_n is the Poincaré bundle on $Y_n \times \hat{Y}_n$ and $\chi = (1, \psi) : Y_n \to Y_n \times \hat{Y}_n$, then $\chi^*(P_n) = N_n$ is a line bundle defined over k_0 with $\varphi_N = 2\psi$ (see §13), so that $\chi(N_n)^2 = \deg \varphi_{N_n} = 2^{2g} \deg \psi = 2^{2g} l^{-2ng} \deg \lambda_n^*(L) = 2^{2g} l^{-2ng} \times (\deg \lambda_n)^2 \chi(L)^2 = 2^{2g} \chi(L)^2$.

Anyhow, we deduce from $Hyp(k_0, X, l)$ that there is an infinite set I of natural integers with smallest integer n and isomorphisms $v_i : Y_n \xrightarrow{\sim} Y_i$ for all $i \in I$. Consider the elements $u_i = \lambda_i v_i \lambda_n^{-1} \in \mathrm{End}^0 X$ and their images $u_i' \in \mathrm{End}_{\mathbf{Q}}(V_l)$. We have $u_i'(T_n) = T_i \subset T_n$ for $i \in I$, and
143 since $\mathrm{End}_{\mathbf{Z}}(T_n)$ is compact, we can select a subsequence $(u_j')_{j\in J}$ which converges to a limit u'. Since E is closed in $\mathrm{End}(V_l)$ and $u_j' \in E$, u' also belongs to E. Since T_n is compact, $u'(T_n)$ consists of elements of the form $x = \lim x_j$ where $x_j \in u_j(T_n) = T_j$, and since the sets T_j are decreasing, it follows that

$$u(T_n) = \bigcap_{j\in J} u_j(T_n) = \bigcap_{j\in J} T_j = T \cap W.$$

Hence $u(V) = W$.

This completes the proof of Step V.

Step VI. *Suppose that for any finite algebraic extension k_1 of k_0, $Hyp(k_1, X, l)$ holds, and that F is isomorphic as a $\mathbf{Q}_l$-algebra to a direct product of copies of $\mathbf{Q}_l$. Then* (4) *is an isomorphism.*

Proof. Replacing k_0 by a finite algebraic extension k_1 over which all elements of $\mathrm{End}\, X$ are defined and which is such that $\mathrm{Gal}(\bar{k}_1/k_1)$ generates F in $\mathrm{End}(V_l)$, we may assume that k_0 itself has these properties. Let D be the commutant of E in $\mathrm{End}(V_l)$, so that $D \supset F$. We first show that any isotropic subspace W for E^L which is F-stable is also D-stable. We proceed by downward induction on $\dim W$. If W is maximal isotropic, i.e. if $\dim W = g$, we can by Step V find a $u \in E$ such that $u(V) = W$, and hence $DW = DuV = uDV = uV = W$, which proves the assertion. Suppose then that $\dim W = r < g$ and the assertion holds for F-stable isotropic subspaces of dimension $r+1$. The orthogonal complement $W^\perp$ of W for E^L is also F-stable, since E^L is invariant under the action of $\mathrm{Gal}(\bar{k}_0/k_0)$. Further, since any simple F-module is one-dimensional and $\dim W^\perp - \dim W = 2g - 2\dim W = 2(g-r) \geqslant 2(g-g+1) = 2$, we can find F-stable one-dimensional subspaces L_1 and L_2 of $W^\perp$ such that the sum $W + L_1 + L_2$ is direct. By induction hypothesis, $W + L_1$ and $W + L_2$ are D-stable, hence so is their intersection W. This completes

the induction. We deduce that any eigen-vector for F in V is also an eigen-vector for D. It follows that $D \subset F$. (The decomposition of V into factors V_i corresponding to the simple factors of F reduces this assertion to the evident statement that an endomorphism of V_i for which every element of V_i is an eigen-vector is a scalar multiplication). Hence $F = D$, completing the proof of Step VI. □

Step VII. *End of proof of theorem.* **144**

We assume from now on that k_0 is a finite field. By replacing it by a finite extension if necessary, we may assume that every element of End X is defined over k_0. Let π be the Frobenius morphism over k_0. Then π belongs to the center of $\mathrm{End}^0 X$, and hence $\mathbf{Q}[\pi]$ is a commutative semi-simple subalgebra of $\mathrm{End}^0 X$. We shall first show that there are an infinity of primes l for which $\mathbf{Q}_l \otimes_{\mathbf{Q}} \mathbf{Q}[\pi]$ is isomorphic as a $\mathbf{Q}_l$-algebra to a direct product of copies of $\mathbf{Q}_l$.

In fact, writing $\mathbf{Q}[\pi] = K_1 \times \ldots \times K_r$ where K_i are finite extensions of $\mathbf{Q}$, it suffices to show that for an infinity of primes l, each $\mathbf{Q}_l \otimes K_i$ is isomorphic to a product of copies of $\mathbf{Q}_l$. Let K be a finite Galois extension of $\mathbf{Q}$ in which all the K_i are embeddable. Then it suffices to show that for an infinity of l, $K \otimes_{\mathbf{Q}} \mathbf{Q}_l$ splits as a product of copies of $\mathbf{Q}_l$ as a $\mathbf{Q}_l$-algebra. It suffices for this that there is one simple factor of $K \otimes_{\mathbf{Q}} \mathbf{Q}_l$ isomorphic to $\mathbf{Q}_l$. In fact, if $K \otimes_{\mathbf{Q}} Q_l \simeq L_1 \times \ldots \times L_k$, the Galois group π of $K/\mathbf{Q}$ permutes the factors L_i. It also acts transitively on the simple factors. For, if not, suppose $L_1 \times \ldots \times L_r$ is π-stable; then the element $(1, 1, \underset{r \text{ times}}{\ldots}, 1, 0, 0, 0) \in L_1 \times \ldots \times L_k$ is π-stable. On the other hand, since π fixes only the elements of $\mathbf{Q}$ in K, it fixes only the elements of $\mathbf{Q}_l$ in $\mathbf{Q}_l \otimes_{\mathbf{Q}} K$, that is, elements of the form $(\alpha, \alpha, \ldots, \alpha) \in L_1 \times \ldots \times L_k$ with $\alpha \in \mathbf{Q}_l$. This proves the assertion.

Now, choose an algebraic integer α of K generating K over $\mathbf{Q}$, and let $F(X) \in \mathbf{Z}[X]$ be its irreducible monic polynomial over $\mathbf{Q}$. Since $K \otimes_{\mathbf{Q}} \mathbf{Q}_l \simeq \mathbf{Q}_l[X]/(F(X))$, it is enough to find an infinity of l for which $F(X)$ has a zero in $\mathbf{Q}_l$. Let Δ be the discriminant of $F(X)$, and l any prime not dividing Δ such that $F(X) \equiv 0 (\mathrm{mod}\ l)$ has a solution n in $\mathbf{Z}$. Then $F'(n) \not\equiv 0 (\mathrm{mod}\, l)$, so that by Hensel's lemma, n can be refined to a

root of F in $\mathbf{Z}_l$. Thus, we are reduced to proving the following

Lemma 5. *Let $F(X) \in \mathbf{Z}[X]$ be a non-constant polynomial. Then there are an infinity of primes l for which $F(X) \equiv 0 (\mathrm{mod}\ l)$ has a solution in $\mathbf{Z}$.*

145 *Proof.* Let $F(X) = a_0X^n + a_1X^{n-1} + \ldots + a_n$. The lemma being trivial when $a_n = 0$, since X is a factor of $F(X)$ in this case, we may assume $a_n \neq 0$. Further, by substituting a_nX for X in F and removing the common factor a_n, we may assume $a_n = 1$. Let S be a finite set of primes p. If $N = \prod_{p \in S} p$, then

$$F(\nu N) = a_0\nu^n N^n + \ldots + a_{n-1}\nu N + 1 \equiv 1 (\mathrm{mod}\ N)$$

so that no prime of S divides $F(\nu N)$. On the other hand, $F(\nu N) \neq \pm 1$ for ν large, hence has a prime factor l not belonging to S.

Next, we show that for all $l \neq$ char. k_0, the dimension of $\mathrm{End}_{\mathbf{Q}_l} V_l^{(G)}$ is the same. Again, assume that every element of End X is defined over k_0, so that the Frobenius π belongs to the center of $\mathrm{End}^0 X$, hence to the centre of $\mathbf{Q}_l \otimes_{\mathbf{Q}} \mathrm{End}^0 X$. Then $\mathbf{Q}_l[\pi]$ is semi-simple and V_l is a $\mathbf{Q}_l[\pi]$-module, so that the image π' of π in $\mathrm{End}_{\mathbf{Q}_l} V_l$ is semi-simple. The characteristic polynomial $P(t)$ of π' in $\mathrm{End}_{\mathbf{Q}_l} V_l$ has coefficients in $\mathbf{Z}$ independent of l. Further, the closed subgroups of $\mathrm{Gal}(\bar{k}_0/k_0)$ generated by $\pi^{n!}$ form a fundamental system of neighborhoods of e in this group, so that $\mathrm{End}_{\mathbf{Q}_l} V_l^{(G)}$ is the commutant of $\pi'^{n!}$ in $\mathrm{End}_{\mathbf{Q}_l} V_l$ for n large. The characteristic polynomial of $\pi'^{n!}$ has for roots $\theta_1^{n!}, \ldots, \theta_{2g}^{n!}$ where $\theta_1, \ldots, \theta_{2g}$ are the roots of $P(t)$ repeated with multiplicity. For all n large, the number of distinct elements of $\theta_1^{n!}, \ldots, \theta_{2g}^{n!}$ as well as their multiplicities is the same. Thus, our assertion is a consequence of the following lemma, applied to an algebraic closure of $\mathbf{Q}_l$. □

Lemma 6. *Let A and B be absolutely semi-simple endomorphisms of two vector spaces V and W of finite dimensions over a field k respectively, with characteristic polynomials P_A and P_B. Let*

$$P_A = \prod_p p^{m(p)},$$

$$P_B = \prod_p p^{n(p)}$$

be the decompositions of P_A and P_B as products of powers of distinct irreducible monic polynomials p. Then the vector space

$$E = \{\varphi \in \mathrm{Hom}_k(V, W) | \varphi A = B_\varphi\}$$

has dimension **146**

$$r(P_A, P_B) = \sum_p m(p)n(p) \deg p,$$

and this integer is invariant under any extension of the base field k.

Proof. Make V (resp. W) into a $k[X]$-module by making X act through A (resp. . B). Denote the $k[X]$-module $k[X]/(p(X))$ by M_p. Because of our assumption of semi-simplicity, we have isomorphisms of $k[X]$-modules

$$V \simeq \prod_p M_p^{m(p)},$$
$$W \simeq \prod_p M_p^{n(p)}.$$

Now, the M_p are non-isomorphic simple $k[X]$-modules for distinct p, and E is nothing but $\mathrm{Hom}_{k[X]}(V, W)$. Since $\dim_k \mathrm{Hom}_{k(X)}(M_p, M_p) = \dim_k M_p = \deg p$, and since E clearly 'commutes with base extension', the lemma follows.

The main Theorem 1 is a consequence of what we have proved, combined with Steps III and VI. □

Remark. The theorem can be stated in the following seemingly stronger form:

Theorem 1′. *Let X and Y be abelian varieties defined over a finite field k_0, and let $\mathrm{Hom}_{k_0}(X, Y)$ be the group of homomorphisms of X into Y defined over k_0. If G is the Galois group of the algebraic closure of k_0 over k_0, we have an isomorphism*

$$Z_l \otimes_{\mathbf{Z}} \mathrm{Hom}_{k_0}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}_l}(T_l(X), T_l(Y))^G.$$

Proof. Take G-invariants on both sides of the isomorphism (2). □

Applications. We give some easy consequences of Theorem 1′. We shall make our statements over a fixed finite field. The 'geometric' statements over the algebraic closure are easily obtained from these. We shall consistently use the notation $r(f, g)$ for two polynomials f and g, introduced in Lemma 6. Further, as we have shown above that if X is an abelian variety defined over a finite field k_0 with Frobenius morphism
147 π, then the Frobenius morphism π^n over a finite extension of k_0 induces semi-simple endomorphisms of $V_l(X)$ over $\mathbf{Q}_l$, it follows since $\mathbf{Q}_l$ is of characteristic zero that π itself induces (absolutely) semi-simple endomorphisms of $V_l(X)$ for all $l \neq \text{char.}\, k_0$. Thus, the structure of $V_l(X)$ as a module over the Galois group of $\bar{k}_0$ over k_0 is uniquely determined by the characteristic polynomial of π, as explained in the proof of Lemma 6.

We now have

Theorem 2. *Let X and Y be abelian varieties defined over a finite field k_0, and let P_X and P_Y be the characteristic polynomials of their Frobenius endomorphisms relative to k_0. Then*

(a) *we have*

$$\text{rank}(\text{Hom}_{k_0}(X, Y)) = r(P_X, P_Y);$$

(b) *the following statements are equivalent:*

- (b_1) *Y is k_0-isogenous to an abelian subvariety of X defined over k_0,*
- (b_2) *$V_l(Y)$ is G-isomorphic to a G-subspace of $V_l(X)$ for some l,*
- (b_3) *P_Y divides P_X;*

(c) *the following statements are equivalent:*

- (c_1) *X and Y are k_0-isogenous,*
- (c_2) *$P_X = P_Y$,*
- (c_3) *X and Y have the same number of k_1-rational points for every finite extension k_1 of k_0.*

Proof. (a) follows from Theorem 1′ and Lemma 6, since the Frobenius morphism generates the Galois group in the topological sense. The implications $(b_1) \Rightarrow (b_2) \Leftrightarrow (b_3)$ are clear, in view of our earlier remarks. We show that $(b_2) \Rightarrow (b_1)$. If (b_2) holds, we can find an injective G-homomorphism φ of $T_l(Y)$ into $T_l(X)$. Then φ is in the image of $Z_l \otimes_{\mathbf{Z}} \mathrm{Hom}_{k_0}(X, Y)$, so that we can find $\psi \in \mathrm{Hom}_{k_0}(X, Y)$ such that $T_l(\psi)$ approximates arbitrarily closely to φ, in particular with $T_l(\psi)$ injective. If ψ is not an isogeny, we can find an abelian subvariety Z of Y in the **148**
kernel of ψ, and the submodule $T_l(Z)$ of $T_1(Y)$ would be in the kernel of $T^l(\psi)$. Hence ψ is an isogeny, proving (b).

The equivalence $(c_1) \Leftrightarrow (c_2)$ is a special case of $(b_1) \Leftrightarrow (b_3)$ when $\dim X = \dim Y$. Further, we have seen during the proof of the Riemann hypothesis in §20 that if $\omega_i (i \in I)$ and $\omega'_j (j \in J)$ are the roots of P_X and P_Y respectively and N_n and N'_n are the number of rational points of X and Y respectively in the extension of degree n of k_0, we have

$$N_n = \prod_i (1 - \omega_i^n),$$
$$N'_n = \prod_j (1 - \omega_j'^n);$$

thus, we have to show that $P_X = P_Y \Leftrightarrow \prod_i (1-\omega_i^n) = \prod_j (1-\omega_j'^n)$ for every $n \geqslant 0$. The implication $\Rightarrow$ is obvious. To prove the other implication, note that

$$\prod_{i \in I} (1 - \omega_i^n) = \sum_{S \subset I} (-1)^{|S|} \omega_S^n,$$
$$\prod_{j \in J} (1 - \omega_j'^n) = \prod_{T \subset J} (-1)^{|T|} \omega_T'^n$$

where $|S|$, $|T|$ denote the respective cardinalities and $\omega_S = \prod_{i \in S} \omega_i$, $\omega'_T = \prod_{j \in T} \omega'_j$. Multiplying the given equation by t^n, where t is a variable, and summing as formal power series, we obtain

$$\sum_{S \subset I} (-1)^{|S|} \frac{1}{1 - t\omega_S} = \sum_{T \subset J} (-1)^{|T|} \frac{1}{1 - t\omega_T}$$

Since $|\omega_i| = |\omega'_j| = q^{1/2}$, comparing the poles on both sides on the circle $|t| = q^{1/2}$, we obtain that there is a bijection $\sigma : I \to J$ such that $\omega_i = \omega'_{\sigma(j)}$. Hence, $P_X = P_Y$.

Before we come to the next theorem, we need some preliminaries. Let X be an abelian variety defined over k_0, so that $X = X_0 \otimes_{k_0} k$ for some group-scheme X_0 over k_0. Let Y be an abelian subvariety of X, which
149 is a k_0-closed subset, so that if $\lambda : X \to X_0$ is the natural morphism, there is a closed subset Y_0 of X_0 with $\lambda^{-1}(Y_0) = Y$ in the set-theoretic sense. We give Y_0 the structure of a reduced subscheme of X_0. If $m_0 : X_0 \times_{k_0} X_0 \to X_0$ is the multiplication morphism, our hypothesis implies the *set-theoretic inclusion* $m_0(Y_0 \times_{k_0} Y_0) \subset Y$. Hence m_0 restricts to a morphism $m_0 : (Y_0 \times_{k_0} Y_0)_{\text{red.}} \to Y_0$. If we can assert that $Y = Y_0 \otimes_{k_0} k$ and $Y_0 \times_{k_0} Y_0$ are reduced, it would follow that Y is an abelian variety defined over k_0. Both of these are consequences of the assertion that the function field $R_{k_0}(Y_0)$ is a regular extension of k_0, or equivalently that $R_{k_0}(Y_0)$ is a separable extension of k_0. This is always true (vide S. Lang, *Abelian varieties,* Chap. I), but we shall not prove this, since we shall need it only when k_0 is a finite (hence perfect) field, so that this is trivially satisfied.

Next, suppose X is an abelian variety defined over k_0, and Y an abelian subvariety which is k_0-closed. We want to show that there is an abelian subvariety Z of X defined over k_0 such that $Y + Z = X$ and $Y \cap Z$ is finite. We know (vide §18, proof of Theorem 1) that if L is an ample line bundle on X, we can take Z to be the connected component of 0 of the group $Z' = \{z \in X |\ T_Z^*(L) \otimes L^{-1}|_Y \text{ is trivial}\}$, so that if we can ensure that Z is defined over k_0 for a suitable choice of L, we are through. Now if we choose L to be a line bundle defined over k_0, Z' is defined over the algebraic closure $\bar{k}_0$ of k_0 and is stable for all automorphisms of $\bar{k}_0$ over k_0. Hence Z' is k_0-closed. Further, the conjugations of $\bar{k}_0$ over k_0 permute the components of Z', and since Z is a component of Z' containing the k_0-rational point 0, Z is also stable under these conjugations. Hence Z is k_0-closed, and it follows from the comments of the earlier paragraph that Z is an abelian variety defined over k_0.

If follows from this by repeating the arguments of §18 that if we call

an abelian variety X defined over k_0 to be k_0-simple if it does not contain an abelian subvariety Y defined over k_0 with $Y \neq \{0\}$, $Y \neq X$, then (i) any abelian variety defined over k_0 is k_0-isogenous to a product of k_0-simple abelian varieties, and (ii) if X is k_0-isogenous to a product $X_1^{n_1} \times \ldots \times X_r^{n^r}$, where X_i are k_0-simple and X_i and X_j are not k_0-isogenous if $i \neq j$, then $\mathrm{End}_{k_0}^0 X \simeq M_{n_1}(D_1) \times \ldots \times M_{n_r}(D_k)$ where $D_i = \mathrm{End}_{k_0}^0(X_i)$ are division algebras of finite rank over $\mathbf{Q}$. □

We now have **150**

Theorem 3. *Let X be an abelian variety of dimension g defined over a finite field k_0. Let π be the Frobenius endomorphism of X relative to k_0 and P its characteristic polynomial. We then have the following statements:*

(a) *The algebra $F = \mathbf{Q}[\pi]$ is the center of the semi-simple algebra $E = \mathrm{End}_{k_0}^0(X)$;*

(b) $\mathrm{End}_{k_0}^0(X)$ *contains a semi-simple* $\mathbf{Q}$*-subalgebra A of rank $2g$ which is maximal commutative;*

(c) *the following statements are equivalent:*

- (c_1) $[E : \mathbf{Q}] = 2g$,
- (c_2) *P has no multiple root,*
- (c_3) $E = F$,
- (c_4) *E is commutative;*

(d) *the following statements are equivalent:*

- (d_1) $[E : \mathbf{Q}] = (2g)^2$,
- (d_2) *P is a power of a linear polynomial,*
- (d_3) $F = \mathbf{Q}$,
- (d_4) *E is isomorphic to the algebra of g by g matrices over the quaternion division algebra D_p over* $\mathbf{Q}(P = \mathrm{char}.k_0)$ *which splits at all primes $l \neq p, \infty$,*

(d_5) *X is k_0-isogenous to the g-th power of a super-singular curve, all of whose endomorphisms are defined over k_0;*

(e) *X is k_0-isogenous to a power of a k_0-simple abelian variety if and only if P is a power of a $\mathbf{Q}$-irreducible polynomial. When this is the case, E is a central simple algebra over F which splits at all finite primes v of F not dividing p, but does not split at any real prime of F.*

Proof. If follows from the main theorem that $F_l = \mathbf{Q}_l \otimes_{\mathbf{Q}} F$ is the center of $E_l = \mathbf{Q}_l \otimes_{\mathbf{Q}} E$, which proves that F is the center of E.

151 Suppose $E = A_1 \times \ldots \times A_r$ is the expression of E as a product of simple algebras A_i with centers K_i. Let $[K_i : \mathbf{Q}] = a_i$, and $[A_i : K_i] = b_i^2$. We can choose subrings L_i of A_i containing K_i with L_i semi-simple and maximal commutative, $[L_i : K_i] = b$. Then $L = L_1 \times \ldots \times L_r$ is a semi-simple $\mathbf{Q}$-subalgebra of E which is maximal commutative, and $[L : \mathbf{Q}] = \sum_1^r a_i b_i$. Now, for any l, we have

$$E \otimes_{\mathbf{Q}} \mathbf{Q}_l = \prod_1^r (A_i \otimes_{\mathbf{Q}} \mathbf{Q}_l) = \prod_1^r (A_i \otimes_{K_i} (K_i \otimes_{\mathbf{Q}} \mathbf{Q}_l))$$
$$= \prod_{i=1}^r \prod_{j=1}^{n_i} A_i \otimes_{K_i} K'_{ij}$$

where $K_i \otimes_{\mathbf{Q}} \mathbf{Q}_l = \prod_{j=1}^{n_i} K'_{ij}$ with K'_{ij} fields. On the other hand, if $P = \prod_1^S P_i^{m_i}$ is the decomposition of P over $\mathbf{Q}_l$ into a product of powers of irreducible polynomials over $\mathbf{Q}_l$, and if we consider $T_l(X)$ as a $\mathbf{Q}_l[T]$-module by making T act via π, we have an isomorphism of $\mathbf{Q}_l[T]$-modules

$$T_l(X) \simeq \prod_{v=1}^S \left(\frac{\mathbf{Q}_l[T]}{(P_v)} \right)^{m_v} = \prod_{v=1}^S S_v^{m_v}, S_v = \frac{\mathbf{Q}_l[T]}{(P_v)},$$

so that $E \otimes_{\mathbf{Q}} \mathbf{Q}_l$, being the commutant of π in $\operatorname{End} T_l(X)$, is isomorphic to

$$\prod_{v=1}^S M_{m_v}(S_v).$$

Comparing the two factorisations of $E \otimes_{\mathbf{Q}} \mathbf{Q}_l$ and keeping in mind that K'_{ij} is the center of $A_i \otimes_{K_i} K'_{ij}$, we deduce that (i) for any prime $l \neq p$ and any prime v of K_i lying over l, A_i splits at v and (ii) there is a partition of $[1, s]$ into r disjoint subsets $I_1, \ldots, I_r$ such that

$$A_i \otimes_{\mathbf{Q}} \mathbf{Q}_l \simeq \prod_{j=1}^{n_i} A_i \otimes_{K_i} K'_{ij} \simeq \prod_{v \in I_i} M_{m_v}(S_v).$$

It follows that $m_v = b_i$ for $v \in I_i$ and $\sum_{v \in I_i} [S_v : \mathbf{Q}_l] = \sum_{j=1}^{n_i} [K'_{ij} : \mathbf{Q}_l] = [K_i :$ **152**
$Q] = a_i$, so that

$$\sum_1^r a_i b_i = \sum_{i=1}^r \sum_{v \in I_i} m_v [S_v : \mathbf{Q}_l] = \sum_{v=1}^s m_v [S_v : \mathbf{Q}_1]$$
$$= \sum_{v=1}^s m_v \deg P_v = \deg P = 2g.$$

This proves (b).

Since F is the center of E and E contains a maximal commutative subring of rank $2g$, (c_1), (c_3) and (c_4) are equivalent, and since E commutative $\Leftrightarrow$ E_l commutative $\Leftrightarrow$ $m_v = 1$ with the above notations, these are also equivalent to (c_2). This proves (c).

Now, $[E : \mathbf{Q}] = (2g)^2$ if and only if $\mathbf{Q}_l \otimes_{\mathbf{Q}} E \simeq M_{2g}(\mathbf{Q}_l)$, hence if and only if $s = 1$, $S_v = \mathbf{Q}_l$ or equivalently, P is a power of a linear polynomial. In this case, $\mathbf{Q}_l$ is the center of $\mathbf{Q}_l \otimes_{\mathbf{Q}} E$, so that $\mathbf{Q}$ is the center of E, and conversely, if this holds, $\mathbf{Q}_l \otimes_{\mathbf{Q}} E$ is the commutant of $\mathbf{Q}_l$ in $\operatorname{End} V_l$, so that it is the whole of $\operatorname{End} V_l$. Thus (d_1), (d_2) and (d_3) are equivalent. If $\mathbf{Q}_l \otimes_{\mathbf{Q}} E = M_{2g}(\mathbf{Q}_l)$, E is a central simple algebra over $\mathbf{Q}$ whose invariants at all finite primes $l \neq p$ are 0. Since its invariant at the infinite prime is 0 or $\frac{1}{2}$ and the sum of invariants at all primes is 0, E is either $M_{2g}(\mathbf{Q})$ or $M_g(D_p)$ where D_p is the quaternion algebra over $\mathbf{Q}$ splitting at all finite primes $l \neq p$. The first possibility is ruled out, since X cannot be a product of $2g$ abelian varieties. This proves that $(d_1) \iff (d_4)$. In view of our remarks preceding the theorem, (d_4) is equivalent to saying that $X \simeq C^q$, where C is an elliptic curve with

$\mathrm{End}_{k_0} C \simeq D_p$. We have then shown that C is supersingular (§22). This proves (d).

Let Q be the product of the distinct irreducible factors of P. Since $F = \mathbf{Q}[\pi]$ is semi-simple, and $P(\pi) = 0$, we have $Q(\pi) = 0$. Further, π acts as an endomorphism of V_l, any irreducible factor over $\mathbf{Q}_l$ of the characteristic polynomial P divides the minimal polynomial of π, so that Q is the minimal polynomial of π over $\mathbf{Q}$. Now, X is k_0-isogenous to a
153 power of a k_0-simple abelian variety if and only if E is simple, hence if and only if the center $F = \mathbf{Q}[\pi]$ of E is a field. Since $F \simeq \mathbf{Q}[X]/(Q(X))$, F is a field if and only if Q is irreducible, or equivalently, P is the power of an irreducible polynomial Q. If F is the center of E, we have shown earlier that E splits at any finite prime v of F not dividing l. Suppose v is a real imbedding of F, so that $v(\pi)$ is a real number. Since $v(\pi)$ satisfies $P(v(\pi)) = 0$ and the roots of P have absolute value $\sqrt{q}$, we must have $v(\pi) = \pm\sqrt{q}$. If q is a square, $v(\pi) \in \mathbf{Q}$ and $F = \mathbf{Q}$, so that the equivalent condition of (d_3) and (d_4) implies that E does not split at ∞. If q is not a square, $F = \mathbf{Q}(\sqrt{p})$. Let k_1 be the quadratic extension of k_0 and $\pi' = \pi^2$ the Frobenius over k_1. Then $\pi^2 \in \mathbf{Q}$, so that the center F' of $E' = \mathrm{End}_{k_l} X$ is $\mathbf{Q}$. Appealing to (d), we conclude that $E' = M_g(D_p)$. On the other hand, we have $F' \subset F \subset E \subset E'$, and E is the commutant of F in E'. By a known result on central simple algebras, we see that E and $F \otimes_{F'} E'$ define the same element of the Brauer group over F, that is, E is the image of E' under the natural map $Br(F') \to Br(F)$. Since both the real primes of $\mathbf{Q}(\sqrt{p})$ lie over the real prime ∞ of $\mathbf{Q}$ and E' has invariant $\frac{1}{2}$ at ∞ with respect to $F' = \mathbf{Q}$, E has invariant $\frac{1}{2}$ at either of the real primes of $F = \mathbf{Q}(\sqrt{p})$. This completes the proof of (e).

□

Corollary . *Any two elliptic curves defined over finite fields with isomorphic algebras of complex multiplications are isogenous (over the algebraically closed field k).*

In particular, any two supersingular elliptic curves are isogenous.

Proof. Suppose X, Y are supersingular elliptic curves. We can choose a common finite field of definition k_0 such that $\mathrm{End}_{k_0} X$ and $\mathrm{End}_{k_0} Y$ are quaternion algebras over $\mathbf{Q}$, so that they have $\mathbf{Q}$ for center. Thus,

their Frobenius morphisms π_X and π_Y lie in $\mathbf{Q}$. Since they must both have absolute value $\sqrt{q}$ where $q = \mathrm{card}(k_0)$, we see that $\pi_X^2 = \pi_Y^2 = q$. Thus, if k_1 is the quadratic extension of k_0, there is an isomorphism $T_l(X) \xrightarrow{\sim} T_l(Y)$ carrying the action of π'_X into π'_Y, where $\pi'_X = \pi_X^2$ and $\pi'_Y = \pi_Y^2$ are the Frobenius morphisms over k_1. By Theorem 2, X and Y are isogenous over k_1.

Next suppose $\mathrm{End}^0 X \simeq K$, $\mathrm{End}^0 Y \simeq K$ for some imaginary quadratic **154**
extension K of $\mathbf{Q}$. Choose a common finite field of definition of X and Y over which all their endomorphisms are defined and all the points of order p are rational. Now, $\mathbf{Q}_p \otimes_{\mathbf{Q}} \mathrm{End}^0 X$ admits a one-dimensional representation in $T_p(X)$. Hence, p splits into a product of two distinct primes $\mathfrak{p}$ and $\mathfrak{p}'$ in K which are conjugate, and $Q_p \otimes_Q \mathrm{End}^0 X \simeq K_{\mathfrak{p}} \times K_{\mathfrak{p}'}$. Suppose for instance that $Q_p \otimes_Q \mathrm{End}^0 X$ acts on $T_p(X)$ via $K_{\mathfrak{p}}$. By what we have said, it follows that $\pi_X \equiv 1(\mathfrak{p})$, and since $Nm\pi_X$ is a power of p, (π_X) has to be a power of $\mathfrak{p}'$ in the ring of integers of K. A similar assertion (possibly with $\mathfrak{p}$ replacing $\mathfrak{p}'$) holds for π_Y. By altering the isomorphism $\mathrm{End}^0 Y \simeq K$ by the conjugation of K if necessary, we may assume that (π_Y) is also a power of $\mathfrak{p}'$. Since $Nm\pi_X = Nm\pi_Y = q = p^f$, we see that in the ring of integers of K, $(\pi_X) = (\pi_Y) = \mathfrak{p}'^f$ so that π_X and π_Y differ by a unit, i.e., a root of unity since K is imaginary quadratic. Thus $\pi_X^n = \pi_Y^n$ in K for suitable n, and they have the same minimal equation over $\mathbf{Q}$, of degree 2. Since this has to be their characteristic polynomial, X and Y are isogenous over an extension of degree n of k_0. □

The Geometry of the Monodromy Theorem*

By Lê Dũng Tráng

Introduction. Let (X, x) be the germ of an analytic space and $f : (X, x) \to (\mathbf{C}, 0)$ be the germ of an analytic function. We shall still denote X and f representants of the corresponding germs. Let us suppose $X \subset U \subset \mathbf{C}^N$ where U is an open subset of $\mathbf{C}^N$ and X is closed in U.

Let us suppose that for any $t \neq 0$ sufficiently small, $f = t$ is smooth. It has been proved in [3] or [6] that if $\epsilon > 0$ is small enough and $0 < \eta \ll \epsilon$, we have a smooth (i.e. C^∞) fibration:

$$\varphi_{\epsilon,\eta} : B_\epsilon \cap f^{-1}(D^*_\eta) \to D^*_\eta$$

induced by f, where B_ϵ is the closed ball of $\mathbf{C}^N$ centered at x with radius $\epsilon > 0$ and $D^*_\eta = \{z \in \mathbf{C} | 0 < |z| \leqslant \eta\}$. We shall call this fibration the Milnor fibration of f at x (cf. [11] when $X = \mathbf{C}^N$).

The monodromy of this fibration is called the local monodromy of f at x (cf. [6]). We prove under the preceding hypothesis:

Theorem. (the Monodromy Theorem): *The local monodromy of f at x is quasi-unipotent, i.e. its eigenvalues are roots of unity.*

In [1] this theorem has been proved using the resolution of singularities. In [2], A. Grothendieck has given a proof which applies in the case we have a proper mapping $f : X \to C$; this proof actually uses the resolution of singularities and applies in any characteristic when there is a resolution of singularities. Another proof with a topological flavour,

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but still using the resolution of singularities is given by A. Landman in [5].

In this paper we are giving a topological proof of the monodromy theorem which does not use the resolution of singularities and which gives a topological interpretation to the eigenvalues. The proof uses **158** the notion of relative monodromy (cf. [6]) which is an avatar of the Lefschetz method for the study of the monodromy. Its strong interest comes from the use of a precise geometric description of the singularity. This leads to an interesting filtration of the Milnor fiber of f at x.

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1 Basic facts about the relative monodromy

Now we recall some results about the relative monodromy. cf. [6]. We still use the notations defined in the introduction. Let z_1 be a linear form of $\mathbf{C}^N$, say the first coordinate of $\mathbf{C}^N$. The restriction of z_1 to X and f defines a mapping (z_1, f) of X into $\mathbf{C}^2$. From [7] and [6], because $X - f^{-1}(0)$ is smooth, we have the following theorem:

Theorem 1.1. *Suppose that $z_1 = 0$ be sufficiently general, then we have polydiscs $D \times P \subset \mathbf{C}^N$ centered at x and $D \times D' \subset \mathbf{C}^2$ centered at $(z_1(x), 0)$, a curve $\Gamma \subset (D \times P) \cap X$ and a curve $\Delta \subset D \times D'$ such that:*

(1) *the restriction Φ of (z_1, f) to $(D \times P) \cap X$ is critical along Γ outside $f^{-1}(0)$, moreover it induces a smooth fibration of $\Phi^{-1}(D \times D' - \Delta)$ onto $D \times D' - \Delta$;*

(2) *f induces a smooth fibration $\Phi^{-1}(D \times D' - D \times \{0\})$ onto $D' - \{0\}$ which is fiber isomorphic to the Milnor fibration of f at x;*

(3) *f induces a smooth fibration of $(\{0\} \times P) \cap \Phi^{-1}(D \times D' - \{0\})$ onto $D' - \{0\}$ which is fiber isomorphic to the Milnor fibration of the restriction of f to $X \cap \{z_1 = 0\}$ at x;*

(4) *for any* $x \in \Delta - \{0\}$ *the fibers* $\Phi^{-1}(x)$ *have only one quadratic ordinary singular point at* $y = \Gamma \cap \Phi^{-1}(x)$.

In [8] and [6] we have called Γ the polar curve of f relatively to $z_1 = 0$ and $\Phi(\Gamma) = \Delta_0$ its Cerf diagram. Notice that it may happen that $\Delta = \Delta_0$, but in general $\Delta = \Delta_0 \cup (D \times \{0\})$.

In [6] we have seen that in this situation we have defined a smooth **159**
mapping h of the Milnor fiber $\Phi(D \times \{\eta\})$ onto itself where $\eta \in \partial D'$ which is a characteristic diffeomorphism of Milnor fibration of f at x. Moreover h lifts a diffeomorphism Ψ of $D \times \{\eta\}$ onto itself such that the following diagram is commutative:

$$\begin{array}{ccc} \Phi^{-1}(D \times \{\eta\}) & \xrightarrow{h} & \Phi^{-1}(D \times \{\eta\}) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ D \times \{\eta\} & \xrightarrow{\Psi} & D \times \{\eta\} \end{array} \tag{1.2}$$

where φ is induced by Φ and Ψ satisfies the following properties:

(1) $\Psi(0, \eta) = (0, \eta)$

(2) $\Psi(D \times \{\eta\}) \cap \Delta = (D \times \{\eta\}) \cap \Delta$.

Further properties of Ψ will be given in the next paragraph. In [6] it has been seen that h induces a smooth mapping of $\Phi^{-1}(0, \eta)$ onto itself which is a characteristic diffeomorphism of the Milnor fibration of the restriction of f to $X \cap \{z_1 = 0\}$. Then h induces an automorphism of $H_*(\Phi^{-1}(D \times \{\eta\}), \Phi^{-1}((0, \eta)))$ which defines the relative local monodromy of f at x. From the fact that $X - f^{-1}(0)$ is smooth and the (4) of the Theorem (1.1) cited above, a reasoning as in [8] leads to:

Lemma 1.3. $H_k(\Phi^{-1}(D \times \{\eta\}), \Phi^{-1}((0, \eta))) = \begin{cases} 0 \text{ if } k \neq \dim_C f^{-1}(t) \\ \mathbf{Z}^r \text{ if } k = \dim_C f^{-1}(t) \end{cases}$

with t *small enough and* $t \neq 0$, *and* r *equal to the number of points of* $\Delta \cap (D \times \{\eta\})$ *or* $\Gamma \cap \Phi^{-1}(D \times \{\eta\})$.

Notice that Γ may be $\oslash$ and thus $H_*(\Phi^{-1}(D \times \{\eta\}), \Phi^{-1}((0, \eta)))$ may entirely vanish.

Now we are going to give a more precise description of Ψ which will allow use to prove the monodromy theorem.

2 The Carrousel

We are going to recall the results of [6] and [7] which describe Ψ. **160**

Let

$$\begin{cases} \lambda = t^r \\ z_1 = \sum_{j \geqslant r_i} a_{ij} t^j \quad a_{ir_i} \neq 0 \end{cases} \tag{2.1}$$

be a parametrization of Δ_i.

Let us index the Δ_1, $i = 1, \ldots, k$, such that for any $1 \leqslant i \leqslant i_1 - 1$, we have

$$\begin{cases} \dfrac{r_1}{r} = \dfrac{r_i}{r} = \dfrac{m'_1}{n'_1} \text{ with } (m'_1, n'_1) = 1 \\ a_{1r_1} = a_{ir_i} = \alpha_1 \\ \quad \ldots\ldots\ldots\ldots \end{cases} \tag{2.2}$$

for any $i_l \leqslant i \leqslant i_{l+1} - 1 = k$, we have

$$\begin{cases} \dfrac{r_{i_l}}{r} = \dfrac{r_i}{r} = \dfrac{m'_l}{n'_l} \text{ with } (m'_l, n'_l) = 1 \\ a_{i_l r_{i_l}} = a_{ir_i} = \alpha_l \end{cases}$$

and moreover

$$\frac{m'_1}{n'_1} \geqslant \frac{m'_2}{n'_2} \geqslant \cdots \geqslant \frac{m'_l}{n'_1}$$

all couples $\left(\dfrac{m'_1}{n'_1}, \alpha_i\right)$ are distinct and if $\dfrac{m'_j}{n'_j} = \dfrac{m'_{j+1}}{n'_{j+1}}$, we have $|\alpha_j| \leqslant |\alpha_{j+1}|$.

Let us call $C_1, \ldots, C_l$ the curves defined by

$$\begin{cases} \lambda = t^{n' j} \\ z_1 = \alpha_j t^{m' j} \end{cases} \qquad j = 1, \ldots, l. \tag{2.3}$$

If η is small enough, it is clear that the points of $(D \times \{\eta\}) \cap (\Delta_{i_j} \cup \ldots \cup \Delta_{i_{j+1}-1})$ lie in a neighbourhood of $(D \times \{\eta\}) \cap C_j$. More precisely

Lemma 2.4. *If η is small enough, in $D \times \{\eta\}$ there are disjoint discs* 161
$\delta_{j\mu}(j = 1, \ldots, l$ and $\mu = 1, \ldots, n'_j)$ centered at the n'_j points of $(D \times \{\eta\}) \cap C_j$, when $j = 1, \ldots, l$, such that

$$(D \times \{\eta\}) \cap (\Delta_{i_j} \cup \ldots \cup \Delta_{i_{j+1}-1}) \subset \bigcup_{\mu=1}^{n'_j} \delta_{j\mu}.$$

Moreover the number of points of $\delta_{j\mu} \cap (\Delta_{ij} \cup \ldots \cup \Delta_{i_{j+1}-1})$ is the same for $\mu = 1, \ldots, n'_j$ and the $\delta_{j\mu}$ have the same radius for $\mu = 1, \ldots, n'_j$.

Now let us suppose that:

$$\left|\begin{array}{l} \dfrac{m'_1}{n'_1} = \ldots = \dfrac{m'_{j_1-1}}{n'_{j_1-1}} \\ \dfrac{m'_{j_1}}{n'_{j_1}} = \ldots = \dfrac{m'_{j_2-1}}{n'_{j_2-1}} \\ \ldots\ldots\ldots \\ \dfrac{m'_{j_s}}{n'_{j_s}} = \ldots = \dfrac{m'_l}{n'_l} \end{array}\right. \tag{2.5}$$

and

$$\frac{m'_1}{n'_1} > \frac{m'_{j_1}}{n'_{j_1}} > \ldots > \frac{m'_{j_s}}{n'_{j_s}}.$$

Lemma 2.6. *If η is small enough, in $D \times \{\eta\}$ there are closed discs centered at $(0, \eta)$, say $D_1 \subset D'_1 \subset \ldots \subset D'_{s-1} \subset D_s \subset D$ such that*

1) *the interior $\mathring{D}_1$ contains the $\delta_{j\mu}$ with $j = 1, \ldots, j_1 - 1$ and $\mu = 1, \ldots, n'_j$;*

2) *the open annulus $\mathring{D}_2 - D'_1$ contains the $\delta_{j\mu}$ with $j = j_1, \ldots, j_2 - 1$ and $\mu = 1, \ldots, n'_j$;*

   .....

s) *the open annulus* $\mathring{D}_s - D'_{s-1}$ *contains the* $\delta_{j\mu}$ *with* $j = j_s, \ldots, l$ *and* $\mu = 1, \ldots, n'_j$.

Now we can summarize the properties of Ψ (cf. [6] and [8]) as follows:

2.7

1) Ψ is a smooth diffeomorphism of $D \times \{\eta\}$ onto itself;

2) its restriction to $D_1 - \bigcup_{j=1}^{j_1-1} \bigcup_{\mu=1}^{n'_j} \delta_{j\mu}$ is a rotation of angle $2\pi \frac{m'_1}{n'_1}$
. . . . . **162**
its restriction to $(\mathring{D}_s - D'_{s-1}) - \bigcup_{j=j_s}^{l} \bigcup_{\mu=1}^{n'j} \delta_{j\mu}$ is a rotation of angle $2\pi \frac{m'_{js}}{n'_{js}}$;

3) its restriction to $\partial D \times \{\eta\}$ is the identity;

4) its restriction to each circle centered at $(0, \eta)$ and contained in $\mathring{D}_i - D'_{i-1}$ and $D - D_s$ is a rotation.

The we define a class of smooth diffeomorphisms of a disc D onto itself to which Ψ will belong.

Definition 2.8. *Any diffeomorphism of* D *onto itself which maps the center of* D *on itself, which has a fixed point* $x \notin \partial D$ *and the restriction of which is the identity on* ∂D *is called a carrousel with one distinguished point* $x \in \mathring{D}$.

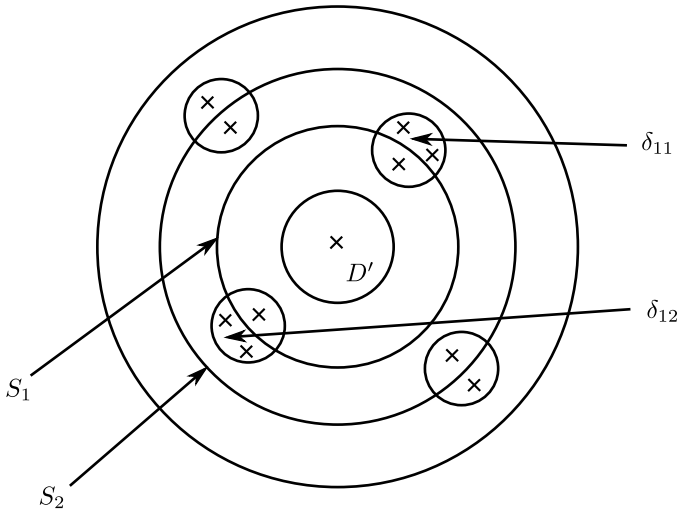
Definition 2.9. *A roll is a diffeomorphism* Ψ *of an open annulus* $\mathring{D} - D'$ *onto itself such that:*

1) there are circles $S_1, \ldots, S_m$ *contained in* $\mathring{D} - D'$ *and discs* $\delta_{ij} \subset \mathring{D} - D'(i = 1, \ldots, m, j = 1, \ldots, q)$ *centered at points* x_{ij} $(j = 1, \ldots, q)$ *equidistributed on* $S_i (i = 1, \ldots, m)$ *and of the same radius for any fixed* i;

2) *there are the same number of distinguished points in each* δ_{ij} $(j = 1, \ldots, q)$ *for a given* i*, such that* Ψ *maps* δ_{ij} *onto* δ_{ij+1} *for* $j = 1, \ldots, q^{-1}$ *(resp.* δ_{iq} *onto* δ_{i1}*) and the distinguished points of* δ_{ij} *onto those of* δ_{ij+1} *for* $j = 1, \ldots, q^{-1}$ *(resp. those of* δ_{iq} *onto those of* δ_{i1}*);*

3) *the restriction of* Ψ *to* $\mathring{D} - D' - \bigcup_{i,j} \delta_{ij}$ *is a rotation of angle* $2\pi\dfrac{p}{q}$, $(p, q) = 1$.

If Ψ is a roll, Ψ^q induces a diffeomorphism Ψ_{ij} of δ_{ij} onto itself which maps the distinguished points of δ_{ij} onto themselves. We shall call the diffeomorphisms Ψ_{ij} the *distinguished diffeomorphisms of the roll*.



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We have defined a carrousel with one distinguished point, we suppose we have defined carrousels with n' distinguished points, $1 \leqslant n' < n$, we define what is a carrousel with n distinguished points.

Definition 2.10. *A 'carrousel with n distinguished points' is a diffeomorphism* Ψ *of* D *onto itself such that:*

1) *there are discs* $D_1 \subset D_1' \subset D_2 \subset \ldots \subset D_k \subset D$ *such that the restriction* Ψ_i *of* Ψ *to* $\mathring{D}_i - D_{i-1}'$ *is a roll;*

2) *the n distinguished points of* Ψ *are the distinguished points of the* Ψ_i*;*

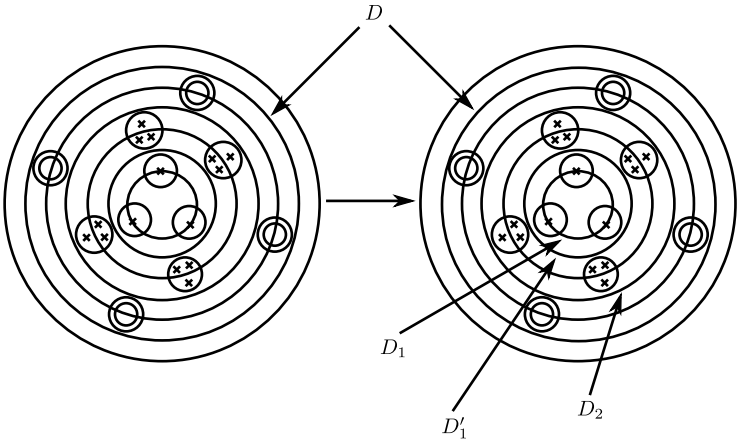
3) *the distinguished diffeomorphisms of each roll are*

   a) *either carrousels with* n' *distinguished with* $1 \leqslant n' < n$*;*

   b) *or* $k = 1$ *and the roll* Ψ_1 *has only one distinguished diffeomorphism which satisfies 1) and 2); then if it satisfies 3)-b) this process will stop after l steps with carrousels with* n' *distinguished points with* $1 \leqslant n' < n$*. Such a carrousel is called a carrousel which simplifies after l steps.*

Example. (cf. [6] and [7]). **164**

Lemma 2.11. *The smooth mapping* $\Psi : D \times \{\eta\} \to D \times \{\eta\}$ *of (2.7) is actually a carrousel the distinguished points of which are* $(\Delta_1 \cup \ldots \cup \Delta_k) \cap (D \times \{\eta\})$.



3 The Monodromy Theorem:

We can summarize the preceding results by

Theorem 3.1. *$\eta > 0$ is small enough, there is a commutative diagram (cf. (1.2)):*

$$\begin{array}{ccc} \Phi^{-1}(D \times \{\eta\}) & \xrightarrow{h} & \Phi^{-1}(D \times \{\eta\}) \\ \downarrow & & \downarrow \\ D \times \{\eta\} & \xrightarrow{\Psi} & D \times \{\eta\} \end{array}$$

such that:

1) *Ψ is a carrousel the distinguished points of which are $(D \times \{\eta\}) \cap \Delta$;*

2) *h is a characteristic diffeomorphism of the Milnor fibration of f at x;* **165**

3) *h induces a diffeomorphism of $\Phi^{-1}(0, \eta)$ onto itself which is a characteristic diffeomorphism of the Milnor fibration of the restriction of f to $X \cap \{z_1 = 0\}$ at x;*

4) *the fibers of φ over the distinguished points of Ψ are singular and have only one ordinary quadratic singular point, the other fibers of φ are smooth.*

Moreover, it is clear that if $\dim_{\mathbf{C}} f^{-1}(t) = 0$ when $t \neq 0$ is small enough, the monodromy theorem is true.

(H) *Let us assume it is true when* $1 \leqslant \dim_{\mathbf{C}} f^{-1}(t) < n$.

Suppose now that $\dim_{\mathbf{C}} f^{-1}(t) = n$.

From the induction hypothesis we know that for any parametrization $\pi : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ of a germ of curve $(C, 0)$ in $(\mathbf{C}^2, 0)$, such that $(C, 0)$ does not lie inside $(\mathbf{C} \times \{0\}0)$ the monodromy of the pullback f_π of Φ is quasi-unipotent:

$$\begin{array}{ccc} X_\pi, 0 & \longrightarrow & X, 0 \\ \downarrow f_\pi & & \downarrow \Phi \\ (\mathbf{C}, 0) & \xrightarrow{\pi} & (\mathbf{C}^2, 0) \end{array}$$

Then we obtain:

Lemma 3.2. *For any point* $x \in D_i - (\mathring{D}'_{i-1} \bigcup_{j,k} \delta_{ijk})$, $h^{n'_{ji}}$ *induces an automorphism of* $H_*(\Phi^{-1}(x))$ *which is quasi-unipotent.*

Proof. Consider a curve C_x parametrized by

$$\begin{cases} \lambda = t^{n'_{ji}} \\ z_0 = \alpha(x) t^{m'_{ji}} \end{cases} \qquad (*)$$

such that $x \in C_x$. It is clear that such a curve exists and does not contain any point of $\Delta \cap (D \times \{\eta\})$ as

$$C_x \cap (D \times \{\eta\}) = C_x \cap (D_i - (\mathring{D}'_{i-1} \bigcup_{j,k} \delta_{ijk})).$$

It is enough now to notice that Ψ induces a bijection of $C_x \cap (D \times \{\eta\})$ **166** onto itself which is lifted in a characteristic diffeomorphism of the fibration $\Phi^{-1}(C_x \cap (D \times \partial D')) \to \partial D'$ induced by f. Finally the monodromy of this fibration is quasi-unipotent as its pull back by (*) is quasi-unipotent according to the induction hypothesis.

Actually we have a more precise result which will allow us to prove the monodromy theorem by induction on the dimension of the fiber □

Lemma 3.3. *Let* $\Psi_{ijk} : \delta_{ijk} \to \delta_{ijk}$ *a distinguished carrousel of the carrousel* Ψ. *Let* x *be a point of* $\delta_{ijk} \to \Delta$ *on which the action of* Ψ_{ijk} *is one of the rotations of this carrousel. Then there is a power* h^q *of* h *which induces a quasi-unipotent automorphism of* $H_*(\Phi^{-1}(x))$. *This property is true for the distinguished carrousels of* Ψ_{ijk} *and so on.*

Let $2\pi p/q$ be the angle of the rotation induced by Ψ_{ijk} on x. We may choose q multiple of n'_{j_i}. The proof is the same as the one of (3.2) by considering a curve parametrized by

$$\begin{cases} \lambda = t^q \\ z_0 = \alpha_i t^{m'_j \frac{q}{n'_{j_i}}} + \beta(x) t^{m'_j \frac{q}{n'_{j_i}} + p} \end{cases}$$

which goes through x.

Because of the specific character of the points x which appear in (3.2) and (3.3) we shall call the points of $D_i - (\mathring{D}'_{i-1} \bigcup_{j,k} \delta_{ijk})$ the *rotating points* of the carrousel Ψ. The distinguished carrousels of distinguished carrousels of Ψ and so on are called *successive distinguished carrousels of* Ψ. The rotating points of the successive distinguished carrousels of Ψ are called the *rolling points* of Ψ.

Notice that if Ψ is a carrousel with one distinguished point the centre of D might be the only rolling point of Ψ.

We can prove the theorem (3.1) now by using the following **167**

Theorem 3.3. *Let $\varphi : X \to D$ be an analytic morphism of a connected analytic manifold X of dimension n onto a disc D. Suppose that*

1) *there are a carrousel $\Psi : D \to D$ and a smooth diffeomorphism $h : X \to X$ such that:*

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ D & \xrightarrow{\Psi} & D \end{array}$$

is commutative, φ is a smooth fibration outside the distinguished points of the carrousel Ψ and the fibers of φ over the distinguished points of Ψ have only one ordinary quadratic singular point;

2) *for any rolling point x of Ψ the restriction to $\varphi^{-1}(x)$ of some power h^q of q such that $h^q(\varphi^{-1}(x)) = \varphi^{-1}(x)$ induces a quasi-unipotent automorphism of $H_*(\varphi^{-1}(x))$.*

Then h induces an automorphism of $H_*(X)$ which is quasi-unipotent.

Proof of the Theorem (3.3). We prove this theorem by induction on the number of distinguished points of the carrousel. If $n = 1$, the theorem is obvious as

$$H_k(X, \varphi^{-1}(x)) = \begin{cases} 0 \text{ if } k \neq n \\ \mathbf{Z} \text{ if } k = n \end{cases}$$

if x is a point which is not the distinguished point of Ψ, i.e. $\varphi^{-1}(x)$ is smooth.

Then if, moreover, x is a rolling point such that there is an integer q such that $\Psi^q(x) = x$, say $x = 0$ and then $q = 1$, then we have the exact sequence:

$$0 \to H_n(X) \to H_n(X, \varphi^{-1}(0)) \to H_{n-1}(\varphi^{-1}(0)) \to H_{n-1}(X) \to 0$$

and isomorphisms **168**

$$H_k(\varphi^{-1}(0)) \xrightarrow{\simeq} H_k(X) \qquad \text{for} \quad 0 \leqslant k \leqslant n-2.$$

As h induces quasi-unipotent automorphisms on $H_n(X, \varphi^{-1}(0))$ and on $H_{n-1}((\varphi^{-1})(0))$ from the commutativity of the corresponding diagrams, we obtain: the automorphism induced by h^q on $H_*(X)$ is quasi-unipotent.

Suppose $n > 1$, and that for any $1 \leqslant n' < n$ the Theorem (3.3) is true for carrousels with n' distinguished points.

Now let us call $U_i = \varphi^{-1}(D_i)$, $i = 1, \ldots, k$ and $U_i' = \varphi^{-1}(D_i')$, $i = 1, \ldots, k-1$.

Notice that $U_i \subset u_i'$ is an equivalence of homotopy.

Then

Lemma 3.4. *The diffeomorphism h induces a quasi-unipotent automorphism of* $H_*(U_i)$.

Proof. $i = 1$. If there is only one distinguished point of Ψ_1 which is therefore a carrousel with one distinguished point, we just apply the reasoning above and our lemma is proved.

If Ψ_1 has n_1 distinguished points, then there are three cases:

a) either there are strictly more than one distinguished diffeomorphism which is a carrousel with n' points, $n' < n_1$, thus with $n' < n$;

b) either there is only one distinguished diffeomorphism which is a carrousel with n' points, $n' = n_1$ and $n_1 < n$;

c) either there is only one distinguished diffeomorphism which is a carrousel with $n_1 = n$ points but this carrousel simplifies after $k-1$ steps.

We may consider the cases a), b) and c) together by doing the inductive hypothesis:

(H') The Theorem (3.3) is true when Ψ is a carrousel with n' distinguished points and which simplifies after l, steps with $n' < n$ or $n' = n$ and $l' < l$.

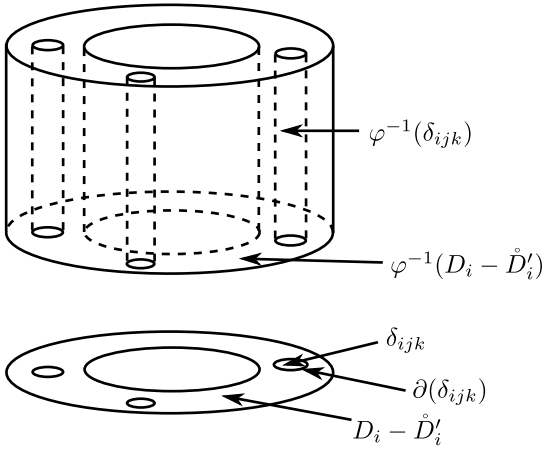
Now we suppose that Ψ is a carrousel with n distinguished points **169**
and which simplifies after l steps.

Then the diffeomorphism h induces an automorphism of $H_*(\varphi^{-1}(\delta_{1jk}))$ which is quasi-unipotent according to the induction hypothesis as we have the commutative diagram.

$$\begin{array}{ccc} \varphi^{-1}(\delta_{1jk}) & \xrightarrow{h_{1jk}} & \varphi^{-1}(\delta_{1jk}) \\ \downarrow{\scriptstyle \delta_{1jk}} & & \downarrow{\scriptstyle \varphi_{1jk}} \\ \delta_{1jk} & \xrightarrow{\Psi_{1jk}} & \delta_{1jk} \end{array}$$

where Ψ_{1jk} is the distinguished carrousel of Ψ_1 induced by $\Psi_1^{q_1}$ and h_{1jk} is induced by h^{q_1}.

$$\begin{aligned} &\text{Notice that :} && U_1 = \varphi^{-1}(U_1 - \bigcup_{j,k} \mathring{\delta}_{1jk}) \bigcup_{j,k} \varphi^{-1}(\delta_{1jk}). \\ &\text{We denote :} && \varphi^{-1}(U_1 - \bigcup_{j,k} \mathring{\delta}_{1jk}) = V_1. \\ &\text{We have :} && V_1 \cap (\bigcup_{j,k} \varphi^{-1}(\delta_{1jk})) = \bigcup_{j,k} \varphi^{-1}(\partial\delta_{1jk}). \end{aligned}$$



□

We prove **170**

Lemma 3.5. *h^{q_1} induces a quasi-unipotent automorphism of $H_*(V_1)$ and of $H_*(\partial\delta_{1jk})$.*

Actually this lemma is a consequence of

Lemma 3.6. *Let $\varphi : X \to Y$ be a smooth fibration of a smooth manifold X onto a relatively compact manifold Y and $\Psi : Y \to Y$ a mapping such that $\Psi^q = IdY$ and $h : X \to X$ such that we have the commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow{\Psi} & Y \end{array}$$

Suppose that the restriction of h^q to $\varphi^{-1}(x)$ for any $x \in Y$ induces a quasi-unipotent automorphism of $H_*(\varphi^{-1}(x))$, then h induces a quasi-unipotent automorphism of $H_*(X)$.

Proof. Considering Ψ^q and h^q instead of Ψ and h we may suppose that $\psi = \mathrm{Id}\, Y$.

Then our result follows from an obvious spectral sequence reasoning or we can prove it by considering a covering $U_1, \ldots, U_s$ of Y by contractible open sets such that $U_i \cap U_j$ are contractible. Then prove by induction that the restriction of h to $\varphi^{-1}(U_1 \cup \ldots \cup U_i)$, $i = 1, \ldots, s$ induces a quasi-unipotent automorphism of $H_*(\varphi^{-1}(U_1 \cup \ldots \cup U_i))$.

Now we have the Mayer-Vietoris sequence

$$\to \bigoplus_{j,k} H_k(\partial\delta_{1jk}) \to \bigoplus_{j,k} H_k(\delta_{1jk}) \oplus H_k(V_1) \to H_k(U_1) \to .$$

The actions of h^{q_1} on $\bigoplus_{j,k} H_k(\partial\delta_{1jk})$ and $H_k(V_1)$ are quasi-unipotent because the points of $\partial\delta_{1jk}$ and V_1 are rolling points. Thus this comes from our hypothesis 2) in (3.3) and the Lemma (3.6). □

Thus we have proved (3.4) when $i = 1$.

Let us suppose we have proved (3.4) for $1 \leqslant i < i_0$. We shall prove **171** it for $H_*(U_{i_0})$. Notice that:

$$U_{i_0} = U'_{i_0-1} U \varphi^{-1}(D_{i_0} - \mathring{D}'_{i_0-1}). \tag{3.7}$$

As $U_{i_0-1} \subset U'_{i_0-1}$ is a homotopy equivalence, the action of h on $H_*(U'_{i_0-1})$ is quasi-unipotent. From Lemma (3.6) and the hypothesis 2) of (3.3) we obtain that the action on $H_*(\varphi^{-1}(\partial D'_{i_0-1}))$ is quasi-unipotent. We obtain our result by using again the Mayer-Vietoris sequence coming from (3.7) if we prove that the action of h on $H_*(\varphi^{-1}(D_{i_0} - \mathring{D}'_{i_0-1}))$ is quasi-unipotent.

Let $\Psi_{i_0 jk} : \delta_{i_0 jk} \to \delta_{i_0 jk}$ be a distinguished carrousel associated to the roll Ψ_{i_0}. From the inductive hypothesis $h^{q_{i_0}}$ induces a quasi-unipotent automorphism of $H_*(\varphi^{-1}(\delta_{i_0 jk}))$. The Lemma (3.6) applies to $\varphi^{-1}(D_{i_0} - (\mathring{D}'_{i_0-1} \bigcup_{j,k} \delta_{i_0 jk}))$ which is a smooth fibration onto $D_{i_0} - D'_{i_0-1} \bigcup_{j,k} \delta_{i_0 jk}$ by φ. Thus because of (3.6) and 2) of (3.3)$h^{q_{i_0}}$ induces a quasi-unipotent automorphism of $H_*(\varphi^{-1}(D_{i_0} - (D'_{i_0-1} \bigcup_{j,k} \delta_{i_0 jk}))$. For the same reason (application of the Lemma (3.6)), $h^{q_{i_0}}$ induces a quasi-unipotent automorphism of $H_*(\varphi^{-1}(\bigcup_{j,k} \partial\delta_{i_0 jk}))$. Using again the Mayer-Vietoris ar-

gument, h^{qi_0}, thus h itself, induces a quasi-unipotent automorphism of $H_*(\varphi^{-1}(D_{i_0} - \mathring{D}'_{i_0-1}))$.

Now our Theorem (3.3) is proved as $U_k \subset X$ is a homotopy equivalence, as $D - \mathring{D}_k$ does not contain any critical values of φ.

Corollary 3.8 (Monodromy theorem)**.** *The local monodromy of f at x is quasi-unipotent.*

Conclusion. The preceding proof has explicited a filtration of $H_*(F)$ where F is the Milnor fibration and $F' = F \cap \{z_1 = 0\}$:

$$H_*(F') \subset H_*(U_1) \subset H_*(U_2) \subset \ldots \subset H_*(U_k) = H_*(F)$$

which is invariant under the action of the local monodromy.

One can prove that

$$H_*(F, F') \cong \bigoplus_{i=1}^{k} H_*(U_i, U_{i-1}) \text{ with } U_0 = F'.$$

The relative monodromy on $H_*(F, F')$ is then the direct sum of the **172**
actions of h on $H_*(U_i, U'_{i-1})$. By excision it is clear that $H_*(\varphi^{-1}(D_i - \mathring{D}'_{i-1}), \varphi^{-1}(\partial D'_{i-1}))$ is isomorphic to $H_*(U_i, U'_{i-1})$.

Now in the case of complex hypersurfaces of $\mathbf{C}^{n+1}$, one can prove that $H_n(\varphi^{-1}(\delta_{ijk})) = 0$. Using a Mayer-Vietoris type argument as in the proof of the Lemma (3.6) above, we obtain that the relative monodromy induces an automorphism of $H_n(U_i, U_{i-1})$ the eigen values of which are p_i-th power of the eigenvalues of the local monodromy of the restriction of z_1 to the hypersurface $f^{q_i} - z_1^{p_i} = 0$, where p_i and q_i are integers, $(p_i, q_i) = 1$ and $2\pi\dfrac{p_1}{q_1}$ is the rotation angle of the roll Ψ_i.

By a more precise study of this geometry this should lead to an exact knowledge of the eigenvalues of the local monodromy by induction on the dimension. As it is seen the rotation angles $2\pi\dfrac{p_i}{q_i}$ interferes in this study. We want to point out that the interest of these $\dfrac{p_i}{q_i}$ have been arisen algebraically by B. Teissier in [12] and [13] in the case when $f = 0$ is

a complex hypersurface with isolated singularity. In [10], M. Merle has computed these numbers in terms of Puiseux exponents of the curve when $n = 1$.

In fact it seems that one can likely study the Gauss-Manin connection of the singularity of f at x by using this geometry. This should lead to a relative Gauss-Manin connection. One expects a knowledge about the link between the roots of the b-polynomial (Bernstein polynomial) of f and the eigenvalues of its local monodromy at x as it has been done by B. Malgrange in [9]. This should give another proof of the rationality of the roots of the b-polynomial which does not use the resolution of the singularities of f (cf. [4]).

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On Euclid Algorithm

By Masayoshi Nagata

THROUGHOUT THIS article, we shall mean by a ring a commutative ring with identity. **175**

Roughly speaking, an integral domain which admits the Euclid algorithm is called a Euclid ring. But there are some distinctions among the ways conditioning the Euclid algorithm, and we want to discuss the subject. Since the Euclid algorithm can be observed independently of the property that the ring is an integral domain, we dare define a Euclid ring without assuming the integrity. Thus:

Definition. *A ring E is called a Euclid ring if there is a mapping ρ of $E-\{0\}$ into a suitable well-ordered set W so that it holds that if $a, b \in E$, $a \neq 0$, then there are $q, r \in E$ such that*

$$b = aq + r, \quad \text{and either} \quad \rho r < \rho a \quad \text{or} \quad r = 0. \tag{*}$$

In this case, we say also that (E, W, ρ) is a Euclid ring.

Of course, main importance is in the case of a usual Euclid ring, for which we add the assumption:

(1) E is an integral domain.

Now, the distinction of the definitions of a Euclid ring comes from whether or not we assume some of the following two conditions:

(2) If b is a multiple of $a (a, b \in E, b \neq 0)$, then $\rho b \geqslant \rho a$.

(3) W is isomorphic to the set **N** of natural numbers.

Our main results are the following:

Theorem 1.2. *If (E, W, ρ) is a Euclid ring, then there is a mapping $\tau : E - \{0\} \to W$ such that (E, W, τ) is a Euclid ring satisfying the condition (2).*

Thus, though (2) restricts the choice of the mapping ρ, it does not restrict the class of Euclid rings.

Theorem 2.3. *There is a Euclid ring (E, W, ρ) such that (i) E is an*
176 *integral domain and (ii) for any mapping $\tau : E - \{0\} \to \mathbf{N}$, $(E, \mathbf{N}, \tau)$ cannot be a Euclid ring.*

Thus, even in the case of integral domains, the condition (3) reduces the class of Euclid rings. As will be seen by Theorem 4.2, a structure theorem of a non-integral Euclid ring, there are easy examples of similar character if we do not assume E to be an integral domain. The existence of such an integral domain was asked by P. Samuel, About Euclidean Rings, *J. Algebra*, 19 (1971), who defined an Euclid ring without assuming integrity. Hiblot [*C. R. Acad. Sc. Paris*, 281 (22 Sept. 1975), ser. A] claimed an example, but his proof contains some errors.

In order to make this article to be selfcontained, we repeat some of the results in the article of Samuel.

1 The condition (2).

A local principal ideal ring is either an Artin local ring or an integral domain. This means that if E is a principal ideal ring and if Q, Q' are mutually distinct primary components of the zero ideal, then there is no maximal ideal containing both of Q, Q'. Thus:

Lemma 1.1. *A principal ideal ring E is the direct sum of certain principal ideal rings $E_1, \ldots, E_m$ such that each E_i is either an Artin local ring or an integral domain.*

Now we prove:

Theorem 1.2. *If (E, W, ρ) is a Euclid ring, then there is a mapping $\tau : E - \{0\} \to W$ such that (E, W, τ) is a Euclid ring satisfying the condition (2) (with τ instead of ρ).*

Proof. As is well known, a Euclid ring is a principal ideal ring. Applying Lemma 1.1 to E, we obtain $E_1, \ldots, E_m$. Let e_i be the identity of E_i. Now, let U be the unit group of E and define τ by: $\tau a = \min\{\rho(au) | u \in U\}$. If $a, b \in E$, $a \neq 0$, then there are $u \in U$ and $q, r \in E$ such that $\tau a = \rho(au)$, $b = auq + r$, $\rho r < \rho(au)$ or $r = 0$. Thus (E, W, τ) is a Euclid ring. Let a be a non-zero element of E, and set $S = \{c \in E | ac \neq 0\}$. Choose $d \in S$ so that $\rho(ad) \leqslant \rho(ac)$ for any $c \in S$. Then there are q, $r \in E$ such that $a = adq + r$, either $\rho r < \rho(ad)$ or $r = 0$. Since r is a mul- **177**
tiple of a, our choice of d shows that $r = 0$. Thus $a(1 - dq) = 0$. In each E_i, $(ae_i)(e_i - de_i qe_i) = 0$. Therefore by the face that E_i is either a local ring or an integral domain with identity e_i, we see that either $ae_i = 0$ or de_i, qe_i are units in E_i. If $ae_i = 0$, then we may change d by adding any element of E_i and therefore there is a unit u such that $ad = au$. By our assumption, $\tau a = \tau(au)$. □

Note that we have proved that if b is a non-zero multiple of a and if $\tau a = \tau b$, then b is an associate of a (i.e., an element of aU). Therefore if $a, b \in E$ and if b is a proper multiple of a (i.e., b is a non-zero multiple of a and is not in aU), then $\tau b > \tau a$. So we have

Corollary 1.3. *If (E, W, ρ) is a Euclid ring satisfying the condition (2), and if b is a proper multiple of a $(a, b \in E)$, then $\rho b > \rho a$.*

2 An example

Before stating our example, we prove some preliminary results.

Lemma 2.1. *Assume that (A, M) is an Artin local ring, and B is an Artin ring containing A, sharing the identity with A and is a finite A-module. Let A^* and B^* be the unit groups of A and B, respectively. If $B \neq A$, then $\sharp(B^*/A^*) \geqslant \sharp(A/M)$.*[*]

[*] If M is a set, $\sharp(M)$ denotes the cardinality of M.

Proof. First we consider the case where B is not a local ring. Let J be the radical of B and consider the natural homomorphism $\varphi : B \to B/J$. The groups B^*, A^* are mapped surjectively to the unit groups of φB, φA, respectively, and therefore we may assume that $J = 0$. Then B is the direct sum of fields $B_1, \ldots, B_s (s \geqslant 2)$. Since each B_i contains Ae_i (e_i being the identity of B_i), we see the assertion easily in this case. Assume next that (B, N) is a local ring. If $B/N \neq A/M$, then by taking an element e of B such that $(e \bmod N) \notin A/M$ and considering cosets of $l + ue (u \in A^*)$, we see the assertion in this case. If $B/N = A/M$, then $N \neq MB$ by the lemma of Krull-Azumaya. Therefore, taking an element e of N outside of MB, we prove the assertion similarly. □

Next we introduce the notion of a canonical structure of a Euclid ring. First we make a well-ordered set W to be canonical. Namely, we
178 take the ordinal number ν which represents the order-type of W; W is isomorphic to the set W' of ordinal numbers smaller than ν. Therefore we identify W with W' at this step. We set $W'' = W \cup \{\nu\}$. Note that the least member of W' is 0 (order-type of the empty set). Now let E be an arbitrary ring. For every $\mu \in W''$, we define R_μ and S_μ inductively as follows: (i) R_0 is the unit group, (ii) if $\mu \in W''$ and if R_λ are defined for all $\lambda < \mu$, then $S_\mu = \{0\} \cup (\bigcup_{\lambda<\mu} R_\lambda)$ and $R_\mu = \left\{a \in E - S_\mu \middle|\right.$ for any $b \in E$, there are $q, r \in E$ such that $\left. b = aq + r, r \in S_\mu \right\}$. Now, our definition of a Euclid ring shows the following fact:

Lemma 2.2. *There is a mapping $\rho : E - \{0\} \to W$ such that (E, W, ρ) is a Euclid ring if and only if $S_\nu = E$. In this case, (E, W, τ) is a Euclid ring with τ such that $\tau x = \mu$ if and only if $x \in R_\mu$. Furthermore if $x \in E - \{0\}$, then $\tau x \leqslant \rho x$.*

Of course, W may be too big. The smallest possible one is $W^* = \tau(E - \{0\})$. This (E, W^*, τ) is called the *canonical structure* of the Euclid ring E.

Now we come to the construction of a Euclid integral domain by which the condition (3) cannot be satisfied.

Consider a set of algebraically independent elements X_t indexed by

the set **R** of real numbers and we set $A = P_S$ where P is the polynomial ring in these X_t over the ring **Z** of rational integers and $S = \{f \in P|$ coefficients of f have no proper common factor $\}$. Then every element a of A is expressed in the form qs with s in the multiplicative group generated by S and q a natural number or zero. This q is uniquely determined by a, called the *absolute value* of a and denoted by $|a|$.

We order prime numbers and denote them by $p_1, p_2, \ldots, (p_1 < p_2 < \ldots)$. For each $n \in \mathbf{N}$, we take natural numbers a_n and e_{in} (for $i = 1, 2, \ldots, n$) and also elements b_0, b_n, b_n^* as follows:

(i) Every e_{in} is a multiple of P_i.

(ii) $a_n = \prod_i p_i^{e_{in}}$, $b_0 = 1$, $b_n = \prod_{i \leqslant n} a_i$ and $b_n^* = \sum_{i=0}^{n-1} b_i x_i$.

Next we consider the polynomial ring $K[T]$ in one variable T over the field K of quotients of A. In this ring, we define T_n by $T = T_0 = b_n^* + b_n T_n$ and we set $B_n = A[T_n]$, $B = \cup_n B_n$. Note that $T_n = X_n + a_{n+1} T_{n+1}$. **179**

Let S^* be the multiplicatively closed subset generated by $T_0, T_1, \ldots, T_n, \ldots$ and set $E = B_{S^*}$. Now we claim:

Theorem 2.3. *The integral domain E is a Euclid ring, but, not under the condition (3).*

The proof proceeds stepwise. First consider $f(T) = d_0 + d_1 T + \ldots + d_s T^s \in A[T] (d_i \in A, d_s \neq 0)$. For each $m \in \mathbf{N}$, f is expressed as the product of a natural number c_m and a primitive polynomial f_m in T_m over A. Obviously c_m divides c_{m+1}. We want to show:

Lemma 2.4. *There is a natural number m such that for every natural number $n \geqslant m$, it holds that (i) $c_n = c_m$ and (ii) the constant term of f_n is a unit in A.*

Proof. Multiplying some element of S, we may assume that all d_i are in P. Then we can take a natural number m such that there is no X_t, with $t \geqslant m - 1$, appearing in some d_i. Then we consider $f = c_{m-1} f_{m-1}$, $f_{m-1} = e_0 + e_1 T_{m-1} + \ldots + e_s T_{m-1}^s$. Then the constant term of f in its expression as a polynomial in T_m is $c_{m-1} f_{m-1}(X_{m-1})$ because $T_{m-1} =$

$X_{m-1} + a_m T_m$. Since $f_{m-1}(T_{m-1})$ is primitive and since X_{m-1} does not appear in the coefficients, we see that $f_{m-1}(X_{m-1}) \in S$. Thus this m is the required natural number.

f_m obtained above enjoys the properties that (i) it is a primitive polynomial in some T_m over A and (ii) for every natural number n not smaller than m, the constant term of f_m as a polynomial in T_n over A is a unit in A. Such an element of B is called a *primitive* element of B.

Lemma 2.4 can be applied to any non-zero element f of $K[T]$, and $f = cf_m$ with a positive rational number c and a primitive element f_m. This c is uniquely determined by f and is called the *content* of f. Note the $f \in B$ if and only if the content is a natural number. Note also that T_n are all primitive. □

Lemma 2.5. *B is a principal ideal domain. As ideal is maximal if and*
180 *only if it is generated either by some $p_i (i \in \mathbf{N})$ or by a primitive element f in B which is irreducible and of positive degree as an element of $K[T]$. The unit group of B is the group generated by S and therefore every residue class of B/p_iB is represented either by 0 or a unit.*

Proof. The last half is obvious. As for the first half, similar statement holds for every B_m and a prime element f in B_m is not a prime element in B_{m+1} if and only if f is a primitive polynomial in T_m but not in T_{m+1}. □

Corollary 2.6. *E is a principal ideal domain and the unit group of E is generated by SS^*.*

If $f \in E - \{0\}$, then with some $s \in S^*$, it holds that $sf \in B$ and we can apply the factorization given in Lemma 2.5 to sf and we obtain $sf = cf_m$ with content c and a primitive element f_m. f_m may contain some factors which are in S^*; taking out all such factors, we have the following factorization:

$f = cpu$; where c is the content, p is primitive and having no factor in S^* and u a unit in E.

These factors are uniquely determined by f (as for p, u, we observe them within unit factors in A) and therefore we denote them by $c(f)$, $p(f)$, $u(f)$, respectively, from now on.

Now we consider $W = \mathbf{N} \times \mathbf{N}$ (with lexicographical order), and we define a mapping $\rho : E - \{0\} \to W$ by $\rho f = (1 + \deg p(f), c(f))$.

We claim:

Lemma 2.7. (E, W, ρ) *is a Euclid ring.*

Proof. Let f, g be non-zero elements of E and we want to show the existence of q, $r \in E$ such that $g = fq + r$ and either $r = 0$ or $\rho r < \rho f$. Since unit factors of f, g can be disregarded (cf. the proof of Theorem 1.2), we may assume that $f = c(f)p(f)$, $g = c(g)p(g)$. We use an induction on ρg. If $\rho g < \rho f$, then $q = 0$, $r = g$. So we assume that $\rho g \geqslant \rho f$. Then either (i) $\deg g > \deg f$ or (ii) $\deg g = \deg f$ and $c(g) \geqslant c(f)$. In the case (i), if we express f and g as polynomials in T_m with sufficiently large m, then the coefficient of the highest degree term in f divides that of g, and we can reduce to the case where $\deg g$ is lower than before. Consider the case (ii). In B_m with sufficiently large m, **181**
the constant terms of f, g are $c(f)$·unit, $c(g)$·unit. Therefore, by some q which is a unit in A, the constant term of r has absolute value $c(g) - c(f)$. If $c(f) \neq c(g)$, then $\deg r \leqslant \deg f$ and $c(r) < c(g)$, and this case is finished by our induction. If $c(f) = c(g)$, then since T_m is a unit in E, we have $\deg p(r) < \deg p(f)$. □

We consider the canonical structure (E, W', τ) of the Euclid ring E. In order to prove Theorem 2.3, it suffices to show:

Proposition 2.8. $W' \simeq W = \mathbf{N} \times \mathbf{N}$.

In order to prove this, we adapt symbols R_μ, S_μ as in the definition of a canonical structure. So, R_0 is the unit group of E. We prove the assertion stepwise.

(i) *If n is a natural number, then $R_n = \bigcup mR_0$ where m runs through $M_n = \left\{\pi_{p_i}^{e_i} \mid \sum e_i = n (e_i \geqslant 0)\right\}$.*

Proof. In proving (i), we use induction arguments on n. First note that every prime number is in R_1 in view of the last half of Lemma 2.5. Note also that if $f (\in E - \{0\})$ has a proper factor which is not in S_n then f

cannot be in R_n in view of Corollary 1.3. By induction, no element of M_n is in S_n. If $m \in M_n$, then in E/mE, an element f such that $c(f)$ is coprime to m is represented by a unit. Therefore, in general, an arbitrary element f is congruent to an element of the form $d \cdot$ unit with $d =$ (the *G.C.M.* of m and $c(f)$) and we see that $m \in R_n$. Thus $\bigcup_{m \in M_n} mR_0 \subseteq R_n$. Conversely, assume that $f \in R_n$. We may assume that $f = c(f)p(f)$. If $f = c(f)\cdot$unit, then the observation above and Corollary 1.3 imply that $c(f) \in M_n$. In order to show that $f = c(f)\cdot$unit, it suffices to show that if f is a prime element of E such that $f = p(f)$, then $f \notin R_n$ (by virtue of Corollary 1.3). Assume first that $\deg f = 1$. Then $f = d_0 + d_1 T_m$ with sufficiently large natural number $m (d_i \in A, |d_0| = 1)$. Then in E/fE, by our choice of a_n, we see that every sufficiently large prime number q appears as a factor of a (unit in E modulo f) only in a power of q^q. Thus q appears as a factor of residue classes of elements of $S_n - \{0\}$ only in
182 the form q^{sq-t} with t such that $0 \leqslant t < n$. Therefore $f \notin R_n$. If $\deg f > 1$, then Lemma 2.1 implies that $f \notin R_n$. □

As a consequence, we have:

(ii) $S_\omega = \bigcup uA$, where ω denotes the ordinal number corresponding to **N**, and u runs through the multiplicative group generated by $T_0, T_1, \ldots$.

(iii) For $n \in \mathbf{N}$ and $m \in \mathbf{N} \cup \{0\}$, it holds that

$$R_{n\omega+m} = \{f \in E - \{0\} | \deg p(f) = n, c(f) \in R_m\}.$$

Proof. By induction, $S_{n\omega} = \{0\} \cup (\cup fR_0)$ where f runs through $F_n = \{f \in E | \deg p(f) < n\}$. Therefore, if f is primitive and if $\deg p(f) = n$, then our proof of Lemma 2.7 shows that $f \in R_{n\omega}$. Conversely, if $f \in R_{n\omega}$, then $\deg p(f) \geqslant n$; if $\deg p(f) > n$, then our method in proving Lemma 2.1 shows that $f \notin R_{n\omega}$. If $c(f) \neq 1$, then Corollary 1.3 shows that $f \notin R_{n\omega}$. Thus we proved the case where $m = 0$. For $m > 0$, our proof (i) above is adapted and we complete the proof. □

Now, this (iii) completes the proof of Proposition 2.8.

3 An operation on ordinal numbers

For each ordinal number α, we denote by W_α the set of ordinal numbers β such that $\beta < \alpha$. For a given cardinality $\mathbf{b}$, we denote by $W_{\mathbf{b}}$ the set of ordinal numbers α such that $\sharp(W_\alpha) < \mathbf{b}$. Note that $\sharp(W_{\mathbf{b}}) = \mathbf{b}$ and the ordinal number representing the order-type of $W_{\mathbf{b}}$ is the beginning ordinal number for the cardinality $\mathbf{b}$ in case $\mathbf{b}$ is an infinite cardinality.

Assuming that $\mathbf{b}$ is an infinite cardinality, we define an operation $*$ on $W_{\mathbf{b}}$ by the following properties:

(i) $\alpha * \beta = \beta * \alpha$, (ii) $0 * 0 = 0$, (iii) $(\alpha + 1) * \beta = (\alpha * \beta) + 1$

(iv) if $(\alpha, \beta) \neq (0, 0)$, then $\alpha * \beta = \sup(\{\alpha * \beta' + 1 | \beta' < \beta\} \cup \{\alpha' * \beta + 1 | \alpha' < \alpha\})$.

One sees easily that this $*$ is well defined by these requirements, if we allow $\alpha * \beta$ to be in a bigger set of ordinal numbers, though we shall show that $\alpha * \beta$ belongs to $W_{\mathbf{b}}$ in Proposition 3.1 below. We shall call $*$ **183** *maximal addition*; $\alpha * \beta$ is called the *maximal sum* of α and β.[†]

The first remark is that $\alpha * \beta$ is independent of $W_{\mathbf{b}}$ containing α, β because of our definition.

We give further results on computing the value of maximal sums. From now on, Greek letters will denote ordinal numbers belonging to $W_{\mathbf{b}}$.

Proposition 3.1. *(i)* $W_{\mathbf{b}} * W_{\mathbf{b}} = W_{\mathbf{b}}$. *(ii)* $\alpha * 0 = \alpha$.

(iii) If m, n are finite, then $(\alpha + m)(\beta + n) = (\alpha * \beta) + m + n$.

(iv) If $\alpha < \alpha'$, *then* $\alpha * \beta < \alpha' * \beta$.

Proof. (ii) ~ (iv) are obvious by the definition. By (ii), we see that $W_{\mathbf{b}} \subseteq W_{\mathbf{b}} * W_{\mathbf{b}}$. By our definition, if $\alpha, \beta \in W_{\mathbf{b}}$, then $W_{\alpha * \beta} \subseteq (W_\alpha \cup \{\alpha\}) * (W_\beta \cup \{\beta\})$. Therefore $\sharp(W_{\alpha * \beta}) \leqslant \sharp(W_\alpha \cup \{\alpha\}) \times \sharp(W_\beta \cup \{\beta\}) < \mathbf{b}$. Thus $W_{\alpha * \beta} \subset W_{\mathbf{b}}$ and $\alpha * \beta \in W_{\mathbf{b}}$. □

[†] One can prove easily by virtue of Theorem 3.4 below that $\alpha * \beta$ is the largest among ordinal numbers which can be obtained in the form $\alpha_1 + \beta_1 + \ldots + \alpha_s + \beta_s$ with α_i, β_i such that $\alpha = \alpha_1 + \ldots + \alpha_s, \beta = \beta_1 + \ldots \beta_s$.

(iii) above shows that in order to compute the value of $\alpha * \beta$, it is essential to know the value in the case where α, β are limit ordinal numbers, and let us investigate the case.

For a limit ordinal number λ, we consider $\Gamma_{\gamma\lambda} = \{\mu \in W_{\mathbf{b}} | \gamma < \mu < \lambda\}$ for every $\gamma < \lambda$ and let τ_λ be min {ordinal number representing the order-type of $\Gamma_{\gamma\lambda} | \gamma < \lambda$}. We call τ_λ the *weight* of λ, and the symbol τ will maintain its meaning in this section.

Lemma 3.2. *(i) If λ is the beginning ordinal number for an infinite cardinality, then the weight of λ is λ itself.*

(ii) If λ is a limit ordinal number, then $\alpha + \lambda$ is also a limit ordinal number and $\tau_\lambda = \tau_{\alpha+\lambda}$.

(iii) If $\alpha = \tau_\lambda$ for some λ, then $\tau_\alpha = \alpha$.

Proof is easy and we omit it.

Theorem 3.3. *Let α be an infinite ordinal number. Then the set $\Gamma = \{\tau_\lambda | \lambda$ limit ordinal $\leqslant \alpha\}$ has a maximal member, say δ, and $\tau_\delta = \delta$.*
184 *Furthermore, $M = \{\beta | \beta$ limit ordinal $\leqslant \alpha, \tau_\beta = \delta\}$ is a finite set and δ is the smallest member in M.*

The maximal member of M is called the *deepest limit ordinal number* in α.

Proof. Consider $\delta = \sup \Gamma$. If $\tau_\delta < \delta$, then there is a γ smaller than δ such that the order-type of $\Gamma_{\gamma\delta}$ is smaller than δ. Then, with $\Gamma' = \{\tau_\beta \in \Gamma | \tau_\beta > \gamma\}$, we have $\delta = \sup \Gamma = \sup \Gamma' < \delta$, a contradiction. Thus $\delta = \tau_\delta$ and δ is the smallest member in M. If M were infinite, then we have an ascending chain $\delta = \beta_0 < \beta_1 < \ldots < \beta_n < \ldots$ in M. Let $\mu = \sup_{n \in \mathbf{N}} \beta_n$. For any $\epsilon < \mu$, $\Gamma_{\epsilon\mu}$ contains infinitely many β_i which are in M; for $\epsilon' < \delta$, $\Gamma_{\epsilon'\mu}$ does not contain any member of M. Thus $\tau_\delta < \tau_\mu$, a contradiction. □

Maximal sums can be computed making use of the following:

Theorem 3.4. *For infinite ordinal numbers α, β, let $\delta_\alpha, \delta_\beta$ be the deepest limit ordinal numbers in α, β, respectively.*

Assume that $\tau_{\delta_\alpha} \geqslant \tau_{\delta_\beta}$. Write $\alpha = \delta_\alpha + \alpha', \beta = \delta_\rho + \beta'$. Then

(i) $\alpha * \beta = \delta_\alpha + (\beta * \alpha')$,

(ii) If $\tau_{\delta_\alpha} = \tau_{\delta_\beta}$, then $\alpha * \beta = (\delta_\alpha + \delta_\beta) + (\alpha' * \beta')$.

Thus the computation can be reduced to the case where $\tau_{\delta_\alpha}, \tau_{\delta_\beta}$ are smaller.

Proof. (ii) follows easily from (i), and it suffices to prove (i). We use an induction on (α, β); namely, we assume the validity of (i) for (α'', β'') such that $\alpha'' \leqslant \alpha$, $\beta'' \leqslant \beta$, $(\alpha'', \beta'') \neq (\alpha, \beta)$. If $\alpha > \delta_\alpha$, $\beta > \delta_\beta$, then (i) is shown easily by induction hypothesis. The most important case is the case where $\alpha = \tau_{\delta_\alpha}$, $\beta = \tau_{\delta_\beta}$. By induction, if $\beta'' < \beta$, then $\alpha * \beta'' = \alpha + \beta''$. If $\tau_{\delta_\alpha} = \tau_{\delta_\beta}$, then by the symmetry, we have $\alpha + \beta = \alpha * \beta$. If $\tau_{\delta_\alpha} > \tau_{\delta_\beta}$ and if $\alpha'' < \alpha$, then $\alpha'' * \beta < \tau_{\delta_\alpha}$ by induction. Therefore $\alpha * \beta = \alpha + \beta$ in this case. The remaining case can be proved similarly. □

Corollary 3.5. *Assume that α, β are limit ordinal numbers and that $\tau_\alpha \geqslant \tau_\beta$. Then $\alpha * \beta = \sup\{\alpha * \beta' | \beta' < \beta\}$ and $\tau_{\alpha * \beta} = \tau_\beta$.*

Theorem 3.6. *The maximal addition satisfies the associativity. Namely,* **185**
$(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

Proof. We use induction on (α, β, γ). If some of α, β, γ is not a limit ordinal number, the assertion is obvious by induction and by (iii) in the definition. If all of α, β, γ are limit ordinal numbers, then we prove the assertion easily by induction and by Corollary 3.5 above. □

4 Structure of generalized Euclid rings

We maintain the notation of §3 except for ρ, ρ', ρ_i, φ (which are mappings) and $W_i (i = 1, \ldots, s)$.

In this section, we want to show the following three theorems.

Theorem 4.1. *Assume that a ring E is the direct sum of rings $E_1, \ldots, E_s$. Then E is a Euclid ring if and only if every E_i is a Euclid ring. Consequently, a Euclid ring is the direct sum of Euclid rings $E_i, \ldots, E_s$ such that each E_i is either an integral domain or an Artin local ring.*

Theorem 4.2. *Assume that $(E_i, W_i, \rho_i)(i = 1, \ldots, s; W_i \subset W_{\mathbf{b}}$ with a sufficiently large cardinality* **b***) are Euclid rings and let λ_i be the ordinal number representing the ordertype of W_i. Extend ρ_i so that $\rho_i 0 = \lambda_i$ and define $\rho : E = E_1 \oplus \ldots \oplus E_s \to W_{\mu+1}$ with $\mu = \lambda_1 * \ldots * \lambda_s$ by:*

$$\rho(a_1 + \ldots + a_s) = (\rho_1 a_1) * \ldots * (\rho_s a_s) \quad (a_i \in E_i).$$

Then (E, W_μ, ρ) is a Euclid ring.

If every (E_i, W_i, ρ_i) is the canonical structure of E_i, then (E, W_μ, ρ) is the canonical structure of E.

Theorem 4.3. *If E is a Euclid ring, then (i) a ring which is a homomorphic image of E is a Euclid ring and (ii) any ring of quotients of E is a Euclid ring.*

As for the proof of Theorem 4.1, the if part follows from Theorem 4.2, its *only if* part follows from Theorem 4.3 (i) and its last part follows from Lemma 1.1.

186 Theorem 4.2 follows from the definition of maximal addition, noting that the condition (*) in the definition of a Euclid ring is equivalent to the following:

(*) If $a, b \in E$, $a \neq 0$, then there are $q, r \in E$ such that $b = aq + r$, and $\rho r < \rho a$ or $r = a$.

Theorem 4.3(i) follows from:

Proposition 4.4. *Let (E, W, ρ) be a Euclid ring. If $\varphi : E \to E'$ is a surjective ring homomorphism, then (E', W, ρ') is a Euclid ring with ρ' defined by:*

$$\rho' x = \min\{ \rho y \mid \varphi y = x\}.$$

Proof. Assume that $a', b' \in E'$, $a' \neq 0$. Take $a, b \in E$ so that $\varphi a = a'$, $\rho a = \rho' a'$, $\varphi b = b'$. Then there are $q, r \in E$ such that $b = aq + r$, either $\rho r < \rho a$ or $r = 0$. Then $b' = a'(\varphi q) + \varphi r$, either $\rho'(\varphi r) \leqslant \rho r < \rho a = \rho' a'$ or $\varphi r = 0$. □

Theorem 4.3(ii) follows from:

Proposition 4.5. *Let* (E, W, ρ) *be a Euclid ring and let* S *be a multiplicatively closed subset of* E *not containing 0. Then* (E_S, W, ρ') *is a Euclid ring with* ρ' *defined by:*

$$\rho' x = \min\{\rho y | y \in E, \text{ there are } s, s' \in S, ys/s' = x\}.$$

This can be proved quite similarly to the proof of Theorem 1.2.

In closing this article, the writer likes to remark that:

Although, in order to state the condition (*) in the definition of a Euclid ring, it is natural to extend ρ so that $\rho 0 < \rho a$ for any $a \neq 0$, in view of Theorem 4.2, Proposition 4.4 etc., the writer feels it to be more natural to extend ρ so that $\rho 0 > \rho a$ for any $a \neq 0$.

Principal Bundles on Affine Space

By M.S. Raghunathan

1 Introduction

The affirmative answer to Serre's question on vector bundles on an affine space (due to Quillen [10] and Sublin) leads one to pose the following question (*): **187**

Let k be a field and $A^n_k = \operatorname{Spec} k[X_1, \ldots, X_n] \to \operatorname{Spec} k$ the affine space of dimension n over k. Let G be an affine group scheme (of finite type) over k and P a principal G-bundle over A^n_k. Is P then isomorphic to a bundle of the form $P' \bigotimes_{\operatorname{Spec} k} A^n_k$ where P' is a principal homogeneous space over k.

(By a principal G-bundle P over a scheme X we mean a scheme $p : P \to X$ over X together with a right action $m : P \times G \to P$ such that the diagram

$$\begin{array}{ccc} P \times G & \xrightarrow{m} & P \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ P & \xrightarrow[p]{} & X \end{array}$$

is commutative and the induced morphism

$$P \times G \to P \underset{X}{\times} P$$

is an isomorphism. (It is known that with this definition, there exists for every point $p \in \bar{X} = X \bigotimes_k \bar{k}$, $\bar{k}$ an algebraic closure of k, a neighbourhood U of p and an etale morphism $q : \tilde{U} \to U$ such that $\tilde{U} \underset{\bar{X}}{\times} \bar{P}$ ($\bar{P} = P \bigotimes_k \bar{k}$) is isomorphic as a G space to $\tilde{U} \underset{\operatorname{Spec} k}{\times} G$).

The immediate expectation that Quillen's theorem would generalise to any reductive group turns out to be false: Ojanguran and Sridharan [8a] and Parimala and Sridharan [8b] have results which show that the answer to the question is in general in the negative. Specifically in [8a]
188 it is shown that when G is the group of norm 1 elements in a *noncommutative* division algebra over k, there exist G-bundles over A^2_k which are not obtained by base change. When $k = \mathbf{R}$ in [8b] it is shown that there are infinitely many inequivalent bundles on $A^2_{\mathbf{R}}$ which are not obtained by a base change from $\mathbf{R}$. Using this, one can also show (see Parimala [9] where the case $SO(4)$ over $\mathbf{R}$ is dealt with) (*) has a negative answer for $SO(3)$ and $SO(4)$ over $\mathbf{R}$. These examples suggested that one should look for an affirmative answer to (*) only under further restrictions on k. H. Bass [1] proved that if k is algebraically closed and of characteristic $\neq 2$ and $n = 2$, (*) has an affirmative answer for $G = SO(n)$ the present work began in fact with providing an alternative (and more direct) proof for Bass's theorem.

Before we can describe the results of the present work-which is mainly concerned with the case $n \geqslant 2$ - we need to briefly outline the situation in the case of the affine line over k. For the sake of brevity in later formulations we make the following:

Definition . *A group G (over k) is 'acceptable' if for every extension $L \supset k$, any principal $G \otimes_k L$-bundle over* Spec $L[X]$ *is obtained from a bundle on* Spec L *by the base change $L \to L[X]$.*

G is known to be acceptable in the following cases:

(i) Char $k = 0$, G any group. This is a recent result due to A. Ramanathan and the author [11].

(ii) $G = 0(n)$ (hence also $SO(n)$), Char $k \neq 2$ (Harder; see [6a])

(iii) $G = SL(n)$, $GL(n)$ or $Sp(n)$ (k arbitrary): these cases are obvious.

(iv) G simply connected of inner-type A_n : this follows from the fact that the projective modules over $D[X]$, D a division algebra, are free (see for instance [15, p. 202].

(v) G simply connected of classical type and Char k 'good' for G(Char $k > 5$ is good for all G, (see [11])).

(vi) G a torus of *any* semisimple group of inner type A_n or a spin group: this follows from the earlier mentioned results and some Galois cohomology arguments. For more details see [11].

We expect the following to hold. **189**

Conjecture. *Any smooth simply 'connected' reductive group is acceptable.*

(The assumption of connectedness is evidently essential in view of the existence of Artin Schrier extensions.)

After this discussion we can now formulate the main results of this work. The results are formulated for acceptable groups. It is possible of course to formulate some sharper results and this will be done in §4.

The first result (which should probably be called a lemma rather than a theorem) is

Theorem A. *Assume that G is acceptable. Let P be a principal G-bundle over A^n. Then there is an open subscheme $U \subset A^n$ such that $P \times_{A^n} U$ is obtained from a G-bundle over k by the base change $U \to$* Spec k.

Theorem B. *Assume that G is connected, smooth, reductive and split. Assume further that the natural map $H^1(k, G) \to H^1(k, AdG)$ is trivial (i.e. for every principal homogeneous space over G the principal homogeneous space over the adjoint group AdG obtained by extension of structure group is trivial). Then a G-bundle P over A^n is trivial if and only if for a non-empty open subscheme $U \subset A^n$, $P \times_{A^n} U$ is obtained from a bundle on* Spec k *by the base change $U \to$* Spec k.

An immediate corollary is

Theorem C. *Assume that G is connected, acceptable and reductive and that k is 'separably closed'. Then any principal G-bundle over A^n is trivial.*

We state one final result which is applicable especially to the case of local fields.

Theorem D. *Assume that G is acceptable, connected, semisimple, simply connected and quasi-split. Assume further that the map $H^1(k, G) \to H^1(k, AdG)$ is trivial. Then any principal G-bundle on A^n is trivial.*

190 The following result for the special case $G = 0(n)$-which has attracted considerable interest-is proved in §4.

Theorem E. *Assume that* Char $k \neq 2$ *and the Brauer group $Br(k)$ of k has no 2-torsion. Then any orthogonal bundle on A^n is obtained from an orthogonal bundle on* Spec k *by the base change $A^n \to$ Spec k.*

The method of proof closely parallels-and needless to say, is suggested by - Quillen's in the case of $GL(n)$. We need an obvious extension of a lemma of Quillen's (Theorem 2, §2). Rigidity of the trivial bundle over $\mathbf{P}^1$ (not surprisingly) serves as a replacement of the Horrocks theorem used in Quillen's work. (Parimala [9] gives an example to show that the theorem of Horrocks does not generalise to $0(n)$-bundles).

My thanks are due to many colleagues who took considerable interest in this work. Among them I should mention: Hyman Bass whose lectures in Bombay on his theorem triggered off the present work; C. S. Seshadri with whom I had many stimulating discussions; also R. C. Cowsik, Mohan Kumar, M. Nori and S. Parimala who had helpful comments and suggestions to offer.

2 Some known results

We make use of three essentially known results in our proofs of the main theorems (Theorems B and C). In this section we collect these known results together. (Sketch proofs are given where no proper references are to be found). The first of these is:

Theorem 1. *Let $\mathscr{P}^1$ denote the projective line over k and X be any k-scheme of finite type. Let E be a principal G-bundle over $\mathscr{P}^1 \times X$.*

Let $\pi : \mathscr{P}^1 \times X \to X$ be the natural projection. Suppose now that $x_0 : \operatorname{Spec} L \to X$ is a closed point such that $E \times_{x_0} \operatorname{Spec} L$ is trivial over $\bar{L}$, the algebraic closure of L. Then there exists an open subscheme U of X containing x_0 and a principal G-bundle E' over U such that $E \underset{(\mathscr{P}^1 \times X)}{\times} \pi^{-1}(U)$ is isomorphic to $E' \underset{U}{\times} \pi^{-1}(U)$.

When E is a $GL(n)$-bundle this is a special case of Grothendieck [4, Corollaire 4.6.4.]: note that for the projective line $\mathscr{P}^1$, $H^1(\mathscr{P}^1, O_{\mathscr{P}^1}) = 0$ **191**
where $O_{\mathscr{P}^1}$ is the structure sheaf. In this case, one may assume that E' is in addition the trivial $GL(n)$-bundle.

For handling the general case we need

Lemma. *Any smooth G is isomorphic to a closed subgroup scheme of some $GL(n)$ such that $G(L)(n)/G$ contains no complete reduced subscheme of positive dimension.*

Assume the lemma for the present. Then we can think of a G-bundle as a $GL(n)$-bundle together with a section in the associated fibre space with $GL(n)/G$ as fibre. From what we have remarked earlier the $GL(n)$-bundle may be assumed trivial so that the section is a morphism $\sigma : \mathscr{P}^1 \times X \to GL(n)/G$. Since $GL(n)/G$ contains no complete subscheme of positive dimension σ factors through $\mathscr{P}^1 \times X \to X$ which shows that the G-bundle E itself is obtained from a bundle on X by base change $P^1 \times X \to X$.

We have to establish the lemma. We may assume that G is a closed (k-) subgroup scheme of $GL(m)$ for some m. Let ρ be a (k-) representation of $GL(m)$ on a vector space V and $0 \neq v \in V(k)$ be such that G is the isotropy group scheme at $\bar{v}$ for the action of $GL(m)$ on the projective space $\mathscr{P}(V)$ associated to V. This means the orbit map $GL(m) \to GL(m)\bar{v}$ is an isomorphism of $GL(m)/G$ on the orbit which is a locally closed subscheme of $\mathscr{P}(V)$. (Here $\bar{v}$ is the closed point of $\mathscr{P}(V)$ determined by v). This determines a morphism $\chi : G \to G_m$ such that for any $g \in G(L)$, L a k-algebra,

$$\rho(g)v = \chi(g) \cdot v.$$

If χ is trivial, G is the isotropy at v for the action of $GL(m)$ on V so that $GL(m)/G$ is locally closed in affine space and the assertion follows. We

assume then that χ is non-trivial. Consider the immersion

$$G \xrightarrow{(i,\chi^{-1})} GL(m) \times GL(1) = H.$$

Let I denote the identity representation of $GL(1)$ and $\rho \otimes I$ the representation of H obtained by forming the tensor product of ρ and I. Then the isotropy group at $v \otimes 1$ is seen to be precisely G so that $(GL(m) \times$
192 $GL(1))/\gamma G$ is a locally closed subscheme of an affine space. On the other hand, $GL(m) \times GL(1)$ is in a natural fashion a closed subscheme of $GL(m+1)$ with affine quotient. It follows that $GL(m+1)/G$ is a fibre space over an affine scheme with fibres $(GL(m) \times GL(1))/\gamma G$. The lemma follows immediately from this.

Theorem 2 (Quillen [10])**.** *Let E be a G-bundle on* $\operatorname{Spec} R[T] (\simeq A^1 \times \operatorname{Spec} R)$*, R a k-algebra. Suppose for every maximal ideal $\mathfrak{m} \subset R$, $E \underset{\operatorname{Spec} R}{\times} \operatorname{Spec} R_{\mathfrak{m}}$ ($R_{\mathfrak{m}}$ = localisation at $\mathfrak{m}$) is trivial, then $E \simeq F \underset{\operatorname{Spec} R}{\times} \operatorname{Spec} R[T]$ for some G-bundle F over* $\operatorname{Spec} R$.

Quillen states and proves this for $G = GL(n)$. The general case follows by simply changing GL(n) to G in his argument.

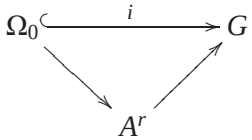
The next result we need is well known.

Theorem 3. *Let G be a 'split' connected semisimple group over k. Let $n = \dim G$ and $l = \operatorname{rank} G$ (= dimension of maximal k-split torus of G). Then G contains an affine open subscheme isomorphic to $A^{n-l} \times \operatorname{Spec} k[T_1, T_1^{-1}, T_2, T_2^{-1} \ldots, T_l, T_l^{-1}]$.*

This is immediate from structure theory: Let T be a maximal k-split torus and $U^{\pm}$ opposing unipotent maximal k-subgroup schemes. Then the morphism $U^+ \times T \times U^- \to G$ given by the multiplication in G is an isomorphism onto the image Ω.

Theorem 4 below must also be regarded as well known: it is a more or less immediate consequence of the work of Chevalley [3] and Steinberg [13]. We give some details, as a proof does not seem to be available explicitly in literature.

Theorem 4. *Let G be a simply connected quasi-split (smooth) group scheme. Then G contains an open (k-) subscheme Ω_0 such that the inclusion $i : \Omega_0 \to G$ factors through an affine space, i.e. there is a commutative diagram of the form*



(for some integer r).

Let T be the centraliser of a k-split torus S in G. Let U and U^- be **193**
opposing maximal unipotent k-subgroup schemes of G normalised by S. Let $N(S)$ be the normaliser of S in G and $\theta \in N(S)(k)$ an element which conjugates U onto U^-. Then the morphism

$$U \times T \times U \to G$$

defined by $(u, t, u') \mapsto u\theta t u'$ on the set of closed points is an open immersion. As a k-scheme U is isomorphic to an affine space A^q, $q = \dim U$. It suffices to show that the inclusion $T' \hookrightarrow G$ for some k-open subset T' of T factors through an affine space. If Δ is the system of simple k-roots determined by U, for each $\alpha \in \Delta$, we have a k-rank 1 quasi split subgroup G_α of G such that $T = \prod_{\alpha \in \Delta} T_\alpha$, a direct product with $T_\alpha \subset G_\alpha$. This shows that the problem immediately reduces to the case when k-rank $G = 1$. The quasi-split groups of k-rank 1 are of one of the following two kinds:

(i) $R_{L/k} SL(2)$,

(ii) $R_{L/K} SU(2,1)$, $SU(2,1)$ being the special unitary group of a hermitian form of Witt index 1 over a quadratic extension of L where L is a finite separable extension of k. Now for the affine n-space $A^n_{L'}$, $R_{L/k} A^n_L$ is isomorphic to A^{nq}_k where $q = \dim_k L$ so that we see that we need only consider the cases of $SL(2)$ and $SU(2,1)$ over k.

In case $G = SL(2)$, take $T' = T$ and the factorisation sought after results from the product decomposition

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Consider now the case $G = SU(2, 1)$: G is the special unitary group of the hermitian quadratic form

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in 3 variables over a quadratic Galois extension K over k. If we denote by $x \mapsto \bar{x}$ the conjugation by the non-trivial element of the Galois group
194 of K/k as well as its extension to $\tilde{L} = L \otimes_k K$ for any $L \supset K$, the L-rational points of a split torus are given by

$$\varphi(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & x^{-1}\bar{x} \end{pmatrix}, \quad x \in \tilde{L} \text{ invertible.}$$

We will deal with the cases $\mathrm{Char}\, k \neq 2$ and $\mathrm{char}\, k = 2$ separately.

Case 1. $\mathrm{Char}\, k \neq 2$. Let T' be the open set whose L-rational points are

$$\{\varphi(x) | x - \bar{x} \text{ is invertible in } \tilde{L}\}.$$

Fix an element $\theta \in K - k$ such that $\theta^2 \in k$. Then any element of K is of the form

$$a + b\theta, a, b \in k,$$

and for any $L \supset k$, $\tilde{L} = L + L\theta$; also for $a + b\theta \in \tilde{L} \supset a$, $b \in L$, $(a + b\theta)^- = a - b\theta$ so that $\varphi(a + b\theta) \in T'(L)$ if and only if b is invertible in L.

Now for $a + b\theta \in \tilde{L}$.

$$\begin{pmatrix} a + b\theta & 0 & 0 \\ 0 & (a - b\theta)^{-1} & 0 \\ 0 & 0 & (a + b\theta)^{-1}(a - b\theta) \end{pmatrix}$$

$$\begin{pmatrix} 0 & -(b^2/2 - ab\theta^{-1}/2 & 0 \\ -(b^2/_2 - ab\theta^{-1}/2 & 0 & 0 \\ 0 & 0 & \delta(a,b) \end{pmatrix}\begin{pmatrix} 0 & +2^{-1}b\theta^{-1} & 0 \\ -2b^{-1}\theta & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= \lambda(a,b) \cdot \mu(a,b)$ say.

We will now give factorisations of $\lambda(a,b)$ and $\mu(a,b)$ as products $\mu(a,b) = \chi'(a,b)\xi'(a,b)\beta'(a,b) \cdot \lambda(a,b) = \alpha(a,b)\xi(a,b)\beta(a,b)$ where α, α' and β, β' morphisms of T' in U and ξ, ξ' are morphism of T' in U^-. For μ this is similar to what we did in the case of $SL(2)$:

$$\mu(a,b) = \begin{pmatrix} 1 & 2^{-1}b\theta^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2b^{-1} & \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2^{-1}b\theta^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As for $\lambda(a,b)$ this is a trifle more complicated. We write down $\alpha(a,b)$; the others are uniquely determined by $\alpha(a,b)$:

$$\alpha(a,b) = \begin{pmatrix} 1 & -b^2/2 - ab\theta^{-1}/2 & b \\ 0 & 1 & 0 \\ 0 & -b & 1 \end{pmatrix}$$

(Given an element $x \in U(k)$, there are unique elements $y \in U^{-1}(k)$ and **195**
$x' \in U(k)$ such that

$$xyx' \in \text{ Normaliser of } S.$$

This is the reason why the argument works.)

Case 2. Char $k = 2$. We pick an element $\theta \in K$ with $\theta + \bar{\theta} = 1$. Then any element of L is of the form $a + b\theta$ with $a, b \in L$. We can then write

$$\begin{pmatrix} a+b\theta & 0 & 0 \\ 0 & \overline{(a+b\theta)}^{-1} & 0 \\ 0 & 0 & \overline{(a+b\theta)}(a+b\theta)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & b^2\theta + ab & 0 \\ \overline{(b^2\theta + ab)^{-1}} & 0 & 0 \\ 0 & 0 & (a,b) \end{pmatrix}\begin{pmatrix} 0 & b & 0 \\ b^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= \lambda(a,b)\,\mu(a,b)$ as before.

Clearly $\mu(a,b) = \begin{pmatrix}1 & b & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -b^{-1} & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & b & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$; and
$\mu(a,b) = \alpha(a,b)\cdot\xi(a,b)\cdot\beta(a,b)$ with $\alpha, \beta \in U$ and $\xi \in U^-$.

As before we write down $\alpha(a,b)$ explicitly:

$$\alpha(a,b) = \begin{pmatrix}1 & b^2\theta + ab & b\\ 0 & 1 & 0\\ 0 & b & 1\end{pmatrix}$$

(β and ξ are determined by α). This completes the proof of Theorem 3.

3 Proofs of the main results (A, B, C, and D).

Throughout this section G will denote an *acceptable* (smooth) group scheme over k.

We prove Theorem A first. We argue by induction on n. When $n = 1$, this is immediate from the definition of acceptability. Assume
196 that the result holds for $n \leqslant m$ and let P be a G-bundle on $A^{m+1} = A \times A^m$. Let $\operatorname{Spec} K \to A^m$ be the generic point of A^m and $P^* = P \times_{A^m} \operatorname{Spec} K : P^*$ is a bundle over $A \underset{k}{\times} \operatorname{Spec} K$. By the definition of acceptability, there is a $G \underset{k}{\times} K$ bundle P' on $\operatorname{Spec} K$ such that $P' \underset{\operatorname{Spec} k}{\times} (A^1 \underset{k}{\bigotimes} K \operatorname{Spec} K) \simeq P^*$. Equivalently there is an open subscheme $U \subset A^m$ and a bundle P'' on U such that

$$P \times_{A^m} U \simeq P'' \underset{U}{\times} (A^1 \underset{k}{\times} U).$$

Let $p : \operatorname{Spec} k \to A$ be any k-point and $\hat{p} : A^m \to A \underset{k}{\times} A^m$ the corresponding immersion. Let

$$P_1 = P \underset{A' \times A^m}{\times} A^m \ \text{(via } \hat{p}).$$

If we let $\hat{p}$ also denote the immersion

$$U \to A \underset{k}{\times} U,$$

clearly

$$P_1 \times_{A^m} U \simeq P''.$$

By induction hypothesis P_1 is obtained by base change $A^m \to \operatorname{Spec} k$ from a principal homogeneous space over k. Thus the same is true of P'' (with respect to the base change $U \to \operatorname{Spec} k$). Since $P \times_{A^m} U \simeq P'' \times_U (A' \underset{k}{\times} U)$ the desired result follows. This concludes the proof of Theorem A.

We proceed to the proofs of Theorems B and C. For the sake of brief formulations we adopt the following definition: A G-bundle P on a k-scheme X is *obtained from* k if P is isomorphic to a bundle of the form $P' \underset{\operatorname{Spec} k}{\times} X$ with P' a principal homogeneous space over k.

Under the hypotheses in Theorems B and D we can find $f \in k[X_1, \ldots, X_n](A^n = \operatorname{Spec} k[X_1, \ldots, X_n])$ such that P restricted to the open subschemes

$$W' = \{p \in A^n | f(p) \neq 0\}$$

is obtained from $\operatorname{Spec} k$. After a change of coordinates if necessary one can assume that f has the form

$$X_1^m + \sum_{i=0}^{m-1} q_i(X') X_1'$$

with $q_i(X') \in k[X'] \overset{def}{=} k[X_2, \ldots, X_n]$. Isolating the variable X_1 gives a product decomposition **197**

$$A^n = A^1 \times A^{n-1}.$$

We want to prove the following.

Assertion 1. *The hypotheses are those of Theorem B or of Theorem D. Then for each closed point $x_0 \in A^{n-1}$, there is an open neighbourhood U_{x_0} of x_0 such that P restricted to $A^1 \times U_{x_0}$ is obtained from a bundle P_{x_0} on U_{x_0} by the base change $A^1 \times U_{x_0} \to U_{x_0}$.*

Observe that the assertion implies Theorems B and D. To see this we argue as follows. Let $p : \operatorname{Spec} k \to A^1$ be a k-point and $\hat{p}$ the corresponding immersion

$$A^{n-1} \hookrightarrow A^1 \times A^{n-1}.$$

Then the restriction of P to A^{n-1} is, by an induction assumption, obtained from $\operatorname{Spec} k$. Using a Galois twist of P we can assume that this bundle is actually trivial. It is then easily seen that the P_{x_0} in the Assertion is necessarily trivial. We can now appeal to Theorem 2 to obtain Theorems B and D.

To prove Assertion 1 we use Theorem 1. Consider the scheme $\mathscr{P}^1 \times A^{n-1}$. This contains $A^1 \times A^{n-1}$ as an open k-subscheme which we note also by U_1 in the sequel. Let

$$U_2' = \mathscr{P}^1 \times A^{n-1} - \{p \in A^n | f(p) = 0\}.$$

Then U_2' is an open k-subscheme as well. Since f is *monic in* X_1, U_1 and U_2' cover $\mathscr{P}^1 \times A^{n-1}$. Moreover if $U_1 \cap U_2' = W'$ and P restricted to W' is *obtained from* $\operatorname{Spec} k$. We may, after a *twist* assume then that P restricted to W' is actually trivial. Since $H^1(k, G) \to H^1(k, AdG)$ is trivial, this *does not change* G. Fix once and for all an isomorphism

$$P \Big|_{U_1 \cap U_2'} \xrightarrow{\Phi_0} 1_{G,W'}$$

198 where $1_{G,W'}$ is the trivial bundle on W'. Now let $x_0 : \operatorname{Spec} L \to A^{n-1}$ be any closed point and $\tilde{x}_0$ the induced morphism

$$\mathscr{P}^1 \times \operatorname{Spec} L \to \mathscr{P}^1 \times A^{n-1}.$$

For an open subscheme V of $\mathscr{P}^1 \times A^{n-1}$ we denote by V_{x_0} the inverse image of V in $\mathscr{P}^1 \underset{k}{\times} \operatorname{Spec} L$. In particular then U_{1x_0} and U_{2x_0}' cover $\mathscr{P}^1 \underset{k}{\times} \operatorname{Spec} L$. Let P_{x_0} denote the bundle on $A^1 \times \operatorname{Spec} L$ induced by P. Since G is acceptable P_{x_0} is obtained from L and, since P is trivial on W' and W'_{x_0} is non-empty (note that f is monic in X_1) P_{x_0} is trivial. Let

$$\theta : P_{x_0} \simeq G \underset{k}{\times} (A^1 \underset{k}{\times} \operatorname{Spec} L)$$

be an isomorphism. On the other hand, Φ_0 by restriction gives an isomorphism

$$\varphi_0 : P_{x_0}|W_{x_0} \simeq G \underset{k}{\times} W'_{x_0}.$$

Let $\alpha = \theta_0 \varphi_0^{-1}$ be the composite isomorphism

$$\alpha : G \underset{k}{\times} W'_{x_0} \to G \underset{k}{\times} W'_{x_0}.$$

α gives rise to and is determined by a morphism

$$\Psi_1 : W'_{x_0} \to G.$$

We will deduce Assertion 1 from

Assertion 2. *Let $x_0 \in A^{n-1}$ be any closed point. Then there is a refinemenet (U_1, U_2) of (U_1, U'_2) with the following properties:*

(i) *there is a neighbourhood V of $x_0 \in A^{n-1}$ such that $U_1 \cup U_2 \supseteq \mathscr{P}^1 \underset{\operatorname{Spec} k}{\times} V$,*

(ii) *if $W = U_1 \cap U_2$, $\psi = \psi_1|W_{x_0}$ extends to a morphism $\overline{\Psi}$ of W in G.*

If we admit Assertion 2, we can construct a bundle $\tilde{P}$ on $\mathscr{P}^1 \underset{\operatorname{Spec} k}{\times} V$ as follows: Let $1_{G,U_2}$ be the trivial G-bundle on U_2. Then $\Psi \circ \Phi_0$ gives an isomorphism of P restricted to W with $1_{G,U_2}$ restricted to W. Patching by this isomorphism we obtain a bundle on $U_1 \cup U_2$ whose restriction to $\mathscr{P}^1 \underset{\operatorname{Spec} k}{\times} V$ we denote $\tilde{P}$. From the fact that $\overline{\Psi}$ extends ψ, it is im- **199**
mediate that the bundle on $\mathscr{P}^1 \underset{k}{\times} \operatorname{Spec} L$ obtained by the base-change $x_0 : \operatorname{Spec} L \to V$ is trivial. By Theorem 1, $\tilde{P}$ is locally obtained by base change from a G-bundle on a neighbourhood of x_0. Since $\tilde{P}$ has P for its restriction to $A^1 \times V$, Assertion 1 follows.

We are still to establish Assertion 2. We consider the hypotheses of the two theorems separately.

Case i (Hypotheses as in Theorem B)**.** Choose Ω as in the proof of Theorem 3 of §2. $\Omega \simeq \operatorname{Spec} k[X_1, \ldots, X_q, T_1 \ldots T_1 T_1^{-1} \ldots T_1^{-1}]$ (1 = k-rank

G, $q^+ = \dim G$). Let $S = \psi_1^{-1}(G - \Omega)$. This is a k-closed subset of W'_{x_0} since $\dim W'_{x_0} = 1$, if S is a proper subset, S in finite hence closed in $U_{1x_0}(\simeq \operatorname{Spec} L[X_1])$; and we assume that $S \neq W'_{x_0}$ by replacing Ω if necessary by a translate by an element of $G(k)$. (This is evident when $G(k)$ is dense in G but one can give an obvious argument using the Bruhat-decomposition even if $G(k)$ were not dense in G). The affine scheme W'_{x_0} is of the form $\operatorname{Spec} L\left[X_1, \dfrac{1}{f^*}\right]$ where $f^* \in L[X_1]$ is the polynomial $X_1^m + \sum_{i=0}^{m-1} q_i(x_0)X_1^i$. The morphism Ψ_1 gives a ring homomorphism

$$\psi_1^* : k[X_j, T_i, T_i^{-1}(1 \leqslant j \leqslant q, 1 \leqslant i \leqslant 1)] \to L(X_1).$$

Let $\psi_1^*(X_j) = (Q_j/R_j)$ and $\psi_1^*(T_i) = A_i/B_i$ where (Q_j, R_j) and (A_i, B_i) are coprime polynomials in $L[X_1]$. Let degree $R_j = r_j$, degree $A_i = a_i$ and degree $B_i = b_i$. Choose elements $(\tilde{A}_i, \tilde{B}_i, \tilde{Q}_j)$ and $\tilde{R}_j$ in $k[X_1, \ldots, X_n]$ such that

(i) $\tilde{A}_i$ (resp. $\tilde{B}_i, \tilde{Q}_j, \tilde{R}_j$) is a lift of A_i (resp. B_i, Q_j, R_j) for the map $\hat{x}_0 : k[X_1, \ldots, X_n] \to L[X_1]$,

(ii) degree $\tilde{A}_i$ (resp. $\tilde{B}_i, \tilde{R}_j$) in X_1 is precisely a_i (resp. b_i, r_j).

Let α_i (resp. β_i, ρ_j) denote the *leading coefficients* of $\tilde{A}_i$ (resp. $\tilde{B}_i, \tilde{R}_j$) as
200 a polynomial in X_1 : $\alpha_i, \beta_i, \rho_j$ belong to $k[X_2, \ldots, X_n]$. These elements do not take the value zero at x_0. Let V be the open neighbourhood of x_0 defined by

$$V = \{p \in A^{n-1} | \alpha_i(p) \neq 0, \beta_i(p) \neq 0, \rho_j(p) \neq 0, 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant q\}.$$

Let $U_2 = U'_2 - \{p \in A^n | \prod_{\substack{1 \leqslant i \leqslant 1 \\ 1 \leqslant j \leqslant q}} A_i B_i R_j(p) = 0\}$. This choice of U_2 and V is easily seen to satisfy the conditions of Assertion 2.

Case ii (Hypotheses as in Theorem D)**.** Since G is quasi split and simply connected we can choose (Theorem 4 of §2) an open subset Ω_0 such that the inclusion $\Omega_0 \subset G$ factors through an affine space : $\Omega_0 \to A^r \to G$.

Let $S = \psi_1^{-1}(G - \Omega_0)$. We may assume that S is finite (if Ω_0 is chosen exactly as in Theorem 4 of §2 this can be secured by translating Ω_0 by an element of $G(k)$). Let $f_0 \in k[X_1]$ be a monic polynomial vanishing on $S \subset \operatorname{Spec} L[X_1])$. Consider f_0 as an element of $k[X_1, \ldots, X_n]$ and let $U_2 = U_2' - \{p \in A^n | f_0(p) = 0\}$. Since f_0 is monic in X_1, $U_1 \cup U_2 = \mathscr{P}^1 \times A^{n-1}$ with this choice of U_2, ψ factors through $\Omega_0 : \psi_1(W_{x_0}) \subset \Omega_0$. Since $\Omega_0 \hookrightarrow G$ factors through A^r, ψ extends (from the closed set W_{x_0}) to all of $W : \psi : W \to G$. This proves Assertion 2 in this case.

4 Complements

We discuss first some special fields.

1. Fields of Dimension $\leqslant 1$. If k is a field of dimension 1, $H^1(k, G) = 0$ for all smooth connected G (Lang [7], Steinberg [14]). Thus Theorem B is applicable to any *acceptable* semisimple G over k. We conclude that *for such k any G-bundle on A^n is trivial*. Among the fields covered are:

   (i) finite fields;

   (ii) $k = L(X)$, the pure transcendental extension of degree 1 over an algebraically closed field L;

   (iii) k is the maximal unramified extension of a local field.

2. Local fields with residue fields of dimension $\leqslant 1$. For such k, it is known (Bruhat-Tits [2]) that $H^1(k, G) = 0$ for any *simply connected G*. Thus Theorem D is applicable: *Any G-bundle with G quasi-split and simply connected on an affine space is trivial.* **201**
   The fields that are covered are:

   (i) finite extensions of p-adic fields,

   (ii) quotient fields of power series rings $k'[[T]]$ in 1 variable with $\dim k' \leqslant 1$.

3. Global fields. Here again it is known that for a simply connected acceptable G, $H^1(k, G) = 0$ except possibly when G has exceptional factors and k is a number field which admits a real place (Harder [5a], [5b], Kneser [6b]). Thus for any quasi-split G, G-bundles on A^n are trivial for simply connected classical G over a number field k.

   The following groups are acceptable for *any* field k : $GL(n)$, $Sp(n)$ Spin f, f a split quadratic form (in more than 3 variables). This is obvious for $GL(n)$ and $Sp(n)$ and can be deduced from Harder's theorem for $SO(n)$ in the case of Spin groups. Theorem D is applicable in all these cases because in addition, $H^1(k, G) = 0$ for all these groups.-Note that for Spin we have taken the split form (see Serre [12], III-25).

   We have therefore

Theorem D′. *Let G be a split simply connected* 'classical' *group over any field k. Then any G-bundle over A^n is trivial.*

Next observe that Theorem C gives the following

Proposition. *If G is an acceptable k-group, these is a bijective correspondence between isomorphism of classes of principal G-bundles on A^n and the Galois cohomology set*

$$H^1(\pi, G(\bar{k}_s[X_1, \ldots, X_n]))$$

where $\bar{k}_s$ = separable closure of k and π is the Galois group of $\bar{k}_s$ over k.

As a corollary we will prove

Theorem E. *Let k be such that $Br(k)$ has no 2-torsion and* Char $k \neq 2$. *Then any orthogonal bundle over A^n is obtained by base change from k.*

202 Theorem E is evidently equivalent to

Theorem E′. *Let k be a field such that $Br(k)$ has no 2-torsion and* Char $k \neq 2$. *Then every non-singular quadratic form over $k[X_1, \ldots, X_n]$ is equivalent to one over k.*

For the proof we need

Lemma. *Let k be a field such that $Br(k)$ has no 2-torsion. Then*

(i) a quadratic form over k is determined by the discriminant;

(ii) if the number of variables is $n \geqslant 3$, the Witt-index is at least $[(n-1)/2]$ (integral part of $(n-1)/2$).

(iii) the spinor norm is surjective for any form f;

(iv) for any quadratic form f, $SO(f)$ is quasi-split over k (if the number of variables is at least 3);

(v) for any form f, $H^1(k, \mathrm{Spin}\, f) = 0$.

Proof. To prove (i) we argue by induction on the number of variables, we may assume that the quadratic form is the orthogonal sum

$$q \perp \underline{\lambda}$$

where q is a form in $(n-1)$ variables $\lambda \in k$ and $\underline{\lambda}$ is the form in 1 variable $x \mapsto \lambda x^2$, $x \in k$. By induction hypothesis q is isomorphic to the orthogonal sum

$$\underline{1} \perp \underline{d}$$

where $\underline{1}$ is the identity form in $(n-2)$ variables, $d \in k$ and $\underline{d}$ is the form $x \to dx^2$ on k. It suffices then to show that $\underline{d} \perp \underline{\lambda}$ is isomorphic to $\underline{1} \perp \lambda\underline{d}$ (with the obvious notation). In order to prove this, we have to show that any quadratic form in 2 variables represents 1. In fact, we prove the following stronger statement which immediately implies (iii) as well:

Let $\alpha \neq 0$ be any element of k; then the quadratic form $\underline{\lambda} \perp \underline{\mu}$ for any pair $\lambda, \mu \in k^*$ represents α.

We have to solve the equation

$$\lambda x^2 + \mu y^2 = \alpha$$

or equivalently

$$x^2 + \frac{\mu}{\lambda}y^2 = \frac{\alpha}{\lambda}.$$

This is the same as proving that α/λ is a norm for the quadratic extension $k(\sqrt{-\mu/\lambda})$. Since there are no quaternion divison algebras over k, this is indeed true (see for instance, O'Meara [15], p. 146).

We have thus proved (i) and (ii). Statement (iii) is immediate from (i): if q is any quadratic form in $m = 2n + 2$ (resp. $2n + 1$) variables and h is the hyperbolic form in $2n$ variables, $q \simeq h \perp \underline{1} \perp \underline{d}$ (resp. $h \perp \underline{d}$) where $d' \in k$ is any representative for the discriminant and $d = \pm d'$.

To prove (iv) again write

$$q \simeq h \perp \underline{d} \quad or$$
$$q \simeq h \perp \underline{1} \perp \underline{d}$$

according as the number of variables is odd or even. Let E be the maximal isotropic subspace for h. Then the subgroup P of $SO(q)$ which fixes E is a parabolic subgroup. Writing q as

$$q \simeq h \perp q'$$

(where $\dim q' \leqslant 2$) we see that the reductive part M of P is isomorphic locally to the product of a torus and $SO(q')$. It follows that (since $\dim q' \leqslant 2$) M is a torus. Thus P is a Borel subgroup. This proves (iv).

Finally (v) follows from (i) and (iii) in view of the following known fact (Serre [12, III-25]).

If f is a quadratic form such that the Spinor norm map is surjective and every form f' with the same Witt-index and discriminant as f (in the same number of variables) is equivalent to f, then $H^1(k, \mathrm{Spin}\ f) = 0$. This proves (v).

□

We proceed to the proof of Theorem E now. When the number of variables is 2 this is trivial. We assume therefore that the number of variables is at least 3. Observe first that for any quadratic form f over k,

$$H^1(\pi, \mathrm{Spin}\ f(k_s[\underline{X}])) = 0$$

204 where k_s = separable closure of k and $\pi = \mathrm{Gal}\,(k_s/k)$. This follows from Theorem C combined with Parts (iv) and (v) of the lemma above. Next consider the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}\, f \to SO(f) \to 1.$$

Since $\mathrm{Char}\, k \neq 2$, one sees immediately that the sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}\, f(k_s[\underline{X}]) \to SO(f)(k_s[\underline{X}]) \to 1$$

is exact. The cohomology sequence gives exactness of

$$\begin{array}{ccccc} H^1(\pi, \mathrm{Spin}\, f(k_s[\underline{X}])) & \longrightarrow & H^1(\pi, SO(f)(k_s[\underline{X}])) & \longrightarrow & H^1(\pi, \pm 1) \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

The last group is trivial since $Br(k)$ has no 2-torsion.

Thus for any f, $H^1(\pi, SO(f)(k_s[\underline{X}])) = 0$. Now consider the split exact sequence for a fixed f_0

$$1 \to SO(f_0)(k_s[\underline{X}]) \to 0(f_0)(k_s[\underline{X}]) \to \{\pm 1\} \to 1.$$

This gives in cohomology the sequence

$$0 \to H^1(\pi, 0(f_0)(k_s[\underline{X}])) \xrightarrow{\eta} H^1(\pi, \{\pm 1\}). \qquad (*)$$

The fibre over the trivial element is thus trivial. We want to show that the fibre over *any* element of $H^1(\pi, \pm 1)$ is trivial. Using the splitting the fibre can be identified (by a standard twisting argument) with the fibre over the trivial element for a twisted form f of f_0:

$$H^1(\pi, O(f)k_s[\underline{X}]) \to H^1(\pi, \pm 1);$$

and since $H^1(\pi, SO(f)(k_s[\underline{X}])) = 0$ it follows that η in $(*)$ is a bijection. Comparing with the Galois cohomology for k_s-points we conclude that $H^1(\pi, O(f_0)(k_s[\underline{X}]))$ is in bijective correspondence with $H^1(\pi, O(f_0)(k_s))$. To conclude the proof of Theorem E, we need only show that $H^1(\pi, O(f_0)$

$k_s[\underline{X}]$) classifies quadratic forms over $k[\underline{X}]$. Equivalently we have to show that all orthogonal bundles over $\operatorname{Spec} k_s[\underline{X}]$ are trivial. And this follows from Theorem B since (in view of our assumption that $\operatorname{Char} k \neq 2$) any $O(n)$-bundle admits an $SO(n)$-reduction.

205 Remarks. (I). Theorems can be formulated for general connected G using the main results of this paper. For this one takes into account the following facts:

(i) For a *split* unipotent G any principal G-bundle is trivial (This is obvious).

(ii) If T is a torus then any torus bundle over A^n is obtained by base change from k, (This can be proved by the following observations. When k is separably closed this amounts to saying that all line bundles are trivial; the general case follows from Galois cohomology once one observes that $T(\bar{k}_s[X]) \xrightarrow{\sim} T(\bar{k})_s$).

(II). Theorem A has the following consequence: Given any G-bundle P, G acceptable, over A^n, there is a product decomposition $A^n \simeq A^1 \times A^{n-1}$ such that P extends to a bundle over $\mathscr{P}^1 \times A^{n-1}$. This suggests that a classification of G-bundles over A^n is closely connected with the study of families of bundles on $\mathscr{P}^1$.

(III). One can prove a sharper result than Theorem D for quasi-split groups. The statement is:

Assume that G is quasi-split acceptable and that the map $H^1(k, G) \to H^1(k, AdG)$ is trivial. Suppose further that the central isogeny $\tilde{G} \to G$ of the simply connected covering group $\tilde{G}$ on G is *separable*. Then any principal G-bundle on A^n is obtained by base change from k.

This is proved by looking at the Galois cohomology exact sequence (for $k_s[\underline{X}]$ rational points) associated to the exact sequence

$$1 \to C \to \tilde{G} \to G \to 1$$

It appears likely that the hypothesis that the isogeny is separable is not really necessary.

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Geometry of G/P-I
Theory of Standard Monomials for Minuscule Representations

By C.S. Seshadri

INTRODUCTION. Let G be a semi-simple simply connected algebraic group defined over an algebraically closed field k. Let P be a maximal parabolic subgroup in G and L the ample line bundle on G/P which generates $\operatorname{Pic} G/P$. We say that P is *minuscule* if the Weyl group W (fixing of course a Borel subgroup, maximal torus etc.) acts transitively on the weight vectors of the irreducible G-module $H^0(G/P, L)$ (k assumed to be of characteristic zero. Note then that $H^0(G/P, L)$ is the irreducible G-module associated to a fundamental weight). We can then index these weight vectors by $\{p_\tau\}$, $\tau \in W/W_{i(P)}$ (where $W_{i(P)}$ denotes the Weyl group of the maximal parabolic subgroup $i(P)$ which is the transform of P under the Weyl involution i) so that p_τ is the highest weight vector, when $\tau \equiv$ Identity (mod $W_{i(P)}$). We say that a monomial in $\{p_\tau\}$, say

$$p_{\tau_1} p_{\tau_2} \dots p_{\tau_m} \in H^0(G/P, L^m)$$

is *standard* if $\tau_1 \leqslant \tau_2 \leqslant \dots \leqslant \tau_m$ (cf. Def. 1). Then the main result of this paper is the following (cf. . Th. 1):

> *Standard monomials of length m, which are distinct, form a basis of* $H^0(G/P, L^m), m \geqslant 0$, *P minuscule.* (*)

It is proved that this result holds also in arbitrary characteristic.

When $G = SL(n)$, every maximal parabolic subgroup is minuscule. In this case, when the base field k is of characteristic zero, (*) is due

to Hodge (cf. [9], [10]) and is related to Young tableaux. When k is of arbitrary characteristic, again for $G = SL(n)$, (*) was proved to be true by many, for example, by Musili (cf. [13]). That the generalisation is to be sought for a minuscule P, was pointed out to the author by A. Borel.

An important consequence of (*) is that it allows one to determine
208 the ideal defining a Schubert variety in G/P, the scheme theoretic unions and intersections of Schubert varieties and the scheme theoretic hyperplane intersection of a Schubert variety, when this intersection decomposes into a union of Schubert varieties (cf. Th. 1). One knows (cf. [12]) that these are the main technical difficulties in trying to prove the following:

$$H^i(G/B, L) = 0, \quad i > 0, \quad L \text{ in the dominant chamber.} \qquad (**)$$

The essential achievement of G. Kempf, in his proof of (**) (cf. [11]), is that he is able to manage these technical difficulties for G of type E_6, E_7, E_8 and F_4 and P associated to what he calls a *distinguished weight*. For the cases E_6 and E_7, there exists a distinguished weight which is also minuscule. Hence from (*), a proof of (**) for G of type E_6 and E_7 follows on the lines as in [12].

In another place, we shall given an application of (*) to the work of C. De Concini and C. Procesi (cf. [7]) on their generalisation (to arbitrary characteristic) of the classical theorems on invariants, which are found, for example, in H. Weyl (cf. [24]).

The proof of Theorem 1 is related to the one in [13]. The two new simple observations which have made the present generalisation possible and which the author missed for a long time, are as follows:

(i) it suffices to do the standard monomial theory when the base field is of characteristic zero (cf. Remark 5);

(ii) it is not necessary to have, *a priori*, the quadratic relations defining G/P in its canonical projective imbedding, in the specific form used in [13]. Such quadratic relations follow, once one has the theorem on standard monomials (cf. Corollary 1, Theorem 1).

The proof of Theorem 1 is somewhat indirect and uses Demazure's results (cf. [8]) that Schubert varieties are Cohen-Macaulay when the

ground field is of characteristic zero. One can give a more direct proof, closer to that of [13], if the following could be checked directly when the ground field is of characteristic zero (cf. Remark 8).

$\dim H^0(G/P, L^2) = \sharp$ (Distinct standard monomials of degree 2). This could perhaps be done by a clever handling of the Weyl's or Demazure's character or dimension formula.

Chapter 1

Standard Monomials

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1 Notations and preliminaries (cf. [1], [2], [3], [5], [12]).

G = a semi-simple, simply-connected, split (Chevalley) group over a field k. Note that G has also a $\mathbf{Z}$-form i.e., G can be considered as $G' \otimes_{\mathbf{Z}} k$, where G is split, semi-simple over $\mathbf{Z}$ etc.

T = a maximal torus of G; B = a Borel subgroup of G, $B \supset T$. Roots and weights indicated below are taken in the usual sense having fixed T and B.

Δ = system of roots, Δ^+ = system of positive roots

S = simple roots: α_i, $1 \leqslant i \leqslant l$, or the set $[1, \ldots, l]$

W = Weyl group with simple reflections $s_i (= s_{\alpha_i})$, $1 \leqslant i \leqslant l$

ϖ_i = fundamental weights, $1 \leqslant i \leqslant l$, i.e., $\langle \varpi_i, \check{\alpha}_j \rangle = \delta_{ij}$ where $\check{\alpha}_j$ denotes the coroot of α_j and $\langle , \rangle$ is as in [4].

$\mathscr{U}_\infty$ = unipotent group $\approx \mathbf{G}_a$ associated to $\alpha \in \Delta$.

ω_0 = the element of maximal length in W

i = the Weyl involution - ω_0.

Let P be a parabolic subgroup of G, $P \supset B$. Then P is *determined* by a subset S_P of S i.e. P is generated by B and $\mathscr{U}_{-\alpha}$, $\alpha = \sum n_i \alpha_i, n_i \geqslant 0$, $\alpha_i \in S_P$. Set

$$\Delta_P = \left\{ \alpha \in \Delta / \alpha = \sum n_i \alpha_i, \alpha_i \in S_P, n_i \in \mathbf{Z} \right\}$$

$$\Delta_P^+ = \left\{\alpha \in \Delta/\alpha = \sum n_i\alpha_i, \alpha_i \in S_P, n_i \geqslant 0, n_i \in \mathbf{Z}\right\}$$
$$\Delta_P^- = \{\text{" " " "} \quad \text{with} \quad n_i \leqslant 0, n_i \in \mathbf{Z}\}.$$

We have

$$P = M_P \cdot U_P$$

M_P being generated by T and $\mathscr{U}_\alpha$, $\alpha \in \Delta_P$ and U_P being generated by $\mathscr{U}_\alpha$, $\alpha \in \Delta^+$, $\alpha \notin \Delta_P^+$. One knows that U_P is the unipotent radical of P and that M_P is reductive so that M_P (resp. U_P) is called the *reductive* (resp. *unipotent*) part of P.

210 The subgroup W_P of W generated by the simple reflections s_α, $\alpha \in S_P$, is called the *Weyl group of P*. One sees that W_P is in fact the Weyl group of the reductive part M_P of P. The radical of M_P is a torus, whose dimension $= \sharp(S - S_P)$. Hence the *character group* (i.e. group of homomorphisms into $\mathbf{G}_m$) of P (or of M_P), considered as an abelian group is free of rank $= \sharp(S - S_P)$. One sees that the character group of P can be identified with the group of characters χ of T such that $\omega(\chi) = \chi \forall \omega \in W_P$. From this fact it is seen easily that $\{\varpi_i\}$, $i \in S - S_P$ form a basis of the character group of P.

In the sequel we will be working with a *maximal parabolic subgroup* P. We write $i_0 = S - S_P$ and $\varpi = \varpi_{i_0}$, $i_0 \in [1, \ldots, l] = S$. We assume that the base field k is algebraically closed.

Let V denote the space of regular functions f on G such that

$$f(gb) = \chi_\varpi(b) f(g)$$

where χ_ϖ denotes the character of B defined by the fundamental weight ϖ. Then V can be identified with the space of sections of the line bundle L on G/B associated to the character $\chi_\varpi^{-1} : B \to k$ of B. Now χ_ϖ extends to a character of P and V can in fact be identified with $H^0(G/P, L)$, where by the same L we denote the line bundle associated to the character $\chi_\varpi^{-1} : P \to k$. Now V has a unique B-fixed line and let us fix a generator F of this line. We have

$$F(b_1 g b_2) = \chi_{i(\varpi)}^{-1}(b_1) F(g) \chi_\varpi(b_2), \ b_i \in B.$$

If $f : G \to k$ is a regular function on G, we define

$L_h f(g) = (h \circ f)(g) = f(h^{-1}g)$, $g, h \in G$. (left regular representation). We find

$$b_1 \cdot F(g) = F(b_1^{-1}g) = \chi_{i(\varpi)} F(g)$$

so that the weight of F is $i(\varpi)$ i.e. the *highest weight of* V is $i(\varpi)$.

For $w \in W$, we denote by $X(w)$ the Schubert variety associated to w i.e. the closure $\overline{BwB}$ of BwB in G. Let X_0 denote the set of zeros of F. Then it is P-stable on the right and B-stable on the left. We denote by the same X_0 the subvariety of G/P defined by X_0. Then X_0 is the unique codimension one Schubert variety in G/P. We claim that

$$X_0 = X(\omega_0 s) = X(i(s)\omega_0), s = s_{i_0}, i_0 = S - S_P, i(s) = \omega_0 s \omega_0.$$

To check this, since $X(\omega_0 s)$ is of codimension one in G/B, it suffices **211**
to check that $X(\omega_0 s)$ is P-stable on the right. This is a consequence of Prop. 1.4, [12] and by this proposition we have only to check that

$$(\omega_0 s)(\alpha) < 0 \quad \forall \alpha \in \Delta_P^+.$$

Hence it suffices to check that

$$s(\alpha) \in \Delta^+, \alpha = \alpha_i, i \in S_P \text{ i.e. } i \neq i_0, s = s_{i_0}.$$

We have $s_{i_0}(\alpha_i) \in \Delta^+$, $i \neq i_0$ and thus the claim that $X_0 = X(\omega_0 s)$ is proved.

We observe now that the set hX_0, $h \in G$, is precisely the zero set of $L_h F = F(h^{-1}g)$. We observe that for $t \in T$, t leaves stable the set X_0 and hence the notation τX_0, $\tau \in W$ makes sense. Now if $\tau \in W$, $\tau \cdot F(g) = F(\tau^{-1}g)$ is well-defined only upto a scalar multiple; however we use this notation as it is convenient for us.

Note that the Weyl group W does not operate even on the projective space $\mathbf{P}(V)$ associated to V (as the torus operates with different weights on the weight vectors). Of course the normaliser $N(T)$ of T operates on V.

2 Preliminaries on the ideals of Schubert varieties in G/P.

Lemma 1. *For $\tau \in W$, we have*

$$\tau X_0 = X_0 \Leftrightarrow \tau \in W_{i(P)}$$

where $i(P)$ denotes the maximal parabolic group $w_0 P \omega_0$ (i.e. the parabolic subgroup associated to the subset $i(S_P)$ of S).

Proof. Observe that the action of W on the characters χ of T is defined as follows:

$$\tau \cdot \chi(t) = \chi(\tau^{-1} t \tau)$$

We have then $(\tau_1 \tau_2) \cdot \chi = \tau_1 \cdot (\tau_2 \chi)$. Define $H(g)$ by

$$H(g) = F(\tau^{-1} g) = L_\tau F(g), g \in G.$$

The zero set of $H(g)$ is τX_0 as mentioned above. We have

$$\begin{aligned} t.H(g) = H(t^{-1}g) &= F(\tau^{-1}t^{-1}g) = F(\tau^{-1}t^{-1}\tau\tau^{-1}g) \\ &= \chi_{i(\overline{\omega})}((\tau^{-1}t^{-1}\tau)^{-1})F(\tau^{-1}g) = \chi_{i(\omega)}(\tau^{-1}t\tau)H(g) \\ &= \chi_{\tau i(\overline{\omega})}(t)H(g). \end{aligned}$$

212 This shows that $H(g)$ is a weight vector under T with weight $\chi_{\tau i}(\overline{\omega})$. By the uniqueness of the codimension one Schubert variety in G/P and the fact that F is the unique (upto scalars) element in V with weight $i(\overline{\omega})$, it follows that

$$\begin{aligned} \tau X_0 = X_0 \Leftrightarrow X_0 = &\text{ zero set of } H(g) \\ \Leftrightarrow \tau i(\overline{\omega}) = i(\overline{\omega}) &\text{ i.e } \tau \in W_{i(P)}. \end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2. *The correspondence*

$$\tau \mapsto X(\tau w_0) = \overline{B\tau w_0 P}, \tau \in W$$

induces a bijection of $W/W_{i(P)}$ onto the set of Schubert varieties in G/P.

Proof. We have the following:

$$X(\tau_1\omega_0) = X(\tau_2\omega_0) \Leftrightarrow \tau_1\omega_0 = \tau_2\omega_0\theta, \theta \in W_P$$
$$\Leftrightarrow \tau_1 = \tau_2(\omega_0\theta w_0) \Leftrightarrow \tau_1 = \tau_2\lambda, \lambda \in W_{i(P)}.$$

This proves Lemma 2. □

Definition 1. *We write* $\tau_1 \leqslant \tau_2$ *for* $\tau_i \in W/W_{i(P)}$, *whenever* $X(\tau_1\omega_0) \supseteq X(\tau_2\omega_0)$. *In a similar fashion we write* $\tau_1 \geqslant \tau_2$, $\tau_1 \not\geqslant \tau_2$ *or* $\tau_1 \not\leqslant \tau_2$.

We see that $\tau_1 \leqslant \tau_2$, $\tau_i \in W/W_{i(P)}$, if and only if we can find representatives of τ_i in W (represented by the same letters τ_i) such that a reduced expression of $\tau_2\omega_0$ (resp. τ_1) can be obtained as a "subword" from a reduced expression of $\tau_1 w_0\theta$ (resp. $\tau_2\lambda$), $\theta \in W_P$ (resp. $\lambda \in W_{i(P)}$) (cf. p. 98, Prop. 1.7, [12]).

Lemma 3. *We have the following:*

(i) $X(\tau_1 w_0) \not\subset \tau_2 X_0$ *if* $\tau_2 \geqslant \tau_1$

(ii) $X(\tau_2 w_0) \subset \tau_2 X_0$ *if* $\tau_2 \not\geqslant \tau_1$

i.e. (i) and (ii) can be written as

$$X(\tau_1 w_0) \subseteq \tau_2 X_0 \Leftrightarrow \tau_2 \not\geqslant \tau_1.$$

Proof. (i) We observe first that

$$X(\tau_1 w_0) \not\subset \tau_1 X_0.$$

To see this, suppose that this is not the case. Then we have

$$\tau_1 w_0 \in \tau_1 X_0 \text{ which implies that } w_0 \in X_0.$$

This leads to a contradiction (for this would mean X_0 = big cell). **213**

Suppose that $\tau_2 \geqslant \tau_1$. Then $X(\tau_2\omega_0) \subset X(\tau_1 w_0)$. We have

$$X(\tau_2\omega_0) \not\subset \tau_2 X_0$$

This implies, *a fortiori*, that $X(\tau_1 w_0) \not\subset \tau_2 X_0$.

(ii) We have to show that

$$\tau_2^{-1} X(\tau_2 w_0) \subset X_0 \text{ if } \tau_2 \not\geqslant \tau_1.$$

Let $C(w) = BwP$ (Bruhat or Schubert cell). It suffices to check that

$$\tau_2^{-1} C(\tau_1 w_0) \subset X_0.$$

Choose a representative of τ_2 in W and a reduced decomposition

$$\tau_2 = s_1 \dots s_p$$

By the axioms (or properties) of a Tits system (axiom T-3 and its easy consequence, cf. [4] pp. 22-23), we deduce that

$$\tau_2^{-1} C(\tau_1 w_0) \subset \bigcup_{(s_{i_1} \dots s_{i_k})} B(s_{i_k} \dots s_{i_1}) \tau_1 w_0 P \qquad (*)$$

where $(s_{i_1} \dots s_{i_k})$ runs through all "subwords" of $s_1 \dots s_p$ (i.e. a word consisting of a subset of $s_1, \dots, s_p$ and written in the same order). Note that $s_{i_k} \dots s_{i_1} = (s_{i_1} \dots s_{i_k})^{-1}$. We claim that to prove (ii), it suffices to check that in (*) every $B(s_{i_k} \dots s_{i_1}) \tau_1 w_0 P (\text{mod } P)$ is not the big cell in G/P. For, if this property is true, then every $B(s_{i_k} \dots s_{i_1}) \tau_1 w_0 P (\text{mod } P)$ in (*) is contained in X_0 (the condimension one Schubert variety in G/P contains any *proper* (i.e. $\neq G/P$) Schubert variety in G/P) and (ii) would then follow. Suppose then that

$$B(s_{i_k} \dots s_{i_1}) \tau_1 w_0 P = Bw_0 P (\text{big cell in } G/P)$$

where $s_{i_1} \dots s_{i_k}$ is a subword of $(s_1 \dots s_p)$. This implies that

$$s_{i_k} \dots s_{i_1} \tau_1 w_0 = w_0 \theta, \theta \in W_p$$

$$\text{i.e.} \quad s_{i_k} \dots s_{i_1} \tau_1 = w_0 \theta w_0 = \theta', \theta' \in W_{i(P)}$$

$$\text{i.e.} \quad (s_{i_1} \dots s_{i_k}) \theta' = \tau_1, \theta' \in W_{i(P)}.$$

From Lemma 2 , it follows then that

$$X(\tau_1 w_0) = X(s_{i_1} \dots s_{i_k} w_0).$$

214 But now $X(s_{i_1} \dots s_{i_k} w_0) \supset X(\tau_2 w_0)$ since $s_{i_1} \dots s_{i_k}$ is a subword of $\tau_2 = s_1 \dots s_p$. Hence $X(\tau_1 w_0) \supset X(\tau_2 w_0)$, which contradicts the assumption that $\tau_2 \not\geqslant \tau_1$. This concludes the proof of (ii) and hence of Lemma 3.

□

Lemma 4. *We have the following:*

$X(\tau w_0) \cap \tau X_0$ = union of all Schubert varieties in $X(\tau w_0)$ of codimension one = union of all proper Schubert subvarieties of $X(\tau w_0)$ ($X(\tau w_0)$ as in Lemma 2).

Equivalently (because of Lemma 3)

(Big cell in $X(\tau w_0)$) $\cap \tau X_0$ is empty.

(The intersections taken in this lemma are set theoretic).

Proof. Now $X(\tau w_0) \cap \tau X_0$ is of pure codimension one in G/P. Hence it suffices to prove that

$X(\tau w_0) \cap \tau X_0$ = union of all proper Schubert subvarieties of $X(\tau w_0)$. By Lemma 3, if $\tau_1 \geqslant \tau$ and $\tau_1 \neq \tau$

$$X(\tau_1 w_0) \subset \tau X_0 (\text{for } \tau \not\geqslant \tau_1).$$

Hence $X(\tau w_0) \cap \tau X_0$ contains all the proper Schubert subvarieties of $X(\tau w_0)$ in G/P. Thus to prove Lemma 4, it suffices to show that

$$(\text{Big cell in } X(\tau w_0)) \cap \tau X_0 \textit{ isempty}.$$

This is easily shown as follows (cf. Prop. A. 4, [12]): we have to show that

$$B\tau w_0 P \cap \tau X_0 \text{ is empty.}$$

Take the action of B on G/B induced by left multiplication and let us compute the isotropy group at $\tau w_0 \in G/B$:

$$b\tau w_0 = \tau w_0 b' (b, b' \in B) \text{ i.e. } (\tau w_0)^{-1} b (\tau w_0) \in B.$$

Hence if B_1 is the isotropy sub-group at (τw_0), we have

$$\begin{aligned} B_1 &= \{b \in B/(\tau w_0)^{-1} b(\tau w_0) \in B\} \\ &= T \times \prod_{\alpha} \mathscr{U}_\alpha/(\tau w_0)^{-1}(\alpha) > 0, \alpha > 0 \\ &= T \times \prod_{\alpha} \mathscr{U}_\alpha/\alpha > 0, \tau^{-1}(-\alpha) > 0 \quad \text{i.e} \quad \tau^{-1}(\alpha) < 0. \end{aligned}$$

215 Now $B = B_1 \times B_2$, where

$$B_2 = \prod_{\alpha} \mathscr{U}_\alpha \begin{cases} \tau^{-1}(\alpha) = \beta > 0, \alpha > 0 \\ \text{i.e. } \alpha = \tau(\beta), \alpha > 0, \beta > 0 \end{cases}$$

Hence

$$T \cdot B_2 = B \cap \tau B \tau^{-1}.$$

Therefore, if $y \in B\tau w_0 B$, we have

$$y = b_1 \tau w_0 b_2, \quad b_i \in B, \quad b_1 = \tau c \tau^{-1}, c \in B.$$

Hence

$$y = \tau c w_0 b_2; \quad b_2, c \in B,$$

This implies that

$$B\tau w_0 B \cap \tau X_0 \text{ is empty}$$

for otherwise $\tau c w_0 b_2 = \tau\theta$, $\theta \in X_0$ which implies that $c w_0 b_2 \in X_0$ and this gives that $w_0 \in X_0$ which is a contradiction. Now it is immediate that

$$B\tau w_0 P \cap \tau X_0 \text{ is empty}$$

for otherwise

$$x = b\tau w_0 p \in \tau X_0, \ b \in B, \ p \in P$$

i.e. $xp^{-1} = b\tau w_0 \in \tau X_0$ i.e. $B\tau w_0 B \cap \tau X_0$ is empty-which is a contradiction. Thus Lemma 4 is proved. □

Lemma 5. *Let L be the line bundle on G/P and F the section of L whose zero set is X_0 as in §1 (the space of sections of L gives a realisation of the irreducible representation of G with highest weight $i(\overline{\omega})$ in characteristic zero). Let A_1 be the linear subspace of $H^0(G/P, L)$ spanned by $\tau \cdot F$, $\tau \in W/W_{i(P)}$. Then the linear system A_1 has no base points.*

Proof. We have to show that given $x \in G/P$, there exists $\tau \in W/W_{i(P)}$ such that $x \notin \tau X_0$. We can suppose that x is in the big cell of a suitable Schubert variety $X(\tau w_0)$, as G/P is the union of Schubert (or Bruhat) **216** cells and distinct Schubert cells are mutually disjoint. By the second part of Lemma 4, it follows that $x \notin \tau X_0$. This proves Lemma 5. □

Remark 1. It is well-known that $\operatorname{Pic} G/P = \mathbf{Z}$, L is ample and generates $\operatorname{Pic} G/P$. Since the linear system A_1 in Lemma 5 has no base points, A_1 defines a *morphism*

$$\varphi : G/P \to \mathbf{P}(A_1^*), A_1^* = \text{ dual of } A_1.$$

We see that φ is a *finite* morphism; this is a consequence of the fact that L is ample, for on a fibre of φ the restriction of L is trivial and one would get a contradiction if the dimension of the fibre is strictly greater than zero. In particular we have

$$\dim . \text{ Image of } \varphi = \dim G/P.$$

3 Standard monomials

Let $L(\tau)$ denote the restriction of L to the Schubert subvariety $X(\tau w_0)$ of G/P (we endow $X(\tau w_0)$ with the canonical structure of a reduced closed subscheme of G/P), L being the ample generator of $\operatorname{Pic} G/P$.

We set:

$$R(\tau) = \bigoplus_{n=0}^{\infty} H^0(X(\tau w_0), L(\tau)^n); R(\tau)_n = H^0(X(\tau w_0), L(\tau)^n)$$

$$R(e) = R = \bigoplus_{n=0}^{\infty} H^0(G/P, L^n), R_n = H^0(G/P, L^n)$$

(e = element of $W/W_{i(P)}$ corresponding to the class $W_{i(P)}$)
$I = W/W_{i(P)}$, $I_r = \{\tau_1 \in I/\tau_1 \geqslant \tau\}$.
p_r = the section $\tau \cdot F$ of L on G/P (F as in §1 and Lemma 5).
$A(\tau)$ = subalgebra of $R(\tau)$ generated by $P_\lambda, \lambda \in I_\tau$
$A = A(e)$, e = the element of $I = W/W_{i(P)}$ defined by $W_{i(P)}$.

Definition 2. *An element $p \in R_m$ (resp. $R(\tau)_m$) of the form*

$$p = p_{\tau_1} \ldots p_{\tau_m}, \tau_i \in I (resp.\ I_r), \tau_1 \leqslant \ldots \leqslant \tau_m)$$

is called a standard monomial in I (resp. I_τ) of degree or length m.

217 **Proposition 1.** *Distinct standard monomials in I_τ are linearly independent (as elements of $R(\tau)$). In particular distinct standard monomials in I are linearly independent.*

Proof. We prove this by induction on $\dim X(\tau w_0)$. By definition $R(\tau)$ is a graded ring and hence it suffices to prove that standard monomials, say in $R(\tau)_m$ are linearly independent. Suppose now that $\dim X(\tau w_0) = 0$ i.e. $X(\tau w_0)$ reduces to a point $x_0 \in G/P$. Then I_τ reduces to one element, namely $I_\tau = \{\tau\}$ and upto a constant multiple, there is only one standard monomial in $R(\tau)_m$ namely p_τ^m. Further $p_\tau^m \neq 0$, for by Lemma 3, $p_\tau(x_0) \neq 0$. Thus the proposition holds when $\dim X(\tau w_0) = 0$.

Let us now pass to the general case. Observe that $R(\tau)$ is an integral domain. Since $p_\tau \neq 0$ in $R(\tau)$ (Lemma 3), it follows in particular that

$$\lambda p_\tau = 0,\ \lambda \in R(\tau) \Rightarrow \lambda = 0.$$

Suppose now that $p \in R(\tau)_m$ can be expressed in the form

$$p = \sum_{(\lambda)} a_{(\lambda)} p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_m} \qquad (*)$$

where $(\lambda) = (\lambda_1 \ldots, \lambda_m)$, $\lambda_1 \leqslant \ldots \leqslant \lambda_m$, runs over *distinct* elements of I_τ^m and $a_{(\lambda)} \neq 0$, $a_{(\lambda)} \in k$. We shall now show that $p = 0$ leads to a contradiction.

Case (i). Suppose that on the right side of (*) we have a (standard) monomial of the form $a_{(\beta)}p_{\beta_1}\dots p_{\beta_m}$, $\beta_1 > \tau(\beta_1 \neq \tau)$ (of course $a_{(\beta)} \neq 0$).

Let q denote the restriction of p to $X(\beta_1 w_0)$. Then q can be expressed as a sum of standard monomials as follows:

$$q = \sum_{(\lambda)} a_{(\lambda)}p_{\lambda_1}\dots p_{\lambda_m}, (\lambda) \text{ as in } (*) \text{ and } \lambda_1 \geqslant \beta_1, \qquad (**)$$

i.e. q is obtained from the right hand side of (*) by cancelling all the terms such that $\lambda_1 \not\geqslant \beta_1$ (by Lemma 3, p_{λ_1} vanishes on $X(\beta_1 w_0)$ when $\lambda_1 \not\geqslant \beta_1$). The right hand side of (**) contains at least one term, namely $a_{(\beta)}p_{\beta_1}\dots p_{\beta_m}$; besides it consists of distinct standard monomials, as it is obtained from the right hand side of (*) by cancelling some terms. We have

$$X(\beta_1 w_0) \subsetneq X(\tau w_0).$$

Hence by our induction hypothesis, $q \neq 0$, so that a fortiori $p \neq 0$. **218**

Case (ii). We have only to consider the case, when in (*) $\lambda_1 = \tau$ for every (λ).

In this case we have $p = p_\tau p'$. Further p' is a sum of distinct standard monomials, in fact in (*) we have only to cancel the first term $p_{\lambda_1} = p_\tau$ of every monomial. To prove that $p \neq 0$, it suffices to prove that $p' \neq 0$ since $R(\tau)$ is an integral domain. Now $p' \in R(\tau)_{m-1}$ and hence by induction on m we can suppose that $p' \neq 0$. Thus $p \neq 0$.

This proves Proposition 1. □

Let us recall that we denoted by $A(\tau)$ (resp. A) the graded subalgebra of $R(\tau)$ (resp. R) generated by p_λ, $\lambda \in I_\tau$ (resp. $\lambda \in I$). Set

$$Y(\tau) = \text{Proj. } A(\tau),\ Y = \text{Proj. } A(Y = Y(e)).$$

Then $Y(\tau)$ is a variety and can be identified with the image of $X(\tau w_0)$ by the canonical morphism into a projective space defined by the linear system $A(\tau)_1 = \{p_\lambda/\lambda \in I_\tau\}$. By Lemma 5, the canonical morphism

$X(\tau w_0) \to Y(\tau)$ is a finite morphism and since it is surjective, it follows that $\dim Y(\tau) = \dim X(\tau w_0)$. The graded algebra $A(\tau)$ is generated by $A(\tau)_1$ (=homogeneous elements of degree one in $A(\tau)$). We have canonical *surjective* homomorphisms

$$A \to A(\tau) \to 0, A(\tau_1) \to A(\tau_2) \to 0, \tau_2 \geqslant \tau_1$$

induced by the canonical (restriction) homomorphisms

$$R \to R(\tau), R(\tau_1) \to R(\tau_2), \tau_2 \geqslant \tau_1.$$

We thus get canonical immersions

$$Y(\tau) \hookrightarrow Y, Y(\tau_2) \hookrightarrow Y(\tau_1), \tau_2 \geqslant \tau_1.$$

We see that $A(\tau)$ is the homogeneous coordinate ring of $Y(\tau)$ in the projective space associated to the linear system $A(\tau)_1$ (on $X(\tau w_0)$).

Definition 3. *A subset T of I is said to be a 'right half space' (resp. 'left half space') if the following property holds:*

$$\lambda \in T, \mu \in I \text{ and } \mu \geqslant \lambda (resp.\ \mu \leqslant \lambda) \Rightarrow \mu \in T.$$

219 We see that intersections and unions of right (resp. left) half spaces are again right (resp. left) half spaces. The complement in I of a right half space is a left half space. If T is a right half space, then

$$T = I_{\tau_1} \cup \ldots \cup I_{\tau_r}$$

where $\tau_1, \ldots, \tau_r$ are the distinct minimal elements of T. We set $X(T)$ to be the *schematic union*

$$X(T) = X(\tau_1 w_0) \cup \ldots \cup X(\tau_r w_0).$$

We have then

$$T = \{\tau \in I / \text{restriction of } p_\tau \text{ to } X(T) \neq 0\}.$$

Since $X(\tau_i w_0)$ is reduced, $X(T)$ is reduced. We speak of *standard monomials in a right half space* T, namely elements of the form

$$p_{\tau_1} p_{\tau_2} \dots p_{\tau_m}, \tau_1 \in T (\text{then all } \tau_i \in T).$$

From Proposition 1, it follows immediately that distinct standard monomials in T are linearly independent (in fact their restrictions to $X(T)$ are linearly independent). We define $Y(T)$ as the *schematic union*

$$Y(T) = \cup Y(\tau_i), \tau_i \text{ minimal elements of } T.$$

We see that $Y(T)$ is *reduced* since $Y(\tau_i)$ is reduced. We note that

$$X(I_\tau) = X(\tau w_0), Y(I_\tau) = Y(\tau), \quad \tau \in I.$$

We denote by Y_0 the subvariety of Y, which is the image of X_0 by the canonical morphism $X \to Y$ i.e. $Y_0 = Y(s)$, $s = S - S_P$. We denote by τY_0 the subvariety of Y, which is the zero set of p_τ, $\tau \in W/W_{i(P)}$. We denote by $R(T)$ the graded algebra

$$R(T) = \bigoplus_{n=0}^{\infty} H^0(X(T), (L/X(T))^n)$$

and by $A(T)$ the graded subalgebra of $R(T)$ spanned by $\{p_\tau\}$, $\tau \in T$ (it suffices to take $\tau \in T$). We see that $A(T)$ is the homogeneous coordinate ring of $Y(T)$ (with respect to the projective imbedding of Y considered above) and that

$$A(T) = A(I_\tau) = A(\tau) \text{ when } T = I_\tau.$$

We have canonical surjective homomorphisms

$A \to A(T) \to 0$, $A(T_1) \to A(T_2) \to 0$; $T_2 \subseteq T_1$ (T_i right half spaces). **220**

Definition 4. *Let T be a right half space. We set*

$\chi(T, m) \neq \sharp$ *(Set of distinct standard monomials in T of degree m)*

$\chi(T, 0) = 1$ *and* $\chi(T, m) = 0$, $m < 0$.

Lemma 6. *(i) $I_\tau - \{\tau\}$ is a right half space; in fact if T is a right half space and $\tau_1, \ldots, \tau_r$ are some distinct element chosen from the minimal elements of T, then*

$T - \{(\tau_1, \ldots, \tau_r)\}$ is again a right half space. We have

$$I_\tau - \{\tau\} = \bigcup_{\tau_i} I_{\tau_i}$$

where τ_i are precisely the minimal elements in I_τ such that $l(\tau_i) = l(\tau) + 1$ (l = length in $W/W_{i(P)}$ or equivalently $l(\tau)$ = codimension of $X(\tau w_0)$ in G/P).

(ii) We have (similar to Lemma 4)

$$Y(\tau) \cap \tau Y_0 = Y(T),\ T = I_\tau - \{\tau\}$$

in the set theoretic sense.

(iii) $\chi(T_1 \cup T_2, m) = \chi(T_1, m) + \chi(T_2, m) - \chi(T_1 \cap T_2, m)$

(iv) $\chi(I_\tau, m) = \chi(I_r - \{\tau\}, m) + \chi(I_\tau, m-1)$

Proof. The proof of (ii) is an immediate consequence of Lemma 4 and the definition of $Y(\tau)$. The proof of the other assertions is straightforward and immediate and is left as an exercise. □

4 Consequences of the hypothesis of generation by standard monomials.

Let T_0 be a right half space in I. If T is a right half space such that $T \subseteq T_0$, we see that $(T_0 - T)$ is a *left half space in* T_0 i.e.

$$\lambda \in (T_0 - T), \mu \leqslant \lambda \Rightarrow \mu \in T_0 - T.$$

We can similarly define *right half spaces T in T_0*, but this is equivalent to the fact that T is a right half space in I, $T \subseteq T_0$. As in the case $T_0 = I$, we note that

T right half space in $T_0 \Leftrightarrow T_0 - T$ is a left half space in T_0.

221 **Proposition 2.** *Let T_0 be a right half space in I. Suppose that $A(T_0)$ is spanned by standard monomials in T_0. Then we have:*

(i) *For every right half space T in T_0, $A(T)$ is spanned by standard monomials in T and the kernel of the canonical surjective homomorphism (of algebras)*

$$A(T_0) \to A(T)$$

is the ideal in $A(T_0)$ generated by $\{p_\tau\}$, $\tau \in T_0 - T$.

(ii) *Let S be a left half space in T_0, $T = T_0 - S$ and $J(S)$ the ideal in $A(T_0)$ generated by $\{p_\tau\}$ $\tau \in S$. Then the map*

$$S \mapsto J(S)$$

from the set of left half spaces in T_0 into the set of ideals in $A(T_0)$ takes set intersection into ideal intersection, set union into ideal sum and preserves distributivity properties.

(iii) *Consider the map*

$$T \mapsto Y(T)$$

from the set of right half spaces in T_0 into the set of closed subschemes of $Y(T_0)$. Then this is a bijective map of the set of right half spaces in T_0 onto the set of closed subschemes of $Y(T_0)$, each member of which is a (scheme theoretic) union of $Y(\tau)$, $\tau \in T_0$. Further this map takes set union into scheme theoretic union, set intersection to scheme theoretic intersection and preserves distributivity properties. Since unions and intersections of right half spaces in T_0 are again right half spaces in T_0 and $Y(T)$ is reduced (T right half space in T_0), it follows that unions and intersections of $Y(T)$ are again reduced.

Proof. (i) Let j be the canonical homomorphism

$$j : A(T_0) \to A(T).$$

Since j is surjective, it is immediate that $A(T)$ is generated by standard monomials, since $A(T_0)$ has this property by hypothesis. Let J be the ideal in $A(T_0)$ generated by $\{p_\tau\}$, $\tau \in T_0 - T$. By (ii) of Lemma 3, it follows that $J \subset \mathrm{Ker}\, j$. Hence j induces a canonical homomorphism

$$j' : A(T_0)/J \to A(T).$$

222 We have to prove that j' is an isomorphism. Let θ_1 be a non-zero element of $A(T_0)/J$. Choose a representative $\theta \in A(T_0)$ of θ_1. Then θ can be written as a linear combination of standard monomials in T_0, but since $p_\tau \in J$ for $\tau \in T_0 - T$, in this sum we can cancel those terms which involve p_τ, $\tau \in T_0 - T$. Thus the representative θ of θ_1 can be chosen so that it is a linear combination of standard monomials in T and $\theta \neq 0$. Now by Prop. 1, it follows that $j(\theta) \neq 0$. This implies that $j'(\theta_1) \neq 0$, which shows that j' is injective. This proves (i).

(ii) The map J in (ii) of Prop. 2, obviously takes set union to ideal sum. On the other hand, to show that $J(S_1) \cap J(S_2) = J(S_1 \cap S_2)$, S_i left half space in T_0, we first observe $J(S_1 \cap S_2) \subset J(S_1) \cap J(S_2)$. If $T_1 = T_0 - S_1$ and $T_2 = T_0 - S_2$, we see immediately that $J(S_1) \cap J(S_2)$ vanishes on $X(T)(= X(T_1) \cup X(T_2))$, $T = T_1 \cup T_2$. On the other hand by (i) $J(S_1 \cap S_2)$ is the ideal of all elements in $A(T_0)$ vanishing on $X(T)$. Hence $J(S_1 \cap S_2) \supset J(S_1) \cap J(S_2)$. This proves the assertion $J(S_1 \cap S_2) = J(S_1) \cap J(S_2)$. The other assertions in (ii) follow easily.

(iii) On account of (i), (iii) is just a reformulation of (ii).
This completes the proof of Prop. 2. □

Corollary 1. *Let T_0 be a right half space in I such that $A(T_0)$ is spanned by standard monomials. Let τ be an element of $W/W_{i(P)}$ such that $I_\tau \subset T_0$. Then the set theoretic intersection in (i), Lemma 6, is in fact scheme theoretic, i.e.*

$$Y(\tau) \cap \tau Y_0 = Y(T), T = I_\tau - \{\tau\}$$

where the intersection is scheme theoretic. In particular, the scheme theoretic intersection $Y(\tau) \cap \tau Y_0$ is reduced.

Proof. This is an immediate consequence of Prop. 2. The scheme theoretic intersection $Y(\tau) \cap \tau Y_0$ is defined by the ideal J in $A(T_0)$ generated by p_λ, $\lambda \in T_0 - I_\tau$ and p_τ. We observe that $(T_0 - I_\tau) \cup \{\tau\}$ is a left half space in T_0, since its complement $I_\tau - \{\tau\}$ is a right half space (cf. (i), Lemma 6). Hence $Y(T)$ is the scheme theoretic intersection $Y(\tau) \cap \tau Y_0$, $T = I_\tau - \{\tau\}$. This proves the corollary. □

Remark 2. Let M be the very ample line bundle on Y (cf. §3 for the **223** definition of Y) induced by the linear system $\{p_\tau\}$, $\tau \in I$. We denote this bundle by $\mathscr{O}_Y$ (1) and we denote the restriction of this line bundle to $Y(T)$ by $\mathscr{O}_{Y(T)}$ (1), T a right half space in I.

Suppose now that T_0 is a right half space in I such that $A(T_0)$ is generated by standard monomials (in T_0). Let $I_\tau \subset T_0$, $\tau \in I$. Then Cor. 1, Prop. 2 gives the exact sequence

$$0 \to \mathscr{O}_{Y(\tau)}(-1) \to \mathscr{O}_{Y(\tau)} \to \mathscr{O}_{Y(T)} \to 0, T = I_\tau - \{\tau\}. \qquad (1)$$

Corollary 2. *Let $\{T_i\}$, $1 \leqslant i \leqslant r$, be a family of right half spaces in I, such that $A(T_i)$ is spanned standard monomials, $1 \leqslant i \leqslant r$. Then if $T = T_1 \cup \ldots \cup T_r$, $A(T)$ is also spanned by standard monomials.*

Proof. By induction on r, obviously it suffices to prove the proposition for the case $r = 2$. First we claim that

$$Y(T_1) \cap Y(T_2) = Y(T_1 \cap T_2) \text{ (scheme theoretically).}$$

To prove this, consider the ideal J in $A(T_1)$ which defines the scheme theoretic intersection $Y(T_1) \cap Y(T_2)$. Obviously J contains all p_τ for $\tau \in (T_1 - T_1 \cap T_2)$. By Prop. 2, since $A(T_1)$ is generated by standard monomials and $T_1 - T_1 \cap T_2$ is a left half space in T_1, it follows that $A(T_1)/J = A(T_1 \cap T_2)$. This is precisely the above claim.

Let now B be a commutative ring and J_1, J_2 two ideals in B. Then one sees easily that the homomorphism $B \to B/J_1 \oplus B/J_2$ defined by

$b \mapsto (b \bmod J_1, b \bmod J_2)$ induces the following exact sequence of B-modules

$$0 \to B/J_1 \cap J_2 \to B/J_1 \oplus B/J_2 \xrightarrow{j} B/J_1 + J_2 \to 0$$

where $j(b_1, b_2) = b_1 - b_2 (\bmod\ J_1 + J_2)$ and (b_1, b_2) in $B \oplus B$ represents an element of $B/J_1 \oplus B/J_2$. If $J_1 \cap J_2 = 0$, then this gives

$$0 \to B \to B/J_1 \oplus B/J_2 \to B/J_1 + J_2 \to 0. \qquad (*)$$

Let $J = \operatorname{Spec} B$ and Z_i the closed subschemes of Z, $Z_i = \operatorname{Spec} B/J_i$, $i = 1, 2$. Then we have

(i) $J_1 \cap J_2 = 0 \Leftrightarrow Z = Z_1 \cup Z_2$ (scheme theoretic union)

224 (ii) $\operatorname{Spec} B/J_1 + J_2 = Z_1 \cap Z_2$.

Thus from (*), it follows that if Z_i, $i = 1, 2$, are closed subschemes of Z such that $Z = Z_1 \cup Z_2$ (scheme theoretic), then we have the following exact sequence of $\mathcal{O}_Z$-modules (patching up Z_1 and Z_2 along $Z_1 \cap Z_2$)

$$0 \to \mathcal{O}_Z \to \mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_2} \to \mathcal{O}_{Z_1 \cap Z_2} \to 0.$$

From the above general remark, it follows that we have an exact sequence of $\mathcal{O}_{Y(T)}$-modules ($T = T_1 \cup T_2$)

$$0 \to \mathcal{O}_{Y(T)} \to \mathcal{O}_{Y(T_1)} \oplus \mathcal{O}_{Y(T_2)} \to \mathcal{O}_{Y(T_1 \cap T_2)} \to 0. \qquad (2)$$

By hypothesis

$$\dim A(T_i)_m = \chi(T_i, m), i = 1, 2$$

and by Prop. 2, since $T_1 \cap T_2 \subset T_1$

$$\dim A(T_1 \cap T_2)_m = \chi(T_1 \cap T_2, m).$$

By (ii) of Lemma 6, it follows that

$$\chi(T, m) = \dim A(T)_m.$$

On the other hand, by Prop. 1, the subsapce in $A(T)_m$ spanned by standard monomials is of dimension $\chi(T, m)$. Hence it follows that $A(T)_m$ is spanned by standard monomials. This proves Corollary 2, Proposition 2. □

Proposition 3. *Let T_0 be a right half space in I such that $A(T_0)$ is spanned by standard monomials. Then for every right half space T in T_0, we have:*

(i) $\dim H^0(Y(T), \mathscr{O}_{Y(T)}(m)) = \chi(T, m)$ *or equivalently (because of Prop. 2)*

$$A(T)_m = H^0(Y(T), \mathscr{O}_{Y(T)}(m)), m \geqslant 0.$$

In particular, if T_1, T_2 are two right half spaces in T_0 such that $T_1 \subset T_2$, then the canonical homomorphism

$$H^0(Y(T_2), \mathscr{O}_{Y(T_2)}(m)) \to H^0(Y(T_1), \mathscr{O}_{Y(T_1)}(m)), m \geqslant 0$$

is surjective

(ii) $H^i(Y(T), \mathscr{O}_{Y(T)}(m)) = 0$, $i > 0$, $m \geqslant 0$

Proof. We prove the proposition by induction on $\dim Y(T)$ (= $\dim X(T)$, **225**
cf. Remark 1). Let $n = \dim Y(T)$. We claim that it suffices to prove the proposition when $Y(T)$ is of the form $Y(\tau)$, $\tau \in I$. To prove this, since $Y(T)$ is the schematic union of $Y(\tau_i)$, $T = \bigcup_{i=1}^{r} I_{\tau_i}$ (τ_i-the minimal elements of T), we see easily that it suffices to prove the following:

$$\left\{\begin{array}{l} \text{Let } T_1 = I_{\tau_1} \text{ and } T_2 = \bigcup_{i=2}^{r} I_{\tau_i}. \text{ Suppose that (i) and} \\ \text{(ii) above hold for } Y(T_1) \text{ and } Y(T_2); \text{ then they hold for} \\ Y(T),\ T = T_1 \cup T_2. \end{array}\right\} \quad (*)$$

The claim (*) is an easy consequence of the exact sequence 2 (cf. Cor. 2, Prop. 2), namely the exact sequence of $\mathscr{O}_{Y(T)}$ modules

$$0 \to \mathscr{O}_{Y(T)} \to \mathscr{O}_{Y(T_1)} \oplus \mathscr{O}_{Y(T_2)} \to \mathscr{O}_{Y(T_1 \cap T_2)} \to 0 \qquad (**)$$

We see that $\dim Y(T_1 \cap T_2) \leqslant (n-1)$. Hence the proposition is true for $Y(T_1 \cap T_2)$. In particular,

$$H^0(\mathscr{O}_{Y(T_1 \cap T_2)}(m)) = A(T_1 \cap T_2)_m$$

and it is spanned by standard monomials. Hence the canonical map

$$H^0(\mathscr{O}_{Y(T_1)}(m) \oplus \mathscr{O}_{Y(T_2)}(m)) \to H^0(\mathscr{O}_{Y(T_1 \cap T_2)}(m)) \qquad (***)$$

is surjective for $m \geqslant 0$. Writing the cohomology exact sequence for (**), we get

$$\dim H^0(\mathscr{O}_{Y(T_1 \cup T_2)}(m)) = \chi(T_1, m) + \chi(T_2, m) - \chi(T_1 \cap T_2, m),$$

which implies (by Lemma 6) that

$$\dim H^0(\mathscr{O}_{Y(T_1 \cup T_2)}(m)) = \chi(T_1 \cup T_2, m).$$

Thus the assertion (1) of Prop. 3 follows.

Since the map (***) is surjective, we see that the sequence

$$0 \to H^1(\mathscr{O}_{Y(T_1 \cup T_2)}(m)) \to H^1(\mathscr{O}_{Y(T_1)}(m)) \oplus H^1(\mathscr{O}_{Y(T_2)}(m)), m \geqslant 0 \quad (3)$$

induced by the cohomology exact sequence of (**) is exact. Further we get the exact sequence

$$0 \to H^{i-1}(\mathscr{O}_{Y(T_1 \cap T_2)}(m)) \to H^i(\mathscr{O}_{Y(T)}(m)) \to H^i(\mathscr{O}_{Y(T_1)}(m)) \oplus H^i$$

226 $(\mathscr{O}_{T(Y_2)}(m))$ for $i \geqslant 2$, $m \geqslant 0$. Since $\dim Y(T_1 \cap T_2) \leqslant (n-1)$, by the induction hypothesis, we have

$$H^{i-1}(\mathscr{O}_{Y(T_1 \cap T_2)}(m)) = 0, \ i \geqslant 2, \ m \geqslant 0.$$

Thus we get the exact sequence

$$0 \to H^i(\mathscr{O}_{Y(T_1 \cup T_2)}(m)) \to H^i(\mathscr{O}_{Y(T_1)}(m)) \oplus H^i(\mathscr{O}_{Y(T_2)}(m)), \begin{cases} i \geqslant 2 \\ m \geqslant 0. \end{cases}$$

Combining this with (3) above, we get

$$H^i(\mathscr{O}_{Y(T)}(m)) = 0, \ i \geqslant 1, m \geqslant 0.$$

This proves the claim (*) above i.e. it suffices to prove the proposition for $Y(\tau)$, $\tau \in I$.

Let $T = I_\tau - \{\tau\}$. Tensoring the exact sequence (1) in Remark 2 by $\mathscr{O}_{Y(\tau)}(m)$, we get the exact sequence

$$0 \to \mathscr{O}_{Y(\tau)}(m-1) \to \mathscr{O}_{Y(\tau)}(m) \to \mathscr{O}_{Y(T)}(m) \to 0, m \geqslant 0. \qquad (4)$$

We have $\dim Y(T) = (n-1)$. By our induction hypothesis, it follows, in particular, that $H^0(\mathscr{O}_{Y(T)}(m)) = A(T)_m$ and that it is spanned by standard monomials. Thus implies that the canonical map

$$H^0(\mathscr{O}_{Y(\tau)}(m)) \to H^0(\mathscr{O}_{Y(T)}(m)), \qquad (5)$$

is surjective. If we set $\lambda_m = \dim H^0(\mathscr{O}_{Y(\tau)}(m))$, writing the cohomology exact sequence for (4) at the H^0 level, we get

$$\lambda_m - \lambda_{m-1} = \chi(T, m) \text{ for all } m (\text{note } \lambda_m = \chi(T, m) = 0, m < 0).$$

On the other hand, by Lemma 6, we get

$$\chi(I_\tau, m) - \chi(I_\tau, m-1) = \chi(T, m) \text{ for all } m.$$

For $m \leqslant 0$, we see immediately that $\chi(I_\tau, m) = \lambda_m$. From this it follows that $\chi(I_\tau, m) = \lambda_m$ i.e.

$$\chi(I_\tau, m) = \dim H^0(\mathscr{O}_{Y(\tau)}(m)) \text{ for all } m.$$

Writing the cohomology exact sequence for (4) and using the fact that the proposition holds for $\mathscr{O}_{Y(T)}(m)$ (on account of the inductive hypothesis) and the surjective map (5) above, we deduce easily the following:

$$0 \to H^i(\mathscr{O}_{Y(\tau)}(m-1)) \to H^i(\mathscr{O}_{Y(\tau)}(m)) \text{ is exact, } i \geqslant 1, m \geqslant 0.$$

One knows that $H^i(\mathscr{O}_{Y(\tau)}(m)) = 0$, $i \geqslant 1$ and m sufficiently large (Serre's theorem). Hence by decreasing induction on m, we deduce immediately that **227**

$$H^i(\mathscr{O}_{Y(\tau)}(m)) = 0, \ i \geqslant 1, \ m \geqslant 0.$$

This concludes the proof of Prop. 3. □

Corollary 1. *Let T_0 be a right half space in I such that $A(T_0)$ is spanned by standard monomials. Let*

$$\hat{Y}(T) = \operatorname{Spec} A(T), T \text{ right half space in } T_0.$$

Then we have the following:

(i) $\hat{Y}(T)$ *is reduced*

(ii) Consider the map

$$T \mapsto \hat{Y}(T)$$

from the set of right half spaces in T_0 into the set of closed subschemes of $\hat{Y}(T_0)$. Then this is a bijective map of the set of right half spaces in T_0 onto the set of closed subschemes of $\hat{Y}(T_0)$, each member of which is a (schematic) union of $\hat{Y}(\tau)$, $\tau \in T_0$. Further this map takes set union into scheme theoretic union, set intersection to scheme theoretic intersection and preserves distributivity properties. Since unions and intersections of right half spaces in T_0 are again right half spaces in T_0 and $\hat{Y}(T)$ is reduced, it follows that unions and intersections of $\hat{Y}(T)$ are again reduced (T right half spaces in T_0).

(iii) $\hat{Y}(\tau)$ *is normal for $\tau \in I$ such that $I_\tau \subset T_0$. In particular, for such a τ, $Y(\tau)$ is normal. We refer to $\hat{Y}(T)$ as cones over $Y(T)$.*

Proof. Since $Y(T)$ is reduced and

$$A(T) = \bigoplus_{m \geqslant 0} H^0(\mathcal{O}_{Y(T)}(m)) \text{ (by Prop. 3),}$$

it follows immediately that $\hat{Y}(T)$ is reduced. The assertion (ii) is merely the assertion (ii) of Prop. 2.

228 Let $\tau \in I$ be such that $I_\tau \subset T_0$. Hence as in the proof of Cor. 1, Prop. 2, it follows that

$$A(\tau)/_{p_\tau A(\tau)} = A(T), T = I_\tau - \{\tau\}.$$

Let $\mathcal{O}_\tau$ be the local ring of $\hat{Y}(\tau)$ at its "*vertex*" i.e. the point corresponding to the irrelevant maximal ideal of the graded ring $A(\tau)$. Since $A(T)$ is reduced by (i) above, it follows that $\mathcal{O}_\tau/_{p_\tau \mathcal{O}_\tau}$ is reduced. This implies that

$$\text{depth } \mathcal{O}_\tau \geqslant 2.$$

On the other hand, it has been shown by Chevalley [6] that the singularity subset of $Y(\tau)$ is of Codim $\geqslant 2$ in $Y(\tau)$. It follows that the singular locus of $\mathcal{O}_\tau$ is of Codim $\geqslant 2$ in Spec $\mathcal{O}_\tau$. It follows then that $\mathcal{O}_\tau$ is normal. i.e. the cone $\hat{Y}(\tau)$ is normal at its vertex. Hence $\hat{Y}(\tau)$ as well as $Y(\tau)$ are normal (note of course that $\hat{Y}(\tau)$ is the *cone over* $Y(\tau)$ in the usual sense for the imbedding of $Y(\tau)$ in the projective space corresponding to the linear system on Y defined by A_1). This proves Cor. 1, Prop. 3. □

Remark 3. Let T_0 and τ be as in Cor. 1, Prop. 3 ($I\tau \subset T_0$). It should be possible to prove that $\hat{Y}(\tau)$ is Cohen-Macaulay by showing that

$$H^i(\mathcal{O}_{Y(\tau)}(m)) = 0, 0 \leqslant i < \dim Y(\tau), m < 0 \qquad (*)$$

Let $T = I_\tau - \{\tau\}$ and τ_i, $1 \leqslant i \leqslant r$, be the minimal element of T so that $Y(\tau_i)$ are the irreducible components of $Y(\tau) \cap \tau Y_0$. For proving (*), following the argument as in Prop. 3 for the case $m \geqslant 0$ (see for example [13]), it is seen easily that one has to show that

$$Y(\tau_i) \cap Y(\tau_j)(i \neq j) \text{ is of pure Codim 1 in } Y(\tau_i)$$

Equivalently one has to show that (cf. Remark 1)

$$X(\tau_i w_0) \cap X(\tau_j w_0)(i \neq j) \text{ is of Codim 1 in } X(\tau_i w_0).$$

When P is *miniscule* (cf. §5, below), this has been proved by Kempf (cf. [11]) and a proof has also been shown to the author by Lakshmibai and Musili.

Remark 4. Suppose that $Y(\tau)(\tau \in I)$ is of dimension n and that $A(T)$ **229** is spanned by standard monomials whenever $\dim Y(T) \leqslant (n-1)$ (T a right half space in I). Then to show that $A(\tau)$ is spanned by standard

monomials, *we claim that it suffices to prove that the scheme theoretic intersection* $Y(\tau) \cap \tau Y_0$ *is reduced*. For, one has the following exact sequence (cf. exact sequence (1) in Remark 2, what is essential for having this exact sequence is that $Y(\tau) \cap \tau Y_0$ is reduced and in Remark 2 this follows from Cor. 1, Prop. 2)

$$0 \to \mathscr{O}_{Y(\tau)}(m-1) \to \mathscr{O}_{Y(\tau)}(m) \to \mathscr{O}_{Y(T)}(m) \to 0 \qquad (*)$$

where $T = I_\tau - \{\tau\}$ (this exact sequence is obtained by tensoring the exact sequence (1) of Remark 2 by $\mathscr{O}_{Y(\tau)}(m)$). Then as in the proof of Prop. 3, writing the cohomology exact sequence for (*), we deduce that $A(\tau)$ is spanned by standard monomials. In particular, if the scheme theoretic intersection $Y(\tau) \cap \tau Y_0$ is reduced for every $\tau \in I$, it will follow that $A(T)$ is generated by standard monomials for every right half space T in I.

Remark 5. Suppose that for a given $\tau \in I$ (or more generally a given right half space T in I) $A(\tau)$ *is spanned by standard monomials when the ground field is of characteristic zero. Then we claim that* $A(\tau)$ *is spanned by monomials when the field is of arbitrary characteristic.*

Let D be a complete discrete valuation ring with quotient field K of characteristic zero and residue field the (algebraically closed) ground field k, assumed to be of characteristic $p > 0$. We know that $G = G' \otimes_D k$, where G' is a *split* semisimple group scheme over D. One knows that the above considerations go through over the base Spec D (cf. [15], esp. Vol. III). To be more precise, let us note the following:

(i) We have a maximal parabolic subgroup P' in G' such that $P' \otimes k = P$ and $G'/P' \otimes k = G/P$. Further Pic $G'/P' \approx \mathbf{Z}$ and it has an ample generator L' such that $L' \otimes k = L$, where L is the ample generator of Pic G/P. We have a canonical section $F' \in H^0(G'/P', L')$ whose zero set is the Schubert scheme X'_0 of codimension one in G'/P' with $X'_0 \otimes k = X_0$. We can also suppose that T' is a maximal split torus in G' such that $T' \otimes k = T$ and that in the $G' - D$ module
230 $H^0(G'/P', L')$ (for the notation $G' - D$ module, see [15]), the D module spanned by F' is T' stable.

(ii) the Weyl group W of G is the Weyl group of the abstract root system associated to the split semi-simple group scheme G'. Then one knows that $W_D = N(T')/T'$, where W_D is the constant group scheme over Spec D associated to the abstract group W (cf. p.168-169, Vol. III, [15]). The parabolic subgroup scheme P' has a structure similar to that of P (cf. §1) and we can talk of the parabolic subgroup scheme $i(P')$, $i = -w_0$ (Weyl involution). The Weyl group of $i(P')$ is the constant group scheme (over Spec D) associated to the abstract group $W_{i(P)}$.

(iii) the elements of W can be canonically identified with the D-valued points of W_D (W_D being a constant group scheme) and these D-valued points of W_D can be lifted (not canonically) to some D-valued points of $N(T')$ since the canonical morphism $N(T') \to W_D$ is smooth and D is a complete discrete valuation ring with algebraically closed residue field. If $\tau \in W$, denote by the same τ the D-valued point of $N(T')$ obtained this way. If the D-valued point τ corresponds to an element of the subgroup $W_{i(P')}$ of W, we see that τ fixes the Schubert subscheme X'_0 or equivalently $\tau \cdot F' = \lambda F'$ (λ=unit in D) (cf. §2 and Lemma 1). Let us denote by p'_τ the element $p'_\tau = \tau \cdot F'$, $\tau \in W$. Since the D-module spanned by F' is T' stable, p'_τ is well-defined upto a unit (i.e. independent of the above lifting to D-valued points in $N(T')$ upto a unit in D) and thus upto a unit in D, the notation

$$p'_\tau, \quad \tau \in W/W_{i(P)}$$

is justified.

Let us now take up the proof of the assertion in Remark 5 above. Let us denote by $X'(\tau w_0)$ the Schubert subscheme in G'/P' associated to $\tau w_0 \in W$ and by $A'(\tau)$, $R'(\tau)$ the algebras over D similar to $A(\tau)$, $R(\tau)$ (cf. §2). We see that $A'(\tau)$ is the subalgebra of $R'(\tau)$ generated by $\{p_\tau\}$, $\tau \in W/W_{i(P)}$. Suppose that $q \in A'(\tau)_m$ is *not standard* (over D). By hypothesis, $A'(\tau) \otimes_D K$ is spanned (over K) by standard monomials.

This implies that

$$q = \sum_{(\alpha)\in I_\tau^m} \lambda_{(\alpha)} p_{\alpha_1} \dots p_{\alpha_m}, \alpha_1 \leqslant \dots \leqslant \alpha_m$$

231 where $\lambda_{(\alpha)} \in K$, $\lambda_{(\alpha)} \neq 0$ and (α) runs over *distinct elements* of I_τ^m. We can write

$$\lambda_{(\alpha)} = a_{(\alpha)}/p^n, \begin{cases} a_{(\alpha)} \in D, n \geqslant 0, \\ p = \text{ generator of maximal ideal of } D \\ \text{and at least one } a_{(\alpha)} \text{ is a } \textit{unit} \text{ in } D. \end{cases}$$

This gives

$$p^n q = \sum_{(\alpha)\in I_\tau^m} a_{(\alpha)} p_{\alpha_1} p_{\alpha_2} \dots p_{\alpha_m}, \begin{cases} \alpha_1 \leqslant \dots \leqslant \alpha_m \\ a_{(\alpha)} \in D, a_{(\alpha)} \neq 0 \end{cases} \tag{1}$$

where the summation on the right side runs over distinct standard monomials. Now $n \neq 0$, since otherwise q is a sum of standard monomials. Hence $n > 0$. Reduce (1) mod p i.e. read (1) by taking the images of the elements by the canonical homomorphism $A'(\tau) \to A(\tau) = A'(\tau) \otimes_D k$. Then we get

$$0 = \sum \bar{a}_{(\alpha)} p_{\alpha_1} \dots p_{\alpha_m}, \bar{a}_{(\alpha)} \neq 0 \text{ at least for one } (\alpha). \tag{2}$$

where $\bar{a}_{(\alpha)}$ denotes the image of $a_{(\alpha)}$ by the canonical homomorphism $D \to k$ and the right side runs over distinct standard monomials. This contradicts the fact that standard monomials are linearly independent over k (cf. Prop. 1). Thus we conclude that $A'(\tau)$ is spanned by standard monomials over D. This implies, a fortiori, that $A(\tau)$ is spanned by standard monomials (over k) as we have $A'(\tau) \otimes k = A(\tau)$. This proves the assertion in Remark 5.

5 Minuscule weights and the main theorem

Definition 5. *A fundamental weight ϖ (or the associated parabolic group P) is said to be* minuscule *(cf. p. 226, exercise 24, [4]) if any*

weight θ of the irreducible representation V_1 with highest weight ϖ in characteristic zero (i.e. if G' is the split group scheme over $\mathbf{Z}$ *such that $G \otimes_{\mathbf{Z}} k = G$, then V_1 is the irreducible representation of $G' \otimes_{\mathbf{Z}} \mathbf{C}$ with highest weight ϖ) is of the form*

$$\theta = \tau(\varpi), \tau \in W, W = \textit{ Weyl group}.$$

Proposition 4 (cf. exercise 24, p. 226, [4])**.** *A fundamental weight ϖ is* **232** *minuscule if and only if*

$$\langle \varpi, \alpha^{\check{}} \rangle = 0, 1 \textit{ or } -1, \forall \alpha \in \Delta : (\alpha^{\check{}} \textit{ coroot of } \alpha).$$

Further, if ϖ is minuscule, $X(\tau\omega_0)$, $\tau \in W/W_{i(P)}$ a Schubert variety in G/P and Y a Schubert variety which is an irreducible component of $X(\tau w_0) \cap \tau X_0$ (cf. Lemma 4), then the multiplicity of this intersection along Y is 1.

Proof. Let $\alpha \in \Delta^+$ (positive root) and ϖ be minuscule. One has to show that $\langle \varpi, \alpha^{\check{}} \rangle = 0$ or 1. One knows that $\langle \varpi, \alpha^{\check{}} \rangle$ is an *integer* $\geqslant 0$ (cf. Théorèm 3, Chap. VII-9, [14]); further by taking the 3-dimensional Lie algebra generated by X_α, Y_α and H_α (notations as in [14]) and using Theorem 1, Chap. IV-3 especially its Corollary 1, (b) (cf. [14]), we see that $(\varpi - r\alpha)$ is also a weight of V_1 where r an integer $0 \leqslant r \leqslant \langle \varpi, \alpha^{\check{}} \rangle$. Suppose now that $\langle \varpi, \alpha^{\check{}} \rangle > 1$. Take r such that $0 < r < \langle \varpi, \alpha^{\check{}} \rangle$ Then $q = \varpi - r\alpha$ is a weight of V_1. We claim that

$$(q|q) < (\varpi/\varpi)(\text{notation as in [4]})$$

for

$$\begin{aligned}(\varpi - r\alpha/\varpi - r\alpha) &= (\varpi/\varpi) + r^2(\alpha/\alpha) - 2(\varpi/\alpha)r \\ &= (\varpi/\varpi) + r^2(\alpha/\alpha) - r(\alpha/\alpha)\langle \varpi, \alpha^{\check{}} \rangle \\ &\qquad \left(\text{for } \langle \varpi, \alpha^{\check{}} \rangle = \frac{2(\varpi/\alpha)}{(\alpha, \alpha)}\right) \\ &= (\varpi/\varpi) + (\alpha/\alpha)[r^2 - r\langle \varpi, \alpha^{\check{}} \rangle].\end{aligned}$$

Since $0 < r < \langle \varpi, \alpha^\vee \rangle$, we deduce that

$$r^2 - r\langle \varpi, \alpha^\vee \rangle < 0$$

This proves the claim that $(q/q) < (\varpi, \varpi)$. But since ϖ is minuscule, $q = \tau(\varpi)$, $\tau \in W$, which implies, in particular, that $(q/q) = (\varpi, \varpi)$. This leads to a contradiction to the hypothesis that $\langle \varpi, \alpha^\vee \rangle > 1$. Thus we see that $\langle \varpi, \alpha^\vee \rangle = 0$ or 1.

Suppose now that ϖ is a fundamental weight, such that

$$\langle \varpi, \alpha^\vee \rangle = 0, 1 \text{ or } -1 \quad \forall \alpha \in \Delta.$$

233 Then we see that for any $\tau \in W$, we have

$$\langle \tau(\varpi), \alpha^\vee \rangle = 0, 1 \text{ or } -1.$$

Let $v \in V_1$ be the element with highest weight ϖ. Then one knows that

$$Y_{\beta_1}^{m_1} \ldots Y_{\beta_k}^{m_k} v, m_i \in N, \beta_i \in \Delta^+, (cf.\ Prop. 2, VII - 3, [14])$$

generate V_1 and one has to show that if $Y_{\beta_1}^{m_1} \ldots Y_{\beta_k}^{m_k} v \neq 0$, then the weight of this element is of the form $\tau(\varpi)$, $\tau \in W$. By a simple induction argument, we see that it suffices to show that if $v' \in V_1$, $v' \neq 0$ and of weight $\tau(\varpi)$, $\tau \in W$ and if $Y_{\beta_i} v' \neq 0$, then $Y_{\beta_i} v'$ is of weight $\tau'(\varpi)$ for some $\tau' \in W$. Now v' is the highest weight vector for a suitable conjugate of the Borel subalgebra and β_i can be supposed to be positive with respect to this Borel subalgebra. Thus we can suppose without loss of generality, that $\tau =$ Identity i.e. $v = v'$. We see that $\langle \varpi, \beta_i^\vee \rangle \neq 0$, for if $\langle \varpi, \beta_i^\vee \rangle = 0$, then by Theorem 1, Chap. IV-3, especially its Cor. 1, (b), [5], the 3-dimensional Lie algebra X_{β_i}, Y_{β_i}, H_{β_i} annihilates v (weight of $v = \lambda = \langle \varpi, \beta_i^\vee \rangle = 0$) and this contradicts the assumption $Y_{\beta_i} v \neq 0$. We see then that $\langle \varpi, \beta_i^\vee \rangle = 1$ and then the weight of $Y_{\beta_i} v$ is $\varpi - \beta_i = \varpi - \langle \varpi, \beta_i^\vee \rangle \beta_i = s_{\beta_i}(\varpi)$. This proves the first assertion of (Prop. 4).

The last assertion of Prop. 4 is an immediate consequence of Chevalley's multiplicity formula (cf. [8], p. 78, Cor. 1, Prop. 1.2), since by this formula, this required multiplicity is of the form

$$\langle \varpi, \alpha^\vee \rangle (\text{or perhaps } \langle i(\varpi), \alpha^\vee \rangle),\ \alpha \in \Delta^+$$

and ϖ is minuscule if and only if $i(\varpi)$ is minuscule (i = Weyl involution).

This proves Prop. 4. □

Remark 6. Suppose that the base field is of *characteristic zero* and ϖ is minuscule. Then

$$R(\tau) = A(\tau), \ \forall \tau \in W/W_{i(P)}.$$

In particular, the canonical morphism $X(\tau w_0) \to Y(\tau)$ is an *isomorphism* (cf. beginning of §3 for the definition of $R(\tau)$, $A(\tau)$).

Proof. Since $R_m = H^0(G/P, L^m)$ is known to be irreducible with highest **234**
weight $mi(\varpi)$ (Borel-Weil theorem), the canonical G-homomorphism

$$\underbrace{R_1 \otimes \ldots \otimes R_1}_{m \text{ times}} \to R_m$$

is surjective. This shows that R_1 generates R. Since $i(\varpi)$ is minuscule, the weights of R_1 are of the form $\tau i(\varpi)$, $\tau \in W/W_{i(P)}$. Now $\tau i(\varpi)$ is the "highest weight" for a suitable conjugate of the Borel subgroup B of G, which shows that the linear subspace of R_1 with weight $\tau i(\varpi)$ is of dimension one and hence coincides with the one dimensional subspace spanned by p_τ. This shows that $R = A$. The general case follows from a result of Demazure (cf. Theorem 1, [8], p. 84), namely that the canonical map

$$H^0(G/P, \mathscr{O}_{G/P}(m)) \to H^0(X(\tau w_0), \mathscr{O}_{X(\tau w_0)}(m))$$

is surjective. □

Remark 7. We see that if G is type A_n, every fundamental weight is minuscule. Looking at the tables in [4], we have the following:

(i) G of type B_n, ϖ is minuscule $\Leftrightarrow$ $\varpi = \varpi_n$ i.e. ϖ corresponds to the "right end root" (in the Dynkin diagram)

(ii) G of type C_n, ϖ is minuscule $\Leftrightarrow$ $\varpi = \varpi_1$ i.e. ϖ corresponds to the "left end root".

(iii) G of type D_n, ϖ is minuscule $\Leftrightarrow$ $\varpi = \varpi_1$ or ϖ_{n-1} or ϖ_n i.e. ϖ corresponds to the "extreme end roots".

(iv) G of type E_6, ϖ is minuscule $\Leftrightarrow$ $\varpi = \varpi_1$ or ϖ_6 i.e. ϖ corresponds to the "left or right end root".

(v) G of type E_7, ϖ is minuscule $\Leftrightarrow$ $\varpi = \varpi_7$ i.e. ϖ corresponds to the "right end root".

(vi) there are no minuscule weights, when G is of type E_8, F_4 or G_2.

Proposition 5. *Suppose that the base field is of characteristic zero and ϖ is minuscule. Then $R(\tau) = (A(\tau))$ is spanned by standard monomials in I_τ, $\tau \in W/W_{i(P)}$.*

235 *Proof.* We prove this by induction on $\dim X(\tau w_0)$. When $\dim X(\tau w_0) = 0$, the required fact is immediate. Now by Remark 4, it suffices to check that the scheme-theoretic intersection $X(\tau w_0) \cap \tau X_0$ is *reduced*. Now by Demazure's results, $X(\tau w_0)$ is Cohen-Macaulay (cf. Cor. 2, Theorem 1, [8], p. 84-85). Hence the scheme theoretic intersection $X(\tau w_0) \cap \tau X_0$ is also Cohen-Macaulay. Now by (ii) of Prop. 4, the scheme theoretic intersection $X(\tau w_0) \cap \tau X_0$ is reduced at the generic points of its irreducible components. It follows that $X(\tau w_0) \cap \tau X_0$ is reduced and hence Prop. 5 follows. □

Theorem 1. *Suppose that the fundamental weigh ϖ is minuscule. Then if T is a right half space in I, we have (the characteristic of the ground field k being arbitrary):*

(i) *$A(T)$ is spanned by standard monomials in T;*

(ii) *$A(T) = R(T)$;*

(iii) *the line bundle L on G/P is very ample;*

(iv) *$H^i(X(T), \mathcal{O}_{X(T)}(m)) = 0$, $i > 0$, $m \geqslant 0$;*

(v) Consider the map

$$T \mapsto X(T)(resp.\ \hat{X}(T) - \ cone\ over\ X(T))$$

from the set of right half spaces in I into the set of closed subschemes of G/P (resp. $\hat{G}/P$). This is a bijective map of the set of right half spaces in I onto the set of closed subschemes of G/P (resp. $\hat{G}/P$), each member of which is a schematic union of $X(\tau)$ (resp. $\hat{X}(\tau)$), $\tau \in I$. Further this map takes set union into scheme theoretic union, set intersection into scheme theoretic intersection and preserves distributivity properties. Scheme theoretic unions and intersections of $X(T)$ (resp. $\hat{X}(T)$) are reduced;

(vi) $X(\tau w_0)$ (in fact the cone $\hat{X}(\tau w_0)$) is normal (in fact, $\hat{X}(\tau w_0)$ is also Cohen–Macaulay, cf. Remark 3 above);

(vii) $X(\tau w_0) \cap \tau X_0$ is reduced.

Proof. By Prop. 5, $A(\tau)$ is spanned by standard monomials in I_τ, $\tau \in W/W_{i(P)}$, when the ground field is of characteristic zero. Hence by Re- **236**
mark 5, the same holds when the ground field is of arbitrary characteristic. Now by Cor. 2, Prop. 2, it follows that $A(T)$ is spanned by standard monomials in T, T being any right half space in I. This proves (i).

Now $H^1(X(T), \mathcal{O}_{X(T)}(m)) = 0$, when m is sufficiently large. Hence $R(T)_m = H^0(X(T), \mathcal{O}_{X(T)}(m))$ is obtained by "reduction mod p" of the same space in characteristic zero (note that $X(\tau w_0)$, $X(T)$ can be constructed as schemes over $\mathbf{Z}$), when m is sufficiently large. In particular, we get

$$\dim R(T)_m = \chi(T, m), m \gg 0$$

since by Prop. 5 and Cor. 2, Prop. 2, $R(T)_m$ is spanned by standard monomials in T when the ground field is of characteristic zero. On the other hand

$$\dim A(T)_m = \chi(T, m), \forall m$$

because of (i). Hence, we get

$$A(T)_m = R(T)_m, m \gg 0.$$

Since $A(T)_1$ generates $A(T)$, it follows that the canonical morphism

$$X(T) = \mathrm{Proj}R(T) \to Y(T) = \mathrm{Proj}A(T)$$

is an isomorphism; in particular $L|_{X(T)}$ is very ample. Since (i) holds and $X(T) = Y(T)$, we deduce that

$$A(T)_m = R(T)_m$$

by (i), Prop. 3. This proves (ii) and (iii).

The assertions (iv), (v) and (vi) and (vii) are taken from Prop. 1, its Cor. 1, Prop. 3 and its Cor. 1. □

Corollary 1. *Let ϖ be minuscule. Suppose that $p_{\beta_1}p_{\beta_2}$ is a quadratic monomial which is not standard, $\beta_i \in I$. Then we have a unique relation*

$$p_{\beta_1}p_{\beta_2} = \sum_{(\alpha)\in I^2} \lambda_{(\alpha)}p_{\alpha_1}p_{\alpha_2}; \alpha_1 \leqslant \alpha_2 \text{ and } \lambda_{(\alpha)} \in k, \lambda_{(\alpha)} \neq 0 \qquad (*)$$

where on the right side (α) runs over distinct elements of I^2 of the form

237 (i) $\alpha_1 \leqslant \beta_1, \alpha_1 \neq \beta_1$; $\alpha_1 \leqslant \beta_2, \alpha_1 \neq \beta_2$

(ii) $\alpha_2 \geqslant \beta_1, \alpha_2 \neq \beta_2$; $\alpha_2 \geqslant \beta_2, \alpha_2 \neq \beta_2$.

Proof. By Theorem 1, $p_{\beta_1}p_{\beta_2}$ can be expressed uniquely as a sum of standard monomials. We shall now show that if a non-standard monomial $p_{\beta_1}p_{\beta_2}$ is expressed as a sum of standard monomials (even without assuming that ϖ is minuscule), then the properties (i) and (ii) above are satisfied.

Suppose then $\alpha_1 \not\leqslant \beta_1$. Restrict (*) to the Schubert variety $X(\alpha_1 w_0)$. Then the restriction of p_{β_1} to $X(\alpha_1 w_0)$ is zero (cf. Lemma 3) and we get

$$0 = \lambda_{(\alpha)}p_{\alpha_1}p_{\alpha_2} + \sum \ldots$$

which contradicts the linear independence of standard monomials (cf. Prop. 1.). Thus we see that $\alpha_1 \leqslant \beta_1$. Suppose that $\alpha_1 = \beta_1$; since β_1, β_2 are *not* comparable, we deduce that $\beta_2 \not\geqslant \alpha_1$. Again the restriction of

p_{β_2} to $X(\alpha_1 w_0)$ is zero and restricting (*) to $X(\alpha_1 w_0)$ we get a contradiction. Thus we deduce that $\alpha_1 \leqslant \beta_1$, $\alpha_1 \neq \beta_1$. Similarly, we deduce that $\alpha_1 \leqslant \beta_2$; $\alpha_1 \neq \beta_2$.

To prove (ii), we observe that $\tau_i \cdot p_{\tau_2} = p_{\tau_1\tau_2}, \tau_i \in W$, (this is well defined only upto a constant, see the discussion preceding Lemma 1). Transforming (*) by left action by the element w_0 (or to be precise by a representative of w_0 in $N(T)$), we get

$$p_{w_0\beta_1} p_{w_0\beta_2} = \sum_{(\alpha)\in I^2} \mu_{(\alpha)} p_{w_0\alpha_2} p_{w_0\alpha_1}, \qquad (**)$$

where (α) runs over the same set of elements as in (*) and $\mu_{(\alpha)} \neq 0$. We see easily that

$$\alpha_1 \leqslant \alpha_2 \Leftrightarrow w_0\alpha_1 \geqslant w_0\alpha_2 \text{ (cf. Definition 1).}$$

Hence the right side of (**) also runs over distinct standard monomials and the left side of (**) is not standard. Hence applying (i), we get for example

$$w_0\alpha_2 \leqslant w_0\beta_1; w_0\alpha_2 \neq w_0\beta_1$$

and this implies $\alpha_2 \geqslant \beta_1$, $\alpha_2 \neq \beta_1$. This proves (ii) and completes the proof of the corollary. □

Remark 8. In the proof of Theorem 1, the work of Demazure [8], especially his result that the Schubert varieties are Cohen-Macaulay in **238** characteristic zero, has been used. A more direct proof would be along the following lines: When ϖ is minuscule and the base field is of characteristic zero, suppose one is able to check (probably using Weyl's or Demazure's character or dimension formula) directly that

$$\dim H^0(G/P, \mathcal{O}_{G/P}(2)) = \chi(I, 2).$$

Then this implies that given a non-standard quadratic monomial $p_{\beta_1} p_{\beta_2}$, this can be expressed as in (*), Cor. 1, Theorem 1 as a linear combination of standard monomials (base field is of characteristic zero) and we have seen in the proof of this corollary that the properties (i) and (ii) of

Cor. 1, Theorem 1, follow then automatically. Then by the argument as in [10] or (Prop. 3. 1, p. 153 [13]), we conclude that the standard monomials span R, and by Remark 5 the same conclusion holds in arbitrary characteristic. Once we have this, Theorem 1, follows from Prop. 2 and Prop. 3.

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Varieties with No Smooth Embeddings

By M.V. Nori

THIS PAPER IS based essentially on the following idea: A scheme X **241** which has no effective Cartier divisors cannot be embedded in a smooth scheme, or for that matter, even in a smooth algebraic space ([1], p. 326). For, given any such embedding of X, one could find plenty of such divisors on the ambient space which do not contain X, and hence their restrictions to X would be effective and locally principal again. However, this says nothing about embedding X in a complex manifold (not necessarily algebraic).

I thank Simha very heartily for providing the incentive to look for examples of such varieties; also for discussing the problem with me in some detail.

One may construct such a scheme X quite simply as follows:

Take irreducible curves C_1 and C_2 of different degrees in $\mathbf{P}^n$, such that there is a birational isomorphism $f : C_1 \to C_2$, and identify the points x and $f(x)$, where $x \in C_1$, to get a quotient variety X of $\mathbf{P}^n$ with the following properties:

(A) X is a reduced irreducible scheme and $\varphi : \mathbf{P}^n \to X$ is the normalisation map,

(B) $\varphi = \varphi \circ f$ for all points of C_1, and φ maps C_1 birationally onto its image $C = \varphi(C_1)$,

(C) any line bundle L on X lifts to the trivial line bundle on $\mathbf{P}^n$, and therefore X is not projective,

(D) however, any finite set of points of X is contained in an affine open set, and finally,

(E) X has no smooth embeddings.

To prove (C), note that the degrees of $\varphi^*(L)|C_1$ and $\varphi^*(L)|C_2$ both coincide with the degree of $L|C$, and are therefore equal. But $\varphi^*(L)$ is $O_{\mathbf{P}^n}(k)$ for some integer k, and therefore the degrees in question are kd_1 and kd_2 respectively, where d_1 and d_2 are the degrees of C_1 and C_2. Now, $d_1 \neq d_2$ by assumption, implying that k is equal to zero and thereby settling the fact that $\varphi^*(L)$ is trivial.

242 Now (E) follows. Because, otherwise, X possesses an effective line bundle L which lifts to $\varphi^*(L)$, an effective line bundle!

The rest of the properties follow from the explicit construction of X, the details of which are probably well-known (compare with Theorem 6.1 of 'Algebraization of formal moduli...' by M. Artin, *Annals of Math.* (1970), vol. 91) but follow nevertheless.

All schemes considered are of finite type over an algebraically closed field k.

Call a scheme a F-scheme if every finite set of (closed) points is contained in an affine open set.

Lemma 1. *If $Y \to Y'$ is a finite morphism and Y' is an F-scheme, so is Y.*

Proof. Obvious. □

Lemma 2. *If $Y \to Z$ is a closed immersion, $g : Y \to Y'$ a finite morphism,and Z and Y' are F-schemes, then given any finite set $S \subset Z$, there exists an affine open subset U of Z which contains S, such that g restricts to a finite morphism from $Y \cap U$ onto its image.*

Proof. Replace S by $T = S \cup g^{-1}g(S \cap Y)$ which is also finite and let W_1 be an affine open set that contains it. Because Y' is an F-scheme, there is an affine open set V' that contains $g(S \cap Y)$. Now, $g^{-1}(V') \cap W_1$ is a neighbourhood of $T \cap Y$ in Y; so it follows that there is an affine open $W_2 \subset W_1$ such that $W_2 \cap Y$ is contained in $g^{-1}(V')$, and $T \subset W_2$. Also, $g^{-1}g(S \cap Y)$ is contained in W_2, which means that there is a h in the co-ordinate ring of V' such that g restricts to a finite morphism from $D(h \circ g) \subset W_2 \to D(g) \subset V'$. Also, there exists f defined on W_2

such that $f = h \circ g$ when restricted to $Y \cap W_2$, and $T \subset D(f)$. Putting $D(f) = U$ proves the lemma. □

Lemma 3. *If $g : Y \to Y'$ is a finite morphism, and Y is an F-scheme, so is Y'.*

Proof. Assume, by induction, that the lemma has been proved when **243**
$\dim Y \leqslant n - 1$. □

Step 1. Y, Y' irreducible, reduced, and have the same quotient field, with $\dim Y = n$.

Let I be the conductor of the morphism; denote by A and A' the closed subschemes defined by I in Y and Y' respectively. Then A is an F-scheme of dimension $\leqslant n - 1$; by induction, A' is an F-scheme too, so one may appeal to Lemma 2 with A', A, Y in place of Y', Y, Z. Now, let S' be any finite set in Y', S its inverse image in Y and U an affine open set containing S such that $U \cap A \to g(U \cap A) \subset A'$ is a finite morphism. This is merely equivalent to saying that $g^{-1}g(U \cap A) = U \cap A$, from which it follows that $g^{-1}g(U) = U$, so that $U \to g(U)$ is a finite morphism, proving that $g(U)$ is affine. Obviously, $g(U)$ contains S', finishing the proof that Y' is an F-scheme.

Step 2. Y and Y' both irreducible and normal. There is a factoring $Y \to Y'' \to Y'$ with $Y \to Y''$ purely inseparable and $Y'' \to Y'$ separable, with Y'' normal too. That Y'' an F-scheme implies Y an F-scheme is classical (it involves identifying Y'' with the quotient of a certain F-scheme by a finite group G) and we omit it. Also, $Y \to Y''$ is a homeomorphism in the Zariski topology, proving that Y' is an F-scheme too.

Step 3. Y and Y' irreducible.

This is proved by replacing Y by its normalisation and putting step 1 and step 2 together.

Step 4. The general case.

Let Z' be any irreducible component of Y' and Z any irreducible component of $g^{-1}(Z')$ that maps onto Z'. Because Z is an F-scheme, by step 3, Z' is a F-scheme. So the lemma would be proved if one has

Sub-Lemma. *If $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are closed subschemes, then Y is an F-scheme if and only if Y_1 and Y_2 are both F-schemes.*

244 *Proof.* Assume that Y_1 and Y_2 are F-schemes, and let S be any finite set in Y. With the given information, it is a trivial matter to find affine open subsets U_i of Y_i containing $S \cap Y_i$ for $i = 1, 2$, such that $U_1 \cap Y_2 = U_2 \cap Y_1$, by choosing affine open V_i in Y_i containing $S \cap Y_i$ and then taking convenient principal affine open subsets of each. Now, U_1 and U_2 are closed in $U = U_1 \cup U_2$, proving that U is affine (because a scheme is affine if and only if each irreducible component is affine). □

Proposition. *Given an F-scheme Z, a closed subscheme Y, a finite surjective morphism $g : Y \to Y'$ which induces a monomorphism on co-ordinate rings, there is a unique commutative diagram*

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Z' \end{array}$$

with:

(a) *Z' is an F-scheme, f is finite and induces a monomorphism on co-ordinate rings, $Y' \to Z'$ is a closed immersion, and*

(b) *the ideal I that defines Y' in Z' remains an ideal in $f_*(\mathcal{O}_Z)$, in fact the ideal that defines Y in Z.*

Proof. First assume that $\mathbf{Z}$ is affine, in which case Y and Y' are affine too. Let A, A/I, B be their co-ordinate rings, and then $B \subset A/I$ in a natural way. Let $j : A \to A/I$ be the standard map, and put $A' = j^{-1}(B)$, spec $A' = Z'$. That Z' has the required properties follows immediately, as does the fact that if Z were replaced by an open subset U such that

$g^{-1}g(U \cap Y) = U \cap Y$, Z' would be replaced by $U' = f(U)$ which is an open subset.

This guarantees the existence of Z' once it has been shown that Z can be covered by affine open subsets U such that $g^{-1}g(U \cap Y) = U \cap Y'$. By Lemma 3, Y' is an F-scheme, so it follows by Lemma 2 that such open sets cover Z, and in fact can be chosen to contain any finite set of points, proving that Z' exists and is an F-scheme.

To come back to the previous problem, put $\mathbf{P}^n = Z$, and $Y = C_1 \cup C_2$ **245**
in $\mathbf{P}^n$. Let D_i be the normalisation of C_i and $\tilde{f} : D_1 \to D_2$ be the lift of the rational map $f : C_1 \to C_2$. There are several candidates for a commutative diagram:

$$\begin{array}{ccc} D_1 \amalg D_2 & \xrightarrow{h'} & Y \\ \downarrow & & \downarrow \\ D_2 & \xrightarrow{h} & Y' \end{array}$$

among which there is a best one, for which,

(I) $h_*(\mathcal{O}_{D_2}/\mathcal{O}_{Y'})$ injects into $h'_*(\mathcal{O}_{D_1 \amalg D_2})/\mathcal{O}_{\gamma'}$,

(II) h is birational.

Applying the above proposition, with Z, Y and Y' as above, one gets the variety X mentioned in the introduction.

A final remark: X has no line bundles at all if C_1 and C_2 are chosen suitably!

Let i_1 and i_2 be the composites $D_1 \to C_1 \to \mathbf{P}^n$ and $D_2 \to C_2 \to \mathbf{P}^n$ respectively. Assume that there is a point P of D_1 such that $i_1(P) = i_s f(P)$. Then

(F) X is simply connected (follows from a careful application of Van Kampen's theorem), and

(G) $\operatorname{Pic} X = 0$.

A rather neat example is provided by taking a conic C_1 and a tangential line C_2 in the projective plane and associating to a point P on

C_1 the point of intersection Q of C_2 and the tangent line to C_1 at P; the properties (A) to (G) can be verified directly in this case.

[Added in Proof: Such varieties were also constructed by G. Horrocks on slightly different lines; see: 'Birationally ruled surfaces without embeddings in regular schemes', by G. Horrocks, *J. Lond, Math. Soc.*, Vol. III (1971)] □

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Some Footnotes to the Work of C. P. Ramanujam

By D. Mumford

THIS PAPER consists of a series of remarks, each of which is connected in some way with the work of Ramanujam. Quite often, in the last few years, I have been thinking on some topic, and suddenly I realize–Yes, Ramanujam thought about this too–or–This really links up with his point of view. It is uncanny to see how his ideas continue to work after his death. It is with the thought of embellishing some of his favourite topics that I write down these rather disconnected series of results. **247**

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The first remark is a very simple example relevant to the purity conjecture (sometimes called Lang's conjecture) discussed in Ramanujam's paper [10]. The conjecture was–let

$$f : X^n \to Y^m$$

be a proper map of an n-dimensional smooth variety onto an m-dimensional smooth variety with all fibres of dimension $n - m$. Assume the characteristic is zero. Then show

$$\{y \in Y | f^{-1}(y) \quad \text{is singular}\}$$

has codimension one in Y. When $n = m$, this result is true and is known as "purity of the branch locus"; when $n = m + 1$, it is also true and

was proven by Dolgačev, Simha and Ramanujam. When $n = m + 2$, Ramanujam describes in [10] a counter-example due to us jointly. Here is another counter-example for certain large values of $n - m$.

We consider the following very special case for f. Start with $\mathbf{Z}^r \subset \mathbf{P}^m$ an arbitrary subvariety. Let $\check{\mathbf{P}}^m$ be the dual projective space–the space of hyperplanes in $\mathbf{P}^m$. The dual variety $\check{\mathbf{Z}} \subset \check{\mathbf{P}}^m$ is, by definition the Zariski-closure of the locus of hyperplanes H such that, at some smooth point $x \in \mathbf{Z}^r$, $T_{x,H} \supset T_{x,Z}$. It is apparently well known, although I don't know a reference, that in characteristic 0,

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$$\check{\check{\mathbf{Z}}} = \mathbf{Z}.$$

Consider the special case where $\mathbf{Z}$ is smooth and spans $\mathbf{P}^m$. Then we don't need to take the Zariski-closure in the above definition and, in fact, the definition of $\check{\mathbf{Z}}$ can be reformulated like this:

Let

$$I \subset \mathbf{P}^m \times \check{\mathbf{P}}^m$$

be the universal family of hyperplanes, i.e., if $(X_0, \ldots, X_m)$, resp. $(\xi_0, \ldots, \xi_m)$ are coordinates in $\mathbf{P}^m$, resp. $\check{\mathbf{P}}^m$, then I is given by

$$\sum \xi_i X_i = 0.$$

Let

$$X = I \cap (\mathbf{Z}^r \times \check{\mathbf{P}}^m).$$

Note that I and $\mathbf{Z}^r \times \check{\mathbf{P}}^m$ are smooth subvarieties of $\mathbf{P}^m \times \check{\mathbf{P}}^m$ of codimension 1 and $m - r$ respectively. One sees immediately that they meet transversely, so X is smooth of dimension $m + r - 1$. Consider

$$p_2 : X \to \check{\mathbf{P}}^m.$$

Its fibres are the hyperplane sections of $\mathbf{Z}$, all of which have dimension $r - 1$. Thus p_2 is a morphism of the type considered in the conjecture. In this case

$$\{\xi \in \check{\mathbf{P}}^m | p_2^{-1}(\xi) \text{ singular}\} = \{\xi \in \check{\mathbf{P}}^m | \text{ if } \xi \text{ corresponds to } H \subset \mathbf{P}^m,$$

then $\mathbf{Z}.\mathbf{H}$ is singular}

$$= \check{\mathbf{Z}}.$$

Thus the conjecture would say that the dual $\check{\mathbf{Z}}$ of a smooth variety $\mathbf{Z}$ spanning $\mathbf{P}^m$ is a hypersurface.

I claim this is false, although I feel sure it can only be false in very rare circumstances. In fact, I don't know any cases other than the following example where it is false.† Simply take **249**

$$\mathbf{Z}^r = [\text{Grassmannian of lines in } \mathbf{P}^{2k}, k > 1].$$

Here $r = 2(2k-1)$, $m = k(2k+1) - 1$ and the embedding $i : \mathbf{Z}^r \subset \mathbf{P}^m$ is the usual Plücker embedding. In vector space form, let

V = a complex vector space of dimension $2k+1$

$\mathbf{Z}$ = set of 2-dimensional subspaces $W_2 \subset V$

$\mathbf{P}^m$ = set of 1-dimensional subspaces $W_1 \subset \Lambda^2 V$

i = map taking W_2 to $W_1 = \Lambda^2 W_2$.

Note that we may identify

$\check{\mathbf{P}}^m$ = set of 1-dimensional subspaces $W_1' \subset \Lambda^2 V^*$, where

$\Lambda^2 V^*$ = space of skew-symmetric 2-forms $A : V \times V \to \mathbf{C}$.

Write $[W_2] \in Z$ for the point defined by W_2, and $H_A \subset \mathbf{P}^m$ for the hyperplane defined by a 2-form A. Then it is immediate from the definitions that

$$i([W_2]) \in H_A \Leftrightarrow \operatorname{res}_{W_2} A \quad \text{is zero.}$$

To determine when moreover,

$$i_*(T_{W_2,\mathbf{Z}}) \subset T_{i(W_2),H_A},$$

†M. Reid has indicated to me another set of examples: Suppose

$$\mathbf{Z} = \mathbf{P}(E)$$

where E is a vector bundle of rank s on y^t, so $r = s + t - 1$, and the fibres of $\mathbf{P}(E)$ are embedded linearly. Then if $s \geq t + 2$, $\check{\mathbf{Z}}$ is not a hypersurface.

let $v_1, v_2 \in W_2$ be a basis, and make a small deformation of W_2 by taking $v_1 + \epsilon v_1', v_2 + \epsilon v_2'$ to be a basis of $\widetilde{W}_2 \subset V \otimes \mathbf{C}[\epsilon]$. The $\widetilde{W}_2$ represents a tangent vector t to $\mathbf{Z}$ at $[W_2]$ and

$$\begin{aligned} i_*(t) \subset T_{i(W_2),H_A} &\Leftrightarrow A(v_1 + \epsilon v_1', v_2 + \epsilon v_2') \equiv 0 \pmod{\epsilon^2} \\ &\Leftrightarrow A(v_1', v_2) + A(v_1, v_2') = 0, \end{aligned}$$

Thus:

$$\begin{aligned} i_*(T_{W_2,\mathbf{Z}}) \subset T_{i(W_2),H_A} \Leftrightarrow \ & \text{for all } v_1', v_2', \in V, \\ & A(v_1, v_2') + A(v_1', v_2) = 0 \\ \Leftrightarrow\ & W_2 \subset \text{(nullspace of } A). \end{aligned}$$

250 Therefore

$$H_A \text{ is tangent to } i(\mathbf{Z}) \Leftrightarrow \dim \text{(nullspace)} \geq 2.$$

Now the nullspace of A has odd dimension, and if it is 3, one counts the dimension of the space of such A as follows:

$$\begin{aligned} \dim \left(\begin{array}{l} \text{space of } A\text{'s with} \\ \dim(\text{nullspace}) = 3 \end{array} \right) &= \dim \left(\begin{array}{c} \text{space of} \\ W_3 \subset V \end{array} \right) + \dim \Lambda^2(V/W_3) \\ &= 3(2k-2) + \frac{(2k-2)(2k-3)}{2} \\ &= 2k^2 + k - 3. \end{aligned}$$

Thus $\dim \check{\mathbf{Z}} = m - 3$, and $\operatorname{codim} \check{\mathbf{Z}} = 3$! (Compare this with Buchsbaum-Eisenbud [3], where it is shown that $\check{\mathbf{Z}} \subset \check{\mathbf{P}}^m$ is a "universal codimension 3 Gorenstein scheme".)

II

The second remark concerns the Kodaira Vanishing Theorem. We want to show that Ramanujam's strong form of Kodaira Vanishing for surfaces of Char. 0 is a consequence of a recent result of F. Bogomolov.

In particular, this is interesting because it gives a new completely algebraic proof of this result, and one which uses the Char. 0 hypothesis in a new way (it is used deep in Bogomolov's proof, where one notes that if $V^3 \to F^2$ is a ruled 3-fold and $D \subset V^3$ is an irreducible divisor meeting the generic fibre set-theoretically in one point, then D is birational to F). Ramanujam's result [11] is this: let F be a smooth surface of Char. 0, D a divisor on F.

Then

$$\left.\begin{array}{l}(D^2) > 0\\ (D.C) \geq 0, \quad \text{all curves} \quad C \subset F\end{array}\right\} \Rightarrow H^1(F, \mathscr{O}(-D)) = (0) \qquad (1)$$

Bogomolov's theorem is that if F is a smooth surface of char. 0, E a rank 2 vector bundle on F, then

$$\begin{aligned} C_1(E)^2 > 4C_2(E) \Rightarrow & E \text{ is unstable, meaning } \exists \text{ an extension}\\ & 0 \to L(D) \to E \to I_Z L \to 0,\\ & I_Z = \text{ ideal sheaf of a 0-dim. subscheme } Z \subset F,\\ & L \text{ invertible sheaf, } D \text{ a divisor}\\ & D \in [\text{num. pos. cone}, \ (D^2) > 0, (D.H) > 0]. \quad (2)\end{aligned}$$

(See Bogomolov [2], Reid [13]; another proof using reduction mod p **251**
instead of invariant theory has been found by D. Gieseker.)

To prove (2) $\Rightarrow$ (1), suppose D_1 is given satisfying the conditions of (1). Take any element $\alpha \in H^1(F, \mathscr{O}(-D_1))$ and via α, form an extension

$$0 \to \mathscr{O}_F \xrightarrow{\mu} E \xrightarrow{\nu} \mathscr{O}_F(D_1) \to 0.$$

Note that $C_1(E) = D_1$, $C_2(E) = 0$, $(D_1^2) > 0$, so E satisfies the conditions of (2). Therefore, by Bogomolov's theorem, E is unstable: this gives an exact sequence

$$\begin{aligned} & 0 \to L(D_2) \xrightarrow{\sigma} E \xrightarrow{\tau} I_Z L \to 0\\ & D_2 \in (\text{num. pos. cone}).\end{aligned}$$

Note that the subsheaf $\sigma(L(D_2))$ of E cannot equal the subsheaf $\mu(\mathscr{O}_F)$ in the definition of E, because this would imply, comparing the 2 sequences, that $D_2 \equiv -D_1$ whereas both D_1, D_2 are in the numerically positive cone. Therefore, the composition

$$L(D_2) \xrightarrow{\sigma} E \xrightarrow{\nu} \mathscr{O}_F(D_1)$$

is not zero, hence

$$L \cong \mathscr{O}_F(D_1 - D_2 - D_3), D_3 \quad \text{an effective divisor.}$$

Next, comparing Chern classes of E in its 2 presentations, we find

$$2C_1(L) + D_2 \equiv C_1(E) \equiv D_1 \tag{3a}$$

$$(C_1(L) + D_2) \cdot C_1(L) + \deg Z = C_2(E) = 0. \tag{3b}$$

252 By (3a), we find $D_1 - D_2 - 2D_3 \equiv 0$, hence $L \simeq \mathscr{O}_F(+D_3)$; by (3b), we find $(D_1 - D_3) \cdot D_3 \leq 0$. But

$$\det \begin{vmatrix} (D_1^2) & (D_1 \cdot D_3) \\ (D_1 \cdot D_3) & (D_3^2) \end{vmatrix} = (D_1^2)[(D_3^2) - (D_1 \cdot D_3)] + (D_1 \cdot D_3)[(D_1^2) - 2(D_1 \cdot D_3)] + (D_1 \cdot D_3)^2$$

while

$(D_3^2) - (D_1 \cdot D_3) \geq 0$	(by 3*b*)
$(D_1^2) - 2(D_1 \cdot D_3) = (D_1 \cdot D_2) > 0$	(since D_1, D_2 num. pos.)
$(D_1 \cdot D_3) \geq 0$	(by the assumptions on D_1).

On the other hand, this det is ≤ 0 by Hodge's Index Theorem. Therefore $(D_1 \cdot D_3) = 0$ and det = 0. From the latter, D_3 is numerically equivalent to λD_1, $\lambda \in \mathbf{Q}$, hence $(D_1 \cdot D_3) = \lambda(D_1^2)$. Thus $\lambda = 0$ and since D_3 is effective, $D_3 = 0$. Therefore the sub-sheaf $\sigma(L(D_2))$ is isomorphic to $\mathscr{O}_F(D_1)$ and defines a splitting of the original exact sequence. Therefore the extension class $\alpha \in H^1(\mathscr{O}_F(-D_1))$ is 0, so $H^1(\mathscr{O}_F(-D_1)) = (0)$.

III

The last two remarks are applications of Kodaira's Vanishing Theorem. To me it is quite amazing how this cohomological assertion has such strong consequences, both for geometry and for local algebra. Here is a geometric application. This application is a link between the recent paper of Arakelov [1] (proving Shafarevich's finiteness conjecture on the existence of families of curves over a fixed base curve, with prescribed degenerations), and Raynaud's counter-example [12] to Kodaira Vanishing for smooth surfaces in char. p. What I claim is this (this remark has been observed by L. Szpiro also):

Proposition. *Let $p : F \to C$ be a proper morphism of a smooth surface F onto a smooth curve C over a field k of arbitrary characteristic. Let $E \subset F$ be a section of p and assume the fibres of p have positive arithmetic genus. Let F_0 be the normal surface obtained by blowing down all components of fibres of p not meeting E. Then:*

Kodaira's Vanishing Theorem $\Longrightarrow (E^2) \leq 0$. *for ample divisors on F_0* **253**

If Char(k) = 0, then Kodaira's Vanishing Theorem holds for F_0 (cf. [9]), so $(E^2) \leq 0$ follows. This result, and its refinement – $(E^2) < 0$ unless all the smooth fibres of p are isomorphic – are due to Arakelov [1], who proved them by a very ingenious use of the Weierstrass points of the fibres $p^{-1}(x)$. On the other hand, if char(k) = p, Raynaud has shown how to construct examples of morphisms $p : F \to C$ and sections $E \subset F$, where all the fibres of p are irreducible but singular (thus $F = F_0$), and $(E^2) > 0$. Thus Kodaira Vanishing is false for this F. If char(k) = 2 or 3, he finds in fact quasi-elliptic surfaces F of this type. This Proposition is, in fact, merely an elaboration of the last part of Raynaud's example.

Proof of Proposition: Suppose $(E^2) > 0$. Let $p_0 : F_0 \to C$ be the projection and let E stand for the image of E in F_0 too. Consider divisors on F_0 of the form

$$H = E + p_0^{-1}(\mathfrak{U}), \quad \deg \mathfrak{U} > 0.$$

Then $(H^2) > 0$ and $(H \cdot C) > 0$ for all curves C on F_0, so H is ample by the Nakai-Moisezon criterion. On the other hand, let's calculate $H^1(F_0, \mathscr{O}(-H))$. We have

$$0 \to H^1(C, p_0, *\mathscr{O}(-H)) \to H^1(F_0, \mathscr{O}(-H)) \\ \to H^0(C, R^1 p_0, *\mathscr{O}(-H)) \to 0.$$

Clearly

$$p_0, *\mathscr{O}(-H) = (0) \text{ and } R^1 p_0, *\mathscr{O}(-H) \cong (R^1 p_0, *\mathscr{O}(-E)) \otimes \mathscr{O}_c(-\mathfrak{U}).$$

Now using the sequences:

$$0 \to \mathscr{O}_{F_0}(-E) \to \mathscr{O}_{F_0} \to \mathscr{O}_E \to 0$$
$$0 \to \mathscr{O}_{F_0} \to \mathscr{O}_{F_0}(E) \to \mathscr{O}_E((E^2)) \to 0$$

we find

$$\begin{array}{ccccccccc} p_0, *\mathscr{O}_{F_0} & \xrightarrow{\alpha} & p_0, *\mathscr{O}_E & \longrightarrow & R^1 p_0, *\mathscr{O}(-E) & \xrightarrow{\beta} & R^1 p_0, *\mathscr{O}_{F_0} & \longrightarrow & 0 \\ \| & & \| & & & & & & \\ \mathscr{O}_C & & \mathscr{O}_C & & & & & & \end{array}$$

254 so α and β are isomorphisms, and (using the fact that the genus of the fibres is positive):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_0, *\mathscr{O}_{F_0} & \xrightarrow{\gamma} & p_0, *\mathscr{O}(E) & \longrightarrow & p_0, *\mathscr{O}_E((E^2)) & \xrightarrow{\delta} & R^1 p_0, *\mathscr{O}_{F_0} \\ & & \| & & \| & & & & \\ & & \mathscr{O}_C & & \mathscr{O}_C & & & & \end{array}$$

so γ is an isomorphism, and δ is injective. Now via the isomorphism resp: $E \to C$, let the divisor class (E^2) on E correspond to the divisor class $\mathfrak{U}$ on C. Then $p_0, *\mathscr{O}_E((E^2)) \cong \mathscr{O}_C(\mathfrak{U})$, and we see that

$$\mathscr{O}_C(\mathfrak{U}) \subset R^1 p_0, *\mathscr{O}_{F_0} \cong R^1 p_0, *\mathscr{O}_{F_0}(-E)$$

hence

$$\mathcal{O}_C \subset R^1 p_0, *\mathcal{O}_{F_0}(-H)$$

hence

$$H^1(C, \mathcal{O}(-H)) \neq (0). \qquad \text{Q.E.D.}$$

IV

The last remark is an application of Kodaira's Vanishing Theorem to local algebra. It seems to me remarkable that such a global result should be useful to prove local statements about the non-existence of local rings, but this is the case. The question I want to study is that of the *smoothability of non-Cohen-Macauley surface singularities*. In other words, given a surface F, $P \in F$ a non-CM-singular point, when does there exist a flat family of surfaces F_t parametrized by $k[[t]]$ such that $F_0 = F$ while the generic F_t is smooth. More locally, the problem is:

Given a complete non-CM-purely[†]-2-dimensional local ring $\mathcal{O}$ without nilpotents, when does there exist a complete 3-dimensional local ring $\mathcal{O}'$ and a non-zero divisor $t \in \mathcal{O}'$ such that

(a) $\mathcal{O} \cong \mathcal{O}'/t\mathcal{O}'$

(b) $\mathcal{O}_{\mathcal{P}}$ regular for all prime ideals $\mathcal{P} \subset \mathcal{O}$ with $t \notin \mathcal{P}$.

First of all, let **255**

$$\mathcal{O}^* = \bigoplus_{\substack{I \subset \mathcal{O} \\ \text{minimal} \\ \text{prime ideals}}} \left(\begin{array}{l} \text{integral closure of } \mathcal{O}/I \\ \text{in its fraction field} \end{array} \right)$$

and let

$$\widetilde{\mathcal{O}} = \left\{ a \in \mathcal{O}^* \;\middle|\; \begin{array}{l} m^n a \subset \mathcal{O} \quad \text{for some} \quad n \geq 1 \\ m = \quad \text{maximal ideal in} \quad \mathcal{O} \end{array} \right\}$$

[†]i.e. $\mathcal{O}/I$ is 2-dimensional for all minimal prime ideals of $I \subset \mathcal{O}$.

$$= \Gamma(\text{Spec}\,\mathcal{O} - \text{closed pt.},\ \mathcal{O})$$

Note that $\widetilde{\mathcal{O}}$ is a finite $\mathcal{O}$-module and $m^n \cdot \widetilde{\mathcal{O}} \subset \mathcal{O}$ for some large n, so that $\widetilde{\mathcal{O}}/\mathcal{O}$ is an $\mathcal{O}$-module of finite length. Moreover, it is easy to see that $\widetilde{\mathcal{O}}$ is a semi-local Cohen-Macauley ring. It has been proven by Rim [14] (cf. also Hartshorne [6], Theorem 2.1, for another proof) that:

$$\mathcal{O} \text{ smoothable } \Rightarrow \widetilde{\mathcal{O}} \text{ local.}$$

The result we want to prove is:

Theorem. *Assume* $\text{char}(\mathcal{O}/m) = 0$, Spec $\mathcal{O}$ *has an isolated singularity at its closed point and that* $\mathcal{O}$ *is smoothable, so that, by the remarks above,* $\widetilde{\mathcal{O}}$ *is a normal local ring. Let* $\pi : X^* \to \text{Spec}\,\mathcal{O}$ *be a resolution and let*

$$p_a(\widetilde{\mathcal{O}}) = l(R^1\pi_*\mathcal{O}_X)$$

be the genus of the singularity $\widetilde{\mathcal{O}}$. *Then*

$$l(\widetilde{\mathcal{O}}/\mathcal{O}) \leq p_a(\widetilde{\mathcal{O}}).$$

Actually, for our applications, we want to know this result for rings $\mathcal{O}$ where Spec $\mathcal{O}$ has ordinary double curves too, with a suitable definition of p_a. We will treat this rather technical generalization in an appendix.

For example, the theorem shows:

Corollary. *Let* $\widetilde{\mathcal{O}} = k[[x,y]]$, $\text{char}\,k = 0$. *Let* $I \subsetneq (x,y)$ *be an ideal of finite codimension. Then if* $\mathcal{O} = k + I$, $\mathcal{O}$ *is not smoothable.*

256 On the other hand, if $F \subset \mathbf{P}^n$ is an elliptic ruled surface and $\mathcal{O}'$ is the completion of the local ring of the cone over F at its apex, then $\mathcal{O}'$ is a normal 3-dimensional ring which is not Cohen-Macauley. If $C = V(t) = (F \cdot \mathbf{P}^{n-1})$ is a generic hyperplane section of F, then $t \in \mathcal{O}'$ and $\mathcal{O} = \mathcal{O}'/t\mathcal{O}'$ is the completion of the local ring of the cone over C at its apex. Now C is an elliptic curve, but embedded by an incomplete

linear system – in fact, C is a projection of an elliptic curve $\widetilde{C}$ in $\mathbf{P}^n$ from a point not on $\widetilde{C}$ – this follows from the exact sequence:

$$0 \longrightarrow H^0(\mathscr{O}_F) \longrightarrow H^0(\mathscr{O}_F(1)) \longrightarrow H^0(\mathscr{O}_C(1)) \longrightarrow H^1(\mathscr{O}_F) \longrightarrow 0$$
$$\Big\| \mathbf{C}$$

Let $\widetilde{\mathscr{O}}$ be the completion of the local ring of the cone over $\widetilde{C}$ at its apex. Then $\widetilde{\mathscr{O}}$ is a normal 2-dimensional ring, in fact an "elliptic singularity", i.e., $p_a(\widetilde{\mathscr{O}}) = 1$; moreover $\widetilde{\mathscr{O}} \supset \mathscr{O}$ and $\dim \widetilde{\mathscr{O}}/\mathscr{O} = 1$. This shows that there are smoothable singularities $\mathscr{O}$ with

$$l(\widetilde{\mathscr{O}}/\mathscr{O}) = p_a(\widetilde{\mathscr{O}}) = 1.$$

Proof of Theorem. Let $\mathscr{O} \cong \mathscr{O}'/t\mathscr{O}'$ give the smoothing of $\mathscr{O}$. The proof is based on an examination of the exact sequence of local cohomology groups:

$$(*) \ldots \to H^1_{\{x\}}(\mathscr{O}') \to H^1_{\{x\}}(\mathscr{O}) \to H^2_{\{x\}}(\mathscr{O}') \xrightarrow{t} H^2_{\{x\}}(\mathscr{O}') \xrightarrow{\alpha} H^2_{\{x\}}(\mathscr{O}) \to \ldots$$

where $x \in \operatorname{Spec} \mathscr{O} \subset \operatorname{Spec} \mathscr{O}'$ represents the closed point.

What can we say about each of these groups?

(a) $H^1_{\{x\}}(\mathscr{O}')$ is zero since $\mathscr{O}'$ is an integrally closed ring of dimension 3, hence has depth at least 2.

(b) To compute $H^1_{\{x\}}(\mathscr{O})$, use

$$H^0(\operatorname{Spec} \mathscr{O}, \mathscr{O}) \to H^0(\operatorname{Spec} \mathscr{O} - \{x\}, \mathscr{O}) \to H^1_{\{x\}}(\mathscr{O}) \to 0$$

which gives us:

$$H^1_{\{x\}}(\mathscr{O}) \cong \widetilde{\mathscr{O}}/\mathscr{O}.$$

(c) As for $H^2_{\{x\}}(\mathscr{O}')$, it measures the degree to which $\mathscr{O}'$ is not Cohen-Macauley. A fundamental fact is that it is of finite length–cf. Theoreme de finitude, p. 89, in Grothendieck's seminar [15]. **257**

(d) As for $H^2_{\{x\}}(\mathcal{O})$, we can say at least:

$$\begin{aligned} H^2_{\{x\}}(\mathcal{O}) &\cong H^1(\operatorname{Spec}\mathcal{O} - \{x\}, \mathcal{O}) \\ &\cong H^1(\operatorname{Spec}\widetilde{\mathcal{O}} - \{x\}, \widetilde{\mathcal{O}}) \\ &\cong H^2_{\{x\}}(\widetilde{\mathcal{O}}) \end{aligned}$$

but unfortunately this group is huge: it is not even an $\widetilde{\mathcal{O}}$-module of finite type.

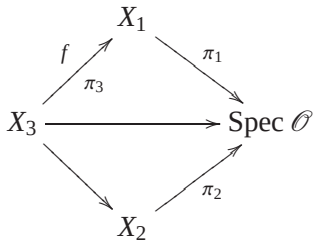
However, for any local ring $\mathcal{O}$ with residue characteristic O and with isolated singularity, we can define, by using a resolution of Spec $\mathcal{O}$, important subgroups:

$$H^i_{\{x\},\text{int}}(\mathcal{O}).$$

Namely, let $\pi : X \to \operatorname{Spec}\mathcal{O}$ be a resolution and set

$$H^i_{\{x\},\text{int}}(\mathcal{O}) = \operatorname{Ker}[\pi^* : H^i_{\{x\}}(\mathcal{O}) \to H^i_{\pi^{-1}x}(\mathcal{O}_X)].$$

This is independent of the resolution, as one sees by comparing any 2 resolutions $\pi_i : X_i \to \operatorname{Spec}\mathcal{O}$, $i = 1, 2$, via a 3rd:



and using the Leray spectral sequence

$$H^p_{\pi_1^{-1}(x)}(X_1, R^q f_*(\mathcal{O}_{X_3})) \Rightarrow H^*_{\pi_3^{-1}(x)}(X_3, \mathcal{O}_{X_3})$$

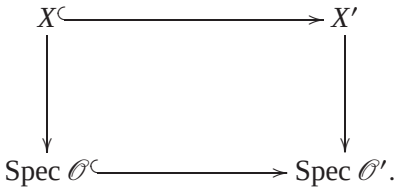
plus Matsumura's result $R^q f_* \mathcal{O}_{X_3} = (0)$, $q > 0$ where X_1 and X_3 are smooth and characteristic zero. Moreover, when $\mathcal{O} \cong \mathcal{O}'/I$, then the restriction map

$$H^i_{\{x\}}(\mathcal{O}') \to H^i_{\{x\}}(\mathcal{O})$$

258 gives

$$H^i_{\{x\},\mathrm{int}}(\mathscr{O}') \to H^i_{\{x\},\mathrm{int}}(\mathscr{O})$$

because we can find resolutions fitting into a diagram:



Next, we prove using the Kodaira Vanishing Theorem and following Hartshorne and Ogus ([16]p. 424):

Lemma. *Assume x is the only singularity of $\mathscr{O}$,* $\dim \mathscr{O} = n$ *and* $\pi : X \to \operatorname{Spec} \mathscr{O}$ *is a resolution. Then*

$$H^i_{\{x\}' int}(\mathscr{O}) \cong H^i_{\{x\}}(\mathscr{O}), \quad 0 \le i \le n-1$$

and

$$H^i_{\{x\}' int}(\mathscr{O}) \cong R^{i-1}\pi_*(\mathscr{O}_X)_x, \quad 2 \le i \le n.$$

Proof. Because $\mathscr{O}$ has an isolated singularity, we may assume $\mathscr{O} \cong \widehat{\mathscr{O}}_{x,X_0}$, where X_0 is an n-dimensional projective variety with x its only singular point. We may assume our resolution is global:

$$\pi : X \to X_0.$$

Let $\widehat{X} = X \times_{X_0} \operatorname{Spec} \mathscr{O}$ and let I be the injective hull of $\mathscr{O}_{x,X_0}/m_{x,X_0}$ as $\mathscr{O}_{x,X_0}$-module. Then according to Hartshorne's formal duality theorem (cf. [7]p. 94), for all coherent sheaves $\mathscr{F}$ on X, the 2 $\mathscr{O}$-modules

$$H^i_{\pi^{-1}x}(\mathscr{F}), \quad \operatorname{Ext}^{n-i}_{\mathscr{O}_{\widehat{X}}}(\widehat{F}, \Omega^n_{\widehat{X}})$$

are dual via $\operatorname{Hom}(-, I)$. In particular,

$$H^i_{\pi^{-1}x}(\mathscr{O}_X), \quad H^{n-i}(\Omega^n_{\widehat{X}})$$

are dual. But

$$H^{n-i}(\Omega^n_{\widehat{X}}) \cong R^{n-i}\pi_*(\Omega^n_X) \otimes_{\mathscr{O}_{X_0}} \mathscr{O}$$

and it has been shown by Grauert and Riemenschneider [5] that $R_i\pi_*(\Omega^n_X) = (0)$, $i > 0$. (This is a simple consequence of Kodaira's Van-
259 ishing Theorem because if L_0 is an ample invertible sheaf on X_0 with $H^i(X_0, L_0 \otimes \pi_*\Omega^n_X) = (0)$, $i > 0$, then by the Leray Spectral Sequence:

$$\begin{array}{ccc} H^i(X, \pi^*L_0 \otimes \Omega^n_X) & \cong & H^0(X_0, L_0 \otimes R^i\pi_*\Omega^n_X) \\ & \updownarrow \text{dual} & \\ H^{n-i}(X, (\pi^*L_0)^{-1}) & & \end{array}$$

and Kodaira's Vanishing Theorem applies to all invertible sheaves M such that $\Gamma(X, M^n)$ is base point free and defines a birational morphism, $n \geq 0$ (cf. [9]).) Recapitulating, this shows $H^{n-i}(\Omega^n_{\widehat{X}}) = (0)$, $i < n$, hence $H^i_{\pi^{-1}x}(\mathscr{O}_X) = (0)$, $i < n$, hence $H^i_{\{x\}\text{int}}(\mathscr{O}) \xrightarrow{\approx} H^i_{\{x\}}(\mathscr{O})$ is an isomorphism.

To get the second set of isomorphisms, we use the Leray Spectral Sequence:

$$H^p_{\{x\}}(X_0, R^q\pi_*\mathscr{O}_X) \Rightarrow H^*_{\pi^{-1}x}(X, \mathscr{O}_X).$$

The only non-zero terms occur for $p = 0$ or $q = 0$, so we get a long exact sequence

$$\begin{aligned} \ldots \to H^0_{\{x\}}(R^{i-1}\pi_*\mathscr{O}_x) \to H^i_{\{x\}}(\pi_*\mathscr{O}_x) \to H^i_{\pi^{-1}x}(\mathscr{O}_X) \to H^0_{\{x\}}(R^i_{\pi_*}\mathscr{O}_x) \\ \to H^{i+1}_{\{x\}}(\pi_*\mathscr{O}_X) \to \ldots \end{aligned}$$

Using the first part, plus the isomorphism:

$$\begin{aligned} H^i_{\{x\}}(\pi_*\mathscr{O}_X) &\cong H^{i-1}(\operatorname{Spec}\mathscr{O} - \{x\}, \pi_*\mathscr{O}_X) \\ &\cong H^{i-1}(\operatorname{Spec}\mathscr{O} - \{x\}, \mathscr{O}) \\ &\cong H^i_{\{x\}}(\mathscr{O}),\ i \geq 2, \end{aligned}$$

we get the results. Q.E.D. □

We now go back to the sequence (*). It gives us:

$$0 \to \widetilde{\mathcal{O}}/\mathcal{O} \to H^2_{\{x\},\mathrm{int}}(\mathcal{O}') \xrightarrow{t} H^2_{\{x\},\mathrm{int}}(\mathcal{O}') \to R^1\pi_*(\mathcal{O}_X)_x \to \dots$$

where $\pi : X \to \operatorname{Spec} \mathcal{O}$ is a resolution. Therefore

$$\begin{aligned} l(\widetilde{\mathcal{O}}/\mathcal{O}) &= l(\text{ker of } t \text{ in } H^2_{\{x\},\mathrm{int}}(\mathcal{O}')) \\ &= l(\text{Coker of } t \text{ in } H^2_{\{x\},\mathrm{int}}(\mathcal{O}')) \\ &\le l(R^1\pi_*(\mathcal{O}_X)_x) = p_a(\widetilde{\mathcal{O}}). \end{aligned}$$

Q.E.D.

Appendix

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The purpose of this appendix is to make a rather technical extension of the result in § IV, which seems to be better for use in applications. Let X be an affine surface, reduced, with at most ordinary double curves, plus one point $P \in X$ about which we know nothing. Let

$$\widetilde{X} = \operatorname{Spec} \Gamma(X - P, \mathcal{O}_X)$$

so that we get

$$\pi_1 : \widetilde{X} \to X,$$

an isomorphism outside P, everywhere a finite morphism, with $\widetilde{X}$ Cohen-Macauley. Our goal is to show that in certain cases X is not smoothable near P, i.e., $\nexists$ an analytic family

$$f : X' \to \Delta = \text{ disc in the } t\text{-plane},$$

where $f^{-1}(0) \approx$ (neighborhood of P in X), and $f^{-1}(t)$ is smooth, $t \neq 0$. (Here we work in the analytic setting rather than the formal one to be able below to take an exponential.) We know that a necessary condition for X' to exist is that $\pi_1^{-1}(P)$ is one point $\widetilde{P}$, so henceforth we assume this too. Next blow up $\widetilde{X}$, but only at $\widetilde{P}$ and at centers lying over $\widetilde{P}$: it is not hard to see that we arrive in this way at a birational proper morphism

$$\pi_2 : X^* \to \widetilde{X}$$

such that X^* has at most ordinary double curves and pinch points (points like $z^2 = x^2y$), these pinch points moreover lying over $\widetilde{P}$. Define

$$p_a(\mathscr{O}_{\widetilde{P}}) = \dim_C[R^1\pi_{2,*}(\mathscr{O}_{X^*})_{\widetilde{P}}].$$

It is easy to verify that this number is independent of the choice of X^*. (However, this would *not* be true if $\widetilde{X}$ had cuspidal lines–in this case, there is no bound on dim $R^1\pi_*$ as you blow up $\widetilde{X}$ more and more!) We claim the following

261 **Theorem.** *If X is smoothable near P, then $l(\mathscr{O}_{\widetilde{P}}/\mathscr{O}_P) \leq p_a(\mathscr{O}_{\widetilde{P}})$.*

Proof. We follow the same plan as in the case where $\mathscr{O}$ has an isolated singularity, except that, for an arbitrary local ring $\mathscr{O}$, we set

$$H^i_{\{x\},\text{int}}(\mathscr{O}) = \bigcup_{\substack{\text{modifications} \\ \pi: X \to \operatorname{Spec}\mathscr{O} \\ \text{where} \\ X - \pi^{-1}(x) \xrightarrow{\approx} \operatorname{Spec}\mathscr{O} - \{x\}}} [\ker : H^i_{\{x\}}(\mathscr{O}) \to H^i_{\pi^{-1}1_{(x)}}(\mathscr{O}_X)]$$

The proof is then the same as before except that we cite only the following case of the lemma: □

Theorem (Boutot [17]). *Let $\mathscr{O}$ be a normal excellent local k-algebra, with residue field k, and* $\operatorname{char}(k) = 0$. *Then:*

$$H^2_{\{x\},int}(\mathscr{O}) = H^2_{\{x\}}(\mathscr{O}).$$

This result is a Corollary of Proposition 2.6, Chapter V [17]. Since $\operatorname{char}(k) = 0$, we may disregard "red" in that Proposition and apply it to the values of the functor on the dual numbers. It tells us that there is a blow-up $\pi : X \to \operatorname{Spec}(\mathscr{O})$ concentrated at the origin such that

$$\operatorname{Pic}_{X/k}(k[\epsilon]/(\epsilon^2)) \to \operatorname{Pic}_{\operatorname{Spec}(\mathscr{O})-\{x\}}(k[\epsilon]/(\epsilon^2))$$

is an isomorphism. In other words, in the sequence:

$$\to H^1(X, \mathscr{O}_X) \xrightarrow{\alpha} H^1(X - \pi^{-1}(x), \mathscr{O}_X) \xrightarrow{\beta} H^2_{\pi^{-1}(x)}(\mathscr{O}_X) \to$$

α is surjective, hence β is zero.

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Singular Modular Forms of Degree s

By S. Raghavan

1. For a natural nubmer s, let H_s denote the Siegel half-plane of degree s, namely the space of s-rowed complex symmetric matrices $W = U + iV$ with U, V real and V positive-definite. The Siegel modular group $\Gamma_s = \mathrm{Sp}(s, \mathbf{Z}) = \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) | A, B, C, D$ s-rowed integral square matrices with

$$M\begin{pmatrix} 0 & E_s \\ -E_s & 0 \end{pmatrix} M' = \begin{pmatrix} 0 & E_s \\ -E_s & 0 \end{pmatrix},$$

where E_s is the s-rowed identity matrix and for a matrix N, N' denotes the transpose of $N\}$ acts on H_s as a discontinuous group of analytic homeomorphisms $W \mapsto M\langle W\rangle = (AW+B)(CW+D)^{-1}$ of H_s (called modular transformations of degree s). By a modular form of degree s and weight $k(\geq 0)$ we mean a complex-valued function f holomorphic in the $s(s+1)/2$ independent elements of W such that $f(M\langle W\rangle)\det(CW+D)^{-k} = f(W)$ for every $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_s$ and further, for $s = 1$, f is bounded in a fundamental domain for Γ_s in H_s [6]. Let $\{\Gamma_s, k\}$ denote the complex vector space of such modular forms. Every $f \in \{\Gamma_s, k\}$ has the Fourier expansion

$$f(W) = \sum_N a(N)\exp(\pi i\, tr\,(NW)) \tag{1}$$

where tr denotes the trace and N runs over all s-rowed integral non-negative definite matrices with even diagonal elements. If $a(N) = 0$ for every positive definite N, we call f a *singular modular form*. On the other hand, if $a(N) = 0$ for all N which are not positive-definite, f is known as a cusp form. It is immediate that

the only cusp form which is, at the same time, a singular modular form is the constant 0. Writing W in H_s as $\begin{pmatrix} Z & w \\ w' & z \end{pmatrix}$ with $z \in H_1$, $Z \in H_{s-1}$, we know $g = \Phi(f)$ given by $g(Z) = \lim_{\lambda \to \infty} f\left(\begin{pmatrix} Z & 0 \\ 0' & i\lambda \end{pmatrix}\right)$ is in $\{\Gamma_{s-1}, k\}$. This map $\Phi : \{\Gamma_s, k\} \to \{\Gamma_{s-1}, k\}$ (with $\{\Gamma_0, k\} = \mathbf{C}$, by
264 convention) is known as the Siegel operator and its kernel consists precisely of cusp forms in $\{\Gamma_s, k\}$.

It is clear that $\{\Gamma_s, k\}$ contains no singular modular form $\neq 0$, if and only if every f in $\{\Gamma_s, k\}$ is uniquely determined by the set of its Fourier coefficients $\{a(N) | N$ positive definite$\}$ referring to (1).

For $s = 1$, a singular modular form is necessarily a constant. For $s > 1$, we know from Resnikoff ([3], [4]) that singular modular forms can exist only for k of the form $r/2$ with r integral and $0 \leq r < s$; such weights, for given s, are called *singular weights.*

Even for $s > 1$, it turns out, as we shall see in a while, that singular modular forms are particularly simple in their structure; in fact, they are quite arithmetical in nature in as much as they are just linear combinations of theta series associated with "positive-definite even quadratic forms of determinant 1 in r variables" (and with $W \in H_s$). It is well known that such quadratic forms exist only when r is a multiple of 8.

2. Let S be an r-rowed integral positive-definite matrix. Then associated to S and $W \in H_s$, we define the theta series

$$t(W; S) = \sum_G \exp(\pi i \, tr \, (G' \, S G \, W))$$

where the summation is over all integral matrices G of r rows and s columns. The series represents a holomorphic function of W on H_s. From Witt [7], we know that it is a modular form of degree s and weight $r/2$, if S is positive-definite, integral with diagonal elements even and of determinant 1; as already mentioned, 8 has to divide r in this case. Thus, for $s > r$, $t(W; S)$ is always a singular modular form for such S, since, in the expansion (1) for $t(W; S)$ only N of the form $G'SG$ with integral G of

$r(< s)$ rows and s columns can occur. From the analytic theory of quadratic forms [5], we know that, for given r divisible by 8, all such even positive-definite S of determinant 1 constitute a single "genus" consisting of finitely many **Z**-equivalence classes (Two r-rowed matrices A, B are said to be **Z**-*equivalent*, if there exists an integral matrix U of determinant ± 1 such that $U'AU = B$). Let **265**
$S_1, \ldots, S_h$ form a complete set of representatives of these classes. Then $t(W; S_i)$ depends only on the class of S_i for $1 \leq i \leq h$. We claim that $t(W; S_i)$, $1 \leq i \leq h$ are linearly independent over the field of complex numbers for $W \in H_r$. In fact, the Fourier coefficient of $t(W; S_i)$ corresponding to S_j (in place of N) as in (1) is given by $\delta_{ij}E(S_j)$ where δ_{ij} is the Kronecker delta and $E(S_j)$ is the number of integral matrices U with $U'S_jU = S_j$. Any linear relation of the form $\sum_{1 \leq i \leq h} b_i t(W; S_i) = 0$ with b_i not all zero, immediately implies that $b_iE(S_i) = 0$ for every i and yields a contradiction, since we have $E(S_i) \geq 2$, always. It is immediately seen, by applying the Φ-operator, that for $s \geq r$ and $W \in H_s$, $t(W, S_1), \ldots, t(W, S_h)$ are linearly independent.

Theorem. *Every (singular) modular form of degree s and weight* $r/2$ *with r integral and* $0 < r < s$ *vanishes identically unless r is a multiple of* 8. *If* 8 *divides r, every modular form of degree s and weight* $r/2$ *is a linear combination of the theta series* $t(W; S_i)$, $1 \leq i \leq h$, *where* $W \in H_s$ *and* $S_1, \ldots, S_h$ *are representatives of* **Z***-equivalence classes of r-rowed positive-definite integral matrices with even diagonal elements and determinant* 1.

Remark. The theorem does not say anything about the nature of $\{\Gamma_r, r/2\}$ for r divisible by 8. If the Siegel operator from $\{\Gamma_{r+1}, r/2\}$ to $\{\Gamma_r, r/2\}$ were onto, then one can conclude that $\{\Gamma_r, r/2\}$ is generated over **C** by theta series. If $r = 8$, we know from ([1], [2]) that $\{\Gamma_s, 4\}$ has dimension 1 over **C** for $s \leq 4$. It has been conjectured by Freitag that even for $5 \leq s \leq 8$, $\{\Gamma_s, 4\}$ has dimension 1, being generated by a theta series.

Assume that the theorem has been proved for $s = r + 1$. We

then claim that it is valid for $s > r+1$. First, for r not divisible by 8, if $\{\Gamma_{r+1}, r/2\}$ consists only of 0, then the Siegel operator Φ from $\{\Gamma_{r+2}, r/2\}$ is clearly onto $\{\Gamma_{r+1}, r/2\} = \{0\}$ and its kernel is already $\{0\}$ since every cusp form which is a singular form vanishes identically. Thus, for r not divisible by 8, $\{\Gamma_{r+2}, r/2\} = \{0\}$ and a similar argument gives $\{\Gamma_s, r/2\} = \{0\}$ for $s \geq r+2$. On the other hand, let, for r divisible by 8, $\{\Gamma_{r+1}, r/2\}$ be a linear com-
266 bination of theta series $t(W; S_i)$, $1 \leq i \leq h$ with $W \in H_{r+1}$. If $W^* = \left(\begin{smallmatrix} W & * \\ * & * \end{smallmatrix}\right) \in H_{r+2}$, clearly $\Phi(t(W^*; S_i)) = t(W; S_i)$, i.e. the Siegel operator from $\{\Gamma_{r+2}, r/2\}$ is onto. Again by the same argument as above, Φ is an isomorphism, implying that $\{\Gamma_{r+2}, r/2\}$ is generated by theta series for r divisible by 8; the spaces $\{\Gamma_s, r/2\}$ for $s > r+2$ are taken care of in a similar manner.

Therefore, we proceed to prove the theorem for $s = r+1$. Let $\{\Gamma_{r+1}, r/2\}$ *contain an* $f \neq 0$. For $W = \left(\begin{smallmatrix} Z & w \\ w' & z \end{smallmatrix}\right) \in H_{r+1}$ with $z \in H_1$, $Z \in H_r$, let us write (1) as

$$f(W) = \sum_N a(N) \exp(\pi i(mz + 2n'w + tr(TZ)) \tag{1}$$

where N runs over all matrices of the form $\left(\begin{smallmatrix} T & n \\ n' & m \end{smallmatrix}\right)$ which are non-negative definite, $(r+1)$-rowed, integral with even diagonal elements, T being r-rowed. Further, f being a singular modular form, $a(N) = 0$ for $\det N > 0$. If $\det T > 0$, we can write N above as

$$N = \begin{pmatrix} T & 0 \\ 0' & m - T^{-1}[n] \end{pmatrix} \begin{bmatrix} E & T^{-1}n \\ 0' & 1 \end{bmatrix}.$$

Here E_r denotes the r-rowed identity matrix and for two matrices A, B we abbreviate $B'AB$ (when defined) as $A[B]$. Thus, for $\det T > 0$, we have necessarily $m = T^{-1}[n]$.

For integral A with $\det A = 1$, we have $f(W[A]) = f(W)$. This gives $a(N) = a(N[A])$. In particular, for integral r-rowed columns g and

$$N = \begin{pmatrix} T & n \\ n' & T^{-1}[n] \end{pmatrix}, N_1 = \begin{pmatrix} T & Tg + n \\ (Tg + n)' & T^{-1}[Tg + n] \end{pmatrix} = N \begin{bmatrix} E_r & g \\ 0' & 1 \end{bmatrix}$$

the corresponding coefficients $a(N)$ and $a(N_1)$ coincide.

For positive T as above and an integral r-rowed column l with $T^{-1}[l]$ even, let $b(T;l) = a\left(\begin{pmatrix} T & l \\ l' & T^{-1}[l] \end{pmatrix}\right)$ and let $b(T;l) = 0$ otherwise. Then $b(T;Tg+l) = b(T;l)$ for every integral g. Let l run over a complete set of r-rowed integral columns such that no two **267**
distinct columns say l_1, l_2 satisfy $l_1 = l_2 + Tg$ for some integral column g (We write $l \bmod T$, in symbols). We can write

$$f(W) = \sum_{T>0} \exp(\pi i\, tr(TZ)) \sum_{l \bmod T} b(T;l) \sum_{g} \exp(\pi i z\, T^{-1}[Tg+l] + \\ + 2\pi i (Tg+l)'w) + \sum_{T_1 \geq 0, \det T_1 = 0} h(z,w;T_1) \exp(\pi i\, tr(T_1 Z)) \tag{2}$$

where T, T_1 respectively run over positive-definite and singular non-negative-definite r-rowed integral matrices with even diagonal elements and g runs over all r-rowed integral columns. Let us abbreviate the innermost sum in the first term on the right side of [3] as $\vartheta(z,w;T,l)$. It is easy to see that

$$\begin{pmatrix} Z & w \\ w' & z \end{pmatrix} = W \mapsto W_1 = \begin{pmatrix} Z - z^{-1}ww' & z^{-1}w \\ z^{-1}w' & -z^{-1} \end{pmatrix} \\ = \begin{bmatrix} E_r & 0 & 0 & 0 \\ 0' & 0 & 0' & -1 \\ 0 & 0 & E_r & 0 \\ 0' & 1 & 0' & 0 \end{bmatrix} \langle W \rangle$$

is a modular transformation of degree s. Now, since $f \in \{\Gamma_s, r/2\}$, we have

$$f(W_1) z^{-r/2} = f(W) \tag{3}$$

taking that branch of $z^{-r/2}$ assuming the value $\exp(-\pi i r/4)$ corresponding to the point iE_{r+1}. On the other hand, it is easy to check that, for positive-definite T,

$$\vartheta(-z^{-1}, z^{-1}w; T, l) = \exp(\pi i z^{-1} T[w]) \times$$

$$\times \sum_{g \text{ integral}} \exp(-\pi i\, T[g + T^{-1}l - w]z^{-1})$$
$$= \exp(\pi i z^{-1} T[w])(\det(iz^{-1}T))^{-1/2}\times$$
$$\times \sum_{g \text{ integral}} \exp(\pi i z\, T^{-1}[g] + 2\pi i\, g'(T^{-1}l - w))$$

using the theta transformation formula. Let us write $T^{-1}g$ again as $T^{-1}j + p$ where j runs modulo T and p runs over all integral columns as g does so. Then we have

$$\begin{aligned}
&\vartheta(-z^{-1}, z^{-1}w; T, l)\\
&= (i/z)^{-r/2}(\det T)^{-1/2} \exp(\pi i z^{-1} T[w])\times\\
&\quad \times \sum_{j \bmod T} \sum_{p \text{ integral}} \exp(\pi i\, T[p + T^{-1}j]z +\\
&\quad + 2\pi i\, l'(T^{-1}j + p) - 2\pi i(Tp + j)'w)\\
&= (i/z)^{-r/2}(\det T)^{-1/2} \exp(\pi i z^{-1} T[w])\times\\
&\quad \times \sum_{j \bmod T} \exp(-2\pi i\, l' T^{-1} j)\vartheta(z, w; T, j) \qquad (4)
\end{aligned}$$

268 noting that $exp(2\pi i\, l' p) = 1$ and making the change $p \to -p$ and $j \to -j$. From (2), (3) and (4), we obtain

$$\begin{aligned}
&z^{r/2} \sum_{0<T, j \bmod T} b(T, j)\vartheta(z, w; T, j) \exp(\pi i\, tr(TZ)\\
&\quad + z^{r/2} \sum_{T_1} h(z, w; T_1) \times \times \exp(\pi i\, tr(T_1 Z))\\
&= \sum_{T, l \bmod T} b(T; l) \exp(\pi i\, tr(TZ) - \pi i z^{-1} T[w])\\
&\quad \times \vartheta(-z^{-1}, z^{-1}w; T, l) +\\
&\quad + \sum_{T_1} h(-z^{-1}, z^{-1}w; T_1) \exp(\pi i\, tr(T_1 Z) - \pi i z^{-1} T_1[w])\\
&= (i/z)^{-r/2} \sum_{T, l \bmod T} b(T; l)(\det T)^{-1/2} \exp(\pi i\, tr(TZ))\times
\end{aligned}$$

$$\times \sum_{j \bmod T} \exp(-2\pi i\ j' T^{-1} l)\vartheta(z, w; T, j) +$$
$$+ \sum_{T_1} h(-z^{-1}, z^{-1}w; T_1) \exp(-\pi i\, z^{-1} T_1[w]) \exp(\pi i\ tr(T_1 Z))$$
$$= (i/z)^{-r/2} \sum_{0<T,\, j \bmod T} (\det T)^{-1/2} \sum_{l \bmod T} b(T; l) \exp(-2\pi i\ j' T^{-1} l) \times$$
$$\times \exp(\pi i\ tr(TZ))\vartheta(z, w; T, j)$$
$$+ \sum_{T_1} h(-z^{-1}, z^{-1}w; T_1) \exp(-\pi i\, z^{-1} T_1[w]) \exp(\pi i\ tr(T_1 Z)).$$

Comparing the coefficient of $\exp(\pi i\ tr(TZ))$ for fixed positive-definite T on both sides, we get

$$z^{r/2} \sum_{j \bmod T} b(T; j)\vartheta(z; w; T, j) = (i/z)^{-r/2} (\det T)^{-1/2} \times$$
$$\times \sum_{j \bmod T} \vartheta(z; w; T, j) \sum_{l \bmod T} b(T : l) \exp(-2\pi i\ j' T^{-1} l).$$

This implies, in turn, that **269**

$$b(T : j) = \exp(-\pi i r/4)(\det T)^{-1/2} \sum_{l \bmod T} b(T; l) \exp(-2\pi i l' T^{-1} j).$$

In particular, we have for T with $\det T > 0$,

$$a\left(\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}\right) = (\det T)^{-1/2} \exp(-\pi i\ r/4) \sum_{l \bmod T} a\left(\begin{pmatrix} T & l \\ l' & T^{-1}[l] \end{pmatrix}\right) \tag{5}$$

(cf. [1](13), p. 284, [4]§§ 5-6). Note that $(\det T)^{1/2}$ is the positive square root.

For given positive-definite T and $l \bmod T$ as above, there exists an integral matrix A of determinant 1 such that

$$\begin{pmatrix} T & l \\ l' & T^{-1}[l] \end{pmatrix} = A' \begin{pmatrix} T_0 & 0 \\ 0' & 0 \end{pmatrix} A. \tag{6}$$

Since the left hand side is integral with even diagonal elements and of rank r, the same is true of T_0. If $A = \left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$ with r-rowed

square d, we have $T = T_0[d]$ implying that d is nonsingular. Further $l = d'T_0c$. If $\det d = \pm 1$, then $l = d'T_0d$. $d^{-1}c = Td^{-1}c$ ($\equiv 0 \bmod T$, in the sense that $l = Tg$ for an integral column g). Thus, if there exists an integral l not of the form Tg for an integral g such that $T^{-1}[l]$ is even, then necessarily $\det d$ has absolute value atleast equal to 2. On the other hand, there always exists an integral l such that $T^{-1}[l]$ is an even integer.

Let us call T *imprimitive* if there exists an integral matrix d with $\det d \neq 0, \pm 1$, such that $T[d^{-1}]$ is integral with even diagonal elements. If T is not imprimitive, we call T *primitive*.

For primitive T, the only term occurring on the right side of (4) corresponds to $l = 0$ (i.e. $l \equiv 0 (\bmod\ T)$). In this case then, we have

$$\left(1 - \frac{\exp(-\pi ir/4)}{(\det T)^{1/2}}\right) a\left(\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}\right) = 0. \tag{7}$$

Unless

$$\det T = 1 \quad \text{and} \quad r \equiv 0 (\bmod\ 8), \tag{8}$$

(7) implies that $a\left(\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}\right) = 0$, We may rewrite (4) as

$$a\left(\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}\right) = \frac{\exp(-\pi ir/4)}{(\det T)^{1/2}} \sum_{(d\ c)} a\left(\begin{pmatrix} T[d^{-1}] & 0 \\ 0' & 0 \end{pmatrix}\right) \tag{5'}$$

270 where, on the right hand side, the summation is over integral matrices $(d\ c)$ of r rows and $r+1$ columns having discriminant 1, with integral non-singular d not mutually differing by a left sided integral matrix factor of determinant ± 1 and with c running only modulo d. From (5′) and (8), we conclude that

$$\text{“}a\left(\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}\right) \neq 0,\ \det T > 1\text{”} \Rightarrow T \quad \text{is imprimitive.} \tag{9}$$

On the other hand, if there exists T as above with $\det T = 1$ and $a\left(\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}\right) \neq 0$, then 8 necessarily divides r, T being r-rowed, positive-definite, of determinant 1 and integral with even diagonal elements. Thus if 8 does not divide r, repeated application of (5)′

eventually yields a primitive T_0 with non-zero $a\left(\begin{pmatrix} T_0 & 0 \\ 0' & 0 \end{pmatrix}\right)$, since $f \neq 0$ (and $\Phi(f) \neq 0$, as a result). For such a T_0, we deduce from (7) and (8) that $\det T_0 = 1$ and $r \equiv 0 (\mathrm{mod}\ 8)$, which is a contradiction. The first assertion of the theorem is thus proved for $s = r + 1$ and hence for every $s \geq r + 1$.

Let us now suppose that r is a (positive) multiple of 8. For the given $f \in \{\Gamma_{r+1}, r/2\}$, we can always find a constant u_i such that the Fourier coefficient $a\left(\begin{pmatrix} S_i & 0 \\ 0' & 0 \end{pmatrix}\right)$ of f is precisely $u_i E(S_i)$ for $1 \leq i \leq h$. Let, for given r-rowed integral symmetric R, $A(S_i, R)$ denote the number of integral matrices G for which $S_i[G] = R$. It is immediate from the choice of $u_i (1 \leq i \leq h)$ that for $\det T = 1$,

$$a\left(\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}\right) = \sum_{1 \leq i \leq h} u_i A(S_i T) \tag{10}$$

Indeed, T is in the $\mathbf{Z}$-equivalence class of exactly one of $S_1, \ldots, S_n$ and (10) follows from the choice of u_i and from the fact that $f(B'WB) = f(W)$ for integral B of determinant ± 1. Assume that (10) has been proved for all integral positive definite T with even diagonal elements and $\det T < t$. Then from (5)′, if $\det T = t$,

$$(1 - 1/(\det T)^{1/2}) a\left(\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}\right) = \frac{1}{(\det T)^{1/2}} \sum_{\substack{(d\ c) \\ |\det d| \neq 1}} a\left(\begin{pmatrix} T[d^{-1}] & 0 \\ 0' & 0 \end{pmatrix}\right)$$

$$= (1/(\det T)^{1/2}) \sum_{\substack{(d\ c) \\ |\det d| \neq 1}} u_i A(S_i, T[d^{-1}]).$$

using the hypothesis on (10). But the last expression is just **271**

$$(1 - 1/(\det T)^{1/2}) \sum_{1 \leq i \leq h} u_i A(S_i, T)$$

since $\sum_{1 \leq i \leq h} u_i A(S_i, T)$ is itself the Fourier coefficient of

$$\sum_{1 \leq i \leq h} u_i t(W; S_i) \in \{\Gamma_{r+1}, r/2\}$$

corresponding to $\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}$ and consequently satisfies (5)′. As a result, we have

$$a\left(\begin{pmatrix} T & 0 \\ 0' & 0 \end{pmatrix}\right) = \sum_{1 \le i \le h} u_i A(S_i, T)$$

for $\det T \geq 1$. In other words,

$$\begin{aligned} (\Phi(f))(Z) &= \sum_{1 \le i \le h} u_i t(Z, S_i) \\ &= \Phi\left(\sum_{1 \le i \le h} u_i t(W, S_i) \right) \\ \text{i.e.} \qquad f(W) &= \sum_{1 \le i \le h} u_i t(W, S_i) \end{aligned}$$

which completes the proof of the theorem for $s = r + 1$.

Acknowledgement. We are grateful to Prof. H.L. Resnikoff for having sent a preprint entitled ‘Stable spaces of modular forms’. It may be remarked that relations (3.7) and (3.8) on page 8 of that preprint do not seem to be correct.

We learned from Prof. H. Klingen in December 1976 that Freitag and Resnikoff have independently proved the assertion: “singular modular forms are generated by theta series”. Perhaps, their proof is also on the same lines as above. We received recently a preprint of D. M. Cohen and H. Resnikoff entitled “Hermitian quadratic forms and hermitian modular forms”; this contains a reference to a preprint of H. L. Resnikoff entitled “The structure
272 of spaces of singular automorphic forms” the contents of which are, however, not known to us.

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Contre-Exemple Au "Vanishing Theorem" En Caractéristique $p > 0$

par M. Raynaud

Sorr k un corps algébriquement clos de caractéristique $p >$ 273
0. Nous allons construire une surface X, propre et lisse sur k, et un faisceau inversible ample $\mathscr{I}$ sur X, tel que $H^1(X, \mathscr{I}^{-1}) \neq 0$. ⟷ Ainsi le théoréme de Kodaira [2] ⟷, ne s'etend pas en caractéristique $p > 0$. Ce contre-exemple est à rapprocher de celui obtenu par Mumford avec une surface normale, non lisse [5].

On va construire la surface X comme fibrée sur une courbe C, avec des fibres integres et une ligne horizontale de "cusps". Ces fibres sont les complétions projectives de courbes affines d'équation $y^2 = x^p + a$, si $p \neq 2$ et $y^3 = x^2 + a$ si $p = 2$.

Je tiens à remercier messieurs Oda et Spiro pour l'aide précieuse qu'ils m'ont apportée dans l'élaboration de ce travail.

1 Frobenius sur les courbes

On reprend ici une petite partie des résultats de Tango [6].

Soit C une courbe propre et lisse sur k, de genre g, de corps des fractions K et soit $F : C' \to C$ le k-morphisme radiciel de degré p, correspondant à la fermeture intégrale de C dans $K' = K^{1/p}$. La différentielle

$$d : \mathscr{O}_{C'} \to \Omega^1_{C'}$$

donne des suites exactes de $\mathscr{O}_C$-modules:

$$0 \to \mathscr{O}_C \to F_*(\mathscr{O}_{C'}) \xrightarrow{\alpha} \mathscr{B}^1 \to 0 \tag{1}$$

$$0 \to \mathscr{B}^1 \to F_*(\Omega^1_{C'}) \xrightarrow{c} \Omega^1_C \to 0 \tag{2}$$

ou c est l'opération de Cartier.

Soit $\mathscr{L} \subset \mathscr{B}^1$ un $\mathscr{O}_C$-module inversible et l son degré. Si $\mathscr{E} = \alpha^{-1}(\mathscr{L})$, on a une suite exacte:

$$0 \to \mathscr{O}_C \to \mathscr{E} \to \mathscr{L} \to 0, \tag{3}$$

274 d'ou un morphisme de $\mathscr{O}_C$-algèbres

$$S(\mathscr{E}) \xrightarrow{\beta} F_*(\mathscr{O}_{C'})$$

(on désigne par $S(\mathscr{E})$ l'algèbre symètrique de $\mathscr{E}$, par $S^n(\mathscr{E})$, sa partie homogène de degré n).

Proposition 1. (i) *On a $l \leq 2(g-1)/p$, avec égalité si et seulement si β est surjectif.*

(ii) *Pour qu'il existe un diviseur D sur C avec $\mathscr{L} \simeq \mathscr{O}_C(D)$, il faut et il suffit qu'il existe $f \in K$ avec $(df) \geq pD$, $df \neq 0$.*

Démonstration. Les courbes C et C' ayant même genre, la caractéristique d'Euler-Poincaré de $\mathscr{B}^1$ est nulle, d'où, par Riemann Roch:

$$\deg(\mathscr{B}^1) + (p-1)(1-g) = 0.$$

Par ailleurs, $\beta(S(\mathscr{E}))$ est isomorphe à $S^{p-1}(\mathscr{E})$, donc $\alpha \circ \beta(S(\mathscr{E}))$ est un sous-faisceau de $\mathscr{B}^1$ qui admet une filtration, a quotients successifs isomorphes à $\mathscr{L}^{\otimes i}$, $1 \leq i \leq (p-1)$. On a donc $lp(p-1)/2 \leq (p-1)(g-1)$, soit $l \leq 2(g-1)/p$, avec égalité si et seulement si β est surjectif, d'ou (i).

Prouvons (ii). On a $\mathscr{L} \simeq \mathscr{O}_C(D) \Leftrightarrow H^0(C, \mathscr{B}^1(-D)) \neq 0$. En tensorisant (2) par $\mathscr{O}_C(-D)$, on trouve la suite exacte:

$$0 \to \mathscr{B}^1(-D) \to F_*(\Omega^1_{C'}(-F^*(D)) \xrightarrow{c(-D)} \Omega^1_C(-D) \to 0$$

Par suite, $H^0(C, \mathscr{B}^1(-D)) = \{df', f' \in K', (df') \geq F^*(D).\}$ Par l'élévation à la puissance p, cet ensemble est en bijection avec

$$\{df, f \in K, (df) \geq pD\}, \quad d'ou(ii).$$

Corollaire. *Pour qu'il existe $\mathscr{L}$ dans $\mathscr{B}^1$, avec $\mathscr{L} \simeq \mathscr{O}_C(D)$* deg $(D) = 2(g-1)/p$, *il faut et il suffit qu'il existe $f \in K$, avec $(df) = pD$.*

Exemple. Soit h un entier > 0 et considérons le revetement d'Artin-Schreier de la droite affine, d'équation:

$$X^p - X = T^{hp-1}.$$

Soit C la complétion projective de ce revetêment, munie de son point à l'infini. La différente $\mathscr{D}$ est égale à $hp(p-1)\infty$, la courbe C est de genre g avec $2(g-1) = p(h(p-1)-2)$ et $(dT) = p(h(p-1)-2)\infty$. On peut donc prendre $\mathscr{L} = \mathscr{O}_C(D)$ avec $D = (h(p-1)-2)\infty$. **275**

2 Construction de $(X, \mathscr{I})$

Désormais, on suppose donnés C de genre $g > 1$ et $\mathscr{L}$ satisfaisant aux conditions énoncées dans le corollaire, donc de degré $1 = 2(g-1)/p > 0$.

Soit $P = P(\mathscr{E})$ le fibré en droites projectives sur C, défini par $\mathscr{E}$. Notons $f : P \to C$ le morphisme structural et $\mathscr{O}_P(1)$ le faisceau inversible relativement ample canonique. On a $f_*(\mathscr{O}_P(n)) = S^n(\mathscr{E})$. Enfin, le faisceau dualisant relatif est $\omega_{P/C} = \mathscr{O}_P(-2) \otimes \Lambda^2(\mathscr{E}) = \mathscr{O}_P(-2) \otimes \mathscr{L}$.

On a deux diviseurs horizontaux naturels sur P. Tout d'abord, le quotient $\mathscr{L}$ de $\mathscr{E}$, définit une section de f; soit $E \simeq C$ son image. L'élément 1 de $\mathscr{O}_C$, vu comme élément de $H^0(C, \mathscr{E}) = H^0(P, \mathscr{O}_P(1))$, donne une section s de $\mathscr{O}_P(1)$, qui s'annule sur E et fournit un isomorphisme de $\mathscr{O}_P(1)$ avec $\mathscr{O}_P(E)$. Par ailleurs, si on tensorise (3) avec le morphisme de Frobénius absolu sur C, on

obtient la suite exacte:

$$0 \to \mathscr{O}_C \to \mathscr{E}^{(p)} \to \mathscr{L}^{\otimes p} \to 0. \tag{4}$$

Comme $\mathscr{E} \subset \mathscr{O}_{C'}$, l'élévation à la puissance p dans $\mathscr{O}_{C'}$, permet de définir un morphisme $\mathscr{O}_C$-linéaire surjectif $\mathscr{E}^{(p)} \to \mathscr{O}_C$; son noyau est isomorphe à $\mathscr{L}^{\oplus p}$ et fournit un scindage de (4). On a donc une droite canonique $\mathscr{L}^{\oplus p}$ dans $\mathscr{E}^{(p)} \subset S^p(\mathscr{E}) = p_x(\mathscr{O}_P(p))$. D'ou une section $t \in H^0(P, \mathscr{O}_p(p) \otimes \mathscr{L}^{\oplus -p})$; t s'annule sur une courbe de P, de degré p sur C qui, compte tenu du choix de $\mathscr{L}$, est C-isomorphe à C' (et on la note C' dans la suite), donc est lisse sur k. Enfin on a $C' \cap E = \phi$.

Examinons d'abord le cas $p \neq 2$. On va construire un revêtement X de P, de degré 2, ramifié le long de $C' \cup E$. Pour celà, on choisit $\mathscr{N}$ inversible sur C, tel que $\mathscr{N}^{\otimes 2} = \mathscr{L}$ (noter que l est pair). Pour décrire le morphisme $\pi : X \to P$, on doit se donner un faisceau inversible $\mathscr{M}$ sur P et un isomorphisme de
276 $\mathscr{M}^{\otimes 2}$ avec $\mathscr{O}_P(-E - C')$; on a alors $\pi_*(\mathscr{O}_X) = \mathscr{O}_P \oplus \mathscr{M}$. On prend $\mathscr{M} = \mathscr{O}_P(-(p+1)/2) \otimes N^{\otimes p}$ et le morphisme de $\mathscr{O}_P$ dans $\mathscr{M}^{\otimes -2} = \mathscr{O}_P(p+1) \otimes \mathscr{L}^{\otimes -p} = \mathscr{O}_P(E + C')$ défini par $s \otimes t$. Comme C' et E sont lisses sur k, et disjointes, X est lisse sur k. Les fibres de $g = f \circ \pi$ sont de genre $(p-1)/2$; un calcul local montre que X possede une ligne de cusps $\widetilde{C}$, au-dessus de C' ($\widetilde{C}$ isomorphe à C'). En dehors de $\widetilde{C}$, X est lisse sur C. Enfin E se dédouble sur X en $\pi_*(E) = 2\widetilde{E}$.

Lemme. *Il existe* $n > 0$ *tel que* $\mathscr{O}_X(n\widetilde{E})$ *soit engendré par ses sections. Le système linéaire correspondant contracte une seule courbe: la ligne de cusps* $\widetilde{C}$.

Evidemment, la sectionl de $\mathscr{O}_X(\widetilde{E})$ engendre $\mathscr{O}_X(\widetilde{E})$ en dehors de $\widetilde{E}$. Donc $\widetilde{C}$, qui est contenue dans $X - \widetilde{E}$, sera nécessairement contractée. Par ailleurs on a $\pi_*(\mathscr{O}_X(\widetilde{E})) = \mathscr{O}_P \oplus \mathscr{M}(E)$ et plus généralement, pour $n \geq 0$, on a

$$\pi_* \mathscr{O}_X(2n\widetilde{E}) = \mathscr{O}_P(nE) \oplus \mathscr{M}(nE) \text{ et } \pi_* \mathscr{O}_X((2n+1)\widetilde{E})$$

$$= \mathscr{O}_P(nE) \oplus \mathscr{M}((n \oplus 1)E).$$

En particulier

$$g_*(\mathscr{O}_X(p\widetilde{E})) = p_*(\mathscr{O}_P((p-1/2)E) \oplus \mathscr{M}((p+1)/2\ E))$$
$$= p_*(\mathscr{O}_P((p-1)/2)) \oplus \mathscr{N}^{\otimes n}.$$

Comme $\mathscr{N}$ est de degré > 0, $\mathscr{N}^{\otimes pm}$ est engendré par ses sections pour $m \gg 0$. Si alors σ engendre $N^{\otimes pm}$ au-dessus d'un ouvert affine U de C, il correspond à σ une section de $\mathscr{O}_X(pm\ \widetilde{E})$, qui engendre ce faisceau au-dessus de U, en dehors de $\widetilde{C}$, donc sur un ouvert affine de X, d'ou le lemme.

Il résulte du lemme que, si $\mathscr{Q}$ est un faisceau inversible sur C, de degré > 0, alors $\mathscr{L} = \mathscr{O}_X(\widetilde{E}) \otimes g(\mathscr{Q})$ est ample sur X. Il nous reste à voir que l'on peut choisir $\mathscr{Q}$, de degré > 0, de facon que $H^1(X, \mathscr{I}^{-1})$ soit $\neq 0$. Or on a $H^1(X, \mathscr{O}_X(-\widetilde{E}) \otimes \mathscr{Q}^{-1}) = H^0(C, R^1g_*(\mathscr{O}_X(-\widetilde{E})) \otimes \mathscr{Q}^{-1})$; $R^1g_*(\mathscr{O}_X(-\widetilde{E})) = R^1p_*(\mathscr{O}_p(-E) \oplus \mathscr{M}) = R^1p_*(\mathscr{M})$. Vu la dualité de Serre, $R^1p_*(\mathscr{M})$ est dual de 277
$p_*(\mathscr{M}^{-1} \otimes \omega_{P/C}) = p_*(\mathscr{O}_P(p-3)/2) \otimes N^{2-p} = S^{(p-3)/2}(\mathscr{E}) \otimes N^{2-p}$. Ce dernier a pour quotient $\mathscr{L}^{\otimes(p-3)/2} \otimes \mathscr{N}^{\otimes(2-p)} = \mathscr{N}^{-1}$, donc $R^1g_*(\mathscr{O}_X(-E)) \otimes \mathscr{Q}^{-1}$ contient $\mathscr{N} \otimes \mathscr{Q}^{-1}$. Il suffit donc de prendre $\mathscr{Q}$ de degré > 0, tel que $H^0(C, \mathscr{N} \otimes \mathscr{Q}^{-1}) \neq 0$, par exemple $\mathscr{Q} = \mathscr{N}$.

Dans le cas $p = 2$, on choisit $l = g - 1$ multiple de 3 et un faisceau inversible $\mathscr{N}$ sur C tel que $\mathscr{N}^{\otimes 3} = \mathscr{L}$. On prend pour $\mathscr{M}$ un revêtement cyclique de degré 3 de P, ramifié le long de $E \cup C'$, défini par le faisceau inversible $\mathscr{M} = \mathscr{O}_P(-1) \otimes \mathscr{N}^2$ et le morphisme

$$\mathscr{O}_P \to \mathscr{M}^{\otimes -3} = \mathscr{O}_P(3) \otimes \mathscr{L}^{-2} = \mathscr{O}_P(E + C')$$

défini par $s \otimes t$. La fin de la démonstration est analogue a celle du cas $p \neq 2$.

3 Remarques et questions

1. La surface X que nous avons construite est un revêtement radi-

ciel de degré p d'une surface réglée de base C; en particulier, elle a pour nombres de Bette, $b_1 = 2g$, $b_2 = 2$. Néanmoins, du point de vue de la classification d'Enriques, Bombiéri, Mumford [1] et [4], elle est de type général pour $p \geq 5$, quasi-elliptique (avec $\chi(\mathcal{O}_X) < 0$) pour $p = 2$ et 3.

2. Comme E est une section de f, $\omega_{P/C}(E)|E$ est trivial, done $\mathcal{O}_P(E)|E$ est isomorphe à $\mathcal{L}|C$; en particulier $E^2 = l > 0$. Il en résulte que l'on a $\widetilde{E}^2 = l/2$ pour $p \neq 2$ et $\widetilde{E}^2 = l/3$ pour $p = 2$. Mumford et Spiro ont remarqué que, des que l'on avait une surface lisse X, fibrée sur une courbe C, à fibres integres de genre ≥ 1 munie d'une section E telle que $E^2 > 0$, on pouvait trouver sur X, un faisceau ample $\mathcal{L}$ tel que $H^1(X, \mathcal{L}^{-1}) \neq 0$.

3. Dans le revêtement d'Artin-schreier cité plus haut prenons, $h = ap - 2$ avec $a \geq 1$ si $p \geq 5$, $a \geq 2$ si $p = 3$ et prenons $h = 8$ si $p = 2$. On peut alors choisir $\mathcal{N} = \mathcal{O}_C(a(p-1/2)-1)p\infty)$ is $p \geq 3$ et $\mathcal{N} = \mathcal{O}_C(2\infty)$ si $p = 2$ et prendre pour faisceau ample $\mathcal{I} = \mathcal{O}_X(\widetilde{E}) \otimes \mathcal{N}$. Comme $\mathcal{N}$ est engendré par ses sections sur C (car image réciproque sur C d'un faisceau sur la droite projective),
278 $\mathcal{L}$ est alors engendré par ses sections en dehors de $\widetilde{E}$. Peut-on contre-exempler le théorème de Kodaira avec un faisceau ample, engendré par ses sections, voir très ample?

4. Soient X une variété propre et lisse sur k, $\mathcal{L}$ un faisceau ample sur X et ω le faisceau dualisant. Supposons que X et $\mathcal{L}$ se relèvent en caractéristique zéro. Il résulte alors du théorème de Kodaira et des propriétés de spécialisation de la cohomologie que pout tout entier $i \geq 0$, on a:

$$\chi^i(X, \mathcal{L} \otimes \omega) = \dim H^i(X, \mathcal{L} \otimes \omega) - \dim H^{i+1}(X, \mathcal{L} \otimes \omega) + \cdot \geq 0.$$

Ces propriétés restent-elles varies sans hypothèses de relèvement? Notons qu'un tel résultat; bien que nettement plus faible que le théorème de Kodaira, suffirait pour étendre à la caractéristique p, le théorème de finitude de Matsuaka [3].

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The Tricanonical Map of a Surface with $K^2 = 2, P_g = 0$

By E. Bombieri and F. Catanese

1 Introduction

In this paper we prove the following result.

Theorem 1. *Let S be a minimal surface of general type with $K^2 = 2$, $p_g = 0$, over an algebraically closed field k,* $\mathrm{char}(k) = 0$. *Then the tricanonial map Φ_{3K} of S is a birational morphism.*

If we combine Theorem 1 with the results of Bombieri ([1] referred in the sequel as [*CM*]) and Miyaoka [4] we deduce

Theorem 2. *If S is a minimal surface of general type over an algebraically closed field k,* $\mathrm{char}(k) = 0$ *and $m \geq 3$, then Φ_{mK} is a birational map with exactly the following exceptions:*

(a) $m = 3$, $K^2 = 1$ *and* $p_g = 2$, $K^2 = 2$ *and* $p_g = 3$

(b) $m = 4$, $K^2 = 1$ *and* $p_g = 2$.

We refer to Horikawa [3] for a detailed and exhaustive study of the surfaces in (a) and (b).[†]

Our notation is as follows:

S a minimal surface of general type, with $K^2 = 2$, $p_g = 0$ (non-singular),

[†] Added in Proof. The result of Theorem 2 has been independently obtained by X. Benveniste, with a similar method.

K a canonical divisor on S,
ω_D the dualizing sheaf of a divisor $D > 0$ on S, hence

$$\omega_D \cong \mathcal{O}_D(D + K),$$

$\mathcal{O}$ the structure sheaf of S,
$\mathcal{F}(-C - a_1x_1 - \cdots a_rx_r)$ the sheaf of germs of sections of $\mathcal{F}$ vanishing on C and on x_i with multiplicity a_i,

$$\mathcal{F}_C(-a_1x_1 - \cdots - a_rx_r) = \mathcal{F}/\mathcal{F}(-C - a_1x_1 - \cdots - a_rx_r),$$
$$|D - C - a_1x_1 - \cdots - a_rx_r|$$
$$= \text{Proj } H^0(\mathcal{O}(D - C - a_1x_1 - \cdots - a_rx_r)) \subseteq |D|p(D)$$
$$p(D) = \frac{1}{2}(D^2 + KD) + 1$$

280 the arithmetic genus of D; P_m the plurigenera of S.

2 Two auxiliary results

The first result of this section gives a necessary condition for an invertible sheaf on a curve C on a surface X to have a section. The technique of proof is one invented by C.P. Ramanujam in the study of numerically connected divisors.

Proposition A. *Let $C > 0$ be a divisor on a smooth surface X and let $\mathcal{L}$ be an invertible sheaf on C with $H^0(C, \mathcal{L}) \neq (0)$.*

Then either

(i) $\deg_C \mathcal{L} \geqq 0$,

or

(ii) *we have $C = C_1 + C_2$, $C_i > 0$ with*

$$C_1C_2 \leqq \deg_{C_2}(\mathcal{L} \otimes \mathcal{O}_{C_2}).$$

Moreover, (ii) *holds whenever there is a section s of $\mathcal{L}$ vanishing on C_1 but not on any C' with $C_1 < C' < C$.*

Proof. Let $s \in H^0(C, \mathscr{L})$ and suppose that the restriction of s to any irreducible component of C is never identically 0. Let $C = \sum n_i \Gamma_i$, Γ_i irreducible, so that

$$\deg_C \mathscr{L} = \sum n_i \deg_{\Gamma_i}(\mathscr{L} \otimes \mathscr{O}_{\Gamma_i}).$$

If $\deg_C \mathscr{L} < 0$, we would get $\deg_{\Gamma_i}(\mathscr{L} \otimes \mathscr{O}_{\Gamma_i}) < 0$ for some i, hence $\mathscr{L} \otimes \mathscr{O}_{\Gamma_i}$ would have only the 0-section since Γ_i is irreducible. This however contradicts our initial assumption that the restriction of s to Γ_i is not identically 0, i.e. that s is not in the kernel of the restriction map

$$H^0(C, \mathscr{L}) \xrightarrow{\text{res}} H^0(\Gamma_i, \mathscr{L} \otimes \mathscr{O}_{\Gamma_i}).$$

It follows that if conclusion (i) does not hold then there is C_1, with $0 < C_1 < C$ such that s is in the kernel of the restriction map

$$H^0(C, \mathscr{L}) \xrightarrow{\text{res}} H^0(C_1, \mathscr{L} \otimes \mathscr{O}_{C_1}).$$

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According to Ramanujam's idea, we take a *maximal* C_1 and verify two exact sequences of sheaves

$$0 \to \mathscr{O}_{C_2} \xrightarrow{s} \mathscr{L} \to \mathscr{L}/s\mathscr{O}_C \to 0$$
$$0 \to \mathscr{F} \to \mathscr{L}/s\mathscr{O}_C \to \mathscr{L} \otimes \mathscr{O}_{C_1} \to 0$$

where $C_1 + C_2 = C$ and where the sheaf $\mathscr{F}$ is supported at finitely many points. Now we take the total Chern class on X of the above sequences and find

$$c(\mathscr{L}) = c(\mathscr{O}_{C_2})c(\mathscr{L}/s\mathscr{O}_C) = c(\mathscr{O}_{C_2})c(\mathscr{L} \otimes \mathscr{O}_{C_1})c(\mathscr{F})$$

whence the equation

$$1 + C + (C^2 - \deg_C \mathscr{L})$$
$$= (1 + C_2 + C_2^2)(1 - \text{length } \mathscr{F})(1 + C_1 + C_1^2 - \deg_{C_1} \mathscr{L})$$

in the Chow ring of X. The equation in degree 2 is simply

$$C_1C_2 + \text{ length } \mathscr{F} = \deg_{C_2} \mathscr{L}$$

and our result is prove. (QED) □

Our next result, the fact that the canonical system of an irreducible curve has no base points is very classical, but since we could not find an adequate reference we give a proof here.

Proposition B. *If Γ is an irreducible Gorenstein curve and $|\omega_\Gamma| \neq \emptyset$, then $|\omega_\Gamma|$ has no base points. More generally, a reduced point p on a reducible curve C on a smooth surface is not a base point of $|\omega_C|$ if either*

(i) *p is simple on C and belongs to a component Γ with $p(\Gamma) \geqq 1$ or*

(ii) *p is singular and for every decomposition $C = C_1 + C_2$ one has*

$$C_1C_2 > (C_1 \cdot C_2)_p,$$

where $(C_1 \cdot C_2)_p$ is the intersection multiplicity of C_1, C_2 at p.

We begin by proving

282 **Lemma B′.** *Let p be a singular point of a curve C lying on a smooth surface S, let m_p be the maximal ideal of p and let $\pi : \widetilde{C} \to C$ be a normalization of C at p. Then* $\mathrm{Hom}(m_p, \mathscr{O}_{C,p})$, *can be embedded in the ring A of regular functions on $\pi^{-\perp}(p)$, i.e. $A = \oplus_{\pi(p')=p} \mathscr{O}_{\widetilde{C},p'}$.*

Proof. Let $\varphi \in \mathrm{Hom}(m_p, \mathscr{O}_{C,p})$ and let x, y be local parameters for S at p. We denote by f a local equation for C at p and for $u \in \mathscr{O}_p$ we denote by $[u]$ its image in $\mathscr{O}_{C,p} = \mathscr{O}_p/(f)$. We may and shall assume that $[x]$ and $[y]$ are not 0-divisors in $\mathscr{O}_{C,p}$. The

homomorphism φ is determined by the knowledge of $\varphi([x])$ and $\varphi([y])$; let ξ and η be such that

$$[\xi] = \varphi([x]), \quad [\eta] = \varphi([y]).$$

We have $\varphi([xy]) = [x][\eta] = [\xi][y]$ thus $[\eta] = [\xi][y]/[x]$ in the ring of quotients of $\mathcal{O}_{C,p}$. It follows that we have a bijection between φ's and classes $[\xi]$ such that

$$[\xi][y]/[x] \in \mathcal{O}_{C,p},$$

since for every $[z] \in m_p$ we may set

$$\varphi([z]) = [z][\xi]/[x].$$

Let $p' \in \pi^{-1}(p)$ and let t be a local parameter on $\widetilde{C}$ at p'. Then we claim that

$$\operatorname{ord}_t \pi^*([\xi]/[x]) \geqq 0.$$

In fact, suppose that $\operatorname{ord}_t \pi^*[\xi] < \operatorname{ord}_t \pi^*[x] \leqq \operatorname{ord}_t \pi^*[y]$; then we cannot have $\xi \in m_p$, hence $[\xi]$ is a unit and $[y] = [x][\eta]/[\xi]$, which shows that the maximal ideal of $\mathcal{O}_{C,p}$ is generated by $[x]$, i.e. p is a regular point of C. If instead $\operatorname{ord}_t \pi^*[y] < \operatorname{ord}_t \pi^*[x]$, there is the same reasoning with η and y instead of ξ and x, because $[\eta]/[y] = [\xi]/[x]$. Q.E.D. □

Now we can prove Proposition B.

If p is a base point of $|\omega_C|$ the exact sequence

$$0 \to \omega_C m_p \to \omega_C \to k_p \to 0$$

yields

$$0 \to k \to H^1(C, \omega_C m_p) \to H^1(C, \omega_C) \to 0.$$

By Grothendieck's Deuality we obtain **283**

$$0 \leftarrow k \leftarrow \operatorname{Hom}(m_p, \mathcal{O}_C) \leftarrow H^0(C, \mathcal{O}_C) \leftarrow 0$$

and

$$\dim \mathrm{Hom}(m_p, \mathcal{O}_C) \geqq 2.$$

In Case (i), $\mathrm{Hom}(m_p, \mathcal{O}_C) \cong H^0(C, \mathcal{O}_C(p))$ hence dim. $H^0(\Gamma, \mathcal{O}_C(p)) \geq 2$ and $p(\Gamma) = 0$, that is Γ is rational non-singular.

In Case (ii), by Lemma B′, $\mathrm{Hom}(m_p, \mathcal{O}_C)$ embeds into $H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$, and the condition of (ii) implies that $\widetilde{C}$ is numerically connected on a smooth surface, thus $\dim H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) = 1$ by a result of C.P. Ramanujam [6]; this contradicts $\dim \mathrm{Hom}(m_p, \mathcal{O}_C) = 2$. Q.E.D.

3 The linear system

$|2K-x-y|$. It is clear, since $P_2 = 3$, that for any two points $x, y \in S$ the linear system $|2K - x - y|$ is non-empty. We denote by Φ the tricanonical map Φ_{3K} and *assume that* Φ *is not birational.* Hence let x be a general point of S and let y be such that $\Phi(x) = \Phi(y)$. We denote by C a divisor in $|2K-x-y|$ and, in case $\dim |2K-x-y| = 1$, we also choose C as a general element.

Lemma 1. *We have* $h^1(\mathcal{O}_C(3K - x - y)) = 2$.

Proof. Since $\Phi(x) = \Phi(y)$ the cohomology sequences of

$$0 \to \mathcal{O}(3K - x - y) \to \mathcal{O}(3K) \to k_x \oplus k_y \to 0$$
$$0 \to \mathcal{O}(3K - C) \to \mathcal{O}(3K - x - y) \to \mathcal{O}_C(3K - x - y) \to 0$$

yield

$$h^1(\mathcal{O}_C(3K - x - y)) = h^1(\mathcal{O}(3K - x - y)) + h^2(\mathcal{O}(3K - C))$$
$$= 1 + h^2(\mathcal{O}(K)) = 2,$$

because $h^1(\mathcal{O}(3K - C)) = h^1(\mathcal{O}(K)) = 0$. Q.E.D. □

The curve C may be reducible and we shall denote by Γ an irreducible component containing x or y. We then have that $K\Gamma \geqq$

2, because $K\Gamma \leqq 1$ would imply $\Gamma^2 < 0$ by the Index Theorem, while there are only finitely many such curves on S ([CM], page 176). Since $KC = 4$ we deduce that either x and y lie on only one component Γ of C or that they belong to exactly two components. **284** Thus we distinguish two cases:

(A) $C = \Gamma_1 + \Gamma_2 + M$, where $K\Gamma_i = 2$, $\Gamma_i^2 \geqq 0$, $KM = 0$ and $x \in \Gamma_1, y \in \Gamma_2$ (possibly $\Gamma_1 = \Gamma_2$);

(B) $C = \Gamma + M$ where $x, y \in \Gamma$ and $x, y \notin M$.

Lemma 2. *If case* (A) *holds then* $C = C_1 + C_2$ *with* $C_1 \sim C_2 \sim K$, $\Gamma_1 \leqq C_1, \Gamma_2 \leqq C_2$ *and either*

(a) $C_1 \neq C_2$, *in which case* Γ_1 *and* Γ_2 *intersect transversally exactly at* x *and* y, *or*

(b) $C_1 = C_2$.

Proof. We shall prove Lemma 2 in several steps.

Step 1. x and y belong to both Γ_1 and Γ_2.

In order to prove this, we shall assume that $x \notin \Gamma_2$ and derive a contradiction.

First of all, we have

$$h^0(\mathcal{O}(3K - \Gamma_2 - M - x)) = h^0(\mathcal{O}(3K - \Gamma_2 - M)) - 1.$$

In fact, since $x \notin \Gamma_2 + M$, if the above assertion were not true then we would have

$$h^0(\mathcal{O}(K + \Gamma_1 - x)) = h^0(\mathcal{O}(K + \Gamma_1))$$

i.e., x would be a base point of $|K + \Gamma_1|$. Since $p_g = 0$, the restriction map gives an isomorphism

$$H^0(\mathcal{O}(K + \Gamma_1)) \xrightarrow{\sim} H^0(\omega_{\Gamma_1})$$

and noting that $|K + \Gamma_1|$ is non-empty ($p(\Gamma_1) > 0$) and that Γ_1 is never a component of elements of $|K + \Gamma_1|$ (since $p_g = 0$) we deduce that x is a base point of $H^0(\omega_{\Gamma_1})$; this however contradicts Proposition B.

Next, we note that

$$h^i(\mathcal{O}(3K - \Gamma_2 - M)) = 0$$

for $i = 1, 2$, because

$$h^i(\mathcal{O}(3K - \Gamma_2 - M)) = h^i(\mathcal{O}(K + \Gamma_1)) = h^{2-i}(\mathcal{O}(-\Gamma_1)) = 0$$

285 for $i = 1, 2$ (in fact, Γ_1 is connected and S has irregularity 0; then use ([CM], page 177). It now follows from the cohomology sequence of

$$0 \to \mathcal{O}(3K - \Gamma_2 - M - x) \to \mathcal{O}(3K - \Gamma_2 - M) \to k_x \to 0$$

that

$$h^i(\mathcal{O}(3K - \Gamma_2 - M - x)) = 0 \quad \text{for} \quad i = 1, 2.$$

Using this result and the cohomology sequence of

$$0 \to \mathcal{O}(3K - \Gamma_2 - M - x) \to \mathcal{O}(3K - x - y) \to \mathcal{O}_{\Gamma_2 + M}(3K - y) \to 0$$

we deduce that

$$h^1(\mathcal{O}_{\Gamma_2+M}(3K - y)) = h^1(\mathcal{O}(3K - x - y)) = 1,$$

the latter equality because $\Phi(x) = \Phi(y)$ ([CM], p. 187).

By Grothendieck's duality we get

$$\dim \mathrm{Hom}(\mathcal{O}_{\Gamma_2+M}(\Gamma_1) \cdot m_y, \mathcal{O}_{\Gamma_2+M}) = 1.$$

If $\pi : \widetilde{S} \to S$ gives an embedded resolution of singularities of $\Gamma_2 + M$ at y and if $\widetilde{\Gamma}_2 + M$ is the corresponding curve on $\widetilde{S}$ then denoting by $\mathcal{L}$ the sheaf

$$\mathcal{L} = \begin{cases} \mathcal{O}_{\widetilde{\Gamma}_2+M}(-\pi^*\Gamma_1) & \text{if } y \text{ is singular} \\ \mathcal{O}_{\widetilde{\Gamma}_2+M}(-\Gamma_1 + y) & \text{if } y \text{ is simple} \end{cases}$$

we obtain $H^0(\widetilde{\Gamma}_2 + M, \mathscr{L}) \neq (0)$, as one can see using Lamme B′. Since $\deg \mathscr{L} \leqq 1 - \Gamma_1(\Gamma_2 + M) < 0$ (note that $\Gamma_1 + \Gamma_2 + M \in |2K|$ is 2-connected, [CM], p. 181) we can apply Proposition A and obtain a decomposition

$$\widetilde{\Gamma}_2 + M = A + B$$

where

$$\begin{aligned} AB &\leqq \deg_B(\mathscr{L} \otimes \mathscr{O}_B) \\ &\leqq 1 - B\pi^*\Gamma_1. \end{aligned}$$

This leads to a contradiction: in fact, let $\pi^*\Gamma_2 = \widetilde{\Gamma}_2 + H$; if $\widetilde{\Gamma}_2 \subset B$ **286** we get $(B + H)(A + \pi^*\Gamma_1) \leq 1$, while if $\widetilde{\Gamma}_2 \subset A$, then $B(A + H + \pi^*\Gamma_1) \leqq 1$ and in both cases one violates the 2-connectedness of $\Gamma_1 + \Gamma_2 + M$ ([CM], p. 81).

Step 2. The cohomology sequence of

$$0 \to \mathscr{O}(3K - \Gamma_2 - M) \to \mathscr{O}(3K - x - y) \to \mathscr{O}_{\Gamma_2 + M}(3K - x - y) \to 0$$

together with

$$h^i(\mathscr{O}(3K - \Gamma_2 - M)) = h^i(\mathscr{O}(K + \Gamma_1)) = h^{2-i}(\mathscr{O}(-\Gamma_1)) = 0$$

for $i = 1, 2$ gives

$$h^1(\mathscr{O}_{\Gamma_2 + M}(3K - x - y)) = 1.$$

By Grothendieck's duality one deduces

$$\dim \operatorname{Hom}(\mathscr{O}_{\Gamma_2 + M}(\Gamma_1)m_x m_y, \mathscr{O}_{\Gamma_2 + M}) = 1$$

and, again by Lemma B′, we find that if $\pi : \widetilde{S} \to S$ is an embedded resolution of singularities of $\Gamma_2 + M$ *st* x and y then $H^0(\widetilde{\Gamma}_2 + M, \mathscr{L}) \neq (0)$ where $\mathscr{L}$ is the sheaf

$$\mathscr{L} = \mathscr{O}_{\widetilde{\Gamma}_2 + M}(-\pi^*\Gamma_1 + ax + by)$$

where a, $b = 1$ or 0 according as whether x, y are simple or singular on Γ_2. Since $\Gamma_1 + \Gamma_2 + M$ is 2-connected we have

$$\deg \mathscr{L} \leqq a + b - 2.$$

Two cases can occur:

(A) $\deg \mathscr{L} = 0$.

Then $a = b = 1$ and x, y are simple on Γ_2 and $\Gamma_1(\Gamma_2 + M) = 2$; by Lemma 2 of [CM], p. 181 this implies

$$\Gamma_2 \sim \Gamma_1 + M \sim K.$$

Moreover in this case $\Gamma_1\Gamma_2 = 2$ hence if $\Gamma_1 \neq \Gamma_2$ the two curves Γ_1, Γ_2 intersect transversally exactly at x and y.

(B) $\deg \mathscr{L} < 0$.

Now we apply Proposition A and deduce that there is a decomposition

$$\widetilde{\Gamma}_2 + M = A + B$$

287 where

$$\begin{aligned} AB &\leqq \deg_B(\mathscr{L} \otimes \mathscr{O}_B) \\ &\leqq -B\pi^*\Gamma_2 + a + b. \end{aligned}$$

This however implies, exactly as at the end of Step 1, that $a = b = 1$ and

$$(A + \Gamma_2)B = 2,$$

whence by Lemma 2 of [CM], p. 181 one finds again that x, y are simple points of Γ_2 and

$$A + \Gamma_2 \sim B \sim K.$$

Since $KB = 2$ this implies that Γ_1 is a component of B, x, y are simple on Γ_1 and $\Gamma_1\Gamma_2 = 2$. Finally, if $\Gamma_1 = \Gamma_2$ and if $B = B' + \Gamma_1$ then $A + \Gamma_2 \sim B' + \Gamma_1$, hence $A \sim B'$ and $A = B'$ ([CM], p. 175). Q.E.D. □

Lemma 3. *Case* (A) *does not hold.*

Proof. Let x, y and $C = C_1 + C_2 \in |2K - x - y|$ be as in Lemma 2. Recall that x, y was a general pair of points with $\Phi_{3K}(x) = \Phi_{3K}(y)$ and C a general element with $C \in |2K - x - y|$. By Lemma 2, there is a torsion class τ such that $C_1 \in |K + \tau|$; since the number of torsion classes is finite, and since x can be taken as a general point of the surface S we conclude that

$$\dim |K \pm \tau| \geqq 1.$$

Now $|K + \tau| + |K - \tau| \subseteq |2K|$ and $\dim |2K| = P_2 - 1 = 2$, thus we deduce that $|2K|$ is composite of a pencil and $|K + \tau| = |K - \tau|$, i.e. τ is a 2-torsion class. We have shown that $C_1, C_2 \in |K + \tau|$, since $x, y \in C_1$ by Lemma 2, we see that if C_2' is a general element of $|K + \tau|$ then $C_1 + C_2'$ is a general element in $|2K - x - y|$. By Lemma 2 again, we obtain that $x, y \in C_2'$, i.e. x, y are base points of $|K + \tau|$. This is plainly impossible because x was a general point on S. □

Lemma 4. *The points x, y are simple points of C.*

Proof. By Lemma 1, $h^1(\mathcal{O}_C(3K - x - y)) = 2$ and Grothendieck's **288**
duality yields

$$\dim \mathrm{Hom}(m_x m_y, \mathcal{O}_C) = 2.$$

Denoting by $\pi : \widetilde{S} \to S$ an embedded resolution of singularities of C at x and y, by Lemma B′ we see that x, y cannot both be singular, otherwise we would have $h^0(\mathcal{O}_{\widetilde{C}}) = 2$, while $\widetilde{C}$ is connected exactly as C (and now use Ramanujam's result in [CM], p. 177).

If, say, x is simple and y singular we get $h^0(\mathcal{O}_{\widetilde{C}}(x)) = 2$ and this implies that Γ, the component of C with $x \in \Gamma$, is a rational curve. This also is impossible, because S is of general type. Q.E.D. □

From now onwards we shall suppose that $C \in |2K - x - y|$ satisfies the requirements of Lemma 3 and Lemma 4, and write $h_C = \mathcal{O}_C(x + y)$ for the hyperelliptic sheaf of C.

Lemma 5. *We have* $\omega_C \cong h_C^{\otimes 6}$.

Proof. Obvious, because C is hyperelliptic. □

4 Proof of Theorem 1

As $P_3 = 7$ we have $\Phi = \Phi_{|3K|} : S \to I\mathbf{P}^6$. We write $V = \Phi(S)$. $d = \deg V$, $m = \deg \Phi$ and note that, since $|3K|$ has no base points ([5]Th. A; see also [2]Th. 5.1) we have

$$dm = (3K)^2 = 18.$$

Also $d \geqq 5$ because V is not contained in any hyperplane and V is not a curve, the latter because otherwise the general element of $|3K|$ would be decomposable in d components, while the curves D on S with $KD \leqq 1$ cannot move. This leads to the two cases $\deg \Phi = 2$ and $\deg \Phi = 3$.

Case I. $\deg \Phi = 2$.

Now Φ determines an involution σ on S, which is everywhere regular because S is of general type. We have

$$\sigma^* \mathcal{O}(K) \cong \mathcal{O}(K)$$

289 (this clearly holds for every surface of general type) and we remark that, C being as in Section III, we also have

$$\sigma(C) = C.$$

In fact, σ identifies pairs of points x, y such that $\Phi(x) = \Phi(y)$; hence the above remark.

Lemma 6. *We have* $\mathcal{O}_C(K) \cong h_C^{\otimes 2}$.

Proof. By Lemma 5 we have $\mathcal{O}_C(3K) = \omega_C \cong h_C^{\otimes 6}$ therefore it is sufficient to prove that $\mathcal{O}_C(8K) \cong h_C^{\otimes 16}$. In fact, since $|4K|$ is free

from base points, we can find a section $s \in H^0(S, \mathcal{O}(4K))$ with 16 zeros $a_1, \ldots, a_{16}$ on C. Then $\sigma^* s$ has 16 zeros $\sigma(a_1), \ldots, \sigma(a_{16})$ on C and the section s. $\sigma^* s$ of $\mathcal{O}(8K)$ has the zeros a_i, $\sigma(a_i)$ on C. It follows that the section $s\sigma^* s|_C$ of $\mathcal{O}_C(8K)$ has divisor

$$\operatorname{div}(s \cdot \sigma^* s|_C) = \sum_{i=1}^{16} (a_i + \sigma(a_i)).$$

Since each $a_i + \sigma(a_i)$ is the divisor of a section of h_C, the result follows. Q.E.D. □

By Lemma 6, we have an exact sequence

$$0 \to \mathcal{O}(-K) \to \mathcal{O}(K) \to h_C^{\otimes 2} \to 0$$

which implies $p_g = 3$, a contradiction. This settles Case I.

Case II. $\deg \Phi = 3$.

Let A be a general point on C and let

$$a + \overline{a} \in |h_C|.$$

Then $\Phi(\overline{a}) = \Phi(\overline{a})$ and, if a' is the third point with $\Phi(a) = \Phi(\overline{a}) = \Phi(a')$ then $a' \notin C$, as one immediately sees by considering $\overline{a}'$ in case that $a' \in C$. Consider $\Phi(C) = N$. Now $\Phi^{-1}(N)$ has a component $N' = \text{locus } (a')$; $\Phi|_{N'} : N' \to N$ is a birational map. However, points in $N' = \text{locus } (a')$ are parametrized by $|h_C| \cong \mathbf{P}^1$, because $a + \overline{a}$ determines a' uniquely; hence N' is rational and S contains a continuous family of rational curves, which is absurd. This settles Case II and completes the proof of Theorem 1.

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Geometry of Hecke Cycles - I

By M.S. Narasimhan and S. Ramanan

1 Introduction

In the course of our study of vector bundles over a smooth projective curve we considered [5] a correspondence between the space vector bundles of rank n and degree d on the one hand and that of vector bundles of rank n and degree $(1-d)$ on the other. This is the geometric analogue of the Hecke Correspondence which is defined in the case of a curve defined over a finite field. Let ξ be a line bundle on the curve X and $U(n,\xi)$ (resp. $U(n,\xi^{-1}X)$) be the moduli space of vector bundles on X of rank n with determinant isomorphic to ξ (resp. isomorphic to $\xi^{-1}\otimes L_x$, $x\in X$), $n\geq 2$. For a general vector bundle $E\in U(n,\xi)$, the subscheme corresponding to E in $U(n,\xi^{-1}X)$ under this correspondence is isomorphic to the projective bundle $P(E^*)$. We call these subschemes of $U(n,\xi^{-1}X)$ good Hecke cycles. This identifies a suitable open subset of $U(n,\xi)$ with an open subset of the Hilbert scheme of $U(n,\xi^{-1}X)$. Results of this type were announced in [5§ 8] and are proved here in § 5, which can be read independently of the rest of the paper.

In the case $n=2$, and ξ is trivial we prove that the irreducible component (which we shall call for convenience the Hecke component) of the Hilbert scheme of $U(2,X)$ containing the good Hecke cycles is smooth and provides a non-singular model for $U(2,\xi)$. This is the main result of the paper (Theorem 8.14).

Now the Kummer variety associated to the Jacobian J of X is the set of non-stable (and even singular if the genus $g\geq 3$)

points of $U(2,\xi)$. The possible elements of the Hecke component corresponding to points of the Kummer variety can be listed (§ 7) and they all turn out to be conic bundles over X. The fibre in the non-singular model over a non-nodal point k of the Kummer variety is isomorphic to $PH^1(X, j^2) \times PH^1(X, j^{-2})$, where $j \in J$ lies above k, and the corresponding subschemes in $U(2, X)$ can be described as the union of two projective line bundles on X corresponding to non-trivial extensions

$$0 \to j^{-1} \to E \to j \to 0$$
$$0 \to j \to E' \to j^{-1} \to 0$$

292 identified along the sections given by j^{-1} and j respectively. Over a node of the Kummer variety, the elements are trivial conic bundles contained in $X \times P(H^1(X, \mathcal{O}))_t$ (imbedded in $U(2, X)$), where $P(H^1(X, \mathcal{O}))_t$ is the thickening of $PH^1(X, \mathcal{O})$ corresponding to the universal quotient bundle Q of $PH^1(X, \mathcal{O})$. (See 4.4 iii). Thus we get here not only conics contained in $PH^1(X, \mathcal{O})$ but also lines in it which are thickened within this Q^*-thickening. (The latter will be referred to as 'outside thickenings'). It is proved that the Hilbert scheme is itself smooth at all these points except at 1) a pair of intersecting lines in $PH^1(X, \mathcal{O}), 2)$ double lines contained schematically in $PH^1(X, \mathcal{O})$ (§ 8). At points of the above type another component of the Hilbert scheme hits the Hecke component. We show that there is a natural morphism of the union of these two components into the Jacobian of X, which is constant on the Hecke component. (This morphism should be thought of as playing the role of the Weil morphism into the intermediary Jacobian). By studying the differential of this morphism we show that the Hecke component is smooth also at these points. (See Lemmas 8.10, 8.11).

The conics contained schematically in $PH^1(X, \mathcal{O})$ is a $\mathbf{P}^5$ bundle over the Grassmannian of planes in $PH^1(X, \mathcal{O})$. It will be shown in a later paper that the non-singular model considered above can be blown down along these fibrations (one for each

node) to another non-singular model.

It turns out that all the subschemes described above consist of bundles which are non-trivial extensions of line bundles of a particular kind. Such (triangular) bundles are parametrised by a projective bundle over $X \times J$. We have a morphism from this space into $U(2, X)$, which may also be looked upon as a family $\{P(D_j)\}_{j\in J}$ of smooth subvarieties of $U(2, X)$ parametrised by J, where each $P(D_j)$ is a projective bundle over X of dimension $(g-1)$. One of the essential points in the proof is to study the first and second order differentials of this morphism and in particular to compute its Hessian at the critical points (§ 6). The necessary **293** preliminaries for this are discussed in § 2. This study is necessary to prove the non-singularity of the Hilbert scheme at a point given by an 'outside thickening'. For this we need information about the conormal sheaf of the thickening $X \times PH^1(X, \mathcal{O})_t$ of $X \times PH^1(X\mathcal{O})$ in $U(2, X)$. Now this thickening is a special case of thickenings which arise in the study of a family of smooth subvarieties (in our case, the family $j \mapsto P(D_j)$ mentioned above). In this situation the conormal sheaf of the thickened scheme can be described in terms of the Hessian (see Lemma 3.7 and Remark 3.9).

A different approach for the desingularisation of $U(2, \xi)$ has been found by C.S. Seshadri [10].

Notation. All schemes will be of finite type over an algebraically closed field k of characteristic $\neq 2$. From § 5 on, X will denote a smooth projective curve of genus $g \geq 2$. If S is a subscheme of Pic X, $U(n, S)$ will denote the subscheme of the moduli scheme $U(n, d)$ of S-equivalence classes of semistable vector bundles of rank n of degree d obtained as the inverse image of S by the morphism det : $U(n, d) \to$ Pic. The Jacobian of X will be denoted J. If E is a family of semistable vector bundles over X, then there is a canonical morphisms θ_E of the parameter space into the moduli space.

If X is a subscheme of Y, we denote by $\check{N}_{X,Y}$ the conormal

sheaf of X in Y. In good cases, e.g. when X is regularly imbedded in Y, the sheaf $\check{N}_{X,Y}$ is locally free and its dual is the normal bundle denoted $N_{X,Y}$ so that in that case, $\check{N}_{X,Y} = N^*_{X,Y}$. If $\pi : X \to Y$ is a smooth morphism, we denote by T_π, the tangent bundle along the fibres of π. Its dual is sometimes denoted by Ω^1_π as well. If x is a (closed) point of X, the tangent space at x is denoted T_x.

If D is a Cartier divisor in a scheme X, then L_D will denote the line bundle defined by it. If L is a line bundle generically generated by sections, then the quotient sheaf of $H^0(X, L)^*_X$ by the subsheaf L^* will be called the quotient sheaf of the linear system defined by L. If $\mathfrak{F}$ is a coherent sheaf on a closed subscheme $i : Y \to X$, the sheaf $i_*(F)$ will be denoted by $\widetilde{\mathfrak{F}}$.

294 If $\pi : X \to Y$ is a projective morphism with a relatively ample sheaf then $\mathrm{Hilb}(X, Y, P)$ will denote the relative Hilbert scheme over Y with Hilbert polynomial P. If $Y = \mathrm{Spec}\, k$, we simply write $\mathrm{Hilb}(X, P)$.

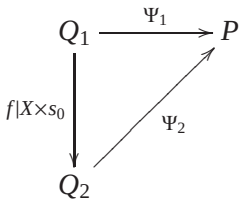
The projections from $X \times Y \times Z$ to the component schemes will be denoted p_1, p_2, p_3, p_{12}, p_{23}, $p_{1,3}$. When we are dealing with closed subschemes or closed imbeddings we usually omit the word "closed".

2 Deformations of Principal bundles

We wish to recall certain facts concerning deformations of locally trivial principal G-bundles, where G is an algebraic group, and in particular study the second order infinitesimal deformations of such bundles.

Definition 2.1. *Let P be a principal G-bundle on X and (S, s_0) a pointed scheme. A deformation of P parametrised by S is a principal G-bundle Q over $X \times S$, and an ismorphism Ψ of $Q|X \times s_0$ with P. Two deformations Q_1, Q_2 are said to be equivalent if for every $s \in S$, there exists a neighbourhood U and an isomorphism*

$f : Q_1|X \times U \to Q_2|X \times U$ *such that the diagram*



commutes if $s_0 \in U'$.

Remark 2.2. (i) If the only automorphisms of P are given by morphisms of X into the centre of G, then it is clear that the equivalence class of a deformation Q of P is independent of Ψ.

(ii) If P is a principal G-bundle on X, then we have obviously the deformation functor δ_P on the category of Artinian local algebras which associates to A, the equivalence class deformations of P parametrised by A. This functor can be seen to satisfy the conditions H_1 and H_2 of Schlessinger [7, Theorem 2.11]. If X is proper k, it satisfies H_3 as well. Moreover, if P is such that $H^0(X, Z_X) \to H^0(X, \mathrm{Ad}\, P)$ is an isomorphism, where Z denotes the centre of the Lie algebra of G, then δ_P also satisfies H_4 and hence is prorepresentable. We proceed to describe this functor a little more explicitly.

Proposition 2.3. *Let S be an Artinian local algebra, and s_0 its* **295**
closed point. Then $\delta_P(S)$ is canonically bijective with the set $H^1(X, N)$ where N is the sheaf associated to the group scheme $N = \ker p_* p^* G(\overline{P}) \to G(\overline{P})$, where this map is given by restriction to $X \times s_0$, $G(P)$ is the group scheme Aut P *and $p : X \times S \to X$ is the projection.*

Proof. The set of principal G-bundles on $X \times S$ are in $(1, 1)$ correspondence with the set $H^1(X \times S, p^*G(P)) \simeq H^1(X, p_* p^* G(P))$.

Thus any deformation of P gives rise to an element in $H^1(X, p_*p^*G(P))$ which is in the 'kernel' of $H^1(X, p_*p^*G(P)) \to H^1(X, G(P))$. But the prescription of Ψ allows one to describe the deformation as a 1-cocycle for $p_*p^*G(P)$ and a 0-cochain for $G(P)$ of which its image in $G(P)$ is the coboundary. This proves the assertion. □

Proposition 2.4. *Let* $(A, \mathbf{m})$ *be an Artinian local algebra with* $\mathbf{m}^2 = 0$ *(resp.* $\mathbf{m}^3 = 0$*). Then the group scheme* N *of* 2.3 *is isomorphic to the vector bundle* $\mathrm{Ad}\,P \otimes \mathbf{m}$ *(resp. the group scheme* $\mathrm{Ad}\,P \otimes \mathbf{m}$*, the group structure being defined by* $(X, Y) \to X + Y + \frac{1}{2}[X, Y]$*).*

Proof. The exponential map yields a bijection of $\mathrm{Ad}\,P \times \mathfrak{m}$ onto N, and the transfer of the group structure on N to $\mathrm{Ad}\,P \otimes \mathbf{m}$ is as given above, by virtue of the Campbell-Hausdorff formula [9, SGA 3, Expose VII, 3.1]. □

Remark 2.5. If the field k is of characteristic 0, we have an obvious generalisation of 2.4 to all Artinian local algebras using the Campbell-Hausdorff formula.

Example 2.6. Let A be the algebra $k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)$. Then $H^1(X, \mathrm{Ad}\,P \otimes \mathbf{m})$ can be described with respect to a suitable open covering (U_i) of X as follows. Any 1-cochain can be described by $f_{ij}\epsilon_1 + g_{ij}\epsilon_2 + h_{ij}\epsilon_1\epsilon_2$ where f_{ij}, g_{ij}, h_{ij} belong to $H^0(U_i \cap U_j, \mathrm{Ad}\,P)$. The cocycle condition with respect to the above group structure on $\mathrm{Ad}\,P \otimes \mathfrak{m}$ is that, for every i, j, k, we have

$$f_{ik}\epsilon_1 + g_{ik}\epsilon_2 + h_{ik}\epsilon_1\epsilon_2 = (f_{ij}\epsilon_1 + g_{ij}\epsilon_2 + h_{ij}\epsilon_1\epsilon_2)(f_{jk}\epsilon_1 + g_{jk}\epsilon_2 + h_{jk}\epsilon_1\epsilon_2)$$
$$= (f_{ij} + f_{jk})\epsilon_1 + (g_{ij} + g_{jk})\epsilon_2 + \{\frac{1}{2}([f_{ij}, g_{jk}] + [g_{ij}, f_{jk}]) + h_{ij} + h_{jk}\}\epsilon_1\epsilon_2$$

or, what is the same, f_{ij}, g_{ij} are 1-cocycles of $\mathrm{Ad}\,P$, and h_{ij} satisfies

$$h_{ij} + h_{jk} = h_{ik} - \frac{1}{2}([f_{ij}, g_{jk}] + [g_{ij}, f_{jk}]). \tag{2.7}$$

296 On the other hand, two cocycles (f_{ij}, g_{ij}, h_{ij}) and $(f'_{ij}, g'_{ij}, h'_{ij})$ are

cohomologous if and only if there exist sections (a_i, b_i, c_i) over U_i of ad P with

$$(f_{ji}\epsilon_1 + g_{ij}\epsilon_2 + h_{ij}\epsilon_1\epsilon_2)(a_j\epsilon_1 + b_j\epsilon_2 + c_j\epsilon_1\epsilon_2)$$
$$= (a_i\epsilon_1 + b_i\epsilon_2 + c_i\epsilon_1\epsilon_2) \times \times (f'_{ij}\epsilon_1 + g'_{ij}\epsilon_2 + h'_{ij}\epsilon_1\epsilon_2);$$

namely, $f'_{ij} - f_{ij}$ is the coboundary of (a_i) in $\mathrm{Ad}\, P$, $g'_{ij} - g_{ij}$ is the coboundary of (b_i), and

$$h'_{ij} - h_{ij} = c_j - c_i + \frac{1}{2}([f_{ij}, b_j] + [g_{ij}, a_j] - [b_i, f'_{ij}] - [a_i, g'_{ij}]) \quad (2.8)$$

2.9 Hessian.

Let f be a morphism of a smooth scheme X into a scheme Y and $df : T_x \to T_y$, $y = f(x)$ be its differential. Then the *Hessian* $h(f)$ of f is defined to be a map $\ker df \otimes T_x \to \operatorname{coker} df$. It is more convenient to define the dual map $\check{h}(f) : \ker(\check{df}) \otimes T_x \to \operatorname{coker} \check{df}$. Let $(\mathscr{O}_x, \mathbf{m}_x)$, $(\mathscr{O}_y, \mathbf{m}_y)$ be the local rings at x, y. If $a \in \mathbf{m}_y$, with $a \circ f \in \mathbf{m}_x^2$ and $t \in T_x$, then we get an element of T_y, by contracting with t the element in $S^2(\mathbf{m}_x/\mathbf{m}_x^2) = \mathbf{m}_x^2/\mathbf{m}_x^3$ given rise to by $a \circ f$. Its image in $\operatorname{coker} df$ depends only on the class of a modulo $\mathfrak{m}_y^2$ and is defined to be $\check{h}(f)(a, t)$.

2.10 Functorial Description of Hessian.

In terms of A-valued points, the Hessian can be described as follows. Consider the ring

$$A = k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)$$

and the quotient rings B_1, B_2 and C given by the ideals (ϵ_2), (ϵ_1) and $(\epsilon_1\epsilon_2)$. Giving vectors a, t in $\ker df$ and T_x respectively is the same as giving a C-valued point p of X at x such that the corresponding B_1-valued point is mapped by f on the 0-vector at y. Let q be an A-valued point extending p; there exists one such since X is smooth. By assumption, the image $f(q)$ actually

yields a $(k + k\epsilon_2 + k\epsilon_1\epsilon_2)$-valued point at $y \in Y$. By restriction to $\mathrm{Spec}(k + k\epsilon_1\epsilon_2)$, we get a vector at y. Its image modulo Image (df) is independent of the extension q of p and gives $h(f)(a, t)$.

Remark 2.11. The above definition of Hessian in terms of A-valued points enables one to define the Hessian of a morphism of a smooth, prorepresentable functor on the category of Artinian local algebras into another prorepresentable functor. In particular,
297 if P is a principal G-bundle with $H^2(X, \mathrm{Ad}\, P) = 0$, then it is easy to check that the functor δ_P defined in 2.2, (ii) is smooth. Hence if P satisfies in addition the conditions of 2.2, (ii), then for any homomorphism of G into another algebraic group H, the notion of Hessian of the morphism $\delta_P \to \delta_Q$ makes sense, where Q is the principal H-bundle associated to P. We proceed to compute this in the case when $G \subset H$.

Proposition 2.12. (i) *Let P be a principal G-bundle on a scheme X, proper over k. For any homomorphism $G \to H$, the induced morphism $e_{G,H} : \delta_P \to \delta_Q$ of the deformation functors has as differential, the natural map $H^1(X, \mathrm{Ad}\, P) \to H^1(X, \mathrm{Ad}\, Q)$, where Q is the principal H-bundle associated to P.*

(ii) *If $H^2(X, \mathrm{Ad}\, P) = 0$ and $G \subset H$, then the cokernel of $de_{G,H}$ is canonically isomorphic to $H^1(X, E)$ where E is the vector bundle associated to P for the linear isotropy action of G on the tangent space at e of H/G.*

(iii) *Assume that Q (and hence P) satisfies the condition in 2.2,* (ii). *Under the identification in* (ii), *the Hessian of $e_{G,H}$ can be described as follows. Let $t_1 \in \ker de_{G,H}$ and $t_2 \in H^1(X, \mathrm{Ad}\, P)$. Then t_1 is in the image under the boundary homomorphism of an element $s_1 \in H^0(X, E)$ associated to the exact sequence*

$$0 \to \mathrm{Ad}\, P \to \mathrm{Ad}\, Q \to E \to 0.$$

Then $h(t_1, t_2) = [s_1, t_2]$, *where the bracket is the cup product associated to the natural action of* $\operatorname{Ad} P$ *on* E.

Proof. (i) is obvious from 2.4.

(ii) is a trivial consequence of (i) and the cohomology sequence of $0 \to \operatorname{Ad} P \to \operatorname{Ad} Q \to E \to 0$.

(iii) We will use the notation of 2.10. By the description of the Hessian given there and of the functors δ_P, δ_Q given in 2.4, $h(t_1, t_2)$ is obtained as follows. The vectors t_1, t_2 give an element of $H^1(X, \operatorname{Ad} P \otimes \mathfrak{m}/(\epsilon_1, \epsilon_2))$ where $\mathfrak{m}$ is the maximal ideal of A. This is the image of an element in $H^1(X, \operatorname{Ad} P \otimes \mathfrak{m})$, the group structure in $\operatorname{Ad} P \otimes \mathfrak{m}$ being given in 2.4. Its image in $H^1(X, \operatorname{Ad} Q \otimes \mathfrak{m})$ goes to zero in **298**
$H^1(X, \operatorname{Ad} Q \otimes \mathfrak{m}/(\epsilon_2))$ and hence comes from an element in $H^1(X, \operatorname{Ad} Q \otimes (\epsilon_2))$. The image of this element in $H^1(X, E \otimes (\epsilon_2)/(k\epsilon_2))$ is $h(t_1, t_2)$. In terms of cocycles for a suitable covering (U_i) of X (as in 2.6) this means the following. Let f_{ij}, g_{ij} be cocycles representing t_1, t_2, in $H^1(X, \operatorname{Ad} P)$. Then there exists $h_{ij} \in H^0(U_i \cap U_j, \operatorname{Ad} P)$ such that (f_{ij}, g_{ij}, h_{ij}) satisfy 2.7. Reading these as sections over $U_i \cap U_j$ of $\operatorname{Ad} Q$, we get a corresponding element of $H^1(X, \operatorname{Ad} Q \otimes \mathfrak{m})$. We are given that f_{ij} is a coboundary for $\operatorname{Ad} Q$, namely, there exist $\lambda_i \in H^0(U_i, \operatorname{Ad} Q)$ with $f_{ij} = \lambda_j - \lambda_i$. Then the cocycle (f_{ij}, g_{ij}, h_{ij}) is cohomologous by 2.8 to $(0, g_{ij}, h_{ij} + \frac{1}{2}([\lambda_i, g_{ij}] - [g_{ij}, \lambda_j])$, taking $(a_i, b_i, c_i) = (\lambda_i, 0, 0)$. Thus the element in $H^1(X, \operatorname{Ad} Q \otimes (k\epsilon_2 + k\epsilon_1\epsilon_2))$ is given by the cocycle $g_{ij}\epsilon_2 + h_{ji} + \frac{1}{2}([\lambda_1 g_{ij}] - [g_{ij}, \lambda_j])\epsilon_1\epsilon_2$. Finally, the Hessian $h(t_1, t_2)$ is given by the cocycle $\frac{1}{2}([\lambda_i g_{ij}] - [g_{ij}, \lambda_j])]$ of E, since h_{ij} is a section of $\operatorname{Ad} P$. Notice that since $\lambda_i - \lambda_j$ is a section of $\operatorname{Ad} P$, $\lambda_i = \lambda_j$ as sections of E on $U_i \cap U_j$ and hence (λ_i) determines a section λ of E. Clearly λ maps on (f_{ij}) under the boundary homomorphism and the cocycle $\frac{1}{2}([\lambda, g_{ij}] - [g_{ij}, \lambda]) = [\lambda, g_{ij}]$ represents the cup product of λ and the class of g_{ij} as was to be proved.

□

Remarks 2.13. (i) We will apply Proposition 2.12 to the case when G is 2×2 triangular (Borel) subgroup of $H = GL(2, k)$, and P is a principal bundle over the curve X. Then P is described by an exact sequence

$$0 \to L_1 \to W \to L_2 \to 0,$$

where L_1 and L_2 are line bundles. If W is simple, then the conditions in Prop. 2.12 are satisfied. The bundle $\mathrm{Ad}\, P$ is the bundle $\Delta(W)$ of endomorphisms of W leaving L_1 invariant, $\mathrm{Ad}\, Q = \mathrm{End}\, W$, and the vector bundle E can be identified with $\mathrm{Hom}(L_1, L_2)$. Moreover, the bundle $\mathrm{Ad}\, P$ consisting of endomorphisms of trace 0 is isomorphic to $\mathrm{Hom}(L_2, W)$. In particular, we have a map η of $\mathrm{Ad}\, P$ onto the sheaf $\mathscr{O}_X$, obtained by composing with the map $W \to L_2$. With these identifications, the action of $\mathrm{Ad}\, P$ on E is simply multiplication by its image in $\mathscr{O}$. Thus if $t_1 \in \ker H^1(X, \mathrm{Ad}\, P) \to H^1(X, \mathrm{Ad}\, Q)$, and $t_2 \in H^1(X, \mathrm{Ad}\, P)$, then $h(t_1, t_2) \in H^1(X, E)$ is the cup product of $\eta t_2 \in H^1(X, \mathscr{O})$
299 and the element $s_1 \in H^0(X, \mathrm{Hom}(L_1, L_2))$ of which t_1 is the image. In other words, $h(t_1, t_2)$ is simply the image of $\eta t_2 \in H^1(X, \mathscr{O})$ in $H^1(X, \mathrm{Hom}(L_1, L_2))$ by the map $\mathscr{O} \to \mathrm{Hom}(L_1, L_2)$ given by t_1.

(ii) If global fine moduli schemes for principal G and H bundles exist, then clearly Proposition 2.12 enables one to compute the Hessian of the morphism $e_{G,H}$ induced on the moduli schemes by extension of structure group.

3 Thickenings

Let X be a scheme and $\mathscr{F}$ a coherent sheaf of $\mathscr{O}$-Modules on X. Let $\varphi : \mathscr{F} \to \Omega^1_X$ be a surjective $\mathscr{O}_X$-homomorphism of $\mathscr{O}_X$-Modules. Then one can define a ring structure on the subsheaf of

$\mathcal{O} \oplus \mathcal{F}$ consisting of $\{(f, x) : df = \varphi x\}$. This is a subsheaf $\mathcal{O}_\varphi$ of rings if we consider $\mathcal{O} \oplus \mathcal{F}$ as an $\mathcal{O}$-Algebra with $\mathcal{F}^2 = 0$.

Lemma 3.1. *$(X, \mathcal{O}_\varphi)$ is a scheme.*

Proof. If $X = \operatorname{Spec} A$ is affine, then $(X, \mathcal{O}_\varphi) \simeq \operatorname{Spec}(M + {}_\varphi\Omega^1)$ where $\widetilde{M} = \mathcal{F}$. This shows in general that $(X, \mathcal{O}_\varphi)$ is a prescheme and since $(X, \mathcal{O}_\varphi)_{\text{red}} = (X, \mathcal{O})_{\text{red}}$ is a scheme, the lemma is proved. □

If $I = \ker \varphi$, then we have an exact sequence of $\mathcal{O}_\varphi$-Modules (I being considered as an $\mathcal{O}_\varphi$-Module)

$$0 \to I \to \mathcal{O}_\varphi \to \mathcal{O} \to 0.$$

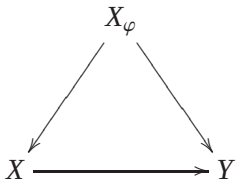
Lemma 3.2. *Let X be a smooth irreducible scheme. The scheme $X_\varphi = (X, \mathcal{O}_\varphi)$ is Cohen-Macaulay if and only if I is a locally free $\mathcal{O}_X$-Module. In this case, the dualising sheaf of X_φ restricts to X as $\underline{\operatorname{Hom}}(I, \omega_X)$. Moreover, X_φ is a local complete intersection if and only if I is locally free of rank* 1 *or* 0.

Proof. The problem being local, we may assume that $X = \operatorname{Spec} A$ and $X_\varphi = \operatorname{Spec} A_\varphi$, and since Ω^1_X is locally free, we may also assume that $A_\varphi = A \oplus I$, with $I^2 = 0$. In the local case, an A-sequence is an A_φ-sequence if and only if it is an I-sequence as well. Hence the first assertion. Now, if I is free, then $A_\varphi = A \otimes (k \oplus V)$ with $V^2 = 0$, where V is a finite dimensional k-vector space. Thus, our assertion on the dualising sheaf needs only to be proved when $X_\varphi = \operatorname{Spec}(k \otimes V)$. In this case, the dualising sheaf is seen to be actually $k \oplus V^*$, in which V operates trivially on k and by duality on V^*. To prove the last assertion, we note that if **300**
A_φ is a complete intersection, then so is $k \oplus V$, and hence $k \oplus V^*$ is free over $k \oplus V$. This clearly implies that $\dim V \leq 1$, while, on the other hand, it is obvious that $\dim V = 1$ implies $k \oplus V$ is a complete intersection. □

Remark 3.3. Let X be a smooth subscheme of a scheme Y. Then we have an exact sequence

$$0 \to \check{N}_{X,Y} \to \Omega^1_Y|_X \to \Omega^1_X \to 0.$$

If $\check{N}_{X,Y} \to I$ is an $\mathcal{O}_X$-homomorphism into an $\mathcal{O}_X$-Module I, then by the push-out construction, we obtain a sheaf $\mathcal{F}$ and a surjective homomorphism $\varphi : \mathcal{F} \to \Omega^1_X$. In particular, this gives rise to the scheme X_φ. Moreover, there is a natural morphism $X_\varphi \to Y$ making the diagram



commutative. It is easy to see that this morphism $X_\varphi \to Y$ is a closed immersion if and only if $\check{N}_{X,Y} \to I$ is surjective. (In this case, the thickened subscheme X_φ of Y will be denoted by X_I). In other words, all the thickenings of X 'inside' Y are given by Quot $\check{N}_{X,Y}$. In the case when $Y = Z_\Psi$, with Z smooth and ψ a surjective homomorphism $\mathcal{F} \to \Omega^1_Z$, then $\check{N}_{X,Y}$ is the kernel of the map $\mathcal{F}|X \to \Omega^1_X$, obtained as the composite of $\psi|X$ and the map $\Omega^1_Z|X \to \Omega^1_X$.

Lemma 3.4. *Let X be a smooth subscheme of a smooth scheme Z, and let Q be a locally free quotient of $N^*_{X,Z}$. If $\eta : N^*_{X,Z} \to Q$ is the canonical map, then the restriction of $N_{X_Q,Z}$ to X is locally free and fits in the exact sequence*

$$0 \to S^2(Q) \to N_{X_Q,Z}|X \to \ker \eta \to 0.$$

Proof. Let I (resp. J) be the ideal sheaf of X (resp. X_Q) in Z. We may choose local parameters $(x_1, \ldots, x_l, y_1, \ldots, y_m, z_1, \ldots, z_n)$ at a point of X so that

$$I = \{x_1, \ldots, x_l, y_1, \ldots, y_m\} \quad \text{and} \quad J = \{x_i x_j, y_1, \ldots, y_n\}, i, j, = 1, \ldots l.$$

301 From this it is clear that I^2/IJ is locally free and the natural map $S^2(Q) = S^2(I/J) \to I^2/IJ$ is an isomorphism. On the other hand $N_{X_Q,Z}|X \cong J|IJ$ fits in the exact sequence

$$0 \to I^2/IJ \to J|IJ \to J/I^2 \to 0.$$

Now $N^*_{X,Z} \cong I/I^2$ and the map η is the natural projection $I/I^2 \to I/J$ and hence $\ker \eta = J|I^2$, proving the lemma. □

The following functorial remark on the extension 3.4 is an immediate consequence of the definition.

Lemma 3.5. *If X is a smooth subscheme of a smooth scheme Z and $\eta_1 : N^*_{X,Z} \to Q_1$, $\zeta : Q_1 \to Q_2$ are surjections, then the extensions 3.4 corresponding to X_{Q_1} and X_{Q_2} fit in a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^2(Q_1) & \longrightarrow & N_{X_{Q_1},Z}|X & \longrightarrow & \ker \eta_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S^2(Q_2) & \longrightarrow & \check{N}_{X_{Q_2},Z}|X & \longrightarrow & \ker \zeta \circ \eta_1 & \longrightarrow & 0 \end{array}$$

Remark 3.6. We will not determine the extension involved in 3.4 in the following situation. Let $\varphi : Y \to Z$ be a morphism of smooth schemes and X a smooth subscheme of Y on which φ is an imbedding. Then the Hessian (see §2.9) of the map φ along X goes down to a map $K \otimes N_{X,Y} \to \operatorname{Coker} d\varphi$ where $K = \ker d\varphi$. Now $d\varphi$ induces a map $N_{X,Y} \to N_{X,Z}$ and let Q be the image of its transpose. We wish to consider the thickening of X in Z given by $\eta : N^*_{X,Z} \to Q$. It is easily seen that this thickening is in fact the schematic image (by φ) of the total normal thickening of X in Y.

Lemma 3.7. *In the situation of 3.6, we further assume that $K = \ker d\varphi$ has rank 1 at all points of X. Then the extension in Lemma 3.4 is the pull back by means of the transpose of the Hessian : $\ker \eta \to K^* \otimes N^*_{X,Y}$ of the exact sequence*

$$0 \to S^2(Q) \to S^2(N^*_{X,Y}) \to K^* \otimes N^*_{X,Y} \to 0$$

obtained by symmetrising the exact sequence

$$0 \to Q \to N^*_{X,Y} \to K^* \to 0.$$

Proof. We use the notation of 3.4. Let I' be the ideal of X in Y. Then notice that $f \in I \Rightarrow f \circ \varphi \in I'$ and $f \in J \Rightarrow f \circ \varphi \in I'^2$.
302 Hence we have a map $J/_{IJ} \to I'^2/I'^3 = S^2(N^*_{X,Y})$ induced by φ. The resulting map $J/I'^2 = (N/Q)^* \to K^* \otimes N^*_{X,Y}$ is easily seen to be the transpose of the Hessian. □

Putting together Lemmas 3.5 and 3.7, we obtain

Proposition 3.8. *In the situation of 3.7, let L be a locally free quotient of Q. Then the pullback of the extension 3.4 for X_L by the inclusion* $\ker\eta \to \ker(N^*_{X,Z} \to L)$ *is isomorphic to the pullback of the sequence*

$$0 \to S^2(L) \to S^2(E) \to K^* \otimes E \to 0$$

by the map $K^* \otimes N^*_{X,Y} \to K^* \otimes E$, where this latter extension is obtained by symmetrising the sequence

$$0 \to L \to E \to K^* \to 0,$$

E being the push-out $N^*_{X,Y} \amalg_Q L$.

Proof. From Lemma 3.5, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^2(L) & \longrightarrow & \check{N}_{X_L,Z}|X & \longrightarrow & \ker(N^*_{X,Z} \to L) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S^2(Q) & \longrightarrow & \check{N}_{X_Q,Z}|X & \longrightarrow & \ker\eta & \longrightarrow & 0
\end{array}$$

This shows that the required pullback is the push-out of the sequence

$$0 \to S^2(Q) \to \check{N}_{X_Q,Z}|X \to \ker\eta \to 0$$

by the map $S^2(Q) \to S^2(L)$. Now the proposition follows from the description of this sequence in Lemma 3.7 and the obvious commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^2(L) & \longrightarrow & S^2(E) & \longrightarrow & K^* \otimes E & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & S^2(Q) & \longrightarrow & S^2(N^*_{X,Y}) & \longrightarrow & K^* \otimes N^*_{X,Y} & \longrightarrow & 0 \end{array}$$

□

Remark 3.9. In our applications, we will be only considering the special case of (3.6), where we have $\psi : Y \to T$ is a proper smooth morphism and φ realises the fibres $\{X\}$ of ψ as a family of smooth subschemes of Z.

4 Conics

Definition 4.1. *A scheme C together with a very amply line bundle h is said to be a conic if its Hilbert polynomial is* $(2m + 1)$. *A* **303**
scheme C over T is said to be a conic over T *if with respect to a line bundle on C its fibres are conics and the morphism $C \to T$ is faithfully flat.*

We will show in 4.2 and 4.3 that this coincides with the usual notion of a conic.

Lemma 4.2. *If C is a reduced scheme and h a very ample line bundle on C with Hilbert polynomial of the form $2m + \delta$, $\delta \leq 1$, then $\delta = 1$, C is isomorphic to a (reduced) conic in $\mathbf{P}^2$, and h is the restriction to C of the hyperplane bundle on $\mathbf{P}^2$.*

Proof. It is obvious that we may assume without loss of generality that C is pure 1-dimensional. Clearly then, either C is irreducible, or C has two components C_1, C_2 with Hilbert polynomials $m + d_1$, $m + d_2$ with $d_1 + d_2 - l = \delta$, where l is the length

of the intersection $C_1 \cap C_2$. In the former case the normalisation of C has to be rational; for, otherwise, a linear system of degree 2 is at most one dimensional and cannot imbed C. Moreover, if C is not normal, the linear system of h on C will have dimenion ≤ 1, which cannot imbed C. Thus if C is irreducible, it must be isomorphic to $\mathbf{P}^1$ and the bundle h is the square of the hyperplane bundle on $\mathbf{P}^1$. In the case when C has two components C_1 and C_2, we conclude as above that C_1, C_2 are both isomorphic to $\mathbf{P}^1$ and h restricts to each of these as the hyperplane bundle. Thus we have $d_1 = d_2 = 1$. From the nature of the Hilbert polynomial, it is clear that C_1 and C_2 intersect. But $\dim H^0(C, h) = 4 - 1 \geq 3$ since h is very ample. Hence $l = 1$, *i.e.* C is a pair of intersection lines in $\mathbf{P}^2$. □

Lemma 4.3. *Let C be a nonreduced scheme and h an ample bundle on C with Hilbert polynomial of the form $2m + \delta$, $\delta \leq 1$. If $h|C_{\text{red}}$ is very ample, then C contains a unique subscheme C' which is obtained by thickening $\mathbf{P}^1$ by a line bundle L of degree ≤ -1. If degree $L = -1$, then L must be the dual of the hyperplane bundle and $C = C'$. In otherwords, C is isomorphic to the total thickening of a line in $\mathbf{P}^2$. In particular, a nonreduced conic (in the sense of 4.1) is such a thickened line.*

304 *Proof.* Clearly, we may assume C has no zero-dimensional components. Consider the exact sequence

$$0 \to I \to \mathcal{O} \to \mathcal{O}_{\text{red}} \to 0$$

on C, where I is the ideal of nilpotents in $\mathcal{O}$. If I has finite support, the Hilbert polynomial of $\mathcal{O}_{\text{red}}$ is of the same form and hence by 4.2, must be $2m + 1$. This shows that $\mathcal{O} = \mathcal{O}_{\text{red}}$, contrary to assumption. Thus the support of I is 1-dimensional. It follows that the Hilbert polynomials of I and $\mathcal{O}_{\text{red}}$ are of the form $m + d_1$, $m + d_2$. Since $h|C_{\text{red}}$ is very ample, we conclude as in 4.2 that C_{red} is isomorphic to $\mathbf{P}^1$, and h restricts to $\mathbf{P}^1$ as the hyperplane bundle. Now consider the filtration

$$\mathcal{O} \supset I \supset I^2 \ldots \supset I^n = (0).$$

Since the Hilbert polynomial $m + d_1$ of I is the sum of the Hilbert polynomials of the sheaves I^k/I^{k+1} on $\mathbf{P}^1$, it follows that $rk\, I/I^2 = 1$ and I^k/I^{k+1} are all torsion sheaves for $k \geq 2$. If L is the line bundle on $\mathbf{P}^1$ obtained as the free quotient of I/I^2, we get an L-thickening C' of $\mathbf{P}^1$ as a subscheme of C. Since now the Hilbert polynomial of $\mathcal{O}_{\text{red}}$ is $m+1$, the Hilbert polynomial of I is $m+\delta-1$ and hence that of L is $m + d$, with $d \leq 0$. In particular, the degree of $L \geq -1$. If now $\deg L = -1$, then L is the dual of the hyperplane bundle on $\mathbf{P}^1$ whose Hilbert polynomial is m, and hence that of C' is $2m + 1$. This proves that $\delta = 1$ and $C = C'$. It is clear that the L-thickening of $\mathbf{P}^1$ is the normal thickening of a line in $\mathbf{P}^2$. Finally, if h is already very ample on C, then from the exact sequence

$$0 \to L \to \mathcal{O}_{C'} \to \mathcal{O}_{\text{red}} \to 0$$

tensored with h, we obtain

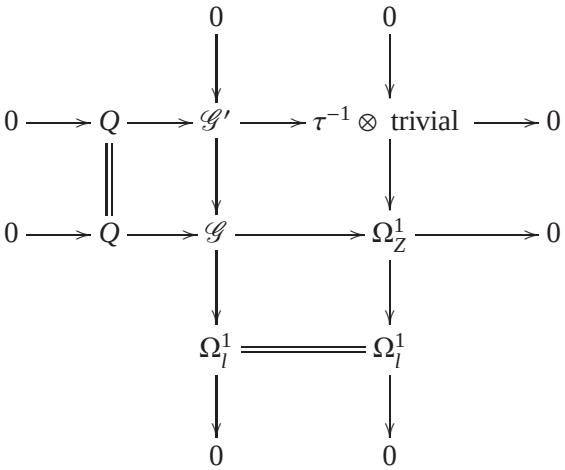
$$0 \to H^0(\mathbf{P}^1, L \otimes h) \to H^0(C', h) \to H^0(\mathbf{P}^1, h).$$

Since h is very ample on C', we have $\dim H^0(C', h) \geq 3$, and $H^0(\mathbf{P}^1, h)$ being 2-dimensional, it follows that $H^0(\mathbf{P}^1, L \otimes h) \neq 0$, *i.e.* $\deg(L \otimes h) \geq 0$, proving that $\deg L \geq -1$. □

Remarks 4.4. (i) From what we have seen above, it is clear **305**
that if (C, h) is a conic as in 4.1, then the linear system of h in C is 2-dimensional and imbeds C as a conic in $\mathbf{P}^2$. In particular, if C is a subscheme of $\mathbf{P}^n$ and C is a conic with respect to the restriction of the hyperplance bundle, then C is contained in a unique plane in $\mathbf{P}^n$ as a conic.

(ii) Also, if C is a subscheme of the one-point union of two projective spaces, and C is a conic with respect to the restriction of the natural very ample (hyperplane) bundle on this union, then C is either a conic in one of the projective spaces, or is a pair of lines, one in each of these spaces passing through their point of intersection.

(iii) Let Y be the thickening of a projective space Z by a surjection $\varphi : \mathscr{G} \to \Omega^1$, with $\ker \varphi$ isomorphic to the dual of the universal quotient bundle Q on Z. In this case, the hyperplane bundle on Z extends uniquely to a line bundle h on Y which is very ample. If C is a conic subscheme (with respect to h) of Y, then either (a) C is a subscheme of Z or (b) C_{red} is a line l in Z and C is obtained as a τ^{-1}-thickening of this line in Y. By 3.3, the latter is obtained as a quotient of $N^*_{l,Z}$ which is isomorphic to τ^{-1}. Consider the diagram on l:



Now $H^0(l, \mathrm{Hom}(Q, \tau^{-1}))$ is 1-dimensional, since $Q|l \simeq \tau^{-1} +$ trivial. Hence any map $\mathscr{G}' \to \tau^{-1}$ when restricted to Q must coincide (upto a scalar factor) with the natural map $Q|l \to \tau^{-1}$. If this map is zero, then the map $\mathscr{G}' \to \tau^{-1}$ factors through ($\tau^{-1} \otimes$ trivial) and in this case, C is actually contained in Z itself. All other τ^{-1} thickenings in Y are parametrised by an affine space of dim = codim of l in Z.

306 (iv) If C is a conic over T, by [9, SGA 6, VII], the morphism $\pi : C \to T$ is a complete intersection. Let ω be the relative

dualising sheaf over T. Then clearly C is imbedded as a T-scheme in the projective plane bundle S over T associated to $E = R^1(\pi)_*\omega^2$. Moreover, if C is imbedded as a T-scheme in a projective bundle P over T such that the fibres of π are imbedded as conics, then the inclusion $C \hookrightarrow P$ factors through an imbedding $S \subset P$, linear on fibres. On E, there is on everywhere nontrivial quadratic form with values in a line bundle L such that C is the associated conic bundle over T.

5 Good Hecke Cycles

If E is a vector bundle ($\neq 0$) on X and $k \in Z$, we denote by $\mu_k(E)$ the rational number $(\deg E + k)/rk\, E$.

Definition 5.1. *A vector bundle E on X is said to be (k, l)-stable (resp. (k, l)-semistable) if, for every proper subbundle F of E, we have*

$$\mu_k(F) < \mu_{-l}(E/F) (\textit{resp.}\ \mu_k(F) \leq \mu_{-l}(E/F)).$$

Remark 5.2. (i) The condition 5.1 is equivalent to $\mu_k(F) < \mu_{k-l}(E)$ or $\mu_{k-l}(E) < \mu_{-l}(E/F)$.

(ii) If E is (k, l)-stable and L any line bundle, then $E \otimes L$ is also (k, l)-stable.

(iii) If E is (k, l)-stable, then E^* is (l, k)-stable.

(iv) A vector bundle of degree 0 is stable if and only if it is $(0, 1)$-stable.

(v) A vector bundles of degree 1 is stable if and only if it is $(0, 1)$-semi-stable.

Proposition 5.3. *In any family E of vector bundles on X, parametrised by T, the set of (k, l)-stable points is open in T.*

Proof. Consider the Quot-scheme of E over T. The set of non-(k,l)-stable points is characterised as the union of the images of the components of the Quot scheme whose Hilbert polynomials satisfy an inequality which constrain them to vary over a finite set. Now our assertion follows from the properness of the Hilbert scheme over T, with a fixed Hilbert polynomial. □

307 **Proposition 5.4.** (i) *Except when $g = 2$, $n = 2$ and d odd, there always exist $(0,1)$-stable (and $(1,0)$-stable) bundles of rank n and degree d.*

(ii) *There exist $(1,1)$-stable bundles of rank n and degree d, except in the following cases.*

(a) $g = 3$, *and d both even*

(b) $g = 2$, $d \equiv 0, \pm 1 (\mathrm{mod}\ n)$

(c) $g = 2$, $n = 4$, $d \equiv 2 (\mathrm{mod}\ 4)$.

Proof. It is enough to estimate the dimension of the subvariety of $U(n,d)$, consisting of non-$(0,1)$-stable (resp. non-$(1,1)$-stable) points and prove that it is a proper subvariety. Clearly any such bundle E contains a subbundle F satisfying the inequality

$$\frac{\deg F}{rk\ F} \geq \frac{\deg E - 1}{rk\ E}$$

$$\text{(resp.)} \qquad \frac{\deg F + 1}{rk\ F} \geq \frac{\deg E}{rk\ E}$$

By using [5, Proposition 2.6] as in [5, Lemma 6.7] we may as well assume that F and E/F are stable and compute the dimension of such bundles E. The dimension of a component corresponding to a fixed rank r and degree δ of F, is majorised by $\dim U(r,\delta) + \dim U(n-r, d-\delta) + \dim H^1(X, \mathrm{Hom}(E/F, F)) - 1 = (g-1)(n^2 - nr + r^2) + 1 + (dr - \delta n)$. In order to show this is $< \dim U(n,d) = n^2(g-1) + 1$ we have only to verify that $(g-1)r(n-r) > dr - \delta n$. But this is a simple consequence of the inequality $dr - \delta n \leq r$ in the first case and $\leq n$ in the second case, taking into account the exceptions mentioned in the proposition. □

Lemma 5.5. *Let $x \in X$ and $0 \to \underline{E'} \to \underline{E} \to \mathscr{O}_X \to 0$ be an exact sequence of sheaves with $\underline{E}$, $\underline{E'}$ locally free. If E is (k,l)-stable then E' is $(k, l-1)$ stable. In particular, if E is $(0,1)$-stable then E' is stable. Similar statements are valid when stable is replaced by semistable.*

Proof. Let F be a subbundle on E' and F' the subbundle of E generated by the map $F \to E$. Then $F \to F'$ is of maximal rank and hence $\deg F' \geq \deg F$. Now $\mu_k(F) \leq \mu_k(F') < \mu_{k-l}(E) =$ **308**
$\mu_{k-l+1}(E')$ as $\deg E = \deg E' + 1$. This proves that E' is $(k, l-1)$ stable. □

Lemma 5.6. (i) *Let E be a $(0,1)$-stable vector bundle of rank n and E' a stable vector bundle of rank n and determinant isomorphic to $\det E \otimes L_x^{-1}$. If $f : E' \to E$ is a non-zero homomorphism, we have an exact sequence*

$$0 \to \underline{E'} \to \underline{E} \to \mathscr{O}_x \to 0.$$

(ii) *Moreover* $\dim H^0(X, \mathrm{Hom}(E', E)) \leq 1$.

Proof. The map f is of maximal rank. For, otherwise, let

$$\begin{array}{ccccc} E' & \longrightarrow & G' & \longrightarrow & 0 \\ & & \downarrow & & \\ E & \longleftarrow & G & \longleftarrow & 0 \end{array}$$

be the canonical factorisation of f, where $rk\, G' < n$ and $G' \to G$ is of maximal rank. We then have $\mu(G) \geq \mu(G') > \mu(E') = \mu_{-1}(E)$ which contradicts the $(0,1)$-stability of E. Now the induced map $\overset{n}{\wedge} f : \overset{n}{\wedge} E' \to \overset{n}{\wedge} E$ is non-zero and can vanish only at x (with multiplicity 1). Thus f is of maximal rank at all points except x and is of rank $(n-1)$ at x. This proves the first part.

If f and g are two linearly independent homomorphisms $E' \to E$, then for $y \neq x$, we can find a suitable (nontrivial) linear combination of f and g which is singular at y, see [4, Lemma 7.1]. This contradicts i) and completes the proof of Lemma 5.6. □

Let now $W \to X \times S$ be a family of $(0,1)$-stable vector bundles on X, with fixed determinant ξ, parametrised by S. Then we may construct a family $H(W)$ of vector bundles on X, parametrised by the associated projective bundle $\pi : P(W^*) \to X \times S$ as follows. It is easy to construct as in [5§ 4] a canonical surjection of the bundle $p_2^*\pi^*W$ on $X \times P(W^*)$ onto $p_2^*\tau \otimes \mathscr{O}_{P(W)^*}$, where $P(W)^*$ is considered a divisor in $X \times P(W^*)$, by the inclusion $(p_1 \circ \pi, Id)$. The kernel of this homomorphism is the vector bundle $H(W)$. Let $K(W)$ be its dual.

Remark 5.7. This construction is the same as that in [5], except that here we have allowed x also to vary on the curve. From this
309 point of view, the families constructed in [5] may properly be denoted $H_x(V)$, $K_x(V)$, etc.

By Lemma 5.5, $K(W)$ is a family of stable vector bundles on X with determinant of the form $\xi^{-1} \otimes L_x$, $x \in X$. More precisely, we have a morphism $\theta_{K(W)} : P(W^*) \to U(n, \xi^{-1}X)$ with a commutative diagram

$$\begin{array}{ccc} P(W^*) & \xrightarrow{(\theta_{K(W)}, p_2 \circ \pi)} & U(n,\xi^{-1}X) \times S \\ \downarrow & & \downarrow{\scriptstyle (\alpha, Id)} \\ X \times S & \xrightarrow{Id} & X \times S \end{array} \tag{5.8}$$

where $\alpha : U(n, \xi^{-1}X) \to X$ is given by $L_{\alpha(E)} = (\det E) \otimes \xi^{-1}$ for $E \in U(n, \xi^{-1}X)$.

Lemma 5.9. *The morphism* $(\theta_{K(W)}, p_2 \circ \pi)$ *of* $P(W)^*$ *in* (5.8) *is a closed immersion.*

To prove this we will need the following

Lemma 5.10. *Let* X *be a scheme proper/*k *and* $\mathscr{F}$ *a coherent sheaf of* $\mathscr{O}_X$*-Modules. Let* S *be a scheme and*

$$0 \to \mathscr{H} \to p_1^*\mathscr{F} \to \mathscr{G} \to 0$$

be an exact sequence of coherent sheaves over $X \times S$ with $\mathscr{G}$ flat over S. Then $\mathscr{H}$ is a flat family of $\mathscr{O}_X$-Modules parametrised by S and its infinitesimal deformation map at $s_0 \in S$, is the negative of the composite of the natural map $T_{s_0} \to H^0(X, \mathrm{Hom}(\mathscr{H}_0, \mathscr{G}_0))$ (the differential of the morphism $S \to$ Quot) and the boundary homomorphism $H^0(X, \mathrm{Hom}(\mathscr{H}_0, \mathscr{G}_0)) \to \mathrm{Ext}^1(X, \mathscr{H}_0, \mathscr{H}_0)$ associated to the sequence $0 \to \mathscr{H}_0 \to \mathscr{F} \to \mathscr{G}_0 \to 0$. Here $\mathscr{H}_0, \mathscr{G}_0$ denote the restrictions of $\mathscr{H}, \mathscr{G}$ to $X \times s_0$.

Proof. It is clearly sufficient to consider the case $S = \mathrm{Spec}\, k[\epsilon]$. To give a sheaf $\mathscr{L}$ on $X \times S$ flat over S which extends a given sheaf $\mathscr{L}_0$ on $X \times s_0$, is the same as giving an extension

$$0 \to \epsilon\mathscr{L}_0 \to \mathscr{L} \to \mathscr{L}_0 \to 0$$

on X, where $\epsilon\mathscr{L}_0$ is another copy of $\mathscr{L}_0$. This identifies the space of infinitesimal deformations of $\mathscr{L}_0$ with $\mathrm{Ext}^1(X, \mathscr{L}_0, \mathscr{L}_0)$. In particular, $p_1^*\mathscr{F}$ is given by $\epsilon\mathscr{F} \oplus \mathscr{F}$. Now the map $p_1^*\mathscr{F} \to \mathscr{G}$ gives **310**
rise to a map $\mathscr{H}_0 \to \mathscr{F} \to \epsilon\mathscr{F} \oplus \mathscr{F} \to \mathscr{G}$. This map goes into $\epsilon\mathscr{G}_0$ and hence induces an element t of $H^0(X, \mathrm{Hom}(\mathscr{H}_0, \mathscr{G}_0))$ and thus identifies this space with the fibre of $\mathrm{Quot}_{\mathscr{F}}(k[\epsilon]) \to \mathrm{Quot}_{\mathscr{F}}(k)$ over $\mathscr{G}_0$. We have a map $\mathscr{H} \to \epsilon\mathscr{F}$ obtained by composing $\mathscr{H} \to p_1^*\mathscr{F}$ with the projection $p_1^*\mathscr{F} \to \epsilon\mathscr{F}$. This fits in a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \epsilon\mathscr{H}_0 & \longrightarrow & \mathscr{H} & \longrightarrow & \mathscr{H}_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \epsilon\mathscr{H}_0 & \longrightarrow & \epsilon\mathscr{F} & \longrightarrow & \epsilon\mathscr{G}_0 & \longrightarrow & 0
\end{array}$$

The last vertical map in the diagram is easily checked to be $-t$. Hence the element of $\mathrm{Ext}^1(X, \mathscr{H}_0, \mathscr{H}_0)$ giving the extension $\mathscr{H}$ is the image of $-t$ under the connecting homomorphism

$$\mathrm{Hom}(X, \mathscr{H}_0, \mathscr{G}_0) \to \mathrm{Ext}^1(X, \mathscr{H}_0, \mathscr{H}_0).$$

□

Remark 5.11. Proposition 3.3 of [5] is a particular case of 5.10, where $\mathscr{F}$ and $\mathscr{G}$ are locally free, in which case $\mathrm{Ext}^1(X, \mathscr{H}_0, \mathscr{H}_0)$ is the same as $H^1(X, \mathrm{End}\, \mathscr{H}_0)$.

Proof of Lemma 5.9. In view of the diagram 5.8, we may assume that T is a point. Thus we have a bundle W on X and the exact sequence

$$0 \to H(W) \to p_1^* W \to p_2^* \tau \otimes \mathscr{O}_{P(W^*)} \to 0$$

on $X \times P(W^*)$. Now it is easy to see that $P(W^*)$ is a component of $\mathrm{Quot}\, E$ and hence its tangent bundle can be identified as in Lemma 5.10, with

$$(p_2)_*(\mathrm{Hom}(H(W), \tau \otimes \mathscr{O}_{P(W^*)}) = \mathrm{Hom}(H(W)|P(W^*), \tau).$$

Moreover, by Lemma 5.10, the infinitesimal deformation map for the family $H(W)$ of bundles on X is given, upto sign, by the connecting homomorphism

$$\mathrm{Hom}(H(W)|P(W^*), \tau) \to R^1(p_2)_* \,\mathrm{End}\, H(W).$$

But this fits in the exact sequence

$$0 \to (p_2)_*(\mathrm{End}\, H(W)) \to (p_2)_*(\mathrm{Hom}(H(W), p_1^* W)) \to \mathrm{Hom}(H(W)|P(W^*), \tau) \to R^1(p_2)_* \,\mathrm{End}\, H(W).$$

Now since $H(W)$ is a family of stable bundles,

$$(p_2)_*(\mathrm{End}(H(W)) \simeq \mathscr{O}_{P(W^*)}$$

and by Lemma 5.6, (ii), the same is true of

$$(p_2)_*(\mathrm{Hom}(H(W), p_2^*(W)).$$

311 This proves that the differential of the map $\theta_{K(E)} : P(W^*) \to U(n, \xi^{-1}X)$ is injective. On the other hand, $\theta_{K(E)}$ is itself injective by Lemma 5.6, (ii), thus proving Lemma 5.9.

Now, for each family of $(0,1)$-stable bundles we have a morphism of the parameter space into $U(n,\xi^{-1}X)$, by Lemma 5.9. Since these morphisms are clearly functorial we get a morphism Φ on the open subscheme of $U(n,\xi)$ consisting of $(0,1)$-stable points into the Hilbert scheme $\mathrm{Hilb}(U(n,\xi^{-1}X))$.

Definition 5.12. *The cycles in $U(n,\xi^{-1}X)$ corresponding to $(0,1)$-stable points of $U(n,\xi)$ will be called* good Hecke cycles. *Any subscheme in the same irreducible component of the Hilbert scheme will be called a* Hecke cycle.

Theorem 5.13. *Let ξ be a line bundle on X. The open subscheme of $U(n,\xi)$ consisting of $(0,1)$-stable points is isomorphic (by means of the map associating to a bundle W the corresponding good Hecke cycle $\Phi(W)$) to an open subscheme of the Hilbert scheme of $U(n,\xi^{-1}X)$.*

Proof. To prove the injectivity, we will check the stronger assertion that for $W, W' \in U(n,\xi)$, both $(0,1)$-stable, $\theta_{K_x(W)}(P(W_x^*)) = \theta_{K_x(W')}(P(W'^*_x))$ if and only if W and W' are isomorphic. Indeed, by [6, Lemma 2.5], such an assumption would yield an isomorphism $\varphi : P(W_x^*) \to P(W'^*_x)$ such that $\varphi^\sharp K_x(W') \simeq K_x(W) \otimes p_2^* L$ on $X \times P(W_x^*)$, for some line bundle L on $P(W_x^*)$. Now by [5, Remark 4.7], we see, on restricting this isomorphism to $P(W_x^*)\times(y)$, $y \neq x$, that L is trivial. But then, by [5, Lemma 4.3] we have $(p_1)_*\varphi^\sharp K_x(W')^* \simeq (p_1)_* K_x(W')^* \simeq L_x^{-1} \otimes W'$ on the one hand, and $(p_1)_* K_x(W')^* \simeq L_x^{-1} \otimes W$ on the other. This proves that $W \simeq W'$.

Next we proceed to show that, if $W \in U(n,\xi)$ is $(0,1)$-stable, the infinitesimal deformation map at W of the family of good Hecke cycles from $T_W \simeq H^1(X, \mathrm{Ad}\, W)$ to $H^0(P(W^*), N)$ is an isomorphism, where N is the normal bundle of $P(W^*)$ in $U(n,\xi^{-1}X)$. This would prove that the Hilbert scheme is smooth at $\Phi(W)$ and that Φ is an isomorphism onto to an open subset, as claimed. To complete the proof of the theorem we need two lemmas.

Lemma 5.14. *Let $\pi_1 : P_1 \to U(n,\xi)\times X$ be the dual of the projective Poincaré bundle on $U(n,\xi)\times X$. Let $\pi_2 : P_2 \to U(n,\xi^{-1}X)$ be* **312**

the restriction of the dual projective Poincaré bundle on $U(n,\xi^{-1}X)\times X$ *to the divisor* $U(n,\xi^{-1}X)$. *Let* $\Omega_1 \subset P_1$ *and* $\Omega_2 \subset P_2$ *be the open subsets of points at which the families K are stable. We then have an isomorphism* $\widetilde{\theta} : \Omega_1 \to \Omega_2$ *such that* $\pi_2 \circ \widetilde{\theta} = \theta$.

Proof. A point p of Ω_1 is described by a $(0,1)$-stable vector bundle in $U(n,\xi)$, a point $x \in X$ and an element of $P(E^*)_x$. Now this can be looked upon as a map $E \to \mathscr{O}_x$ with kernel F, where F^* belongs to $U(n,\xi^{-1}X)$. By looking at the map $F_x \to E_x$ of the fibres at x, we can define a map $\Omega_1 \to P_2$ by associating the $\ker F_x$ to p. It is easy to see that this lift of θ maps Ω_1 isomorphically onto Ω_2. In fact this map is induced by the section σ [5, p. 402]. □

Lemma 5.15. *Let* $W \in U(n,\xi)$ *be* $(0,1)$*-stable. Then the normal bundle* $N = N_{P(W^*),U(n,\xi^{-1}X)}$ *of* $P(W^*)$ *in* $U(n,\xi^{-1}X)$ *fits into an exact sequence*

$$0 \to p_X^* T_X \otimes T_\pi^* \to P(W^*) \times T_W \to N \to 0.$$

Proof. Since W is $(0,1)$-stable, $P(W^*) = (p_2 \circ \pi_1)^{-1} W$ is contained in Ω_1 and $N_{P(W^*),\Omega_1} \simeq P(W^*) \times T_W$. Moreover $N_{P(W^*),\Omega_1} \simeq N_{\widetilde{\theta}P(W^*),\Omega_2}$. Since $\pi_2 \circ \widetilde{\theta} = \theta$, and $\theta | P(W^*)$ is an imbedding (Lemma 5.9), we see that $\widetilde{\theta}(P(W^*))$ is transversal to the fibres of π_2. Hence we have an exact sequence (on $P(W^*)$)

$$0 \to \widetilde{\theta}^* T_{\pi_2} \to N_{\widetilde{\theta}(P(W^*)),\Omega_2} \to N_{(\theta(P(W^*))),U(n,\xi^{-1}X)} \to 0.$$

But by Proposition 4.12 in [5], we have $\widetilde{\theta}^* T_{\pi_2} \simeq (p_2 \circ \pi_1)^* T_X \otimes T_\pi^*$. This proves the Lemma. □

Completion of the Proof of Theorem 5.13. Taking the direct image of the exact sequence in Lemma 5.15, and noting that, $H^0(X, T_X) = 0$, we see that the natural map $T_W \to H^0(P(W^*), N)$ is an isomorphism. But this is clearly the differential of Φ at W.

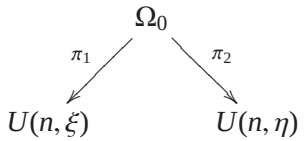
□

Taking determinants in the exact sequence 5.15, we obtain

Corollary 5.16. *Let $P(W^*)$ be a good Hecke cycle in $U(n, \xi^{-1}X)$. Then the line bundle $K_{\det}$ on U_X restricts to $P(W^*)$ as $\pi^* K_X^{-(n-1)} \otimes K_\pi^2$.*

Remarks 5.17. (i) Although the Hilbert scheme in Theorem 5.11 is smooth, $H^1(P(W^*), N) \neq 0$ unlike the case discussed below. **313**

(ii) Let us now fix $x \in X$. The open subset Ω_0 of points in the projective dual Poincaré bundle over $U(n, \xi) \times (x)$, corresponding to the family K_x, is isomorphic to a similar open subset of the projective Poincaré bundle over $U(n, \eta) \times (x)$ where $\xi \otimes \eta = L_x$. Thus we have two maps



which are projective fibrations over the open subsets of $(0,1)$-stable poins of $U(n, \xi)$, $U(n, \eta)$ respectively. Now the content of [5, Proposition 4.12] is that $T^*_{\pi_1} \simeq T_{\pi_2}$. (See also [8a, Ch 5, § 1]). If W is a $(0,1)$-stable point of $U(n, \xi)$, then $\pi_1^{-1}(W)$ is imbedded by π_2, thus giving a family of cycles in $U(n, \eta)$ parametrised by the $(0,1)$-stable points of $U(n, \xi^{-1}X)$. For the normal bundle N of the cycle $\Phi(W)$ corresponding to W we have the exact sequence

$$0 \to \Omega^1_{P(W^*_x)} \to H^1(X \operatorname{Ad} W)_{P(W^*)} \to N \to 0.$$

This shows that $H^1(\Phi(W), N) = 0$ for $i \geq 1$ and

$$\dim H^0(\Phi(W), N) = \dim H^1(X, \operatorname{Ad} W) + 1.$$

As in Theorem 5.13 we see that the open subset of $(0,1)$-stable points of $U(n, \xi^{-1}X)$ can be identified with an open subset of a component of the Hilbert scheme of $U(n, \xi)$.

(iii) If W is a $(1,1)$-stable bundle in $U(n,\xi)$, the corresponding cycle $\Phi(W)$ in $U(n,\xi^{-1}X)$ which is isomorphic to $P(W^*)$, is contained entirely in the set of $(0,1)$-stable points (Lemma 5.5 and Remark 5.2, (iii)). By the reverse construction in (ii), $P(W^*)$ is contained in the Hilbert scheme of $U(n,\xi)$ consisting of good Hecke cycles (this time, isomorphic to a projective space) passing through W. Thus $P(W^*)$ can be identified with this subscheme of the Hilbert scheme. From this one may conclude that any continuous group of automorphism of $U(n,\xi)$ lifts to the projective Poincaré bundle over some open subset of $U(n,\xi)$ consisting of $(1,1)$ stable
314 points. Thus, if $(1,1)$-stable points exist, then the automorphism group of $U(n,\xi)$ is finite. This was our original approach to the Theorems 1, 2 of [5].

6 Triangular bundles

We will deal with the case of rank 2 bundles in the rest of the paper. We will denote $U(2,\xi)$ (resp. $U(2,L_x)$) by U_ξ (resp. U_x). When ξ is trivial we write $U_\xi = U_0$.

We wish to study the limits of good Hecke cycles, i.e., points in the component of the Hilbert scheme, containing good Hecke cycles. It will turn out (§ 7) that these cycles consist of bundles which are extensions of line bundles of a particular kind. These bundles have been studied in [6] and [8] and the computations made in this article result from a closer analysis of the situation.

Let R be a Poincaré bundle on $X \times J$, where J is the Jacobian of X, Consider on $X\times X\times J$ the bundle $p_{13}^*R^2\otimes p_{12}^*L_\Delta^{-1}$, where Δ is the diagonal in $X\times X$. Then the first direct image $R^1(p_{23})_*(p_{13}^*R^2\otimes p_{12}^*L_\Delta^{-1})$ on $X\times J$ is clearly a vector bundle D. For each $j\in J$, the restriction of D to $X\times j$ gives rise to a bundle D_j on X.

Lemma 6.1. *If $j^2 \neq 1$, then the bundle D_j fits in an exact sequence*

$$0 \to j^2 \to D_j \to H^1(X, j^2)_X \to 0$$

given by the identity element of $H^1(X\,\mathrm{Hom}(H^1(X, j^2)_X, j^2))$. *If* $j^2 = 1$, *then* D_j *is isomorphic, upto tensorisation by a line bundle, to* $H^1(X, \mathscr{O})_X$. *In particular,* D_j *is a semistable vector bundle of degree* 0 *and any nonzero subbundle of* D_j *of degree* 0 *is the inverse image of a trivial subbundle of* $H^1(X, j^2)_X$.

Proof. By the base change theorem, the bundle D_j is isomorphic to $R^1(p_2)_*(p_1^* j^2 \otimes L_\Delta^{-1})$, where p_1, p_2 are the projections $X \times X \to X$. Hence it is dual to $(p_2)_*(p_1^*(K \otimes j^{-2}) \otimes L_\Delta)$. Consider on $X \times X$ the exact sequence

$$0 \to p_1^*(K \otimes j^{-2}) \to p_1^*(K \otimes j^{-2}) \otimes L_\Delta \to \widetilde{j}^{-2} \to 0, \qquad (6.2)$$

where the sheaf $\widetilde{L}$ denotes the extension of a line bundle L on Δ to $X \times X$ and we have used the fact that $L_\Delta|_\Delta \simeq K_X^{-1}$. We will show that by applying $(p_2)_*$ to this sequence and dualising, we get the **315**
canonical extension in 6.1. Now the pullback by $p_2^* j^{-2} \to \widetilde{j}^{-2}$ of this sequence yields an extension of $p_2^* j^{-2}$ by $p_1^*(K \otimes j^{-2})$. The corresponding element in $H^1(X \times X, p_2^* j^2 \otimes p_1^*(K \otimes j^{-2})) = H^0(X, K \otimes j^{-2}) \otimes H^1(X, j^2)$ (since $j^2 \neq 1$) is the canonical element given by the duality theorem. Thus it is clear that this pulled-back extension is also the push-out by the evaluation map $H^0(K \otimes j^{-2})_{X \times X} \to p_1^*(K \otimes j^{-2})$ of the extension

$$0 \to H^0(X, K \otimes j^{-2})_{X \times X} \to \ldots \to p_2^* j^{-2} \to 0$$

given by the canonical element of $H^1(X \times X, p_2^* j^2 \otimes H^0(X, K \otimes j^{-2}))$. Hence the exact sequence obtained by taking direct image by p_2, namely,

$$0 \to H^0(X, K \otimes j^{-2}) \to \ldots \to j^{-2} \to 0$$

is again given by the canonical element. But this is also obtained by taking the direct image of the sequence (6.2), thus proving the first part of Lemma 6.1. If $j^2 = 1$, then $D_j = R^1(p_2)_*(L_\Delta^{-1}) \simeq H^1(X, \mathscr{O})_X$ as is seen from the exact sequence

$$0 \to L_\Delta^{-1} \to \mathscr{O} \to \widetilde{\mathscr{O}} \to 0.$$

If F is a subbundle of degree 0, then $F \cap j^2 = 0$ or $F \supset j^2$. In either case, the image of F in $H^1(X, j^2)_X$ is a subbundle of degree 0 and hence trivial. If $F \cap j^2 = 0$, this would imply that the extension $0 \to j^2 \to D_j \to H^1(X, j^2)_X \to 0$ splits over a trivial subbundle of $H^1(X, j^2)_X$ which is seen to be not possible, in view of our description of this extension. □

Lemma 6.3. *Let* $C = \{(x, j) \in X \times J : j^{-2} \otimes L_x \in X\}$. *Then*

(i) *the bundle* $P(D)$ *is trivial on* C.

(ii) $X \times j \subset C$ *if and only if* j *is an element of order* 2.

Proof. (i) On $X \times C$, the bundle $(p_{13})^* R^2 \otimes (p_{12})^* L_\Delta^{-1}$ is isomorpic to $p_C^* \zeta \otimes (1 \times \alpha)^* L_\Delta^{-1}$ where ζ is some line bundle on C and $\alpha : C \to X$ is the map defined by $L_{\alpha(x,j)} = j^{-2} \otimes L_x$. Hence $D|C$ is isomorphic (by base change theorem) to $\zeta \otimes \alpha^*(R^1(p_2)_* L_\Delta^{-1})$. But clearly $R^1(p_2)_* L_\Delta^- \simeq H^1(X, \mathcal{O})_X$.

(ii) If j is an element of order 2, clearly $X \times j \subset C$. On the other hand, if $X \times j \subset C$, then $D_j = D|_{X \times j} \simeq (\zeta | X \times j) \otimes$ trivial
316 bundle, and hence by Lemma 6.1, this is possible only if j is an element of order 2.

We now have a family E of (triangular) vector bundles on $X \times P(D)$ given by [6, Lemma 2,4]:

$$0 \to (1 \times \pi)^* p_{13}^* R \otimes p_2^* \tau \to E \to (1 \times \pi)^* p_{13}^* R^{-1} \otimes (1 \times \pi)^* p_{12}^* L_\Delta \to 0 \tag{6.4}$$

where $\pi : P(D) \to X \times J$ is the projective fibration associated to D.

The bundle E on $X \times P(D)$ is a family of stable vector bundles [3, Lemma 10.1 (i)] on X of rank 2 and determinants of the form L_x, $x \in X$. Thus there is an induced classifying morphism θ_E :

$P(D) \to U_X$ fitting in an obvious commutative diagram

$$\begin{array}{ccc} P(D) & \xrightarrow{\theta_E} & U_X \\ {\scriptstyle p_1\circ\pi}\downarrow & & \downarrow{\scriptstyle \det} \\ X & \xrightarrow{\mathrm{Id}} & X \end{array} \tag{6.5}$$

We now wish to study the morphism θ_E and, in particular, its differential. Clearly E is a family of bundles with the triangular group as structure group. As such, there is an infinitesimal deformation map $T_{P(D)} \to R^1(p_2)_*(\Delta(E))$ where $\Delta(E)$ is the bundle of endomorphisms of E preserving the exact sequence (6.4). This is because the associated bundle with the Lie algebra of the triangular group as fibre is $\Delta(E)$. (Remark 2, 13, i). □

Lemma 6.6. *The above infinitesimal deformation map induces an isomorphism of $T_{p_1\circ\pi}$ with the kernel of $R^1(p_2)_*(\Delta(E)) \to R^1(p_2)_*(\mathscr{O})$ given by the trace map $\Delta(E) \to \mathscr{O}$ and hence with $R^1(p_2)_*(S\Delta(E))$, where $S\Delta(E)$ is the subbundle of $\Delta(E)$ consisting of endomorphisms of trace* 0.

Proof. In view of the diagram 6.5 it is clear that $T_{p_1\circ\pi}$ is mapped into the kernel. From the diagram of exact rows (on $P(D)$),

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_\pi & \longrightarrow & T_{p_1\circ\pi} & \longrightarrow & \pi^* p_2^* T_J & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R^1(p_2)_*(E\otimes(1\times\pi)^*(p_{13}^*R\otimes p_{12}^*L_\Delta^{-1})) & \to & R^1(p_2)_*(\Delta(E)) & \to & H^1(X,\mathscr{O})_{P(D)} & \to & 0 \end{array}$$

we see that it is enough to show that the infinitesimal deformation map maps T_π isomorphically on $R^1(p_2)_*(E\otimes R\otimes L_\Delta^{-1})$, since it is obvious that the last vertical map is an isomorphism. Indeed, we have □ **317**

Lemma 6.7. *Let V, W be two simple vector bundles on X. Let $P = PH^1(X, \mathrm{Hom}(W,V))$, and*

$$0 \to p_1^*V \otimes p_2^*\tau \to E \to p_1^*W \to 0$$

be the universal exact sequence on $X \times P$ (see [6, Lemma 2.3]). Then the infinitesimal deformation map of the family E of bundles with the evident parabolic group as structure group, maps T_P isomorphically onto the kernel of $R^1(p_2)_*(E \otimes p_1^* W^*) \to R^1(p_2)_*(\mathrm{End}\ W)$.

Proof. From the definition of the universal extension, it follows that the map $\mathscr{O}_P = (p_2)_*(W \otimes W^*) \to R^1(p_2)_*(V \otimes \tau \otimes W^*) = H^1(X, V \otimes W^*) \otimes \tau$ is the natural inclusion. Hence its cokernel is isomorphic to the tangent bundle T_P of P. From the cohomology exact sequence obtained from the given sequence tensored with W^*, we obtain an isomorphism of T_P with the kernel of $R^1(p_2)_*(E \otimes p_1^* W^*) \to R^1(p_2)_*(\mathrm{End}\, W)$. We have to check that this identification is the infinitesimal deformation map. This verification can be done by choosing splittings for the given sequence over $U_i \times P$ where (U_i) is an open covering of X and expressing the infinitesimal deformation map in terms of Čech cocycles with respect to this covering. □

Proposition 6.8. *Recall that $C = \{(x, j) \in X \times J : j^{-2} \otimes L_x \in X\}$. The differential $d\theta_E$ is injective outside $\pi^{-1}(C)$ and $\ker d\theta_E$ is a line bundle on $\pi^{-1}(C)$.*

Proof. Consider the exact sequence obtained from 6.4:

$$0 \to S\Delta(E) \to \mathrm{Ad}\, E \to (1 \times \pi)^*(p_{13}^* R^{-2} \otimes p_{12}^* L_\Delta) \otimes p_2^* \tau^{-1} \to 0. \tag{6.9}$$

Take the direct image on $P(D)$. The (zeroth) direct image of the bundle on the right is zero since it is clearly torsion free and zero outside $\pi^{-1}(C)$. Thus we get a short exact sequence (using Lemma 6.6)

$$0 \to T_{p_1 \circ \pi} \to R^1(p_2)_*(\mathrm{Ad}\, E) \to R^1(p_2)_*(R^{-2} \otimes L_\Delta) \otimes \tau^{-1} \to 0.$$

From [6, Lemma 2.6], we have $R^1(p_2)_*(\mathrm{Ad}\, E) = \theta_E^* T_{\mathrm{det}}$ and it is easy to see that the first map is $d\theta_E$. Moreover, outside $\pi^{-1}(C)$, the last term in this sequence is clearly locally free of rank $(g-2)$
318 while its restriction to $\pi^{-1}(C)$ is also locally free of rank $(g-1)$. This proves the proposition. □

Identifying $\pi^{-1}(C)$ with $C \times P$ where $P = PH^1(X, \mathscr{O})$, the restriction of E to $X \times C \times P$ can be described as follows.

Lemma 6.10. (i) *There is a natural isomorphism* $H^1(X \times C \times P, p_3^*\tau \otimes p_{12}^*(1 \times \alpha)^* L_\Delta^{-1})$ *with* $H^1(X, \mathscr{O}) \otimes H^1(X, \mathscr{O})^*$.

(ii) *The family* $E \otimes p_{13}^* R^{-1}$ *resticts to* $X \times C \times P$ *as an extension*

$$0 \to p_3^*\tau \to E \otimes p_{13}^* R^{-1} \to p_{12}^*(1 \times \alpha)^* L_\Delta \to 0$$

given by the canonical element in $H^1(X, \mathscr{O}) \otimes H^1(X, \mathscr{O})^*$, *using the isomorphism in* (i).

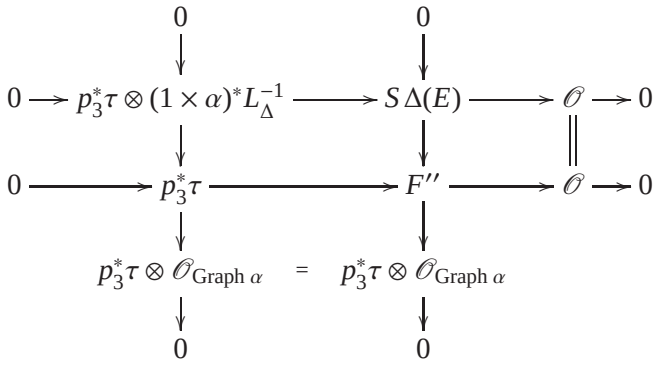
Proof. Note that $H^1(X \times C, (1 \times \alpha)^* L_\Delta^{-1}) \simeq H^0(C, R^1(p_2)_*(1 \times \alpha)^* L_\Delta^{-1}) \simeq H^0(C, \alpha^* R^1(p_2)_* L_\Delta^{-1}) \simeq H^0(C, \alpha^* R^1(p_2)_* \mathscr{O}_{X \times X}) \simeq H^1(X, \mathscr{O})_C$. This proves (i). That the canonical element gives the required extension follows from the definition of E and the base change theorem. □

Proposition 6.11. *Let* $0 \to p_2^*\tau \to F' \to \mathscr{O} \to 0$ *be the universal extension of* $\mathscr{O}$ *by* τ *on* $X \times PH^1(X, \mathscr{O})$. *Then the restriction of* $T_{p_1 \circ \pi}$ *to* $\pi^{-1}(C) = C \times P$ *is isomorphic to the quotient of* $F' \otimes H^1(X, \mathscr{O})$ *by the trivial subbundle contained in* $p_2^*\tau \otimes H^1(X, \mathscr{O})$.

Proof. We will apply Lemma 6.6, and use the description of $E|X \times C \times P$ given in Lemma 6.10. Thus we have the extension

$$0 \to p_3^*\tau \otimes p_{12}^*(1 \times \alpha)^* L_\Delta^{-1} \to S\Delta(E) \to \mathscr{O} \to 0.$$

This may be embedded in the commutative diagram



(6.12)

where the middle line is obtained from the top line as the push-out by means of the map $p_3^*\tau \otimes (1\times\alpha)^* L_\Delta^{-1} \to p_3^*\tau$. By Lemma
319 6.6, $T_{p_1\circ\pi}|C\times P$ is isomorphic to the first direct image on $C\times P$ of $S\Delta(E)$. The middle vertical line of the above diagram yields an isomorphism of this with the first direct image of F''. For, $R^1(p_{23})_*(p_3^*\tau\otimes\mathscr{O}_{\text{Graph } \alpha})$ is clearly 0. Since $(p_{23})_*(p_3^*\tau) \to (p_{23})_*(\tau\otimes \mathscr{O}_{\text{Graph } \alpha})$ is clearly an isomorphism, it follows that $(p_{23})_*F'' \to (p_{23})_*(p_3^*\tau \otimes \mathscr{O}_{\text{Graph } \alpha})$ is surjective, proving our assertion above. It remains to compute $R^1(p_{23})_*(F'')$. From the description of the extension in Lemma 6.10 and hence that in the top horizontal line of the diagram 6.12, it can be checked that the element in $H^1(X\times C\times P, p_3^*\tau) = H^1(X,\mathscr{O})\otimes H^1(X,\mathscr{O})^* \oplus H^1(C,\mathscr{O})\otimes H^1(X,\mathscr{O})^*$ given by the extension F'' is the element $(Id, -\alpha^*(Id))$. Indeed, this follows on remarking that the map $H^1(X,\mathscr{O}) \simeq H^1(X\times X, L_\Delta^{-1}\Delta) \to H^1(X\times X,\mathscr{O}) \to H^1(X,\mathscr{O})\oplus H^1(X,\mathscr{O})$ is given by $(Id, -Id)$. Now our assertion follows from □

Lemma 6.13. *Consider on $X\times X\times P$, the exact sequence*

$$0 \to p_3^*\tau \to F_0 \to \mathscr{O} \to 0$$

given by the element $(Id, -Id)$ in $H^1(X\times X,\mathscr{O})\otimes H^0(P,\tau) \simeq V\otimes V^*\oplus V\otimes V^*$, with $V = H^1(X,\mathscr{O})$. The direct image on $X\times P$ by

p_{23} of this sequence fits in a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p_2^*T_P & \longrightarrow & R^1(p_{23})_*F_0 & \longrightarrow & H^1(X,\mathscr{O})_{X\times P} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & p_2^*\tau\otimes H^1(X,\mathscr{O}) & \longrightarrow & F'\otimes H^1(X,\mathscr{O}) & \longrightarrow & H^1(X,\mathscr{O})_{X\times P} & \longrightarrow & 0
\end{array}$$

where $0 \to p_2^*\tau \to F' \to \mathscr{O} \to 0$ is the universal extension on $X \times P$ and the first vertical map is induced by the natural map $\tau \otimes H^1(X,\mathscr{O}) \to T_P$ on P.

Proof. In the following Lemma, take $L = K_X$, $L' = \mathscr{O}$, $T = X\times P$, and apply duality to prove the Lemma. □

Lemma 6.14. *Let*

$$0 \to p_1^*L \to M \to p_1^*L \otimes p_2^*L' \to 0$$

be an exact sequence of vector bundles on the variety $X \times T$. Assume that $\dim H^0(X\times t, M)$ *is independent of t. Then the extension*

$$0 \to H^0(X,L)_T \to (p_T)_*M \to \ker\delta \to 0,$$

where δ is the connecting homomorphism **320**

$$H^0(X,L)\otimes L' \to H^1(X,L),$$

is given by the image under the natural map

$$H^1(T, L'^{-1}) \to H^1(T, (\ker\delta)^* \otimes H^0(X,L))$$

of the Künneth component in $H^1(T, L'^{-1})$ of the element of $H^1(X\times T, \mathrm{Hom}(L,L)\otimes L'^{-1})$ determined by the extension M.

Proof. Let $x_1, \ldots, x_N$ be points of X such that the evaluation map $H^0(X,L) \to \sum\limits_{\oplus} L_{x_i}$ is an isomorphism. This fits in a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X,L)_T & \longrightarrow & (p_T)_*M & \longrightarrow & \ker\delta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sum L_{x_i} & \longrightarrow & \sum M_{x_i\times T} & \longrightarrow & L'\otimes\sum L_{x_i} & \longrightarrow & 0
\end{array}$$

The latter sequence is clearly defined by the required element in $H^1(T, L'^{-1})$. □

Proposition 6.15. *The differential $d\theta_E$ on $\pi^{-1}C = C \times P$ fits in an exact sequence*

$$0 \to \tau^{-1} \to T_{p_1 \circ \pi}|\pi^{-1}C \xrightarrow{d\Theta_E} \theta_E^* T_{\det} \to \tau^{-1} \otimes \pi^* \alpha^* F \to 0$$

where F is the vector bundle on X defined by the exact sequence

$$0 \to T_X \to H^1(X, \mathscr{O}) \to F \to 0$$

corresponding to the canonical linear system and $\alpha : C \to X$ is the map defined by $j^{-2} \otimes L_x = L_{\alpha(j,x)}$.

Proof. Note that if Δ denotes the diagonal divisor in $X \times X$, then $(p_2)_*(L_\Delta)$
$\simeq \mathscr{O}_X$ and $R^1(p_2)_*(L_\Delta) \simeq F$. In fact, the first of these isomorphisms is obvious and the second follows from the exact sequence (on $X \times X$)

$$0 \to \mathscr{O}_{X \times X} \to L_\Delta \to \widetilde{T}_X \to 0$$

where $\widetilde{T}_X$ is the sheaf on $X \times X$ supported on Δ and inducing T_X on it. Restricting 6.9 to $X \times C \times P$, we get the sequence

$$0 \to S\Delta(E) \to \operatorname{Ad} E \to p_3^* \tau^{-1} \otimes p_{12}^* (1 \times \alpha)^* L_\Delta \to 0.$$

The direct image of this sequence, using base change for α, yields the lemma, in view of Lemma 6.6.

□

321 We will now compute the Hessian of the map θ_E restricted to $\pi^{-1}C = C \times P$. By Proposition 6.8, $\pi^{-1}C$ is the critical set for θ_E. Moreover, $\ker d\theta_E$ on $C \times P$ is isomorphic to τ^{-1} by Proposition 6.15. The Hessian is thus a map $\tau^{-1} \otimes N_{C \times P, P(D)} \to \operatorname{coker} d\theta_E | C \times P$. (See Remark 3.6). Now $N_{C \times P, P(D)} \simeq \pi^* N_{C, X \times J} \simeq \pi^* \alpha^* N_{X, J^1} \simeq \pi^* \alpha^* F$ where F is the bundle defined in Proposition 6.15. On the other hand, by Proposition 6.15, $\operatorname{coker} d\theta_E | C \times P$ is isomorphic to $\tau^{-1} \otimes \pi^* \alpha^* F$. With these identifications we have

Proposition 6.16. *The Hessian on $\pi^{-1}C = C \times P$ of the map θ_E considered as a map $\tau^{-1} \otimes \pi^*\alpha^*F \to \tau^{-1} \otimes \pi^*\alpha^*F$ is the identity map.*

Proof. By Lemma 6.6, the computation of the required Hessian may be made using Remark 2.13, i). From the description of the family E over $X \times \pi^{-1}C$ given in Lemma 6.10, we see that the Hessian of θ_E is simply the map

$$(p_{23})_*(\tau^{-1} \otimes \alpha^*L_\Delta) \otimes R^1(p_{23})_*(\alpha^*L_\Delta^{-1} \otimes E \otimes R^{-1}) \to R^1(p_{23})_*(\tau^{-1} \otimes \alpha^*L_\Delta)$$

given by the cup product for the natural map

$$(\tau^{-1} \otimes \alpha^*L_\Delta) \otimes (\alpha^*L_\Delta^{-1} \otimes E \otimes R^{-1}) \to \tau^{-1} \otimes E \otimes R^{-1} \to \tau^{-1} \otimes \alpha^*L_\Delta.$$

From this it is clear that this map is the identity on τ^{-1} tensored with the induced map

$$(p_{23})_*(\alpha^*L_\Delta) \otimes R^1(p_{23})_*(\mathscr{O}) \to R^1(p_{23})_*(\alpha^*L_\Delta).$$

Now since $(p_{23})_*(\alpha^*L_\Delta)$ is canonically trivial, this is simply the inverse image by $\alpha \circ p_1$ of the map $H^1(X, \mathscr{O})_X \to R^1(p_2)_*(L_\Delta)$. This is clearly the natural map $H^1(X, \mathscr{O})_X \to F$, proving our assertion. □

Remark 6.17. From Proposition 6.16, we see that $\pi^{-1}C$ is a 'non-degenerate' critical manifold for θ_E.

Proposition 6.18. *For every $j \in J$, the map $\theta_E : \pi^{-1}(X \times j) \to U_X$ is an imbedding.*

Proof. We first remark that in view of 6.5, it is enough to prove that $\theta_E : \pi^{-1}(x, j) \to U_X$ is an imbedding, for every $x \in X$ and $j \in J$. Secondly, from [3, Lemma 10.1], it follows that this map is injective. Moreover, if $(x, j) \in C$, then by Proposition 6.8, $\theta_E : \pi^{-1}(x, j) \to U_x$ is an imbedding. Finally, let $(x, j) \in C$. In **322**
order to show that $d\theta_E|T_\pi$ is injective, it is enough to prove that the composite of the inclusion $\tau^1 \to T_{p_1 \circ \pi}|_{C \times P}$ with the differential

$d\pi : T_{p_1\circ\pi} \to \pi^* T_{p_1} = H^1(X, \mathscr{O})_{C\times P}$ is injective. Consider the commutative diagram (on $\pi^{-1}C$)

$$\begin{array}{ccccccccc} 0 & \to & S\Delta(E) & \longrightarrow & \Delta(E) & \longrightarrow & (1\times\pi)^*(p_{13}^* R^{-2} \otimes p_{12}^* L_\Delta) \otimes p_2^* \tau^{-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathscr{O} & \to & E \otimes (1\times\pi)^* p_{13}^* R^{-1} \otimes p_2^* \tau^{-1} & \longrightarrow & '' & \longrightarrow & 0 \end{array}$$

From this we conclude that this composite is given by the boundary homomorphism of the lower sequence. Now our assertion follows from the definition of the extension E. □

The nonsingular subvarieties $\theta_E(\pi^{-1}(X \times j)) \subset U_X$ will also be denoted $P(D_j)$.

Proposition 6.19. *If $j_1 \neq j_2$ or j_2^{-1}, then $P(D_{j_1})$ intersects $P(D_{j_2})$ in only finitely many poins. If $j^2 \neq 1$, then $P(D_j)$ and $P(D_{j^{-1}})$ intersect transversally along the sections given by the exact sequence in Lemma 6.1.*

Proof. Let $j_1 \neq j_2$, j_2^{-1} and $W \in P(D_{j_1}) \cap P(D_{j_2})$. Then W is given by an extension

$$0 \to j_1 \to W \to j_1^{-1} \otimes L_x \to 0.$$

and W contains j_2 also. The existence of a nonzero homomorphism $j_2 \to j_1^{-1} \otimes L_x$ implies that $j_1 \otimes j_2 = L_x \otimes L_y^{-1}$ for some $y \in X$. This proves that there are (if at all) only finitely many solutions for x and y. If (x, y) is one such, then E has to be in the kernel of the map $H^1(j_1^2 \otimes L_x^{-1}) \to H^1(j_1^2 \otimes L_x^{-1} \otimes L_y)$ which is at most 1-dimensional. Thus W is uniquely determined by such a choice of x and y. This proves our first assertion.

Now let $W \in P(D_j) \cap P(D_{j^{-1}})$, $j^2 \neq 1$. As above, we see that W is the unique bundle given by the kernel of $H^1(X, j^2 \otimes L_x^{-1}) \to H^1(X, j^2)$. Clearly, this is the kernel defined in Lemma 6.1. It remains to prove the transversality of the intersection of

$P(D_j)$ and $P(D_{j^{-1}})$. Now W is given simultaneously by two exact sequences

$$0 \to j \to W \to j^{-1} \otimes L_x \to 0$$
$$\text{and} \qquad 0 \to j^{-1} \to W \to j \otimes L_x \to 0.$$

Consider the map $j \oplus j^{-1} \to W$. Since $j \neq j^{-1}$, this map is a generic isomorphism and hence fits in an exact sequence **323**

$$0 \to j \oplus j^{-1} \to W \to \mathscr{O}_x \to 0.$$

Let $M \subset \operatorname{Ad} W$ be the bundle of endomorphisms (with trace 0) of W taking j into j^{-1}. Then we have the exact sequence

$$0 \to M \to \operatorname{Ad} W \to L_x \to 0. \tag{6.20}$$

On the other hand, both $\operatorname{Hom}(j^{-1} \otimes L_x, j) = j^2 \otimes L_x^{-1}$ and $\operatorname{Hom}(j \otimes L_x, j^{-1}) = j^{-2} \otimes L_x^{-1}$ are clearly subbundles of M. At any $y \neq x$, we have $W_y = j_y \oplus j_y^{-1}$ and these two subspaces of M_y may be represented in matrix form by $\left(\begin{smallmatrix} 0 & 0 \\ * & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$ and hence are linearly independent. In other words, the map $j^2 \otimes L_x^{-1} \oplus j^{-2} \otimes L_x^{-1} \to M$ fails to be of maximal rank only at x. Since $\det M = L_x^{-1}$, we have the exact sequence

$$0 \to j^2 \otimes L_x^{-1} \oplus j^{-2} \otimes L_x^{-1} \to M \to \mathscr{O}_x \to 0.$$

In particular, the natural map $H^1(j^2 \otimes L_x^{-1}) \oplus H^1(j^{-2} \otimes L_x^{-1}) \to H^1(M)$ is surjective. From (6.20), we conclude that the map $H^1(M) \to H^1(\operatorname{Ad} W)$ has as cokernel $H^1(X, L_x)$ which is of dimension $(g-1)$. Thus the images of $H^1(j^2 \otimes L_x^{-1})$, $H^1(j^{-2} \otimes L_x^{-1})$ in $H^1(\operatorname{Ad} W)$ by the natural inclusions have sum of dimension $(2g-2)$. But the image of $H^1(j^2 \otimes L_x^{-1})$ in $H^1(\operatorname{Ad} W)$ is the tangent space at W to the cycle $P(W_x^*)$ in U_x. This proves the transversality, since $\dim U_x = 3g-3$. □

Lemma 6.21. *Let $j \in J$ with $j^2 \neq 1$. Then identifying X with the intersection of $P(D_j)$ and $P(D_{j^{-1}})$, we have the exact sequence*

$$0 \to T_{\pi j} \oplus T_{\pi j^{-1}} \to T_{\det}|X \to F \to 0$$

where $T_{\pi j}$ is the restriction to X of the tangent bundle along the fibres of $P(D_j)$ and F is the quotient bundle associated to the canonical linear system on X.

Proof. The exact sequence 6.20 when x is also varied, yields a sequence on $X \times X$

$$0 \to M \to \mathrm{Ad}\, W \to L_\Delta \to 0.$$

324 Taking direct image on X, we get

$$R^1(p_1)_* M \to T_{\det}|X \to F \to 0.$$

On the other hand, the map $R^1(p_1)_*(j^2 \otimes L_\Delta^{-1}) \oplus R^1(p_1)_*(j^{-2} \otimes L_\Delta^{-1}) \to R^1(p_1)_* M$ has been proved to be surjective. As in Proposition 6.19, we see that the image of $R^1(p_1)_*(j^2 \otimes L_\Delta^{-1})$ is $T_{\pi j}$, proving our assertion. □

Lemma 6.22. (i) *If $j^2 \neq 1$, then we have the exact sequence on $P(D_j)$:*

$$0 \to H^1(X, \mathcal{O})_{P(D_j)} \to N_{P(D_j),U_X} \to \tau^{-1} \otimes \pi^* F(j) \to 0$$

where $F(j)$ is the quotient sheaf defined by the linear system $K \otimes j^2$.

(ii) *If $j^2 = 1$, then we have the exact sequence on $P(D_j) = X \times P$ with $P = PH^1(X, \mathcal{O})$*

$$0 \to \mathrm{Im}\, d\theta_E | P(D_j) \to T_{\det}|P(D_j) \to \tau^{-1} \otimes \pi^* F \to 0$$

where F is the quotient bundle on X defined by the canonical linear system. Moreover, we have a commutative diagram on $P(D_j)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau \otimes H^1(X, \mathscr{O}) & \longrightarrow & F' \otimes H^1(X, \mathscr{O}) & \longrightarrow & H^1(X, \mathscr{O}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & p_2^* T_P & \longrightarrow & \operatorname{Im} d\theta_E & \longrightarrow & \tau^{-1} \otimes p_2^* T_p & \longrightarrow & 0 \end{array}$$

where the extreme vertical maps are the natural ones and the first extension is obtained by tensoring the universal extension on $X \times P$ with $H^1(X, \mathscr{O})$.

Proof. (i) This follows from the sequence (6.9) and its direct image sequence noting that since $X \times j \not\subset C$ by Lemma 6.3 ii) and Proposition 6.8, the map $T_{p_1 \circ \pi} \to T_{U_X}$ restricted to $P(D_j)$ is still (sheaf theoretically) injective. The cokernel is clearly the tensor product of τ^{-1} and the sheaf $R^1(p_1)_*(p_2^* j^{-2} \otimes L_\Delta)$. From the exact sequence associated to

$$0 \to p_2^* j^{-2} \to p_2^* j^{-2} \otimes L_\Delta \to j^{-2} \widetilde{\otimes} K_X^* \to 0$$

we identify this sheaf with $F(j)$.

(ii) This is an immediate consequence of Propositions 6.15 and 6.11.

□

Corollary 6.23. *For any $j \in J$, we have* **325**

$$K_{\det}^* | P(D_j) \simeq \tau^2 \otimes \pi^*(j^4 \otimes K).$$

Proof. From the exact sequence

$$0 \to K^* \otimes j^{-2} \to H^1(X, j^{-2})_X \to F(j) \to 0$$

we see that $\det F(j) = K \otimes j^2$ and now from Lemma 6.22, i) we get $\det T_{\det} | P(D_j) = \tau^{-(g-2)} \otimes \det F(j) \otimes \det T_\pi = \tau^{-(g-2)} \otimes K \otimes j^2 \otimes \tau^g \otimes j^2 = \tau^2 \otimes j^4 \otimes K$, as claimed. The case $j^2 = 1$ follows similarly from 6.22, ii). □

7 Limits of good Hecke cycles

In this article, we study the limis of good Hecke cycles and their normal bundles in U_X. We first wish to fix an ample line bundle $\mathcal{O}(1)$ on U_X.

Lemma 7.1. *If $K_{\det}$ denotes the canonical line bundle along the fibres of the fibration* $\det : U_X \to X$, *then the line bundle $\mathcal{O}(1) = K_{\det}^* \otimes (\det)^* K_X$ is an ample bundle on U_X.*

Proof. Fix $\alpha \in J^1$. Consider the map $f : J \to J^1$ given by $f(j) = j^2 \otimes \alpha$. Then we have a commutative diagram

$$\begin{array}{ccc} f^*U(2,1) & \longrightarrow & U(2,1) \\ \downarrow & & \downarrow \\ J & \xrightarrow{f} & J^1 \end{array}$$

Now $f^*(U(2,1)) \to J$ is isomorphic to the product $U(2,\alpha) \times J \to J$. This gives in turn a commutative diagram

$$\begin{array}{ccc} U(2,\alpha) \times \widetilde{X} & \xrightarrow{\tilde{f}} & U_X \\ \downarrow & & \downarrow \\ \widetilde{X} = f^{-1}(X) & \xrightarrow{f} & X \end{array}$$

Since f is étale surjective, X is nonsingular and $f^*K_X = K_X$. On the other hand, $f^*K_{\det} \simeq p_1^* K_{U_\alpha}$, so that our assertion follows from [2, Proposition 4.4] and the ampleness of K^* on U_α (See [6, Theorem 1]) and of K on X. □

326 **Lemma 7.2.** *Let Z be a good Hecke cycle. Then the polynomial $P(m,n) = \chi(Z, K_{\det}^{-m} \otimes dp^* K_X^n)$ is given by $(4m+1)(2m+2n-1) \times (g-1)$. In particular, the Hilbert polynomial of a good Hecke cycle (with respect to $\mathcal{O}(1)$) is $P(n) = (4n+1)(4n-1)(g-1)$.*

Proof. If W is the vector bundle to which the Hecke cycle Z corresponds, then by 5.16, we have $(K_{\det}^{-m} \otimes \det^* K_X^n)|_Z \simeq K_\pi^{-2m} \otimes \pi^* K_X^{m+n}$. Now

$$\pi_*(K_\pi^{-2m} \otimes \pi^* K_X^{m+n}) \simeq (\det W)^{2m} \otimes S^{4m}(W^*) \otimes K_X^{m+n}.$$

Since the higher direct images are zero for $m \geq 0$, we have

$$P(m,n) = \chi(X, (\det W)^{2m} \otimes S^{4m}(W^*) \otimes K_X^{m+n})$$

and the latter is computed to be as claimed, using the Riemann-Roch theorem. □

Let $p : H \to U_X$ be the restriction of the dual Poincaré bundle on $X \times U_X$ to U_X considered as a divisor by the map (det, Id). By 5.2, v) and Lemma 5.5, we obtain a map $h : H \to U_0$ which is a projective fibration over the set of stable points of U_0 by 5.2, iv) and Lemma 5.14. The morphism $\theta_E : P(D) \to U_X$ (in the notation of § 6) lifts to a map $\widetilde{\theta}_E : P(D) \to H$ since $\theta_E^* H \simeq P(E)^*$ and E^* has a natural line subbundle by construction. Let $\mathscr{K}$ be the variety of nonstable points of U_0, namely the Kummer variety $\mathscr{K} = J/i$, where i is the involution $j \mapsto j^{-1}$.

Lemma 7.3. *We have a commutative diagram*

$$\begin{array}{ccc} P(D) & \xrightarrow{\theta_E} & h^{-1}(\mathscr{K}) \subset H \\ \downarrow{\scriptstyle p_2 \circ \pi} & & \downarrow{\scriptstyle h} \\ J & \longrightarrow & \mathscr{K} \end{array}$$

Moreover $\widetilde{\theta}_E$ is onto $h^{-1}(\mathscr{K})$.

Proof. A point of $P(D)$ is represented by an exact sequence

$$0 \to j \to E \to j^{-1} \otimes L_X \to 0$$

Its image in H is given by the element $E \in U_X$ and the one- 327
dimensional space of linear forms on E_x vanishing on j_x. Now the construction K on H which defines the morphism h associates to it the bundle F obtained by

$$0 \to F \to E \to \mathscr{O}_x \to 0$$

given by the above linear form. It is obvious j is a subbundle of F and hence F is S-equivalent to $j \oplus j^{-1}$ proving the commutativity of the diagram. Now a point of $h^{-1}(\mathscr{K})$ is given by a stable bundle $E \in U_x$, for some $x \in X$, and a linear form on E_x such that the bundle F defined by the sequence

$$0 \to F \to E \to \mathscr{O}_x \to 0$$

is non-stable. Let $j \subset F$. Clearly then j is also contained in E, for otherwise some line bundle of the form $j \otimes L_D$, $D > 0$ will be contained in E contradicting its stability. Thus E can be written as an extension

$$0 \to j \to E \to j^{-1} \otimes L_x \to 0$$

proving the lemma. □

Lemma 7.4. (i) *If $k \in \mathscr{K}$ is not a node, then the reduced fibre of h over k is the push-out of $P(D_j)$ and $P(D_{j^{-1}})$ along the natural section X, where j, j^{-1} are points in J over k.*

(ii) *If $k \in \mathscr{K}$ is the image of an element $j \in J$ of order 2, then the reduced fibre of h over k is $X \times P$ where $P = PH^1(X, \mathscr{O})$. Moreover the schematic fibre of h contains as a subscheme $X \times P_t$ where P_t is the thickening of P by $0 \to Q^* \to \mathscr{F} \to \Omega^1_P \to 0$, where Q is the universal, quotient bundle of P.*

Proof. (i) follows from diagram 7.3 and the fact that images by $\widetilde{\theta}_E$ of $(P(D_j))$ and $P(D_{j^{-1}})$ intersect transversally along X since transversatility is true of θ_E (Lemma 6.19).

(ii) Clearly the total thickening Z of $P(D_j)$ in $P(D)$ is in the fibre over k of the morphism $h \circ \widetilde{\theta}_E$. Thus we have only to check that θ_E induces an epimorphism of Z onto $X \times P_t$ and that the latter is a subscheme of H. But this follows from Remark 3.6.

□

Remark 7.5. The reduced scheme defined in 7.4, (i) will be de- **328**
noted F_k, if k is not the image of an element of order 2. If it is, then F_k will denote the subscheme $X \times PH^1(X, \mathscr{O})_t$ of U_X. These are contained in the fibres of $H \to U_0$. Probably, these are precisely the schematic fibres, and some of the technical difficulties in this article can be obviated if one could prove this directly.

Lemma 7.6. *The natural morphism of the relative Hilbert scheme* $\mathrm{Hilb}(H, U_0, P(n))$ *into* $\mathrm{Hilb}(U_X, P(n))$ *is injective.*

Proof. Since $P(n)$ is of degree 2 in n, any element of $\mathrm{Hilb}(U_X, P(n))$ represents a subscheme of U_X of dimension 2. Hence our assertion follows from Proposition 6.19. □

Let $\mathrm{Hilb}(H, U_0, P(m, n))$ be the relative Hilbert scheme of the map $H \to U_0$ of cycles Z satisfying $\chi(Z, p^*(K_{\det}^{-m} \otimes \det^* K_X^n) = P(m, n)$. If H_2 is the image of $\mathrm{Hilb}(H, U_0, P(m, n))$ in $\mathrm{Hilb}(U_X, P(n))$ then H_2 contains good Hecke cycles. Now if $Z \in H_2$ is not a good Hecke cycle, then $Z_{\mathrm{red}} \subset F_k$ for a unique $k \in \mathscr{K}$. For a fixed $k \in \mathscr{K}$ the set of elements of H_2 which are contained set theoretically in F_k is denoted $H_{2,k}$.

7.7 Definition of The Varieties Q_j, Q_k, R_k.

Now we will define for k not a node, three subvarieties Q_j, $Q_{j^{-1}}$ and R_k (with j, $j^{-1} \in J$ lying over k) of $H_{2,k}$. Consider the Grassmannian of lines in $PH^1(X, j^2)$; the symmetric product of the projective line bundle over it will be denoted Q_j. Also, we denote by R_k the product of $PH^1(X, j^2)$ and $P(H^1(X, j^{-2}))$ for j lying over

$k \in \mathscr{K}$. Now points of Q_j, $Q_{j^{-1}}$ and R_k can be considered as subschemes of F_k. In fact, consider the exact sequences

$$0 \to j^2 \to D_j \to H^1(X, j^2)_X \to 0$$
$$0 \to j^{-2} \to D_{j^{-1}} \to H^1(X, j^{-2})_X \to 0.$$

Any line in $PH^1(X, j^2)$ gives rise to a plane subbundle of $P(D_j)$ over X. A point of Q_j gives rise a pair of projective line subbundles (possibly identical) of this plane bundle. Thus we obtain a subscheme of F_k. Similarly a point of R_k gives rise to projective line subbundles of $P(D_j)$, $P(D_{j^{-1}})$ containing the sections
329 of $P(D_j)$, $P(D_{j^{-1}})$ given by j^2 and j^{-2} respectively. To compute the Hilbert polynomials of these schemes, we may assume without loss of generality that the scheme is obtained by gluing two projective line bundles $P(F_1)$, $P(F_2)$ on X where F_1 and F_2 are of degree zero and contain either j^2 or j^{-2} as line subbundles, along the sections given by these subbundles. Now, from Corollary 6.22, it follows that $\mathscr{O}(1)$ restricts to $P(D_j)$ as $K_X^2 \otimes j^4 \otimes \tau^2$. Its restriction to the section of $P(D_j)$ is therefore seen to be K_X^2. Hence the Hilbert polynomial of the subschemes corresponding to points of Q_j, R_k is, on using the Mayer-Vietoris sequence,

$$2\chi(X, K_X^{m+n} \otimes S^{2m}(F^*)) - \chi(X, K_X^{m+n}) = (4m+1)(2m+2n-1)(g-1).$$

Thus, we have identified the sets Q_j, $Q_{j^{-1}}$, R_k with subsets of $H_{2,k}$.

If k is a node, Q_k is defined to be the variety of subschemes of the form $X \times C$, where C is a conic in $P = PH^1(X, \mathscr{O})$, while R_k denotes those of the form $X \times C$ where C is a line thickened by τ^{-1} contained in P_t but not in P. As above Q_k, R_k are checked to be subsets of $H_{2,k}$.

Proposition 7.8. *If k is not a node of $\mathscr{K}$, and j, $j^{-1} \in J$ are points over k, then $H_{2,k} = Q_j \cup Q_{j^{-1}} \cup R_k$. If k is a node of $\mathscr{K}$, then $H_{2,k} = Q_k \cup R_k$.*

The rest of this section will be devoted to the proof of Proposition 7.8.

Lemma 7.9. *If $Z \in H_2$, then the map $p : Z \to X$ is surjective and for generic $x \in X$, the fibre Z_x has $(2m + 1)$ as the Hilbert polynomial with respect to the ample generator h of* Pic U_x.

Proof. Since the polynomial $P(m, n)$ is not independent of n, it follows that the map $p : Z \to X$ is surjective. For large m, we have

$$P(m, n) = \chi(X, p_*(K_{\det}^{-m}) \otimes K_X^n).$$

In other words, $P(m, n)$ is the Hilbert polynomial of $p_*(K_{\det}^{-m})$ with respect to the ample line bundle K_X on X. Hence the rank of the sheaf $p_*(K_{\det}^{-m})$ is $\dfrac{1}{\deg K}$ (the coefficient of n in $P(m, n)$), namely $\dfrac{1}{2g-2}(8m + 2)(g - 1) = 4m + 1$. But $K_{\det}^*$ restricts to U_x as $K_{U_x}^*$, which is twice the ample generator of Pic U_x. Hence the lemma. □

Lemma 7.10. *Let S be a subscheme of U_x having $2m + 1$ as Hilbert polynomial. If $S_{\text{red}} \subset (F_k)_{\text{red}}$, for some $k \in \mathscr{K}$, then S is a conic contained schematically in F_k.* **330**

Remark 7.11. The proof of Lemma 7.10 given below can be simplified if one knew a priori that $h|S$ is very ample. This would follow for instance if we could show that h itself is very ample which is very likely to be true and indeed so at least if X is hyperelliptic [1, 5.10, II]. However, since the irreducible components of $(F_k)_{\text{red}} \cap U_x$ are projective spaces, the restriction of h to these is very ample, as ample bundles on a projective space are very ample. From this it is easy to see that $h|(F_k)_{\text{red}} \cap U_x$ is very ample and hence $h|S_{\text{red}}$ is very ample. This fact will be used in the proof.

Proof of Lemma 7.10. Let S be a subscheme of U_x having $2m+1$ as Hilbert polynomial. If S is reduced, our assertion follows from Lemma 4.2 and Remark 4.4. If S is not reduced, we shall use Lemma 4.3. Let S' be the thickening of $S_{\text{red}} = \mathbf{P}^1$ by the line

bundle L mentioned in that lemma. We first show that $\deg L \geq -1$. Since S' is a subscheme of U_x, the bundle L is a quotient of the conormal bundle of S_{red} in U_x. Now we have the exact sequences (Lemma 6.21)

$$0 \to \tau \otimes \text{ trivial } \to N^*_{P(D)_{(j,x)},U_x} \to \text{ trivial } \to 0$$

and

$$0 \to \tau \otimes \text{ trivial } \to N^*_{P(D)_{(j,x)},U_x} \to \tau \otimes \Omega^1 \to 0$$

according as $j^2 \neq 1$ or $j^2 = 1$. On the other hand, we have the exact sequence

$$0 \to N^*_{P(D)_{(j,x)},U_x} \to N^*_{\mathbf{P}^1,U_x} \to \tau^{-1} \otimes \text{ trivial } \to 0.$$

Since $\tau \otimes \Omega^1$ restricts to $\mathbf{P}^1$ as $\tau^{-1} \oplus$ trivial, it follows that any quotient line bundle of $N^*_{\mathbf{P}^1,U_x}$ is of deg ≥ -1. Now if $j^2 \neq 1$, any quotient of $N^*_{\mathbf{P}^1,U_x}$ isomorphic to τ^{-1} is actually a quotient of $N^*_{\mathbf{P}^1,P(D)_{(j,x)}}$ so that $S \subset P(D)_{(j,x)}$. If $j^2 = 1$, we similarly conclude that $S \subset F_k = PH^1(X, \mathcal{O})_t$. This proves the lemma.

Lemma 7.12. *If $Z \in H_{2,k}$, then the morphism $p : Z \to X$ is flat.*

331 *Proof.* Using generic flatness, we see that on an open subset of X, p itself defines a section of a suitable relative Hilbert scheme for the morphism $Z \to X$. This extends to a section over the whole of X and thus gives a closed subscheme Z' of Z flat over X. Now from Lemma 7.9, it follows that *all* the fibres Z'_x of Z' have Hilbert polynomial $2m+1$ with respect to h. Now by Lemma 7.10, it follows that $Z'_x \subset F_k$ for each $x \in X$. Since tha map $p : Z' \to X$ is flat, it follows that $Z' \subset F_k$. Now suppose we show that $K^m_{\text{det}}|F_k$ has a direct image V on X of the form $K_X^{4m} \otimes$ (a semistable vector bundle of degree 0), at least for large m. Then we would have $p_*(K^m_{\text{det}}|Z')$ is a quotient of V for large m and hence would be a vector bundle of rank $4m + 1$ and degree $\geq 2m(g-1)(4m+1)$. Hence

$$\chi(Z', K^m_{\text{det}} \otimes p^* K^n_X) = \chi(X, p^*(K^m_{\text{det}}) \otimes K^n_X)$$

$$\geq (4m+1)(2m+2n-1)(g-1) = P(m,n)$$

by Riemann-Roch theorem. On the other hand, if I is the sheaf of ideals defining Z' in Z, then from the exact sequence

$$0 \to I \to \mathscr{O}_Z \to \mathscr{O}_{Z'} \to 0,$$

we conclude that $\chi(Z, \mathscr{O}(n)) \geq \chi(Z', \mathscr{O}(n))$ for large n, and that equality holds only if $Z = Z'$. It remains to prove □

Lemma 7.13. *The direct image of* $K_{\det}^m|F_K$ *on* X *is of the form* $K_X^m \otimes$ *(a semistable vector bundle of degree* 0*).*

Proof. First assume that k is not a node. Then $F_k = P(D)_j \cup P(D)_{j^{-1}}$ with $P(D)_j \cap (PD)_{j^{-1}} = X$. The restriction of $K_{\det}$ to $P(D)_j$ (resp. X) is isomorphic to $K_X \otimes j^4 \otimes \tau_{D_j}^2$ (resp. K_X) (Corollary 6.22). Now the direct image of $j^4 \otimes \tau_{D_j}^2$ is isomorphic to $j^4 \otimes S^2(D_j^*)$, and the restriction map to the direct image of $j^4 \otimes \tau_{D_j}^2$ on X (namely, the trivial bundle) is induced by the natural map $S^2(D^*) \to j^{-4}$, given by the inclusion $j^2 \to D_j$. Hence, by the Mayer-Vietoris sequence, we see that $p^*(K_{\det}^m|F_k)$ is the tensor product of K_X^m and the fibre product of $j^{4m} \otimes S^{2m}(D_j^*)$ and $j^{-4m} \otimes S^{2m}(D_j^*)$ over the trivial bundle. Since $S^{2m}(D_j^*)$ is obtainable (6.1) as successive extension of line bundles of degree 0, it is semistable of degree 0. Hence so is the fibre product mentioned above proving Lemma 7.13 in this case.

Now let k be a node. Then we have $\mathrm{Pic}(X \times PH^1(X, \mathscr{O})_t) \to$ **332**
$\mathrm{Pic}(X \times PH^1(X, \mathscr{O}))$ is an isomorphism. In particular from 6.22, we see that the restriction of $K_{\det}^m$ to $X \times PH^1(X, \mathscr{O})_t$ is of the form $K_X^m \otimes$ a line bundle coming from $PH^1(X, \mathscr{O})_t$. Hence its direct image is of the form $K_X^m \otimes$ a trivial bundle, completing the proof of Lemma 7.13.

□

Proof of Proposition 7.9. Let $Z \in H_{2,k}$. By lemma 7.12, the map $p : Z \to X$ is flat, and by Lemma 7.10, Z is schematically

contained in F_k and all the fibres are conics. Let us first consider the case when Z is a subscheme of $P(D)_j$ for some j over k. By (4.4, iv) there exists a vector subbundle E of D_j of rank 3, such that $Z \subset P(E)$. Now Z defines a divisor in $P(E)$ with $L_Z \simeq \tau_E^2 \otimes L$ for some line bundle L on X. We have the exact sequence

$$0 \to \tau_E^{-2} \otimes L^* \otimes K_{\det}^{-m}|_{P(E)} \otimes K_X^n \to K_{\det}^{-m}|_{P(E)} \otimes K_X^n \to K_{\det}^{-m}|_Z \otimes K_X^n \to 0.$$

Substituting $K_{\det}^*|_{P(E)} \simeq \tau_E^2 \otimes K_X \otimes j^4$, and taking direct image on X, we obtain

$$\chi(Z, K_{\det}^{-m} \otimes K_X^n) = \chi(X, K_X^{m+n} \otimes S^{2m}(E^*)) - \chi(X, L^* \otimes K_X^{m+n} \otimes S^{2m-2}(E^*)).$$

The right side can be computed in terms of the degrees of L and E by the Riemann-Roch theorem. But the left side is given to be $P(m, n) = (4m + 1)(2m + 2n - 1)(g - 1)$. Equating coefficients of these two polynomials, we check that $\deg L = 0$ and $\deg E = 0$. In other words, when $Z \subset P(D)_j$, we have shown that Z is a divisor in $P(E)$ given by a nonzero quadratic form $E \to E^* \otimes L$, where L is of degree 0 and E a subbundle of D_j, also of degree 0. Now, if k is a node, clearly E is also trivial since D_j is trivial in this case. Moreover the quadratic form is a nonzero section of $S^2(D_j^*) \otimes L$, with $\deg L = 0$, and hence it follows (a) that L is trivial and (b) that the quadratic form is a constant. Thus if $Z \subset P(D)_j$ with $j^2 = 1$, then $Z \in R_k$. On the other hand, if k is not a node, by § 6.1, E is the inverse image of a trivial vector subbundle of $H^1(X, j^2)_X$ of rank 2. Notice first that the map $E \to E^* \otimes L$ cannot be an isomorphism since $E^* \otimes L$ contains a direct sum of two line bundles of degree 0 while E does not. Moreover the kernel of $E \to E^* \otimes L$ is a proper subbundle of degree 0 and, again
333 by § 6.1, it must contain j^2. Thus the quadratic form on E is induced by an L-valued quadratic form on the trivial subbundle $E/_{j^2} \subset H^1(X, j^2)_X$ viz. a nonzero section of $S^2((E/j^2)^* \otimes L$. As before, since L is of degree 0, it follows (a) that L is trivial, and (b) that the section of $S^2(E/j^2)^*$ is constant. This proves that if Z is contained in $P(D)_j$ then $Z \in Q_j$.

It remains to consider the case when $Z \not\subset P(D)_j$ for any j over k. Clearly if k is not a node, Z consists of two irreducible components Z_1 and Z_2 with $Z_1 \subset P(D)_j$, $Z_2 \subset P(D)_{j^{-1}}$. Moreover, Z_1 (resp. Z_2) defines a subbundle E_1 (resp. E_2) of D_j (resp. $D_{j^{-1}}$) of rank 2, containing j^2 (resp. j^{-2}). Let their degrees be d_1, d_2. By the Mayer-Vietoris sequence, we see as before,

$$\begin{aligned} P(m,n) = \chi(Z, K_{\det}^{-m} \otimes K_X^n) = \chi(X, S^{2m}(E_1^*) \otimes K_X^{m+n} \otimes j^{4m}) \\ + \chi(X, S^{2m}(E_2^*) \otimes K_X^{m+n} \otimes j^{-4m}) \\ - \chi(X, K_X^{m+n}). \end{aligned}$$

Computing the right side by the Riemann-Roch theorem, and substituting for $P(m,n)$, we obtain $d_1 = d_2 = 0$. Thus E_1, E_2 are inverse images of line subbundle of the trivial bundles $H^1(X, j^2)_X$, $H^1(X, j^{-2})_X$ and since these subbundles are of degree 0, they must themselves be trivial. In other words, $Z \in R_k$.

Finally let k be a node and $Z \in H_{2,k}$, $Z \not\subset P(D)_j = (F_k)_{\text{red}}$. By 7.10, $Z \subset F_k$. In this case, Z_{red} is projective bundle associated to a rank 2 vector subbundle E of $D_j = H^1(X, \mathcal{O})_X$. Moreover Z is given by thickening within $X \times F_k$ by a line bundle of the form $\tau_E^{-1} \otimes L$, where L is a line bundle on X. From the exact sequence

$$0 \to \tau_E^{-1} \otimes L \to \mathcal{O}_Z \to \mathcal{O}_{Z_{\text{red}}} \to 0$$

and the fact that $K_{\det}|X \times PH^1(X, \mathcal{O}) \simeq \tau^2 \otimes K_X$ we obtain

$$\begin{aligned} \chi(Z, K_{\det}^{-m} \otimes K_X^n) = \chi(X, S^{2m}(E^*) \otimes K_X^{m+n}) \\ + \chi(X, S^{2m-1}(E^*) \otimes K_X^{m+n} \otimes L). \end{aligned}$$

Equating this expression with $P(m,n)$, we obtain $\deg E = 0$, $\deg L = 0$. Since E is contained in the trivial bundle $H^1(X, \mathcal{O})_X$, E itself is trivial. Now we claim that if L is nontrivial, $\text{Hom}(\tau_E \otimes$ **334** $L^{-1}, N_{X \times P(E), U_X}) = 0$. This follows from sequence

$$0 \to N_{X \times P(E), X \times P} \to N_{X \times P(E), UX} \to N_{X \times P, U_X}|_{X \times P(E)} \to 0$$

and the sequences in 6.21, ii), since

(a) $H^0(\mathrm{Hom}(\tau_E \otimes L^{-1}, N_{X\times P(E),X\times P})) = H^0(\mathrm{Hom}(\tau_E \otimes L^{-1}, \tau \otimes$ trivial $)) = 0$.

(b) $H^0(\mathrm{Hom}(\tau_E \otimes L^{-1}, \tau^{-1} \otimes F)) = 0$.

(c) $H^0(\mathrm{Hom}(\tau_E \otimes L^{-1}, \tau^{-1} \otimes T_P)) = 0$, and

(d) $H^0(\mathrm{Hom}(\tau_E \otimes L^{-1}, T_P)) = 0$.

It follows that L is trivial, since $\tau^{-1} \otimes L \subset N_{X\times P(E),U_X}$. Thus the scheme Z is of the form $X \times C$, where C is a conic contained in P_t, and hence $Z \in R_k$. This completes the proof of Proposition 7.9.

8 Nonsingularity of the Hecke component

We will now prove that H_0, the irreducible component of $\mathrm{Hilb}(U_X)$ containing Hecke cycles, is nonsingular. Since H_0 is a variety of dimension $3g-3$, it is enough to show that the Zariski tangent space at any point of H_0 is of dimension $3g-3$. This is done in Propositions 8.1, 8.5, 8.9 and 8.12. From the description in § 7 of the nature of the subschemes of U_X occurring in H_2 (and hence H_0), it follows that all these schemes are local complete intersections and hence the Zariski tangent space at any $Z \in H_0$ to the Hilbert scheme itself is given by $H^0(Z, N_{Z,U_X})$ [9a]. Thus the nonsingularity of H_0 at some $Z \in H_0$ would follow if $\dim H^0(Z, N_{Z,U_X}) \leq 3g-3$. However, it turns out that there are points in H_0 at which the Hilbert scheme itself is not smooth. We will first compute $H^0(Z, N_{Z,U_X})$ for $Z \in H_0$.

We have to discuss several cases.

Proposition 8.1. *Let $Z \in R_k$, k not a node and $Z \not\subset P(D)_j$, for any j over k. Then the Hilbert scheme is smooth at Z.*

The proof is completed in 8.4.

By the definition of R_k, there exist points p, p' in

$$PH^1(X, j^2),\ P(H^1(X, j^{-2}))$$

such that if E, E' are the inverse images in D_j, $D_{j^{-1}}$ of the trivial line bundles corresponding to these two points, then $Z = P(E) \cap$ **335**
$P(E')$. Recall also that $P(E) \cap P(E') = X$ imbedded in $P(D_j)$, $P(D_{j^{-1}})$ by means of the subbundles j^2, j^{-2} of D_j, $D_{j^{-1}}$ respectively.

Lemma 8.2. *Let U be a nonsingular variety and l_1, l_2 be nonsingular subvarieties intersecting (schematically) in a nonsingular subvariety X which is a divisor in both l_1 and l_2.*

(i) *Then the reduced scheme $l = l_1 \cup l_2$ is a local complete intersection.*

(ii) *Moreover, we have the exact sequence*

$$0 \to N^*_{l,U} \to \widetilde{N}_1 \oplus \widetilde{N}_2 \to N^*_{l,U}|X \to 0$$

where N_1, N_2 are the restrictions of $N^*_{l,U}$ to l_1, l_2 respectively.

(iii) *N_1 fits in an exact sequence*

$$0 \to N_1 \to N^*_{l_1,U} \to N^*_{X,l_2} \to 0, \quad \textit{while}$$

(iv) *$N_{l,U}|X$ fits in the sequence*

$$0 \to N_{X,U}/_{N_X, l_1 \oplus N_X, l_2} \to N_{l,U}|X \to N_{X,l_1} \otimes N_{X,l_2} \to 0.$$

Proof. At any point of l, we can choose a regular system of parameters $(x_1, \ldots, x_r, y_1, y_2, z_1, \ldots, z_s)$ for $\mathcal{O}_U$ with $(x_1, \ldots, x_r, y_1) = I_1$ and $(x_1, \ldots, x_r, y_2) = I_2$ being the ideals defining l_1, l_2 respectively. Since l is then defined by $I_1 \cap I_2 = (x_1, \ldots, x_r, y_1, y_2)$, it follows that l is a complete intersection. The sequence (ii) is obtained by tensoring with $N^*_{l,U}$ the basic exact sequence

$$0 \to \mathcal{O}_l \to \widetilde{\mathcal{O}}_{l_1} \oplus \widetilde{\mathcal{O}}_{l_2} \to \widetilde{\mathcal{O}}_X \to 0.$$

We have now the isomorphism $N_l = I_1 \cap I_2/_{(I_1 \cap I_2)^2} \simeq I_1 \cap I_2/_{I_1(I_1 \cap I_2)}$ and an exact sequence

$$0 \to I_1 \cap I_2/_{I_1(I_1 \cap I_2)} \to I_1/_{I_1^2} \to I_1/_{I_1^2 + I_1 \cap I_2} \to 0.$$

But

$$I_1/_{I_1^2 + I_1 \cap I_2} \simeq \frac{I_1/_{I_1 \cap I_2}}{(I_1/I_1 \cap I_2)^2} \simeq \frac{I_1 + I_2/_{I_2}}{(I_1 + I_2/_{I_2})^2} \simeq N^*_{X,l_2}$$

proving (iii).

Finally, restrict (iii) to X to get the four term sequence

$$0 \to \underline{\mathrm{Tor}}^1_{l_1}(N^*_{X,l_2}, \mathscr{O}_X) \to N^*_{l,U}|X \to N^*_{l_1,U}|X \to N^*_{X,l_2} \to 0.$$

From the resolution

$$0 \to L_X^{-1} \to \mathscr{O}_{l_1} \to \mathscr{O}_X \to 0$$

336 and the isomorphism $L_X^{-1}|X \simeq N^*_{X,l_1}$, we evaluate the Tor-term to be $N^*_{X,l_2} \otimes N^*_{X,l_1}$. On the other hand, it is clear that $N^*_{l_1,U}|X \to N^*_{X,l_2}$ has kernel $\simeq (N_{X,U}/_{N_X,l_1 \otimes N_X,l_2})^*$. Dualising, we obtain (iv). □

We will apply 8.2 to the case $U = U_X$, $l_1 = P(E)$, $l_2 = P(E')$ and $X = l_1 \cap l_2$. In this case we have $N_{X,l_2} \simeq \mathrm{Hom}(j^{-2}, \mathscr{O}) = j^2$ from the exact sequence

$$0 \to j^{-2} \to E' \to \mathscr{O} \to 0.$$

From the definition of N_1, we get

$$\det N_1 \simeq \det N^*_{l_1,U_X} \otimes (\det \widetilde{N}^*_{X,l_2})^{-1}.$$

But $\det \widetilde{N}^*_{X,l_2} \simeq L_X$, where L_X is the line bundle on l_1 associated to the divisor X, as is seen from the fact that the bundle j^{-2} on X extends to l_1 and from the resolution

$$0 \to L_X^{-1} \to \mathscr{O}_{l_1} \to \mathscr{O}_X \to 0$$

for the sheaf $\mathscr{O}_X$ on l_1. But L_X is easily seen to be τ_E. Thus if ω^l is the dualishing sheaf on l, we get $\omega_l|l_1 \simeq \omega_{U_X} \otimes \det N_1^* \simeq \omega_{U_X} \otimes \det N_{l_1,U_X} \otimes (\det N_{X,l_2}^*) \simeq \omega_{l_1} \otimes \tau_E$. But $\omega_{l_1} \simeq K_X \otimes K_\pi$, where K_π is the canonical bundle along the fibres of $P(E) \to X$ and hence $\simeq \tau_E^{-2} \otimes \det E^* \simeq \tau_E^{-2} \otimes j^{-2}$. This proves

Lemma 8.3. *The dualising sheaf ω_l of $l = l_1 \cup l_2$ restricts to l_1 as $\tau_E^{-1} \otimes j^2 \otimes K_X$. In particular, $\omega_l|_X \simeq K_X$.*

Proof of Proposition 8.1. We wish to compute $H^2(l, \omega_l \otimes N_{l,U_X}^*)$ which is dual to the space $H^0(l, N_{l,U_X})$ required. To this end we first compute $H^2(l_1, \omega_l \otimes N_l)$. From the exact sequence $0 \to N_1 \to N_{l_1,U_X}^* \to \widetilde{j^{-2}} \to 0$, we obtain $H^2(l_1\omega_l \otimes N_1) \simeq H^2(l_1, \omega_l \otimes N_{l_1,U_X}^*)$ since $H^i(X, K_X \otimes j^{-2}) = 0$ for $i = 1, 2$. But $H^2(l_1, \omega_l \otimes N_{l_1,U_X}^*) = H^2(l_1, \omega_{l_1} \otimes \tau \otimes N_{l_1,U_X}^*)$ by Lemma 8.3, which in turn is dual to $H^2(l_1, \tau^{-1} \otimes N_{l_1,U_X})$. To compute this, we use the exact sequence

$$0 \to N_{l_1,P(D_j)} \to N_{l_1,U_X} \to N_{P(D_j),U_X}|_{l_1} \to 0$$

and observe that $N_{l_1,P(D_j)} \simeq \tau \otimes V'$, where V' is a vector space of rank $(g-2)$. Hence $\dim H^0(l_1, \tau^{-1} \otimes N_{l_1,P(D_j)}) = g-2$. To compute $H^0(l_1, \tau^{-1} \otimes N_{P(D_j),U_X})$, we appeal to Lemma 6.22, 1). Since $H^0(l_1, \tau^{-1} \otimes H^1(X, \mathscr{O})) = 0$ and $H^0(l_1, \tau^{-2} \otimes \pi^* F(j))$ is also seen to be zero, we get $H^0(l_1, \tau^{-1} \otimes N_{P(D_j),U_X}) = 0$ and hence **337**
$\dim H^2(l_1, \omega_l \otimes N_1) = g-2$. Now Proposition 8.1 will follow from the exact sequence in 8.2 tensored with ω_l, if we prove

Lemma 8.4. $\dim H^1(X, N_{l,U_X}^* \otimes \omega_l) \leq g+1$.

Proof. Since $\omega_l|_X \simeq K_X$ by Lemma 8.3, we have only to show by duality, that $\dim H^0(X, N_{l,U_X}) \leq g+1$. For this, we will use Lemma 8.2, (iv). In fact, $N_{X,l_1} \simeq j^{-2}$ and $N_{X,l_2} \simeq j^2$ and hence $H^0(X, N_{X,l_2} \otimes N_{X,l_2})$ is of dimension 1. To compute

$$H^0(X, N_{X,U_X}/N_{X,l_1} \oplus N_{X,l_2}),$$

we use the exact sequence in 6.21, namely

$$0 \to N_{X,P(D_j)} \oplus N_{X,P(D_{j^{-1}})} \to N_{X,U_X} \to F \to 0.$$

Thus we get

$$0 \to (N_{l_1,P(D_j)} \oplus N_{l_2,P(D_{j-1})})|X \to N_{X,U_X/N_{X,l_1}\oplus N_X,l_2} \to F \to 0.$$

But $N_{l_1,P(D_j)}|X \simeq \tau \otimes (D_j/_{l_1})|X \simeq j^{-2}\otimes$ trivial. Hence H^0 of the first term above is zero while $H^0(X,F)$ is clearly of dimension g. This proves Lemma 8.4, and hence completes the proof of Proposition 8.1. □

Proposition 8.5. *Let $Z \in Q_k$, k a node. Assume that $Z = X \times C$, where C is a conic in $P = PH^1(X,\mathscr{O})$. Then* $\dim H^0(Z, N_{Z,U_X}) \leq 3g-3$, $3g-2$ *or* $3g-1$ *according as C is nondegenerate, is a pair of intersecting lines, or is a double line. In particular, if C is nondegenerate, the Hilbert scheme is smooth at Z.*

Proof. We first recall [9, SGA 6, VII, Proposition 1.7] that if $X_1 \subset X_2 \subset X_3$ with X_2, X_3 smooth and X_1 a local complete intersection, then by pushing out the restriction to X_1 of the exact sequence

$$0 \to T_{X_2} \to T_{X_3}|X_2 \to N_{X_2,X_3} \to 0$$

by means of the map $T_{X_2}|X_1 \to N_{X_1,X_2}$, we obtain the exact sequence

$$0 \to N_{X_1,X_2} \to X_{X_1,X_3} \to N_{X_2,X_3}|X_1 \to 0.$$

We wish to apply this to the case $X_1 = Z$, $X_2 = X\times P$ and $X_3 = U_X$, and use the description of $T_{U_X}|X \times P$ given in 6.22, (ii). Then we see that N_{Z,U_X} fits in exact sequence

$$0 \to N' \to N_{Z,U_X} \to \tau^{-1} \otimes F|Z \to 0 \tag{8.6}$$

338 where N' is obtained as the push-out of the sequence

$$0 \to T_P|Z \to \operatorname{Im} d\theta_E|Z \to \tau^{-1} \otimes T_P|Z \to 0$$

by means of the map $T_P|Z \to N_{Z,X\times P} = N_{C,P}$. Since the direct image of $\tau^{-1} \otimes F|Z$ on X is zero, it follows that $H^0(Z, \tau^{-1} \otimes F) =$

0 and hence that $H^0(Z, N') \to H^0(Z, N_{Z,U_X})$ is an isomorphism. Again by 6.22, (ii), we have the commutative diagram (on Z)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau \otimes H^1(X, \mathscr{O}) & \longrightarrow & F' \otimes H^1(X, \mathscr{O}) & \longrightarrow & H^1(X, \mathscr{O})_Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{C,P} & \longrightarrow & N' & \longrightarrow & \tau^{-1} \otimes T_P & \longrightarrow & 0 \end{array} \tag{8.7}$$

where the top sequence is obtained by tensoring the universal extension F' on $X \times P$ with $H^1(X, \mathscr{O})$. Now $H^0(N_{C,P})$ is easily computed to be $3g - 4$, for instance by imbedding C as a divisor in a plane ϖ and proving $\dim H^0(C, N_{C,\widetilde{\omega}}) = 5$ and $\dim H^0(C, N_{\widetilde{\omega}P}) = 3(g-3)$. It remains therefore to compute the kernel of the boundary homomorphism $H^0(X \times C, \tau^{-1} \otimes T_P) \to H^1(X \times C, N_{C,P})$. But now it is easy to see that $H^0(P, \tau^{-1} \otimes T_P) \to H^0(C, \tau^{-1} \otimes T_P)$ is surjective by checking $H^0(P, \tau^{-1} \otimes T^P) \to H^0(\varpi, \tau^{-1} \otimes T_P)$ and $H^0(\varpi, \tau^{-1} \otimes T_P) \to H^0(C, \tau^{-1} \otimes T_P)$ are both surjective. Thus from (8.7), we see that the required boundary homomorphism is the composite of that of the top sequence $H^1(X, \mathscr{O}) \to H^1(X \times P, H^1(X, \mathscr{O}) \otimes \tau)$ and the natural map $H^1(X \times P, H^1(X, \mathscr{O}) \otimes \tau) \to H^1(C, N_{C,P})$. Again we have the diagram (on Z).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau \otimes H^1(X, \mathscr{O}) & \longrightarrow & F' \otimes H^1(X, \mathscr{O}) & \longrightarrow & H^1(X, \mathscr{O})_Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & \searrow & \\ & & & & & & & & \tau^{-1} \otimes T_P \\ & & & & & & & \swarrow & \\ 0 & \longrightarrow & N_{C,P} & \longrightarrow & F' \otimes \tau^{-1} \otimes N_{C,P} & \longrightarrow & \tau^{-1} \otimes N_{C,P} & \longrightarrow & 0 \end{array} \tag{8.8}$$

where the lower sequence is obtained as the tensor product on $X \times C$ of the universal extension F' by $\tau^{-1} \otimes N_{C,P}$. From (8.8) we conclude that the required map is the composite of the natural map $H^0(C, \tau^{-1} \otimes T_P) \to H^0(C, \tau^{-1} \otimes N_{C,P})$ and the boundary homomorphism $H^0(C, \tau^{-1} \otimes N_{C,P}) \to H^1(X \times C, N_{C,P})$. From the definition of the universal extension, we conclude that the latter **339** map is injective. Thus finally we have

$$\dim H^0(X \times C, N_{Z,U_X}) = \dim H^0(X \times C, N')$$

$$= 3g - 4 + \dim \ker H^0(C, \tau^{-1} \otimes T_P) \to H^0(C, \tau^{-1} \otimes N_{C,P})$$

so that Proposition 8.5 would follow from □

Lemma 8.9. *If C is a conic in a projective space P, then*

$$\ker H^0(C, \tau^{-1} \otimes T_P) \to H^0(C, \tau^{-1} \otimes N_{C,P})$$

is of dimension 1, 2, *or* 3 *according as C is nondegenerate, is a pair of lines, or is a double line.*

Proof. By duality, and noting that $\tau^{-1}|C$ is the dualising sheaf, we have to compute the dimension of the cokernel of $H^1(C, N^*_{C,P}) \to H^1(C, \Omega^1_P)$. From the exact sequence

$$N^*_{C,P} \to \Omega^1_P|_C \to \Omega^1_C \to 0$$

we see that this cokernel is isomorphic to $H^1(C, \Omega^1_C)$ since $H^2(C, \mathscr{F}) = 0$ for any coherent sheaf $\mathscr{F}$ on C. For a nondegenerate conic, its dimension is clearly 1. On the other hand, if C is a pair of lines l_1, l_2, we have a surjection $\Omega^1_C \to \widetilde{\Omega}^1_{l_1} \oplus \widetilde{\Omega}^1_{l_2}$ with the kernel sheaf supported at the point of interesection of l_1 and l_2. Hence $\dim H^1(\Omega^1_C) = 2$. If C is the τ^{-1}-thickening of the projective line l, then we also have a projection $p : C \to l$. From this we get the exact sequence

$$0 \to p^*\Omega^1_l \to \Omega^1_C \to \Omega^1_p \to 0.$$

Now Ω^1_p is easily seen to be $i_*(\tau^{-1})$ where $i : l \to C$ is the inclusion, and hence $H^i(C, \Omega^1_p) = 0$ for all i. On the other hand,

$$H^1(C, p^*\Omega^1_l) \simeq H^1(l, \Omega^1_l) \oplus H^1(l, \tau^{-1} \otimes \Omega^1_l)$$

is of dimension 3. This proves Lemma 8.9.

□

We will in fact see that if $Z \in H_{2,k}$ corresponds to a degenerate conic $C \subset P = PH^1(X, \mathscr{O})$, then Hilb$(P(n))$ is not smooth at Z.

To show that H_0 is nonsingular at such a Z, we proceed as follows. Let $\widetilde{Z} \subset H_2 \times U_X$ be the universal subscheme representing H_2. Then the map $\widetilde{Z} \to H_2 \times X$ given by $(p_{H_2}, \det)$ is flat. Since the fibres are conics, it follows that this is a morphism of complete intersection [9, SGA 6, VII]. Moreover, the direct image on $H_2 \times X$ of the dual of the relative dualising sheaf is locally free of rank 3. The determinant of this direct image is a line bundle and hence **340** induces a map $\varphi : H_2 \to \operatorname{Pic} X$. If $Z = P(E)$, is a good Hecke cycle, its image under φ is given as follows: If $\pi : P(E) \to X$ is the projective fibration, $\varphi(Z) = \det \pi_*(T_\pi) \simeq \det \operatorname{Ad} E$ is the trivial bundle. Hence φ is a constant on H_0. On the other hand, if $Z \in Q_j$ is given by (rank 2) subbundles F_1, F_2 of D_j containing j^2 with $F_1/_{j^2}$, $F_2/_{j^2}$ trivial, then $\omega_\pi^* \simeq \tau_E \otimes (F_1 \cap F_2)|_Z$ where $E = F_1 + F_2$. Now from the exact sequence (on $P(E)$)

$$0 \to \tau_E^{-1} \otimes F_1 \cap F_2 \to \tau_E \otimes F_1 \cap F_2 \to \omega_\pi^* \to 0$$

we get, on taking direct images on X,

$$\begin{aligned} \varphi(Z) &= \det \pi_*(\tau_E \otimes F_1 \cap F_2) = \det(E^* \otimes F_1 \cap F_2) \\ &= (F_1 \cap F_2)^3 \otimes \det E^* = j^4. \end{aligned}$$

since $F_1 \cap F_2 = j^2$ and $\det E = j^2$. Thus we have

Lemma 8.10. *If $Z \in H_0$, then $\varphi(Z)$ is trivial, while if $Z \in Q_j$ is given by two subbundles F_1, F_2 of D_j containing j^2, then $\varphi(Z) = j^4$.*

Now in order to prove that H_0 is smooth at points $Z \in Q_k$, k a node, we will show

Lemma 8.11. Im$(d\varphi)_Z$ *has dimension* ≥ 1 *(resp. .* ≥ 2*) if Z is represented by a pair of lines (resp. a double line).*

Assume the Lemma for the moment. Since φ is a constant on H_0, it follows that $(d\varphi)_Z$ is zero on $T_Z(H_0)$. But by Proposition

8.5, $\dim T_Z(H_2) \leq 3g-2$ (resp. $3g-1$) in these cases. Now the map $(d\varphi)_Z : T_Z(H_2)/T_Z(H_0) \to T_{\varphi(Z)}(J)$ has image of dimension ≥ 1 (resp. ≥ 2). Hence $\dim T_Z(H_0) \leq (3g-3)$ in both these cases, proving H_0 is nonsingular at these points.

Proof of Lemma 8.11. Let P be a Poincaré bundle on $J \times X$. The sheaf $\mathscr{F} = R^1(p_j)_*(P^2)$ on J is locally free, outside elements of order 2, of rank $(g-1)$. It is easily seen that $\mathrm{Grass}_{g-1}(\mathscr{F})$ is isomorphic to the blow up $\pi : \widetilde{J} \to J$ at all elements of order 2. Hence on $\widetilde{J}$ we have a surjection $\pi^*\mathscr{F} \to Q \to 0$ where Q is the tautological quotient bundle or rank $(g-1)$. On the other hand, on
341 $X \times J$ we have a surjection of the vector bundle D onto $p_J^*\mathscr{F}$. This gives rise to a surjection $(1\times\pi)^*D \to p_{\widetilde{J}}^*Q$ (on $X\times\widetilde{J}$) where now Q is also locally free. Thus we get a family of subschemes of $P(D)$, parametrised by $P(Q)$. These subschemes are projective bundles associated to subbundles of D_j of rank 2 containing the kernel of $D_j \to Q_j$ given by the surjection above. Thus $P(Q) \times_{\widetilde{J}} P(Q)$ parametrises two families of projective line subbundles. It is easy to see that there is a flat family of schemes over $P(Q) \times_{\widetilde{J}} P(Q)$ which is obtained as the union of these two schemes and that this family is a family of subschemes of U_X with Hilbert polynomial $P(n)$. Thus we have a morphism $P(Q)\times_{\widetilde{J}}P(Q) \to H_2$. By Lemma 8.10, we have the commutative diagram

$$\begin{array}{ccc} P(Q) \times_{\widetilde{J}} P(Q) & \longrightarrow & H_2 \\ \downarrow & & \downarrow \\ \widetilde{J} & & \\ \downarrow & & \\ J & \xrightarrow{4_J} & J \end{array}$$

where 4_J is the map $j \mapsto j^4$ of J. If $Z \in H_2$ is represented by $X\times$ a pair of lines in $PH^1(X, \mathscr{O})$, then it is the image of some point in $P(Q) \times_{\widetilde{J}} P(Q)$ over an element of order 2 of J. Since the map $\widetilde{J} \to J$ has nonzero differential at any point, and since $P(Q) \times_{\widetilde{J}}$

$P(Q) \to \widetilde{J}$ is a fibration, it follows that $\mathrm{Im}(d\varphi)_Z$ has dimension ≥ 1. If Z is represented by $X\times$ a double line in $PH^1(X, \mathscr{O})$, then Z can be obtained as the image of two points of $P(Q) \times_{\widetilde{J}} P(Q)$ whose images x, x' in $\widetilde{J}$ are two different points over the given node. In fact, x and x' could be taken as any point on the line in $PH^1(X, \mathscr{O})$, to which Z corresponds. Hence

$$\dim \mathrm{Im}(d\varphi)_Z \geq \dim(\mathrm{Im}(d\pi)_x + \mathrm{Im}(d\pi)_x) \geq 2.$$

This completes the proof of 8.11 and hence the nonsingularity of H_0 at $Z \in Q_k$, k a node.

It remains to consider the case $Z \in R_k$, k a node.

Proposition 8.12. *Let $Z \in R_k$, k a node. Then* $\dim H^0(Z, N_{Z,U_X}) \leq$ **342** $3g - 3$.

Proof. Let $Z = X \times l_L$, where l is a projective line in $PH^1(X, \mathscr{O})$ and L is a subbundle of $N = N_{X\times l,U_X}$ isomorphic to the hyperplane bundle on l. Then we have the exact sequence

$$0 \to L^{-1} \otimes N_{Z,U_X}|X \times l \to N_{Z,U_X} \to N_{Z,U_X}|X \times l \to 0.$$

Now Proposition 8.12 follows from □

Lemma 8.13. (i) $\dim H^0(X \times l, N_{Z,U_X}) \leq 2g - 1$.

(ii) $\dim H^0(X \times l, L^{-1} \otimes N_{Z,U_X}) \leq g - 2$.

Proof of (i). By § 3.4 we have the exact sequence

$$0 \to N/L \to N_{Z,U_X}|_{X\times l} \to L^2 \to 0. \tag{8.14}$$

Since $H^0(X \times l, L^2) \simeq H^0(l, \tau^2)$ is of dimension 3, (i) will be proved if we show that $\dim H^0(X \times l, N/L) \leq 2(g - 2)$. Now for this computation, we need the exact sequences (Lemma 6.22. (ii))

$$0 \to \tau \otimes V/W \to N'/L \to V/W \to 0 \tag{8.15}$$

$$0 \to N'/L \to N/L \to \tau^{-1} \otimes F \to 0 \tag{8.16}$$

where $V = H^1(X, \mathcal{O})$, W is the two dimensional subspace of V corresponding to l, and N' is the kernel of the surjection $N \to \tau^{-1} \otimes F$. From (8.15), we conclude that $H^0(X \times l, N'/L) \simeq H^0(X \times l, N/L)$, since $H^0(X \times l, \tau^{-1} \otimes F) = 0$. To compute $H^0(X \times l, N'/L)$, we note that $\dim H^0(X \times l, \tau \otimes V/W) = 2(g-2)$ and hence i), Lemma 8.13 will be proved if we can show that the boundary homomorphism $H^0(V/W) \to H^1(\tau \otimes V/W)$ is injective. From 6.22 (ii), we have the commutative diagram

$$\begin{array}{ccc} V/W \simeq H^0(V/W) & \longrightarrow & H^1(\tau \otimes V/W) \simeq V \otimes V^* \otimes V/W \\ \uparrow & & \uparrow \\ V \simeq H^0(V) & \longrightarrow & H^1(\tau \otimes V) \simeq V \otimes V^* \otimes V \end{array}$$

where the lower map is the boundary homomorphism given by the universal extension F', and hence is the map $v \mapsto \mathrm{Id}_V \otimes v$. From this it is easy to conclude that the top horizontal map is injective. This proves (i).

343 **Proof of (ii).** We proceed as in (i). By tensoring 8.14 with L^{-1}, we are reduced to computing a) $H^0(X \times l, L^{-1} \otimes N/L)$ and b) the boundary homomorphism $H^0(X \times l, L) \to H^1(X \times l, L^{-1} \otimes N/L)$. As for a), we have from 8.16, $H^0(L^{-1} \otimes N/L) \simeq H^0(L^{-1} \otimes N'/L)$ and from 8.15, $H^0(L^{-1} \otimes N'/L) \simeq H^0(V/W)$ which has dimension $g-2$. Now (ii) will be proved if we can show that the boundary homomorphism b) is injective. To prove this, we need a description of the extension 8.14 (which is the dual of the extension 3.4 for l_L). In other words, for $N_{X \times P, U_X}$ we have the exact sequence (6.22. (ii))

$$0 \to \tau^{-1} \otimes T_P \to N_{X \times P, U_X} \to \tau^{-1} \otimes F \to 0.$$

Now $L = \tau^{-1} \times T_l \subset \tau^{-1} \otimes T_P | X \times l$. Any line subbundle of $N_{X \times l, U_X}$ mapping isomorphically on L gives rise to a thickening $X \times l_L$. From Proposition 3.8, we get the following information about 8.14. The map $d\theta_E | X \times l$ induces a map $N_{X \times P, P(D)} =$

$H^1(X, \mathcal{O})_{X\times P} \to \tau^{-1} \otimes T_P$. Let W be the inverse image of $L = \tau^{-1} \otimes T_l$ so that we have the exact sequence (on $X \times l$)

$$0 \to \tau^{-1} \to W \to \tau^{-1} \otimes T_l = L \to 0.$$

Of course, W is actually the trivial bundle with fibre = the subspace of P corresponding to l. Symmetrising this, we get the sequence

$$0 \to \tau^{-1} \otimes W \to S^2(W) \to L^2 \to 0. \tag{8.17}$$

Taking into account the description of the Hessian of $\theta_E|X\times P$, we see that the push-out of (8.14) by the map $N/L \to N_{X\times P, U_X}/\tau^{-1} \otimes T_P = \tau^{-1}\otimes F$ is isomorphic to the push-out of (8.17) by the natural map $\tau^{-1} \otimes W \to \tau^{-1} \otimes H^1(X, \mathcal{O}) \to \tau^{-1} \otimes F$. In view of this, we have a commutative diagram

$$\begin{array}{ccc} H^0(X \times l, L) & \to & H^1(X \times l, L^{-1} \otimes \tau^{-1} \otimes W) \\ \downarrow & & \downarrow \\ H^1(X \times l, L^{-1} \otimes N/L) & \longrightarrow & H^1(X \times l, L^{-1} \otimes \tau^{-1} \otimes F) \end{array} \quad H^1(X \times l, L^{-1} \otimes \tau^{-1} \otimes H^1(X, \mathcal{O}))$$

We have to show that the left vertical map is injective. The second top horizontal map is clearly injective, while the right vertical map is even an isomorphism. Thus our assertion will be proved if the map

$$H^0(l, L) \to H^1(l, L^{-1} \otimes \tau^{-1} \otimes W),$$

associated to (8.17) tensored with L^{-1}, is injective. But this is **344**
clear since $H^0(l, L^{-1} \otimes S^2(W)) = 0$, thus proving Lemma 8.13 and hence Proposition 8.12.

We now have

Theorem 8.14. *There is a natural morphism of the Hecke component H_0 into the moduli space U_0 giving a non-singular model of U_0, which is an isomorphism over the set of stable points. The reduced fibre over a non-nodal point of the Kummer variety is isomorphic to $\mathbf{P}^{g-2} \times \mathbf{P}^{g-2}$ while that over the node is isomorphic to the union of a $\mathbf{P}^5$ bundle over the Grassmannian of planes in $\mathbf{P}^{g-1}$ and a $\mathbf{P}^{g-2}$ bundle over the Grassmannian of lines in $\mathbf{P}^{g-1}$.*

Proof. The morphism $\pi : \mathrm{Hilb}(H, U_0, P(m,n)) \to U_0$ is an isomorphism over the open set U of stable points. Moreover the restriction of the morphism $\mathrm{Hilb}(H, U_0, P(m,n)) \to \mathrm{Hilb}(U_X, P(n))$ to the (schematic) closure of $p^{-1}(U)$ is an injective (Lemma 7.6) and birational (Theorem 5.13) morphism onto H_0. Since H_0 is smooth, this morphism is an isomorphism onto H_0. This proves the first part of the theorem.

The fibre over a non-nodal point of $\mathscr{K}$ corresponding to $j \in J$, $j^2 \neq 1$, is isomorphic to $PH^1(X, j^2) \times PH^1(X, j^2)$ (Proposition 7.8). The fibre over a node is isomorphic to the union of the space of conics which are schematically contained in $PH^1(X, \mathscr{O})$ and the space of τ^{-1}-thickenings of lines in $PH^1(X, \mathscr{O})$ which are contained in the thickening $PH^1(X, \mathscr{O})_t$ (Proposition 7.8). The first variety is a $\mathbf{P}^5$-bundle over the Grassmannian of planes in $PH^1(X, \mathscr{O})$ while the second is a $\mathbf{P}^{g-2}$ bundle over the Grassmannian of lines in $PH^1(X, \mathscr{O})$ (See Remark 4.4. iii). □

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On a Minkowski-Type Inequality for Multiplicities-II

By B. Teissier

INTRODUCTION. In the note [M.I.], to which this paper is a sequel, **347**
I proved the following Minkowski-type inequality, for multiplicities of primary ideals in a local noetherian Cohen-Macaulay algebra over an algebraically closed field; setting $d = \dim \mathcal{O}$, we have:

$$e(\mathfrak{n}_1 \cdot \mathfrak{n}_2)^{1/d} \leqslant e(\mathfrak{n}_1)^{1/d} + e(\mathfrak{n}_2)^{1/d} \qquad (*)$$

for any two primary ideals $\mathfrak{n}_1$, $\mathfrak{n}_2$ in $\mathcal{O}$, where $e(\mathfrak{n})$ denotes the multiplicity.

In this paper I will prove:

Theorem 1. *Let $\mathcal{O}$ be a Cohen-Macaulay normal complex analytic algebra. Then, given two ideals $\mathfrak{n}_1, \mathfrak{n}_2$ of $\mathcal{O}$ which are primary for the maximal ideal, we have the equality*

$$e(\mathfrak{n}_1 \cdot \mathfrak{n}_2)^{1/d} = e(\mathfrak{n}_1)^{1/d} + e(\mathfrak{n}_2)^{1/d}$$

if and only if there exist positive integers a, b such that

$$\overline{\mathfrak{n}_1^a} = \overline{\mathfrak{n}_2^b}$$

(where the bar means the integral closure of ideals, for the properties of which see [C.E.W.] *and* [L.T.]*).*

As will be clear from the proof, the conjunction of this result and (*) constitutes a generalization to arbitrary dimension of the classical result asserting the negative definiteness of the intersection matrix of the

components of the exceptional divisor of a resolution of singularities of a germ of a normal surface (see [DV], [M], [Li]).

This paper also contains two rather different applications of the above result. The first is a lightning proof of a special case of a theorem of Rees ([Re]) according to which, given two primary ideals $\mathfrak{n}_1$, $\mathfrak{n}_2$ in an equidimensional ring $\mathcal{O}$ such that $\mathfrak{n}_1 \subset \mathfrak{n}_2$ and $e(\mathfrak{n}_1) = e(\mathfrak{n}_2)$, we have $\overline{\mathfrak{n}_1} = \overline{\mathfrak{n}_2}$.

348 This result now plays a rather important role in the study of the geometry of singularities (cf. [C.E.W.] and [H.L.]) and is also used in the proof of the author's "principle of specialization of integral dependence" (cf. [D.C.N.] App. I) which is at the source of several results in the theory of equisingularity. The other application is a numerical characterization of those germs of real-analytic maps $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ which are isotopic (in a strong sense) to a germ of a holomorphic map $(\mathbf{C}, 0) \to (\mathbf{C}, 0)$: they are exactly those maps such that their local topological degree is equal to the square root of the degree of their complexification (= $\dim_{\mathbf{R}} \mathcal{O}_{f^{-1}(0),0}$). This result, which is apparently new, makes use of the algebraic description of the local topological degree, given by Eisenbud and Levine in [E.L.].

Remarks. (1) The equivalence relation: $\mathfrak{n}_1 \sim \mathfrak{n}_2$ if and only if there exist $a, b \in \mathbf{N}$ such that $\overline{\mathfrak{n}_1^a} = \overline{\mathfrak{n}_2^b}$ has been studied by Samuel in [Sa_1], from a different viewpoint, under the name "projective equivalence".

(2) The proof of Theorem 1 above uses only the Cohen-Macaulay property of $\mathcal{O}$, and the possiblity of resolving singularities of 2-dimensional quotients of $\mathcal{O}$. Indeed it seems to be valid under some mere "excellence" assumption on $\mathcal{O}$, (see [Li]) and the assumption that $\mathcal{O}$ is Cohen-Macaulay and normal. However, I have in this redaction once again sacrificed generality to geometry, and restricted the presentation to complex analytic geometry to be able to present the ideas in their naivety. In any case, the absence of a base field would indeed make the proofs a lot more cumbersome.

1 A geometric result

The essential technical ingredient of the proof of Theorem 1 is the following result, in itself rather useful (see [T]).

Proposition. *Let O be a reduced Cohen-Macaulay complex analytic algebra and $\mathfrak{n}$ an ideal of O primary for the maximal ideal. Set $d = \dim O$ and let as usual $\mathfrak{n}^{[i]}$ denote an ideal O generated by i "sufficiently general" linear combinations of the elements of a fixed system of generators of $\mathfrak{n}$. Then we have:*

(1) $\overline{\mathfrak{n}^{[d]}} = \overline{\mathfrak{n}}$

(2) *Given $f \in O$, $f \in \overline{\mathfrak{n}}$ if and only if* **349**

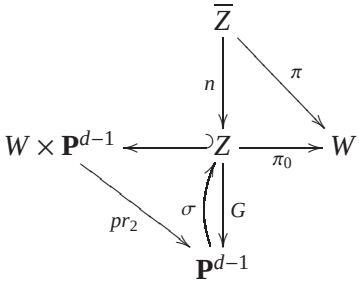
$$f \cdot O/\mathfrak{n}^{[d-1]} \subset \overline{\mathfrak{n} \cdot O/\mathfrak{n}^{[d-1]}}$$

with the necessary precision that the meaning of "sufficiently general" in the notation $\mathfrak{n}^{[d-1]}$ occurring in (2) depends on f, in other words, we have (2) for a Zariski-open dense subset U_f of the space of coefficients of $d-1$ linear combinations of generators of $\mathfrak{n}$, and U_f depends on f.

Proof. (1), which is due to Samuel [Sa_2], is a consequence of Noether's normalization theorem.

Let us prove (2): let $(W, 0)$ be a representative of the germ of complex space corresponding to O, and such that f and $\mathfrak{n}$ are the germs of a globally defined function and ideal on W. Let $\pi : \overline{Z} \to W$ be the normalized blowing up of $\mathfrak{n}$ in W. It follows from (1) and the results in ([H_1] Lecture 7, [L.T.]) that π is also the normalized blowing up of an $\mathfrak{n}^{[d]} = (f_1, \ldots, f_d) \subset \mathfrak{n}$. Let $\pi_0 : Z \to W$ be the blowing up of this $\mathfrak{n}^{[d]}$. Since O is Cohen-Macaulay, $(f_1, \ldots, f_d)$ is a regular sequence and hence (Lemma 1.9 of [H_2]) we can describe π_0 as the restriction of the first projection to the subspace of $W \times \mathbf{P}^{d-1}$ defined by the ideal $(f_i T_j - f_j T_i)(1 \leqslant i < j \leqslant d)$, where $(T_1 : \ldots : T_d)$ is a system of homo-

geneous coordinates on $\mathbf{P}^{d-1}$. Let us consider the following diagram:



350 where n denotes the normalization map and σ is the section of $G = pr_2|Z$ defined by $\sigma(\mathbf{P}^{d-1}) = \{0\} \times \mathbf{P}^{d-1}$, which is in Z since $\mathfrak{n}^{[d]} \subset \mathfrak{m}$, the maximal ideal of $\mathcal{O}$.

Now we consider $Z \underset{G}{\overset{\sigma}{\curvearrowright}} \mathbf{P}^{d-1}$ as a family of germs of curves, which like any family of curves admits a simultaneous normalization ([D.C.N.]) over an open-analytic (i.e. complement of a closed analytic subset) dense subset of its parameter space hence, here, over a Zariski open dense subset $U \subset \mathbf{P}^{d-1}$. This means that for $p \in U$, the composed map $\overline{G} : \overline{Z} \to Z \to \mathbf{P}^{d-1}$ is flat in a neighbourhood of the finite set $n^{-1}(\sigma(p))$, and the multi-germ $((\overline{Z})_p, n^{-1}(\sigma(p)))$ where $(\overline{Z})_p = \overline{G}^{-1}(p)$, is the union of a finite set of germs of non-singular curves, each being the normalization of an irreducible component of the curve $(Z_p, \sigma(p))$, where $Z_p = G^{-1}(p)$. Furthermore since clearly Z_p is "transversal" to $\{0\} \times \mathbf{P}^{d-1}$ which is set-theoretically the exceptional divisor of π_0, $(\overline{Z}_p, z_i)$ will be a germ of a non-singular curve transversal to the exceptional divisor D of π, for $z_i \in n^{-1}(\sigma(p))$. The idea is, that these germs of curves *can be used to compute multiplicities along the exceptional divisor D of* π. Indeed, since n is a finite morphism, each irreducible component D_i of D, which is purely of codimension 1 in $\overline{Z}$ by the hauptidealsatz, is mapped surjectively onto $\{0\} \times \mathbf{P}^{d-1}$ by n. Let us consider the decomposition $D = \bigcup_{i=1}^{l} D_i$ of D in its irreducible components. Then by the properties of normal spaces, for each given $f \in \mathcal{O}$ we can find a dense open analytic subset $U_i \subset D_i$ such that at each point $z \in U_i$ we have:

(1) $(D_z)_{\text{red}}$ is non-singular and coincides with $(D_{i,z})_{\text{red}}$.

(2) $\overline{Z}$ is non-singular at z.

(3) $f \cdot \mathcal{O}_{\overline{Z},z}$ defines a subspace having a reduced associated subspace which coincides with $(D_{i,z})_{\text{red}}$, in other words: the strict transform by π of $f = 0$ is empty near z.

By diminishing U if necessary, in a way which depends upon f, since the U_i do, we can assume:

$$D_i \cap n^{-1}(\sigma(U)) \subset U_j \qquad (1 \leqslant i \leqslant l).$$

Now for $z \in D_i \cap n^{-1}(\sigma(U))$ we have by (1), (2), (3): **351**

(i) $\mathfrak{n} \cdot \mathcal{O}_{\overline{Z},z} \simeq v_1^{\nu_i} \cdot \mathcal{O}_{\overline{Z},z}$, where

(ii) $\mathcal{O}_{\overline{Z},z} \simeq \mathbf{C}\{v_i, \ldots, v_d\}$ by (2).

(iii) $f \cdot \mathcal{O}_{\overline{Z},z} \simeq v_1^{\mu i} \cdot \mathcal{O}_{\overline{Z},z}$,

where ν_i, (resp. μ_i) is the order with which $\mathfrak{n} \cdot \mathcal{O}_{\overline{Z}}$ (resp. $f \circ \pi$) vanishes along D_i. This order being locally constant is constant on U_i since D_i is irreducible.

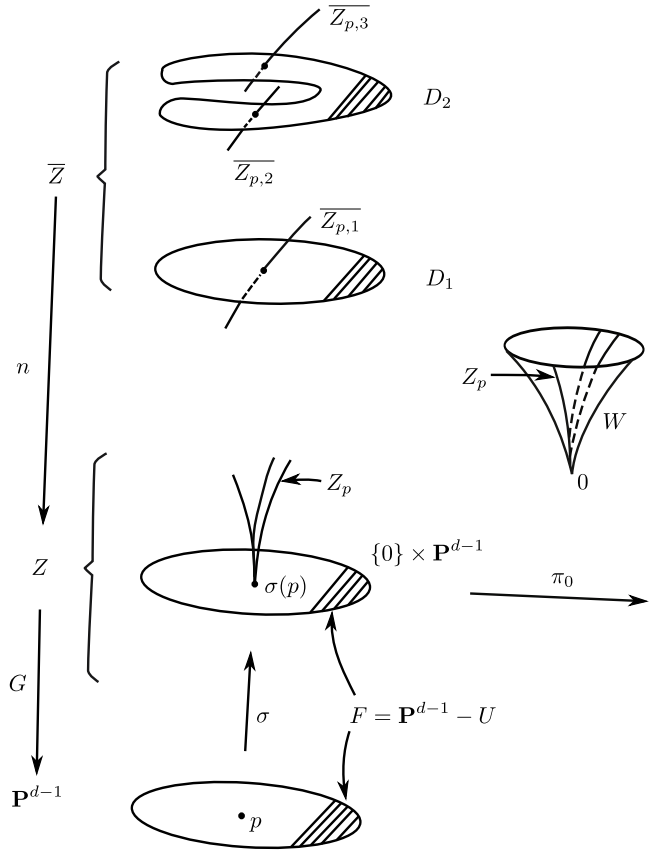
Setting $p = G(n(z))$, certainly one of the components of the curve $(Z_p, \sigma(p))$ has a normalization $((\overline{Z})_p, z)$ which goes through z and is transversal to D_i at z. Therefore, since the algebra $\mathcal{O}_{(\overline{Z})_p,z}$ is isomorphic to $\mathbf{C}\{t\}$, and $v_1 \cdot \mathcal{O}_{(\overline{Z})_p,z} = a_1 t + a_2 t^2 + \ldots$ with $a_i \in \mathbf{C}$, $a_1 \neq 0$, we have, denoting by v the order in t of elements of $\mathbf{C}\{t\}$, i.e. the natural valuation on $\mathcal{O}_{(\overline{Z})_p,z}$, that

$$\begin{cases} \nu_i = v(\mathfrak{n} \cdot \mathcal{O}_{(\overline{Z})_p,z}) \quad (p = G(n(z)), z \in U_i) \\ \mu_i = v(f \cdot \mathcal{O}_{(\overline{Z}_p,z)}). \end{cases}$$

Since the polar locus of a meromorphic function in a normal space is either of codimension 1 or empty, and since by ([H_1], [L.T.]) $f \in \overline{\mathfrak{n}}$ if and only if $f \cdot \mathcal{O}_{\overline{Z},z} \subset \mathfrak{n}$, $\mathcal{O}_{\overline{Z},z}$ for all $z \in \pi^{-1}(0)$, we see that $f \in \overline{\mathfrak{n}}$ if and only if $\mu_i \geqslant \nu_i$ $((1 \leqslant i \leqslant l))$ and this is *equivalent* to the fact that for

some $p \in U$, and any irreducible component $Z_{p,i}$ of $Z_p = G^{-1}(p)$, we have $f \cdot \overline{\mathcal{O}_{Z_{p,i},\sigma(p)}} \subset \mathfrak{n} \cdot \overline{\mathcal{O}_{Z_{p,i},\sigma(p)}}$ for all i, that is: $f \cdot \mathcal{O}_{Z_p,\sigma(p)} \subset \overline{\mathfrak{n} \cdot \mathcal{O}_{Z_p,\sigma(p)}}$ (by the valuative criterion for integral dependance, see [H_1], [L.T.]). But now it is clear that Z_p, $p \in U$ is isomorphic to the curve in $(W, 0)$ defined by the $d-1$ equations $\frac{f_1}{t_1} = \ldots = \frac{f_d}{t_d}$ where $(t_1 : \ldots : t_d)$ are the coordinates of p, i.e. the equations $f_i t_{i+1} - f_{i+1} t_i = 0 (1 \leqslant i \leqslant d-1)$, and this means that $(Z_p, \sigma(p))$ is the curve defined by a $\mathfrak{n}^{[d]}$ according to our conventions. This ends the proof of Proposition 1. □

Note. The following diagram may perhaps help the reader.



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Remark. There is not necessarily a bijection between the irreducible components of Z_p, $p \in U$ and those of D. What counts is that each (μ_i, ν_i) appears at least once as the valuation of $(\mathfrak{n}, f)$ given by an irreducible component of Z_p and that all these valuations occur in this way. This fact is important in the construction of invariants; see ([T], proof of Theorem 2). 353

2 Proof of Theorem 1

The assertion is obvious when $d = 0, 1$ since O is then $\mathbf{C}$, $\mathbf{C}\{t\}$. We therefore assume $d \geqslant 2$. Let us recall from [M.I.] that given two primary ideals $\mathfrak{n}_1$ and $\mathfrak{n}_2$ of O, we set

$$e_i = e\left(\mathfrak{n}_1^{[i]} + \mathfrak{n}_2^{[d-i]}\right)$$

and showed that (*) was a consequence of inequalities

$$e_i^d \leqslant e_d^i \cdot e_0^{d-i} \quad (1 \leqslant i \leqslant d) \tag{I.2}$$

themselves consequences of:

$$\frac{e_i}{e_{i-1}} \geqslant \frac{e_{i-1}}{e_{i-2}} \quad (2 \leqslant i \leqslant d) \tag{I.3}$$

All this in view of the equality

$$e(\mathfrak{n}_1 \cdot \mathfrak{n}_2) = \sum_{i=0}^{d} \binom{d}{i} e_i \tag{E.1}$$

of ([C.E.W., Chap. I § 2]).

The idea being that by induction on d it is enough to prove

$$\frac{e_d}{e_{d-1}} \geqslant \frac{e_{d-1}}{e_{d-2}}$$

and that this is in fact a result on surfaces: setting $O = O/\mathfrak{n}_1^{[d-2]}$, $\tilde{\mathfrak{n}}_i = \mathfrak{n} \cdot \tilde{O}$, $i = 1, 2$ we saw, thanks to the Cohen-Macaulay property of O, that

$e_d = \tilde{e}_d$, $e_{d-1} = \tilde{e}_1$, $e_{d-2} = \tilde{e}_0$ where $\tilde{e}_i = e(\tilde{\mathfrak{n}}_1^{[i]} + \tilde{\mathfrak{n}}_2^{[2-i]})$. We considered a well-chosen resolution of singularities of the germ of surface S_0 corresponding to $\tilde{O}$, say $r : S \to S_0$ such that in particular $\tilde{\mathfrak{n}}_1^{[2]}$, $\tilde{\mathfrak{n}}_1^{[1]} + \tilde{\mathfrak{n}}_2^{[1]}$, $\tilde{\mathfrak{n}}_2^{[2]}$ and the maximal ideal $\mathfrak{m}$ of $\tilde{O}$ all become invertible on S, $\mathfrak{m}O_S$ defining a divisor with normal crossings. Then, denoting by u_k (resp. v_k) the order of vanishing of $\tilde{\mathfrak{n}}_1 \cdot O_S$ (resp. $\tilde{\mathfrak{n}}_2 \cdot O_S$) along the k-th component E_k of $r^{-1}(0)$, we had, using a result of C. P. Ramanujam in [R]

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$$\left.\begin{aligned} \tilde{e}_2 &= -\sum_{k,k'} \langle E_k, E_{k'} \rangle u_k u_{k'} \\ \tilde{e}_1 &= -\sum_{k,k'} \langle E_k, E_{k'} \rangle u_k v_{k'} \\ \tilde{e}_0 &= -\sum_{k,k'} \langle E_k, E_{k'} \rangle v_k v_{k'} \end{aligned}\right| \qquad (1)$$

where $\langle , \rangle$ denotes the intersection multiplicities of divisors on S *supported in* $r^{-1}(0)$, which is easy to define (see [R]).

Now let us remark that, in view of the inequalities mentioned above, if equality holds in (*) then necessarily we have

$$\frac{e_d}{e_{d-1}} = \ldots = \frac{e_1}{e_0}.$$

Let $\frac{a}{b}$ denote the common value of these ratios. It follows easily from the results in ([C.E.W.] Chap. I §2) that if we replace $\mathfrak{n}_1$ by $\mathfrak{n}_1^q$ and $\mathfrak{n}_2$ by $\mathfrak{n}_2^b$, e_i is replaced by $a^i b^{d-i} e_i$. After this substitution we see that the proof of Theorem 1 is reduced to proving that if $e_d = e_{d-1} = \ldots = e_0$, then $\overline{\mathfrak{n}}_1 = \overline{\mathfrak{n}}_2$.

Now by the theorem of Bertini for normality (see [F]) we have that $O/\mathfrak{n}_1^{[d-2]}$ is a normal analytic algebra if O is so, and then by the classical result ([DV], [Li], [M]) the matrix of the $\langle E_k, E_{k'} \rangle$ is negative definite, from which follows immediately in view of (1) that if $\tilde{e}_2 = \tilde{e}_1 = \tilde{e}_0$ we have $u_k = v_k$ for all k, hence $\tilde{\mathfrak{n}}_1 \cdot O_S = \tilde{\mathfrak{n}}_2 \cdot O_S$ (since $\tilde{\mathfrak{n}}_1 \cdot O_S$ and $\tilde{\mathfrak{n}}_2 \cdot O_S$ are invertible on S) and from this follows $\overline{\tilde{\mathfrak{n}}_1} = \overline{\tilde{\mathfrak{n}}_2}$ in $O/\mathfrak{n}_1^{[d-2]}$. Let us now show that $\mathfrak{n}_1 \subset \overline{\mathfrak{n}}_2$ in O: given $f \in \mathfrak{n}_1$, to show that $f \in \overline{\mathfrak{n}}_2$ it is sufficient, in view of Proposition 1, to show that $f \cdot O/\mathfrak{n}_1^{[d-1]} \subset \overline{\mathfrak{n}_2 \cdot O/\mathfrak{n}_1^{[d-1]}}$, but

certainly a $O/\mathfrak{n}_1^{[d-1]}$ is a quotient of a $O/\mathfrak{n}_1^{[d-2]}$ as above, and since $\overline{\tilde{\mathfrak{n}}_1} = \overline{\tilde{\mathfrak{n}}_2}$ we have

$$f \cdot O/\mathfrak{n}_1^{[d-2]} \subset \overline{\mathfrak{n}_2 \cdot O/\mathfrak{n}_1^{[d-2]}}$$

hence *a fortiori* **355**

$$f \cdot O/\mathfrak{n}_1^{[d-1]} \subset \overline{\mathfrak{n}_2 \cdot O/\mathfrak{n}_1^{[d-1]}}.$$

This shows that for any $f \in \mathfrak{n}_1$, $f \in \overline{\mathfrak{n}}_2$, i.e., $\mathfrak{n}_1 \subset \overline{\mathfrak{n}}_2$ whence $\overline{\mathfrak{n}}_1 = \overline{\mathfrak{n}}_2$ by symmetry. This proves that if $e(\mathfrak{n}_1 \cdot \mathfrak{n}_2)^{1/d} = e(\mathfrak{n}_1)^{1/d} + e(\mathfrak{n}_2)^{1/d}$ we have $\overline{\mathfrak{n}_1^a} = \overline{\mathfrak{n}_2^b}$. The converse is an immediate consequence of the fact that $e(\mathfrak{n}) = e(\overline{\mathfrak{n}})$ ([C.E.W.] Chap. 0) and the remark made about the behaviour of the e_i under the operation $\mathfrak{n} \to \mathfrak{n}^a$. This ends the proof of Theorem 1.

Remark. We have seen in the course of the proof that, thanks to (1), when $d = 2$, Theorem 1 and (*) together follow from the negative definiteness of $(\langle E_k, E_{k'} \rangle)$.

3 Applications

3.1 First application: the Theorem of Rees (in a special case).

Theorem (Rees). *Let O be a Cohen-Macaulay normal analytic algebra, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ two ideals of O primary for the maximal ideal and such that $\overline{\mathfrak{n}}_1 \subseteq \overline{\mathfrak{n}}_2$ and $e(\mathfrak{n}_1) = e(\mathfrak{n}_2)$. Then we have $\overline{\mathfrak{n}}_1 = \overline{\mathfrak{n}}_2$.*

Proof. It is easy to see that for any positive integer a, we have $\overline{\mathfrak{n}_1} = \overline{\mathfrak{n}_2} \Leftrightarrow \overline{\mathfrak{n}_1^a} = \overline{\mathfrak{n}_2^a}$ (e.g., use the valuative criterion for integral dependence, [H_1], [L.T.]). Now let us set $e(\mathfrak{n}_1) = e(\mathfrak{n}_2) = e$. Since for any two primary ideals, $\mathfrak{n} \subseteq \mathfrak{n}'$ implies $e(\mathfrak{n}) \geqslant e(\mathfrak{n}')$, $\mathfrak{n}_1 \subseteq \mathfrak{n}_2$ implies $e(\mathfrak{n}_1 \cdot \mathfrak{n}_2) \geqslant e(\mathfrak{n}_2^2) = 2^d e$. Using (*) now we see that

$$2 \cdot e^{1/d} \leqslant e(\mathfrak{n}_1 \cdot \mathfrak{n}_2)^{1/d} \leqslant e^{1/d} + e^{1/d} = 2 \cdot e^{1/d},$$

hence we have equality, and by Theorem 1 there exist $a, b \in \mathbf{N}$ such that $\overline{\mathfrak{n}_1^a} = \overline{\mathfrak{n}_2^b}$. But we saw that $e(\mathfrak{n}^a) = a^d e(\mathfrak{n})$ and $e(\overline{\mathfrak{n}}) = e(\mathfrak{n})$. Since $e(\mathfrak{n}_1) = e(\mathfrak{n}_2)$, the above equality therefore implies $a = b$, from which the theorem follows. □

3.2 Second Application Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a germ of a real analytic mapping, described by $f_1, \ldots, f_n$ in $\mathbf{R}\{\underline{x}\} = \mathbf{R}\{x_1, \ldots, x_n\}$ and
356 such that $Q(f) = O_{f^{-1}(0),0} = \mathbf{R}\{\underline{x}\}/(f_1, \ldots, f_n)$ is a finite dimensional vector space over $\mathbf{R}$. Set $q = \dim_{\mathbf{R}} Q(f)$; q is also the topological degree of the complexified map $f^{\mathbf{C}} : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$. It was proved in [E.L.] that the topological degree $\deg f$ of f satisfies

$$|\deg f| \leqslant q^{1-1/n}$$

and this inequality was proved as follows: first one proved that $|\deg f| \leqslant e(\mathfrak{n}^{[n-1]} + \mathfrak{m}^{[1]})$ where $\mathfrak{n}$ is the ideal generated by $(f_1, \ldots, f_n)$ in $\mathbf{C}\{\underline{x}\}$ and $\mathfrak{m}$ is the maximal ideal, then one used the inequality [1.2] from [M.I.] quoted above to show that

$$e(\mathfrak{n}^{[n-1]} + \mathfrak{m}^{[1]}) \leqslant e(\mathfrak{n})^{1-1/n} = q^{1-1/n}.$$

Let us consider the special case $n = 2$. We have $|\deg f| \leqslant q^{1/2}$ and there is at least one case where we have equality: let f be a holomorphic map $(\mathbf{C}, 0) \to (\mathbf{C}, 0)$ given by $z \mapsto z^k$. Setting $z = x_1 + ix_2$ we see that the components of f, $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are homogeneous polynomials of degree k. From the additivity of intersection multiplicities follows that in this case $q = k^2$. By writing $z = \rho \cdot e^{i\theta}$ we see that all the zeroes of f_1 and f_2 are real, and the k lines in $\mathbf{R}^2$ where f_1 vanishes *alternate* with the k lines where f_2 vanishes, being obtained from them by a rotation of $\pi/2k$. Furthermore each of the sets of lines divides $\mathbf{R}^2$ in $2k$ sectors. One readily checks that $\operatorname{Re} z^k = t$, $\operatorname{Im} z^k = 0$ is a regular fibre of f, at each point of which f preserves the orientation, and which contains k points, since $\operatorname{Re} z^k = t$ defines a curve in every second sector among those defined by $\operatorname{Re} z^k = 0$, which is asymptotic to the walls of this sector and meets the lines $\operatorname{Im} z^k = 0$ transversally. We remark that the fact that f is orientation preserving comes from the following property of $f_2 = \operatorname{Im} z^k$, $f_1 = \operatorname{Re} z^k$:

(OR) In each sector where $f_1 > 0$, the sign of f_2 changes from $-$ to $+$ as we cross the line $f_2 = 0$ circulating in the trigonometric direction (counterclockwise).

Having seen this, we have no difficulty in proving

Lemma 1. *Let* $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ *be given by two homogeneous polynomials* f_1, f_2 *of the same degree* k. *Then we have* $\deg f = k$ *if, and only if,* f_1 *and* f_2 *both have all their roots real, these roots alternate and* 357 *the condition (OR) is satisfied.*

Indeed, if they do not both have all their roots real, we can find a regular fibre with less than k points, hence $|\deg f| < k$. If they do not alternate, then we can find two points in a fibre with k points, where the orientation is not the same, hence again $|\deg f| < k$ and finally if the first two conditions are satisfied we have $\deg f = \pm k$, and condition (OR) implies $\deg f - k$ (and conversely). We now prove:

Lemma 2. *Let* $f = (f_1, f_2)$ *and* $f' = (f_1', f_2')$ be two mapping satisfying *the condition of Lemma 1. Then there is a 1-parameter family* $(f_t)_{t\in[0,1]}$ *of mappings, all satisfying the condition of Lemma 1, such that* $f_0 \simeq f$ *and* $f_1 \simeq f'$.

Proof. After a suitable choice of coordinates, we can write (up to the isomorphism corresponding to a constant factor)

$$f_1 = \prod_{i=1}^{k}(x_1 - \alpha_i x_2) \quad f_2 = \prod_{i=1}^{k}(x_1 - \beta_i x_2) \quad \alpha_i, \beta_i \in \mathbf{R}$$

with $\alpha_1 < \beta_1 < \alpha_2 < \ldots < \alpha_k < \beta_k$, and similarly

$$f_1' = \prod_{s=1}^{k}(x_1 - \alpha_i' x_2), \quad f_2' = \prod_{i=1}^{k}(x_1 - \beta_i' x_2)$$

with $\alpha_1', < \beta_1' < \alpha_2' < \ldots < \alpha_k' < \beta_k'$.

Then the family given by

$$f_{t,1} = \prod_{i=1}^{k}(x_1 - (t\alpha_i' + (1-t)\alpha_i)x_2)$$

$$f_{t,2} = \prod_{i=1}^{k}(x_1 - (t\beta_i' + (1-t)\beta_i)x_2)$$

for $0 \leqslant t \leqslant 1$, obviously gives the answer. □

Theorem 2. *Let* $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ *be a germ of a real-analytic mapping with algebraic degree* $q (q = \dim_{\mathbf{R}} \mathcal{O}_{f^{-1}(0),0})$. *Then* f *can be continuously deformed, with degree and algebraic degree both constant, to a holomorphic mapping* $(\mathbf{C}, 0) \to (\mathbf{C}, 0)$ *if and only if* $\deg f = q^{1/2}$.

358 *Proof.* The condition is obviously necessary, after what we have just seen. Let us prove it sufficient. From what we saw above, the equality $\deg f = q^{1/2}$ implies $e(\mathfrak{n}^{[1]} + \mathfrak{m}^{[1]}) = e(\mathfrak{n})^{1/2}$ which, since $e(\mathfrak{m}) = 1$, and in view of the equality $(E.\ 1)$ quoted in §2, gives $e(\mathfrak{n} \cdot \mathfrak{m})^{1/2} = e(\mathfrak{n})^{1/2} + e(\mathfrak{m})^{1/2}$. From Theorem 1 we deduce the existence of integers a, b such that $\overline{\mathfrak{n}^a} = \overline{\mathfrak{m}^b}$ and from the properties of multiplicities we deduce that in fact we must have $\overline{\mathfrak{n}} = \mathfrak{m}^k$ in $\mathbf{C}\{x_1, x_2\}$, where $k = q^{1/2} = \deg f$. We are going to show that this implies that the components f_1, f_2 of f can be taken (up to isotopy) to be homogeneous polynomials of degree k. Let $\mathfrak{n}'$ be the ideal in $\mathbf{C}\{x_1, x_2\}$ generated by those among (f_1, f_2) which are in $\mathfrak{m}^k - \mathfrak{m}^{k+1}$: we have $\mathfrak{n}' \subset \mathfrak{n} \subset \mathfrak{n}' + \mathfrak{m}^{k+1}$ hence, since $\overline{\mathfrak{n}} = \mathfrak{m}^k$, we have $\mathfrak{n}' + \mathfrak{m} \cdot \mathfrak{m}^k = \mathfrak{m}^k$ which by the integral Nakayama Lemma ([C.E.W.] Chap. 11, 2.4) implies $\overline{\mathfrak{n}'} = \mathfrak{m}^k$, which in turn implies that $\mathfrak{n}'$ is generated by at least two elements, whence $\mathfrak{n}' = \mathfrak{n}$. Now we know $f_i \in \mathfrak{m}^k - \mathfrak{m}^{k+1}$, $i = 1, 2$. We can set

$$f_1 = P_k + P_{k+1} + \dots$$
$$f_2 = Q_k + Q_{k+1} + \dots$$

where P_i (or Q_i) is an homogeneous polynomial of degree i in x_1 and x_2. We know furthermore that since $\mathfrak{n}' = (P_k, Q_k)$ has at its integral closure $\mathfrak{m}^k$, $\dim_{\mathbf{R}} \mathbf{R}\{x_1, x_2\}/(P_k, Q_k) = \dim_{\mathbf{R}} \mathbf{R}\{x_1, x_2\}/(f_1, f_2) = q$. Consider the family of functions

$$f_{1,t} = P_k + tP_{k+1} + \dots + t^l P_{k+l} + \dots$$
$$f_{2,t} = Q_k + tQ_{k+1} + \dots + t^l Q_{k+l} + \dots$$

the family of ideals $\mathscr{F}_t = (f_{1,t}, f_{2,t}) \cdot \mathbf{R}\{x_1, x_2\}$ and the family of algebras $Q_t = \mathbf{R}\{x_1, x_2\}/\mathscr{F}_t$. For any $t \neq 0$, Q_t is isomorphic to $Q(f)$ as an $\mathbf{R}$-algebra (change x_i to tx_i in f, and divide by t^k), and $Q_0 = \mathbf{R}\{x_1, x_2\}/(P_k, Q_k)$.

We now use the main result of [E.L.]: all the Q_t are isomorphic as vector spaces; choosing a linear form $l : Q_0 \to \mathbf{R}$ such that $l(J_0) > 0$, where J_0 is the Jacobian determinant of (P_k, Q_k), we can extend l to Q_t and denoting by J_t the Jacobian determinant of $(f_{1,t}, f_{2,t})$, we get $l(J_t) > 0$ for t sufficiently small. According to Theorem 1.2 of [E.L.], **359** we have:

(1) the bilinear form on Q_t defined by $\langle p, q\rangle = l(p \cdot q)$ is non-singular for all sufficiently small t, therefore its signature is independent of t near $t = 0$.

(2) This signature is equal to deg f for $t \neq 0$ and to the degree of the map $f_0 : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ defined by (P_k, Q_k) for $t = 0$.

Hence deg $f_0 = k = q^{1/2}$ and Theorem 2 now follows easily from Lemmas 1 and 2. □

Remarks.

(1) The key point is to check that the assumption deg $f = q^{1/2}$ implies that the tangent cones at 0 of $f_1 = 0$ and $f_2 = 0$ have the same degree, and no common component. This is what Theorem 1 does for us in the above proof.

(2) Theorem 2 above is valid also for C^∞ maps $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that $\dim \mathscr{E}_2/(f_1, f_2) < \infty$ where $\mathscr{E}_2$ is the ring of germs of C^∞ function on $(\mathbf{R}^2, 0)$, since the finiteness assumption implies that in those problems we have finite determinacy (see [E.L.]).

(3) It is an interesting problem to find invariants of algebras of the form $\mathbf{R}\{x_1, \ldots, x_n\}/(f_1, \ldots, f_n)$ of finite dimension over $\mathbf{R}$ which allow one to determine when two such algebras (or the corresponding germs of mappings $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$) can be continuously deformed into one another with constant dimension of the algebras (i.e. algebraic degree) and degree.

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