Number Theory
And Related Topics

NUMBER THEORY AND RELATED TOPICS

Papers presented at the Ramanujan Colloquium, Bombay 1988, by

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Ramanujan Birth Centenary International Colloquium on Number Theory and Related Topics

Bombay, 4-11 January 1988

REPORT

An International Colloquium on Number Theory and related topics 1
was held at the Tata Institute of Fundamental Research, Bombay during 4-11 January, 1988, to mark the birth centenary of Srinivasa Ramanujan. The purpose of the Colloquium was to highlight recent developments in Number Theory and related topics, especially those related to the work of Ramanujan "such as the Circle method, Sieve methods and Combinatorial techniques in Number theory, Partition congruences, Rogers - Ramanujan identities, Lacunarity of power series, Hypergeometric series and Special functions, Complex multiplication, Hecke theory etc."

The Colloquium was organized by the Tata Institute of Fundamental Research with co-sponsorship from the International Mathematical Union. Financial support was received from the International Mathematical Union and the Sir Dorabji Tata Trust, as in former years. The organizing committee of the Colloquium consisted of Professors M.S. Narasimhan, S. Raghavan, M.S. Raghunathan, K. Ramachandra and C.S. Seshadri and Dr. S.S. Rangachari. The International Mathematical Union was represented on the committee by Professors M.S. Narasimhan and C.S. Seshadri.

The following mathematicians delivered one-hour addresses at the Colloquium:

G.E. Andrews, R. Askey, B. C. Berndt, D. M. Bressoud, D. R. Heath-Brown, N. V. Kuznetsov, K. Ramachandra, K. G. Ramanathan, S. S. Rangachari, R. A. Rankin, I. Satake, W. M. Schmidt, A. Selberg, J. P. Serre, T. N. Shorey and D. Zagier.

Professor H. Iwaniec could not attend the Colloquim but sent a paper for inclusion in the Proceedings.

Besides members of the School of Mathematics of the Tata Institute of Fundamental Research, mathematicians from universities and educational institutions in India, France, Canada, Japan and the United States of America were also invited to attend the Colloquium.

The social programme for the Colloquium included a tea-party on 4 January, a classical Indian dance performance (Bharatanatyam) on 6 January, a film show and a dinner-party at the Institute on 7 January, a violin recital (Hindustani music) on 8 January, an excursion to the Elephanta Caves on 9 January and a farewell dinner-party on 10 January 1988.

Contents

VARIANTS OF CLAUSEN'S FORMULA FOR THE SQUARE OF A SPECIAL $_2F_1$

By RICHARD ASKEY*

1 Introduction

One of the most striking series Ramanujan [10] found is 1

$$\frac{9801}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty}[1103 + 26390n]\frac{(4n)!}{[n!]^4(4.99)^{4n}}. \tag{1.1}$$

The first proofs of 1.1 have been given recently by Jonathan and Peter Borwein [3] and by David and Gregory Chudnovsky [5]. They have also found other identities of a similar nature, [4], [5]. As they remark, Clausen's identity [6]

$$\left[{}_2F_1\left(\begin{matrix}a, b\\ a+b+\frac{1}{2}\end{matrix}; x\right)\right]^2 = {}_3F_2\left(\begin{matrix}2a, 2b, a+b\\ a+b+\frac{1}{2},\ 2a+2b\end{matrix}; x\right) \tag{1.2}$$

plays a central role in the derivation of (1.1). Here

$${}_pF_p\left(\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}; x\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n \ldots (a_p)_n x^n}{(b_1)_n \ldots (b_q)_n n!} \tag{1.3}$$

with

$$(a)_n = \Gamma(n+a)/\Gamma(a). \tag{1.4}$$

*Supported in part by an NSF grant, in part by a sabbatical leave from the University of Wisconsin, and in part by funds the Graduate School of the University of Wisconsin.

Ramanujan [11] stated an extension of Clausen's formula

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; \frac{1-\sqrt{1-x}}{2}\right) {}_2F_1\left(\begin{matrix} a, b \\ d \end{matrix}; \frac{1-\sqrt{1-x}}{2}\right) \tag{1.5} $$
$$ = {}_4F_3\left(\begin{matrix} a, b, (a+b)/2, (c+d)/2 \\ c, d, a+b \end{matrix}; x\right) $$

when $c + d = a + b + 1$. When $c = d$ and the quadratic transformation

$$ {}_2F_1\left(\begin{matrix} a, b \\ (a+b+1)/2 \end{matrix}; \frac{1-\sqrt{1-x}}{2}\right) = {}_2F_1\left(\begin{matrix} a/2, b/2 \\ (a+b+1)/2 \end{matrix}; x\right) $$

is used, the result is (1.2). The first published proof of (1.5) is due to Bailey [1].

2 David and Gregory Chudnovsky have been asking me if there are other results like Clausen's formula, where the square of a ${}_2F_1$ is represented as a generalized hypergeometric series. There are other instances, ans one will be given explicitly. The method of deriving it is probably similar to Ramanujan's method of deriving Clausen's formula. As a warm up, here is how I think Ramanujan derived (1.2).

There are two chapters in Ramanujan's Second Notebook devoted to hypergeometric series. The first formula in this first of these two chapters is the sum of the 2-balanced very well posited ${}_7F_6$. This is a fundamental formula, as Ramanujan knew, since he started with it. This sum is

$$ {}_7F_6\left(\begin{matrix} a, 1+(a/2), b, c, d, e, -n \\ a/2,\ a+1-b,\ a+1-c,\ a+1-d,\ a+1-e,\ a+1-n \end{matrix}; 1\right) \tag{1.6} $$
$$ = \frac{(a+1)_n(a+1-b-c)_n(a+1-b-d)_n(a+1-c-d)_n}{(a+1-b)_n(a+1-c)_n(a+1-d)_n(a+1-b-c-d)_n} $$

and

$$ e = 2a + 1 + n - b - c - d, \tag{1.7} $$

The phrases *very well poised* and 2-*balanced* are defined as follows. A series

$$ {}_{p+1}F_p\left(\begin{matrix} a_0, a_1, \dots, a_p \\ b_1, \dots, b_p \end{matrix}; x\right) \tag{1.8} $$

is said to be *k-balanced* if $x = 1$, if one of the numerator parameters is a negative integer, and if

$$ k + \sum_{j=0}^{p} a_j = \sum_{j=1}^{p} b_j. $$

The series 1.8 is said to be *well poised* if $a_0 + 1 = a_1 + b_1 = \dots = a_p + b_p$. It is *very well poised* if it is well poised and if $a_1 = b_1 + 1$. Observe that the condition (1.7) comes from the series being 2-balanced.

Dougall [7] published the first derivation of (1.6). Ramanujan's discovery was probably later, but not much later.

To derive Clausen's formula, first consider

$$ \left[{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)\right]^2 = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{(a)_k(b)_k(a)_{n-k}(b)_{n-k}}{(c)_k k!(c)_{n-k}(n-k)!} \tag{1.9} $$

$$ = \sum_{n-0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} {}_4F_3\left(\begin{matrix} -n, a, b, 1-n-c \\ 1-n-a\ 1-n-b,\ c \end{matrix}; 1\right) x^n. $$

The ${}_4F_3$ series that multiplies x^n in the expression in (1.9) is well **3**
poised. While a well poised ${}_3F_2$ at $x = 1$ can be summed, and a very well poised ${}_5F_4$ can be summed when $x = 1$, a general well poised ${}_4F_3$ at $x = 1$ cannot be summed. However when the series is 2-balanced it can be summed. To see this, first reduce the very well poised ${}_7F_6$ to a well poised ${}_4F_3$. This is done by setting $d = a/2$, $c = (a + 1)/2$. Then (1.6) becomes

$$ {}_4F_3\left(\begin{matrix} a, b, e, -n \\ a+1-b, a+1-e, a+1+n \end{matrix}; 1\right) \tag{1.10} $$

$$ = \frac{(a+1)_n((a+1-2b)/2)_n((a+2-2b)/2)_n(1/2)_n}{(a+1-b)_n((a+1)/2)_n((a+2)/2)_n((1-2b)/2)_n} $$

$$= \frac{(a+1)_n(a+1-2b)_{2n}(1/2)_n}{(a+1-b)_n(a+1)_{2n}((1-2b)/2)_n}$$
$$= \frac{(a+1-2b)_{2n}(1/2)_n}{(a+1-b)_n(a+n+1)_n((1-2b)/2)_n}$$
$$= \frac{\Gamma(a+1-2b+2n)\Gamma(n+1/2)\Gamma(a+1-b)\Gamma(a+n+1)\Gamma(1/2-b)}{\Gamma(a+1-2b)\Gamma(1/2)\Gamma(a+1-b+n)\Gamma(a+2n+1)\Gamma((1/2)-b+n)}$$

This last expression can be used when $a = -k$. Then

$$_4F_3\left(\begin{matrix} -k, b, e, -n \\ 1-b-k, 1-e-k, 1+n-k \end{matrix}; 1\right)$$
$$= \frac{\Gamma(1-k-2b+2n)\Gamma(1/2+n)\Gamma((1/2)-b)\Gamma(1-b-k)\Gamma(1+n-k)}{\Gamma(1-k-2b)\Gamma(1-k+2n)\Gamma(1/2)\Gamma(1-k-b+n)\Gamma((1/2)-b+n)}.$$

holds for $n = k, k+1, \ldots$, and is a rational function of n, so it holds when n is replaced by continuous parameter $-a$. The result is

$$_4F_3\left(\begin{matrix} -k, a, b, e \\ 1-a-k,\ 1-b-k,\ 1-e-k \end{matrix}; 1\right) = \frac{(2a)_k(2b)_k(a+b)_k}{(a)_k(b)_k(2a+2b)_k} \quad (1.11)$$

after simplification. Recall that this series is 2-balanced, so $e = -a-b-k+(1/2)$.

One can take $a = -k$ in (1.6) and then remove the restriction that one of the other parameters is a negative integer. However setting $c = (1-k)/2$, $d = -k/2$ to obtain the $_4F_3$ leads to an indeterminate form, so
4 it is better to reduce to a $_4F_3$ initially before letting $a \to -k$.

Both (1.10) and (1.11) are 2-balanced well poised series, but they are different in that different parameters are used to terminate the series. When (1.11) is used in (1.9), he result is Clausen's formula (1.2).

2 The four balanced very well poised $_7F_6$

To find another formula like Clausen's identity, we can look for another well poised series that can be summed. The obvious candidate is the 4-balanced very well poised $_7\mathrm{F}_6$. There are two natural ways to sum this series. One is an easy consequence of (1.6), so it is a derivation Ramanujan could have easily given. We start with it. Set

$$f_k(b) = \frac{(b)_k(e)_k}{(a+1-b)_k(a+1-e)_k} \tag{2.1}$$

and use the 2-balanced condition

$$e = 2a + 1 + n - b - c - d. \tag{2.2}$$

A routine calculation gives

$$\begin{aligned} &b(a-b)f_k(b+1) - (e-1)(a+1-e)f_k(b) \\ &= \frac{(b)_k(e-1)_k}{(a+1-b)_k(a+2-e)_k}[b(a-b) - (e-1)(a+1-e)]. \end{aligned}$$

Observe that the last factor is

$$\begin{aligned} &b(a-b) - (2a+n-b-c-d)(b+c+d-n-a) \\ &= (n + \frac{3a}{2} - b - c - d + \frac{a}{2})(n + \frac{3a}{2} - b - c - d - \frac{a}{2}) - (b - \frac{a}{2} - \frac{a}{2})(b - \frac{a}{2} + \frac{a}{2}) \\ &= (n + \frac{3a}{2} - b - c - d)^2 - (b - \frac{a}{2})^2 = (n + 2a - 2b - c - d)(n + a - c - d); \end{aligned}$$

so

$$\begin{aligned} &(n + 2a - 2b - c - d)(n + a - c - d)\times \\ &\quad \times {}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b, c, d, e-1, -n \\ \frac{a}{2}, a+1-b, a+1-c, a+1-d, a+2-e, a+1+n \end{matrix}; 1\right) \\ &= b(a-b){}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b+1, c, d, e-1, -n \\ \frac{a}{2}, a-b, a+1-c, a+1-d, a+2-e, a+1+n \end{matrix}; 1\right) \\ &- (e-1)(a+1-e)\times \\ &\quad \times {}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, -n \\ \frac{a}{2}, a+1-b, a+1-c, a+1-d, a+1-e, a+1+n \end{matrix}; 1\right) \end{aligned}$$

$$= \frac{b(a-b+n)(a+1)_n(a-b-c)_n(a-b-d)_n(a+1-c-d)_n}{(a+1-b)_n(a+1-c)_n(a+1-d)_n(a-b-c-d)_n}$$
$$-(2a+n-b-c-d)(b+c+d-a)\times$$
$$\times \frac{(a+1)_n(a+1-b-c)_n(a+1-b-d)_n(a+1-c-d)_n}{(a+1-b)_n(a+1-c)_n(a+1-d)_n(a-b-c-d)_n}$$

or shifting e up by 1 and doing some algebra:

$$ {}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, -n \\ \frac{a}{2}, a+1-b,\ a+1-c,\ a+1-d,\ a+1-e, a+1+n \end{matrix}; 1\right) \tag{2.3}$$
$$= \frac{(a+1)_n(a-b-c)_n(a-b-d)_n(a-c-d)_n}{(a+1-b)_n(a+1-c)_n(a+1-d)_n(a-b-c-d)_n}\times$$
$$\times \left[1+\frac{n(n+2a-b-c-d)(a-b-c-d)}{(a-b-c)(a-b-d)(a-c-d)}\right]$$

when the series is 4-balanced, or equivalently when

$$e = 2a+n-b-c-d. \tag{2.4}$$

The second natural way to derive (2.3) uses a more complicated formula than (1.6), but the calculations from the starting formula are easier, and one can see how to extend the sum to the very well poised $2k$-balanced series. The starting formula is Whipple's transformation [14] between a very well poised ${}_7F_6$ and a balanced ${}_4F_3$:

$$ {}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, -n \\ \frac{a}{2}, a+1-b, a+1-c, a+1-d, a+1-e, a+1+n \end{matrix}; 1\right) \tag{2.5}$$
$$= \frac{(a+1)_n(a+1-b-c)_n}{(a+1-b)_n(a+1-c)_n}\,{}_4F_3\left(\begin{matrix} -n, a+1-d-e, b, c \\ b+c-n-a, a+1-d, a+1-e \end{matrix}; 1\right).$$

When $e = 2a+n-b-c-d$, the ${}_4F_3$ on the right is

$$ {}_4F_3\left(\begin{matrix} -n, b+c+1-n-a, b, c \\ b+c-n-a, a+1-d,\ b+c+d+1-n-a \end{matrix}; 1\right)$$

$$= \sum_{k=0}^{n} \frac{(-n)_k(b)_k(c)_k}{(a+1-d)_k(b+c+d+1-n-a)_k k!} \cdot \frac{(k+b+c-n-a)}{(b+c-n-a)}$$

$$= {}_3F_2\left(\begin{matrix} -n, b, c \\ a+1-d, b+c+d+1-n-a \end{matrix}; 1\right) +$$

$$+ \frac{(-n)bc}{(a+1-d)(b+c-n-a)(b+c+d+1-n-a)} \times$$

$$\times {}_3F_2\left(\begin{matrix} 1-n, b+1, c+1 \\ a+2-d, b+c+d+2-n-a \end{matrix}\right); 1$$

The second ${}_3F_2$ is balanced, and so can be summed using the Pfaff- 6
Saalschütz sum

$${}_3F_2\left(\begin{matrix} -n, b, c \\ d, 1+b+c-n-d \end{matrix}\right); 1 = \frac{(d-b)_n(d-c)_n}{(d)_n(d-b-c)_n}. \tag{2.6}$$

The first ${}_3F_2$ is two balanced, and so can be written as the sum of 2 terms by use of the transformation formula:

$${}_3F_2\left(\begin{matrix} -n, a, b \\ c, d \end{matrix}; 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \times \tag{2.7}$$

$$\times \quad {}_3F_2\left(\begin{matrix} -n, a, a+b+1-n-c-d \\ a+1-n-c, a+1-n-d \end{matrix}; 1\right).$$

For, when the series on the left of (2.7) is k-balanced, the third numerator parameter in the series on the right is $1-k$; so the series can be written as the sum of k terms when $k = 1, 2, \ldots$

For those unacquainted with (2.7), an argument giving a q-extension is in the last section.

These series combine to give another derivation of (2.3) when (2.4) has been assumed. This method clearly extends to give the sum of the $2k$-balanced very well poised ${}_7F_6$, but the resulting identity is too messy to be worth stating until it is needed.

3 Another Clausen type identity.

To obtain the next Clausen type identity take the ${}_4F_3$ in (1.9) to be 4-balanced, or take $c = a + b + 3/2$. As before, specialize (2.3) by taking $c = a/2$, $d = (a+1)/2$ and make the series on the left 4-balanced. The resulting series is

$$
{}_4F_3\left(\begin{matrix} -n, a, b, e \\ a+1+n, a+1-b, a+1-e \end{matrix}; 1\right) \tag{3.1}
$$
$$
= \frac{(a+1)_n((a-2b)/2)_n((a-2b-1)/2)_n(-\frac{1}{2})_n}{(a+1-b)_n((a+2)/2)_n((a+2)/2)_n(-\frac{1}{2}-b)_n} \times
$$
$$
\times \left(1 + \frac{n(n+a-b-\frac{1}{2})}{[(a-2b)/2][(a-2b-1)/2](-\frac{1}{2})}\right)
$$
$$
= \frac{(a-2b-1)_{2n}(-\frac{1}{2})_n}{(a+1-b)_n(a+n+1)_n(-\frac{1}{2}-b)_n}\left[1 + \frac{4n(n+a-b-\frac{1}{2})(2b+1)}{(a+2b)(a-2b-1)}\right].
$$

7 The replace a by $-k$ and after the same argument given above, replace $-n$ by a. The result is

$$
{}_4F_3\left(\begin{matrix} -k, a, b, e \\ 1-a-k, 1-b-k, 1-e-k \end{matrix}; 1\right) = \frac{(2a)_k(2b)_k(a+b)_k}{(a)_k(b)_k(2a+2b+2)_k} \times A \tag{3.2}
$$

with A given by

$$
A = 1 + \frac{(2+4a+4b+8ab)k + k^2 - k}{2(a+b)(2a+1)(2b+1)} \tag{3.3}
$$

or by

$$
A = \frac{k^2 + (8ab+4a+4b+1)k + 2(a+b)(2a+1)(2b+1)}{2(a+b)(2a+1)(2b+1)} \tag{3.4}
$$

and

$$
e = -k - a - b - \frac{1}{2}. \tag{3.5}
$$

Using (3.2) with A given by (3.3) in (1.9), we obtain

$$\left[{}_2F_2\left(\begin{matrix}a, b\\ a+b+\frac{3}{2}\end{matrix};x\right)\right]^2 = {}_3F_2\left(\begin{matrix}2a, 2b, a+b\\ a+b+\frac{3}{2}, 2a+2b+2\end{matrix};x\right) \tag{3.6}$$

$$+\frac{2ab\ x}{(a+b+1)(a+b+3/2)}\,{}_3F_2\left(\begin{matrix}2a+1, 2b+1, a+b+1\\ a+b+\frac{5}{2}, 2a+2b+3\end{matrix};x\right)$$

$$+\frac{abx^2}{2(a+b+3/2)^2(a+b+5/2)}\,{}_3F_2\left(\begin{matrix}2a+2, 2b+2, a+b+2\\ a+b+\frac{7}{2}, 2a+2b+4\end{matrix};x\right)$$

Using (3.2) with A given by (3.4) gives

$$\left[{}_2F_1\left(\begin{matrix}a, b\\ a+b+\frac{3}{2}\end{matrix};x\right)\right]^2 = {}_5F_4\left(\begin{matrix}2a, 2b, a+b, c+1, d+1\\ a+b+\frac{3}{2}, 2a+2b+2, c, d\end{matrix};x\right) \tag{3.7}$$

where c and d are determined by

$$x^2+(8ab+4a+4b+1)x+2(a+b)(2a+1)(2b+1) = (x+c)(x+d). \tag{3.8}$$

4 Comments.

After working out the above results, I went to a library to see if they **8**
were new. The fact that

$$\left[{}_2F_1\left(\begin{matrix}a, b\\ a+b+n+\frac{1}{2}\end{matrix};x\right)\right]^2, \quad n = 0, 1, \ldots, \tag{4.1}$$

is a generalized hypergeomatric series was proved by Goursat [8]. He also showed that

$$\left[{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix};x\right)\right]^2$$

is a generalized hypergeometric series only when $c = a + b + n + \frac{1}{2}$, $n = 0, 1, \ldots$. His proof that (4.1) is a generalized hypergeometric series uses Clausen's formula (1.2), its derivative

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; x\right) {}_2F_1\left(\begin{matrix} a+1, b+1 \\ a+b+\frac{3}{2} \end{matrix}; x\right) \\ &= {}_3F_2\left(\begin{matrix} 2a+1, 2b+1, a+b+1 \\ a+b+\frac{3}{2}, 2a+2b+1 \end{matrix}; x\right) \end{aligned} \tag{4.2}$$

and the transformation

$$ {}_2F_1\left(\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; x\right) = (1-x)^{1/2} {}_2F_1\left(\begin{matrix} a+\frac{1}{2}, b+\frac{1}{2} \\ a+b+\frac{1}{2} \end{matrix}; x\right). $$

Of course Ramanujan knew all of those facts. Goursat also used some recurrence relations. Ramanujan knew about some of the recurrence relations hypergeometric series satisfy, and almost surely derived some of his continued fractions from these recurrence relations. However Ramanujan did not use recurrence relations as much as he could have, or as often as he used other properties of hypergeometric series. While Ramanujan almost surely could have rediscovered Goursat's result if he had needed it, it is more likely he would have used an argument like the one given above. Ramanujan does not seem to have found Whipple's transformation formula (2.5). He did find a limiting case with one parameter missing, but we have not found (2.5) in any of the sheets of his. If there is another treasure like the sheets in Trinity College, I would not be surprised in (2.5) is there.

Actually, I would be surprised if Ramanujan was very interested in Goursat's result. What he really loved was not general results that could not be made very explicit, but beautiful formulas. I could imagine Ra-
9 manujan working out the details in section 3, but the resulting formulas are already starting to be messier than those he loved.

I sent an outline of the results in sections 2 and 3 to a couple of people, and George Andrews wrote back that the 4-balanced very well poised ${}_7F_6$ sum was found by Lakin [9]. The two proofs given in section 2 are easier than the two Lakin gave, so it is worth including them above. Lakin also found a basic hypergeometric extension of this sum. The derivation of his result from the q-extension of Whipple's formula is the most natural one, so it will be given in the next section.

5 The 3-balanced very well poised ${}_8\varphi_7$.

The analogue of Whipple's transformation formula (2.5) was found by Watson [13]. It is

$$
{}_8\varphi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, aq^{n+1} \end{matrix}; q, \frac{a^2q^{n+1}}{bcde}\right) \tag{5.1}
$$
$$
= \frac{(aq;q)_n(\frac{aq}{bc};q)_n}{(\frac{aq}{b};q)_n(\frac{aq}{c};q)_n}\, {}_4\varphi_3\left(\begin{matrix} q^{-n}, \frac{aq}{de}, b, c, \\ \frac{aq}{d}, \frac{aq}{e}, \frac{bcq^{-n}}{a} \end{matrix}; q, q\right)
$$

where

$$
(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}) \tag{5.2}
$$

and

$$
{}_{p+1}\varphi_p\left(\begin{matrix} a_0, \dots, a_p \\ b_1, \dots, b_p \end{matrix}; q, x\right) = \sum_{k=0}^{\infty} \frac{(a_0;q)_k \dots (a_p;q)_k x^k}{(b_1;q)_k \dots (b_p;q)_k (q;q)_k}. \tag{5.3}
$$

The series (5.3) is called k-balanced at q^j if $x = q^j$, one of the numerator parameters is q^{-n} and $a_0a_1\dots a_pq^k = b_1\dots b_p$. It is called balanced if $k = 1$ and $j = 1$. The series (5.3) is well poised if $a_0q = a_1b_1 = \dots = a_pb_p$, and very well poised if it is well poised and if $a_1 = qb_1, a_2 = -a_1$.

The sum that corresponds to (1.6) occurs when the ${}_4\varphi_3$ in (5.1) becomes a ${}_3\varphi_2$ bey setting $a^2q^{n+1} = bcde$, and using

$$ {}_3\varphi_2\left(\begin{matrix} q^n, a, b \\ c, q^{1-n}abc^{-1} \end{matrix}; q, q\right) = \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n} \tag{5.4} $$

to sum the resulting balanced ${}_3\varphi_2$. Observe that the balancing condition is now 1-balanced in the q-case as opposed to 2-balanced in the hypergeometric case.

The analogue of (2.3) requires a 3-balanced very well poised ${}_8\varphi_7$ at q^2. To obtain this sum, use (5.1) with

$$ \frac{aq}{de} = \frac{bcq^{1-n}}{a} \tag{5.5} $$

The ${}_4\varphi_3$ becomes

$$ \sum_{k=0}^{n} \frac{(q^{-n};a)_k(b;q)_k(c;q)_k q^k}{\frac{aq}{d};q)_k(\frac{aq}{e};q)_k(q;q)_k} \frac{(1-bcq^{k-n}/a)}{1-bcq^{-n}/a} \tag{5.6} $$

$$ = {}_3\varphi_2\left(\begin{matrix} q^{-n}, b, c, \\ aq/d, aq/e \end{matrix}; q, q\right) + \frac{bc(1-q^{-n})(1-b)(1-c)q}{(aq^n-bc)(1-aq/d)(1-aq/e)} \times $$

$$ \times\, {}_3\varphi_2\left(\begin{matrix} q^{1-n}, bq, cq \\ aq^2/d, aq^2/e \end{matrix}; q, q\right) $$

where

$$ 1-bcq^{k-n}/a = 1-bcq^{-n}/a + bcq^{-n}(1-q^k)a^{-1} $$

was used to break the series into two sums. The second sum on the right in (5.6) is balanced, and so can be summed by (5.4). The first is 2-balanced at q, and a q-extension of (2.7) can be sued to sum this series. To obtain this transformation, recall a transformation of Sears [12]:

$$ {}_4\varphi_3\left(\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix}; q, q,\right) = \left(\frac{bc}{d}\right)^n \frac{(aq^{1-n}/e;q)(aq^{1-n}/f;q)_n}{(e;q)_n(f;q)_n} \times \tag{5.7} $$

$$ \times {}_4\varphi_3\left(\begin{matrix} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/e, aq^{1-n}/f \end{matrix}; q, q\right) $$

when $q^{1-n}abc = def$.

Let b, $d \to 0$ in (**??**). The result is

$$ {}_3\varphi_2\left(\begin{matrix} q^{-n}, a, c \\ e, f \end{matrix}; q, q\right) = \left(\frac{efq^{n-1}}{a}\right)^n \frac{(aq^{1-n}/e;q)_n(aq^{1-n}/f;q)_n}{(e;q)_n(f;q)_n} \times \tag{5.8} $$

$$ \times\ {}_3\varphi_2\left(\begin{matrix} q^{-n}, a, q^{1-n}ac/ef \\ aq^{1-n}/e, aq^{1-n}/f \end{matrix}; q, q\right) $$

When the left hand side is k-balanced, $q^n ac = efq^{-k}$; so that right hand side is the sum of k terms.

Formula (5.8) with $k = 2$ reduces to formula (27) in [9]. The result obtained when the series on the right in (5.6) are summed is equivalent to (29) in [9] comes from the series on the left in (5.6) when $1 - bcq^{k-n}a^{-1}$ is broken into the two parts 1 and $-bcq^{k-n}a^{-1}$. Since these identities and the sum of (5.1) when it is 3-balaced at q^2 and very well poised are given by Lakin [9], they will not be repeated here.

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TITCHMARSH'S PHENOMENON FOR $\zeta(s)$

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1 Introduction

Under the title "On the frequency of Titchmarsh's phenomenon for $\zeta(s)$" **13**
we have written seven papers [11, 12, 2, 1, 3, 4, 13] sometimes individually and sometimes jointly. the present article is a summary of these results. The function $\zeta(s)(s = \sigma + it)$ is defined in $\sigma > 0$ by

$$\zeta(s) = \sum_{n=1}^{\infty}\left(\frac{1}{n^s} - \int\limits_{n}^{n+1}\frac{du}{u^s}\right) + \frac{1}{s-1}.$$

The sum on the right can be easily shown to be an entire function by a repetition of the trick which we have employed to prove that this is an analytic continuation of $\zeta(s) = \sum\limits_{n-1}^{\infty} n^{-s}(\sigma > 1)$ in $\sigma > 0$. Thus the serious problem about $\zeta(s)$ is not the analytic continuation. But the conjecture $\zeta(s) \neq 0$ in $\sigma > 1/2$ is really a very serious problem. [This is called Riemann hypothesis (R.H.)]. To serve as an introduction to our results we will first state some results (free from any hypothesis). We next recall some well-known consequences of Riemann hypothesis for comparison with these results. We will be concerned with the size of $|\zeta(\sigma + it)|$ in $1/2 \leqslant \sigma \leqslant 1$, $t \geqslant t_0$ where t_0 is a large positive constant which may depend on parameters like σ and other constants like (arbitrarily small positive) ϵ when they appear. The letter A will denote an absolute positive constant and C will denote a positive constant independent of t

but may depend on other parameters. There may not be the same at each occurrence.

$$|\zeta(1/2) + it| < t^{\mu+\epsilon} \tag{1}$$

($\mu = 1/2$ is easy; $\mu = 1/4$ is a little more difficult, the fundamental result $\mu = 1/6$ is due to G. H. Hardy, J.E. Littlewood and H. Weyl [19]. There have been a number of important papers by various authors which reduce $\mu = 1/6$, the latest being $\mu = 9/56$ due to E. Bombieri and H. Iwaniec [7] and a further reduction by $\frac{1}{560}$ due to M. N. Huxley and N. Watt. A. result of N. V. Kuznetsov proved by him in a paper presented by him in this conference implies that we can take $\mu = 1/8$).

$$|\zeta(\sigma + it)| < (t^{(1-\sigma)^{3/2}} \log t)^A \tag{2}$$

14 (due to the ideas of I.M. Vinogradov, see A. Walfisz's book [20]; see also H.-E. Richert [18]).

$$|\zeta(\sigma + it)| < t^{\mu(\sigma)+\epsilon} \tag{2'}$$

(various values of $\mu(\sigma)$ are obtained by various methods by various authors; see E. C. Titchmarsh's book [19].)

$$|\zeta(1 + it)| < A(\log t)^{2/3} \tag{3}$$

(due to the ideas of I.M. Vinogradov, see A. Walfisz's book [20].)

We now state consequences of R.H.

$$|\zeta(1/2 + it)| < \mathrm{Exp}(A \log t / \log\log t) \tag{4}$$

(due to J.E. Littlewood, see [19])

$$|\zeta(\sigma + it)| \, \mathrm{Exp}(C(\log t)^{2(1-\sigma)} / \log\log t), C = C(\delta), \tag{5}$$

uniformly in $1/2 \leqslant \sigma \leqslant 1 - \delta (\delta > 0)$. (This is due to J.E. Littlewood, E.C. Titchmarsh and others, see [19])

$$|\zeta(1 + it)| < (2e^{\gamma} + \epsilon) \log\log t \tag{6}$$

(due to J.E. Littlewood, see [19]. Littlewood's method shows that if θ defined as the least upper bound of the real parts of the zeros of $\zeta(s)$ is less than 1, then (6) would follow with some positive constant in place of $2e^{\gamma}$. Here as elsewhere we denote by γ the Euler's constant).

If we compare (1), (2), (3) with (4), (5), (6) we see how much has been achieved in the direction of Lindelöf hypothesis (L. H.) (which is a consequence of R.H.) with states that in (1) we can take $\mu = 0$. A consequence of L.H. is that we can take $\mu(\sigma) = 0$ in $(2')$. The L.H. has also remained unsolved for a long time. We do not know whether we can take $\mu(\sigma) = 0$ for any value of σ in $1/2 \leqslant \sigma < 1$. These results seem to be out of reach for many centuries to come. Also we do not know whether the results (4), (5), (6) can be improved on the assumption of R.H.. However we can show that in (6) $2e^{\gamma}$ cannot be replaced by any constant less than e^{γ}. The corresponding results regarding (4) and (5) are not so satisfactory. In (4), we can show that we cannot replace the right hand side by $\text{Exp}((\log t)^{\frac{1}{2}-\epsilon})$ and that the right hand side in (5) cannot be replaced by $\text{Exp}((\log t)^{1-\sigma-\epsilon})$ in $1/2 + \delta \leqslant \sigma \leqslant 1 - \delta$. These results (which are called Ω results) are due to J.E. Littlewood and E.C. Titchmarsh. Littlewood generally assumes R.H. and Titchmarsh's 15
results are independent of any hypothesis. For references to the work of Littlewood and Titchmarsh see [19].

THE PROBLEM. Let σ be fixed in $\sigma \geqslant 1/2$ (σ may depend on T and H to follow). Let I denote an interval of length H contained in $[T, 2T]$, where $H > 1000$. (We may also make σ depend on T and H; for example, we can take $\sigma = 1 + 1/\log H$). In the first two papers [11, 12] of the series max second author investigated $\max\limits_{t \text{ in } I} |\zeta(\sigma + it)|$ and also $\max\limits_{\alpha \geqslant \sigma(1)} \max\limits_{t \text{ in } I} |\zeta(\alpha + it)|$ and other problems like $\max\limits_{t \text{ in } I} |\zeta(\sigma + it)|$ where the minimum is taken over all intervals I of length H contained in $[T, 2T]$. (He has also improved Theorem **??** of [11] as follows. Let $(\zeta_K(s))^{1/2} = \sum\limits_{n=1}^{\infty} \dfrac{a_n(K)}{n^s}$). Then the RHS in Theorem **??** can be replaced by $U \times \sum_{n \leqslant U} \frac{|a_n(K)|^2}{n^{2\sigma_0}} (\log n)^l$ with the condition $1000 \log\log X \leqslant U \leqslant X$). These and other problems were studied further in papers [2, 1, 3, 4, 13].

2 Key result and its applications.

The method employed in the first paper of series was systematized by the second of us in [14]. This was improved by us in [5]; but this improvement (though a significant progress) does not give any new Ω results. The net result is as follows.

Theorem 1. *Let* $\{a_n\}$ *be a sequence of complex numbers satisfying* $a_1 = 1$ *and for* $n \geqslant 2$, $|a_n| < (n(H+2))^A$, *where* $H > 0$ *and* A *is an arbitrary positive constant. Let* $0 < H \leqslant T$ *and* $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *be analytically continuable in* $\sigma > 0$ *and continuous in* $\sigma \geqslant 0$. *Then*

$$\max_{\sigma \geqslant 0} \left(\frac{1}{H} \int_I |F(\sigma + it)|^2 dt \right) \geqslant$$

$$C_A \sum_{n \leqslant \frac{H}{100}+1} |a_n|^2 \left(1 - \frac{\log n}{\log H} + \frac{1}{\log\log H} \right)$$

where $C_A \geqslant 0$ *is effective and depends only on* A.

Corollary. *If* $\max_{\sigma \geqslant 0,\, t \text{ in } I} |F(\sigma+it)| < \operatorname{Exp}\operatorname{Exp} \frac{H}{100}$, *then* $\frac{1}{H} \int_I |F(it)|^2 dt > \frac{1}{2} C_A \sum_{n \leqslant \frac{H}{100}+1} |a_n|^2 \left(1 - \frac{\log n}{\log H} + \frac{1}{\log\log H} \right)$.

16 Proof of the corollary. One method of deduction of the corollary is by Gabriel's two variable two variable convexity theorem coupled with the kernel $\operatorname{Exp}.(\sin(z/100)^2)$, (see the appendix to [15]). For another method see [14]. Note that this kernel decays in $|\operatorname{Re} z| \leqslant 1/2$ uniformly at most like a constant multiple of $(\operatorname{Exp}\operatorname{Exp}(c|\operatorname{Im} z|))^{-1}$ where $c > 0$ is a constant. Given faster decaying kernels we can deduce the Corollary

with more relaxed conditions. The same applies to all applications of the key theorem. It may be remarked that we do not know any kernel which decays (uniformly in $|z| \leqslant 1/2$) at most like a constant multiple of $(\operatorname{Exp}\operatorname{Exp}(c|\operatorname{Im} z|))^{-1}$ where c is any large positive constant.

We begin the applications by the following remark, which follows from The theorem by putting $F(s) = (\zeta(\alpha + s))^k$ where $\alpha \geqslant 1/2$ (we may assume without loss of generality that H exceeds a large positive constant) and k is a positive integer which may depend on H.

Theorem 2. *We have, for $\alpha \geqslant 1/2$,*

$$\max_{\sigma \geqslant \alpha,\, t \text{ in } I} |\zeta(\sigma+it)| > \left(C_A \sum_{n \leqslant \frac{H}{100}} \frac{(d_k(n))^2}{n^{2\alpha}} \left(1 - \frac{\log n}{\log H} + \frac{1}{\log\log H} \right) \right)^{1/2k},$$

where $k \leqslant \log H$ so that the condition $d_k(n) \leqslant (n(H+2))^A$ is satisfied. (In fact it may be noted that the maximum of RHS *is attained in $k \leqslant \log H$ itself).*

Remark. We may also state a similar theorem for

$$\max_{\sigma \geqslant \alpha,\, t \in I} |\zeta(\sigma + it)|^{-1}$$

where $\alpha \geqslant 1/2 + \delta$ and $(\sigma \geqslant \alpha - \delta/2, t \text{ in } I)$ is free from zeros of $\zeta(s)$. Here δ is an arbitrary positive constant.

It is not very difficult to investigate the order of magnitude of $\max_{k \geqslant 1}$ (R.H.S.) in Theorem 2. By a very ingenious argument, R. Balasubramanian has shown (see [1]) that its logarithm is asymptotic to

$$C_0(\log H/\log\log H)^{1/2}$$

where $C_0 = 0.75\ldots$, when $\alpha = 1/2$. This gives the best known Ω result $\max_{\sigma \geqslant \frac{1}{2},\, t \text{ in } I} |\zeta(\sigma + it)| > \operatorname{Exp}\left[\frac{3}{4}\left(\frac{\log H}{\log\log H}\right)^{1/2}\right]$.

Earlier in [2], we had obtained a small positive constant in place of

3/4. Balasubramanian's asymptotic formula shows that it is not possible to replace 3/4 by even 0.76 by our method. Earlier to our result [2], nearly around the same time, H.L. Montgomery [9] had obtained the constant 1/20 (in place of 3/4) on the assumption of Riemann hypothesis. We would also like to remark that in order to obtain the maximum order (of RHS in Theorem 2) as k varies, we have to take (in case $\alpha = 1/2$) a large number of terms of the sum and ignore the rest. One the contrary, when $\alpha \geqslant 1/2 + \delta$ it is enough to take a particular term, namely, the maximum term of the sum. If $1/2 + \delta \leqslant \alpha \leqslant 1 - \delta$ then it is enough to take n to be the biggest square-free product of the first k primes, which does not exceed H. If $\alpha = 1$, then for each p we select that prime power p^m for which $(d_k(p^m))^2 p^{-2m\alpha}$ is the largest and then take n to be the product of the first k prime powers p^m which does not exceed H (for details see [2] or [17].) If $\alpha = 1 + (1/\log H)$ for instance we may take out from $(d_k(n))^2 n^{-2\alpha}$ the portion $n^{-(2/\log H)}$ (which certainly exceeds e^{-2} for $n \leqslant H$). In [2] the first of us has shown that even if we take all the terms of the sum we do not get a better result. In fact following the method of [1] we may show that

$$\log\left[\max_{k \geqslant 1}\left(\sum_{n \leqslant x} \frac{[d_k(n)]^2}{n^{2\alpha}}\right)^{1/2k}\right](\log x)^{\alpha-1}(\log\log x)$$

tends to a positive constant if $1/2 + \delta \leqslant \alpha \leqslant 1 - \delta$. These results show the limitations of our method. However out net result are

Theorem 3(A). *We have, if* $1/2 + \delta \leqslant \alpha \leqslant 1 - \delta$,

$$\max_{\sigma \geqslant \alpha,\ t \text{ in } I} |\zeta(\sigma + it)| > \operatorname{Exp}\left(\frac{C(\log H)^{1-\alpha}}{\log\log H}\right).$$

Theorem 3(B). *We have, if* $1 - \frac{1}{\log H} \leqslant \alpha \leqslant 1 + \frac{1}{\log H}$,

$$\max_{\sigma \geqslant \alpha,\ t \text{ in } I} |\zeta(\sigma + it)| \geqslant e^{\gamma}(\log\log H - \log\log\log H + O(1)).$$

Remark 1. H. L. Montgomery [9] has shown that if $1/2+\delta \leqslant \alpha \leqslant 1-\delta$ then $|\zeta(a+it)|$ exceeds $\mathrm{Exp}[C(\log t)^{1-\alpha}]/(\log\log t)^{\alpha}]$ for a sequence of values of t tending to infinity by a different method. But this method does not enable one to conclude that the maximum of $|\zeta(\alpha+it)|$ as t varies for example, over $[T, 2T]$ exceeds $\mathrm{Exp}[C(\log T)^{1-\alpha}]/(\log\log T)^{\alpha}]$. The lower bound $\mathrm{Exp}[C(\log T)^{1-\alpha}/(\log\log T)]$ given by our method seems to be the best known till today.

Remark 2. N. Levinson [8] has shown by a different method that $|\zeta(1+it)|$ exceeds $e^{\gamma}\log\log t + O(1)$ for a sequence of values of t tending to infinity. But, for short intervals like $[T, 2T]$, ours is the only result known. 18

Remark 3. Let $1/2+\delta \leqslant \alpha \leqslant 1-\delta$. I suspected that if we take $F(s) = (\log\zeta(\alpha+s))^k$ then we might get a better result Theorem 3(A). But it was shown by H. L. Montgomery [10] that if $(\log\zeta(s))^k = \sum\limits_{n=1}^{\infty} a_k(n)n^{-s}$, then

$$\max_{k \geqslant 1}\left(\sum_{n\leqslant x}(a_k(n))^2 n^{-2\alpha}\right)^{1/2k}$$

lies between two constant multiples of $(\log x)^{1-\alpha}(\log\log x)^{-1}$.

Remark 4. H.L. Montgomery has conjectured [9] that if $1/2 \leqslant \alpha \leqslant 1-\delta$ then $|\zeta(\alpha+it)|$ does not exceed $\mathrm{Exp}([C(\log t)^{1-\alpha}]/[\log\log t]^{\alpha})$

3 Further study of the maximum in 1/2 $+\delta \leqslant \alpha \leqslant 1-\delta$ by other methods.

In the second paper of the series [12], the second of us proved that if $1/2+\delta \leqslant \alpha \leqslant 1-\delta$ and I runs over all intervals, of fixed length H, contained in $[T, 2T]$ then

$$\log\log\left(\min_{I}\max_{t \text{ in } I}|\zeta(\alpha+it)|\right) \sim (1-\alpha)\log\log H,$$

provided $C < 100\log\log T \leqslant H \leqslant \mathrm{Exp}[D\log T]/[\log\log T])$ where C is a large positive constant and D a positive constant depending only on

α. This aspect of the problem has been studied further by us in [2]. The method is very closely related to a principle which we formulated and employed in [6]. The main result of [3] is as follows:

Theorem 3. *let α be as above, $E > 1$ an arbitrary constant, $C \leqslant H \leqslant \operatorname{Exp}\left(\dfrac{D\log T}{\log\log T}\right)$ where C is a large positive constant and D an arbitrary positive constant. Then there are $\geqslant TH^{-E}$ disjoint open intervals I (of fixed length H) all contained in $[T, 2T]$, such that,*

$$\frac{(\log H)^{1-\alpha}}{(\log\log H)^{\alpha}} \ll \max_{t \text{ in } I} |\log\zeta(\alpha + it)| \ll \frac{(\log\log H)^{1-\alpha}}{(\log\log H)^{\alpha}}.$$

Here $\log\zeta(s)$ is the analytic continuation in $t \geqslant 2$ along lines parallel to the real axis (and free from zeros of $\zeta(s)$) from $\sigma \geqslant 1$. The Vinogradov symbol $\ll$ means "less than a positive constant times".

4 Study of the maximum on $\sigma = 1$.

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As a corollary to Theorem 3, we deduced in [4] the following

Theorem 4. *Let J denote the interval I (of Theorem 3) with intervals of length $(\log H)^2$ removed from both extremities. Then*

$$\max_{t \text{ in } J} |\zeta(1 + it)| \leqslant e^{\gamma}[\log\log H + \log\log\log H + O(1)].$$

Note that LHS *is $\geqslant e^{\gamma}(\log\log H - \log\log\log H + O(1))$ by applying the Corollary to the key theorem. (The conditions for deriving this lower bound from the Corollary to the key theorem are satisfied in the course of the proof of Theorem 3).*

The key result of §2 can be used to obtain lower bounds for $\max_{\sigma \geqslant 1,\ t \text{ in } I} |\zeta(\sigma + it)|$ and also a similar result (the lower bound gets multiplied by the factor $(6/\pi^2)$ for $|\zeta(\sigma + it)|^{-1}$. But to obtain lower bounds for $\max_{t \text{ in } I} |\zeta(1+it)|$ and for $\max_{t \text{ in } i} |\zeta(1+it)|^{-1}$ we need conditions looking like $100 \log\log\log T \leqslant H \leqslant T$. But by a somewhat complicated application of the key result and other techniques the second of us [13] proved the following theorem. To state the theorem it is better to introduce some notation. The letter θ will, as before, denote the least upper bound of the real parts of the zeros of $\zeta(s)$ (we do not know whether $\theta < 1$ or not). For $x \geqslant 1$ we define $\log_1 x = \log x$ and for $n \geqslant 2$ we define $\log_n x$ to be $\log(\log_{n-1} x)$; similarly we define for real x, $\operatorname{Exp}_1(x) = \operatorname{Exp}(x)$ and for $n \geqslant 2$ we define $\operatorname{Exp}_n(x) = \operatorname{Exp}(\operatorname{Exp}_{n-1}(x))$.

Theorem 5. *Consider for open intervals I (for t, of length $H \geqslant 100$) contained in $[T, 2T]$ where $T \geqslant T_0$, a large positive constant, the inequality*

$$\max_{t \text{ in } I} |\zeta(1+it)| \geqslant e^{\gamma}(\log\log H - \log\log\log H - \rho), \tag{$*$}$$

where ρ is a certain real constant which is effective. Then we have the following four results:

(1) *$(*)$ holds for all I for which $T \geqslant H \geqslant A_1 \log_4 T$*

(2) *If $\theta < 1$ then $(*)$ holds for all I for which $T \geqslant H \geqslant A_2 \log_5 T$.*

(3) *Let now $H < A_1 \log_4 T$. Consider a set of disjoint intervals I (of fixed length H) for which $(*)$ is false. Then the number of such intervals I does not exceed TX_1^{-1} where $X_1 = \operatorname{Exp}_4(\beta H)$ where β is a certain positive constant less than A_1^{-1}.*

(4) *Let now $H < A_2 \log_5 T$. Consider a set of disjoint intervals I (of*
fixed length H) for which $(*)$ is false. Then the number of such 20
intervals I does not exceed TX_2^{-1} where $X_2 = \operatorname{Exp}_5(\beta' H)$ where β'
is a certain positive constant which is less than A_2^{-1}.

5 An announcement

In this section the length of the interval will not be denoted by H. We wish to announce a result [16] due to the second author which is obtained by quite a different method.

Theorem 6. *Let ϵ be a constant satisfying $0 < \epsilon < 1$, $T \geqslant T_0(\epsilon)$, a constant depending only on ϵ, $X = \text{Exp}\left(\dfrac{\log T}{\log\log T}\right)$. If from the intervals $T \leqslant t \leqslant T + e^X$ we exclude certain (boundedly many depending on ϵ) disjoint open intervals I each of length at most X^{-1}, then in the remaining portions of the interval, we have,*

$$|\log \zeta(1 + it)| \leqslant \epsilon \log\log T.$$

Further put $\beta_0 = A(\log T)^{-mu}(\log\log T)^{-2\mu}$ where $\mu = 2/3$ and A is any positive constant. Consider the rectangle R defined by $\sigma \geqslant 1 - \beta_0$, $T \leqslant t \leqslant T + e^X$. Let I denote an open interval for t of length $1/X$ and let J denote the corresponding rectangle $\sigma > 1 - \beta_0$, t in I. Then with the exception of certain boundedly many (depending on ϵ and A) disjoint rectangles J we have for s in R,

$$|\log \zeta(s)| \leqslant \epsilon \log\log T$$

where $T \geqslant T_0(\epsilon, A)$

Remark . The first result can be proved without assuming the Vinogradov's zero free region. But if we assume the Vinogradov's zero free region, we get a better upper bound for the number of intervals which have to be excluded. However, for the proof of the second part, the Vinogradov zero free region is essential.

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RAMANUJAN'S FORMULAS FOR EISENSTEIN SERIES

By Bruce C. Berndt*

As is customary, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathscr{H} = \{\tau : \operatorname{Im}\tau > 0\}$, and 23

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 (\operatorname{mod} n) \right\},$$

where $n \in \mathbb{N}$. If $n = 1$, $\Gamma_0(1)$ is the full modular group $\Gamma(1)$.

Let

$$E_2(\tau) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^{2k}}{1 - q^{2k}}$$

and

$$F_n(\tau) = E_2(\tau) - nE_2(n\tau),$$

where $q = e^{\pi i \tau}$, $\tau \in \mathscr{H}$, and $n \in \mathbb{N}$. Although $E_2(\tau)$ is not a modular form, it can be easily shown that $F_n(\tau)$ is a modular form of weight 2 and trivial multiplier system on $\Gamma_0(n)$.

In a very famous paper [8, pp. 23-39], Ramanujan gave formulas for F_n when $n = 2, 3, 4, 5, 7, 11, 15, 17, 19, 23, 31, 35$. However, no proofs are indicated. Furthermore, in Chapter 21 of his second notebook [9], Ramanujan offers, without proofs, formulas for F_n when $n = 3, 5, 7, 9, 11, 15, 17, 19, 23, 25, 31, 35$. In contrast to [8] where only one formula is given for each value of n, in [9] several formulas are stated for most values of n.

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Part of Ramanujan's motivation in calculating F_n arose from its appearance in certain approximations to π found by Ramanujan [8]. J.M. and P.B. Borwein [6] have extensively developed Ramanujan's ideas. Using their work, we shall very briefly indicate how these approximations are obtained. Let K denote the complete elliptic integral of the first kind associated with the modulus k, where $0 < k < 1$, and let E' denote the complete elliptic integral of the second kind associated with the complementary modulus $k' = \sqrt{1-k^2}$. For $r > 0$, define

$$\alpha(r) = \frac{E'}{K} - \frac{\pi}{4K^2},$$

24 where $k = k(r) = \theta_2^2(e^{-\pi\sqrt{r}})/\theta_3^2(e^{-\pi\sqrt{r}})$, where θ_2 and θ_3 are the classical theta-functions, usually so denoted. Put $\alpha_m = \alpha(n^{2m}r)$, where $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. There exists a recursion formula for α_m in terms of F_n [6, p. 158]. This leads to an approximation of for $1/\pi$ given by

$$0 < \alpha_m - 1/\pi < 16n^m\sqrt{r}e^{-n^m\sqrt{r}\pi}$$

provided that $rn^{2m} \geqslant 1$ [6, p. 169]. For complete details, see [6].

The Borweins leave the calculation of F_n for $n = 2, 3, 4$ as exercises [6, p. 161]. In fact, they [6, 9. 158] state that "The verification... is tedious but straightforward for small n. For larger n, we rely on Ramanujan." The Surpose of this paper is to indicate how Ramanujan's formulas for F_n can be proved. Complete proofs for all of Ramanujan's formulas for F_n can be found in the author's forthcoming book [2]. We offer two general approaches. The first is probably similar to that employed by Ramanujan, while the second depends upon the theory of modular forms.

The first method rests upon modular equations. Thus, we need to give the definition of a modular equation, as understood by Ramanujan.

Definition. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, K', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1}$$

holds for some $n \in \mathbb{N}$. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (1).

Ramanujan sets $\alpha = k^2$ and $\beta = l^2$.
If $q = \exp(-\pi K'/K)$ and

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}$$

then it is well known that

$$K = \frac{\pi}{2}\varphi^2(q).$$

Furthermore, set $z_n = \varphi^2(q^n)$.

Definition. The multiplier m for a modular equation of degree n is defined by

$$m = \frac{K}{L} = \frac{\varphi^2(q)}{\varphi^2(q^n)} = \frac{z_1}{z_n}.$$

In his notebooks [9], Ramanujan devotes more space to modular 25
equations than to any other topic. Despite this, Ramanujan never published any of his work on modular equations, except for the aforementioned formulas for Eisenstein series in [8]. For an expository account of Ramanujan's discoveries on modular equations, see our paper [1]. Some of Ramanujan's modular equations have been proved in three papers [3], [4], [5] that we have coauthored with A. J. Biagioli and J. M. Purtilo. For proofs of all of Ramanujan's modular equations, see the author's forthcoming book [2].

We now state perhaps the primary formula that Ramanujan employed in establishing formulas for $F_n(\tau)$. He has not stated this formula in either [8] or [9]. However, some cryptic remarks on p. 253 of his second notebook [9] point to a result such as that given below.

Theorem 1. *Let q, F_n, α, β, m, and z_1 be as given above. Then*

$$F_n(\tau) = -\alpha(1-\alpha)z_1^2 \frac{d}{d\alpha} \operatorname{Log}\left(\frac{\beta(1-\beta)}{m^6\alpha(1-\alpha)}\right).$$

We now sketch proofs for three of seven formulas for $F_3(\tau)$ found in Entry 3 of Chapter 21 in Ramanujan's second notebook [9].

Theorem 2. *Let φ, α, and β be as given above. Put*

$$\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}.$$

Then

$$S_3(\tau) := -\frac{1}{2}F_3(\tau) = \left\{\frac{\varphi^4(q) + 3\varphi^4(q^3)}{4\varphi(q)\varphi(q^3)}\right\}^2 \tag{2}$$

$$= \varphi^2(q)\varphi^2(q^3) - 4q\psi^2(-q)\psi^2(-q^3) \tag{3}$$

$$= \frac{1}{2}\varphi^2(q)\varphi^2(q^3)\left\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\right\}. \tag{4}$$

The last formula was stated by Ramanujan in [8], [10, p. 33].

Proof. Letting $n = 3$ in Theorem 1, we find that

$$S_3(\tau) = \frac{1}{2}\alpha(1-\alpha)z_1^2\frac{d}{d\alpha}\operatorname{Log}\left(\frac{\beta(1-\beta)}{m^6\alpha(1-\alpha)}\right). \tag{5}$$

We need to determine the interdependence of α, β and m in order to calculate the derivative above. From our work [2] on modular equations of degree 3 in Section 5 of Chapter 19 in Ramanujan's second notebook [9],

$$\alpha(1-\alpha) = \frac{(m^2-1)(9-m^2)^3}{256m^6}, \tag{6}$$

$$\frac{\beta(1-\beta)}{\alpha(1-\alpha)} = \frac{m^4(m^2-1)^2}{(9-m^2)^2}, \tag{7}$$

26 and

$$\frac{dm}{d\alpha} = \frac{16m^4}{(9-m^2)^2}. \tag{8}$$

Substituting (6)-(8) into (5) and employing the chain rule, we deduce that

$$S_3(\tau) = \frac{(m^2-1)(9-m^2)}{16m^2} z_1^2 \frac{d}{dm} \operatorname{Log}\left(\frac{m^2-1}{m(9-m^2)}\right) \tag{9}$$
$$= \frac{z_1^2}{16m^3}(m^2+3)^2.$$

If we now use the definition of m, we find that (2) readily follows.

Using again the definition of m, we may rewrite (9) in the form

$$S_3(\tau) = z_1 z_3 \left(1 - \frac{(9-m^2)(m^2-1)}{16m^2}\right) \tag{10}$$
$$= z_1 z_3 (1 - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}),$$

where we have employed (6) and (7). Now in Chapter 17 of his second notebook [9], Ramanujan offers a "catalogue" of evaluations of theta-functions in terms of $q(q^n)$, $\alpha(\beta)$, and $z_1(z_n)$. In particular, from Entry 11,

$$\psi(-q) = (\tfrac{1}{2}z_1)^{1/2}\{\alpha(1-\alpha)/q\}^{1/8}$$

and

$$\psi(-q^3) = (\tfrac{1}{2}z_3)^{1/2}\{\beta(1-\beta)/q^3\}^{1/8}.$$

Solving these two equalities for $\alpha(1-\alpha)$ and $\beta(1-\beta)$, respectively, and substituting them in (10), we immediately deduce (3).

The simplest modular equation of degree 3 is given by

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{11}$$

This was first discovered by Legendre and may be found in Cayley's book [7, p. 196], for example. Ramanujan [9, chpater 19, Entry 5(ii)] rediscovered (11). If we square both sides of (11) and substitute in (10), we immediately deduce (4).

Unfortunately, we have been unsuccessful in using Theorem 1 to establish certain formulas of Ramanujan for $F_n(\tau)$. We thus have had

to invoke the theory of modular forms in these cases. In order to offer 27
one such example, we need to make an additional definition. Let, in the notation of Ramanujan,

$$f(-q) = \prod_{j=1}^{\infty}(1 - q^j),$$

where, as above, $q = e^{\pi i \tau}$. Note that $f(-q^2) = q^{-1/12}\eta(\tau)$, where η denotes the Dedekind eta-function. We now state Entry 8(i) in Chapter 21 of Ramanujan's second notebook [9]. □

Theorem 3. *Let φ, ψ, and f be defined as above. Then*

$$\begin{aligned} -\frac{1}{2}F_{11}(\tau) &= 5\varphi^2(q)\varphi^2(q^{11}) - 20qf^2(q)f^2(q^{11}) \\ &\quad + 32q^2f^2(-q^2)f^2(-q^{22}) - 20q^3\psi^2(-q)\psi^2(-q^{11}). \end{aligned} \tag{12}$$

We now briefly describe how the theory of modular forms can be used to prove Theorem 3. The functions $\varphi(q)$, $\psi(q)$, and $f(-q)$ are associated with modular forms of weight 1/2 on

$$\Gamma(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : a \equiv d \equiv 1 (\text{mod } 2), b \equiv c \equiv 0 (\text{mod } 2)\}.$$

Thus, (12) is first converted into an equality relating modular forms. Each of the five expressions in (12) is a modular form of weight 2 on $\Gamma(2) \cap \Gamma_0(11)$. We have already mentioned that the multiplier system of $F_{11}(\tau)$ is trivial. By employing the multiplier system of $\eta(\tau)$, we can show that each of the four expressions on the right side of (12) also has a trivial multiplier system.

Let $\Gamma = \Gamma(2) \cap \Gamma_0(p)$, where p is an odd prime. Let $\mathscr{F}$ be a fundamental set for Γ. If F is a nonconstant modular form of weight r on Γ, then the valence formula

$$\sum_{z \in \mathscr{F}} \text{Ord}_\Gamma(F; z) = \frac{1}{2}r(p + 1) \tag{13}$$

is valid, where $\mathrm{Ord}_\Gamma(F : z)$ is the invariant order of F at z. Suppose that we can show that the coefficients of $q^0, q^1, q^2, \ldots, q^\mu$ in F are equal to 0, i.e. $\mathrm{Ord}_\Gamma(F; \infty) \geqslant \mu + 1$. Suppose furthermore that $\mu + 1 > \frac{1}{2}r(p+1)$. Then if $\mathrm{Ord}_\Gamma(F : z) \geqslant 0$ for each $z \in \mathscr{F}$,

$$\sum_{z \in \mathscr{F}} \mathrm{Ord}_\Gamma(F; z) \geqslant \mathrm{Ord}_\Gamma(F; \infty) \geqslant \mu + 1 > \frac{1}{2}r(p+1).$$

Hence $F(\tau) \equiv 0$, for otherwise, we could have a contradiction to the valence formula (13).

Now write the proposed identity (12) in the form 28

$$F := F_1 + \ldots + F_5 = 0. \tag{14}$$

We have shown that F is a modular form of weight 2 and trivial multiplier system on $\Gamma = \Gamma(2) \cap \Gamma_0(11)$. Moreover, $\mathrm{Ord}_\Gamma(F : z) \geqslant 0$ for each $z \in \mathscr{F}$. Since $(1/2)r(p+q) = 12$, it suffices to show that the coefficients of q^j, $0 \leqslant j \leqslant 12$, in F are equal to 0 in order to prove (14), and hence also (12). Using MACSYMA, we have indeed done this, and so the proof of Theorem (3) has been completed.

More complete details on the use of modular forms and MACSYMA in proving modular equations may be found in [2] and [4].

We are grateful to A.J. Biagioli and J.M. Purtilo for their collaboration on modular forms and MACSYMA, respectively.

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ON THE PROOF OF ANDREWS' q-DYSON CONJECTURE

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This is a breif sketch of work done by Doron Zeilberger, Ian Goulden 31
and myself in late 1983 and early 1984 which settled in the affirmative a conjecture made by George Andrews [1] as well as more detailed conjectures made by Kevin Kadell [12].

The problem has its origins in the evaluation of a definite integral which arose in a physical problem [5], its solution has given evaluations for other definite integrals arising in physics [4]. The original integral was discovered by Freeman Dyson [5]:

$$I(n,z) = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} |\Delta_n(e^{i\theta})|^{2z} d\theta_1 \ldots d\theta_n, \tag{1}$$

where

$$\Delta_n(e^{i\theta}) = \Pi(e^{i\theta_j} - e^{i\theta_k}),\ \ 1 \leqslant j < k \leqslant n.$$

Dyson conjectured that

$$I(n,z) = \frac{\Gamma(nz+1)}{\Gamma^n(z+1)}, \tag{2}$$

a conjecture which was simultaneously and independently proved by Gunson [7] and Wilson [23].

It is sufficient to prove that conjecture for positive integral integral z. In this case, we can use the following equality:

$$|\Delta_n(e^{i\theta})|^2 = \Pi(e^{i\theta_j} - e^{i\theta_k})(e^{-i\theta_j} - e^{-i\theta_k}). \tag{3}$$

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$$= \Pi(1 - e^{i(\theta_j - \theta_k)})(1 - e^{i(\theta_k - \theta_j)}).$$

If we set $x_j = e^{i\theta_j}$, then the integral picks out the constant term in a polynomial in $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$. Given a monomial, M, in the x_i's, let [M] denote the coefficient of M in the succeeding polynomial. Let x^0 denote the monomial in which each x_i appears to the power 0. Equation (2) for $z \in \mathbb{N}$ can be restated as

$$[x^0]\Pi(1 - x_j/x_k)^z(1 - x_k/x_j)^z = \frac{(nz)!}{(z!)^n}, \; 1 \leqslant j < k \leqslant n. \tag{4}$$

32

Dyson discovered that more was probably true, and actually stated his conjecture in the following form:

$$\begin{aligned} &[x^0]\Pi(1 - x_j/x_k)^{a_k}(1 - x_k/x_j)^{a_j} \\ &= \frac{(a_1 + \ldots + a_n)}{a_1! \ldots a_n!} \end{aligned} \tag{5}$$

In 1975, Andrews [1] noted that equation (5) seemed to have a nice generalization in which the product $(1 - x)^a$ could be replaced by

$$(x)_a = (1 - x)(1 - xq)(1 - xq^2)\ldots(1 - xq^{a-1})$$

Specifically, Andrews conjectured the following:

$$\begin{aligned} &[x^0]\Pi(x_j/x_k)_{a_k}(qx_k/x_j)_{a_j}, \; 1 \leqslant j < k \leqslant n, \\ &= \frac{(q)_{a_1+\ldots+a_n}}{(q)_{a_1}\ldots(q)_{q_n}}. \end{aligned} \tag{6}$$

On reason for the interest in equations (5) and (6) is the intractability of the blunt approach. If one expands the binomials in equations (5), the constant term is a simple summation when $n = 3$, and Dyson's conjecture is the classical identity:

$$\sum_i (-1)^i \binom{a_1 + a_2}{i}\binom{a_2 + a_3}{i - a_2 + a_3}\binom{a_1 + a_3}{i + a_1 - a_2} \tag{7}$$

$$= (-1)^{a_2}\binom{a_1 + a_2 + a_3}{a_1, a_2, a_3}.$$

For larger n, however, the constant term is an $\binom{n-1}{2}$-fold summation, and virtually nothing is known about such non-trivial multiple summations.

The same situation applies to Andrews' conjecture, except that instead of multiple hypergeometric series we get multiple basic hypergeometric series.

To understand how equation (6) was first proved, one must understand an ingenious combinatorial proof of Dyson's equation (5) which was found by Zeilberger [24] a few years earlier. Equation (5) is equivalent to

$$\begin{aligned}&[(x_1^{a_1} \dots x_n^{a_n})^{n-1}]\Pi(x_j - x_k)^{a_j+a_k} \\ &= (-1)^{a_2+2a_3+\dots+(n-1)a_n}\frac{(a_1 + \dots + a_n)!}{a_1! \dots a_n!}.\end{aligned} \tag{8}$$

We can formally expand the product of binomials in equation (8): 33

$$\Pi(x_j - x_k)^{a_j+a_k} = \sum_{T\in\mathscr{T}^*}(-1)^{u(T)}x_1^{w_1(T)} \dots x_n^{w_n(T)} \tag{9}$$

where $\mathscr{T}^*$ is the set of "multi-tournaments" in which each pair of players, say j and k, meet a total of $a_j + a_k$ times and the "winner" of each game is recorded. The exponent $w_i(T)$ is the number of games won by player i, and $u(T)$ records the number of "upsets" : $k > j$ and k beats j.

If we let $\mathscr{T} \subseteq \mathscr{T}^*$ be the subset of multi-tournaments in which each player j wins $(n-1)a_j$ games, then equation (8) can be restated as:

$$\sum_{T\in\mathscr{T}}(-1)^{u(T)} = (-1)^{a_2+\dots+(n-1)a_n}\binom{a_1 + \dots + a_n}{a_1, \dots, a_n}. \tag{10}$$

The right side of equation (10) involves the multinomial coefficient which counts the number of "words" which can be constructed with $a_1 1's$, $a_2 2$' $s, \dots, a_n n's$. Each such word corresponds to a multi-tournament in a natural way. Given j and k, remove the subword of length

$a_j + a_k$ in the letters j and k. The winners in order are read off left to right.

As an example, if $n = 4$, $a_1 = a_2 = a_3 = a_4 = 2$, the word 32114243 corresponds to the multi-tournament:

2112

3113

1144

3223

2424

3443

We observe that the number of upsets is always $a_2 + 2a_3 + \ldots + (n-1)a_n$.

If we let $\mathscr{T}' \subseteq \mathscr{F}$ be the subset of multi-tournaments which do not correspond to a word, then equation (10) can be further simplified to

$$\sum_{T \in \mathscr{T}'} (-1)^{\mu(T)} = 0. \tag{11}$$

Zeilberger showed how to prove this by establishing a bijection between the set of $T \in \mathscr{T}'$ for which $u(T)$ is even and the set of $T \in \mathscr{T}'$ for
34 which $u(T)$ is odd. We shall demonstrate the bijection with an example. Let T be

2111

3133

1144

2232

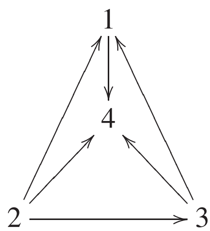
2424

3443

Inspection immediately shows us that while this element is in $\mathscr{T}$, it cannot be the two letter subwords of a single word. Nevertheless, we shall attempt to construct a word to which this multi-tournament corresponds.

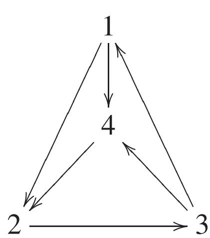
The leading entries of each row define a tournement:

2 beats 1, 3 beats 1, 1 beats 4, etc. Schematically, this tournament is given by:



We call a tournament "transitive" if it contains no cycles, "non-transitive" otherwise. If our multi-tournament arose from a single word, then this tournament is transitive and the player beating everyone else is the first letter of the word. Since our tournament is transitive, it is possible at this stage that it comes from a single word. We record the first letter : 2, and modify the tournement by looking at the next outcome of the games of player 2: 1 beats 2, 2 beats 3, 4 beats 2.

The tournament becomes :



Our tournament is now non-transitive which will eventually happen if and only if T is in $\mathscr{T}'$.

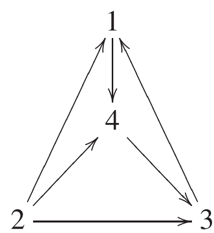
Every non-transitive tournament contains a 3-cycle and reversing the arrows in a 3-cycle will change the parity of the number of upsets in the tournament. We have two 3-cycles in this tournament. which one we choose to reverse is significant. 35

If we reverse $2 \to 3 \to 4 \to 2$ and then restore the first letter, 2, we get the multi-tournament

2111

3133
1144
2332
2224
4443

But the leading entries of this multi-tournament give us a non-transitive tournament:



An iteration of our procedure would not take us back to the original multitournament.

If no letters of the word have been recorded, then it doesn't matter which 3-cycle we reverse as long as we are consistent. If at least one letter has been recorded, then we are in a peculiar situation. Let v_1 be the last letter recorded. Since we have only changed the arrows connected to v_1, all cycles of the non-transitive tournament include vertex v_1.

Let the remaining vertices be labelled $v_2, v_3, \ldots, v_n$ where v_2 beats v_3 beats ... beats v_n, and choose the smallest i for which v_1 beats v_i and v_{i+1} beats v_1. It is the 3-cycle $v_1 \to v_i \to v_{i+1} \to v_1$ that we reverse.

it is exactly this procedure that was used to prove Andrews' conjecture, except that the details are more complicated because the parameter q introduces an additional weight on the multi-tournaments.

The proof first demonstrates that

$$[x^0]\Pi(x_j/x_k)_{a_k}(qx_k/x_j)_{a_j} \tag{12}$$
$$= (-1)^{a_2+\ldots+(n-1)a_n} \sum_{t\in\mathscr{F}} (-1)^{\mu(T)} q^{wt(T)}, \tag{13}$$

where $wt(T)$ is the sum of the "Major Indices" of all the two letter words in the multi-tournament. The Major Index of a word is the sum of the number of letters to the left of each "descent" in the word. Thus **36**

$$32114243$$

has four descents : (32, 21, 42, 43) , and its major index is $1+2+5+7 = 15$.

On the other hand, if we sum the major indices of the two letter subwords of 32114243, we get $1 + 1 + 0 + 1 + 2 + 3 = 8$. This sum of Major Indices is called the Z-statistic, denoted $Z(T)$. The second part of the proof involves showing that the sum of $q^{Z(T)}$ over all multi-tournaments corresponding to a single word is equal to

$$\frac{(q)_{a_1+\ldots+a_n}}{(q)_{a_1}\ldots(q)_{a_n}}$$

Equation (6) now reduces to verifying that

$$\sum_{T\in\mathscr{T}'}(-1)^{u(T)}q^{wt(T)} = 0. \tag{14}$$

The bijection given above does not preserve weights. The last and most elaborate part of the proof involves finding and verifying a bijection which does.

It is curious that this combinatorial approach is still the only known proof of equation (6).

Goulden and I[3] generalized this proof of yield a more useful identity. In the following we let A be an arbitrary set of unordered pairs (j,k), $1 \leqslant j \neq k \leqslant n$, $\chi(S)$ is 1 if S is true, 0 otherwise, $\mathfrak{S}_A$ is the set of permutations of $\{1,\ldots,n\}$ for which $j > i$ and $\sigma^{-1}(i)$ implies $(i,j) \notin A$, and $wt(\sigma)$ is the sum over all j of a_j times the number of $k < j$ for which $\sigma^{-1}(j) < \sigma^{-1}(k)$.

$$[x^0]\Pi(x_j/x_k)_{a_j}(qx_k/x_j)_{a_k-\chi((j,k)\in A)} \tag{15}$$

$$= \frac{(q)_{a_1+\ldots+a_n}}{(q)_{a_1}\ldots(q)_{a_n}}\sum_{\sigma\in\mathfrak{S}_A} q^{wt(\sigma)}\Pi_j\frac{1-q^{a_{\sigma(j)}}}{1-q^{a_{\sigma(1)}+\ldots+a_{\sigma(j)}}} \tag{16}$$

This identity implies several conjectures of Kadell [12] and has had applications in studying the characters of $SL(n,\mathbb{C})$[21] and in evaluating definite integrals arising in statistical mechanics [4].

The theorems first conjectured by Dyson and Andrews are only the tip of the iceberg of a very extensive theory. These identities are related to the Vandermonde determinant formula which is Weyl's denominator formula for the root system A_n. Macdonald [15] conjectured the appro-
37 priate generalizations to arbitrary root systems and he and W.G. Morris [16] gave conjectures and some proofs for the basic analogs.

Macdonald's conjecture for the root system BC_n was discovered to be equivalent to a multi-dimensional beta integral evaluation of Selberg [18, 19]. A basic analog of this was conjectured by Askey [2]. Habsieger [8] and Kadell [13] independently proved Askey's conjecture and then Habsieger [9] and Zeilberger [25] showed that this integral evaluation implied some of Morris' conjectures.

Most recently, Kadell [14] has proved Macdonal's conjecture for the basic analog of the BC_n conjecture, Garvan [6] has done the same for F_4, and E.M. Opdam [17] has proved the original Macdonald conjecture for arbitrary root systems. Only the basic analogs for the special root systems E_6, E_7 and E_8 are unproven at the moment.

Stembridge [22] has found a strikingly simple proof of Andrews' conjecture in the case where the parameters are equal. He has also found formulas for some of the non-constant terms [21]. Connections with representation theory can be found in an article by Stanley [20]. Hanlon has pursued the connections between these identities and cyclic homology [10, 11].

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WEYL'S INEQUALITY, WARING'S PROBLEM AND DIOPHANTINE APPROXIMATION

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For fixed positive integers s and k, we define 41

$$R_{s,k}(N) = \#\{(n_1, \ldots, n_2) \in \mathbb{N}^s : \sum_1^s n_j^k = N\}$$

One central question in Waring's problem is to prove the Hardy-Littlewood asymptotic formula

$$R_{s,k}(N) = \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)} \mathfrak{S}(N) N^{(s/k)-1} + O(N^{(s/k)-1-\delta}) \tag{1}$$

for as large a range of s as possible. To tackle this, one uses an exponential sum

$$S(\alpha) = \sum_{n=1}^{P} e(\alpha n^k),$$

where $P = [N^{1/k}]$. One then has

$$R_{s,k}(N) = \int_0^1 S(\alpha)^s e(-\alpha N) dx. \tag{2}$$

The trivial bound for $S(\alpha)$ is $|S(\alpha)| \leqslant P$. However, one can improve on this for suitable α, by using the following estimate.

Weyl's inequality. *Let* $|\alpha - a/q| \leqslant q^{-2}$, *with* $(a, q) = 1$. *Then*

$$S(\alpha) \ll_\epsilon P^{1+\epsilon}(q^{-1} + P^{-1} + qP^{-k})^{2^{1-k}},$$

for any $\epsilon > 0$.

Thus if α can be approximated with $P \leqslant q \leqslant P^{k-1}$ one has

$$S(\alpha) \ll P^{1-2^{1-k}+\epsilon}, \tag{3}$$

and the corresponding contribution to (2) is

$$\ll P^{s(1-2^{1-k}+\epsilon)} \ll N^{s/k-1-\delta},$$

provided that $s > 2^{k-1}k$. Those α which have no useable approximation produce the main term of (1). Thus one obtains (1) for $s \geqslant 1 + 2^{k-1}k$.

42 One can improve on this agrument by using an average bound.

Hua's inequality. *For any* $\epsilon > 0$, *one has*

$$\int_0^1 |S(\alpha)|^{2^k} d\alpha \ll_\epsilon P^{2^k-k+\epsilon}.$$

This leads to (1) for $s \geqslant 1 + 2^k$. Until recently, this was the best known range for (1), for small $k \geqslant 3$.

The sum $S(\alpha)$ may also be used in Diophantine approximation problems. It was shown by Danicic [2] that if $\epsilon > 0$ and $k \in N$ are given, then there exists $P(\epsilon, k)$ as follows. For any $P \geqslant P(\epsilon, k)$ and any $\alpha \in \mathbb{R}$, one can find $n \leqslant P$ with

$$||\alpha n^k|| \leqslant P^{\epsilon-2^{1-k}}. \tag{4}$$

This generalizes Dirichlet's approximation Theorem, when $k = 1$, and a result of Heilbronn (4), for $k = 2$. To prove Danicic's theorem one can use a result of Montgomery (see Baker [1, Theorem 2.2]): If $||a_n|| > \Delta$ for $1 \leqslant n \leqslant P$, then

$$\sum_{1 \leqslant h \leqslant \Delta^{-1}} |\sum_{n \leqslant P} e(ha_n)| \geqslant P/6.$$

We therefore wish to estimate

$$\sum_{h \leqslant \Delta^{-1}} |S(\alpha h)|. \tag{5}$$

As with Weyl's inequality, this can be done satisfactorily, with a relative saving of $P^{-2^{1-k}+\epsilon}$, unless α has an approximation

$$|\alpha - a/q| \leqslant \frac{\Delta}{qP^{k-1}}, q \leqslant P. \tag{6}$$

Thus Montgomery's result allows us to take $\Delta \approx P^{\epsilon - 2^{1-k}}$. Of course, if (6) holds then $\|\alpha q^k\| \leqslant \Delta$, and $q \leqslant P$.

A sharpening of Weyl's inequality has recently been obtained (Heath-Brown [3]).

Theorem 1. *Let* $|\alpha - a/q| \leqslant q^{-2}$ *with* $(a, q) = 1$, *and suppose that* $k \geqslant 6$. *Then*

$$S(\alpha) \ll_\epsilon P^{1+\epsilon}(Pq^{-1} + P^{-2} + qP^{1-k})^{(4+3)2^{-k}}$$

for any $\epsilon > 0$.

Thus 43

$$S(\alpha) \ll P^{1-(4/3)2^{1-k}+\epsilon},$$

if $P^3 \leqslant q \leqslant P^{k-3}$. One therefore has a sharper bound than (3), but for a shorter range of q, (and only for $k \geqslant 6$). Closely related to Theorem 1 is an improvement on Hua's Inequality (Heath-Brown [3]).

Theorem 2. *Let* $k \geqslant 6$ *and* $\epsilon > 0$. *Then*

$$\int_0^1 |S(\alpha)|^{(7/8)2^k}\, d\alpha \ll P^{(7/8)2^k - k + \epsilon}.$$

As before one may deduce :

Corollary. *The Hardy-Littlewood asymptotic formula [1] holds for* $k \geqslant 6$ *and* $s \geqslant 1 + \frac{7}{8}2^k$.

One may also try to sharpen Danicic's result. One obtains a saving in (5) of

$$P^{-(4/3)2^{1-k}+\epsilon}$$

relative to the trivial estimate, unless

$$|\alpha - a/q| \leqslant \frac{\Delta}{qP^{k-3}}, \ q \leqslant P^3.$$

Unfortunately in this latter case, one gets no useable bound for $\|aq^k\|$. The attempt to improve on (4) therefore fails. However, if one starts with an approximation $|\alpha - a/q| \leqslant q^{-2}$ and fixes $P = [q^{(1/3)}]$, for example, one is led to an "unlocalized" result (Heath-Brown [3]).

Theorem 3. *Let $\alpha \in \mathbb{R}$ and $\epsilon > 0$ be given. For any integer $k \geqslant 6$, there are infinitely many $n \in \mathbb{N}$ with*

$$\|\alpha n^k\| \leqslant n^{\epsilon-(4/3)2^{1-k}}.$$

Let us now look at the proof of Theorem 1. One uses Weyl's "square and difference" trick, but with the symmetric difference

$$(\nabla_h)(x) = (x+h) - (x-h)$$

in place of the forward difference. After j steps, one has

$$|S(\alpha)|^{2^j} \ll P^{2^j-j-1} \sum_{h_1,\ldots,h_j} |R(\alpha)|, \tag{7}$$

44 where $|h_i| < P/2$ and

$$R(\alpha) = R(\alpha; h_1, \ldots, h_j) = \sum_{n \in I} e(\alpha \nabla_{h_1} \ldots \nabla_{h_j}(n^k)).$$

Here I is a subinterval of [1, P], depending on $h_1, \ldots, h_j$. As a function of n, the polynomial

$$\nabla_{h_1} \ldots \nabla_{h_j}(n^k)$$

has degree $k - j$. An appropriate version of Weyl's Inequality would therefore give

$$R(\alpha) \ll P^{1-2^{1-(k-j)}+\epsilon}, \tag{8}$$

for suitable α. in conjunction with (7) we would then obtain

$$|S(\alpha)|^{2^j} \ll P^{2^j-j-1} \cdot P^j \cdot P^{1-2^{1-(k-j)+\epsilon}},$$

whence

$$S(\alpha) \ll P^{1-2^{1-k}+\epsilon},$$

for suitable α. One thus merely recovers Weyl's Inequality again.

To improve on this, we replace (8) by a mean-value bound, where one averages over the parameters h_i. If one takes $j = k - 1$ or $k - 2$ then $R(\alpha)$ is a linear or quadratic sum, and the bound (8) is essentially best possible. Thus nothing can be gained by averaging. One therefore chooses $j = k - 3$, in which case $R(\alpha)$ is a cubic sum of the form

$$R(\alpha) = \sum_{n \in 1} e(An^3 + Bn).$$

Here the interval I and the coefficients A and B depend on the h_i. In fact, A takes the form

$$A = \frac{k!}{6} 2^{k-3} h_1 \dots h_{k-3}.$$

Had one used forward differences in deriving (7) rather than symmetric differences, there would have been a term in n^2 appearing in $R(\alpha)$, and so one would have to average over three coefficients, rather than two. With $j = k - 3$, the Weyl bound now takes the form

$$|R(\alpha)| \ll P^{3/4+\epsilon}, \tag{9}$$

for suitable α, whereas one would conjecture that

$$|R(\alpha)| \ll P^{1/2+\epsilon},$$

in general. In fact, one can easily prove that

$$\int_0^1\int_0^1 |\sum_{n\leqslant P} e(An^3+Bn)|^6 dAdB \ll P^{3+\epsilon}, \tag{10}$$

by counting the number of solutions of the simultaneous equations

$$\begin{aligned} n_1^3+n_2^3+n_3^3 &= n_4^3+n_4^3+n_6^3 \\ n_1+n_2+n_3 &= n_4+n_5+n_6. \end{aligned} \qquad (1\leqslant n_i\leqslant P)$$

To pass from the sum on the right hand side of (7) to the mean value (10), one uses the bound

$$\sum_{h_i\ldots,h_{k-3}}{}' |R(\alpha)|^6 \ll P^{4+\epsilon}\mathcal{N}\int_0^1\int_0^1 |\sum_{n\leqslant P} e(An^3+Bn)|^6 dAdB,$$

where

$$\mathcal{N} = \max_{A\in[0,1]} \#\{(h_1,\ldots,h_{k-3}) : \|\frac{k!}{6}2^{k-3}h_1\ldots h_{k-3}\alpha - A\| \leqslant P^{-3}\}.$$

Here we exclude the possibility that any h_i vanishes, both in the sum $\sum'$ and in the maximum occurring in the definition of $\mathcal{N}$. It is apparent that there is a loss of a factor P in passing from the discrete average of $R(\alpha)$ over the h_i to the mean-value (10). Nonetheless, one finds that $R(\alpha)$ is $O(P^{2/3+\epsilon})$ on average, and this is a sufficient improvement on (9) for the proof of Theorem 1.

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THE CIRCLE METHOD AND THE FOURIER COEFFICIENTS OF MODULAR FORMS

By Henryk Iwaniec

To the memory of Srinivasa Ramanujan

1 Introduction.

47 The circle method was first used in number theory by G. Hardy and S. Ramanujan [2] to establish an asymptotic formula for the partition function (see also [7]) and it was applied extensively in the series of papers under the common title : Some problems of "Partitio Numerorum" by G. Hardy and J.E. Littlewood to study additive problems such as the Waring problem or the Goldbach problem (see for example [1]). The method was particularly interesting for additive problems with many summands. Yet at that time the important results were conditional subject to sharp estimates for the relevant extponential sums.

Perhaps the most ambitious are the binary problems, i.e. the problems of evaluating the number of solutions to the equation

$$a + b = n,$$

where a, b range over finite sets of integers A, B respectively and n is a fixed integer. Clearly, the number of solutions is given by the integral (Vinogradov's modification)

$$\int_0^1 e(-\alpha n)\left(\sum_{a\in A} e(\alpha a)\right)\left(\sum_{b\in B} e(\alpha b)\right) d\alpha.$$

The Hardy-Littlewood arguments fail to handle the binary problem for a fundamental reason–the use of Parseval's identity

$$\int_0^1 |\sum_{a\in A} e(\alpha a)|^2 d\alpha = |A|.$$

It is evident that when dealing with a binary problem one cannot ignore a cancellation in the integration over any set of positive constant measure. Taking this into account in 1926, H. D. Kloosterman [4] introduced a brilliant refinement which is described by Yu. V. Linnik in [6] as the **48**
process of levelling (a sophisticated partition of the segment $0 < \alpha < 1$ by means of Farey's points). Kloosterman's method was originally used for the binary problem in which a and b assume values of some quadratic forms. The important point should be mentioned that the exponential sum

$$\sum_{a\in A} e(\alpha a)$$

is evaluated precisely enough to control the oscillatory behaviour of the remainder term which is usually of the order of magnitude $|A|^{1/2}$. Both the partition of the segment $0 < \alpha < 1$ and the nature of the remainder term comprise the appearance of the Kloosterman sums

$$S(\dot{m}, n : c) = \sum_{d(\mathrm{mod}\ c)} e\left(\frac{md + n\bar{d}}{c}\right),$$

where $\sum^*$ means that the summation ranges over d prime to c and $\bar{d}$ is the multiplicative inverse to $d(\mathrm{mod}\ c)$. Then a non-trivial bound for $S(m, n; c)$ yields a cancellation of the remainder terms and consequently one breaks the barrier set by the use of Parseval's identity. The Kloosterman device enables one to handle a large class of binary problems. Moreover it turns out to be successful in answering various questions about the Fourier coefficients of modular forms (see for example [5]).

Kloosterman did not exploit a cancellation of terms of summation over the moduli c that exists due to the variation of sign of the Kloosterman sum $S(m, n; c)$. In this connection Linnik [6] was led to formulate the following hypothesis

$$\sum_{c\leqslant C} c^{-1} S(m, n; c) \ll C^{\epsilon}$$

and he said : "This hypothesis can be considered as a certain analogy to the well-known hypothesis of Hasse on the behaviour of congruence zeta-functions arising by the reduction of a given curve with respect to all prime moduli." A somewhat stronger statement was expressed by A. Selberg [8] in the context of estimating the Fourier coefficients of modular forms. The recent developments in the spectral theory of automorphic functions brought a remarkable progress towards the Linnik-Selberg hypothesis.

If one sequence A or B in the binary problem has no reference to the modular forms, then naturally other exponential sums emerge in place
49 of the Kloosterman sums. For example see the paper by C. Hooley [3] in which the Kloosterman refinement is applied to advance in the Waring problem for cubes under the Riemann hypothesis for certain Hasse-Weil L-Functions.

In this paper we elaborate the Kloosterman ideas in a general context. We shall express the distribution

$$\delta(n) = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n \neq 0 \end{cases}$$

in terms of the Ramanujan sums

$$S(n; c) = \sum_{d(\mathrm{mod}\ c)}^{*} e\left(n\frac{d}{c}\right)$$

and of new sums of type

$$S_{\nu}(n; c) = \sum_{d(\mathrm{mod}\ c)}^{*} \left(\left(\frac{d+\nu}{c}\right)\right) e\left(n\frac{d}{c}\right),$$

where $((\zeta)) = \zeta - [\zeta] - 1/2$. We shall establish a formula for the Fourier coefficient of a cusp form in terms of the Kloosterman sums $S(m, n; c)$

and of the new Kloosterman type sums

$$S_\nu(m,n;c) = \sum_{d(\mathrm{mod}\ c)}^{*} \left(\left(\frac{d+\nu}{c}\right)\right) e\left(\frac{md+n\bar{d}}{c}\right).$$

These sums are closely related. Indeed, by the Fourier expansion (boundedly convergent)

$$(\zeta)) = \sum_{0<|h|<H} (2\pi ih)^{-1} e(\xi h) + O[(1+\|\zeta\|H)^{-1},$$

where $\|\zeta\|$ is the distance of ζ to the nearest integer, it follows that

$$S_\nu(m,n;c) = \sum_{0<|h|<H} (2\pi ih)^{-1} e\left(\frac{h\nu}{c}\right) S(m+h,\ n;c) + O\left(1+\frac{c\log 2c}{H}\right).$$

We expect, but were not able to prove it, that the error term above should be

$$O\left[c^{\epsilon}\left(1+\frac{c}{H}\right)^{\frac{1}{2}}\right].$$

2 A general result.

We begin by considering a periodic function $f(x)$ of period 1 with the aim of evaluating its mean value 50

$$\mu(f) = \int_0^1 f(x)dx.$$

Divide the range of integration by Farey's points of order C, i.e. by the rational numbers d/c with

$$1 \leqslant c \leqslant C,\ 0 \leqslant d < c,\ (d,c) = 1$$

For $2 \leqslant c \leqslant C$ let $\mathfrak{M}(d/c)$ stand for the interval whose endpoints are Farey's mediants, i.e.

$$\mathfrak{M}\left(\frac{d}{c}\right) = \left[\frac{d+d_-}{c+c_-}, \frac{d+d_+}{c+c_+}\right] = \left[\frac{d}{c} - \frac{1}{c(c+c_-)}, \frac{d}{c} + \frac{1}{c(c+c_+)}\right]$$

where $d_-/c_- < d/c < d_+/c_+$ are the adjacent points. For $c = 1$ we have $d = 0$ and we set $c_\mp = C$, $d_\mp = \mp 1$,

$$\mathfrak{M}\left(\frac{0}{1}\right) = \left[\frac{-1}{C+1}, \frac{1}{C+1}\right].$$

We obtain

$$\mu(f) = \sum_{1\leqslant c\leqslant C} \sum_{0\leqslant d<c} \int_{\mathfrak{M}(d/c)} f(x)dx$$

$$= \sum_{1\leqslant c\leqslant C} c^{-1} \sideset{}{^*}\sum_{0\leqslant d<c} \int_{-(c+c_-)^{-1}}^{(c+c_+)^{-1}} f\left(\frac{d+x}{c}\right)dx$$

For notational simplicity, put $F(x) = f((d+x)/c)$. We have

$$\int_{-(c+c_-)^{-1}}^{(c+c_+)^{-1}} F(x)dx = \int_{c+c_-}^{\infty} F\left(\frac{-1}{t}\right)\frac{dt}{t^2} + \int_{c+c_+}^{\infty} F\left(\frac{1}{t}\right)\frac{dt}{t^2}$$

$$= \int_C^{\infty} \left[X_-(t)F\left(\frac{-1}{t}\right) + X_+(t)F\left(\frac{1}{t}\right)\right]\frac{dt}{t^2},$$

where

$$X_\mp(t) = \begin{cases} 1 \text{ if } t \geqslant c + c_\mp \\ 0 \text{ if } C \leqslant t < c + c_\mp. \end{cases}$$

51 We find that

$$X_\mp(t) = \frac{t-C}{c} + \left(\left(\frac{C-c_\mp}{c}\right)\right) - \left(\left(\frac{t-c_\mp}{c}\right)\right)$$

in $C \leqslant t < c + C$ and clearly $X_\mp(t) = 1$ if $t \geqslant c + C$. Since $C < c + c_\mp \leqslant c + C$ and $cd_\mp - c_\mp d = \mp 1$ we have

$$c_\mp = \left[\frac{C \mp d}{c}\right]c \pm \bar{d} \equiv \mp\bar{d} (\text{mod } c).$$

Hence we obtain

$$\int_{-(c+c_-)^{-1}}^{(c+c_+)^{-1}} F(x)dx = \int_C^\infty \min\left\{1, \frac{t-C}{c}\right\}\left[F\left(\frac{-1}{t}\right) + F\left(\frac{1}{t}\right)\right]\frac{dt}{t^2}$$

$$+ \int_C^{c+C} \left\{\left(\left(\frac{C-d}{c}\right)\right) - \left(\left(\frac{t-d}{c}\right)\right)\right\} F\left(-\frac{1}{t}\right)\frac{dt}{t^2}$$

$$+ \int_C^{c+C} \left\{\left(\left(\frac{C+\bar{d}}{c}\right)\right) - \left(\left(\frac{t+\bar{d}}{c}\right)\right)\right\} F\left(\frac{1}{t}\right)\frac{dt}{t^2}.$$

Setting

$$G_{tv}(f;c) = \sideset{}{^*}\sum_{d(\mathrm{mod}\ c)} f\left(\frac{\bar{d}+t^{-1}}{c}\right)$$

and

$$G_{tv}(f;c) = \sideset{}{^*}\sum_{d(\mathrm{mod}\ c)} \left(\left(\frac{\bar{d}+v}{c}\right)\right) f\left(\frac{\bar{d}+t^{-1}}{c}\right)$$

we conclude

Theorem 1. *We have*

$$\mu(f) = \sum_1^C c^{-1} \int_C^\infty \min\left\{1, \frac{t-C}{c}\right\}(G_t + G_{-t})(f;c)\frac{dt}{t^2}$$

$$+ \sum_1^C c^{-1} \int_C^{c+C} (G_{tC} - G_{-t-C} + G_{-t-t} - G_{tt})(f;c)\frac{dt}{t^2}$$

Now suppose $f(x)$ is the additive character, **52**

$$f(x) = e(nx).$$

We then have

$$\mu(f) = \delta(n), G_t(f;c) = e\left(\frac{n}{ct}\right)S(n;c),$$

and

$$G_{tv}(f;c) = e\left(\frac{n}{ct}\right) S_v(n;c),$$

so Theorem 1 turns into

Theorem 2. *Let C be a positive integer. We have*

$$\delta(n) = D(n) + \overline{D}(n) + E(n) + \overline{E}(n)$$

with

$$D(n) = \sum_1^C c^{-1} S(n;c) \int_C^\infty e\left(\frac{n}{ct}\right) \min\left\{1, \frac{t-C}{c}\right\} \frac{dt}{t^2}.$$

and

$$E(n) = \sum_1^C c^{-1} \int_C^{c+C} e\left(\frac{n}{ct}\right) \{S_C(n;c) - S_t(n;c)\} \frac{dt}{t^2}.$$

3 A formula for the Fourier coefficients of a cusp form.

Now let f be given by

$$f(x) = e(-nx)u(x+iy),$$

where $u(z)$ is an automorphic function with respect to the modular group $\Gamma = SL_2(\mathbb{Z})$. Thus we have

$$u\left(\frac{d+t^{-1}}{c} + iy\right) = u\left(\gamma\left(\frac{d+t^{-1}}{c} + iy\right)\right) = u\left(\frac{-\bar{d}}{c} - \frac{c^{-1}t}{1+icty}\right)$$

for some $\gamma = \begin{pmatrix} * & * \\ c & -d \end{pmatrix} \in \Gamma$. Hence

$$G_t(f;c) = e\left(\frac{-n}{ct}\right) \sideset{}{^*}\sum_{d \pmod c} e\left(-n\frac{\bar{d}}{c}\right) u\left(\frac{-d}{c} - \frac{c^{-1}t}{1+icty}\right)$$

and

$$G_{tv}(f;c) = e\left(\frac{-n}{ct}\right) \sum_{d(\text{mod } c)}^{*} \left(\left(\frac{d+v}{c}\right)\right) e\left(-n\frac{\bar{d}}{c}\right) u\left(\frac{-d}{c} - \frac{c^{-1}t}{1+icty}\right).$$

Suppose $u(z)$ is a Maass cusp form, so it has the Fourier expansion 53

$$u(z) = \sum_{m\neq 0} \sigma_m W(mz),$$

where $W(z)$ is the Whittaker function defined on $\mathbb{C}\ \mathbb{R}$ that satisfies the rules

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} z\right) = e(x)W(z)$$

and

$$\overline{W}(z) = W(-\bar{z}),\ \ W(z) = W(\bar{z}).$$

Hence it follows that

$$\mu(f) = \sigma_n W(iny),$$

$$G_t(f;c) = e\left(\frac{-n}{ct}\right) \sum_{m\neq 0}^{*} \sigma_m S(m,n;c) W\left(\frac{-mc^{-1}t}{1+icty}\right)$$

and

$$G_{tv}(f;c) = e\left(-\frac{n}{ct}\right) \sum_{m\neq 0}^{*} \sigma_m S_v(-m,-n;c) W\left(\frac{-mc^{-1}t}{1+icty}\right)$$

Combining the above evaluations with Theorem 1 and using the properties $S_v(-m,-n;c) = \overline{S}_v(m,n;c)$ and $S_{-v}(m,n;c) = -S_v(-m,-n;c)$ we conclude

Theorem 3. *Let $u(z)$ be a Maass cusp form on the modular group whose Fourier coefficients σ_m are real. Let C be a positive integer. Then, for any $n \neq 0$, and $y > 0$, we have*

$$\sigma_n W(iny) = U(n,u) + \overline{U}(n,y) + V(n,y) + \overline{V}(n,y),$$

where

$$U(n,y)=\sum_{m\neq 0}\sigma_m\sum_{1}^{C}c^{-1}S(m,n;c)\int\limits_{c}^{\infty}e\left(\frac{n}{ct}\right)W\left(\frac{mc^{-1}t}{1-icty}\right)\min\left\{1,\frac{t-C}{c}\right\}\frac{dt}{t^2}$$

and

$$V(n,y)=\sum_{m\neq 0}\sigma_m\sum_{1}^{C}c^{-1}\times$$
$$\times\int\limits_{C}^{c+C}e\left(\frac{n}{ct}\right)W\left(\frac{mc^{-1}t}{1-icty}\right)\{S_C(m,n;c)-S_t(m,n;c)\}\frac{dt}{t^2}.$$

54 **Remarks.** The convergenece of both series is very rapid. Indeed, if $u(z)$ is an eigenform of the Laplace-Beltrami operator

$$\Delta=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

with eigenvalue $\lambda=s(1-s)$, Re $s=1/2$, i.e. if

$$(\Delta+\lambda)u(z)=0$$

then the Whittaker function is given by

$$W(z)=2|y|^{1/2}K_{s-1/2}(2\pi|y|)e(x),$$

where $K_\nu(y)$ is the Macdonald Bessel function. By the ingegral representation of Poisson for $K_\nu(y)$ we obtain for $z=x+iy$ with $y>0$,

$$W(z)=e(z)\frac{1}{\Gamma(s)}\int\limits_{\Gamma(s)}^{1}\int\limits_{0}^{\infty}e^{-}\xi\left[\xi\left(1+\frac{\xi}{4\pi y}\right)\right]^{s-1}d\xi.$$

In particular for s on the line Re $s=1/2$ it gives

$$|W(z)|\leqslant e^{-2\pi|y|}\frac{\Gamma(\frac{1}{2})}{|\Gamma(s)|},$$

so the terms of the series $U(n, y)$ and $V(n, y)$ decay exponentially as $|m| \to \infty$. In practice only the few first terms matter.

There is a great flexibility in choosing C and y. A good choice is $C = \sqrt{n}$ and $y = n^{-1}$ for $n > 0$, giving the upper bound $\sigma_n \ll n^{-1/4} + \epsilon$ by Weil's estimate for Kloosterman sums. Other applications will be discussed elsewhere.

I would like to thank Prof. S. Raghavan for interesting mathematical comments on this paper.

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SUMS OF KLOOSTERMAN SUMS AND THE EIGHTH POWER MOMENT OF THE RIEMANN ZETA-FUNCTION

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Dedicated to Atle Selberg

0 Introduction.

The domain of mathematics which will be discussed here was called 57
"Kloostermania" by M. Huxley. This name outlined (but not too sharply) the boundary between number theory, the theory of the modular and automorphic functions, spectral theory and geometry.

The beginning was due to Poincaré. The contribution which defined the base of this theory was given by Petersson, Hecke, Rankin and Maass. In the last few decades, its development was stimulated by the famous talk of Atle Selberg at Tata Institute and by L. Faddeev's work which clarified the spectral expansion.

It was ten years ago when I found the explicit form of the connection between sums of Kloosterman sums ("known quantities") and Fourier coefficients of cusp forms (unknown quantities which are very mysterious up to this day). In the next year, R. Bruggeman rediscovered (independently) a part of these results. From that time, the number of publications is increasing rapidly in this domain.

So it happened that the Kloosterman sums (which will be defined below) arose firstly for improving the Hardy-Littlewood "circle method".

But these sums would have arisen earlier, if Poincare had wished to calculated the Fourier coefficients of the series which are called today "the Poincaré series".

The goal of this paper is to make more popular this dynamic branch of Mathematics and to demonstrate new possibilities for the Riemann zeta-functions.

The first part of the paper (the short subsection 1.1-1.13) contains known results. The second part gives, as a new consequence of the "Kuznetsov trace formula", the exact order for the eighth power moment of the Riemann zeta-function. Namely. we have, for some absolute constant $B > 4$.

$$\int_0^T |\zeta(\frac{1}{2} + it)^8 dt \ll T(\log T)^{16+B}, T \to +\infty.$$

58 From various consequences of this estimate, one can derive the conclusion: there is a fixed constant B so that

$$|\zeta(\frac{1}{2} + it)| \ll |t|^{1/8} (\log |t|)^B, |t| \to \infty, t \text{ is real.}$$

Part I. Sums of Kloosterman sums

1.1 The Lobatcevskii plane.

This plane will be considered as the upper half plane $\mathbb{H}$ of the complex variable $z = x + iy$, $x, y \in \mathbb{R}$, $y > 0$, with the metric

$$ds^2 = y^{-2}(dx^2 + dy^2), \tag{1.1}$$

measure

$$d\mu(z) = y^{-2} dx\, dy \tag{1.2}$$

and with the corresponding Laplace operator

$$L = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2} \right). \tag{1.3}$$

The full modular group acts on this plane in the natural way:

$$z \mapsto \gamma z = \frac{ax+b}{cz+d}, a, b, c, d \in \mathbb{Z},\ ad - bc = 1. \tag{1.4}$$

Most of the results may be developed for certain Fuchsian groups but there are no essentially new ideas for these cases; so, I restrict myself to the full modular group Γ only.

1.2 The appearance of Kloosterman sums.

Their appearance is inescapable, if one calculates the Fourier coefficients of an automorphic function which is defined as a sum over a gorup.

For example, let us define the classical Poincaré series by

$$P_n(z;k) = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \sum_{\gamma \in \Gamma_\infty / \Gamma} j^{-k}(\gamma, z) e(n\gamma z), n \geqslant 1 \tag{1.5}$$

(where Γ_∞ is generated by the translation $z \mapsto z+1$, $j(\gamma, z) = cz + d$ if the transformation γ is defined by a matrix $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ and we assume that k is an even integer and $k \geqslant 4$). Then, for the m-th Fourier coefficient fo this series we have an almost obvious formula (the so-called "Petersson formula"):

$$p_{n,m}(k) := \int_0^1 P_n(z,k) e(-mx) dx\ e^{2\pi my} \tag{1.6}$$

$$= \frac{(4\pi\sqrt{nm})^{k-1}}{\Gamma(k-1)} (\delta_{n,m} + 2\pi i^{-k} \sum_{c \geqslant 1} \frac{1}{c} S(n,m;c) J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right)), n, m \geqslant 1,$$

where J_{k-1} is the Bessel function of the order $k-1$ and S is the Kloost- 59
erman sum

$$S(n,m;c) := \sum_{\substack{1 \leqslant d \leqslant |c|, (d,c)=1 \\ dd' \equiv 1 (\mathrm{mod}\ c)}} e\left(\frac{nd + md'}{c}\right). \tag{1.7}$$

We have a similar (but more complicated) representation, for the Fourier coefficients of the non-holomorphic Poincaré-Selberg series which, for Re $s > 1$, is defined by

$$U_n(z, s) := \sum_{\gamma \in \Gamma_\infty/\gamma} (\operatorname{Im} \gamma z)^s e(n\gamma z), \ n \geqslant 1, \tag{1.8}$$

(For $n = 0$, it is the Eisenstein series.)

For the Kloosterman sums, we have the famous estimate due to A. Weil:

$$|S(n, m; c)| \leqslant (2n, 2m, c) d^2(c) c^{1/2}. \tag{1.9}$$

But, for the applications, we need estimates for the averages of these sums. Yu V. Linnik was the first to conjecture that Kloosterman sums oscillate regularly; his conjecture is that

$$\left| \sum_{c \leqslant X} \frac{1}{c} S(n, m; c) \right| \ll_{n,m} X^\epsilon \tag{1.10}$$

for every $\epsilon > 0$ as $X \to +\infty$.

It is obvious that A-Weil's estimate give only $O(X^{(1/2)+\epsilon})$ on the right side and A. Selberg destroyed the hopes of any near progress in this conjecture when he constructed the counterexamples of groups for which Linnik conjecture is not valid (1963).

At this point, there was a nice result from my first paper on this subject (1977): for every fixed $\epsilon > 0$, we have

$$\left| \sum_{1 \leqslant c \leqslant X} \frac{1}{c} S(n, m; c) \right| \ll_{n,m} X^{(1/6)+\epsilon} \tag{1.11}$$

At the same time, for the "smoothing" average, we have a stronger estimate : if $\varphi \in C^\infty(0, \infty)$, $\varphi = 0$ outside the interval $(a, 2a)$ and if, for every fixed integer $r \geqslant 0$, we have $\left(\frac{\partial}{\partial x}\right)^r \varphi(x) \ll a^{-r}$, then, for every fixed $A > 0$, the following estimate is valid:

$$\left| \sum_{a \leqslant c \leqslant 2a} \varphi(c) S(n, m; c) \right| \ll a^{-A}, a \to +\infty. \tag{1.12}$$

Thus it is a confirmation of the Linnik conjecture.

1.3 The eigenfunctions of the automorphic Laplacian.

As a generalization of the classical cusp forms of even integral weight **60**
k (which are regular functions on the upper half plane such that $f(\gamma z) = j^k(\gamma, z) f(z)$ for any $\gamma \in \Gamma$ and $y^{k/2}|f(z)|$ is bounded for $y > 0$, the Poincaré series $P_n(z,k), n \geqslant 1$, being an example of a cusp form of weight k with respect to full modular group), Maass introduced the non-holomorphic cusp forms (the so called Maass waves).

The Laplace operator L in $L^2(\Gamma/\mathbb{H})$ has a continuous spectrum on the half axis $\lambda \geqslant \frac{1}{4}$ and a discrete spectrum $\lambda_0 = 0, 0 < \lambda_1 < \lambda_2 \leqslant \dots$ with limit point at ∞ (note that $\lambda_1 \simeq 91.07\dots$). For the case of the full modular group, there are no exceptional eigenvalues in the interval $(0, \frac{1}{4})$; (Huxley proved that the same is true for any congruence subgroup with the level $\leqslant 19$). So $L^2(\Gamma/\mathbb{H})$ decomposes as $L^2_{\text{eis}}(\Gamma/\mathbb{H}) \oplus L^2_{\text{cusp}}(\Gamma/\mathbb{H})$ where L^2_{eis} is the continuous direct sum of $E(z, \frac{1}{2} + it)$, $t \in \mathbb{R}$ (E being the Eisenstein series) and L^2_{cusp} is spanned by the eigenfunctions $u_j(z)$ given by

$$Lu_j \equiv -y^2 \left(\frac{\partial^2 u_j}{\partial x^2} + \frac{\partial^2 u_j}{\partial y^2} \right) = \lambda_j u_j, \tag{1.13}$$

$$u_j(\gamma z) = u_j(z),\ \ \gamma \in \Gamma; (u_j, u_j) = \int\limits_{\Gamma/\mathbb{H}} |u_j|^2 d\mu(z) < \infty.$$

Any $f \in L^2(\Gamma/\mathbb{H})$ which is smooth enough can be expanded into eigenfunctions of L and we have

$$f(z) = \sum_{j \geqslant 0} (f, u_j) u_j(z) + \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} (f, E(., \frac{1}{2} + ir)) E(z, \frac{1}{2} + ir) dr \tag{1.14}$$

if we choose u_j so that we have an orthonormal basis $\{u_j\}_{j \geqslant 0}$.

Note that the Eisenstein series has the Fourier expansion

$$E(z, s) = y^2 + \frac{\xi(1-s)}{\xi(s)} y^{1-s} + \tag{1.15}$$

$$+ \frac{2\sqrt{y}}{\xi(x)} \sum_{n \neq 0} \tau_s(n) e(nx) K_{s-1/2}(2\pi |n| y), \xi(s) := \pi^{-s} \Gamma(s) \zeta(2s),$$

where $K_{s-1/2}(.)$ is the modified Bessel function of order $s - \frac{1}{2}$ and

$$\tau_s(n) = |n|^{s-1/2}\sigma_{1-2s(n)} = \sum_{\substack{d|n \\ d>0}} \left(\frac{|n|}{d^2}\right)^{s-1/2} \tag{1.16}$$

61 The eigenfunctions of a point λ_j of the discrete spectrum have a similar Fourier expansion

$$u_j = \sum_{n \neq 0} \rho_j(n) e(nx) \sqrt{y} K_{i\chi_j}(2\pi|n|y) \tag{1.17}$$

with $\chi_j = \sqrt{\lambda_j - \frac{1}{4}}, \lambda_j > \frac{1}{4}$.

1.4 The Hecke operators.

The ideas behind Hecke operators go back to Poincare and Mordell used them to prove that Ramanujan's τ-function was multiplicative.

The main observation is a simple fact that if H is a subgroup if Γ with finite index, so that Γ is a finite coset union $\bigcup_j H_{\gamma_j}$, and f is automorphic on H, then $\sum_j f(\gamma_j z)$ is automorphic on Γ.

By appropriately choosing the set of representatives, we can define the n-th Hecke operator as the average

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ d>0}} \sum_{b (\bmod\ d)} f\left(\frac{az+b}{d}\right) \tag{1.18}$$

For this normalization, we have

$$T_n T_m = \sum_{d|(n,m)} T_{nm/d^2} \tag{1.19}$$

and all these operators commute.

Now we have a set of commuting Hermitian operators, with the same set of eigenfunctions that arose for the Laplace operator. Thus we can choose the eigenfunctions of the Laplace operator, so that in the basis which was constructed from these, each Hecke operator has a diagonal form. Then these eigenfunctions will be called “Maass waves”. For that, choose

$$T_n u_j = t_j(n) u_j,\ \ n \geqslant 1, j \geqslant 0, \tag{1.20}$$

$$T_n E(., s) = \tau_s(n) E(., s). \tag{1.21}$$

The eigenvalues $t_j(n)$ of the discrete spectrum of the n-th Hecke operator T_n are connected with the Fourier coefficients of u_j by the equalities

$$\rho_j(1) t_j(n) = \rho_j(n),\ \ n \geqslant 1, j \geqslant 1. \tag{1.22}$$

It is convenient to choose the eigenfunctions so that they will be eigenfunctions of the operator T_{-1}:

$$(T_{-1} f)(z) = f(-\bar{z}).$$

Then $T_{-1} u_j = \epsilon_j u_j$ with $\epsilon_j = \pm 1$ and we have 62

$$\rho_j(-n) = \epsilon_j \rho_j(1) t_j(n),\ \ n \geqslant 1, j \geqslant 1. \tag{1.23}$$

1.5 The sum formulae for Kloosterman sums.

The natural generalization of the classical Petersson formula

$$\begin{aligned}(f, P_n) &= \int\limits_{\Gamma/\mathbb{H}} f(z) \overline{P_n(z, k)} y^k d\mu(z) \qquad (1.24)\\ &= a_f(n) (= n - \text{th Fourier coefficient})\end{aligned}$$

for any f from the space M_k of the regular cusp forms of weight k is the same formula for the inner product $(f, U_n(., \bar{s}))$ for an automorphic f from the space of cusp forms M_0 of weight zero.

It is not hard to show that the non-holomorphic Poincaré series $U_n(z, s)$ may be continued analytically (with its Fourier expansion) in the half plane $\operatorname{Re} s > \frac{3}{4}$ (this being based on A. Weil’s estimate for the Kloosterman sum). So, for $\operatorname{Re} s_1$, $\Re s_2 > \frac{3}{4}$, the inner product $(U_n(., s_1), U_m(., \overline{s_2}))$ is well-defined. Since U_n may be expressed as a sum over a group, this inner product is a sum of Kloosterman sums. On the other hand, the inner product (u_j, U_n) may be calculated explicitly in terms of Γ-functions and the n-th Fourier coefficient of j-th eigenfunction u_j of the automorphic Laplacian, Hence the bilinear form of n-th Fourier coefficients of the eigenfunctions

$$\sum_{j\geqslant 0} \rho_j(n)\overline{\rho_j(m)}h(\chi_j)$$

for a certain test function h, may be expressed as a sum of Kloosterman sums.

Of course, we have, for the case $(U_n(., s_1), U_m(., \bar{s}_2))$, two free variables s_1, s_2 and we can try to construct an arbitrary test function in our bilinear form by integration with respect to these variables.

This plan was fulfilled in my first paper and in this way, we have following sum formula (referred to by some authors as the "Kuznetsov trace formula").

Theorem 1. *Let us assume that the function $h(r)$ of the complex variable r is regular in the strip $|\operatorname{Im} r| \leqslant \delta$ with some $\delta > \frac{1}{2}$ and $|h(r)| \ll |r|^{-B}$ for some $B > 2$ when $r \to \infty$ in this strip.*

63 *Then, for any integers $n, m \geqslant 1$, we have*

$$\sum_{j=1}^{\infty} \alpha_j t_j(n) t_j(m) h(\chi_j) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \tau_{(1/2)+ir}(n) \tau_{(1/2)+ir}(m) \times \tag{1.25}$$

$$\times \frac{h(r)dr}{|\zeta(1+2ir)|^2} =$$

$$= \frac{\delta_{n,m}}{\pi^2} \int_{-\infty}^{\infty} r \operatorname{th}(\pi r) h(r) dr + \sum_{c\geqslant 1} \frac{1}{c} S(n, m; c) \varphi\left(\frac{4\pi\sqrt{nm}}{c}\right),$$

where

$$\alpha_j = (\operatorname{ch}(\pi\chi_j))^{-1} |\rho_j(1)|^2, \tag{1.26}$$

ζ is the Riemann zeta function and for $x > 0$, the function $\varphi(x)$ is defined in terms of h by the integral transform

$$\varphi(x) = \frac{2i}{pi} \int_{-\infty}^{\infty} J_{2ir}(x) \frac{rh(r)}{\mathrm{ch}(\pi r)} dr. \tag{1.27}$$

Identity (1.25) is modified in the following manner, if the integers, n, m on the right-hand side have different signs.

Theorem 2. *Assume that the function h satisfies the conditions of the preceding theorem. Then, for any integers n, $m \geqslant 1$, we have*

$$\sum_{j \geqslant 1} \epsilon_j \alpha_j t_j(n) t_j(m) h(\chi_j) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \tau_{(1/2)+ir}(n) \tau_{(1/2)+ir}(m) \frac{h(r)dr}{|\zeta(1+2ir)|^2} = \tag{1.28}$$

$$= \sum_{c \geqslant 1} \frac{1}{c} S(n, -m; c) \psi\left(\frac{4\pi\sqrt{nm}}{c}\right)$$

where $\psi(x)$, for $x > 0$, is defined in terms of h by the integral

$$\psi(x) = \frac{4}{\pi^2} \int_{-\infty}^{\infty} K_{2ir}(x) h(r), sh(\pi r) dr. \tag{1.29}$$

We can invert identities (1.25) and (1.28) and we shall assume that the sum of Kloosterman sums is given rather than the bilinear form in the Fourier coefficients.

Theorem 3. *Assume that to a function $\psi : [0, \infty) \to \mathbb{C}$, the integral transform* **64**

$$h(r) = 2\mathrm{ch}(\pi r) \int_0^{\infty} K_{2ir}(x) \psi(x) \frac{dx}{x} \tag{1.30}$$

associates the functions $h(r)$ satisfying the conditions of Theorem 1. Then, for this ψ and for integers n, $m \geqslant 1$, identity (1.28) *is satisfied, where h is defined by the integral* (1.30).

Theorem 4. Let $\varphi \in C^3(0,\infty)$, $\varphi(0) = \varphi'(0) = 0$ and assume that $\varphi(x)$, together with its derivatives up to the third order, is $O(x^{-B})$ for some $B > 2$, as $x \to \infty$. Then, for any integers $n, m \geqslant 1$, we have

$$\sum_{c\geqslant 1} \frac{1}{c} S(n,m;c)\varphi_H\left(\frac{4\pi\sqrt{nm}}{c}\right) = -\frac{\delta_{n,m}}{2\pi}\int_0^\infty J_0(x)\varphi(x)dx + \tag{1.31}$$

$$+\sum_{j\geqslant 1}\alpha_j t_j(n)t_j(m)h(\chi_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty}\tau_{(1/2)+ir}(n)\tau_{(1/2)+ir}(m)\frac{h(r)dr}{|\zeta(1+2ir)|^2},$$

where the functions $\varphi_H(x)$ and $h(r)$ are defined in terms of φ by the integral transforms

$$\varphi_H(x) = \varphi(x) - 2\sum_{k=1}^{\infty}(2k-1)J_{2k-1}(x)\int_0^\infty J_{2k-1}(y)\varphi(y)\frac{dy}{y}. \tag{1.32}$$

$$h(r) = \frac{i\pi}{2sh(\pi r)}\int_0^\infty (J_{2ir}(x) - J_{-2ir}(X))\varphi(x)\frac{dx}{x}. \tag{1.33}$$

It should be useful to note that the transformation $\varphi \to \varphi_H$ in (1.32) is a projection by which, to a given φ, one associates its component orthogonal on the semiaxis $x \geqslant 0$ (with respect to the measure $x^{-1}dx$) to all the Bessel functions of odd integral order.

Together with (1.32), this projection can be defined by the equality

$$\varphi_H(x) = \varphi(x) - x\int_0^\infty \varphi(u)\left(\int_0^1 \xi J_0(x\xi)J_0(u\xi)d\xi\right)du \tag{1.34}$$

$$= \varphi(x) - x\int_0^\infty \varphi(u)\frac{xJ_0(u)J_1(x) - uJ_0(x)J_1(u)}{x^2 - u^2}du$$

65 and any sufficiently smooth φ admits a decomposition

$$\varphi = \varphi_H + (\varphi - \varphi_H) \tag{1.35}$$

where $\varphi - \varphi_H$ is a combination of the Bessel functions defined by (1.32) while φ_H is equal to integral (1.27), in which by h one means the integral transform (1.33) of the function φ.

The classical Petersson formula

$$\sum_{j=1}^{\nu_k} \|f_{j,k}\|^{-2} t_{j,k}(n) t_{j,k}(m) = \tag{1.36}$$

$$= i^k \delta_{n,m} + 2\pi \sum_{c=1}^{\infty} \frac{1}{c} S(n,m,c) J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right)$$

(where $f_{j,k}$ form an orthogonal basis in the space $\mathscr{M}_k$ of cusp forms of weight k, $\|f_{j,k}\|^2 = (f_{j,k}, f_{j,k})$ and $\nu_k = \dim \mathscr{M}_k$) allows us to represent the sum

$$\sum_{c\geqslant 1} \frac{1}{c} S((n,m;c))\varphi\left(\frac{4\pi\sqrt{nm}}{c}\right) \tag{1.37}$$

as a bilinear form in the eigenvalues of the Hecke operators for the case when φ may be represented by the Neumann series of the Bessel functions of odd order. Together with (1.31) this gives a representation of the sum (1.37) as a bilinear form of the eigenvalues of the Hecke operators (in all $\mathscr{M}_k$ with even k and $\mathscr{M}_0$) for an arbitrary "good" function φ.

1.6 Some relations with Bessel functions.

The special case of the following expansion in terms of Bessel functions is the crucial key to prove the identities of the preceding theorems (Really our identities are consequences of a suitable averaging of the initial identity which results from a comparison of two different expressions for the inner product $(U_n(., 1+it), U_m(., 1-it))$, $t \in \mathbb{R}$).

Theorem 5. *Let $f \in C^2(0,\infty)$, $f(0) = 0$ and $\sum_{r=0}^{2} |f^{(t)}(x)| \ll x^{-B}$ for some $B > 2$, as $x \to +\infty$. Let $\alpha \in \mathbb{R}$ and $F(x,t;\alpha)$ be defined by the equality*

$$F(x,t;\alpha) = J_{it}(x)\cos\frac{\pi}{2}(\alpha - it) - J_{-it}(x)\cos\frac{\pi}{2}(\alpha + it). \tag{1.38}$$

Then we have the representation

$$f(x) = -\int_0^\infty F(x,t;\alpha)\hat{f}(t;\alpha)\frac{t\,dt}{sh(\pi t)(\mathrm{ch}(\pi t)+\cos(\pi\alpha))}+ \tag{1.39}$$
$$+\sum_{n>(\alpha-1)/2} J_{2n+1-\alpha}(x)h_n(f)$$

66 *where*

$$\hat{f}(t;\alpha) = \int_0^\infty F(x,t;\alpha)f(x)\frac{dx}{x}, \tag{1.40}$$

$$h_n(f) = 2(2n+1-\alpha)\int_0^\infty J_{2n+1-\alpha}(x)f(x)\frac{dx}{x}. \tag{1.41}$$

1.7 Some consequences.

We have an explicit from of the connection between $\rho_j(n)$ and the sum of Kloosterman sums. For this reason, we can transform the information about the Kloosterman sums into information about the Fourier coefficients of the eigenfunctions and vice versa.

The first example is the confirmation of the Linnik conjecture. The second is

Theorem 6. *For any $n \geqslant 1$, as $T \to +\infty$, we have*

$$\sum_{\chi_j\leqslant T}\alpha_j t\frac{2}{j}(n) = \frac{T^2}{\pi^2} + O(T(\log T + d^2(n))) + O(\sqrt{nd_3(n)}\log^2 n) \tag{1.42}$$

where $\alpha_j = (\mathrm{ch}(\pi\chi_j))^{-1}|\rho_j(1)|^2$, $d_3(n) = \sum_{d_1d_2d_3=n} = \sum_{d|n} d\left(\frac{n}{d}\right)$

The following (indirect) consequence is due to V. Bykovskij:

$$\sum_{n\leqslant T} d(n^2-D) = T(e_1(D)\log T + c_0(D)) + O_D((T\log T)^{2/3}) \tag{1.43}$$

where D is a fixed non-square and c_1, c_0 are constants.

H. Iwaniec proved the excellent estimate for the number $\pi_\Gamma(X)$ of the conjugate primitive hyperbolic classes $\{P_0\}$ with $NP_0 < X$:

$$\pi_\Gamma(X) = liX + O(X^{(35/48)+\epsilon}) \text{ for any } \epsilon > 0. \tag{1.44}$$

The proof is based essentially on the sum formulae for Kloosterman sums.

We have some progress in the additive divisor problem (H. Iwaniec and J.-M. Deshoouillers and myself):

$$\sum_{n\leqslant T} d(n)d(n+N) = TP_2(\log T, N) + O_N(T\log T)^{2/3}) \tag{1.45}$$

where $P_2(z, N)$ is a polynomial in z of degree 2.

1.8 The Hecke series.

To each eigenfunction of the ring of Hecke operators (regular in the case of $\mathscr{M}_k$ with $k > 0$ and real analytic in the case of $\mathscr{M}_0$), we associate 67
the Dirichlet series whose n-th coefficient is the eigenvalue of the n-th Hecke operator.

As we have relations connecting the spectra of the Hecke operators with the Fourier coefficients of the eigenfunctions, these series differ, only upto normalization, from the series associated by Hecke to regular parabolic cusps by means of the Mellin transform.

We set

$$\mathscr{H}_{j,k}(s;x) = \sum_{n=1}^{\infty} e(nx)n^{-s}t_{j,k}(n), \tag{1.46}$$
$$\mathscr{H}_j(s;x) = \sum_{n=1}^{\infty} e(nx)n^{-s}t_j(n),$$

and we denote by $\mathscr{L}_v(s;x)$ the Hecke series associated with the Eisenstein-Maass series $E(z, v)$,

$$\mathscr{L}_v(s;x) = \sum_{n\geqslant 1} e(nx)n^{-s}\tau_v(n). \tag{1.47}$$

For $x = 0$, these series are denoted by $\mathscr{H}_{j,k}(s)$, $\mathscr{H}_j(s)$, $\mathscr{L}_v(s)$ respectively.

Theorem 7. *Let x be rational, $x = \frac{d}{c}$ with $(d, c) = 1$, $c \geqslant 1$. Then*

(1) *$\mathscr{H}_{j,k}(s, d/c)$, $\mathscr{H}_j(s, d/c)$ are entire functions of s,*

(2) *for $v \neq \frac{1}{2}$, the only singularities of $\mathscr{L}_v(s, d/c)$ are simple poles at the point $s_1 = v + \frac{1}{2}$ and $s_2 = \frac{3}{2} - v$ with residues $c^{-2v}\zeta(2v)$ and $c^{2v-2}\zeta(2-2v)$; the function $((S-1)^2 - (v-\frac{1}{2})^2)\mathscr{L}_v(s, d/c)$ is an entire function of s.*

For what follows, it is convenient to set

$$\gamma(u, v) = \frac{2^{2\mu-1}}{\pi}\Gamma(u + v - \frac{1}{2})\Gamma(u - v + \frac{1}{2}); \tag{1.48}$$

as a consequence of the functional equation for the gamma function, this function for any $u, v \in C$ satisfies the relation

$$\gamma(u, v)\gamma(1-u, v) = -(\cos^2 \pi u - \sin^2 \pi v)^{-1}. \tag{1.49}$$

Theorem 8. The Hecke series have functional equations of the Riemann type; moreover

1) *for even integers $k \geqslant 12$ and for $(d, c) = 1$, $c \geqslant 1$, we have*

$$\mathscr{H}_{j,k}(s, d/c) = -(4\pi/c)^{2s-1}\gamma(1-s, k/2)\cos(\pi s)\mathscr{H}_{j,k}(1-s, -d'/c) \tag{1.50}$$

where d' is defined by the congruence $dd' \equiv 1 (\text{mod } c)$

2) *with the same d',*

$$\mathscr{L}_v(s, d/c) = (4\pi/c)^{2s-1}\gamma(1-s, v)\{-\cos(\pi s)\mathscr{L}_v(1-s, -d/c) + \tag{1.51}$$

$$+ \sin(\pi v)\mathscr{L}_v(1-s, d'/c)\},$$

$$\mathscr{H}_j(s, d/c) = (4\pi/c)^{2c-1}\gamma(1-s, \frac{1}{2} + i\chi_j) \tag{1.52}$$

$$\{-\cos(\pi s)\mathscr{H}_j(1-s, d'/c) + \epsilon_j \text{ch}(\pi\chi_j)\mathscr{H}_j(1-s, d/c)\}.$$

We conclude with the simple but important consequence of the multiplicative relations (1.19) for Hecke operators : for $\operatorname{Re} s > 1 + |\operatorname{Re} \nu - \frac{1}{2}|$, we have

$$\sum_{n=1}^{\infty} \frac{\tau_\nu(n) t_j(n)}{n^s} = \frac{1}{\zeta(2s)} \mathscr{H}_j(s+\nu-\frac{1}{2}) \mathscr{H}_j(s-\nu+\frac{1}{2}). \tag{1.53}$$

If we replace $t_j(n)$ by $\tau_\mu(n)$ (that corresponds to the continuous spectrum of the Hecke operators) then the well-known Ramanujan identity will arise instead of (1.53):

$$\sum_{n=1}^{\infty} \frac{\tau_\nu(n) \tau_\mu(n)}{n^s} = \tag{1.54}$$

$$= \frac{1}{\zeta(2s)} \zeta(s+\nu-\mu)\zeta(s+\mu-\nu)\zeta(s-\nu-\mu+1)\zeta(s+\nu+\mu-1).$$

For this reason, equality (1.53) is a direct generalization of the Ramanujan identity; both will be essential for the estimate of the eighth moment of the Riemann zeta-function.

1.9 The spectral mean of Hecke series.

Let $N \geqslant 1$ be an integer and let s, ν be complex variables. We set

$$Z_N^{(d)}(s,\nu;h) = \sum_{j \geqslant 1} \alpha_j t_j(N) \mathscr{H}_j(s+\nu-\frac{1}{2}) \mathscr{H}_j(s-\nu+\frac{1}{2}) h(\chi_j) \tag{1.55}$$

$$Z_N^{(d)}(s,\nu;h) = \sum_{j \geqslant 1} \epsilon_j \alpha_j t_j(N) \mathscr{H}_j(s+\nu-\frac{1}{2}) \mathscr{H}_j(s-\nu+\frac{1}{2}) h(\chi_j) \tag{1.56}$$

(with $\alpha_j = (\operatorname{ch}(\pi\chi_j))^{-1} |\rho_j(1)|^2$). Here the summation is over the positive discrete spectrum of the automorphic Laplacian and one assumes that its eigenfunctions have been selected in such a manner that they are at the same time eigenfunctions of the ring of Hecke operators and of the reflection operator T_{-1}($\epsilon_j = \pm 1$ are the eigenvalues of T_{-1}).

Further, we define the square mean of the Hecke series over the continuous spectrum by the equality

$$Z_N^{(c)}(s,\nu;h) = \tag{1.57}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\zeta(s+v-\frac{1}{2}+ir)\zeta(s+v-\frac{1}{2}-ir)\zeta(s-v+\frac{1}{2}+ir)\zeta(s-v+\frac{1}{2}-ir)}{\zeta(1+2ir)\zeta(1-2ir)}$$

$$\times \tau_{(1/2)+ir}(N)h(r)dr$$

69 with the stipulation that, by means of integral (1.57), the function $Z_N^{(c)}$ is defined under the conditions

$$\mathrm{Re}(s+v-\frac{1}{2}) < 1, \mathrm{Re}(s-v+\frac{1}{2}) < 1.$$

If any one of the points $s \pm (v - \frac{1}{2})$ lies to the right hand side of the unit line, then the integral (1.57) defines another function, connected with $Z_N^{(c)}$ by the Sokhotskii formulae. Fro example, if by $\tilde{Z}_N^{(c)}$, we denote the function which is defined by (1.57) with $\mathrm{Re}\, s > 1$, $\mathrm{Re}\, v = \frac{1}{2}$, then a simple computation gives

$$\tilde{Z}_N^{(c)}(s,v;h) = Z_N^{(c)}(s,v;h) + 4\zeta_N(s,v)h(i(s-v-\frac{1}{2}))+$$
$$+4\zeta_N(s,1-v)h(i(s+v-\frac{3}{2}))$$

where we have introduced the notation

$$\xi_N(s,v) = \frac{\zeta(2s-1)\zeta(2v)}{\zeta(2-2s+2v)}\tau_{s-v}(N)$$

and the regularity strip of h is assumed to be sufficiently wide for the right hand side to make sense.

Now we need the mean with respect to the weights of the Hecke series associated with regular cusp forms. For an integer $k \geqslant 1$, we set

$$Z_{N,k}(s,v) = 2(-1)^k \frac{\Gamma(2k-1)}{(4\pi)^{2k}} \sum_{j=1}^{v_{2k}} |\alpha_{j,2k}(1)|^2 t_{j,2k}(N) \tag{1.58}$$
$$\times \mathscr{H}_{j,2k}(s+v-\frac{1}{2})\mathscr{H}_{j,2k}(s-v+\frac{1}{2})$$

where $t_{j,2k}(N)$ is the eigenvalue of the N-th Hecke operator in the space $\mathscr{M}_{2k}$ of regular cusp forms of weight $2k$, $v_{2k} = \dim \mathscr{M}_{2k}$; the empty sum for $1 \leqslant k \leqslant 5$ and $k = 7$ is assumed to be equal to zero.

Assume now that $h^* = \{h_{2k-1}\}_{k=1}^{\infty}$ is a sequence of sufficiently fast decreasing numbers; we define the mean of the Hecke series with respect to weights by the equality

$$Z_N^{(p)}(s, v; h^*) = \sum_{k \geqslant 6} h_{2k-1} Z_{N,k}(s, v) \tag{1.59}$$

1.10 The convolution formula.

Some of the consequences of the algebra of modular forms are the so **70**
called “exact formulae”, an example of which is the identity

$$\sum_{n=1}^{N-1} \sigma_3(n)\sigma_3(N-n) = \frac{1}{120}(\sigma_7(N) - \sigma_3(N)), \sigma_a(n) = \sum_{d|n} d! \tag{1.60}$$

A source of similar identities is the obvious assertion that the product of modular forms of weight k and l is a modular form of weight $k + l$.

There are analogues of these identities for the real analytic Eisenstein series of weight zero. For an integer $N \geqslant 1$, we associate to a pair of series $E(z, s)$ and $E(z, v)$ the expression of convolution type

$$W_N(s, v; w_0, w_1) = N^{s-1} \sum_{n=1}^{\infty} \tau_v(n)(\sigma_{1-2s}(n-N)w_0(\sqrt{n/N}) \tag{1.61}$$
$$+\sigma_{1-2s}(n+N)w_1(\sqrt{n/N}))$$

where $\sigma_{1-2s}(0)$ means $\zeta(2s-1)$ and w_0, w_1 are assumed to be sufficiently smooth and sufficiently fast decreasing for $x \to +\infty$.

Theorem 9. *Assume that the functions w_0, w_1 are continuous on the semiaxis $x \geqslant 0$ together with derivatives up to the fourth order, $w_j(0) = w'_j(0) = 0$ for $j = 0, 1$ and that, for $x \to +\infty$, the functions $w_j(x)$ as well as their derivatives up to the third order are $O(x^{-B})$ for some $B > 4$.*

Then, for any integer $N \geqslant 1$ and s, $v \in \mathbb{C}$ satisfying $\operatorname{Re} v = \frac{1}{2}, \frac{1}{2} < \operatorname{Re} S < 1$, *we have*

$$W_N(s, v; w_0; w_1) = Z_N^{(d)}(s, v; h_0) + Z_N^{(d)}(s, v; h_1)+ \tag{1.62}$$

$$Z_N^{(c)}(s,v;h_0+h_1)+Z_N^{(p)}(s,v;h^*)+\zeta_N(s,v)V(\frac{1}{2},v)+$$
$$\zeta_N(s,1-v)V(\frac{1}{2},1-v)+\zeta_N(1-s,v)V(s,v)+\zeta_N(1-s,1-v)|\times$$
$$V(s,1-v)$$

where

$$\zeta_N(s,v)=\frac{\zeta(2s)\zeta(2v)}{\zeta(2s+2v)}\tau_{s+v}(N), \tag{1.63}$$

$$V(s,v)=2\int_0^\infty(|1-x^2|^{1-2s})w_0(x)\to(1+x^2)^{1-2s}w_1(x))x^{2v}dx \tag{1.64}$$

and the column vector $h(r;s,v)=\begin{pmatrix}h_0\\h_1\end{pmatrix}$ *is defined in terms of* $w=\begin{pmatrix}w_0\\w_1\end{pmatrix}$ *by the integral transform*

$$h(r;s,v)=\pi\int_0^\infty\begin{pmatrix}k_0(x,\frac{1}{2}+ir) & 0\\ 0 & k_1(x,\frac{1}{2}+ir)\end{pmatrix}\left(\frac{x}{4\pi}\right)^{2s-1}\times \tag{1.65}$$
$$\int_0^\infty\begin{pmatrix}k_0(xy,v) & k_1(xy,v)\\ k_1(xy,v) & k_0(xy,v)\end{pmatrix}w(y)ydydx$$

71 *with the kernels*

$$k_0(x,v)=\frac{J_{2v-1}(x)-J_{1-2v}(x)}{2\cos(\pi v)}, \tag{1.66}$$
$$k_1(x,v)=\frac{2}{\pi}\sin(\pi v)K_{2v-1}(x).$$

Finally, the coefficients of the mean of the regular forms $Z_N^{(\rho)}$ *are given by the relations*

$$h_{2k-1}=2(2k-1)\times$$
$$\times\int_0^\infty J_{2k-1}(x)\left(\frac{x}{4\pi}\right)^{2s-1}\int_0^\infty(k_0(xy,v)w_0(y)+k_1(xy,v)w_1(y))ydy\,dx.$$

1.11 Some consequences of the convolution formula.

The first example of the use of (1.62) is the additive divisor problem; if we choose $s = v = 1/2$, $w_1 = 0$ and w_0 so that it is close to 1 in the interval $(0, \sqrt{T/N})$ (so that $w_0(\sqrt{n/N})$ will be close to 1 for $n \leqslant T$), then the left hand side of (1.62) gives the sum on the left side of (1.45). Terms with the integral (1.64) are leading terms and all other terms give the remainder term.

Of course, the asymptotic formula for the additive divisor problem is crucial for the investigation of the fourth power moment of the Riemann zeta-function. A consequence of (1.62) in this direction is the following

Theorem. *(N. Zavorotnyi, 1987). Let $T \to +\infty$; then, for any $\epsilon > 0$, we have*

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt = TP_4(\log T) + O(T^{2/3+\epsilon}) \tag{1.67}$$

where $P_4(z)$ is a polynomial in z of the fourth degree with constant coefficients.

We can consider the functions h_0 and h_1 in (1.62) as given; the fol- **72**
lowing unusual integral transform is useful to invert (1.62).

Let us define the matrix kernel $\mathbb{K}(x, v)$ by the equality

$$\mathbb{K}(x, v) = \begin{pmatrix} k_0(x, v) & k_1(x, v) \\ k_1(x, v) & k_0(x, v) \end{pmatrix} \tag{1.68}$$

with k_0, k_1 from (1.66). Now we shall consider the matrix equation

$$w(x) = \int_0^\infty \mathbb{K}(xy, v)u(y)\sqrt{xy}dy \tag{1.69}$$

where $w = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}(x)$, $u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(x)$.

Theorem 10. *Let* $\operatorname{Re} v = \frac{1}{2}$ *and* $w \in L^2(0,\infty)$ *in the sense that* $w_0, w_1' \in L^2(0,\infty)$. *Then there exists a unique solution* u *in* $L^2(0,\infty)$ *of the equation* (1.69) *and this solution is given by the formula*

$$u(x) = \int_0^\infty \mathbb{K}(xy, u) w(y) \sqrt{xy} dy \tag{1.70}$$

where the integral is understood in the mean-square sense.

Now, as a special case of the convolution formula (1.62), we have the following asymptotic formulae.

Theorem 11. *Let* $T \to +\infty$. *Then for a fixed* σ *and* $t \in \mathbb{R}$ *with* $\frac{1}{2} < \sigma < 1$, *we have*

$$\sum_{\chi_j \leqslant T} \alpha_j |\mathscr{H}_j(\sigma + it)|^2 = \frac{T^2}{\pi^2}(\zeta(2\sigma) + \frac{\zeta(2-2\sigma)}{2(1-\sigma)} \left(\frac{T^2}{2\pi}\right)^{1-2\sigma}) + O(T \log T) \tag{1.71}$$

while, for $\sigma = \frac{1}{2}$ *the right-hand side has to be replaced by*

$$\frac{2T^2}{\pi^2}(\log T + 2\gamma - 1 + 2log(2\pi)) + O(T \log T). \tag{1.72}$$

1.12 The explicit formulae for the transformation (1.65)

We rewrite equality (1.65) in the form

$$h = \int_0^\infty A(r, y; s, v) \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} (y) dy, A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}, \tag{1.73}$$

where the matrix kernel A, under the conditions $\operatorname{Re} v = \frac{1}{2}$, $\frac{1}{2} < \operatorname{Re} s < 1$,
73 $|Iimr| < \operatorname{Re} s$, is determined by the integrals that appear in the term-by-term integration in (1.65). All these integrals are given in tables (these are the Weber-Schafheitlin integrals); we will be needing the following explicit form for the kernels $A_{j,l}$, $j, l = 0, 1$.

Proposition 1. *Let us denote, for* $\mathrm{Re}\, v = \frac{1}{2}, \frac{1}{2} < \mathrm{Re}\, s < 1$, $|\mathrm{Im}\, r| < \mathrm{Re}\, s$

$$a = s + v - \frac{1}{2} + ir,\ b = s - v + \frac{1}{2} + ir,\ c = 1 + 2ir, \tag{1.74}$$

$$a' = s + v = -\frac{1}{2} - ir,\ b' = s - v + \frac{1}{2} - ir,\ c' = 1 - 2ir. \tag{1.75}$$

Then, for $0 < y < 1$*, we have*

$$(2\pi)^{2s-1} A_{00}(r, y; s, v) := \pi y \int_0^\infty k_0(x, \frac{1}{2} + ir) k_0(xy, v) (\frac{x}{2})^{2s-1} dx \tag{1.76}$$

$$= \frac{1}{2\cos(\pi v)} \left\{ \frac{\Gamma(a)\Gamma(a')}{\Gamma(2v)} y^{2v} F(a, a'; 2v; y^2) \sin \pi(s + v) + \right.$$
$$\left. + \frac{\Gamma(b)\Gamma(b')}{\Gamma(2 - 2v)} y^{2-2v} F(b, b'; 2 - 2v; y^2) \sin \pi(s - v) \right\}$$

and for $y > 1$*,*

$$(2\pi)^{2s-1} A_{00}(r, y; s, v) = \tag{1.77}$$
$$= \frac{iy^{1-2s}}{2sh(\pi r)} \left\{ \frac{\Gamma(a)\Gamma(b)}{y^{c-1}\Gamma(c)} F\left(a, b; c; \frac{1}{y^2}\right) \cos \pi(s + ir) - \right.$$
$$\left. - \frac{\Gamma(a')\Gamma(b')}{y^{c'-1}\Gamma(c')} F\left(a', b'; c'; \frac{1}{y^2}\right) \cos \pi(s - ir) \right\}$$

At the same time, for all $y > 0$*, we have*

$$(2\pi)^{2s-1} A_{00}(r, y; s, v) = \sin(\pi s)\Gamma(2s - 1) y^{2v} |1 - y^2|^{1-2s} \times \tag{1.78}$$
$$F(1 - b, 1 - b'; 2 - 2s; 1 - y^2) + \frac{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}{2\pi\Gamma(2s)\cos(\pi s)} \times$$
$$(\mathrm{ch}^2 \pi r + \sin^2 \pi v - \sin^2 \pi s) y^{2v} F(a, a'; 2s; 1 - y^2)$$

where, in the first term, the absolute value sign combines the two cases $y < 1$ *and* $y > 1$.

Proposition 2. With the same parameters, we have 74

$$(2\pi)^{2s-1}A_{01}(r,y;s,v) := \pi y \int_0^\infty k_0(x,\frac{1}{2}+ir)k_1(xy,v)(\frac{x}{2})^{2s-1}dx \tag{1.79}$$

$$= \frac{iy^{1-2s}\sin(\pi v)}{2\mathrm{sh}(\pi r)}\left\{\frac{\Gamma(a)\Gamma(b)}{y^{c-1}\Gamma(c)}F\left(a,b;c;-\frac{1}{y^2}\right)\right.$$

$$\left.-\frac{\Gamma(a')\Gamma(b')}{y^{c'-1}\Gamma(c')}F\left(a',b';c';-\frac{1}{y^2}\right)\right\}$$

Proposition 3. *The kernel A_{10} is defined by the relation*

$$(2\pi)^{2s-1}A_{10}(r,y;s,v) := \pi y \int_0^\infty k_1(x,\frac{1}{2}+ir)k_1(xy,v)(\frac{x}{2})^{2s-1}dx \tag{1.80}$$

$$= \frac{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}{\pi\Gamma(2s)}\,\mathrm{ch}(\pi r)\sin(\pi v)y^{2v}F(a,a';2s;1-y^2).$$

Proposition 4. *With the parameters* (1.74)-(1.75), *we have*

$$(2\pi)^{2s-1}A_{11}(r,y;s,v) := \pi y \int_0^\infty k_1(x,\frac{1}{2}+ir)k_0(xy,v)(\frac{x}{2})^{2s-1}dx =$$

$$= \frac{\mathrm{ch}(\pi r)}{2\cos(\pi v)}\left\{\frac{\Gamma(a)\Gamma(a')}{\Gamma(2v)}y^{2v}F(a,a';2v;-y^2)-\right.$$

$$\left.-\frac{\Gamma(b)\Gamma(b')}{\Gamma(2-2v)}y^{2-2v}F(b,b';2-2v;-y^2)\right\} \tag{1.81}$$

and, at the same time,

$$(2\pi)^{2s-1}A_{11}(r,y;s,v) = \tag{1.82}$$

$$= \frac{iy^{1-2s}}{2\mathrm{sh}(\pi r)}\left\{\frac{\Gamma(a)\Gamma(b)}{y^{c-1}\Gamma(c)}F\left(a,b;c;-\frac{1}{y^2}\right)\cos\pi(s+ir)-\right.$$

$$\left.-\frac{\Gamma(a')\Gamma(b')}{y^{c'-1}\Gamma(c')}F\left(a';b';c';-\frac{1}{y^2}\right)\cos\pi(s-ir)\right\}.$$

Proposition 5. *Let us write the quantities* h_{2k-1} *in* (1.62) *as*

$$h_{2k-1}(s,v) = \int_0^\infty (A_k^0(y;s,v)w_0(y) + A_k^1(y;s,v)w_1(y))dy. \tag{1.83}$$

Then, for $0 < y < 1$, 75

$$(2\pi)^{2s-1}A_k^0(y;s,v) = \tag{1.84}$$
$$\frac{2k-1}{\cos(\pi v)}\left\{\frac{\Gamma(s+v-1+k)}{\Gamma(2v)\Gamma(1-s-v+k)}y^{2v}F(k+s+v-1,s+v-k;2v;y^2)-\right.$$
$$\left.-\frac{\Gamma(k+s-v)}{\Gamma(2-2v)\Gamma(v-s+k)}y^{2-2v}F(k+s-v,s+1-v-k;2-2v;y^2)\right\}$$

and, for $y > 1$,

$$(2\pi)^{2s-1}A_k^0(y;s,n) = \tag{1.85}$$
$$\frac{(-1)^{k-1}\sin(\pi s)}{\pi y^{2k+2s-2}}\frac{\Gamma(k+s+v-1)\Gamma(k+s-v)}{\Gamma(2k-1)}\times$$
$$F\left(k+s+v-1,k+s-v;2k;\frac{1}{y^2}\right).$$

For the second kernel in (1.83), *we have*

$$(2\pi)^{2s-1}A_k^1(y;s,v) = \tag{1.86}$$
$$= -\frac{\sin(\pi v)}{\pi y^{2k+2s-2}}\frac{\Gamma(k+s+v-1)\Gamma(k+s-v)}{\Gamma(2k-1)}\times$$
$$F\left(k+s+v-1,\ k+s-v;2k;-\frac{1}{y^2}\right)$$

Part II. The eighth moment of the Riemann zeta-function.

2.1 The result and a rough sketch of the proof.

Since the question about the true order of zeta-function on the critical line is open even today - and it will be so in the foreseeable future-, a sizeable part of the theory of the Riemann zeta-function is an attempt to present the asymptotic mean value

$$\frac{1}{T}\int_0^T |\zeta(\frac{1}{2}+it)|^{2k} dt, \ k = 1, 2, \ldots \tag{2.1}$$

The case $k = 4$ will be investigated here; for this case, the following new estimate will be given.

76 **Theorem 12.** *There is an absolute constant B such that*

$$\int_0^T |\zeta(\frac{1}{2}+it)|^8 dt \ll T(\log T)^B \tag{2.2}$$

when $T \to +\infty$.

Furthermore, the same estimate is valid for the fourth power moment of the Hecke series of the discrete spectrum of the automorphic Laplacian. Namely, we have

Theorem 13. *For every fixed $j \geqslant 1$ with the same B as in* (2.2)*, we have*

$$\alpha_j \int_0^T |\mathscr{H}_j(\frac{1}{2}+it)|^4 dt \ll T(\log T)^{B-6}, T \to +\infty. \tag{2.3}$$

To give these estimates we shall consider the fourth spectral moment of the Hecke series over the discrete and continuous spectrum. The one over the discrete spectrum is defined by

$$Z^{\text{dis}}\left(\begin{matrix} s, \nu \\ \rho, \mu \end{matrix} \middle| h\right) = \tag{2.4}$$

$$\sum_{j \geqslant 1} \alpha_j \frac{\mathscr{H}_j(s+\nu-\frac{1}{2})\mathscr{H}_j(s-\nu+\frac{1}{2})\mathscr{H}_j(\rho+\mu-\frac{1}{2})\mathscr{H}_j(\rho-\mu+\frac{1}{2})}{\zeta(2s)\zeta(2\rho)} h(\chi_j)$$

with

$$\alpha_j = (\mathrm{ch}(\pi\chi_j))^{-1}|\rho_j(1)|^2. \tag{2.5}$$

Of course, this function results from the following summation (see the generalized Ramanujan identity (1.53)):

$$Z^{\mathrm{dis}}\left(\begin{matrix} s, \nu \\ \rho, \mu \end{matrix}\middle| h\right) = \sum_{n,m\geqslant 1} \frac{\tau_\nu(n)}{n^s}\frac{\tau_\mu(m)}{m^\rho}\left(\sum_{j\geqslant 1} \alpha_j t_j(n) t_j(m) h(\chi_j)\right) \tag{2.6}$$

when $\mathrm{Re}\, s > \mathrm{Re}\, \nu - \frac{1}{2}| + 1$, $\mathrm{Re}\, \rho > |\mathrm{Re}\, \mu - \frac{1}{2}| + 1$. The function

$$\zeta(2s)\zeta(2\rho)Z^{\mathrm{dis}}\left(\begin{matrix} s, \nu \\ \rho, \mu \end{matrix}\middle| h\right)$$

is regular in some domain of the kind $\mathrm{Re}\, s > s_0$' $\mathrm{Re}\, \rho > \rho_0$, where $s_0'\rho_0$ depend on the order of decay of the function $|h(r)|$ for $|r| \to +\infty$.

We shall denote by $\tilde{Z}^{\mathrm{dis}}\left(\begin{matrix} s, \nu \\ \rho, \mu \end{matrix}\middle| h\right)$ the expression obtained on replacing α_j by $\epsilon_j\alpha_j$ in (2.6):

$$\tilde{Z}^{\mathrm{dis}}\left(\begin{matrix} s, \nu \\ \rho, \mu \end{matrix}\middle| h\right) = \sum_{n,m\geqslant 1} \frac{\tau_\nu(n)}{n^2}\frac{\tau_\mu(m)}{m^\rho}\left(\sum_{j\geqslant 1} \epsilon_j\alpha_j t_j(n) t_j(m) h(\chi_j)\right) \tag{2.7}$$

In the same manner as in (2.6), we define the fourth spectral moment **77**
over the continuous spectrum by

$$Z_0^{\mathrm{con}}\left(\begin{matrix} s, \nu \\ \rho, \mu \end{matrix}\middle| h\right) = \frac{1}{4\pi}\int_{-\infty}^{\infty} Z(s; \nu, \frac{1}{2} + ir)Z(\rho; \mu, \frac{1}{2} + ir)\frac{h(r)dr}{\zeta(1+2ir)\zeta(1-2ir)} \tag{2.8}$$

where we assume $\mathrm{Re}(s \pm (\nu - \frac{1}{2})) > 1$, $\mathrm{Re}(\rho \pm (\mu - \frac{1}{2})) > 1$ and the notation z is introduced for the right side of the well-known Ramanujan identity:

$$z(s; \nu, \mu) = \sum_{n=1}^{\infty} \frac{\tau_\nu(n)\tau_\mu(n)}{n^s} \tag{2.9}$$

$$= \frac{1}{\zeta(2s)} \zeta(s+\nu-\mu)\zeta(s-\nu+\mu)\zeta(s+\nu+\mu-1)\times$$

$\zeta(s-\nu-\mu+1)$.

Finally, for the given sequence $h^* = \{h_{2k-1}\}_{k=1}^{\infty}$ we define the fourth moment of the Hecke series over regular cusp forms

$$Z^{\text{cusp}}\left(\begin{matrix} s, \nu \\ \rho, \mu \end{matrix}\middle| h^*\right) = 2 \sum_{\substack{k=2 \\ k\equiv 0(\text{mod } 2)}}^{\infty} (-1)^{k/2} h_{k-1} \tag{2.10}$$

$$\times \sum_{j=1}^{\nu_k} \alpha_{j,k} \sum_{n,m\geqslant 1} \frac{\tau_\nu(n)\tau_\mu(m)}{n^s m^\rho} t_{j,k}(n) t_{j,k}(m),$$

where $\nu_k = \dim \mathscr{M}_k$ is the dimension of the space $\mathscr{M}_k$ of the regular cusp forms of weight k and $t_{j,k}(n)$ are the eigenvalues of the n-th Hecke operators in this space $\mathscr{M}_k$. The quantities $\alpha_{j,k}$ are the normalized coefficients; if the functions $f_{j,k}$ form an orthonormal basis in $\mathscr{M}_k$, then

$$\alpha_{j,k} = \frac{\Gamma(K-1)}{(4\pi)^k} |a_{j,k}(1)|^2, \tag{2.11}$$

where $a_{j,k}(1)$ is the first Fourier coefficient of $f_{j,k}$. Together with (2.10), we have

$$Z^{\text{cusp}}\left(\begin{matrix} s, & \nu \\ \rho, & \nu \end{matrix}\middle| h^*\right) = 2 \sum_{\substack{k\geqslant 2 \\ k\equiv 0(\text{mod } 2)}} (-1)^{k/2} h_{k-1} \sum_{j=1}^{\nu_k} \alpha_{j,k}\times \tag{2.12}$$

$$\frac{\mathscr{H}_{j,k}(s+\nu-\frac{1}{2})\mathscr{H}_{j,k}(s-\nu+\frac{1}{2})\mathscr{H}_{j,k}(\rho+\mu-\frac{1}{2})\mathscr{H}_{j,k}(\rho-\mu+\frac{1}{2})}{\zeta(2s)\zeta(2\rho)};$$

78 this equality is quite similar to (2.4).

Now we shall describe the main idea. We shall consider the double Dirichlet series

$$L^{(\pm)}\left(\begin{matrix} s, & \nu \\ \rho, & \nu \end{matrix}\middle| \varphi\right) = \sum_{n,m=1}^{\infty} \frac{\tau_\nu(n)\tau_\mu(m)}{n^s m^\rho} K_{m,n}^{(\pm)}(\varphi), \tag{2.13}$$

where the coefficients are sums of Kloosterman sums with the smooth "test" function φ:

$$K^{(\pm)}_{m,n}(\varphi) = \sum_{c\geqslant 1} \frac{1}{c} S(n, \pm m; c)\varphi\left(\frac{4\pi\sqrt{nm}}{c}\right). \tag{2.14}$$

Since there is a functional equation of the Riemann type for the Dirichlet series

$$\mathcal{L}_\nu\left(S, \frac{a}{c}\right) = \sum_{n=1}^{\infty} n^{-s}\tau_\nu(n)e(n\frac{a}{c}), e(x) := e^{2\pi i x},$$

when a, $c \in \mathbb{Z}$ are coprime and this equation connects $\mathcal{L}_\nu(S, \frac{a}{c})$ with $\mathcal{L}_\nu(1-s, \pm\frac{a'}{c'})$, $aa' \equiv 1(\text{mod } c)$, a functional equation has to exist for the functions $L^{(\pm)}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle|\varphi\right)$; it will connect this function with the functions of the same kind $L^{(\pm)}\left(\begin{matrix} \rho, & \nu \\ s, & \mu \end{matrix}\middle|\varphi\right)$ for an appropriate φ. As a consequence of the sum formula for Kloosterman sums, it means that there is a functional equation for the function

$$Z\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle|h\right) = Z^{\text{dis}}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle|h\right) + Z^{\text{con}}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle|h\right); \tag{2.15}$$

roughly speaking, this equation (which will be written below in detail) is the result of the exchange of s and ρ and the replacement of h by some integral transform.

Now, on the left side of this functional equation, for the special case

$$s = \mu = \frac{1}{2}, \rho = \nu = \frac{1}{2} + it, t \in \mathbb{R} \text{ is large and positive,} \tag{2.16}$$

we have in the continuous spectrum the product

$$\zeta^3(\frac{1}{2} + it + ir)\zeta^3(\frac{1}{2} + it - ir)\zeta(\frac{1}{2} - it + ir)\zeta(\frac{1}{2} - it - ir) \tag{2.17}$$

and the other product will be on the right side; namely, we have therein

$$\zeta^3(\frac{1}{2} + ir)\zeta^3(\frac{1}{2} - ir)\zeta(\frac{1}{2} + 2it + 2ir)\zeta(\frac{1}{2} + 2it - 2ir). \tag{2.18}$$

After this specialization, we shall choose the special function h essentially as $\exp(-\alpha r^2)$ with a fixed positive α. Then the essential part of the interval of the integration is $|r| \ll (\log t)^{(1/2)}$. The length of this interval is small in comparison to the large t and for this reason, (using the Riemann functional equation), we can reduce the main term of our product on the left side to the form

$$|\zeta^4(\frac{1}{2}+it+ir)\zeta^4(\frac{1}{2}+it-ir)|.$$

79 It means that we may hope to estimate the integral

$$\int_{-\epsilon}^{\epsilon}\int_{0}^{\infty}\omega_T(t)|\zeta^4(\frac{1}{2}+it+ir)\zeta^4(\frac{1}{2}+it-ir)|dtr^2dr \tag{2.19}$$

for arbitrary small positive ϵ, if $\omega_T(t)$ is the smooth function which is not zero for $t \in (T, 2T)$ only and close to 1 when $\frac{5}{4}T \leqslant t \leqslant \frac{7}{4}T$ (see picture).

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The main term will be close to the integral

$$\epsilon^3\int_{0}^{\infty}\omega_T(t)|\zeta(\frac{1}{2}+it)|^{\infty}dt$$

if $\epsilon \ll (\log T)^{-2}$ (see subsection (2.4)); so we have the eighth moment of the Riemann zeta-function here. At the same time, the contribution of the discrete spectrum is positive too. Hence the desired conclusion follows if the integrals on the right side can be estimated with sufficient accuracy.

But the integrand for these integrals contains one Hecke series only; so the integration may be done asymptotically. As a result, we shall reduce the problem of the estimate of the eighth power moment to the problem of the estimate for the fourth spectral moment. It is sufficient to prove our theorem for the latter.

In the conclusion of the introduction of the second part, we shall note that the estimate

$$|\zeta(\frac{1}{2}+it)| \ll |t|^{1/8}(\log|t|)^{B_1},, t \to \pm\infty,$$

follows from (2.2), $B_1 = \frac{1}{8}B + \frac{1}{2}$.

It is preceptibly better than the last achievement in the long chain of the results of the kind $|\zeta(\frac{1}{2}+it) \ll |t|^{\gamma}$.

2.2 The first functional equation

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Since the sum $Z^{\text{dis}} + Z^{\text{con}}$ is connected with the sum of Kloosterman sums, we shall consider the triple sum

$$L^{(\pm)}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle| \varphi\right) = \sum_{n,m=1}^{\infty} \frac{\tau_\nu(n)\tau_\mu(m)}{n^s m^\rho} K_{n,m}^{(\pm)}(\varphi). \tag{3.1}$$

Here the notation (2.14) is used and we assume that a function φ is "good" namely, the Mellin transform $\hat{\varphi}(u)$

$$\hat{\varphi}(u) = \int_0^\infty \varphi(x)x^{u-1}dx, \tag{3.2}$$

is regular in the strip $-\alpha_0 \leqslant \operatorname{Re} u \leqslant \alpha_1$ with positive α_0, α_1 and $|\hat{\varphi}(u)|$ decreases sufficiently rapidly in this strip. For this reason, we can write

$$\varphi(x) = \frac{1}{i\pi}\int_{i\pi}^{\infty}\int_{(\alpha)}^{\infty} \hat{\varphi}(2u)x^{-2u}du, x > 0, \tag{3.3}$$

where $\int_{(\alpha)}^{\infty}$ stands for the integral over the line $\operatorname{Re} u = \alpha$. As we have

$$|S(n,m;c)| \ll c^{1/2}(n,m,c)d(c),$$

the triple sum (3.1) converges absolutely if $\alpha < -\frac{1}{4}$ and both $\mathrm{Re}(s+\alpha)$, $\mathrm{Re}(\rho+\alpha)$ are larger than 1. For this case, we have, for the sum (3.1), the following expression

$$L^{(\pm)}\left(\begin{matrix} s, & v \\ \rho, & \mu \end{matrix}\middle|\varphi\right) = \frac{1}{i\pi}\sum_{c\geqslant 1}\frac{1}{c}\times \tag{3.4}$$
$$\sum_{ad\equiv 1 (\mathrm{mod}\ c)}\int_{(\alpha)} \mathscr{L}_v(s+u, \frac{a}{c})\mathscr{L}_\mu(\rho+\mu, \pm\frac{d}{c})(c/4\pi)^{2u}\hat{\varphi}(2\mu)du.$$

Now we shall integrate over the line $\mathrm{Re}\, u = \alpha_0$ where α_0 will be chosen so that both $\mathrm{Re}(s+\alpha_0)$, $\mathrm{Re}(\rho+\alpha_0)$ are negative. Taking into account the contribution from the poles, we have

$$L^{(\pm)}\left(\begin{matrix} s, & v \\ \rho, & \mu \end{matrix}\middle|\varphi\right) \tag{3.5}$$
$$= 2\sum_{c\geqslant 1}\sum_{ad\equiv 1 (\mathrm{mod}\ c)}\left\{\frac{(4\pi)^{2s-1}}{c^{2s}}\left(\frac{\zeta(2v)}{(4\pi)^{2v}}\mathscr{L}_\mu(\rho - s + v + \frac{1}{2}, \pm d/c)\hat{\varphi}(2v+1-2s)+\right.\right.$$
$$\left.+\frac{\zeta(2-2v)}{(4\pi)^{2-2v}}\mathscr{L}_\mu(\rho - s + \frac{3}{2} - v, \pm d/c)\hat{\varphi}(3-2v-2s)\right)+$$
$$+\frac{(4\pi)^{2\rho-1}}{c^{2\rho}}\left(\left(\frac{\zeta(2\mu)}{(4\pi)^{2\mu}}\mathscr{L}_v(s-\rho+\mu+\frac{1}{2}, a/c)\right)\times\right.$$
$$\left.\left.\hat{\varphi}(2\mu+1-2\rho) + \frac{\zeta(2-2\mu)}{(4\pi)^{2-2\mu}}\mathscr{L}_v(s-\rho+\frac{3}{2}-\mu, a/c)\hat{\varphi}(3-2\mu-2\rho)\right)\right\}+$$
$$+\frac{1}{i\pi}\sum_{c\geqslant 1}\frac{1}{c}\sum_{ad\equiv 1 (\mathrm{mod}\ c)}\int_{\alpha_0}\mathscr{L}_v(s+u, a/c)\mathscr{L}_\mu(\rho+u, \pm d/c)(\frac{c}{4\pi})^{2u}\hat{\varphi}(2u)du.$$

81 In the last term, we shall use the functional equation (1.51). If the sign "plus" is taken, then it gives for our sum the expression

$$\sum_{c\geqslant 1}\frac{1}{c}\sum_{n,m\geqslant 1}\frac{\tau_v(n)\tau_\mu(m)}{n^\rho m^s}(S(n,m;c)\Phi_0\left(\frac{4\pi\sqrt{nm}}{c}\right)+ \tag{3.6}$$
$$+S(n,-m;c)\Phi_1\left(\frac{4\pi\sqrt{nm}}{c}\right))$$

where

$$\Phi_0 = \Phi_0(x; s, \nu; \rho, \mu) = \tag{3.7}$$

$$= \frac{1}{i\pi} x^{2s+2\rho-2} \int_{(\alpha_0)} \gamma(1-s-u,\nu)\gamma(1-\rho-u,\mu)\times$$

$$(\cos\pi(s+\mu)\cos\pi(\rho+u) + \sin(\pi\nu)\sin(\pi\mu))x^{2u}\hat{\varphi}(2u)du,$$

$$\Phi_1 = \Phi_1(x; s, \nu; \rho, \mu) = \tag{3.8}$$

$$= -\frac{1}{i\pi} x^{2s+2\rho-2} \int_{(\alpha_0)} \gamma(1-s-u,\nu)\gamma(1-\rho-u,\mu)(\sin(\pi\mu)\times$$

$$\cos\pi(s+u) + \sin(\pi\nu)\cos\pi(\rho+\mu)x^{2u}\hat{\varphi}(2u)du,$$

Of course, when the sign "minus" is taken in (3.5), then the same function s Φ_0 and Φ_1 are the coefficients, but Φ_0 will occur with $S(n, -m; c)$ and Φ_1 with $S(n, m; c)$.

We have $\Phi_j(x) = O(x^{2\min(\operatorname{Re} s, \operatorname{Re}\rho)})$ as $x \to 0+$ and these functions are bounded when x is large. For this reason, the triple sums in (3.6) converge absolutely and we can again interchange the order of the summations. Hence we have, in (3.7), the sum

$$L^{(+)}\left(\begin{matrix}\rho, & \nu \\ s, & \mu\end{matrix}\middle| \Phi_0\right) + L^{(-)}\left(\begin{matrix}\rho, & \nu \\ s, & \mu\end{matrix}\middle| \Phi_1\right) \tag{3.9}$$

for the case "+" on the left side (3.5) and

$$L^{(+)}\left(\begin{matrix}\rho, & \nu \\ s, & \mu\end{matrix}\middle| \Phi_1\right) + L^{(-)}\left(\begin{matrix}\rho, & \nu \\ s, & \mu\end{matrix}\middle| \Phi_0\right) \tag{3.10}$$

for the other case. **82**

Now we are ready to give the first functional equation.

Theorem 14. *Let* $\operatorname{Re}\nu = \operatorname{Re}\mu = \frac{1}{2}$ *and let, for some positive* $\delta < \frac{1}{4}$, *the variables* s, ρ *satisfy* $\frac{5}{4} < \operatorname{Re} s, \operatorname{Re}\rho < \frac{5}{4}+\delta$. *Let* $\varphi : [0,\infty) \to \mathbb{C}$ *have the Mellin transform* $\hat{\varphi}(u)$ *such that* $\hat{\varphi}(u)$ *is regular for* $-\frac{3}{2} - 2\delta \leqslant \operatorname{Re} u \leqslant 2$. *Then we have*

$$L^{(+)}\left(\begin{matrix}s, & \nu \\ \rho, & \mu\end{matrix}\middle| \varphi\right) = L^{(+)}\left(\begin{matrix}\rho, & \nu \\ s, & \mu\end{matrix}\middle| \varphi_0\right) L^{(-)}\left(\begin{matrix}\rho, & \nu \\ s, & \mu\end{matrix}\middle| \varphi_1\right) + \tag{3.11}$$

$$+\frac{2(4\pi)^{2s-1}}{\zeta(2s)}\left(\frac{\zeta(2\nu)}{(4\pi)^{2\nu}}z(\rho+\nu;s,\mu)\hat{\varphi}(2\nu+1-2s)+\right.$$
$$\left.+\frac{\zeta(2-2\nu)}{(4\pi)^{2-2\nu}}z(\rho+1-\nu;s,\mu)\hat{\varphi}(3-2\nu-2s)\right)+$$
$$+\frac{2(4\pi)^{2\rho-1}}{\zeta(2\rho)}\left(\frac{\zeta(2(\mu)}{(4\pi)^{2\mu}}z(s+\mu;\rho,\nu)\hat{\varphi}(2\mu+1-2\rho)+\right.$$
$$\left.+\frac{\zeta(2-2\mu)}{(4\pi)^{2-2\mu}}z(s+1-\mu;\rho,\nu)\hat{\varphi}(3-2\mu-2rho)\right),$$

where Φ_0, Φ_1 are defined by the following integral transformations

$$\Phi_0(x)\equiv\Phi_0(x;s,\nu;\rho,\mu)=x^{2s+2\rho-2}\iint\limits_0^\infty(k_0(\xi,\nu)k_0(\eta,\mu)+ \tag{3.12}$$
$$k_1(\xi,\nu)k_1(\eta,\mu))\varphi(\xi\eta/x)\xi^{1-2s}\eta^{1-2\rho}d\xi d\eta$$

$$\Phi_1(x)\equiv\Phi_1(x;s,\nu;\rho,\mu)=x^{2s+2\rho-2}\iint\limits_0^\infty(k_0(\xi,\nu)k_0(\eta,\mu)+ \tag{3.13}$$
$$k_1(\xi,\nu)k_0(\eta,\mu))\varphi(\xi\eta/x)\xi^{1-2s}\eta^{1-2\rho}d\xi d\eta.$$

Of course, it is the same as what we have in (3.5). If $\operatorname{Re}\rho>\operatorname{Re}s$, then in the first term on the right side of (3.5), one has the sum

$$\sum_{c\geqslant 1}\frac{1}{c^{2s}}\sum_{n=1}^{\infty}\frac{\tau_\mu(n)}{n^{\rho-s+\nu+(1/2)}}S(0,n;c)= \tag{3.14}$$
$$=\frac{1}{\zeta(2s)}\sum_{n=1}^{\infty}\frac{\tau_\mu(n)\sigma_{1-2s}(n)}{n^{\rho-s+\nu+(1/2)}}=\frac{z(\rho+\nu;s,\mu)}{\zeta(2s)}$$

83 On the right side, we have a meromorphic function of ρ in the half-plane $\operatorname{Re}\rho>\frac{1}{2}$; so this equality holds for the analytic continuation of the initial sum

$$\sum_{c\geqslant 1}\frac{1}{c^{2s}}\sum_{(d,c)=1}\mathscr{L}_\mu\left(\rho-s+\nu+\frac{1}{2},\frac{d}{c}\right) \tag{3.15}$$

if we can be sure that this function is meromorphic not only for $\mathrm{Re}\,\rho >$ $\mathrm{Re}\,s$. It is sufficient for this to know that $\mathscr{L}_\mu(w, \frac{d}{c})$ as a function of c is bounded in the mean when $\mathrm{Re}\,w > \frac{1}{2}$ (except at the poles). But this fact is a consequence of the Bombieri-Vinogradov inequality which asserts that

$$\sum_{1\leqslant c\leqslant M}\sum_{(d,c)=1}\left|\sum^{P+Q})_{n=P}b(n)e(\frac{nd}{c})\right|^2 \ll \max(Q, M^2)\sum_{n=P}^{P+Q}|b(n)|^2 \qquad (3.16)$$

for an arbitrary sequence of complex numbers $b(n)$.

Now one can check that relations (3.12) - (3.13) and (3.7) - (3.8) are idential. We have the tabular integrals

$$\int_0^\infty k_0(x,v)x^{w-1}dx = \gamma\left(\frac{w}{2}, v\right)\cos\left(\frac{\pi w}{2}\right),\ 0 < \mathrm{Re}\,w < \frac{3}{2}, \qquad (3.17)$$

$$\int_0^\infty k_1(x,v)x^{w-1}dx = \gamma\left(\frac{w}{2}, v\right)\sin(\pi v), \mathrm{Re}\,w > 0, \qquad (3.18)$$

After writing φ in (3.12)-(3.13) as the Mellin integral,

$$\varphi\left(\frac{\xi\eta}{x}\right) = \frac{1}{i\pi}\int\hat{\varphi}(2u)\left(\frac{x}{\xi\eta}\right)^{2u}du,$$

we shall come to an absolutely convergent triple integral if

$$\max\left(\frac{3}{4} - \mathrm{Re}\,s, \frac{3}{4} - \mathrm{Re}\,\rho\right) < \mathrm{Re}\,u < \min(1 - \mathrm{Re}\,s, 1 - \mathrm{Re}\,\rho).$$

Hence there is a non-empty strip where we can integrate in any order; this gives our relations for Φ_0 and Φ_1.

2.3 The main functional equation: the preparations.

For the given function $h(r)$ and the sequence $h^* := \{h_{2k-1}\}_{k-1}^\infty$, we shall consider the function

$$Z\left(\begin{matrix} s, & v \\ \rho, & \mu\end{matrix}\middle| h\right) = Z^{\mathrm{dis}}\left(\begin{matrix} s, & v \\ \rho, & \mu\end{matrix}\middle| h\right) + Z_0^{\mathrm{con}}\left(\begin{matrix} s, & v \\ \rho, & \mu\end{matrix}\middle| h\right) + Z^{\mathrm{cusp}}\left(\begin{matrix} s, & v \\ \rho, & \mu\end{matrix}\middle| h^*\right). \qquad (4.1)$$

84 When $\operatorname{Re} s, \operatorname{Re}\rho > 1$, this function is equal to

$$\sum_{n,m=1}^{\infty} \frac{\tau_\nu(n)\tau_\mu(m)}{n^s m^\rho}\{K_{n,m}^{(+)}(\varphi) + \frac{\delta_{n,m}}{\pi^2}\int_{-\infty}^{\infty} r\mathrm{th}(\pi r)h(r)dr\} = \tag{4.2}$$

$$= L^{(+)}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle|\varphi\right) + \frac{1}{\pi^2} z(s+\rho;\nu,\mu)\int_{-\infty}^{\infty} r\mathrm{th}(\pi r)h(r)dr$$

where φ corresponds to h and h^* in the sense of Theorems 1 and 4.

Our intention must be clear now; we shall use the first functional equation for $L^{(+)}$ and after this, the analytic continuation of both sides will be carried out.

Firstly, it is convenient to write the analytic continuation for the function Z_0^{con}. Let us denote by Z^{con} the integral in which (under the usual conditions $\operatorname{Re}\nu = \operatorname{Re}\mu = \frac{1}{2}$) we have $\operatorname{Re} s < 1$, $\operatorname{Re}\rho < 1$. Then Z_0^{con} and Z^{con} are connected by the following relation.

Proposition 6. *Let h be a regular function on the sufficiently wide strip $|\operatorname{Im} r| \leqslant \Delta$, $\Delta > \frac{1}{2}$. Then for $\operatorname{Re} s > 1$, $\operatorname{Re}\rho < 1$, the meromorphic continuation of $Z_0^{con}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle| h\right)$ is given by the equality.*

$$Z_0^{\mathrm{con}}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle| h\right) - Z^{\mathrm{con}}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle| h\right) + \tag{4.3}$$
$$+ \frac{\zeta(2s-1)}{\zeta(2s)}\left\{\frac{\zeta(2\nu)z(\rho;\mu,1-s+\nu)}{\zeta(2+2\nu-2s)} h(i(s-\nu-\frac{1}{2}))\right\} +$$
$$+ \frac{\zeta(2-2\nu)z(\rho;\mu,2-s-\nu)}{\zeta(4-2\nu-2s)} h(i(s+\nu-\frac{3}{2})) +$$
$$+ \frac{\zeta(2\rho-1)}{\zeta(2\rho)}\left\{\frac{\zeta(2\mu)z(s;\nu,1-\rho+\mu)}{\zeta(z+2\mu-2\rho)} h(i(\rho-\mu-\frac{1}{2})) +\right.$$
$$\left. + \frac{\zeta(2-2\mu)z(s;\nu,2-\rho-\mu)}{\zeta(4-2\mu-2\rho)} h(i(\rho+\mu-\frac{3}{2}))\right\}.$$

Really Z_0^{con} is a Cauchy integral, because ζ has only a simple pole.

so the poles of $z(s;\nu,\frac{1}{2}+ir)$ are the points r_j, $1 \leqslant j \leqslant 4$, with

$$ir_1 = \frac{1}{2} - s + \nu,\ ir_2 = \frac{3}{2} - s - \nu,\ r_3 = -r_1,\ r_4 = -r_2.$$

When $\mathrm{Re}\, s > 1$ the points r_1, r_2 are lying above the real axis and if $\mathrm{Re}\, s < 1$, they are below the same. Now one can deform the path of integration (see picture; the deformation must be so small that the functions $\zeta(1+\pm 2ir)$ have no zeros inside the lines; it is possible, since the **85**
Riemann zeta-function has no zeros on the line $\mathrm{Re}\, s = 1$) and the desired conclusion is the result of the direct calculation of the residues.

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The next step is the representation of the functions $L^{(\pm)}\left(\begin{matrix}\rho, & \nu \\ s, & \mu\end{matrix}\middle|\Phi_j\right)$ as a bilinear form in the eigenvalues of the Hecke operators. For this, we need to consider the integral transforms of Theorems (4) and (1). The situation now is the following : for a given h, we define φ by the transformation (1.27), or what is the same, by the equality

$$\varphi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} k_0(x,\frac{1}{2}+iu)u\,\mathrm{th}(\pi u)h(u)du, \tag{4.4}$$

and thereafter, we should calculate the integrals Φ_0 and Φ_1 in (3.12) and (3.13) with this φ and finally the integral transformations

$$h_0(r) \equiv h_0(r;s,\nu;\rho,\mu) = \pi\int_0^{\infty} k_0(x,\frac{1}{2}+ir)\Phi_0(x)\frac{dx}{x}, \tag{4.5}$$

$$h_1(r) \equiv h_1(r;s,\nu;\rho,\mu) = \pi\int_0^{\infty} k_1(x,\frac{1}{2}+ir)\Phi_1(x)\frac{dx}{x}. \tag{4.6}$$

In order to obtain an asymptotic estimate, it is preferable to diminish the length of the sequence of these integral transformations; we give the results in the following

Proposition 7. *Assume that the function $h(u)$ is even and regular in the strip $|\operatorname{Im} u| \leqslant \frac{3}{2}$ and h has zeros at $u = \pm\frac{i}{2}$. Let $|h|$ decrease as $O(|u|^{-B})$ for some $B > 4$ when $|u| \to \infty$ with $|\operatorname{Im} u| \leqslant \frac{3}{2}$. Then the function h_0 is given by the integral transform*

$$h_0(r) = \frac{2}{\pi^2} \int\limits_{-\infty}^{\infty} B_0(r; u; \rho, v, \mu; s) u \operatorname{th}(\pi u) h(u) du \tag{4.7}$$

where, with the notation from (1.78), (1.79)*, we have, for* $\operatorname{Re} v = \operatorname{Re} \mu = \frac{1}{2}$, $\frac{1}{2} \leqslant \operatorname{Re} s$, $\operatorname{Re} \rho < 1$

$$B_0(r, u; \rho, v, \mu; s) = \int\limits_0^{\infty} (A_{00}(r, \xi, \rho, v) A_{00}(u, \frac{1}{\xi}; 1 - \rho, \mu) + \tag{4.8}$$
$$A_{01}(r, \xi; \rho, v) A_{01}(u, \frac{1}{\xi}; 1 - \rho, \mu)) \xi^{2\rho - 2s - 1} d\xi$$

86 and here

$$B_0(r, u; \rho, v, \mu; s) = B_0(r, u; s, \mu, v; \rho). \tag{4.9}$$

Proposition 8. *Under the same conditions*

$$h_1(r) = \frac{2}{\pi^2} \int\limits_{-\infty}^{\infty} B_1(r, u; s, \mu, v; \rho) u \operatorname{th}(\pi u) h(u) du \tag{4.10}$$

where, with the notation (1.78), (1.80)*,*

$$B_1(r, u; s, \mu, v; \rho) = B_1(r, u; \rho, v, \mu; s) = \tag{4.11}$$
$$= \int\limits_0^{\infty} (A_{10}(r, \xi, s, \mu) A_{00}(u, \frac{1}{\xi}; 1 - s, v) +$$
$$+ A_{11}(r, \xi, s, \mu) A_{01}(u, \frac{1}{\xi}; 1 - s, v)) \xi^{2s - 2\rho - 1} d\xi.$$

Both the propositions result from term-by-term integration in the corresponding multiple integrals; it is sufficient to consider the first relation (4.7).

First of all, the function φ in (4.4), for our case, is $O(x^3)$ when $x \to 0$ and $O(x^{-(1/2)})$ for $x \to +\infty$. Furthermore, the Mellin transform of this function, which is defined by the integral

$$\hat{\varphi}(w) := \int_0^\infty \varphi(x) x^{w-1} dx = \tag{4.12}$$

$$= \frac{2}{\pi} \cos\left(\frac{\pi w}{2}\right) \int_{-\infty}^{\infty} \gamma\left(\frac{w}{2}, \frac{1}{2} + iu\right) u \,\mathrm{th}(\pi u) h(u) du$$

is regular for $\mathrm{Re}\, w > -3$ and $|\hat{\varphi}(w)|$ may be estimated as $O(|w|^{\mathrm{Re}\, w - 1})$ when $|w| \to \infty$ and $\mathrm{Re}\, w$ is fixed.

For this reason, the integrals (3.7) - (3.8) are absolutely convergent if $\alpha_0 < \mathrm{Re}(s+\rho) - 1$. At the same time, both the integrals with $k_j(x, \frac{1}{2} + ir) \times x^{2s+2\rho+2u-3}$ for $j = 0, 1$ are absolutely convergent for $1 - \mathrm{Re}(s+\rho) < \mathrm{Re}\, u < \frac{5}{4} - \mathrm{Re}(s+\rho)$. If $\mathrm{Re}(s+\rho) > \frac{1}{2}$, we can choose α_0 in such a manner that the term-by-term integration would be valid in the integrals which will arise on replacing Φ_j in (4.5) and (4.6) by the representations (3.7) and (3.8). In this way, we have

$$h_0(r) = \frac{2i}{\pi^2} \int_{-\infty}^{\infty} u \,\mathrm{th}\, (\pi u) h(u) \times \tag{4.13}$$

$$\times \int_{(\alpha_0)} \gamma(s+\rho+w-1, \frac{1}{2} + ir) \gamma(1-s-w, \nu) \gamma(1-\rho-w, \mu) \gamma(w, \frac{1}{2} + iu) \times$$
$$\times \cos(\pi w) \sin \pi(s+\rho+w) (\cos \pi(s+w) \sin(\pi\nu) +$$
$$+ \cos \pi(\rho+w) \sin(\pi\mu)) dw du.$$

After this, it is sufficient to check that two representations are identical **87**
for $\mathrm{Re}\, s, \mathrm{Re}\, \rho < 1$; but this results immediately from the explicit formulae for the Mellin transforms of the kernels k_j and the definitions of the kernels $A_{k,l}$.

To finish the preparations, it remains to write the coefficients in the sum over the regular cusps for the sum of Kloosterman sums with weight function Φ_0 and, finally, to consider the analytic continuation of the function $Z_0^{\mathrm{con}}\left(\begin{matrix} \rho, & \nu \\ s, & \mu \end{matrix} \middle| h_0 + h_1\right)$.

The first is not difficult; it is sufficient to do the formal substitution $r = i(k - \frac{1}{2})$ in the expression for $h_0(r)$ and to note the well-known limiting case

$$\lim_{c\to -2k} (\Gamma(c))^{-1} F(a,b;c;z) = \frac{\Gamma(a+2k+1)\Gamma(b+2k+1)}{\Gamma(2k+2)} \times \tag{4.14}$$
$$\times z^{2k+1} F(a+2k+1, b+2k+1; 2k+2; z)$$

which holds for a positive integer k.

The analytic continuation is given by the same kind of relation as in (4.3); so it is sufficient to calculate the values $(h_0+h_1)(i(s-1) \pm (\mu - \frac{1}{2}))$ and $(h_0+h_1)(i(\rho-1) \pm (\nu - \frac{1}{2}))$.

Proposition 9. *Let h be the same function as in Proposition* (7)*; then, for $\frac{1}{2} < \operatorname{Re} s$, $\operatorname{Re}\rho < 1$, $\operatorname{Re}\nu = \operatorname{Re}\mu = \frac{1}{2}$, we have*

$$(h_0+h_1)(i(\rho - \nu - \frac{1}{2})) = \frac{2}{\pi^2} \int_{-\infty}^{\infty} u \operatorname{th}(\pi u) h(u) \tilde{B}_0(u;\rho,\mu,s-\nu) du \tag{4.15}$$

where

$$\tilde{B}_0(u;\rho,\mu,w) = (2\pi)^{1-2\rho} \sin(\pi\rho)\Gamma(2\rho-1)\times \tag{4.16}$$
$$\times \int_0^{\infty} (|1-\xi^2|^{1-2\rho} A_{00}(u, 1/\xi; 1-\rho,\mu) +$$
$$+ (1+\xi^2)^{1-2\rho} A_{01}(u.1/\xi; 1-\rho,\mu))\xi^{2\rho-2w-1} d\xi.$$

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This relation is a consequence of the explicit formulae for the kernels $A_{k,l}$. If $r = i(\rho - \nu - \frac{1}{2})$, then, in these formulae, we have

$$a = 2\nu, b = 1, c = 2 - 2\rho + 2\nu; a' = 2\rho - 1, b' = 2\rho - 2\nu, c' = 2\rho - 2\nu.$$

Now, we have, for the special case of the hypergeometric functions, $F(0,b;c;z) \equiv 1$ and $F(a,b;b;z) = (1-z)^{-a}$ and as a result, we have the following equalities

$$A_{00}(i(\rho-\nu-\frac{1}{2}),\xi;\rho,\nu) + A_{10}(i(\rho-\nu-\frac{1}{2}),\xi;\rho,\nu) \tag{4.17}$$

$$= (2\pi)^{1-2\rho} \sin(\pi\rho)\Gamma(2\rho-1)|\xi^2-1|^{1-2\rho}\xi^{2\nu}$$

and

$$A_{01}(i(\rho-\nu-\frac{1}{2}),\xi;\rho,\nu)+A_{11}(i(\rho-\nu-\frac{1}{2}),\xi;\rho,\nu)$$
$$= (2\pi)^{1-2\rho} \sin(\pi\rho)\Gamma(2\rho-1)|\xi^2+1|^{1-2p}\xi^{2\nu};$$

our proposition follows from these expressions.

2.4 The main functional equation and the specialization.

Theorem 15. *Assume that the even function* $h(r)$ *is regular in the strip* $|\operatorname{Im} r| \leqslant \frac{3}{2}$, *decreases as* $O(|r|^{-B})$, $B > 4$, *when* $r \to \infty$ *in this strip and has zeros at* $r = \pm\frac{i}{2}$. *Then we have, for* $\operatorname{Re}\nu = \operatorname{Re}\mu = \frac{1}{2}$, $\frac{1}{2} < \operatorname{Re} s$, $\operatorname{Re}\rho < 1$, *the following functional equation*

$$Z^{\text{dix}}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle| h\right) + Z^{\text{con}}\left(\begin{matrix} s, & \nu \\ \rho, & \mu \end{matrix}\middle| h\right) = \tag{5.1}$$

$$= \frac{z(s+\rho;\nu,\mu)}{\pi^2}\int_{-\infty}^{\infty} r\,\text{th}(\pi r)(h(r)-h_0(r;s,\nu;\rho,\mu))dr+$$

$$+Z^{\text{dis}}\left(\begin{matrix} \rho, & \nu \\ s, & \mu \end{matrix}\middle| h_0\right) + \tilde{Z}^{\text{dis}}\left(\begin{matrix} \rho, & \nu \\ s, & \mu \end{matrix}\middle| h_1\right)+$$

$$+Z^{\text{con}}\left(\begin{matrix} \rho, & \nu \\ s, & \mu \end{matrix}\middle| h_0+h_1\right) + Z^{\text{cusp}}\left(\begin{matrix} \rho, & \nu \\ s, & \mu \end{matrix}\middle| h^*\right).$$

$$+\Phi_\varphi(s,\nu;\rho,\mu)+\Phi_\varphi(s,1-\nu;\rho,\mu)+\Phi_\varphi(\rho,\mu;s,\nu)+$$
$$\Phi_\varphi(\rho,1-\mu;s,\nu)+\vartheta_h(s,\nu;\rho,\mu)+\vartheta_h(s,1-\nu;\rho,\mu)+$$
$$+\vartheta_h(\rho,\mu;s,\nu)+\vartheta_h(\rho,1-\mu;s,\nu)+\vartheta_{h_0+h_1}(s,\nu;\rho,\mu)+$$
$$+\vartheta_{h_0+h_1}(s,1-\nu;\rho,\mu)+\vartheta_{h_0+h_1}(\rho,\mu;s,\nu)+\varphi_{h_0+h_1}(\rho,1-\mu;s,\nu)$$

where h_0 *and* h_1 *are defined in terms of* h *by* (4.7) *and* (4.10), *the se-* **89**
quence h^* *is the result of the formal substitution of* $i(k-\frac{1}{2})$ *in place of* r

in the expression for $h_0(r)$ *and*

$$\vartheta_n(s,v;\rho,\mu)=\frac{\zeta(2s-1)\zeta(2v)}{\zeta(2\rho)\zeta(2+2v-2s)}z(s;\mu,1-\rho+v)h(i(\rho-v-\frac{1}{2})) \tag{5.2}$$

$$\Phi_\varphi(s,v;\rho,\mu)=2\frac{(4\pi)^{2s-2v-1}\zeta(2v)}{\zeta(2s)}z(\rho+v;s,\mu)\hat{\varphi}(2v+1-2s) \tag{5.3}$$

with φ *from* (4.4).

This functional equation follows simply on putting together the preceding considerations.

Of particular interest is the special case when s,v,ρ,μ are chosen as in (2.16) and the function h is positive for real r and decreases very rapidly; namely, we choose

$$h(r)=(r^2+\frac{1}{4})^2(r^2+\frac{9}{4})(r^2+\frac{25}{4})e^{-\alpha r^2},\ \alpha>0. \tag{5.4}$$

Now we have only one variable t and, for brevity, we shall introduce new notation. Let, for $s=\mu=\frac{1}{2}$, $\rho=\mu=\frac{1}{2}+it$ and further, for the function h from (5.4), let

$$Z_c(t)=\zeta(2s)\zeta(2\rho)Z^{\text{con}}\left(\begin{matrix} s, & v \\ \rho, & \mu\end{matrix}\middle| h\right) \tag{5.5}$$

Then we have

$$Z_c(t)= \tag{5.6}$$

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\zeta^3(\frac{1}{2}+it-ir)\zeta^3(\frac{1}{2}+it-ir)\zeta(\frac{1}{2}-it+ir)\zeta(\frac{1}{2}-it-ir)}{|\zeta(1+2ir)|^2}h(r)dr,$$

where the main contribution is determined by the interval $|r|\ll(\log t)^{1/2}$ (we assume that t is a positive large number). For this reason, we can write

$$\zeta(\frac{1}{2}+it+ir)\zeta(\frac{1}{2}+it-ir)=\zeta(\frac{1}{2}-it+ir)\zeta(\frac{1}{2}-it-ir)\chi(t,r), \tag{5.7}$$

where

$$\chi(t,r) = \pi^{2it}\frac{\Gamma(\frac{1}{4}-\frac{it}{2}+\frac{ir}{2})\Gamma(\frac{1}{4}-\frac{it}{2}-\frac{ir}{2})}{\Gamma(\frac{1}{4}+\frac{it}{2}+\frac{ir}{2})\Gamma(\frac{1}{4}+\frac{it}{2}-\frac{ir}{2})} \tag{5.8}$$

$$= i\left(\frac{2\pi}{t}\right)^{2it} e^{2it}\left(1+O\left(\frac{r^2+1}{t}\right)\right).$$

Now we have 90

$$\overline{\chi(t,0)}Z_c(t) = \tag{5.9}$$

$$\frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\frac{|\zeta^4(\frac{1}{2}+it+ir)\zeta^4(\frac{1}{2}+it-ir)|}{|\zeta(1+2ir)|^2}(1+O\left(\frac{1+r^2}{t}\right))h(r)dt$$

and the main term in the integrand is positive. If we estimate the integral

$$\int\limits_0^\infty \omega_T(t)\overline{\chi}(t,0)Z_c(t)dt, T\to+\infty, \tag{5.10}$$

then the desired estimate for the eighth moment will be a consequence of the following simple statement.

Proposition 10. *For $T\to+\infty$, we have, with a fixed positive integer $k\geqslant 1$ and for every fixed $\delta>0$,*

$$\int\limits_T^{2T}|\zeta'(\frac{1}{2}+it)|^{2k}dt \ll (\log T)^{4k}\int\limits_{T(1-\delta)}^{2T(1+\delta)}|\zeta(\frac{1}{2}+it)|^{2k}dt \tag{5.11}$$

To prove this inequality, one can see firstly that

$$|\zeta'(\frac{1}{2}+ut)| \ll \log T\max_{x\geqslant 1}\left|\sum_{n\leqslant t}\frac{1}{n^s}\right|,\ s=\frac{1}{2}+it \tag{5.12}$$

and we have with $4M=T^{2/3}$ and $\epsilon=(\log T)^{-1}$

$$\sum_{n\leqslant x}\frac{1}{n^2} = \frac{1}{2\pi i}\int\limits_{\epsilon-iM}^{\epsilon+iM}\zeta(s+w)x^w\frac{dw}{w}+O(1), x\leqslant t. \tag{5.13}$$

As a consequence of (5.12), (5.13) and Holder's inequality, we have

$$\int\limits_{T}^{2T} |\zeta'(\frac{1}{2}+it)|^{2k} dt \ll (\log T)^{2k} \int\limits_{T-M}^{2T+M} |\zeta(\frac{1}{2}+\epsilon+it)|^{2k} dt \left(\int\limits_{-M}^{M} \frac{d\eta}{|\epsilon+i\eta|} \right)^{2k} \tag{5.14}$$

$$\ll (\log T)^{4k} \int\limits_{T-M}^{2T+M} |\zeta(\frac{1}{2}+\epsilon+it)|^{2k} dt$$

$$\ll (\log T)^{4k} \int\limits_{T-M}^{2T+M} |\zeta(\frac{1}{2}+it)|^{2k} dt, M = T^{2/3},$$

91 since the last integral is non-increasing as a function of ϵ.

Now, for every positive $\epsilon \in (0,1)$ we have

$$|\overline{\chi(t,0)} Z_c(t)| \gg \int\limits_{-\epsilon}^{\epsilon} |\zeta^4(\frac{1}{2}+it+ir)\zeta^4(\frac{1}{2}+it-ir)| r^2 h(r) dr$$

$$\gg \epsilon^3 |\zeta^8(\frac{1}{2}+it)| - \int\limits_{-\epsilon}^{\epsilon} (\epsilon - r) r^2 h(r) \frac{\partial}{\partial r} |\zeta^4(\frac{1}{2}+it+ir)\zeta^4(\frac{1}{2}+it-ir)| dr, \tag{5.15}$$

so that

$$\int \omega_T(t) \overline{\chi(t,0)} Z_c(t) dt \gg \epsilon^3 \int |\zeta^8(\frac{1}{2}+it)| \omega_T(t) dt - \tag{5.16}$$
$$-\epsilon^4 (\int \omega_t(t) |\zeta^8(\frac{1}{2}+it)| dt)^{7/8} (\int (\omega_T(t) |\zeta'(\frac{1}{2}+it)|^8 dt)^{1/8}$$

If $\epsilon = A(\log T)^{-2}$ with some sufficiently small constant A, then the last term of the right side of (5.16) is of a lower order than the first term and so we have the inequality

$$\int \omega_T(t) |\zeta^\infty(\frac{1}{2}+it)| dt \ll (\log T)^6 |\int \omega_T(t) \overline{\chi(t,0)} Z_c(t) dt|. \tag{5.17}$$

For this reason, an estimate for the integral in (5.10) is sufficient for our purpose.

2.5 The functions h_j: the non-essential terms.

To carry out the non-trivial integration over t, we must know the asymptotic behaviour of the functions h_0 and h_1 in the special case (2.16), where $\rho = \nu = \frac{1}{2} + it$ with some large positive t. The plan is simple: instead of the hypergeometric functions we shall use the corresponding asymptotic formulae (these will be written by the asymptotic integration of the differential equation with a large parameter) and after this, using the saddle-point method, we shal integrate over ξ.

2.5.1 The integral with A_{01}in (4.8) 92

We have (see (1.79))

$$A_{01}\left(u, \frac{1}{\xi}; 1-\rho, \frac{1}{2}\right) = \frac{i(\xi/2\pi)^{1-2}\rho}{2\mathrm{sh}(\pi u)}\xi^{2iu}\times \tag{6.1}$$

$$\frac{\Gamma^2(1-\rho+iu)}{\Gamma(1+2iu)}F(1-\rho+iu, 1-\rho+iu; 1+2iu; -\xi^2)+$$

$$+\{\text{the same with } u \to -u\},$$

$$A_{01}(r,\xi,\rho,\rho) = \frac{i(2\pi\xi)^{1-2\rho}\sin(\pi\rho)\Gamma(2\rho-\frac{1}{2}+ir)\Gamma(\frac{1}{2}+ir)}{2\xi^{2ir}\mathrm{sh}(\pi\xi)\Gamma(1+2ir)}\times \tag{6.2}$$

$$F\left(2\rho-\frac{1}{2}+ir, \frac{1}{2}+ir; 1+2ir; -\frac{1}{\xi^2}\right)+\{\text{the same with } r \to -r\},$$

Later, we shall use the following method of considering our integrals. It is well-known that the function

$$w = z^{c/2}(1 \mp z)^{(a+b+1-c)/2}F(a,b;c;\pm z)$$

is a solution of the differential euqation

$$w'' + \left(\frac{c(2-c)}{4z^2} + \frac{1-(a+b-c)^2}{4(1\mp z)^2} \pm \frac{\frac{1}{2}c(a+b+1-c)-ab}{z(1\mp z)}w = 0.\right) \tag{6.3}$$

As a consequence (using an appropriate transformation of the varialbe), we see that the function

$$W(\eta) = (tg\eta/2)^{(1/2)+2iu}(\cos\eta/2)^{2\rho-2}\times \tag{6.4}$$

$$F(1-\rho+iu, 1-\rho+iu; 1+2iu; -tg^2\eta/2)$$

satisfies the differential equation (for $\rho = \frac{1}{2} + it$):

$$\frac{d^2w}{d\eta^2} + \left(-t^2 + \frac{u^2}{\sin^2\eta/2} + \frac{1}{4\sin^2\eta}\right)W = 0. \tag{6.5}$$

Hence, for large t and $0 < \eta < \pi - \delta$ for any fixed $\delta > 0$, we have

$$w = \sqrt{\eta/2}I_{2iu}(t\eta)\frac{\Gamma(1+2iu)}{t^{2iu}}(1 + O(\frac{1}{t})). \tag{6.6}$$

This consequence is the distinctive feature of our method of considereing the asymptotic behaviour for all hypergeometric functions here. This method is based on the principle: "neighbouring equations have
93 neighbouring solutions"; the method of estimation for the corresponding closeness is routine today (see, for example, [5], where the estimates are written for similar equations).

Now the following statment would be obvious for the reader: the contribution of the term with the kernels A_{01} from (4.8) is negligible for large values of t. Indeed,

$$|t^{-2iu}\Gamma^2(1-\rho+it)| \ll e^{-\pi t}$$

and the part of the integral with $\xi \leqslant At(\log t)^{-1}$ with some (small) fixed A is small. But, for large ξ, we have an additional resource. We shall assume that parameter α in the definition of the initial function $h(r)$ will be small; then we can move the path of the integration over u in (4.7) and (4.10) and the factor of the type ξ^{2iu} for $\xi \gg t(\log t)^{-1}$ and $\operatorname{Im} u \geqslant +\Delta$ will give $O(t^{2\Delta}(\log t)^{2\Delta})$. Here Δ is defined by the width of the strip where $(\operatorname{ch}(\pi u))^{-1}h(u)$ is regular; for the function (5.4), one can choose $\Delta = \frac{7}{2}$.

For the same reason, one can reject the term with $A_{01}(u, \frac{1}{\xi}; 1-\rho.\frac{1}{2})$ in the expression (4.10), (4.11) for the function h_1.

Furthermore, we have

$$A_{10}(r, \xi, \rho, \rho) = \frac{\Gamma(2\rho - \frac{1}{2} + ir)\Gamma(2\rho - \frac{1}{2} - ir)}{\Gamma(2\rho)} \times \tag{6.7}$$

$$\sin(\pi\rho)\xi^{2\rho}F(2\rho-\frac{1}{2}+ir,2\rho-\frac{1}{2}-ir;2\rho;1-\xi^2)$$

and the hypergeometric function with these parameters is a solution of the differential equation

$$w''+\left(\frac{\rho(1-\rho)}{z^2(1\pm z)^2}\pm\frac{r^2+\frac{1}{4}}{z(1\pm z)}\right)w=0 \tag{6.8}$$

if $w=z^\rho(1\pm z)^\rho F(2\rho-\frac{1}{2}+ir,2\rho-\frac{1}{2}-ir;2\rho;\mp z)$. For the upper sign (which corresponds to $\xi>1$), all solutions are oscillating; at the same time, we have

$$\left|\frac{\Gamma(2\rho-\frac{1}{2}+ir)\Gamma(2\rho-\frac{1}{2}-ir)}{\Gamma(2\rho)}\sin(\pi\rho)\right|\ll \tag{6.9}$$
$$\ll\exp(-\frac{\pi}{2}(|2t+r|+|2t-r|-3t)$$
$$\ll\exp(-\frac{\pi}{2}(\max(t,2t-3t))$$

So, if $\xi\geqslant 1$, the kernel (6.7) is exponentially small. For the case $\xi<1$ (which corresponds in (6.8) to the case $z\in(0,1)$ and the sign "minus"), the solution (6.8) does not exceed $\exp(r\arcsin\sqrt{1-\xi^2})$ and so we have the factor $e^{\pi r/2}$ for $\xi=0$ only. But the contribution of the interval with small ξ, $\xi\ll t^{-1}\log t$, is small (for the same reason - one 94
can move the path of the integration over u and to render the factor ξ^{2iu} small).

2.6 The integral with A_{00}.

2.6.1 The explicit form.

The unique essential term is the first integral in (4.8) and we shall consider this term in greater detail; in passing, we shall give some examples of the asymptotic integration of the differential equations with a large parameter.

First of all, we shall write the result of substituting the special values for our parameters. Let us introduce the notation

$$v \equiv v(z;\rho,r) = |z|^{1-\rho}(1+z)^{\rho}F(\frac{1}{2}+ir,\frac{1}{2}-ir;2-2\rho;-z) \tag{7.1}$$

and

$$w = w(z;\rho,u) = |z|^{\rho}(1+z)^{(1/2)+iu}F(\rho+iu,\rho+iu;2\rho;-z) \tag{7.2}$$

(here z is a real variable and $z \geqslant -1$). Then we have, for all $\xi > 0$,

$$\begin{aligned}(2\pi)^{2\rho-1}A_{00}(r,\xi;\rho,\rho) = \\ = \sin(\pi\rho)\Gamma(2\rho-1)(v(\xi^2-1;\rho,r)+Av(\xi^2-1;1-\rho,r))\end{aligned} \tag{7.3}$$

where

$$A = \frac{\Gamma(2\rho-\frac{1}{2}+ir)\Gamma(2\rho-\frac{1}{2}-ir)}{\Gamma(2\rho)\Gamma(2\rho b-1)}\frac{\mathrm{ch}(\pi r)}{\sin(2\pi\rho)}. \tag{7.4}$$

This relation is a consequence of (1.78) and the simple relationship $F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z)$. The representation (7.3) is very convenient for $r < 2t$, because, in this case we have $|A| \ll e^{-\pi(2t-r)}$. Hence with exponential accuracy, we can retain just the first term on the right side (7.3) for $r \leqslant 2t(1-\delta)$ with some fixed (small) $\delta > 0$.

The representation (1.78) will be used for the other kernel too; here, we shall use the relation $F(a,b;c,z) = z^{-a}F(a,c-b;c;\frac{z}{z-1})$ and as a consequence we shall obtain the equality

$$\begin{aligned}(2\pi)^{1-2\rho}A_{00}(u,\frac{1}{\xi};1-\rho,\frac{1}{2}) = \\ = \sin(\pi\rho)\Gamma(1-2\rho)(w(\xi^2-1;\rho,u)-Bw(\xi^2-1;1-\rho,u))\end{aligned} \tag{7.5}$$

where

$$B = \frac{\mathrm{ch}^2\pi u+\cos^2\pi\rho}{\pi\sin(2\pi\rho)}\cdot\frac{\Gamma^2(1-\rho+iu)\Gamma^2(1-\rho-iu)}{\Gamma(2-2\rho)\Gamma(1-2\rho)} \tag{7.6}$$

Now, after the change of the variable of integration $\xi^2 - 1 \mapsto z$, we have a representation for the essential part of the function $(h_0 + h_1)$:

$$h^{(0)}(r, u; t) = \int_0^\infty A_{00}(r, \xi; \rho, \rho) A_{00}(u, \frac{1}{\xi}; 1 - \rho, \frac{1}{2}) \xi^{2\rho - 2} d\xi \tag{7.7}$$

$$= \frac{\pi}{8} \frac{tg(\pi\rho)}{2\rho - 1} \int_{-1}^\infty (v(z; \rho, r) + Av(z; 1 - \rho, r))(w(z; \rho, u) -$$

$$-Bw(z; 1 - \rho, u)) \frac{dz}{|z|(1 + z)^{3/2}}$$

and for $r \leqslant 2(1 - \delta)t$ with fixed $\delta > 0$, we can reject the term with A. 95

If $r > 2t$, then the representation (1.78) is not convenient: the bounded function is expressed here as a linear combination of exponentially large terms. The relations (1.76) and (1.77) are more suitable in this case (One can see that (1.78) is the consequence of the preceding equalities and the Kummer relations, which connect the hypergeometric function in z with the functions of the argument $1 - z$).

To write the explicit form of the obtained equality for $h^{(0)}$, we shall introduce the additional notation

$$\tilde{v}(z; \rho, r) = |z|^{(1/2) + ir} (1 - z)^\rho F(2\rho - \frac{1}{2} + ir, \frac{1}{2} + ir; 1 + 2ir; z), \tag{7.8}$$

$$\tilde{w}(z; \rho, u) = z^\rho (1 - z)^{1/2} F(\rho + iu, \rho - iu; 2\rho; z). \tag{7.9}$$

Then, for the function $h^{(0)}(r, u; t)$, we have the representation

$$h^{(0)}(r, u; t) = C_1(r, \rho) \int_0^1 v(-z; 1 - \rho, r)(w(z - 1; \rho, u) - \tag{7.10}$$

$$-Bw(z - 1; 1 - \rho, u)) \frac{dz}{z^{3/2}(1 - z)} + \int_0^1 (C_2(r, \rho)\tilde{v}(z; \rho, r) +$$

$$+C_2(-r, \rho)\tilde{v}(z; \rho, -r))(\tilde{w}(1 - z; \rho, u) - B\tilde{w}(1 - z; 1 - \rho, u)) \frac{dz}{z^{3/2}(1 - z)}$$

where

Now we have 10 integrals V_j, $0 \leqslant j \leqslant 9$; we shall enumerate these integrals so that $h^{(0)}$ is equal to the sum

$$V_0 + AV_1 - BV_2 - ABV_3 \qquad \text{or}$$
$$C_1 V_4 - C_1 BV_5 + C_2(r,\rho)V_6 + C_2(-r,\rho)V_7 - BC_2(r,\rho)V_8 - BC_2(-r,\rho)V_9.$$

96 Later, we shall see that the essential contribution will arise only from the integrals V_0 and V_4.

2.6.2 The Liouville-Green transformation

There is a clear method worked out for the asymptotic integration of the differential equations of the second order with a large parameter. This method is based on the Liouvill-Green transformation. Assume we have a differential equation of the kind

$$v + (t^2 p_0(z) + p_1(z))v = 0, \cdot := \frac{d}{dz}, \tag{7.13}$$

where t is a large positive parameter and p_0, p_1 are real functions. then we can transform the independent variable and the unknown function by the relation

$$v = \dot{\xi}^{-(1/2)}(z)W(\xi(z)), \dot{\xi} := \frac{d\xi}{dz}. \tag{7.14}$$

The formal differentiation gives, for the function W the equation

$$\frac{d^2W}{d\xi^2} + \dot{\xi}^{-2}(t^2 p_0 + p_1 - \frac{1}{2}\{\xi, z\})W = 0. \tag{7.15}$$

where $\{\xi, z\}$ denotes the Schwarzian derivative,

$$\{\xi, z\} = \frac{\dddot{\xi}}{\dot{\xi}} - \frac{3}{2}\frac{\ddot{\xi}^2}{\dot{\xi}^2}.$$

If one can choose the function ξ so that the new equation is close to the equation with the known solution, then we shall be successful in finding the desired asymptotic approximation. The possibility of getting the known functions is explained by the vast set of the investigated equations for the special functions.

The simplest case is one when p_0 has no zeros and p_1 is smooth and bounded. Then we can choose ξ so that $\dot{\xi}^{-2}p_0 = \pm 1$. If p_0 has a zero of the first order, then we can transform our equation, choosing $\dot{\xi}^{-2}p_0 = \xi$ (so that $\dot{\xi}^{-1}$ will be smooth); the Airy function will arise as the main term of the asymptotic formula. For the case when p_0 has two simple zeros nearby, one transforms the initial equation to the Weber equation; if p_0 has a zero and a pole (both simple), then the transformation to the Whittaker equation will be useful and so on.

For the purpose of giving asymptotic forumlae for the four functions v, w, $\tilde{v}$, $\tilde{w}$ in the integrals V_j, it is sufficient to use the inequalities from [5].

The initial differential equations for these functions have the form 97
(7.13); the coefficients p_0, p_1 are given in the following tables where the parameter α is equal to $t^{-1}r$ and $q(z) = z(1+z)$:

Function	Coefficients:	
	p_0	p_1
v	$q^{-2}(z)(1+\alpha^2 q(z))$	$(2q(z))^{-2}(1+q(z))$
w	$(zq(z))^{-1}$	$-u^2((1+z)q(z))^{-1}+$ $+(2q(z))^{-2}(1+q(z))$
$\tilde{v}$	$(q(-z))^{-2}(\alpha^2-\alpha^2 z+z^2)$	$(2q(-z))^{-2}(1+q(-z))$
$\tilde{w}$	$(-zq(-z))^{-1}$	$u^2(q(-z))^{-1}+(2q(-z))^{-2}(1+q(-z))$

2.6.3 The function v, the case $z > 0$ or the case $z \in (-1, 0)$ and $\alpha < 2$.

The function v, the case $z > 0$ or the case $z \in (-1, 0)$ and $\alpha < 2$. For these cases, we shall use the transformation (7.14) by choosing

$$\dot{\xi}^2 = \frac{1+\alpha^2 q}{q^2}, \quad q = z(1+z), \tag{7.16}$$

and therefore we can assume

$$\xi(z) = \log|q| + \alpha \log \frac{2\sqrt{+\alpha^2 1} + \alpha\sqrt{4q+1}}{2+\alpha} \tag{7.17}$$
$$-2\log\frac{\sqrt{1+\alpha^2 q + \sqrt{4q+1}}}{2};$$

so $\xi = \log|z| + O(z)$ when $z \to 0$.

Now equation (7.15), for this case, has the form

$$\frac{d^2W}{d\xi^2} + t^2W = Q_1(\xi.\alpha)W \tag{7.18}$$

where, with $q = q(z(\xi))$, we have

$$Q_1 = -\frac{1}{16}q(1+\alpha^2 q)^{-3}(\alpha^2(\alpha^2-16)q - 4(\alpha^2+2)). \tag{7.19}$$

It is essential that this function tends to zero both when $q \to 0$ and $q \to \infty$. Taking into account the fact $v = |z|^{-it}(1+O(z)) = e^{-it\xi}(1+O(e^{\xi}))$ when $q \to 0$ (it corresponds to $\xi \to -\infty$), we conclude that, for all $z \geqslant 0$, v has the asymptotic expansion

$$\dot{\xi}^{1/2}(z)v(z;\rho,r) = e^{-it\xi}\sum_{n\geqslant 0}\frac{a_n(\xi;Q_1)}{(-2it)^n} \tag{7.20}$$

98 where

$$a_0 = 1, a_1 = \int_{-\infty}^{\xi} Q_1(\eta)d\eta, \ldots, a_{n+1} = a_n' + \int_{-\infty}^{\xi} Q_1(\eta)a_n(\eta)d\eta. \tag{7.21}$$

The polynomial $1+\alpha^2 q(-2) = 1-\alpha^2 z(1-z)$ has no zeros in the interval $z \in (0,1)$ if $\alpha^2 < 4$; for this reason, we can use the same transformation and we have the same expansion (7.20) for $-z \in (0,1)$ if $\alpha^2 \leqslant 4(1-\delta)$.

2.6.4 The function w : z positive.

For the case $z > 0$, we suppose $\xi^2 = z^{-2}(1+z)^{-1}$, so that $z = sh^{-2}\frac{\xi}{2}$ and $\xi = 2\log((1/\sqrt{z}) + (\sqrt{1+(1/z)}))$. The transformed equation has the form

$$\frac{d^2W}{d\xi^2} + \left(t^2 + \frac{1}{4\xi^2}\right)W = Q_2(\xi)W \tag{7.22}$$

with

$$Q_2 = \frac{u^2}{\mathrm{ch}^2\xi/2} + \frac{1}{4}\left(\frac{1}{\xi^2} - \frac{1}{\mathrm{sh}^2\xi}\right). \tag{7.23}$$

when $\xi \to 0$ (which corresponds to $z \to \infty$), we have

$$\xi^{1/2}w(z;\rho,u) = \frac{\Gamma(2\rho)z^{-(1/4)}}{\Gamma(\rho+iu)\Gamma(\rho-iu)}\times \tag{7.24}$$
$$\left(\log z + 2\frac{\Gamma'}{\Gamma}(1) - \frac{\Gamma'}{\Gamma}(\rho+iu) - \frac{\Gamma'}{\Gamma}(\rho-iu) + O\left(\frac{\log z}{z}\right)\right).$$

(Here the analytic continuation of the hypergeometric function is used in the logarithmic case). If $z \to 0$, then $\xi^{1/2}w = z^{it}(1+O(z)) = 2^{2it}e^{-it\xi} \times (1+O(e^{-\xi}))$ and for this reason, our solution must be proportional to $\sqrt{\xi} \times H_0^{(2)}(t\xi)$ (it is the Hankel function). Finally, have the uniform asymptotic expansion

$$\xi^{1/2}w(z;\rho u) = -\frac{i\pi\Gamma(2\rho)}{\sqrt{2}\Gamma(\rho+iu)\Gamma(\rho-iu)}\cdot\left(\sqrt{\xi}H_0^{(2)}(t\xi)\sum_{n\geqslant 0}\frac{b_n(\xi)}{t^{2n}}+\right. \tag{7.25}$$
$$\left. +(\sqrt{\xi}H_0^{(2)}(t\xi))'\sum_{n>1}\frac{C_n(\xi)}{t^{2n}}\right)$$

where $b_0 = 1$, $c_1 = \frac{1}{2}\int_0^\xi Q_2 d\eta$ and for $n \geqslant 1$,

$$b_n(\xi) = -\frac{1}{2}c_n(\xi) - \frac{1}{2}\int^\xi Q_2(x)c_n(x)dx, \tag{7.26}$$

$$c_{n+1}(\xi) = \frac{1}{2}b_n'(\xi) + \frac{1}{2}\int_0^{\xi} Q_2(x)b_n(x)dx - \frac{1}{4}\int_0^{\xi}(x^{-1}c_n(x))\frac{dx}{x}. \qquad (7.27)$$

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The same solution may be expanded again when $\xi \geqslant \xi_0$ with some fixed $\xi_0 > 0$; then we have

$$\dot{\xi}^{1/2}w(z;\rho,u) = 2^{2it}e^{-it\xi}\sum_{n\geqslant 0}\frac{a_n(\xi,\tilde{Q}_2)}{(-2it)^n} \qquad (7.28)$$

where $a_0 = 1$ and the other coefficients are given by the recurrence relations (7.21) with the replacement of Q_1 by $\tilde{Q}_2 = Q_2 - (\frac{1}{4})\xi^{-2}$.

2.6.5 The function w : z negative.

In essence, there is no difference from the previous case. To get an asymptotic formula for $w(-z;\rho,u)$ with $z \in (0,1)$, we choose the new variable $\xi(z) = 2\log(1/\sqrt{z} + (\sqrt{1/z}) - 1)$, so that $\dot{\xi}^2 = z^{-2}(1-z)^{-1}$ and $z = (\mathrm{ch}\frac{\xi}{2})^{-2}$; $z = 0$ corresponds to $\xi = +\infty$. The transformed equation for the function $W = \dot{\xi}^{1/2}W(-z;\rho,u)$ has the form

$$\frac{d^2W}{d\xi^2} + \left(t^2 + \frac{u^2}{\mathrm{sh}^2\xi/2} + \frac{1}{4\mathrm{sh}^2\xi}\right)W = 0. \qquad (7.30)$$

As the initial condition at $z = 0$ is

$$w(-z;\rho,u) = z^{\rho}(1 + O(z)),$$

we have, for $\xi \geqslant \xi_0$ with fixed $\xi_0 > 0$, the expansion

$$\dot{\xi}^{1/2}w(-z;\rho,u) = 2^{2it}e^{-it\xi}\sum_{n\geqslant 0}\frac{a_n(\xi,Q_3)}{(-2it)^n}, \qquad (7.31)$$

where $a_0 = 1$ and a_n, $n \geqslant 1$, are defined by the relations (7.21) with $Q_3 = -u^2(\mathrm{sh}\xi/2)^{-2} - (2\mathrm{sh}\xi)^{-2}$ instead of Q_1.

If ξ were small (which corresponds to a neighbourhood of $z = 1$), then we rewrite equation (7.30) as

$$\begin{aligned} &\frac{d^2W}{d\xi^2} + \left(t^2 + \left(4u^2 + \frac{1}{4}\right)\frac{1}{\xi^2}\right)W = Q_4 W, \\ &Q_4 = u^2\left(\frac{4}{\xi^2} - \frac{1}{\mathrm{sh}^2\xi/2}\right) + \frac{1}{4}\left(\frac{1}{\xi^2} - \frac{1}{\mathrm{sh}^2}\right). \end{aligned} \tag{7.32}$$

When $z \to 1$, we have, as a consequence of the Kummer relation between the hypergeometric function in z and in $(1 - z)$.

$$F(\rho + iu, \rho + iu; 2\rho; z) = \frac{\Gamma(2\rho)\Gamma(-2iu)}{\Gamma^2(\rho - iu)} F(\rho + iu, \rho + iu; 1 + 2iu; 1 - z) +$$
$$+ \frac{\Gamma(2\rho)\Gamma(2iu)}{\Gamma^2(\rho + iu)}(1 - z)^{-2iu} F(\rho - iu, \rho - iu; 1 - 2iu; 1 - z). \tag{7.33}$$

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It gives the initial condition at $\xi = 0$ for our function

$$\xi^{-(1/2)} W(-z; \rho, u) :$$

$$\begin{aligned} W = \Gamma(2\rho)(\frac{\xi}{2})^{1/2} \Bigg(&\frac{\Gamma(-2iu)}{\Gamma^2(\rho - iu)}(\frac{\xi}{2})^{2iu}(1 + O(\xi^2)\Bigg) + \\ &+ \frac{\Gamma(2iu)}{\Gamma^2(\rho + iu)}(\frac{\xi}{2})^{-2iu}(1 + O(\xi^2))). \end{aligned} \tag{7.34}$$

It means that this solution is a linear combination of solutions which are close to $A^{(\pm)}(\rho, u)\sqrt{\xi} J_{\pm 2iu}(t\xi)$ and we have

$$\xi^{1/2} w(-z; \rho, u) = \frac{i\pi}{2} \frac{\Gamma(2\rho) t^{-2iu}}{\mathrm{sh}(2\pi u)\Gamma^2(\rho - iu)} \times \tag{7.35}$$
$$\left\{ \sqrt{\xi} J_{2iu}(t\xi) \sum_{n \geqslant 0} \frac{\tilde{b}_n(\xi)}{t^{2n}} + (\sqrt{\xi} J_{2iu}(t\xi))' \sum_{n \geqslant 1} \frac{\tilde{c}_n(\xi)}{t^{2n}} \right\} +$$
$$+ \{\text{the same with } u \mapsto -u\}$$

where $\tilde{b}_0 \equiv 1$ and the coefficients are defined by relations which are similar to (7.26) and (7.27).

2.6.6 The function $h^{(0)}$ for $\alpha^2 \leqslant 4(1-\delta)$.

We shall use the standard formulae for the method of the stationary phase from [6]. The main principle (not an all-embracing one and nevertheless true for our integrals with hypergeometric functions) is the following statement: if one has an integral without points of the stationary phase, then this integral will be small in a suitable sense.

It is easy to check that there is no point of the stationary phase in the integral with $v(z;\rho,r)w(z;1-\rho,u)$. Furthermore, the coefficient A in (7.7) is exponentially small for $\alpha^2 \leqslant 4(1-\delta)$. For these reasons, the function $h^{(0)}$ is defined by the integral V_0 only.

To distinguish the functions "ξ" in the asymptotic formulae for v and w we shall write ξ_v and ξ_w. With this agreement, the integral V_0 is equal asymptotically to

$$t^{-1}2^{2it}\int_{-1}^{\infty}\frac{\exp(-it(\xi_v(z)+\xi_2(z)))}{(1+\alpha^2 q)^{1/4}(1+z)^{3/4}}\mathscr{E}(z,\alpha)dz \tag{7.36}$$

101 where $\mathscr{E}$ is an asymptotic series in t^{-1} with smooth and bounded coefficients; the main term in $\mathscr{E}$ is equal to $\pi/16$. Now

$$\dot{\xi}_v + \dot{\xi}_w = \frac{\sqrt{1+\alpha^2 q}}{q} - \frac{1}{z\sqrt{1+z}}$$

and the point of the stationary phase is equal to

$$z_0 = \alpha^{-2} - 1 \tag{7.37}$$

At this point, we have

$$2t\log 2 - t\xi_v(z_0) - t\xi_w(z_0) = (2t-r)\log(2t-r) - 2t\log t + r\log r \tag{7.38}$$

and

$$t\ddot{\xi}_v(z_0) + t\ddot{\xi}_w(z_0) = -\frac{1}{2}t\alpha^5. \tag{7.39}$$

The other details may be omitted here; the methods explained in [6] give us the following

Proposition 11. *Let $r \leqslant 2t(1-\delta)$ with some fixed small $\delta > 0$ and t be large. Then the function $h^{(0)}$ can be written as*

$$h^{(0)}(r,u,t) = \frac{1}{t\sqrt{r}} e^{i\psi(t,r)} \mathscr{E}(t,r,u) \tag{7.40}$$

where

$$\psi(t,r) := (2t-r)\log(2t-r) - 2t\log t + r\log r - \frac{\pi}{4} \tag{7.41}$$

and $\mathscr{E}$ is a smooth non-oscillating function, $|\mathscr{E}| \ll 1$ and for any fixed integer $n \geqslant 1$, $|(\partial/\partial t)^n \mathscr{E}| \ll t^{-n}$.

2.6.7 The case $r \geqslant 2(1-\delta)t$.

Now we shall use the representation (7.10). Here $C_2(\pm r, \rho)$ is exponentially small for $2t - |r| \gg 1$. At the same time, for all α^2, there are not turning points in the equation for $\tilde{v}$ and this function has an oscillating nature. For the points of the stationary phase in the integrals with $\tilde{v}$ and $\tilde{w}$, we have the equation

$$(z(1-z))^{-1}\sqrt{\alpha^2(1-z)+z^2} = ((1-z)\sqrt{z})^{-1}, \tag{7.42}$$

or, what is the same, $z_0 = \alpha^2$. So, there are no such points in the interval (0, 1); for this reason, the last integral on the right side of (7.10) can be omitted.

When α is close to 2, the full asymptotic investigation of the function $v(-z; 1-\rho, r)$ is very complicated. But due to a fortunate coincidence, the simplest case is sufficient for our purposes.

The fact of the matter is given by (1) the exponentially small nature
of the coefficient $C_1(r,\rho)$ for $r - 2t \gg 1$ and 2) the absence of points of 102
the stationary phase in the interval $z > \frac{1}{2}$ for $r \geqslant 2(1-\delta)$. Really, the equation for these points is

$$(z(1-z))^{-1}\sqrt{1-\alpha^2 z(1-z)} = ((1-z)\sqrt{z})^{-1}, 1-\alpha^2 z(1-z) > 0, \tag{7.43}$$

and $z_0 = \alpha^{-2}$ is the unique possible solution. For this reason, it is sufficient to know the exact asymptotic formulae for the function $v(-z; 1 -$

$\rho, r)$ in the interval $z \leqslant \alpha^{-2}(1+\delta)$ only. But the turning points of our equation (the zeros of the polynomial $1-\alpha^2 z(1-z)$) are $z^{(\pm)} = 2(\alpha(\alpha \pm \sqrt{\alpha^2-4}))^{-1}$; these points are close to $\frac{1}{2}$ when α is close to 2. So we have the interval $((\frac{1}{8}, \frac{3}{8})$, for example) where the stationary point lies and the positive polynomial $1+\alpha^2 q(-z)$ is strongly separated from zero. For the last reason, we have, in this interval, an asymptotic expansion of the same kind as in (7.20). The unique natural difference is the exchange of the signs, because ρ is replaced by $1-\rho$ here:

$$\dot{\xi}_v^{1/2} v(-z; 1-\rho, r) = e^{it\xi_v} \sum_{n \geqslant 0} \frac{a_n(\xi_v, Q_1)}{(2it)^n}. \tag{7.43}$$

Now one can see that at the stationary point $z_0 = \alpha^{-2}$, we have

$$t(\xi_v(z_0) - \xi_w(z_0)) = -(2t+r)\log(2t+r) + 2t\log(2t) + r\log r \tag{7.44}$$

and $\ddot{\xi}_v - \ddot{\xi}_w = -\frac{1}{2}\alpha^5$ at this point.

As a consequence of the Stirling expansion, in the case $2t - r \gg 1$,

$$C_1(r,\rho) = \frac{\pi}{8t} \exp\big(i(2t+r)\log(2t+t) + \tag{7.45}$$

$$+(2t-r)\log(2t-r) - 4t\log(2t) + O\left(\frac{1}{2t-r}\right)\Big)\Big)$$

so that for $r \leqslant 2t(1-\delta)$ with $\delta > 0$ at the point $z_0 = \alpha^{-2}$

$$C_1(r,\rho) e^{it(\xi_v - \xi_w) - i\pi/4} = e^{i\psi(t,r)} \cdot \frac{\pi}{8t}\left(1 + O\left(\frac{1}{t}\right)\right), \tag{7.46}$$

where ψ is the same phase as in (7.40).

To estimate the contribution of the integration over the complement of the interval $(\alpha^{-2} - \delta, \alpha^{-2} + \delta)$, especially in the transition region $|\alpha^2 - 4| \leqslant \delta$, we shall use approximation by the Weber functions.

Let, for definiteness, $\alpha > 2$ and the quantity $\epsilon^2 = (\frac{1}{4} - \alpha^{-2})$ be small. Then the differential equation for $v(-z;\rho,r)$ may be written in the form

$$v'' + \left(r^2 \frac{16(z^2-\epsilon^2)}{(1-4z^2)^2} + \frac{3+4z^2}{(1-4z^2)^2}\right) v = 0, -\frac{1}{2} < z < \frac{1}{2} \tag{7.47}$$

103 (here z is written instead of $z - \frac{1}{2}$ in the initial equation).

The corresponding Liouville-Green transformation is taken by choosing

$$\dot{\xi}^2(\xi^2 - \gamma^2) = \frac{16(z^2 - \epsilon^2)}{(1 - 4z^2)^2} \tag{7.48}$$

with the conditions $\xi(-\epsilon) = -\gamma$, $\dot{\xi} > 0$. The new parameter γ is chosen so that the equality $\xi(+\epsilon) = \gamma$ is fulfilled. This last condition gives

$$\gamma^2 = 2(1 - \sqrt{1 - 4\epsilon^2}) = 4\epsilon^2(1 + \epsilon^2 + \ldots) \tag{7.49}$$

If we denote $\epsilon^{-1}z$ and $\gamma^{-1}\xi$ as x and $y(x, \epsilon)$, then for the Schwarzian derivative $\{\xi, z\}$, we have the expression $\epsilon^{-2}\{y, x\}$ and the function $y(x, \epsilon)$ is defined by the equation

$$(y^2 - 1)\left(\frac{dy}{dx}\right)^2 = \frac{16\epsilon^4}{\gamma^4}\frac{x^2 - 1}{(1 - 4\epsilon^2 x^2)^2}. \tag{7.50}$$

Here the function on the right hand side is a power series in ϵ^2 with the leading term $(x^2 - 1)$. For this reason, we have a solution of the form

$$y(x, \epsilon) = x + \epsilon^2 y_1(x) + \epsilon^4 y_2(x) + \ldots \tag{7.51}$$

Hence the Schwarzian derivative $\{y, x\}$ is of order $O(\epsilon^2)$ (it being obvious that $\{x, x\} = 0$) and as a result, we have the boundedness of $\{\xi, z\}$ in a certain interval $\epsilon^2 \leqslant \epsilon_0^2$. Now we have the transformed equation for the function $W = \dot{\xi}^{1/2}v$:

$$\frac{d^2W}{d\xi^2} + r^2(\xi^2 - \gamma^2)W = Q_4(\xi, \epsilon)W \tag{7.52}$$

where Q_4 is bounded uniformly (in ϵ) for all $\xi \in (-\infty, \infty)$ and at the same time, this function tends to zero, for $\xi \to \pm\infty$, as $O(\xi^{-2})$.

An estimate of the closeness of the solutions of this equation to the solutions of the equation with $Q_4 \equiv 0$ (the Weber functions is given in [7]). We have useful inequalities for the Weber functions and the full asymptotic expansions due to F. Olver [8]. They allow us to given the asymptotic formulae for v in the transition region $\alpha^2\tilde{4}$. After that, everyone who is a past master in integration by parts will be also to prove the smallness for all integrals, except in the case considered. As a result we have

Proposition 12. *For any r with the condition $1 \ll r \leqslant 2t + B_0 \log t$, for fixed $B_0 \geqslant 1$, we have*

$$h^{(0)}(r,t) = \frac{1}{\sqrt{r}} C_1 \left(r, \frac{1}{2} + it \right) e^{i\psi_0(t,r)} \mathcal{E}(t,r) \tag{7.53}$$

where

$$\psi_0(t,r) = -(2t+r)\log(2t+r) + 2t\log(2t) + r\log r - \frac{\pi}{4} \tag{7.54}$$

104 *and $\mathcal{E}$ is a smooth non-oscillating function,*

$$|\mathcal{E}(t,r)| \ll 1, \left| \left(\frac{\partial}{\partial t} \right)^n \mathcal{E}(t,r) \right| \ll t^{-n}, n = 0, 1, \ldots \tag{7.55}$$

If $r \geqslant 2t + B_0 \log t$ with fixed $B_0 \geqslant 1$, then

$$|h^{(0)}(r,t)|^1 r^{-3B_0}. \tag{7.56}$$

2.7 The integration over t

2.7.1 The summation formulae

The next step is the calculation of the integrals over t, where the integrand contains the Hecke series (associated with the continuous or discrete spectrum) and the function $h^{(0)}(r,t)$. To do this, we need to approximate the corresponding Hecke series by a finite sum; it will be achieved by using the following summation formulae (using other forms of the functional equations for the Hecke series).

Proposition 13. *Assume that $\varphi : [0,\infty) \to \mathbb{C}$ and its Mellin transform $\hat{\varphi}(s)$ satisfies the conditions:*

i) *$\hat{\varphi}(2s)$ is regular in the strip $\alpha_0 \leqslant \operatorname{Re} s \leqslant \alpha_1$ with some $\alpha_0 < 0$ and $\alpha_1 > 1$;*

ii) *for* $\sigma \in [\alpha_0, \alpha_1]$, *the function*

$$((1+|t|)^{-1-2\sigma}+1)^{-1}|\hat{\varphi}(2\sigma+2it)|$$

is integrable on the axis $(-\infty, +\infty)$. *Then, for any* v *with* $\mathrm{Re}\, v = \frac{1}{2}$ *and for any relatively prime integers* c, d *with* $c \geqslant 1$, *one has the identity*

$$\frac{4\pi}{c}\sum_{n\geqslant 1} e\left(\frac{nd}{c}\right)\tau_v(n)\varphi\left(\frac{4\pi\sqrt{n}}{c}\right) = 2\frac{\zeta(2v)}{(4\pi)^{2v}}\hat{\varphi}(2v+1)+ \tag{8.1}$$

$$+2\frac{\zeta(2-2v)}{(4\pi)^{2-2v}}\hat{\varphi}(3-2v)+$$

$$+\sum_{n\geqslant 1}\tau_v(n)\int_0^{\infty}(e(-nd'/c)k_0(x\sqrt{n},v)+e(nd'/c)k_1(x\sqrt{n},v))\varphi(x)x dx$$

where d' *is defined by the congruence* $dd' \equiv 1(\mathrm{mod}\ c)$ *and the kernels* k_0, k_1 *are defined by the relations* (1.66).

Proposition 14. *Let* φ *have the same properties as in* (8.1) *and let* $t_j(n)$, $n = 1, 2, \ldots$, *be the eigenvalues of n-th Hecke operator. Let* $\lambda_j > \frac{1}{4}$ *and let the* j*-th eigenfunction of the automorphic Laplacian be even. Then, for any coprime integers* c, d *with* $c \geqslant 1$, *we have*

$$\frac{4\pi}{c}\sum_{n\geqslant 1} e\left(\frac{nd}{c}\right)t_j(n)\varphi\left(\frac{4\pi\sqrt{n}}{c}\right) = \tag{8.2}$$

$$= \sum_{n\geqslant 1} t_j(n)\mathop{\mathrm{inf}}_{0}^{\infty}\left(e(-nd'/c)k_0(x\sqrt{n},\frac{1}{2}+i_{\chi_j})+\right.$$

$$\left.+e\left(\frac{nd'}{c}\right)k_1\left(x\sqrt{n},\frac{1}{2}+i\chi_j\right)\right)\varphi(x)x\, dx.$$

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2.7.2 The integration over *t*.

Our next problem is the asymptotic calculation of the integral

$$\mathscr{J}(T) = \int \omega_T(t)\mathscr{H}_j(\frac{1}{2}+2it)h^{(0)}(\chi_j,t)\overline{\chi(t,0)}dt \tag{8.3}$$

(where χ is defined by the equality (5.8)) and the similar integral

$$\mathcal{J}(T,r) = \int \omega_T(t)\zeta(\frac{1}{2}+2it+ir)\zeta(\frac{1}{2}+2it-ir)h^{(0)}(r,t)\overline{\chi(t,0)}dt. \quad (8.4)$$

We shall consider the second integral; the first one may be considered in the same manner.

Let $\beta : [0,\infty) \to [0,1]$ be the infinitely smooth monotone function with the conditions

$$\beta(x)+\beta(1/x) \equiv 1,\ \beta(x) \equiv 0\, for\ 0 \leqslant x \leqslant \frac{1}{2} \quad (8.5)$$

(and for this reason $\beta(x) \equiv 1$ if $x \geqslant 2$).

If Re $s > 1$, writing v instead of $\frac{1}{2}+ir$ for brevity, we have, for any positive δ,

$$\zeta(s+v-1/2)\zeta(s-v+1/2) = \sum_{n=1}^{\infty} \frac{\tau_v(n)}{n^s} = \quad (8.6)$$

$$= \sum_{n=1}^{\infty} n^{-s}\beta(\delta n)\tau_v(n) + \sum_{n=1}^{\infty} n^{-s}\beta(1/\delta n)\tau_v(n).$$

Applying, to the second sum, the summation formula (8.1) with $c = d = 1$, we shall obtain the representation

$$\zeta(s+v-1/2)\zeta(s-v+1/2) = \sum_{n\geqslant 1} n^{-s}\beta(\delta n)\tau_v(n)+ \quad (8.7)$$

$$+(4\pi)^{2s-1}\sum_{n\geqslant 1}\tau_v(n)\int_0^{\infty}(k_0(x\sqrt{n},v)+k_1(x\sqrt{n},v))\beta\left(\frac{16\pi^2}{\delta x^2}\right)x^{1-2s}dx-$$

$$-\frac{\delta^{s-v-(1/2)\zeta(2v)}}{s-v-\frac{1}{2}}\int_0^{\infty}\beta'(x)x^{s-v-(1/2)}dx-$$

$$-\frac{\delta^{s+v-(3/2)}\zeta(2-2v)}{s+v-\frac{3}{2}}\int_0^{\infty}\beta'(x)x^{s+v-(3/2)}dx$$

106 (using integration by parts in the terms with the Mellin transform of the function β).

The series on the right side of (8.7) are convergent absolutely for all values of $\operatorname{Re} s$ (as will be obvious after some integrations by parts); so this identity gives the meromorphic continuation of the function on the left hand side in the critical strip $0 < \operatorname{Re} s < 1$.

If we calculate the integrals in (8.7) (using the asymptotic formulae for the Bessel functions of large order) then the so-called "shortened functional equation" will be the result. But there is no need for an explicit asymptotic form, for the purpose of integration over t in our case.

We can do the first integration over t: in the case $2t - r \gg 1$, we have the inner integral

$$\frac{1}{\sqrt{r}} \int \omega_T(t) e^{i\psi(r,t) - 4it \log(x/4\pi) + 2it \log(t/2\pi) - 2it} \mathscr{E}(r,t) \frac{dt}{t} \tag{8.8}$$

with ψ and $\mathscr{E}$ from (7.40). Here the point of the stationary phase is defined by the equation

$$2t - r = \frac{x^2}{8\pi}. \tag{8.9}$$

Note that $t \in (T, 2T)$ and $x^2 \geqslant 8\delta^{-1}\pi^2$ in the integrand. So, for $\delta = \frac{\pi}{8T}$, the derivative of the function in the exponent is $\ll -\log x \ll -\log T$. Hence we have the possibility of integrating by parts any number of times. Each integration by parts will give the additional factor $O(t^{-1})$n in the integrand. After multiple integration by parts over t, we shall do the integration by parts over x (to obtain the absolutely convergent series summed over n)

As a consequence, we can reject the second series in the representation (8.7) and this rejection does not affect any remainder terms in the integral (8.4)

There are some differences in the case when the quantity $2t - r$ may 107
be small, i.e. when $2T \leqslant r \leqslant 4T$ (note that $\omega_T(t) \equiv 0$ for $t \leqslant T$ and $t \geqslant 2T$). For this case, we have a slightly different expression (7.53) instead of (7.40). Using the Stirling expansion for $\Gamma(\frac{1}{2} + 2it + ir)$ and the Binet representation for $\log \Gamma(\frac{1}{2} + 2it - ir)$, we can rewrite this expression

in the form

$$h^{(0)}(r,t)e^{2it\log(t/2\pi)-2it} = \frac{1}{t\sqrt{r}}e^{i\psi(r,t)}\mathscr{E}_1(r,t), \tag{8.10}$$

where $\mathscr{E}_1$ has the same properties as $\mathscr{E}$ and

$$\psi_1(r,t) = (2t-r)(\log(\frac{1}{2}+i(2t-r)) - \frac{i\pi}{2}) - 2t\log(4\pi)- \tag{8.11}$$

$$-2t + r\log r - i\int_0^\infty \left((e^v-1)^{-1} - \frac{1}{v} - \frac{1}{2}\right)e^{-v((1/2)+i(2t-r))}\frac{dv}{v}$$

The zero of the function

$$\frac{\partial\psi_1}{\partial t} - 4\log\frac{x}{4\pi}$$

lies to the right of the interval of integration. So we can carry out integration by parts and this gives us

$$\int_T^{2T} \omega_T(t)\mathscr{E}_1 e^{iG(t)}\frac{dt}{t} = -\int_T^{2t}\left(\frac{\omega_T(t)}{t}\mathscr{E}_1\right)'\int_1^t e^{iG(\tau)}d\tau\, dt. \tag{8.12}$$

where $G(t) = \psi_1(r,t) - 4t\log\dfrac{x}{4\pi}$. Now in the inner integral, we have analytic functions ans for this reason, we can use the saddle-point method. We shall integrate over the curve $\tau = \tau(y;r,t)$, which is parametrized by the real positive new variable y; this curve is defined by the condition that imaginary part of the function in the exponent is constant:

$$iG(\tau) = iG(t) - y,\ \ y > 0. \tag{8.13}$$

Then the inner integral has the form

$$e^{iG(t)}\left(\int_0^r e^{-y}(G'(\tau(y))\tau'(y))^{-1}dy + O(e^{-r})\right) \tag{8.14}$$

(Note that $\mathrm{Re}(i\psi_1(r,\tau)) < \frac{\pi}{2}(r-2\tau)$ if τ is small in comparison with $r/2$).

It means that there is the possibility of repeating the integration by parts and so on.

108 Hence, the contribution of the second sum on the right side of (8.7) in the integral over t is negligible in any case.

In the last terms on the right side of (8.7), the integrand is not zero for $\frac{1}{2} \leqslant x \leqslant 2$ only. For this reason, we have the same integral (with the replacement of $x/(4\pi)$ by $(x\delta)^{-(1/2)}$ and with the additional factor $(i(2t-r)-\frac{1}{2})^{-1}$); these terms contribute to the integral over t, the quantity $O((T\sqrt{r})^{-1}|\zeta(1+2ir)|)$.

Of course, the same considerations are applicable to the integral with $\mathscr{H}_j(\frac{1}{2}+2it)$.

Thus, to calculate the integrals (8.3) and (8.4), we can replace the corresponding Hecke series by the finite sum with the weight function $\beta(\delta n)$ if we choose $\delta = \frac{\pi}{8T}$.

We have $O(T)$ members in these sums and its estimate gives us the following main inequality.

Proposition 15. *The integrals $\mathscr{I}(T,r).\mathscr{I}_j(T)$ are exponentially small for $r-4T \gg \log T$, $\chi_j - 4T \gg \log T$ and for $r, \chi_j \leq 4(T+\log T)$, we have*

$$|\mathscr{I}(T,r)| \ll \frac{|\zeta(1+2ir)| + \log T}{\sqrt{r}}, \quad r \gg 1, \tag{8.15}$$

$$|\mathscr{I}_j(T)| \ll 1/T \tag{8.16}$$

First of all, we have the same integrals as in (8.8) with $2t\log(4\pi n)$ instead of $4t\log\frac{x}{4\pi}$ in the exponent.

There is no large difference between the cases $r < 2T$ and $r \in (2T, 4T)$ and the first case is typical.

It is convenient to use transformation (of the variable of integration) $t = \frac{r}{2} + 4\pi ny$ (with the obvious intention of fixing the position of the point of the stationary phase); then we have the integral

$$\frac{\beta(\delta n)}{\sqrt{r}} \cdot 4\pi\sqrt{n} \cdot e^{-ir\log(4\pi n)+ir\log r-ir-(i\pi/4)} \int\limits_0^\infty g(ny,r)e^{4\pi in(y\log y-y)}dy \tag{8.17}$$

with

$$g(x,r) = \frac{\omega_T(2\pi x + r/2)}{2\pi x + r/2}\mathscr{E}(r, 2\pi x + r/2). \tag{8.18}$$

This integral is the classic example of an exercise in the method of stationary phase. It is essential that the derivatives with the respect to
109 y, of the function $g(ny,r)$ are bounded by $O(T^{-1})$ here. Really, we have $n \ll T$ and for $l = 0, 1, \ldots$

$$\left(\frac{\partial}{\partial t}\right)^l \mathscr{E}(r,t) \ll T^{-1}, \left(\frac{\partial}{\partial t}\right)^l \omega_T \ll T^{-1}.$$

Using the usual formulae, we shall obtain the expression

$$\frac{2\pi\beta(\delta n)}{\sqrt{r}}(4\pi n)^{-ir}e^{ir\log r-ir}\times \tag{8.19}$$

$$\left\{g(n,r) + \frac{1}{4\pi n}\left(g'' + g' + \frac{11}{12g}\right) + O\left(\frac{1}{n^2}\right)\right\}$$

where $' := \dfrac{d}{dy}$ and g, g', g'' are taken at $y = 1$; all these quantities are $O(T^{-1})$.

The summation of the absolute values is not sufficient for our purposes but there is no problem in effecting this with the desired accuracy.

Let us suppose that

$$\tilde{g}(s,r) = \int\limits_0^\infty \beta(\delta x)g(x,r)x^{s-1}dx. \tag{8.20}$$

It is an entire function of s, if $r < 2T$ and meromorphic with simple poles at $s = 0, -1, -2, \ldots$, when $r \in (2T, 4T)$; if $r > 4T$, then this function is identically zero.

First of all, we can write

$$\tilde{g}(s,r) = T^{s-1} \int_0^\infty g_0(x,r) x^{s-1} dx \tag{8.21}$$

with

$$g_0(x,r) = \beta(8\pi x) \frac{\omega_1(2\pi x + r/(2T))}{2\pi x + r/(2T)} \mathscr{E}(r, Tx + r/2) \tag{8.22}$$

If, in the beginning, Re $s > 0$, then

$$\begin{aligned} \tilde{g}(s,r) &= -(1/s) T^{s-1} \int_0^\infty x^s g_0' dx \\ &= \frac{1}{s(s+1)} T^{s-1} \int_0^\infty x^{s+1} g_0'' dx \\ &= \dots \end{aligned} \tag{8.23}$$

This representation gives the meromorphic continuation on the whole **110**
s-plane at the same time, we see that, for any fixed $B \geqslant 1$,

$$\tilde{g}(s,r) = O(T^{\mathrm{Re}\, s-1} |s|^{-B}) \tag{8.24}$$

when $|s| \to \infty$ and Re s is fixed.

Now, with this function, we have, for $v = \frac{1}{2} + ir$,

$$\sum_{n \geqslant 1} \frac{\tau_v(n)}{n^{ir}} g(n,r) = \frac{1}{2\pi i} \int_{(3/2)} \zeta(s)\zeta(s+2ir)\tilde{g}(s,r) ds \tag{8.25}$$

where $\int_{(\sigma)}$ denotes the integral over the line Re $s = \sigma$.

Let us move the path of the integration and integrate over the line Re $s = -\frac{1}{2}$. The poles at $s = 1$, $s = 1 - 2ir$ and $s = 0$ give the terms

$$\ll |\zeta(1+2ir)| + \frac{1}{T} |\zeta(2ir)| \tag{8.26}$$

and the integral over the line $\operatorname{Re} s = -\frac{1}{2}$ contributes $O(T^{-3/2}r)$. Since $\zeta(2ir) = O(r^{1/2})\zeta(1-2ir)$ and $r \ll T$, it proves the inequality (8.15). In the second case, we have the same representation.

$$\sum_{n\geqslant 1} \frac{t_j(n)}{n^{i\chi_j}} g(n,\chi_j) = \frac{1}{2\pi i} \int_{(3/2)} \mathscr{H}_j(s+i\chi_j)\tilde{g}(s,\chi_j)ds \tag{8.27}$$

but without the poles on the line $\operatorname{Re} s = 1$. There is no pole at $s = 0$ also; really, from the functional equation for the Hecke series, we have $\mathscr{H}_j(\pm i\chi_j) = 0$, if this series corresponds to the even eigenfunction ($\epsilon_j = +1$ in (1.52); note that $\mathscr{H}_j(\frac{1}{2}) = 0$ if $\epsilon_j = -1$, so the consideration of the case $\epsilon_j = +1$ is sufficient for our purpose). On the line $\operatorname{Re} s = -|\frac{1}{2}$, for the case $|s| = o(r)$, we have

$$\begin{aligned} |\mathscr{H}_j(s+i\chi_j)| &\ll e^{\pi\chi_j}|\Gamma(1-s)\Gamma(1-s-2i\chi_j)\mathscr{H}_j(1-s-i\chi_j)| \\ &\ll \chi_j|s|e^{-(\pi/2)s} \end{aligned} \tag{8.28}$$

Together with (8.24), it gives the estimate $O(\chi_j T^{-3/2})$ for the sum (8.27).

The other terms from the asymptotic formula (8.19) contribute only smaller quantities and inequality (8.16) is proved.

2.8 The sum over cusps.

2.8.1 The explicit form.

The next step will be to estimate the contribution of the sum

$$Z^{\text{cusp}}\left(\begin{matrix}\frac{1}{2}, & \rho \\ \rho, & \frac{1}{2}\end{matrix}\middle| h^*\right)$$

in the integral over t.

111 Firstly, we shall write the explicit form of the coefficients h_{2k-1} in this sum; these coefficients $h_{2k-1}(t)$, $k = 6, 7, \ldots$, result from the analytic continuation of the integral

$$2(2k-1)\int_0^\infty J_{2k-1}(x)\Phi_0(x; s, v, \rho, \mu)\frac{dx}{x} \tag{9.1}$$

at the point $s = \mu = \frac{1}{2}, \rho = \nu = \frac{1}{2} + it$; here Φ_0 is the same function as in (3.12) with φ from (4.4) (where h means the function (5.4)).

Let us introduce some additional notation:

$$w_k(z) = z^{1/2}(1-z)^{1/2} F(k, 1-k; 1; z), \tag{9.2}$$

$$\tilde{w}_k(z) = \frac{\partial}{\partial \epsilon}\left\{z^{1/2+\epsilon}(1-z)^{1/2} F(k+\epsilon, 1+\epsilon-k; 1+2\epsilon; z)\right\}_{\epsilon=0}. \tag{9.3}$$

Further, let, for real z with $z < 1$, $v_k(z)$ and $y(z; t, u)$ be defined by

$$v_k(z) = |z|^k (1-z)^{1/2} F(k, k; 2k; z), \tag{9.4}$$

$$y(z; t, u) = |z|^{(1/2)+it}(1-z)^{1/2} F(\frac{1}{2} + it + iu, \frac{1}{2} + it - iu; 1 + 2it; z) \tag{9.5}$$

Finally, it is convenient to suppose

$$b(u,t) = \frac{\Gamma(\frac{1}{2} + it + iu)\Gamma(\frac{1}{2} - it + iu)}{\Gamma(1 + 2iu)} \tag{9.6}$$

With this notation, the coefficients $h_{2k-1}(t)$ in the sum Z^{cusp} in the case of our specialization are defined by the following equality.

Proposition 16. *We have*

$$h_{2k-1}(t) = 2(2k-1)\int_{-\infty}^{\infty} (B_{00}^{(k)}(u,t) + B_{01}^{(k)}(u,t) + B_{11}^{(k)}(u,t)) \tag{9.7}$$

$$u \operatorname{th}(\pi u) h(u) du,$$

where the kernels are given by

$$B_{00}^{(k)}(u,t) = \tag{9.8}$$

$$= \frac{1}{2\pi^2}\int_0^1 \left(\tilde{w}_k(x) + 2\left(\frac{\Gamma'}{\Gamma}(k) - \frac{\Gamma'}{\Gamma}(1)\right) w_k(x)\right)(b(u,t)y(x;u,t) +$$

$$+ b(-u,t)y(x;-u,t))x^{-it}\frac{dx}{x^{3/2}(1-x)},$$

$$B_{01}^{(k)}(u,t) = \frac{(-1)^k\Gamma^2(k)}{2\pi^2\Gamma(2k)} \int\limits_0^1 v_k(x)(b(t,u)y(x;t,u)+ \tag{9.9}$$

$$+b(-t,u)y(x;-t,u))x^{it}\frac{dx}{x^{3/2}(1-x)'}$$

$$B_{11}^{(k)}(u,t) = \frac{i\mathrm{cth}(\pi t)\Gamma^2(k)}{2\pi^2\Gamma(2k)} \int\limits_0^{+\infty} v_k(-\frac{1}{x})x^{-it}(b(u,t)y(-x;u,t)- \tag{9.10}$$

$$-b(-u,t)y(-x;-u,t))\frac{dx}{x^{1/2}(1+x)}.$$

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2.8.2 The equations.

To estimate the integrals (9.8)-(9.10), we shall use again the same method of asymptotic integration of the differential equation with a large parameter. This large parameter will be the integer k for the function w_k, $\tilde{w}_k$ and v_k and t for $y(x;u,t)$, $y(x;t,u)$.

It is convenient to write here all equations to clarify the nature (oscillatory or monotonic) of the corresponding functions.

The function $y(x;t,u)$ is a solution of the equation

$$y'' + \left(\frac{t^2}{x^2(1-x)} + \frac{1-x+x^2}{4x^2(1-x)^2} - \frac{u^2}{x(1-x)}\right)y = 0, \tag{9.11}$$

and so this function is oscillatory.

When the parameters t and u are interchanged, we have for the function $y(x;u,t)$ the equation

$$\tilde{y}'' + \left(-\frac{t^2}{x(1-x)} + \frac{1-x+x^2}{4x^2(1-x)^2} + \frac{u^2}{x^2(1-x)}\right)\tilde{y} = 0; \tag{9.12}$$

it is obvious that, for large t, the solutions are non-oscillatory.

Now both the functions $w_k(x)$ and $\tilde{w}_k(x)$ are solutions of the equation

$$w'' + \left(\frac{(k-\frac{1}{2})^2}{x(1-x)} + \frac{1}{4x^2(1-x)^2}\right)w = 0 \tag{9.13}$$

Finally, the function $v_k(x)$ is a solution of the equation

$$v'' + \left(-\frac{(k-\frac{1}{2})^2}{x^2(1-x)} + \frac{1-x+x^2}{4x^2(1-x)^2}\right)v = 0 \tag{9.14}$$

2.8.3 The order of the integrals.

From the Stirling expansion, it follows for a large positive t

$$b(u,t) = \frac{2\pi t^{2iu}}{\Gamma(1+2iu)}e^{-\pi t}\left(1 + O\left(\frac{1+u^2}{t}\right)\right) \tag{9.15}$$

(noting that, in all integrals, we can assume $|u| \ll \log t$) and at the same time,

$$b(t,u) = \left(\frac{\pi}{t}\right)^{1/2} e^{-2it-ipi/4}\left(1 + O\left(\frac{1+u^2}{t}\right)\right) \tag{9.16}$$

So the integrals (9.8) and (9.10) are negligible and it is sufficient to consider (9.9). 113

For both the functions $v_k(x)$ and $y(x;t,u)$ the transformation

$$x = (\text{ch}\,(\xi/2))^{-2},\ \xi \in (0,+\infty) \tag{9.17}$$

of the variable is suitable. Then, for the functions

$$V_k(\xi) = \dot{\xi}^{1/2}v_k(x(\xi)),\ Y(\xi;t,u) = \dot{\xi}^{1/2}y(x(\xi);t,u) \tag{9.18}$$

We have the differential equations

$$\frac{d^2V_k}{d\xi^2} + \left(-\left(k-\frac{1}{2}\right)^2 + \frac{1}{4\text{sh}^2\xi}\right)V_k = 0, \tag{9.19}$$

$$\frac{d^2Y}{d\xi^2} + \left(t^2 + \frac{1}{4\text{sh}^2\xi} - \frac{u^2}{\text{ch}^2\xi/2}\right)Y = 0 \tag{9.20}$$

If we take into account the initial conditions, then we can conclude that V_k must be proportional to

$$(k-\frac{1}{2})^{-1/2}\sqrt{\xi}K_0((k-\frac{1}{2})\xi) \tag{9.21}$$

with an absolute coefficient (not depending on k). At the same time, Y must have the main term

$$(\text{constant}) \cdot t^{-1/2}\sqrt{\xi} H_0^{(2)}(t\xi). \tag{9.22}$$

For this reason, the main term of the asymptotic formula for the integral in (9.9) is defined by the integral

$$(t(k-\frac{1}{2}))^{-1/2}\int\limits_0^{+\infty} \xi g(\xi) K_0((k-\frac{1}{2})\xi) H_0^{(2)}(t\xi)d\xi \tag{9.23}$$

where $g(0) \neq 0$ and $g(\xi)$ in some neighbourhood of $\xi = 0$ is a power series in ξ^2. All such integrals are $O(t^{-2})$ and because of the factor $(\Gamma(2k))^{-1}\Gamma^2(k)$ we have the following estimate.

Proposition 17. *Let h_{2k-1} be defined by the equality* (9.7)*; then*

$$|h_{2k-1}(t)| \ll 2^{-k}t^{-5/2}. \tag{9.24}$$

It means that the trivial estimate for the integral over t is sufficient and we can omit the sum Z^{cusp}.

2.9 The eighth moment.

Let us for brevity, denote by $\tilde{M}(s, v; \rho, \mu)$ the sum of all "main terms" on the right side of (5.1) (excepting the functions Z^{dis}, $\tilde{Z}^{\text{dis}}$, Z^{con} and Z^{cusp}) for the case of the specialization (5.4). Further, let

$$M(t) = \lim_{\substack{s,\mu\to 1/2 \\ \rho,v\Rightarrow(1/2)+it}} \zeta(2s)\zeta(2\rho)\tilde{M}(s, v; \rho, \mu). \tag{10.1}$$

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Then a cumulative result of our preceding considerations is the inequality

$$\sum_{j\geqslant 1}\alpha_j \int\limits_0 \omega_T(t)|\mathscr{H}_j(\frac{1}{2}+it)|^4 dt \cdot \left(1 + O\left(\frac{\chi_j^2}{T}\right)\right) h(\chi_j) + \tag{10.2}$$

$$+(\log T)^{-6}\int_0^\infty \omega_T(t)|\zeta(\frac{1}{2}+it)|^8 dt \ll \frac{1}{T}\sum_{\chi_j\leqslant 4T}\alpha_j|\mathscr{H}_j(\frac{1}{2})|^3+$$

$$+\int_0^{4T}|\zeta(\frac{1}{2}+ir)|^6\frac{dr}{\sqrt{r}|\zeta(1+2ir)|}+\int_0^\infty \omega_T(t)\overline{\chi(t,0)}M(t)dt.$$

As $|\zeta(\frac{1}{2}+ir)| \ll (|r|+1)^{1/6}$, the integral with the sixth power of zeta function is estimated by the quantity $O(T^{1-1/6+\epsilon})$ for any $\epsilon > 0$. The sum over the discrete spectrum on the right side is not larger than

$$\frac{1}{t}\left(\sum_{\chi_j\leqslant 4T}\alpha_j\mathscr{H}_j^4\right)^{1/2}\left(\sum_{\xi_j\leqslant 4T}\alpha_j\mathscr{H}_j^2\left(\frac{1}{2}\right)\right)^{1/2} \qquad (10.3)$$

Here the second sum is $O(T^2\log T)$. For the first sum, the estimate $O(T^{2+\epsilon})$ for any $\epsilon > 0$ due to H. Iwaniec and J. M. Deshouillers is known. But this estimate may be elaborated; if, in the main functional equation, we choose the function h so that h is close to 1 for $-T \leqslant r \leqslant T$ and $s = \mu = \rho = \nu = \frac{1}{2}$, then we should get an asymptotic formula for this fourth spectral moment. A suitable $h(r)$ may be taken, for example,

$$h_\delta(r) = \left(\int_{-\infty}^{\infty}(\mathrm{ch}(\delta r))^{-1}dr\right)^{-1}\int_{-T}^{T}(\mathrm{ch}\delta(r-\eta))^{-1}d\eta \qquad (10.4)$$

with a fixed small positive δ. Then the main term will be

$$\sum_{\chi_j\leqslant T}\alpha_j\mathscr{H}_j^4\left(\frac{1}{2}\right) \gg T^2 \qquad (10.5)$$

and the contribution from the continuous spectrum is

$$\ll (\log T)^2\int_0 |\zeta(\frac{1}{2}+it)|^8 dt \ll T^{5/3}. \qquad (10.6)$$

On the right side of the functional equation, we have the same quantities $\mathscr{H}_j^4(\frac{1}{2})$ but with another weight function $h_0(\chi_j)+h_1(\chi_j)$ (note again 115 that $\mathscr{H}_j(\frac{1}{2}) = 0$ if $\epsilon_j = -1$). This function results from the integration of the initial h with an oscillatory kernel and for this reason, its order must be smaller; hence, an asymptotic formula must exist for the fourth spectral moment with the main term $T^2(\log T)^{n_0}$. So the product (10.3) may be estimated as $O(T(\log T)^B)$ with some fixed B (positive, of course). For this reason, the proof will be complete, if we estimate the integral with the "main" terms; simultaneously, the estimate for the integrals with fourth power of the Hecke series follows.

2.9.1 The integral with h_0.

First of all, we shall calculate the integral with h_0 in the first term on the right side of (5.1).

The key to do this is the equality

$$\int_{-\infty}^{\infty} h(r) r \operatorname{th}(\pi r) dr = -\frac{\pi}{2} \int_{0}^{\infty} J_0(x)\varphi(x)\, dx \tag{10.7}$$

if arbitrary "good" functions h and φ are connected by the transformation (1.27) (compare the coefficients before $\delta_{n,m}$ in (1.25) and (1.31); the proof of (10.7) is contained in [1]). Now, for our function $h_0(r)$, we have the representation (4.7) and it is sufficient to calculate the integral

$$\int_{-\infty}^{\infty} A_{00}(r; \xi; \rho, v) r \operatorname{th}(\pi r) dr. \tag{10.8}$$

It may be interpreted as the main term in the summation formula for the sum of Kloosterman sums with weight function $\varphi(x) = \pi\xi(4\pi)^{1-2\rho}$ $k_0(x\xi, v)$ (ξ being considered as the parameter). It allows us to use (10.7) and using the tabular integrals, we come to the relation

$$\int_{-\infty}^{\infty} h_0(r) r \operatorname{th}(\pi r) dr = \int_{-\infty}^{\infty} h(u) u \operatorname{th}(\pi u) b_0(u; s, v; \rho, \mu) du \tag{10.9}$$

where, with the additional notation,

$$\varphi_0(\xi;\rho,v) = \tag{10.10}$$
$$= \frac{2^{2\rho-1}\Gamma(v+\rho-\frac{1}{2})\xi^v}{\cos(\pi v)\Gamma(2v)\Gamma(3/2-v-\rho)} F(v+\rho-\frac{1}{2}\cdot v+\rho-\frac{1}{2};2v;\xi),$$

$$\varphi_1(\xi;\rho,v) = \tag{10.11}$$
$$= \frac{1}{\pi}\left(\frac{2}{|\xi|}\right)^{2\rho-1}\Gamma(\rho+v-\frac{1}{2})\Gamma(\rho-v+\frac{1}{2})F(\rho+v-\frac{1}{2},\rho-v+\frac{1}{2};1;\frac{1}{\xi})$$

we have 116

$$b_0(u;s,v;\rho,\mu) = -\frac{(4\pi)^{1-2\rho}}{2\pi}\times \tag{10.12}$$
$$\left\{\int_0^1(\varphi_0(\xi^2;\rho,v)+\varphi_0(\xi^2;\rho,1-v))A_{00}(u,\frac{1}{\xi};1-\rho,\mu)\xi^{2\rho-2s-1}d\xi+\right.$$
$$+2\int_1^\infty\varphi_1(\xi^2;\rho,v)A_{00}(u,\frac{1}{\xi};1-\rho,\mu)\xi^{2\rho-2s-1}d\xi+$$
$$\left.+2\int_0^\infty\varphi_1(-\xi^2;\rho,v)A_{01}(u,\frac{1}{\xi};1-\rho,\mu)\xi^{2\rho-2s-1}d\xi\right\}.$$

2.9.2 The transition to the limit.

No problems arises from terms which contain the values of the initial function at points $\rho\pm(v-\frac{1}{2})-1$ and $s\pm(\mu-\frac{1}{2})-1$. Since $h(\pm\frac{i}{2}), h'(\pm\frac{i}{2})$ are zeros, only one term is non-zero when $s=\mu=\frac{1}{2}$. This term contains the value $h(i(2it-\frac{1}{2}))$ which is exponentially small.

Further, taking into account the representations (4.15), (4.12) and (10.9), we can write the sum of all the other terms as

$$\int_{-\infty}^{\infty} u\,\mathrm{th}(\pi u)h(u)\{\zeta(2s)\zeta(2\rho)C_0(u;s,v,\rho,\mu)\}du. \tag{10.13}$$

We know definitely that the function in the brackets must be finite for our specialization, because the left side of the main identity is finite (in addition, for an arbitrary function h). Firstly, we can take $\rho = \nu$; then we come to the limit $\mu \to s$, and finally, take the limiting case $s \to \frac{1}{2}$. The requirement for the result to be finite in this limiting process will give us several identities for the integrals with some hypergeometric functions; after all, we shall come to a linear combination of some terms with the products of the derivatives $\zeta^{(k)}(1 + 2it)$. The number of zeta-functions in each term does not exceed 6 and the order of the differentiation is not larger than 4. The coefficients of this linear combination are some integrals with hypergeometric functions and their derivatives
117 with respect to parameters. So these coefficients may be investigated in the same manner as before; for this reason, the integral with the "main" terms (over t) does not exceed $O(T(\log T)^B)$ with some fixed B.

This would be the end of the proof. Of course, someone can say that, indeed, some final steps are not there. It is true; but I believe that there are no pressing reasons to extend this sufficiently long paper and it would be better to publish the details somewhere else.

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ON RAMANUJAN'S ELLIPTIC INTEGRALS AND MODULAR IDENTITIES

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119 **Introduction** It is known from Hecke ([5], p. 472) that 'special integrals of the third kind' of stufe N (i.e. having logarithmic singularities at most at the cusps of the principal congruence subgroup $\Gamma(N)$ turn out to be of elementary type, namely, logarithms of functions invariant under $\Gamma(N)$. Results of this kind have been discovered much earlier by Ramanujan in [11] as may be seen in the sequel, especially in §2. In this connection, Ramanujan was perhaps, one of the first to have considered the problem of 'evaluating' elliptic integrals associated with modular curves of small level, although elliptic integrals, in general, have been investigated in depth by various mathematicians such as Jacobi, Cayley and others. In the literature, transformations of orders 2, 3, 5 have been employed with a view to reduce formidable elliptic integrals to simpler (or more explicit) form [4].

In [11], Ramanujan has considered elliptic integrals associated with $\Gamma_0(N)$ for $N = 5, 7, 10, 14, 15$ and also a solitary hyperelliptic integral (for $\Gamma_0(35)$). Various such elliptic integrals are found in scattered form in [11], with endeavours, via quadratic and higher order transformations and interesting modular relations, to simplify them.

The principal objective of this paper is to make a systematic study of all the elliptic integrals and associated formulae recorded by Ramanujan in [11] and provide complete proofs.

We shall, in addition, uphold various modular identities involving Eisenstein series stated by Ramanujan in different places in [11] and presumably used by him in computing singular values of modular functions. With the help of such identities, we exhibit a nonlinear differential equation for Eisenstein series denoted in §3 by $\mathscr{E}_p$, for $p = 5, 7$; for $p = 5$, this differential equation is essentially equivalent to the nonlinear differential equation written down by ramanujan [11] for a function $F(\lambda_5)$, where λ_5 is a 'Hauptmodul' for $\Gamma_0(5)$. One has to compare these with nonlinear differential equations obtained by Eichler-Zagier [2] for divisor values of Weierstrass' elliptic function.

1 Notation and preliminary results 120

1.1 Let $\mathfrak{H}$ denote the complex upper half-plane and for $z \in \mathfrak{H}$, let $x = e^{2\pi iz}$, so that $|x| < 1$. By $\Gamma(1) = \Gamma$, we mean the modular group $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) | a, b, c, d \in \mathbb{Z},\ ad - bc = 1\}$, acting on $\mathfrak{H}$ via the analytic homeomorphisms $z \to (az + b)(cz + d)^{-1}$. As usual, $\Gamma(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma | c \equiv 0 (\text{mod } N)\}$. By Q, R we mean the normalized Eisenstein series $E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d^3 \right) e^{2\pi inz}$, $E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d^5 \right) \times e^{2\pi inz}$ of weights 4 and 6 respectively. Further, P stands for $E_2(z) = 1 - 24 \times \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d \right) e^{2\pi inz}$. Connecting E_2, E_4 and E_6, we have the important relations ([9], p. 142) :

$$\begin{aligned} E_4 - E_2^2 &= -12\vartheta(E_2) \\ E_6 - E_2E_4 &= -3\vartheta(E_4) \\ E_4^2 - E_2E_6 &= -2\vartheta(E_6) \end{aligned} \tag{1}$$

where $\vartheta := \dfrac{1}{2\pi i} \dfrac{d}{dz} = x \dfrac{d}{dx}$.

Following Ramanujan, we write for $|x|$, $|x'| < 1$,

$$\begin{aligned} f(x, x') :&= 1 + \sum_{n=1}^{\infty} (xx')^{n(n-1)/2} (x^n + (x')^n) \\ &= \prod_{n=0}^{\infty} [1 + x(xx')^n][1 + x'(xx')^n][1 - (xx')^{n+1}) \end{aligned}$$

(from Gauss and Jacobi).

Further, setting $f(-x) := f(-x, -x^2)$ for $|x| < 1$, we know that $\eta(z) = x^{1/24} f(-x)$ is just Dedekind's η-function; indeed, $\eta(z) = e^{2\pi i z/24} \times \prod_{n=1}^{\infty}(1 - e^{2\pi i n z})$ and moreover, $\eta^{24} = (1/1728)(E_4^3 - E_6^2)$. For rational $r > 0$, let η_r be defined by $\eta_r(z) = \eta(rz)$ for $z \in \mathfrak{H}$. Then the function $x^{-1/8} \times \eta_2^2(z)/\eta(z)$ is just Ramanujan's function $\psi(z)$, as found in [**?**]. Setting

$$u = x^{1/5} f(-x, -x^4)/f(-x^2, -x^3) = x^{1/5} \prod_{n=0}^{\infty} \frac{(1 - x^{5n+1})(1 - x^{5n+4})}{(1 - x^{5n+2})(1 - x^{5n+3})}$$

we know that u is a 'Hauptmodul' for the group $\Gamma(5)$ of genus 0. Also, u, can be represented as an infinite continued fraction continued frac-
121 tion $u = \frac{x^{1/5}}{1+}\frac{x}{1+}\frac{x^2}{1+}\cdots$ Moreover, we know from Ramanujan [10] the modular relations :

$$1/u - 1 - u = \eta_{1/5}/\eta_5, \tag{2}$$

$$1/u^5 - 11 - u^5 = \eta^6/\eta_5^6. \tag{3}$$

Let us note that, on the right hand side of (3), we have just $1/\lambda_5$, where $\lambda_5 := \eta_5^6/\eta^6$ is just a 'Hauptmodul' for the group $\Gamma_0(5)$ of genus 0. Likewise, for the congruence subgroup $\Gamma_0(7)$ of genus 0, we have $\lambda_7 := \eta_7^4/\eta^4$ as a 'Hauptmodul'.

Between u and u_2 defined by $u_2(z) = u(2z)$, we have the modular relation ([10], p. 326) :

$$\begin{aligned} u_2/u^2 &= (1 + uu_2^2)/(1 - uu_2^2) \\ &= (1 + k)/(1 - k) \end{aligned} \tag{4}$$

where we have written k for uu_2^2 following Ramanujan ([11], p. (70)
/78). From (4), we see easily that

$$u^5 = k(1-k)^2/(1+k)^2, \; u_2^5 = k^2(1+k)/(1-k). \tag{5}$$

The group $\Gamma_0(p)$ for odd primes p is known to have two cusps 0 and ∞ and Eisenstein series of Nebentypus $(-l, p, \chi_p)$ corresponding to the two cusps are given. for $l \geq 2$, by

$$E_l^0(z;\chi_p) := \sum_{n=1}^{\infty}\left(\sum_{1\leq d|n} d^{l-1}\chi_p(n/d)\right)e^{2\pi inz} \tag{6}$$

$$E_l^\infty(z;\chi_p) := \frac{(-1)^{[l/2]}p^{[l-1]/2}}{(2\pi)^l}(l-1)!\sum_{n=1}^{\infty}\chi_p(n)n^{-l}+$$
$$\sum_{n=1}^{\infty}\left(\sum_{1<dn} d^{l-1}\chi_p(d)\right)e^{2\pi inz}$$

where $\chi_p(m)$ denotes the Legendre symbol $(\frac{m}{p})$, [ ] denotes the integral part and further, $\chi_p(-1) = (-1)^l$, necessarily ([5], p. 818). There exist two linearly independent series $E_l(z)$, $E_l(pz)$ for even $l > 2$, which are of Haupttypus $(-l, p, 1)$; however, for $l = 2$, there is just one Eisenstein series of Haupttypus $(-2, p, 1)$ and it is given by

$$E_2(z;1;\Gamma_0(p)) = E_2(z) - pE_2(pz)$$
$$= 1 - p - 24\sum_{\substack{n=1\\ p\nmid d}}^{\infty}\left(\sum_{1\leq d|n}\right)e^{2\pi inz}$$

For $p = 5$ and $l = 2$, we know from [8] that $E_2^0(z;\chi_5) = \eta_5^5/\eta$ **122**
and further, denoting simply by $E_2(z;\chi_5)$ obtained by 'normalizing' $E_2^\infty(z;\chi_5)$ so as to have constant term $-(1/5)$, we also have $E_2(z;\chi_5) = -(1/5)\eta^5/\eta_5$.

In the case $p = 7$ and $l = 3$, $E_3^\infty(z;\chi_7)$ is upto a constant factor, just the cube of the Eisenstein series $E_1(z;\chi_7)$ of Nebentypus $(-1, 7, \chi_7)$ given by

$$E_1(z;\chi_7) = 1 + 2\sum_{n+1}^{\infty}\chi_7(n)\frac{x^n}{1-x^n} \quad \text{(see [11], [8]).}$$

Logarithmic differentiation of η with respect to x gives

$$\frac{24}{\eta}x\frac{d\eta}{dx} = \frac{24}{2\pi i\eta}\frac{d\eta}{dz} = E_2(z) \tag{8}$$

Similarly, logarithmic differentiation of $u = x^{1/5}\prod_{n=1}^{\infty}(1 - x^n)^{\chi_5(n)}$ and $u_2 = x^{2/5}\prod_{n=1}^{\infty}(1 - x^{2n})^{\chi_5(n)}$ gives

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{5x} - \sum_{n=1}^{\infty}\frac{n\chi_5(n)x^{n-1}}{1 - x^n} \tag{9}$$

$$= -\frac{1}{x}E_2(z : \chi_5),$$

$$\frac{1}{u_2}\frac{du_2}{dx} = -\frac{2}{x}E_2(2z; \chi_5).$$

For the Hauptmodul $\lambda_p = (\eta_p/\eta)^{24/(p-1)}$ for $p = 5, 7$ we have likewise

$$\frac{1 - p}{2\pi i \cdot \lambda_p}\frac{d\lambda_p}{dz} = E_2(z; 1; \Gamma_0(p)) \tag{10}$$

in view of (8) and (**??**).

Setting $r = \frac{k}{1 - k^2}$, $s = \frac{1 + k - k^2}{1 - 4k - k^2}$ where k is just the function uu_2^2 considered above, we recall from Weber ([14], p. 86) the formulae

$$rs^5 = x\prod_{n=1}^{\infty}(1 + x^n)^{24} = (\eta_2/\eta)^{24}$$

$$r^5 s = x^5\prod_{n=1}^{\infty}(1 + x^{5n})^{24} = (\eta_{10}/\eta_5)^{24}.$$

123 These imply immediately that $r^{24} = (r^5 s^5)/rs^5$ and so

$$\frac{1 - k^2}{k} = \frac{1}{r} = \frac{\eta_2\eta_5^5}{\eta\eta_{10}^5} \tag{11}$$

$$= \frac{1}{x\psi^2(x^5)}\frac{\eta_2\eta_5^3}{x^{1/4}\eta\eta_{10}}$$

$$= \frac{1}{x\psi^2(x^5)} \cdot f(x, x^4)f(x^2, x^3)$$

$$= \frac{\psi^2(x) - x\psi^2(x^5)}{x\psi^2(x^5)}. \tag{12}$$

The last-mentioned equality is a consequence of Ramanujan's identity

$$\psi^2(x) - x\psi^2(x^5) = f(x, x^4)f(x^2, x^3)$$

proved by Watson [13] and called a "rudimentary" example of the use of quadratic forms. (See also [1], pp. 63-65 which provides a proof ostensibly "more difficult" and further "similar to that of Entry 9(iii), which is obviously an analogue of Entry 10(v), a fact made even more transparent by Entry 10(iv)"). In contrast, the following Proposition 1(ii) gives an independent, refreshingly different and perhaps even elegant proof for Ramanujan's identity above; in fact, one needs merely to substitute $\psi(x) = x^{-1/8}\eta_2^2(z)/\eta(z)$ therein and note further that $f(x, x^4)f(x^2, x^3) = x^{-1/4}\eta_2(z)\eta_5^3(z)/(n(z)\eta_{10}(z))$. This proof corroborates G.N. Watson's belief that Ramanujan discovered this formula "not by manipulating quadratic forms but by transforming series of Lambert's type".

Proposition 1.

(i) $\eta_2^4\eta_5^2 - 5\eta^2\eta_{10}^4 = \eta^5\eta_5\eta_{10}/\eta_2$

(ii) $\eta_2^4\eta_5^2 - \eta^2\eta_{10}^4 = \eta\eta_2\eta_5^5.\eta_{10}$.

Proof. We first rewrite these identities in terms of the Eisenstein series $E_2^0(z;\chi_5)$ and $E_2^\infty(z;\chi_5)$ obtained by normalizing $E_2^\infty(z;\chi_5)$, using the relations $E_2^0(z;\chi_5) = \eta_5^5/\eta$, $E_2(z;\chi_5) = -\frac{1}{5}\eta^5/\eta_5$ and the 'Hauptmodul' $\tau : 10\eta_2\eta_{10}^3/(\eta^3\eta_5)$ for $\Gamma_0(10)$, as follows :

(i)′ $E_2(z;\chi_5) - E_2(2z;\chi_5) = -\dfrac{\tau}{2}E_2(z;\chi_5)$

(ii)′ $E_2^0(z;\chi_5) + E_2^0(2z;\chi_5) = \dfrac{\eta_2^4\eta_5^2}{\eta^2\eta_{10}^4}E_2^0(2z;\chi_5)$

□

By direct checking (see also [3], p. 449) we see that the modular function τ has a simple zero at $i\infty$ and a simple pole at 0. On the other hand, $E_2(z;\chi_5)$ is regular at $i\infty$ and in view of the relation $E_2(-1/z;\chi_5) = (z^2;\ \sqrt{5})E_2^0(z/5)$ from ([5], p. 819), $E_2(z;\chi_5)$ has a zero at 0. Consequently, $(\tau/2+1)E_2(z;\chi_5)$ which is regular at $i\infty$ and 0 is indeed an entire modular form of weight 2 (and nebentypus) for $\Gamma_0(10)$. Thus the proof of (i)′ reduces to showing that $(\tau/2+1)E_2(z;\chi_5) = E_2(2z;\chi_5)$. In view of Hecke's result ([5], Satz 2, p. 811 — see also p. 953), it is enough to compare the first $\dfrac{[\Gamma(1):\Gamma_0(10)]\times 2}{12}$ (= 3) coefficients in the Fourier expansions of both sides. That the first three Fourier coefficients agree on both sides is immediate from the following expansions : 124

$$\begin{aligned}
E_2(z;\chi_5) &= -\frac{1}{5} + e^{2\pi iz} - e^{4\pi iz} - 2e^{6\pi iz} + \cdots \\
(\tau/2+1) &= 1 + 5e^{2\pi iz} + 15e^{4\pi iz} + 40e^{6\pi iz} + \cdots \\
(\tau/2+1)E_2(z;\chi_5) &= -\frac{1}{5} + 0\cdot e^{2\pi iz} + e^{4\pi iz} + 0\cdot e^{6\pi iz} + \cdots \\
E_2(2z;\chi_5) &= -\frac{1}{5} + 0\cdot e^{2\pi iz} + e^{4\pi iz} + 0\cdot e^{6\pi iz} + \cdots
\end{aligned}$$

This proves (i)′ and therefore (i). Using (i), we may rewrite the factor $\eta_2^4\eta_5^2/(\eta^2\eta_{10}^4)$ on the right hand side of (ii)′ as $5 + 10/\tau$. Hence (ii)′ will follow, if we establish

(ii)″ $$\frac{\tau}{4\tau+10}E_2^0(z;\chi_5) = E_2^0(2z;\chi_5).$$

Now $\tau/(4\tau+10)$ is seen to be regular at all the cusps of $\Gamma_0(10)$, except those equivalent to $\pm 1/5$ where it has a simple pole. Further, $E_2^0(z;\chi_5)$ of Nebentypus $(-2,5,\chi_5)$ has a zero at ∞ and hence at $\pm 1/5$ (equivalent to ∞ under $\Gamma_5(5)$. Consequently $[\tau/(4\tau+10)]E_2^0(z;\chi_5)$ is an entire modular form of weight 2 (and Nebentypus) for $\Gamma_0(10)$. From the Fourier expansions

$$\frac{\tau}{4\tau+10} = e^{2\pi iz} - e^{4\pi iz} + 0\cdot e^{6\pi iz} + \cdots$$

$$E_2^0(z;\chi_5) = e^{2\pi iz} + e^{4\pi iz} + 2\cdot e^{6\pi iz} + \cdots,$$

it is clear that the first three Fourier coefficients of $[\tau/(4\tau+10)]E_2^0(z;\chi_5)$ coincide with the corresponding coefficients of $E_2^0(2z;\chi_5)$. This proves (ii)″ by Hecke's theorem above and hence (ii) is proved.

Remark. As a further illustration for the utility of Proposition 1, we give an alternative proof for Ramanujan's identity 125

$$\begin{aligned} &x\psi^3(x)\psi(x^5) - 5x^2\psi(x)\psi^3(x^5) \\ &= \frac{x}{1-x^2} - \frac{2x^2}{1-x^4} - \frac{3x^3}{1-x^6} + \frac{4x^4}{1-x^8} + \frac{6x^6}{1-x^{12}} - \cdots \end{aligned}$$

(See [1], pp. 45-49 for a "rather difficult" proof which uses besides "results from Section 13", leading nevertheless to no "circular reasoning" etc.). The right hand side of this identity is just

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{nx^n\chi_5(n)}{1-x^{2n}} &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} n\chi_5(n)x^{n(2m+1)} \\ &= E_2^\infty(z;\chi_5) - E_2^0(2z,\chi_5) \\ &= \frac{1}{5}\frac{\eta^5}{\eta_5} + \frac{1}{5}\frac{\eta_2^5}{\eta_{10}} \quad \text{(by [8], p. 227)} \\ &= \frac{\eta^2}{5\eta_5^2\eta_{10}}\left(\eta_2^4\eta_5^2 - \frac{\eta^5\eta_5\eta_{10}}{\eta_2}\right) \\ &= \eta^2\eta_2\eta_{10}^3/\eta_5^2, \quad \text{by Proposition 1(ii),} \end{aligned}$$

while the left hand side is precisely $\dfrac{\eta_2^6\eta_{10}^2}{\eta^3\eta_5} - 5\dfrac{\eta_2^2\eta_{10}^6}{\eta\eta_5^3} = \dfrac{\eta_2^2\eta_{10}^2}{\eta^3\eta_5^3} \times (\eta_2^4\eta_5^2 - 5\eta^2\eta_{10}^4) = \dfrac{\eta^2\eta_2\eta_{10}^3}{\eta_5^2}$, using Proposition 1(i).

1.2 We have gathered here, from Chapter XIX of the Notebooks, many of Ramanujan's identities involving the functions $\varphi(q) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}$

or $\psi(q) := q^{-1/8}\eta^2(2z)/\eta(z)$ introduced by him in Chapter XVI with $q := \exp(2\pi iz)$ or various transforms of φ and ψ. We shall rewrite the identities (with a view to elucidate them) in terms of the normalized Eisenstein series $E_4(z)$ of weight 4 for $\Gamma_0(1)$, its transforms and the following Eisenstein series of Haupttypus or Nebentypus $(-k, N, \epsilon)$ with ϵ equal to the trivial character modulo N or the real character $\epsilon(n) := (\frac{n}{N})$ modulo N *and* $\epsilon(-1) = (-1)^k$ namely

$$E_{k,N,1}(z) = E_{k,1}(z;\Gamma_0(N);\epsilon) := \sum_{n=1}^{\infty}\left(\sum_{1\leq d|n}\epsilon(n/d)d^{k-1}\right)e^{2\pi inz}$$

$$E_{k,N,z}(z) = E_{k,2}(z;\Gamma_0(N);\epsilon) := \gamma_k(N) + \sum_{n=1}^{\infty}\left(\sum_{1<d|n}\epsilon(d)d^{k-1}\right)e^{2\pi inz}$$

126 with a constant $\gamma_k(N)$ that can be determined. We note that with $q = \exp(2\pi iz)$, φ^2 is a modular form of weight 1 with multipliers for the congruence subgroup $\Gamma_0(4)$. For dealing with products of transforms of ψ, we recall from Honda and Miyawaki (J. Math. Soc. Japan, 26(1974), 362-373) that the power product $\prod_{j=1}^{r}\eta_{t_j}^{n_j}$ with integral $t_1,\dots,t_r,n_1,\dots,n_r$ is a modular form of weight $\frac{1}{2}(n_1+\dots+n_r)$ for the congruence subgroup $\Gamma_0(\underset{j}{l.c.m.}t_j)$ if 24 divides both $n_1t_1 + \dots + j_rt_r$ and $(\underset{j}{l.cm.}\ t_j) \times (n_1/t_1 + \dots + n_r/t_r)$ Ramanujan's identities referred to may now be rewritten as follows.

(1) $\varphi^2(q) = E_{1,2}(z;\Gamma_0(4);\epsilon)$

(2) $\varphi^4(q) = 8(E_{2,2}(z;\Gamma_0(4);1) - 4E_{2,2}(4z;\Gamma_0(4);1))$

(3) $\varphi^8(q) = \frac{1}{15}(E_4(z) - 2E_4(2z) + 16E_4(4z))$

(4) $q\psi^3(q)\psi(q^5) - 5q^2\psi(q)\psi^3(q^5) = E_{2,2}(z;\Gamma_0(5);1) - E_{2,2}(2z;\Gamma_0(5);1)$

(5) $5\varphi(q)\varphi^3(q^5) - \varphi^3(q)\varphi(q^5) = 4(E_{2,5,2}(z) - 2E_{2,5,2}(2z) - 4E_{2,5,2}(4z))$

(6) $25\varphi(q)\varphi^3(q^5) - \varphi^5(q)/\varphi(q^5) = 40(E_{2,5,2}(z) - 4E_{2,5,2}(4z))$

(7) $\psi^5(q)/\psi(q^5) - 25q^2\psi(q)\psi^3(q^5) = 5(E_{2,5,2}(z) - 2E_{2,5,2}(2z))$

(8) $q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3) = E_{3,3,2}(z) - E_{3,3,2}(2z)$

(9) $9\varphi(q)\varphi^5(q^3) - \varphi^5(q)\varphi(q^3) = 8(E_{3,3,2}(z) - 2E_{3,3,2}(2z) - 8E_{3,3,2}(4z))$

(10) $\psi^3(q)/\psi(q^3) = E_{1,3,2}(z) - E_{1,3,2}(2z)$

(11) $\varphi^3(q)/\varphi(q^3) = 6(E_{1,3,2}(z) + 2E_{1,3,2}(2z) - 2E_{1,3,2}(4z))$

(12) $q\psi(q^2)\psi(q^6) = E_{1,3,2}(z) - E_{1,3,2}(4z)$

(13) $\varphi(q)\varphi(q^3) = 2(E_{1,3,2}(z) + 2E_{1,3,2}(4z))$

(14) $q\psi^2(q)\psi^2(q^3) = E_{2,3,2}(z) - E_{2,3,2}(2z)$

(15) $\varphi^2(q)\varphi^2(q^3) = 4(E_{2,3,2}(z) - 2E_{2,3,2}(2z) + 4E_{2,3,2}(4z))$

(16) $q\psi(q)\psi(q^7) = E_{1,7,2}(z) - E_{1,7,2}(2z)$

(17) $\varphi(q)\varphi(q^7) = E_{1,7,2}(z) - 2E_{1,7,2}(2z) + 2E_{1,7,2}(4z)$

The identities are easily proved by noting that both sides are modular forms of weight k for $\Gamma_0(M)$ for the appropriate value of M and comparing their first $1 + \frac{k}{2}(\Gamma_0(1) : \Gamma_0(M)$ corresponding Fourier coefficients in the light of Hecke's theorem ([5], p. 811).

Remark. Using Proposition 1, we can prove the relation $\gamma = \frac{1}{\mu}\left(\frac{\mu-1}{\mu-5}\right)^2$ for the Hauptmoduls $\gamma := \eta_5^6/\eta^6$ and $\mu := \psi^2(q)/\psi^2(q^5)$ for $\Gamma_0(5)$ and $\Gamma_0(10)$ respectively. Substituting this relation between γ and μ, we obtain

$$\begin{aligned}1 + 22\gamma + 125\gamma^2 &= \frac{1}{\mu^2(\mu-5)^4}\\ &\qquad [\mu^2(\mu-5)^4 + 22\mu(\mu-1)^2(\mu-5)^2 + 125(\mu-1)^4]\\ &= \frac{1}{\mu^2(\mu-5)^4}(\mu^2+2\mu+5)^2(\mu^2-2\mu+5)\end{aligned}$$

As a consequence, we obtain Entry 4(i) of Chapter XXI of the Notebooks in the following form :

$$(\eta^5/\eta_5)(1+22\gamma+125\gamma^2)^{1/2} = \frac{\psi^5(q^5)}{\psi(q)}(\mu^2+2\mu+5)(\mu^2-2\mu+5)^{1/2}$$

We have, as an analogue of assertion (i) of Proposition 1, the identity $\varphi^2(q)-\varphi^2(q^5) = 4\eta_2^2\eta_5\eta_{20}/\eta\eta_4$ which is a consequence of the remarks above. We note further that Entry 5(i) in Chapter XXI is equivalent to the easily proved relations :

$$(E_{1,7,2}(z))^2 = \left(1+2\Sigma\left(\frac{n}{7}\right)\frac{q^n}{1-q^n}\right)^2 = E_{2,7,2}(z) = E_2(z)-7E_2(7z)$$

with the usual definition of E_2 and

$$(E_{1,7,2}(z))^3 = \eta^7/\eta_7 + 13\eta^3\eta_7^3 + 49\eta_7^7/\eta.$$

One needs, for these, merely to verify the equality of at most the first 3 corresponding Fourier coefficients on both sides, in view of Hecke's theorem again.

Other relations stated by Ramanujan for $\psi^2, \psi^4, \psi^6, \psi^8, \varphi^3/\varphi_3, \varphi_3^3/\varphi$, $\psi^3/\psi_3, \psi_3^3/\psi, \eta^3/\eta_3, \eta_9^3/\eta_3$ may be derived in the same manner.

The proofs of identities (4)–(11) as indicated above may be seen to be simpler than those in [1] (p. 45-56, 12-16). One may also compare the proofs of identities (12)-(13) with those outlined in Hardy's "Ramanujan" (pp. 220-222)

2 Elliptic integrals considered by Ramanujan

2.1 Let us being with the simplest example of elliptic integrals for which Ramanujan ([11], p. (67) + 1) has written explicit primitives :

$$\int_{e^{-2\pi}}^{x} \sqrt{Q}\frac{dx}{x} = \log\frac{Q^{3/2}-R}{Q^{3/2}+R}. \qquad (13)$$

Applying Ramanujan's "very useful substitution" $Z = R^2/Q^3$, we have $\dfrac{dZ}{Z} = 2\dfrac{dR}{R} - 3\dfrac{dQ}{Q}$ and hence (1) leads to $\dfrac{1}{Z}x\dfrac{dZ}{dx} = \dfrac{R^2 - Q^3}{QR}$. Now

$$x\frac{d}{dx}\log\frac{Q^{3/2} - R}{Q^{3/2} + R} = x\frac{dZ}{dx}\frac{d}{dZ}\log\left(\frac{1 - \sqrt{Z}}{1 + \sqrt{Z}}\right) = x\frac{dZ}{dx}\cdot\frac{1}{\sqrt{Z}(Z - 1)} = \sqrt{Q}.$$

Since $R(\mathrm{i}) = 0$, (13) is proved. **128**

2.2 We take up next Ramanujan's formula on page (70) /78 of [11] explicitly evaluating an elliptic integral in elementary terms :

$$\frac{8}{5}\int\frac{\psi^5(x)}{\psi(x^5)}\frac{dx}{x} = \log u^2u_2^3 + \sqrt{5}\log\frac{1 + \epsilon^{-3}uu_2^2}{1 - \epsilon^2uu_2^2} \quad \left(\text{with } \epsilon = \frac{\sqrt{5} + 1}{2}\right) \tag{14}$$

From (5), $\log(u^2u_2^3) = \dfrac{1}{5}\log\left(k^8\dfrac{(1 - k)}{(1 + k)}\right)$. Also $\dfrac{u}{u_2} = \dfrac{f(x^2, x^3)}{x^{1/5}f(x, x^4)}$ by substituting the infinite product expansion for f. Using Ramanujan's identity ([10] II, p. 234) proved by Berndt ([1], p. 57)

$$\frac{\psi^5(x)}{\psi(x^5)} - 25x^2\psi(x)\psi^3(x^5) = 1 - 5x\frac{d}{dx}\log\frac{f(x^2, x^3)}{f(x, x^4)}$$

we may thus rewrite (14) as

$$\begin{aligned} 8\int_0^x x\psi(x)\psi^3(x^5)dx &= \log\frac{1 - k}{1 + k} + \frac{1}{\sqrt{5}}\log\frac{1 + \epsilon^{-3}k}{1 - \epsilon^3k} \\ &= \int_0^x \frac{d}{dx}\left(\log\frac{1 - k}{1 + k} + \frac{1}{\sqrt{5}}\log\frac{1 + \epsilon^{-3}k}{1 - \epsilon^3k}\right)dx \\ &= \int_0^x \frac{8kk'}{(1 - k^2)(1 - 4k - k^2)}dx \end{aligned}$$

denoting differentiation with respect to x by $'$. Therefore (14) will be established, once we show that

$$\frac{k'}{1-k^2} = \left(\frac{1-k^2}{k} - 4\right) x\psi(x)\psi^3(x^5). \tag{15}$$

Now we obtain from (4), by logarithmic differentiation (with respect to x) that

$$\begin{aligned} k'(1-k^2) &= \frac{1}{2}u_2'/u_2 - u'/u \\ &= \frac{1}{x}(E_2(z;\chi_5) - E_2(2z;\chi_5)), \quad \text{by (9)} \\ &= \frac{1}{5x}(\eta_2^5/\eta_{10} - \eta^5/\eta_5) \\ &= \frac{\eta_2}{5x\eta_5^2\eta_{10}} \cdot 5\eta^2\eta_{10}^4, \quad \text{by (i) of Proposition 1.} \end{aligned}$$

129 Thus using (11), we see that (15) will be proved, if we show that

$$\frac{\eta^2\eta_2\eta_{10}^3}{x\eta_5^2} = \frac{\eta_2\eta_5^5 - 4\eta\eta_{10}^5}{\eta\eta_{10}^5} \cdot x\frac{\eta_2^2\eta_{10}^6}{x^2\eta\eta_5^3}$$

$$\text{i.e.} \quad \frac{\eta^5\eta_5\eta_{10}}{\eta_2} = \frac{\eta\eta_2\eta_5^5}{\eta_{10}} - 4\eta^2\eta_{10}^4. \tag{16}$$

But the right hand side is just $\eta_2^4\eta_5^2 - 5\eta^2\eta_{10}^4$ by (ii) of proposition 1 and so (16) is just identity (i) of the same proposition, proving Ramanujan's assertion (14).

3 Elliptic integrals arising from cusp forms

3.1 On page (67) +1 of [11], Ramanujan writes down the formula

$$\int_0^x \eta\eta_3\eta_5\eta_{15}\frac{dx}{x} = \frac{1}{5}\int_{2\tan^{-1}(1/\sqrt{5})\left(\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\phi}{\sqrt{1-\frac{9}{25}\sin^2\phi}}$$

where $v := \dfrac{\eta^3(z)\eta^3(15z)}{\eta^3(3z)\eta^3(5z)}$. The integrand on the left hand side is the 'unique' holomorphic differential for $\Gamma_0(15)$.

For the proof of (**??**), we need to note the following from Fricke ([3], pp. 438-439). If $\tau := (\eta_3\eta_5/\eta\eta_{15})^3$ and $\sigma = \dfrac{2\pi i}{4\pi^2\eta\eta_3\eta_5\eta_{15}}\dfrac{d\tau}{dz}$, then $\sigma^2 = \tau^4 - 10\tau^3 - 13\tau^2 + 10\tau + 1$; the two modular functions τ and σ generate the field of modular functions for $\Gamma_0(15)$. Note that $\tau = e^{-2\pi iz} + 3 + \cdots$ has a simple pole at $i\infty$. Now

$$\int_0^x \eta\eta_3\eta_5\eta_{15}\frac{dx}{x} = 2\pi i\int_{i\infty}^{z} \eta\eta_3\eta_5\eta_{15}\,dz \quad (z = iy, y > 0)$$
$$= \int_{\tau}^{\infty} \frac{d\tau}{\sqrt{\tau^4 - 10\tau^3 - 13\tau^2 + 10\tau + 1}} \tag{18}$$

Further $X^4 - 10X^3 - 13X^2 + 10X + 1 = (X^2 - 11X - 1)(X^2 + X - 1) =$ **130**
$(X + \epsilon^{-5})(X - \epsilon^5)(X - \epsilon^{-1})(X + \epsilon)$ with $\epsilon = (\sqrt{5} + 1)/2$ and $\epsilon^5 > \epsilon^{-1} > -\epsilon^{-5} > -\epsilon$. To evaluate the (real-valued) integral (18), it is enough to consider the range $\epsilon^5 < \tau < \infty$, since for other values of τ, say $\epsilon^{-1} < \tau < \epsilon^5$, the expression inside the radical sign viz. $(\tau + \epsilon^{-5})(\tau - \epsilon^5)(\tau - \epsilon^{-1}) \times (\tau + \epsilon)$ becomes negative and hence the integrand will be purely imaginary. Using formula (78) in §66 of Greenhill [4] (with $\alpha = \epsilon^5, \beta = \epsilon^{-1}, \gamma = -\epsilon^{-5}, \delta = -\epsilon$) we have, for $\epsilon^5 < \tau < \infty$.

$$\int_{\epsilon^5}^{\tau} \frac{d\tau}{\sqrt{\tau^4 - 10\tau^3 - 13\tau^2 + 10\tau + 1}}$$
$$= \frac{2}{\sqrt{(\epsilon^5 + \epsilon^{-5})(\epsilon^{-1} + \epsilon)}}\,\mathrm{sn}^{-1}\sqrt{\frac{(\epsilon^{-1} + \epsilon)(\tau - \epsilon^5)}{(\epsilon^5 + \epsilon)(\tau - \epsilon^{-1})}}$$
$$= \frac{2}{5}\,\mathrm{sn}^{-1}\sqrt{\frac{\sqrt{5}}{6 + 3\sqrt{5}}\frac{(\tau - \epsilon^5)}{(\tau - \epsilon^{-1})}}$$

$$= \frac{2}{5}\int_0^{\phi} \frac{d\phi}{\sqrt{1-\frac{9}{25}\sin^2\phi}} \tag{19}$$

since $\kappa^2 = \dfrac{(\epsilon^{-1}+\epsilon^{-5})(\epsilon^5+\epsilon)}{(\epsilon^5+\epsilon^{-5})(\epsilon^{-1}+\epsilon)} = \dfrac{9}{25}$. The upper limit ϕ in (19) is given by $\sin^2\phi = \dfrac{\sqrt{5}}{6+3\sqrt{5}}\dfrac{(\tau-\epsilon^5)}{(\tau-\epsilon^{-1})}$ or $\cos^2\phi = \dfrac{6+2\sqrt{5}}{6+3\sqrt{5}}\dfrac{(\tau+\epsilon)}{(\tau-\epsilon^{-1})}$ so that $\phi = \tan^{-1}\left(\dfrac{5^{1/4}}{2\epsilon}\dfrac{\tau-\epsilon^5}{\tau+\epsilon}\right)$. From Ramanujan ([10], p. 208), we know that, for ϕ_1, ϕ_2 with $\cot\phi_1 \cdot \tan(\frac{1}{2}\phi_2) = \sqrt{1-\kappa^2\sin_1^2}$.

$$2\int_0^{\phi_1} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} = \int_0^{\phi_2} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} \tag{20}$$

However, from above, we have

$$\begin{aligned}
(\tan\phi)\sqrt{1-\kappa^2\sin^2\phi} &= \left(\frac{5^{1/4}}{2\epsilon}\sqrt{\frac{\tau-\epsilon^5}{\tau+\epsilon}}\right)\sqrt{1-\frac{9}{25}\frac{\sqrt{5}}{3\epsilon^3}\frac{(\tau-\epsilon^5)}{(\tau-\epsilon^{-1})}} \\
&= \frac{5^{1/4}}{2\epsilon}\sqrt{\frac{\tau-\epsilon^5}{\tau+\epsilon}}\sqrt{\frac{4\epsilon^2}{5^{3/2}}\frac{(\tau+\epsilon^{-5})}{(\tau-\epsilon^{-1})}} \\
&= \frac{1}{\sqrt{5}}\sqrt{\frac{(\tau-\epsilon^5)(\tau+\epsilon^{-5})}{(\tau+\epsilon)(\tau-\epsilon^{-1})}} \\
&= \frac{1}{\sqrt{5}}\sqrt{\frac{(\tau^2-11\tau-1)}{(\tau^2+\tau-1)}}
\end{aligned}$$

131 Thus from (19) and (20), we obtain

$$\int_{\epsilon^5}^{\tau} \frac{d\tau}{\sqrt{\tau^4-10\tau^3-13\tau^2+10\tau+1}}$$

$$= \frac{1}{5} \int_{0}^{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{(\tau^2-11\tau-1)}{(\tau^2+\tau-1)}}\right)} \frac{d\phi}{\sqrt{1-\frac{9}{25}\sin^2\phi}}$$

$$= \frac{1}{5} \int_{0}^{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)} \frac{d\phi}{\sqrt{1-\frac{9}{25}\sin^2\phi}}$$

where $v = \dfrac{1}{\tau} = \dfrac{\eta^3\eta_{15}^3}{\eta_3^3\eta_5^3}$. Hence

$$\int_{\tau}^{\infty} \frac{d\tau}{\sqrt{\tau^4-\cdots+1}} = \int_{\epsilon^5}^{\infty} \frac{d\tau}{\sqrt{\tau^4-\cdots+1}} - \int_{\epsilon^5}^{\tau} \frac{d\tau}{\sqrt{\tau^4-\cdots+1}}$$

$$= \frac{1}{5} \int_{0}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\phi}{\sqrt{1-\frac{9}{25}\sin^2\phi}}$$

$$- \frac{1}{5} \int^{2\tan_0^{-1}\left(\frac{1}{\sqrt{5}}\ \frac{\sqrt{1-11v-v^2}}{1+v-v^2}\right)} \frac{d\phi}{\sqrt{1-\frac{9}{25}\sin^2\phi}}$$

$$= \frac{1}{5} \int_{2\tan^{-1}(1/\sqrt{5})}^{2\tan^{-1}\left(\frac{1}{5}\ \frac{\sqrt{(1-11v-v^2)}}{1+v-v^2}\right)} \frac{d\phi}{\sqrt{1-\frac{9}{25}\sin^2\phi}}$$

which together with (18) gives us Ramanujan's formula (**??**).

The right hand side of (**??**) admits of interesting reduction when one resorts to well-known transformations available for elliptic integrals **132** such as Landen's transformation or Gauss' transformation. On page (67)

+ 1 of [11], Ramanujan also notes that

$$= \frac{1}{5} \int\limits_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{(1-11v-v^2)}{1+v-v^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\phi}{\sqrt{1-\frac{9}{25}\sin^2\phi}}$$

$$= \frac{1}{9} \int\limits_{2\tan^{-1}\left(\frac{(1-v/\epsilon^3)}{(1+v\epsilon^3)}\sqrt{\frac{(1+v\epsilon)(1-v\epsilon^5)}{(1-v/\epsilon)(1+v/\epsilon^5)}}\right)}^{\pi/2} \frac{d\psi}{\sqrt{1-\frac{1}{81}\sin^2\psi}} \tag{21}$$

also equal to

$$\frac{1}{4} \int\limits_{\tan^{-1}\left((3-\sqrt{5})\sqrt{\frac{(1-v/\epsilon)(1-v\epsilon^5)}{(1-v\epsilon)(1+v/\epsilon^5)}}\right)}^{\tan^{-1}(3-\sqrt{5})} \frac{d\psi}{\sqrt{1-\frac{15}{16}\sin^2\psi}} \tag{22}$$

with v and ϵ as above.

To prove (21), let us use Landen's transformation ([12], p. 496) :

$$\int \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} = \frac{1}{1+\sqrt{1-\kappa^2}} \int \frac{d\psi}{\sqrt{1-\left(\frac{1-\sqrt{1-\kappa^2}}{1+\sqrt{1-\kappa^2}}\right)^2\sin^2\psi}}$$

$$\text{for } \tan(\psi-\phi) - (\sqrt{1-\kappa^2})\tan\phi \tag{23}$$

with $\kappa = 3/5$ and determine the limits of integration on the right hand side corresponding to those on the left hand side of (23). We have only to verify that the relations

$$\tan(\psi-\phi) = \frac{4}{5}\tan\phi,\ \tan(\phi/2) = \frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}$$

together imply that

$$\tan(\psi/2) = \frac{1-v/\epsilon^3}{1+v\epsilon^3}\sqrt{\frac{(1+v\epsilon)(1-v\epsilon^5)}{(1-v/\epsilon)(1+v/\epsilon^5)}},$$

so that the upper limit $\pi/2$ for ψ in (21) will correspond to $2\tan^{-1}(1/\sqrt{5})$ arising for $v = 0$. Setting $t_1 = \tan(\phi/2)$, $t_2 = \tan((\psi-\phi)/2)$, we have then

$$\frac{2t_2}{1-t_2^2} = \tan(\psi-\phi) = \frac{4}{5}\frac{2t_1}{1-t_1^2}$$

and so 133

$$t_2 = \frac{5(1-t_1^2)}{4t_1} \pm \frac{1}{2}\sqrt{\frac{25}{16}\frac{(1-t_1^2)^2}{t_1^2}+4}.$$

Since $1-t_1^2 = 1-(1-11v-v^2)/[5(1+v-v^2)] = (4/5)(1+4v-v^2)/(1+v-v^2)$ and $(25/16)(1-t_1^2)^2/t_1^2+4 = 9(1+v^2)^2/\{(1+v-v^2)\times(1-11v-v^2)\}$, we have

$$t_2 = \frac{-\sqrt{5}(1+4v-v^2)+3(1+v^2)}{2\sqrt{(1+v-v^2)(1-11v-v^2)}}$$

noting that only the positive root has to be taken for t_2, in view of t_2 having to be positive for large v. Thus

$$\begin{aligned} t_2 &= \frac{\epsilon^2(\epsilon-v)(\epsilon^{-5}-v)}{\sqrt{(1-v/\epsilon)(1+v\epsilon)(1-v\epsilon^5)(1+v/\epsilon^5)}} \\ &= \epsilon^{-2}\sqrt{\frac{(1-v/\epsilon)(1-v\epsilon^5)}{(1+v\epsilon)(1+v/\epsilon^5)}}. \end{aligned}$$

Finally

$$\begin{aligned} \tan(\psi/2) &= \frac{t_1+t_2}{1-t_1t_2} = \frac{\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}1+v-v^2+\epsilon^{-2}\sqrt{\frac{(1-v/\epsilon)(1-v\epsilon^5)}{(1+v\epsilon)(1+v/\epsilon^5)}}}{1-\frac{\epsilon^{-2}}{\sqrt{5}}\frac{(1-v\epsilon^5)}{(1+v\epsilon)}} \\ &= \frac{\sqrt{\frac{(1-v\epsilon^5)}{(1-v/\epsilon)(1+v\epsilon)(1+v/\epsilon^5)}}}{1/(1+v\epsilon)}\cdot\frac{(1+v/\epsilon^5)+\sqrt{5}\epsilon^{-2}(1-v/\epsilon)}{\sqrt{5}(1+v\epsilon)-\epsilon^{-2}(1-v\epsilon^5)} \\ &= \sqrt{\frac{(1-v\epsilon^5)(1+v\epsilon)}{(1-v/\epsilon)(1+v/\epsilon^5)}}\cdot\frac{(1-v\epsilon^{-3})}{(1+v\epsilon^3)} \end{aligned}$$

establishing the validity of (21).

In order to prove (22), apply Gauss' transformation [12] :

$$\int \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} = \frac{2}{1+\kappa}\int \frac{d\psi}{\sqrt{1-[4\kappa/(1+\kappa)^2]\sin^2\psi}}$$
$$\text{for}\quad \sin(2\psi-\phi)=\kappa\sin\phi$$

134 with $\kappa = 3/5$ again and determine the limits of integration which correspond to each other on either side. From $\sin(2\psi - \phi) = (3/5)\sin\phi$, we get a quadratic equation for $t := \tan\psi$, viz.

$$\frac{5t}{4-t^2} = \tan\phi = \frac{2\tan\phi/2}{1-\tan^2(\phi/2)} = \frac{\sqrt{5}}{2}\frac{\sqrt{(1-11v-v^2)(1+v-v^2)}}{(1+4v-v^2)}$$

proceeding as in the earlier case. Since by the same calculations,

$$\frac{20(1+4v-v^2)^2}{(1-11v-v^2)(1+v-v^2)} + 16 = \frac{36(1+v^2)^2}{(1-11v-v^2)(1+v-v^2)},$$

we get

$$\tan\psi = t = \frac{-\sqrt{5}(1+4v-v^2)+3(1+v^2)}{\sqrt{(1-11v-v^2)(1+v-v^2)}}$$

taking the positive square root as before. Thus

$$t = \frac{2}{\epsilon^2}\sqrt{\frac{(1-v/\epsilon)(1-v\epsilon^5)}{(1+v\epsilon)(1+v/\epsilon^5)}}$$

giving the lower limit for ψ in (22); the upper limit $\tan^{-1}(3-\sqrt{5})$ clearly corresponds to the upper limit $2\tan^{-1}(1/\sqrt{5})$ arising when $v = 0$. Consequently Ramanujan's formula (22) is proved.

Using a different 'uniformiser' $v_1 := \dfrac{\eta^2(3z)\eta^2(15z)}{\eta^2(z)\eta^2(5z)}$ related to $\Gamma_0(15)$, Ramanujan ([11], p. 70/78) also records the formula :

$$\int_0^x \eta\eta_3\eta_5\eta_{15}\frac{dx}{x} = \frac{1}{5}\int_{2\tan^{-1}\frac{(1-3v_1)}{\sqrt{5}(1+3v_1)}}^{2\tan^{-1}(1/\sqrt{5})} \sqrt{d\phi}\sqrt{1-\frac{9}{25}\sin^2\phi} \qquad (24)$$

In view of formula (**??**), it suffices to show that

$$\sqrt{\frac{1-11v-v^2}{1+v-v^2}} = \frac{1-3v_1}{1+3v_1} \tag{25}$$

in order to prove (24). Substituting for v and v_1 on both sides of (25) and simplifying further, (25) will follow from

$$(\eta_3\eta_5)^6-(\eta\eta_5)^5\eta_3\eta_{15}-5(\eta\eta_3\eta_5\eta_{15})^3-9(\eta\eta_5)(\eta_3\eta_{15})^5-(\eta\eta_{15})^6 = 0. \tag{26}$$

Formula (26), however, is a consequence of the following

Proposition 2. **135**

$$\frac{\eta_3^5\eta_5^5}{\eta\eta_{15}} - \eta^4\eta_5^4 - 5(\eta\eta_3\eta_5\eta_{15})^2 - 9(\eta_3\eta_{15})^4 - \frac{\eta^5\eta_{15}^5}{\eta_3\eta_5} = 0. \tag{27}$$

Proof. Each term on the left hand side of (27) can be shown to be a modular form of weight 4 for $\Gamma_0(15)$ and it is not hard to derive the following Fourier expansions (writing x for $e^{2\pi iz}$) :

$$\frac{\eta_3^5\eta_5^5}{\eta\eta_{15}} = x + x^2 + 2x^3 - 2x^4 - 0\cdot x^5 - 8x^6 - 4x^7 - 15x^8 + 7x^9 - 0\cdot x^{10} + \cdots$$

$$-\eta^4\eta_5^4 = -x + 4x^2 - 2x^3 - 8x^4 + 5x^5 + 8x^6 - 6x^7 + 0\cdot x^8 + 23x^9 - 20\cdot x^{10} + \cdots$$

$$-5(\eta\eta_3\eta_5\eta_{15})^2 = -5x^2 + 10x^3 + 5x^4 + 0\cdot x^5 - 25x^6 - 10x^7 + 15x^8 - 10x^9 + 25x^{10} + \cdots$$

$$-9(\eta_3\eta_{15})^4 = -9x^3 + 0\cdot x^4 + 0\cdot x^5 + 36x^6 + 0\cdot x^7 + 0\cdot x^8 - 18x^9 + 0\cdot x^{10} + \cdots$$

$$-\frac{(\eta\eta_{15})^5}{\eta_3\eta_5} = -x^3 + 5\cdot x^4 - 5\cdot x^5 - 11x^6 + 20x^7 + 0\cdot x^8 - 2x^9 - 5x^{10} + \cdots$$

From these expansions, the left hand side of (27) is a modular form of weight 4 for $\Gamma_0(15)$ all of whose Fourier coefficients corresponding to

$e^{2\pi i n z}$ for $0 \leq n \leq 8 = 4[\Gamma(1) : \Gamma_0(15)]/12$ vanish. By Hecke's theorem ([5], p. 811-see also p. 953) again, this modular form has to vanish identically. □

Remark. We may rewrite (27) in terms of the above modular functions v, v_1 for $\Gamma_0(15)$ as

$$\frac{1}{v} - v - 5 - 9v_1 - \frac{1}{v_1} = 0 \tag{28}$$

If we show that the left hand side (which is already regular on all of $\mathfrak{H}$ is also regular at all the four cusps of $\Gamma_0(15)$, we can obtain an alternative proof for (27), via (28).

3.2 Ramanujan has considered on the same page (67) + 1 of [11] an elliptic integral coming now from the group $\Gamma_0(14)$ of genus 1 and has noted the formula

$$\int_0^x \eta\eta_2\eta_7\eta_{14}\frac{dx}{x} = \ldots \int_{\cos^{-1}\left(\frac{1}{7}\frac{1+v_2}{1-v_2}\sqrt{13+16\sqrt{2}}\right)} \frac{d\phi}{\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\phi}} \tag{29}$$

136 where $v_2 := (\eta\eta_{14}/\eta_2\eta_7)^4$.

We know that $\tau := 1/v_2$ and $\sigma := \dfrac{2\pi i}{4\pi^2\eta\eta_2\eta_7\eta_{14}}\dfrac{d\tau}{dz}$ are connected by the relation $\sigma^2 = \tau^4 - 14\tau^3 + 19\tau^2 - 14\tau + 1$, from Fricke ([3], pp. 451-453) and in fact, they generate the field of modular functions for $\Gamma_0(14)$. Proceeding as in §3.1, we have

$$\int_0^x \eta\eta_2\eta_7\eta_{14}\frac{dx}{x} = -\pi i \int_z^{i\infty} \eta\eta_2\eta_7\eta_{14}dz \text{ (with } z = iy, y > 0)$$

$$= \int_\tau^\infty \frac{d\tau}{\sqrt{\tau^4 - 14\tau^3 + 19\tau^2 - 14\tau + 1}} \tag{30}$$

Now $X^4 - 14X^3 + 19X^2 - 14X + 1 = (X-\alpha)(X-\beta)[(X-m)^2 + n^2)$ where $\alpha = \frac{1}{2}(7 + 4\sqrt{2} + \sqrt{7}\sqrt{11 + 8\sqrt{2}})$, $\beta = \frac{1}{2}(7 + 4\sqrt{2} - \sqrt{7}\sqrt{11 + 8\sqrt{2}})$,

$m = \frac{1}{2}(7-4\sqrt{2})$, $n^2 = \frac{7}{4}(8\sqrt{2}-11)$. From Greenhill ([4], p. 61 as quoted from page 23 of "Elliptische Funktionen" by Enneper), we obtain

$$\int_{\alpha}^{x} \frac{dX}{\sqrt{(X-\alpha)(X-\beta)[(X-m)^2+n^2)}}$$
$$= \frac{1}{\sqrt{HK}} \operatorname{cn}^{-1}\left\{\frac{H(X-\beta)-K(X-\alpha)}{H(X-\beta)+K(X-\alpha)}, \kappa\right\}$$
$$= \frac{1}{\sqrt{HK}} \int_{0}^{\phi} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}}$$

with $\phi = \cos^{-1}\left(\frac{H(X-\beta)-K(X-\alpha)}{H(X-\beta)+K(X-\alpha)}\right)$, $H^2 = (\alpha-m)^2+n^2 = 4\sqrt{2}(7+4\sqrt{2}+\sqrt{7}\sqrt{11+8\sqrt{2}})$, $K^2 = (\beta-m)^2+n^2 = 4\sqrt{2}(7+4\sqrt{2}-\sqrt{7}\sqrt{11+8\sqrt{2}})$, and $\kappa^2 = \frac{1}{2} - \frac{1}{4}\{(\alpha-\beta)^2 - H^2 - K^2\}/HK$. Also, $HK = 8\sqrt{2}$ and $\kappa^2 = (16\sqrt{2}-13)/(32\sqrt{2})$. Hence, for the (real-valued) integral (30) 137
wherein $\tau > \alpha$ necessarily, we have the value

$$\left(\int_{\alpha}^{x} - \int_{\alpha}^{\tau}\right) \frac{d\tau}{\sqrt{\tau^4 - \cdots + 1}}$$

$$= \frac{1}{\sqrt{8\sqrt{2}}} \int_{\cos^{-1}\left(\frac{H-K-v_2(H\beta-K\alpha)}{H-K-v_2(H\beta+K\alpha)}\right)}^{\cos^{-1}\left(\frac{H-K}{H+K}\right)} \frac{d\phi}{\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\phi}}$$

$$= \frac{1}{\sqrt{8\sqrt{2}}} \int_{\cos^{-1}\left(\frac{\sqrt{13+16\sqrt{2}}}{7}\frac{1+v_2}{1-v_2}\right)}^{\cos^{-1}\left(\frac{\sqrt{13+16\sqrt{2}}}{7}\right)} \frac{d\phi}{\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\phi}}$$

since $H^2\beta = K^2\alpha = HK$, $(H\beta-K\alpha)/(K-H) = 1 = (H\beta+K\alpha)/(H+K)$

and $(H-K)/(H+K) = (H^2-K^2)/(H^2+K^2+2HK) = \frac{1}{7}\sqrt{13+16\sqrt{2}}$. This proves formula (29).

3.3 An integral of the same form as above but *not* of elliptic type has been mentioned by Ramanujan on page (70) /78 of [11] :

$$\int_0^x \eta\eta_5\eta_7\eta_{35}\frac{dx}{x} = \frac{1}{2}\int_0^{v_3} \frac{dt}{\sqrt{(1-t+\sqrt{t})(1-t^3-\sqrt{t}(5+9t+5t^2))}} \tag{31}$$

where $v_3 = (\eta\eta_{35}\eta_5\eta_7)^2$.

If $\tau := 1/\sqrt{v_3}$ and $\sigma = \dfrac{4\pi i}{2\pi^2\eta^2\eta_{35}^2}\dfrac{d\tau}{dz}$, then by Fricke ([3], pp. 444-445), σ and τ are modular functions for $\Gamma_0(35)$ connected by the relation $\sigma^2 = \tau^8 - 4\tau^7 - 6\tau^6 - 4\tau^5 - 9\tau^4 + 4\tau^3 - 6\tau^2 + 4\tau + 1$. [The right hand side factorizes as $(\tau^2+\tau-1)(\tau^6-5\tau^5-9\tau^3-5\tau-1)$]. As in earlier examples,

$$\int_0^x \eta\eta_5\eta_7\eta_{35}\frac{dx}{x} = \int_\tau^\infty \frac{\tau d\tau}{\sqrt{(\tau^2+\tau-1)(\tau^6-5\tau^5-9\tau^3-5\tau-1)}}$$

$$= \frac{1}{2}\int_0^{v_3} \sqrt{dt}\sqrt{(1-t+\sqrt{t})(1-t^3-\sqrt{t}(5+9t+5t^2))}$$

(setting $\tau = 1/\sqrt{t}$)

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Remarks. The further reduction of this *hyperelliptic* integral can be carried out by known methods in the theory of elliptic functions ([4], pp. 159-160).

It is interesting to note the following relation between $P = \eta/\eta_7$ and $Q = \eta_5/\eta_{35}$ on page 303 of [10] :

$$(PQ)^2 - 5 + 49/(PQ)^2 = (Q/P)^3 - 5(Q/P)^2 - 5(P/Q)^2 - (P/Q)^3$$

which is the same as equation (29) on page 446 of Fricke (3); the latter is itself a consequence of the above relation between σ and τ.

3.4 Ramanujan has also considered elliptic integrals wherein the integrand involves (higher) powers of η. On page (45) /54 of [11], he has written down the following formulae :

$$5^{3/4}\int_0^x \eta^2\eta_5^2\frac{dx}{x} = \int_0^{2\tan^{-1}(5^{3/4}\sqrt{\lambda_k})} \frac{d\phi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\phi}} \tag{32}$$

$$= 2\int_{\cos^{-1}[(\epsilon u)^{5/2}]}^{\pi/2} \frac{d\phi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\phi}} \tag{33}$$

$$= \sqrt{5}\int_0^{2\tan^{-1}[5^{1/4}\sqrt{x}\psi(x^5)/\psi(x)]} \frac{d\phi}{\sqrt{1-\epsilon/\sqrt{5})\sin^2\phi}} \tag{34}$$

recalling that $\lambda_5 = n_5^6/\eta^6$, $u = \dfrac{x^{1/5}}{1+}\,\dfrac{x}{1+}\,\dfrac{x^2}{1+}\ldots$, $\psi(x) = x^{-1/8}\times\eta_2^2(z)/\eta(z)$ and $\epsilon = (\sqrt{5}+1)/2$.

Before proving (32)–(34), we state

Proposition 3.

(i) $E_2(z) - 5E_2(5z) = -4(\eta^5/\eta_5)\sqrt{1+22\lambda_5+125\lambda_5^2}$

(ii) $E_4(z) = \eta^{10}/\eta_5^2 + 250\eta^4\eta_5^4 + 3125\eta_5^{10}/\eta^2$

(iii) $E_4(5z) = \eta^{10}/\eta_5^2 + 10\eta^4\eta_5^4 + 5\eta_5^{10}/\eta^2$.

Proof. We know that η^{10}/η_5^2, $\eta^4\eta_5^4$, η_5^{10}/η^2 form a basis for the space of modular forms of Haupttypus $(-4, 5, 1)$ and their Fourier expansions are given by

$$\eta^{10}/\eta_5^2 = 1 - 10e^{2\pi iz} + 35e^{4\pi iz} + \cdots$$

$$\eta^4\eta_5^4 = e^{2\pi iz} - 4e^{4\pi iz} + \cdots$$
$$\eta_5^{10}/\eta^2 = e^{4\pi iz} + \cdots$$

139 Writing $E_4 = \alpha\eta^{10}/\eta_5^2 + \beta\eta^4\eta_5^4 + \lambda\eta_5^{10}/\eta^2$ and comparing the first three Fourier coefficients, we have $\alpha = 1$, $-10\alpha + \beta = 240$, $35\alpha - 4\beta + \lambda = 2160$ i.e. $\alpha = 1$, $\beta = 250$, $\lambda = 3125$, proving (ii). The proof of (iii) is identical. For proving (i), we have only to argue instead with $[E_2(z) - 5E_2(5z)]^2$ of Haupttypus $(-4, 5, 1)$ and identify it likewise with $16(\eta^{10}/\eta_5^2 + 22\eta^4\eta_5^4 + 125\eta_5^{10}/\eta^2)$. Identity (i) is stated by Ramanujan on page (73) /81 of [11]. □

Corollary. $x(\frac{d}{dx}\lambda_5) = \eta^2\eta_5^2\sqrt{\lambda_5 + 22\lambda_5^2 + 125\lambda_5^3}.$

Proof is immediate from (**??**), (10) and (i) of Proposition 3.

We now proceed to prove (32). In fact, from the Corollary, we have

$$5^{3/4}\int_0^x \eta^2\eta_5^2\frac{dx}{x} = \frac{1}{5^{3/4}}\int_0^{\lambda_5}\frac{d\lambda_5}{\sqrt{\lambda_5^3 + \frac{22}{125}\lambda_5^2 + \frac{1}{125}\lambda_5}} \text{(since } \lambda_5(i\infty) = 0)$$
$$= \frac{1}{5^{3/4}}\int_0^{\lambda}\frac{d\lambda}{\sqrt{\lambda[(\lambda + \frac{11}{125})^2 + (\frac{2}{125})^2]}}, \qquad (35)$$

dropping the suffix 5 from λ_5. Now, using formula (24) on page 40 of Greenhill ([4], §46) with $\alpha = 0$, $m = -11/5^3$, $n = 2/5^3$, $H^2 := (\alpha - m)^2 + n^2 = 1/5^3$, $\kappa^2 := \frac{1}{2}[1 - (\alpha - m)/H] = (5\sqrt{5} - 11)/(2.5^{3/2}) = \epsilon^{-5}/5^{3/2}$, we see that (35) is the same as

$$\frac{1}{5^{3/4}\sqrt{H}}\,\mathrm{cn}^{-1}\left\{\frac{H-\lambda}{H+\lambda}, \frac{\epsilon^{-5}}{5^{3/2}}\right\} = \int_0^{\cos^{-1}\left(\frac{5^{-3/2}-\lambda}{5^{-3/2}+\lambda}\right)}\frac{d\phi}{\sqrt{1 - \epsilon^{-5}5^{-3/2}\sin^2\phi}}$$

But $\cos\phi = (5^{-3/2} - \lambda)/(5^{-3/2} + \lambda)$ implies that $\tan(\phi/2) = 5^{3/4}\sqrt{\lambda}$ and so (32) is proved.

To prove that the right hand side of (32) is the same as (33), let us first invoke the following transformation formulae from Ramanujan ([10], Chapter XVII, 7(vi) and 7(ii), pp. 207-208) :

$$\int_0^{\beta} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} = 2\int_0^{\alpha} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}}$$

$$(\text{where}\quad \tan(\beta/2) = (\tan\alpha)|\sqrt{1-\kappa^2\sin^2\alpha})$$

$$= 2\int_0^{\lambda} \frac{d\phi}{\sqrt{1-\kappa^2\cos^2\phi}}$$

$$(\text{where}\quad \tan\lambda = (\tan\alpha)\sqrt{1-\kappa^2})$$

which together imply that **140**

$$\int_0^{\beta} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} = 2\int_{\pi/2-\lambda}^{\pi/2} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} \tag{36}$$

Now taking

$$\kappa^2 = \epsilon^{-5}/5^{3/2} \quad \text{and} \quad \lambda = (\pi/2) - \cos^{-1}((\epsilon u)^{5/2}),$$

we have

$$\tan\lambda = (\epsilon u)^{5/2}/\sqrt{1-(\epsilon u)^5},$$

$$\tan\alpha = (\tan\lambda)/\sqrt{1-\kappa^2} = 5^{3/4}u^{5/2}/\sqrt{1-(\epsilon u)^5},$$

$$1-\kappa^2\sin^2\alpha = 1/(1+\epsilon^{-5}u^5)$$

and

$$\tan(\beta/2) = (\tan\alpha)\sqrt{1-\kappa^2\sin^2\alpha}$$

$$= 5^{3/4}u^{5/2}/\sqrt{(1-\epsilon^5u^5)(1+\epsilon^{-5}u^5)}$$

$$= 5^{3/4}/\sqrt{u^{-5} - u^5 - 11} = 5^{3/4}\sqrt{\lambda_5},$$

by (3). Our assertion above concerning (32) and (33) is immediate.

Finally, we show that the right hand side of (32) coincides with (34). For this, we need to use Ramanujan's transformation formula ([10], p. 231 – see also Smith [12], p. 469) :

$$\int_0^A \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} = \frac{3}{1+2\alpha}\int_0^B \frac{d\phi}{\sqrt{1-\mu^2\sin^2\phi}} \tag{37}$$

where $\kappa^2 = \dfrac{\alpha^3(2+\alpha)}{1+2\alpha}$, $\mu^2 = \alpha\left(\dfrac{2+\alpha}{1+2a}\right)^3$ and

$$\tan((A-B)/2) = (1-\alpha)/(2\alpha+1)\tan B. \tag{38}$$

Taking $\alpha = 1/(\epsilon^2\sqrt{5})$, we have $1-\alpha = 3/(\epsilon\sqrt{5})$, $2\alpha+1 = 3/\sqrt{5}$, $2+\alpha = 3\epsilon/\sqrt{5}$, $\kappa^2 = \epsilon^{-5}/5^{3/2}$ and $\mu^2 = \epsilon/\sqrt{5}$. Further, in (37), if we take $A = 2\tan^{-1}(5^{3/4}\sqrt{\lambda_5})B = 2\tan^{-1}[5^{1/4}\sqrt{x}\psi(x^5)/\psi(x)]$, we have to verify that (38) holds. But the latter is the same as

$$\frac{\eta_5^3}{\eta^3} = \frac{\eta\eta_{10}^2}{\eta_2^2\eta_5} \times \frac{1-(\eta\eta_{10}^2/\eta_2^2\eta_5)^2}{1-5(\eta\eta_{10}^2/\eta_2^2\eta_5)^2}$$

which, on the other hand, follows at once from Proposition 1. It is now clear that (37) implies our assertion concerning (32) and (34).

Remark. On the same page in [11] wherein Ramanujan has noted down formulae (32)-(34) as well as the integrals in (39)-(41) considered in §3.5, one finds also values of κ^2, $\kappa^2(1-\kappa^2)$ corresponding to two succes-
141 sive "cubic" transformations. We should mention here that classically the object of applying such transformations to elliptic integrals repeatedly was the realisation of the complete integral in the limit ([4], p. 322).

3.5 Formulae (32)-(34) were connected with the modular relation (3) for u. The following formulae due to Ramanujan ([11], p. (45) /54) are analogous and connected with the modular relation (2) instead :

$$5^{-3/4}\int_0^x \frac{\eta^5}{\sqrt{\eta_{1/5}\eta_5}}\frac{dx}{x} = 2\int_{\cos^{-1}(\sqrt{su})}^{\pi/2} \frac{d\phi}{\sqrt{1-(\epsilon^{-1}/\sqrt{5})\sin^2\phi}} \tag{39}$$

$$= \int_0^{2\tan^{-1}(5^{1/4}\sqrt{\eta_5/\eta_{1/5}})} \frac{d\phi}{\sqrt{1-(\epsilon^{-1}/\sqrt{5})\sin^2\phi}} \tag{40}$$

$$= \frac{1}{\sqrt{5}}\int_0^{2\tan^{-1}(5^{3/4}((\eta_{1/5}+\eta_5)/(\eta_{1/5}+5\eta_5))\sqrt{\eta_5/\eta_{1/5}})} \frac{d\phi}{\sqrt{1-(\epsilon^5/5^{3/2})\sin^2\phi}} \tag{41}$$

Before proving (39), we note that, by virtue of (9), $\frac{1}{u}\frac{du}{dx} = \frac{1}{5x}\eta^5/\eta_5$ and so $u = u(1)\exp\left(-\frac{1}{5}\int_x^1 \frac{\eta^5}{\eta_5}\frac{dx}{x}\right)$. But $u(1) = \frac{1}{1+}\frac{1}{1+}\dots = (\sqrt{5}-1)/2 = \epsilon^{-1}$. Moreover, $u < \epsilon^{-1}$ since $x = e^{-2\pi y} < 1$ (for $y > 0$) and $-\frac{1}{5}\int_x^1 \frac{\eta^5}{\eta_5}\frac{dx}{x} < 0$. Using (2) it is now easily seen that the left hand side of (39) is the same as

$$5^{1/4}\int_0^u \frac{du}{\sqrt{u-u^2-u^3}} = 5^{1/4}\int_0^u \frac{du}{\sqrt{-u(u+\epsilon)(u-\epsilon^{-1})}}$$

$$\text{(noting } 0 < u < \epsilon^{-1}\text{)}$$

$$= 5^{1/4}\left(\int_0^{\epsilon^{-1}} - \int_u^{\epsilon^{-1}}\right)\frac{du}{\sqrt{-u(u+\epsilon)(u-\epsilon^{-1})}} \tag{42}$$

But from Greenhill ([4], formula (14), p. 36), we know that 142

$$5^{1/4}\int_u^{\epsilon^{-1}} \frac{du}{\sqrt{-u(u+\epsilon)(u-\epsilon^{-1})}} = 2\int_0^{\cos^{-1}(\sqrt{\epsilon})} \frac{d\phi}{\sqrt{1-(\epsilon^{-1}/\sqrt{5})\sin^2\phi}}$$

taking $M = \frac{1}{2}(\epsilon^{-1} + \epsilon)^{1/2}$ and $(\kappa')^2 = \epsilon^{-1}/(\epsilon + \epsilon^{-1}) = \epsilon^{-1}/\sqrt{5}$ therein. Letting u tend to 0 in (**??**) and substituting into (42), we see that formula (39) is true.

In order to verify that (40) is the same as the right hand side of (39), we have only to use Ramanujan's formula (36) with $\kappa^2 = \epsilon^{-1}/\sqrt{5}$ and $\lambda = \frac{\pi}{2} - \cos^{-1}(\sqrt{\epsilon u})$; indeed, $\tan\lambda = (\epsilon u/(1 - \epsilon u))^{1/2}$, $\tan^2\alpha = \sqrt{5}\, u/(1 - \epsilon u)$, $1 - (\epsilon^{-1}/\sqrt{5})\sin^2\alpha = 1/(1 + \epsilon^{-1}u)$ and consequently $\tan\beta/2 = (\epsilon u/(1-\epsilon u))^{1/2} 5^{1/4}/(\epsilon + u)^{1/2} = 5^{1/4}(\eta_5\eta/\eta_{1/5})$ in view of (2).

For proving that (40) and (41) are the same, we appeal to the Legendre transformation ([4], p. 323) :

$$\int_0^\infty \frac{d\phi}{\sqrt{1 - \kappa^2\sin^2\phi}} = \frac{1}{2\alpha + 1}\int_0^\psi \frac{d\psi}{\sqrt{1 - \mu^2\sin^2\psi}}$$

$$\left(\text{with } \tan\frac{\phi + \psi}{2} = (\alpha + 1)\tan\phi\right)$$

where $\kappa^2 = (\alpha^4 + 2\alpha^3)/(2\alpha + 1)$ and $\mu^2 = \alpha[(\alpha + 2)/(2\alpha + 1)]^3$; we need only to take $\alpha = \epsilon^{-1}$, $\phi = 2\tan^{-1}(5^{1/4}\sqrt{\eta_{1/5}\eta_5})$ and $\psi = 2\tan^{-1}(5^{3/4} \cdot \sqrt{\eta_5/\eta_{1/5}} \cdot [(\eta_{1/5} + \eta_5)/(\eta_{1/5} + 5\eta_5)]$ and verify that $\tan\frac{\phi + \psi}{2} = (\alpha + 1)\tan\phi$.

3.6 Finally, we take up an interesting formula stated by Ramanujan ([11], p. 45 /54) concerning u, namely

$$u^{-5} + u^5 = \frac{\eta^3}{2\eta_5^3}\left\{C + \int_x^1 \frac{\eta^8}{\eta_5^4}\frac{dx}{x} + 125\int_x^x \frac{\eta_5^8}{\eta^4}\frac{dx}{x}\right\} \tag{44}$$

where

$$C = 5^{3/4}\left\{-\pi + 4\int_0^{\pi/2} \sqrt{1 - \epsilon^{-5}5^{-3/2}\sin^2\phi}\, d\phi - \right.$$

$$\left.\int_0^{\pi/2} \sqrt{d\phi}\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\phi}\right\}$$

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To prove (44), we first remark that $G := 2\sqrt{\lambda_5}(u^5+u^{-5})$ is the same as $2\sqrt{125\lambda_5+22+1/\lambda_5}$, in view of the modular relation (3). Hence

$$\begin{aligned}\frac{dG}{dx} &= \frac{125-1/\lambda_5^2}{\sqrt{125\lambda_5+22+1/\lambda_5}}\frac{d\lambda_k}{dx}\\ &= \frac{1}{x}\left(125\frac{\eta_5^8}{\eta^4}-\frac{\eta^8}{\eta_5^4}\right),\ \text{ by Corollary to Proposition 3.}\end{aligned}$$

Now, by the fundamental theorem of the integral calculus,

$$G(x)-G(e^{-2\pi/\theta}) = \int_{e^{-2\pi/\theta}}^{x} 125\frac{\eta_5^8}{\eta^4}\frac{dx}{x} - \int_{e^{-2\pi/\theta}}^{x} \frac{\eta^8}{\eta_5^4}\frac{dx}{x}(\text{for any } \theta > 0)$$

Consequently, we have

$$u^{-5}+u^5 = \frac{\eta^3}{2\eta_5^3}\left(C' + \int_x^1 \frac{\eta^8}{\eta_5^4}\frac{dx}{x} + 125\int_0^x \frac{\eta_5^8}{\eta^4}\frac{dx}{x}\right) \tag{45}$$

where

$$C' := G(e^{-2\pi/\theta}) - 125\int_0^{e^{-2\pi/\theta}} \frac{\eta_5^8}{\eta^4}\frac{dx}{x} - \int_{e^{-2\pi/\theta}}^{1} \frac{\eta^8}{\eta_5^4}\frac{dx}{x}.$$

It is easy to see that C' is independent of θ and moreover $C' < G(e^{-2\pi/\theta})$ for all $\theta > 0$. From the transformation formula for the η-function, we find that

$$125\cdot\frac{\eta^6(-\frac{1}{z})}{\eta^6(-\frac{1}{5z})} = \frac{\eta^6(z)}{\eta_5^6(z)} \tag{46}$$

$$125\cdot\frac{\eta_5^8}{\eta^4}\left(-\frac{1}{5z}\right) = 5\cdot(z/i)^2\cdot\frac{\eta^8(z)}{\eta_5^4(z)}$$

and therefore

$$125 \int_0^{e^{-2\pi/\theta}} \frac{\eta_5^8}{\eta^4} \frac{dx}{x} = \int_{e^{-2\pi\theta/5}}^{1} \frac{\eta^8}{\eta_5^4} \frac{dx}{x} \cdot \left(\text{using } z \to -\frac{1}{5z}\right).$$

As a result,

$$C' = G(e^{-2\pi/\theta}) - 250 \int_0^{e^{-2\pi/\theta}} \frac{\eta_5^8}{\eta^4} \frac{dx}{x} = G(e^{-}2^{\pi/\theta}) - 2 \int_{e^{-2\pi\theta/5}}^{1} \frac{\eta^8}{\eta_5^4} \frac{dx}{x}.$$

144 Formula (45) may now be rewritten as

$$u^{-5} + u^5 = \frac{\eta^3}{2\eta_5^3} \left(C + \int_x^{e^{-2\pi/\theta}} \frac{\eta^8}{\eta_5^4} \frac{dx}{x} + 125 \int_{e^{-2\pi/\theta}}^{x} \frac{\eta_5^8}{\eta^4} \frac{dx}{x} \right),$$

denoting

$$C' + \int_{e^{-2\pi/\theta}}^{1} \frac{\eta^8}{\eta_5^4} \frac{dx}{x} + 125 \int_0^{e^{-2\pi/\theta}} \frac{\eta_5^8}{\eta^4} \frac{dx}{x} (= G(e^{-2\pi/\theta}) \text{ clearly}) \text{ by } C.$$

This last stated formula for $u^{-5} + u^5$ may be deemed to be obtained from (45), merely by replacing the limits of integration 0, 1 therein by $e^{-2\pi/\theta}$ and further the constant of integration C' there by $C(= G(e^{-2\pi/\theta}))$. Ramanujan evaluates the constant C for $\theta = 1, \sqrt{5}$ and 5. Taking $\theta = \sqrt{5}$ first, we obtain from (46) with $z = i/\sqrt{5}$, that $125\lambda_5(-1/(5i/\sqrt{5})) = 1/\lambda_5(i/\sqrt{5})$, i.e. $\lambda_5(i/\sqrt{5}) = 5^{-3/2}$. Hence

$$C = G(e^{-2\pi/\sqrt{5}}) = 2(125 \cdot 5^{-3/2} + 22 + 5^{3/2})^{1/2} = 4(11 + 5\sqrt{5})/2)^{1/2},$$

in agreement with Ramanujan's value for C for this case. Again, using the functional equation for λ_5 implied by (46) with $z = i$, we have $125\lambda_5(i) = 1/\lambda_5(i/5)$ and therefore $G(e^{-2\pi}) = G(e^{-2\pi/5})$. From Ramanujan's identity (iii) in §4, viz.

$$E_6^2 = \eta^{24}(\eta^3/\eta_5^3 - 500\eta_5^3/\eta^3 - 5^6 \cdot \eta_5^9/\eta^9)^2(1 + 22\lambda_5 + 125\lambda_5^2),$$

we infer $\lambda_5(i)$ is a root of the polynomial $1 - 500X - 15625X^2$ since $E_6(i) = 0$ and the polynomial $1 + 22X + 125X^2$ has no real root. Thus $\lambda_5(i) = 1/(5\epsilon)^3$ and consequently, $G(e^{-2\pi/5}) = G(e^{-2\pi}) = 6{\cdot}5^{1/4}(3+\sqrt{5})$; however, this does *not* agree with the value for C given by Ramanujan.

Remark. It is possible to express C' in terms of complete elliptic integrals of the first and the second kind as Ramanujan has noted in connection with C in (44).

4 Identities for Eisenstein series

We discuss in this section several useful identities for $E_4(z)$, $E_4(pz)$, $E_6(z)$ and $E_6(pz)$ for $p = 5, 7$ written down by Ramanujan ([11], p. (70) /78, also p. (67) /75) :

$$\text{(i)}\quad E_4^3(z) = (\eta^{10}/\eta_5^2 + 250\eta^4\eta_5^4 + 5^5\cdot\eta_5^{10}/\eta^2)^3$$

$$\text{(ii)}\quad E_4^3(5z) = (\eta^{10}/\eta_5^2 + 10\eta^4\eta_5^4 + 5\eta_5^{10}/\eta^2)^3$$

$$\text{(iii)}\quad E_6^2(z) = \eta^{24}(\eta^3/\eta_5^3 - 500\eta_5^3/\eta^3 - 5^6\eta_5^9/\eta^9)^2(1+22\eta_5^6/\eta^6+125\eta_5^{12}/\eta^{12})$$

$$\text{(iv)}\quad E_6^2(5z) = \eta_5^{24}(\eta^{15}/\eta_5^{15}+4\eta^9/\eta_5^9-\eta^3/\eta_5^3)^2(1+22\eta_5^6/\eta^6+125\eta_5^{12}/\eta^{12})$$ 145

$$\text{(v)}\quad E_4^3(z) = (\eta^7/\eta_7 + 5\cdot7^2\eta^3\eta_7^3 + 7^4\eta_7^7/\eta)^3(\eta^7/\eta_7 + 13\eta^3\eta_7^3 + 49\eta_7^7/\eta)$$

$$\text{(vi)}\quad E_4^3(7z) = (\eta^7/\eta_7 + 5\eta^3\eta_7^3 + \eta_7^7/\eta)^3(\eta^7/\eta_7 + 13\eta^3\eta_7^3 + 49\eta_7^7/\eta)$$

$$\text{(vii)}\quad E_6^2(z) = (\eta^7/\eta_7 - 7^2(5+2\sqrt{7})\eta^3\eta_7^3 - 7^3(21+8\sqrt{7})\eta_7^7/\eta)^2\times (\eta^7/\eta_7 - 7^2(5-2\sqrt{7})\eta^3\eta_7^3 - 7^3(21-8\sqrt{7})\eta^7/\eta)^2$$

$$\text{(viii)}\quad E_6^2(7z) = (\eta^7/\eta_7 - (7+2\sqrt{7})\eta^3\eta_7^3 + (21+8\sqrt{7}\eta_7^7/\eta)^2\times (\eta^7/\eta_7 - (7-2\sqrt{7})\eta^3\eta_7^3 + (21-8\sqrt{7})\eta_7^7/\eta)^2$$

Identities (i) and (ii) have already been proved as (ii) and (iii) in Proposition 2 in §3.4. One can derive the other identities in a similar fashion. We can also deduce these identities from the results of Klein ([7], p. 46). If $\tau_1 := -\lambda_5^{-1}$ and $\tau_2 := \lambda_7^{-1}$, then for elliptic modular function $j(z)$, we have from Klein (see also [4], p. 329) the following relations :

(i) $1728j(z) = (\tau_1^2 - 250\tau_1 + 5^5)^3/(-\tau_1^5)$

(ii) $1728j(5z) = (\tau_1^2 - 10\tau_1 + 5)^3/(-\tau_1)$

(iii) $1728(j(z) - 1) = (\tau_1^2 + 500\tau_1 - 5^6)^2(\tau_1^2 - 22\tau_1 + 125)/(-(\tau_1^5))$

(iv) $1728(j(5z) - 1) = (\tau_1^2 - 4\tau_1 - 1)^2(\tau_1^2 - 22\tau_1 + 125)/(-\tau_1)$

(v) $1728j(z) = (\tau_2^2 + 5\cdot 7^2\tau_2 + 7^4)^3(\tau^2 + 13\tau_2 + 49)/\tau_2^7$

(vi) $1728j(7z) = (\tau_2^2 + 5\tau_2 + 1)^3(\tau_2^2 + 13\tau_2 + 49)/\tau_2$

(vii) $1728[j(z) - 1) = (\tau_2^4 = 10\cdot 7^2\tau_2^3 - 9\cdot 7^4\tau_2^2 - 2\cdot 7^6\tau_2 - 7^7)^2/\tau_2^7$

(viii) $1728(j(7z) - 1) = (\tau_2^4 + 14\cdot\tau_2^3 + 9\cdot 7\tau_2^2 + 10\cdot 7\tau_2 - 7)^2/\tau_2$

Substituting for $j(z) = E_4^3/(E_4^3 - E_6^2) = E_4^3/(1728\,\eta^{24})$, we deduce Ramanujan's identities (i)-(viii) immediately from the above identities (i)-(viii).

One finds a few more striking identities involving the Eisenstein series E_4 and E_6, stated by Ramanujan ([11], p. 67 + 1) :

$$(E_4^2(z) + 94E_4(z)E_4(5z) + 625E_4^2(5z))^{1/2}$$

$$= 12\sqrt{5}(\eta^{10}/\eta_5^2 + 26\eta^4\eta_5^4 + 125\eta_5^{10}/\eta^2) \tag{47}$$

$$(5(E_6(z) + 125E_6(5z))^2 - (126)^2E_6(z)E_6(5z))^{1/2} \tag{48}$$

$$= 252(\eta^{10}/\eta_5^2 + 62\eta^4\eta_5^4 + 125\eta_5^{10}/\eta^2)(\eta^{10}/\eta_5^2 + 22\eta^4\eta_5^4 + 125\eta_4^{10}/\eta^2)^{1/2}$$

Identity (47) may be verified by squaring both sides, substituting the expressions for $E_4(z)$ and $E_4(5z)$ from the identities (i)-(ii) above and checking that the coefficients corresponding to the monomials η^{24}, $\eta^{18}\eta_5^6$, $\eta^{12}\eta_5^{12}$, $\eta^6\eta_5^{18}$ and η_5^{24} are the same on both sides. A similar remark applies to identity (48), using now identities (iii)-(iv) for E_6 above.

5 Differential equations satisfied by 'Eisenstein series' We discuss, in this section, a differential equation mentioned
146 by Ramanujan ([11], p. (73) /81) for certain 'Eisenstein series'. First, let us recall 'Hecke summation' ([5], p. 469) for Eisenstem series of weight 2 :

$$\begin{aligned}\operatorname*{Lt}_{s\to 0}\sum_{\substack{(c,d)=1\\ c\geq 0}}(cz+d)^{-2}|cz+d|^{-s}\\ &=\frac{-6i}{\pi(z-\overline{z})}+1-24\sum_{n=1}^{\infty}\left(\sum_{0<d|n}d\right)e^{2\pi inz}\\ &=\frac{-6i}{\pi(z-\overline{z})}+E_2(z) \qquad (49)\end{aligned}$$

The left hand side of (49) picks up a factor of automorphy of weight 2 under modular substitutions but is not holomorphic in z, due to the presence of $-6i/(\pi(z-\overline{z}))$ on the right hand side; on the other hand, E_2 is holomorphic but it does not have a nice behaviour under modular transformations. Now associated with $E_2(z)$ and $E_2(5z)$, Ramanujan has introduced on page (73) /81 of [11] a function F through the equations

$$\begin{aligned}E_2(z)&=(\eta^5/\eta_5)[(1+22\lambda_5+125\lambda_5^2)^{1/2}-30F(\lambda_5)]\\ E_2(5z)&=(\eta^5/\eta_5)[(1+22\lambda_5+125\lambda_5^2)^{1/2}-6F(\lambda_5)] \qquad (50)\end{aligned}$$

These two equations for F are certainly consistent by virtue of (i) in Proposition 3 above; further, they clearly imply that

$$\begin{aligned}F(\lambda_5)&=-(1/24)(\eta_5/\eta^5)(E_2(z)-E_2(5z))\\ &=-(\eta_5/\eta^5)\mathscr{E}_5(z) \qquad (51)\end{aligned}$$

on defining $\mathscr{E}_5(z)=(E_2(z)-E_2(5z))/24$. For $F(\lambda_5)$, Ramanujan ([11], p. (73)) has recorded the non-linear differential equation.

$$(1+22\lambda_5+125\lambda_5^2)^{1/2}\left(\frac{d}{d\lambda_5}F\right)=1+(25/2)\lambda_5+[5/(2\lambda_5)]F^2(\lambda_5). \qquad (52)$$

This differential equation may be verified by using the relation $E_4 = E_2^2 - 12\vartheta(E_2)$ from (1) and the identity (i) above, namely

$$E_4 = g_5^2(1 + 250\lambda_5 + 3125\lambda_5^2) = g_5^2(h_5(\lambda_5) + 228\lambda_5 + 3000\lambda_5^2)$$

with

$$g_5 := \eta^5/\eta_5 \quad \text{and} \quad h_5(X) := 1 + 22X + 125X^2.$$

In fact, by (50), $E_2^2 = g_5^2(h_5(\lambda_5) - 60\sqrt{h_5(\lambda_5)}F + 900F^2)$ and further $\vartheta(E_2) = g_5^2(-5\sqrt{h_5(\lambda_5)}F + 150F^2 + 11\lambda_5 + 125\lambda_5^2 - 30\lambda_5\sqrt{h_5(\lambda_5)}\frac{d}{d\lambda_5}F)$ on noting that $\vartheta\lambda_5 = g_5\lambda_5\sqrt{h_5(\lambda_5)}$ in view of the Corollary to Proposition 3. Now $F = -\mathscr{E}_5/g_5$ and logarithmic differentiation with respect to z gives $(\vartheta F)/F = (\vartheta\mathscr{E}_5)/\mathscr{E}_5 - (\vartheta g_5)/g_5 = (\vartheta\mathscr{E}_5)/\mathscr{E}_5 + 5g_5F$, by (8) and (50). This enables us to rewrite (52) as a differential equation for $\mathscr{E}_5$:

$$\vartheta\mathscr{E}_5 - \frac{5}{2}\mathscr{E}_5^2 = -\eta^4\eta_5^4 - \frac{25}{2}\eta_5^{10}/\eta^2. \tag{53}$$

147 The right hand side can be expressed as a linear combination of $E_4(z)$, $E_4(5z)$ and $\eta^4\eta_5^4$ which form a basis for the space of modular forms of Haupttypus $(-4, 5, 1)$.

Remark. The differential equation (53) for $\mathscr{E}_5$ is perhaps the right analogue of the first relation in (1), for the case of $\Gamma_0(5)$. We derive next an analogue for the case of $\Gamma_0(7)$ as follows.

Let us start from Ramanujan’s identity (v) above for E_4, namely

$$E_4 = g_7(1 + 5\cdot 7^2\lambda_7 + 7^4\lambda_7^2)(g_7\cdot h_7(\lambda_7))^{1/3} \tag{54}$$

where $g_7 := \eta^7/\eta_7$ and $h_7(X) := 1 + 13X + 49X^2$. We might mention here that $(g_7\cdot h_7(\lambda_7))^{1/3}$ is just the Eisenstein series $E_1(z;\chi_7) = 1 + 2\sum_{n=1}^{\infty}\chi_7(n)\times x^n/(1-x^n)$ of weight 1 and (real) Nebentypus $(-1, 7, \chi_7)$; further $E_1^2(z, \chi_7)$ happens to be precisely $-(1/6)E_2(z; 1; \Gamma_0(7))$. By analogy with Ramanujan’s function $F(\lambda_5)$ above, let us introduce $F_1 = F_1(\lambda_7)$ by the two equations

$$E_2(z) = g_7^{2/3}((7/6)F_1(\lambda_7) + h_7^{2/3}(\lambda_7))$$

$$E_2(7z) = g_7^{2/3}((1/6)F_1(\lambda_7) + h_7^{2/3}(\lambda_7)) \tag{55}$$

The consistency of (55) follows from the relation

$$E_2(z;1;\Gamma_0(7)) + 6E_1^2(z;\chi_7) = 0.$$

In view of (8), we have

$$F_1(\lambda_7) = g_7^{-2/3}(E_2(z) - E_2(7z)) = (12/7\pi i)g_7^{-2/3}\frac{d}{dz}(\log g_7).$$

Moreover, by (10), (**??**) and (55).

$$(1/\lambda_7)\left(\frac{d}{dz}\lambda_7\right) = (2\pi i/6)(7E_2(7z) - E_2(z)) = 2\pi i g_7^{2/3}h_7^{2/3}(\lambda_7).$$

Now (1), (54) and (55) together imply that

$$\begin{aligned}
&g_7^{4/3}h_7^{1/3}(\lambda_7)(1 + 5\cdot 7^2\lambda_7 + 7^4\cdot\lambda_7^2)\\
&\qquad = g_7^{4/3}((49/36)F_1^2 + (7/3)F_1\cdot h_7^{2/3}(\lambda_7) +\\
&h_7^{4/3}(\lambda_7)) - 12\vartheta E_2 \quad \text{and} \quad \left(\frac{d}{dz}E_2\right)\\
&\qquad = (2/3)g_7^{-1/3}\left(\frac{d}{dz}g_7\right)\left(\frac{7}{6}F_1 + h_7^{2/3}(\lambda_7)\right) +\\
&g_7^{2/3}\left(\frac{d}{dz}\lambda_7\right)\left(\frac{7}{6}F_1'(\lambda_7) + \frac{2}{3}h_7^{-1/3}(\lambda_7)(13 + 98\lambda_7)\right).
\end{aligned}$$

Consequently, we obtain the differential equation

$$F_1'(\lambda_7)h_7^{1/3}(\lambda_7) + [7/(72\lambda_7)]h_7^{-1/3}F_1^2(\lambda_7) + 224\lambda_7 + 24 = 0.$$

Defining $\mathscr{E}_7$ by $\mathscr{E}_7 = (1/24)g_7^{2/3}F_1$, we see that

$$\begin{aligned}
(1/\mathscr{E}_7)\frac{d\mathscr{E}_7}{dz} &= [F_1(\lambda_7)/F_1(\lambda_7)]\frac{d\lambda_7}{dz} + (2/3)\frac{d}{dz}(\log g_7)\\
&= -2\pi i\lambda_7\, g_7^{2/3} \times h_7^{1/3}(\lambda_7) \times [(7/72\lambda_7)]h_7^{-1/3}(\lambda_7)\\
&\qquad F_1 + (224\lambda_7 + 24)/F_1) + (28\pi i/3)\mathscr{E}_7
\end{aligned}$$

$$= (14\pi i/3)\mathscr{E}_7 - (2\pi i/3)(3 + 28\lambda_7)\lambda_7 g_7^{4/3} h_7^{1/3}(\lambda_7)/\mathscr{E}_7.$$

This leads at once to a differential equation for $\mathscr{E}_7$ (analogous to (53)) :

$$\begin{aligned}\vartheta\mathscr{E}_7 - \frac{7}{3}\mathscr{E}\frac{2}{7} &= -\frac{1}{3}(3\lambda_7 + 28\lambda_7^2) g_7^{4/3} h_7^{1/3}(\lambda_7)\\ &= -\frac{1}{3}(3\lambda_7 g_7 + 28\lambda_7^2 g_7) E_1(z;\chi_7)\\ &= -\eta^3\eta_7^3 E_1(z;\chi_7) - (28/3)(\eta_7^7/\eta) E_1(z;\chi_7).\end{aligned}$$

148 The right hand side, being a modular form of weight 4 for $\Gamma_0(7)$ is expressible as a linear combination of $E_4(z)$, $E_4(7z)$ and $(\eta\eta_7)^3 E_1(z;\chi_7)$; note that by Ramanujan's identities (v), (vi) above, $E_1(z;\chi_7)$ divides $E_4(z)$ and $E_4(7z)$ and hence every modular form of Haupttypus $(-4, 7, 1)$.

Remark. Eichler and Zagier [2] have obtained for the N-divisor values of Weierstrass' $\wp$-function a non-linear differential equation of degree 2 with the coefficients independent of N. Eisenstein series of weight 2 for $\Gamma(N)$ are known to be expressible linearly in terms of these Weierstrass' N-divisor values. For the 'Eisenstein series' $\mathscr{E}_5$, $\mathscr{E}_7$ which are analogues of E_2 for $\Gamma_0(5)$, $\Gamma_0(7)$ respectively, we have again a non-linear differential equation whose coefficients however *depend on the stufe*. In any case, it is remarkable that Ramanujan has recorded in [11] the interesting differential equation (52) for $F(\lambda_5)$ which is the same as (53) and which does not seem to have been observed prior to Ramanujan.

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ON SOME THEOREMS STATED BY RAMANUJAN

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151 **1** Ramanujan seems to have been fascinated by the continued fractions

$$R(\tau) = \frac{e^{2\pi i\tau/5}}{1+}\frac{e^{2\pi i\tau}}{1+}\frac{e^{4\pi i\tau}}{1+} \tag{1}$$

and

$$S(\tau) = \frac{e^{\pi i\tau/5}}{1-}\frac{e^{\pi i\tau}}{1+}\frac{e^{2\pi i\tau}}{1-} \tag{2}$$

where $\tau = x + iy$, $y > 0$, $i = \sqrt{-1}$. We discusses them in many places in the Notebooks and more importantly in the 'Lost' Notebook. In particular, he evaluated $R(\tau)$ and $S(\tau)$ for $\tau = i\sqrt{n}$ for many rational values of $n > 0$. Some of these evaluations were sent by him to Hardy in his early letters from India. A number of evaluations of $R(\tau)$ and $S(\tau)$ contained in the 'Lost' Notebook were discussed and upheld by us [4] using the Kronecker limit formula which seems to be well adapted for these problems. We do not, of course, know Ramanujan's methods. They could not be the method using the limit formula. There are two evaluations [7, p. 46] which are particularly intriguing. The are

$$S(i\sqrt{3}) = \frac{(-3+\sqrt{5}) + \sqrt{6(5+\sqrt{5})}}{4} \tag*{(3)$_{\rm R}$}$$

$$S(i/\sqrt{3}) = \frac{-3+\sqrt{5} + \sqrt{6(5-\sqrt{5})}}{4} \tag*{(4)$_{\rm R}$}$$

As far as we know, these two results have not been proved until now. In attempting to prove these, we encountered another of Ramanujan's evaluations. If λ_n for integers $n \geq 1$ is defined by

$$\lambda_n = \frac{e^{(\pi/2)\sqrt{n/3}}}{3\sqrt{3}}\{(1+e^{-\pi\sqrt{n/3}})(1-e^{-2\pi\sqrt{n/3}})(1-e^{-4\pi\sqrt{n/3}})\ldots\}^6$$

the Ramanujan states

$$\begin{aligned}\lambda_1 &= 1, \lambda_9 = 3, \lambda_{17} = 4+\sqrt{17}, \lambda_{25} = (2+\sqrt{5})^2,\\ \lambda_{33} &= 18+3\sqrt{33}, \lambda_{41} = 32+5\sqrt{41},\\ \lambda_{49} &= 55+12\sqrt{21}, \lambda_{89} = 500+53\sqrt{89}, \ldots\end{aligned} \tag{5$_R$}$$

The function λ_n seems to have been introduced earlier in the Notebooks (Vol 2, p. 393) where Ramanujan gives a formula for evaluating λ_n for $\frac{n}{3} = 11, 19, 43, 67, 163$ and others. It is to be noticed that these values of $-\frac{n}{3}$ are precisely the discriminates $\equiv 5 \pmod 8$ of imaginary quadratic fields of class number one. If we use Dedekind's modular form **152**

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau}).$$

then

$$\lambda_n = \frac{1}{3\sqrt{3}}\left\{\frac{\eta[\frac{1}{2}(1+i\sqrt{n/3})]}{\eta[\frac{1}{2}(1+i\sqrt{3n})]}\right\}^6 \tag{6}$$

As was shown by us in [5], if $-3n$ is a fundamental discriminant of an imaginary quadratic field $K = Q(\sqrt{-3n})$ which has only one class in each genus of ideal classes, then λ_n can be evaluated fairly easily using L-series. For example, for $n = 17, 41, 89$ this property is true. However, for $n = 25$, 49 the numbers -3.25 and -3.49 are not fundamental discriminant of an imaginary quadratic field $K = Q(\sqrt{-3n})$ which has only one class in each genus of ideal classes, then λ_n can be evaluated fairly easily using L-series. For example, for $n = 17, 41, 89$ this property is true. However, for $n = 25$, 49 the numbers -3.25 and -3.49 are not fundamental discriminants but nevertheless they are discriminants in the orders in $Q(\sqrt{-3})$ with conductors 5 and 7, with similar properties with regard to genera of ring ideal classes. One has then an analogue of the Kronecker limit formula for the L-series of such ideal classes which leads to the evaluation of λ_{25} and λ_{49} and consequently to the proof of (3) and (4). For $n = 9$ and 33, the subrings of $Q(\sqrt{-3})$ and $Q(\sqrt{-11})$ with conductors 3 have similar properties but the evaluation of λ_9 and λ_{33} depends on different ideas.

Ramanujan, in every case, seems to consider only dicriminants, fundamental or not, which have only two classes. We shall do the same in this note and restrict further to odd discriminants with class number 2 since we are dealing only with $S(\tau)$.

2 Let $K = Q(\sqrt{d})$, $d < 0$ be an imaginary quadratic field with discriminant d and class number $h(d)$. Let R be the maximal order in K and for any rational integer $f \geq 1$, R_f the ring with conductor f. Clearly $R = R_1$. The ring R_f has discriminant df^2 and a minimal basis $(1, \theta)$ where

$$\theta = \begin{cases} \frac{-1+i\sqrt{Df^2}}{2} & , \text{ if } df^2 \equiv 1 (\text{mod } 4) \\ \frac{i\sqrt{Df^2}}{2} & , \text{ if } df^2 \equiv 0 (\text{mod } 4) \end{cases} \tag{7}$$

153 where $D = |d|$.

We consider in R_f only ideals which are prime to f. As is well known, there is a $(1, 1)$ correspondence between ideals in R prime to f and those in R_f prime to f. If a and b are two ideals prime to f, in R_f, then they are said to be in the same ideal class in R_f if there exist λ and μ in R_f both prime to f such that

$$\lambda_a = \mu b$$

This leads to a class division of ideals of R_f into ideal classes. The number $h(df^2)$ of these ideal classes is given by

$$h(df^2) = \frac{h(d) \cdot \varphi([f])}{e \cdot \varphi(f)} \tag{8}$$

where $\varphi([f])$ denotes the Euler function of the ideal $[f]$ in R so that

$$\varphi([f]) = f^2 \prod_{\varkappa / f} (1 - \frac{1}{N_\varkappa}) \tag{9}$$

where $\varkappa$ runs through all prime ideals in R dividing f and $\varphi(f)$ in the denominator is the ordinary totient function. The number e is the index of the group of units in R_f in the group of units of R.

It is to be noted that formula (8) is still true if $d > 0$.

Let C be any ideal class in R_f. The *zeta function* $\zeta_f^*(s, C)$ *of the class* C is defined by

$$\zeta_f^*(S, C) = \sum_{\substack{a \in C \\ (a,f)=1}} (Na)^{-s}, \ \text{Re}\, s > 1 \tag{10}$$

where a runs through all integral ideals in C which are prime to f. If ℓ is an ideal in the class C^{-1} which is prime to f, then

$$\zeta_f^*(s, C) = \frac{(N\ell)^s}{w} \sum_{\substack{0 \neq \alpha \in \ell \\ (\alpha,f)=1}} |N\alpha|^{-s} \tag{11}$$

w being the number of roots of unity in R_f. If $f > 1$, then $w = 2$.

If $f > 1$, because of the restrictive summation in (11), it is not possible to apply at once the Kronecker Limit formula to $\zeta_f^*(s, C)$. We shall see that for our purposes, the *zeta function of the class C in the extended sense* defined below would be sufficient. Put

$$\zeta_f(s, C) = \frac{(N\ell)^s}{w} \sum_{\substack{\alpha \neq 0 \\ \alpha \in \ell}} |N\alpha|^{-s} \tag{12}$$

with α running through all elements of ℓ not equal to zero. The sum in **154**
(12) is then an Epstein zeta function.

We shall choose the ideal class C in a particular way using the (1, 1) correspondence between ring ideal classes and binary, positive, primitive integral quadratic forms with discriminant df^2, d being, of course, a negative fundamental discriminant.

Let p be a prime number dividing d but not f (we assume that such primes exist). We shall construct a binary, primitive, positive form which represents p primitively. Let

$$px^2 + bxy + cy^2$$

be the quadratic form with discriminant df^2 so that

$$b^2 - 4pc = df^2. \tag{13}$$

Clearly $p|b$ and so if $b = pb_1$, $d = pd_1$, then

$$pb_1^2 - 4c = d_1 f^2.$$

Let p be odd. If df^2 is odd, then

$$p - d_1 f^2 \equiv 0 (\text{mod } 4)$$

and so we choose $b_1 = 1$ and

$$c = (p - d_1 f^2)/4.$$

The quadratic form

$$px^2 + pxy + \frac{p - d_1 f^2}{4} y^2$$

is primitive since $p \nmid d_1$ and is odd. It has discriminant df^2. We choose the ideals class C to be the inverse of the ideal class C^{-1} represented by the ideal ℓ with basis $(1, z)$ where

$$z = \frac{-1 + \sqrt{df^2/p}}{2} \tag{14}$$

The ideal clearly has norm equal to $1/p$. Note that $(p, f) = 1$.

If p is odd and df^2 is even, then, by (13), $2p|b_1$. One easily sees that we can again take the quadratic form to be

$$px^2 + 2pxy + (p - d_1 f^2/4)y^2$$

which means that the ideal class C^{-1} is represented by the ideal $(1, z)$ with

$$z = \sqrt{df^2/2p} \tag{15}$$

In a similar way, we obtain for C^{-1} the ideal class represented by the ideal $(1, z)$ with

$$z = \begin{cases} \frac{1}{2}(1 + \sqrt{df^2}/2) & , \ p = 2, (d/4) \text{ odd} \\ \sqrt{df^2}/4 & , \ p = 2, (d/4) \text{ even} \end{cases} \tag{16}$$

If we go back to formula (12) and take $C = C_0$ as the principal class and 155
apply the Kronecker limit formula ([5, formula 6]), we have

$$\begin{gathered} - \lim_{s \to 1} [\zeta_f(s, C_0) - \zeta_f(s, C) = \\ = (4\pi/w\sqrt{Df^2}) \log(N[1, z])^{1/2} |\eta(z)/\eta(\theta)|^2) \end{gathered} \tag{17}$$

where $w = w_f$ is the number of roots of unity in R_f, z is given by (14), (15) and (16) and θ by (7). The two functions $\zeta_f(s, C_0)$ and $\zeta_f(s, C)$ are the zeta functions of the classes C_0 and C respectively in the extended sense.

3 In order to proceed further, it is necessary to obtain another expression for the left side of (17).

Let χ be any character of the ring ideal class group of R_f. We define the L-function

$$L_f(s, \chi) = \sum_{\substack{a \in R_f \\ (a, f_1) = 1}} \frac{\chi(a)}{(Na)^s}, \ \text{Re}\, s > 1. \tag{18}$$

since χ is a multiplicative function on the ideal of R_f prime to f.

$$L_f(s, \chi) = \prod_{\varkappa | f} (1 - \chi(\varkappa) N\varkappa^{-s})^{-1} \tag{19}$$

Furthermore

$$L_f(s, \chi) = \sum_C \chi(C) \zeta_f^*(s, C) \tag{20}$$

where C runs through all ring ideal classes of R_f.

If χ is a non-principal character, it is shown by Meyer that we have even the relation

$$L_f(s,\chi) = \sum_C \chi(C)\zeta_f(s,C) \tag{21}$$

in terms of the zeta functions of classes in the extended sense. Formula (21) has the advantage that one can apply the Kronecker limit formula. If now we assume that every genus of ring ideal classes of R_f has only one class in it, then one has

$$-2^{r-2}\left[\zeta_f(s,C_0) - \zeta_f(s,C) = \sum_{\chi(c)=-1} L(s,\chi)\right] \tag{22}$$

where the sum runs through all characters which take the value -1 on C, 2^{r-1} being the number of genera.

We now define the genus characters.

Let df^2 have the decomposition

$$df^2 = d_0 d_0^*$$

156 where d_0 is a fundamental discriminant and d_0^* a discriminant. For such a decomposition, we have a character of the class group of R_f

$$\chi_{d_0}(\varkappa) = \left(\frac{d_0}{N\varkappa}\right)$$

for all prime ideals $\varkappa$ which do not divide df^2; $\left(\frac{d_0}{}\right)$ being the Kronecker symbol ([9, p. 380 et seq]). For prime ideals not dividing d_0, (**??**) also makes sense. If $\varkappa$ divides d_0, then take

$$\chi_{d_0}(\varkappa) = \left(\frac{d_0^*}{N\varkappa}\right)$$

It is to be noted that d_0 and d_0^* have only divisors of f as common divisor.

We shall now confine ourselves to the case

$$df^2 \text{ odd, } h(df^2) = 2, (d,f) = 1. \tag{23}$$

Since d is a fundamental discriminant,

$$df^2 = -pf^2,\ p \equiv -1 (\text{mod } 4) \tag{24}$$

Further df^2 has only one non-trivial decomposition

$$df^2 = \begin{cases} -pf \cdot f & , \text{ if } f \equiv 1 (\text{mod } 4) \\ -f \cdot pf & , \text{ if } f \equiv -1 (\text{mod } 4) \end{cases} \tag{25}$$

Following Siegel [8], we see that there is only one L-series and

$$L_f(s, \chi) = \begin{cases} L_{-pf}(s) \cdot L_f(s), \text{ if } f \equiv 1 (\text{mod } 4) \\ L_{-f}(s) \cdot L_{pf}(s), \text{ if } f \equiv -1 (\text{mod } 4) \end{cases}$$

where $L_*(s)$ is the ordinary Dirichlet L series.

From (22), we get

$$-\lim_{s \to 1} (\zeta_f(s, C_0) - \zeta_f(s, C)) = L_f(1, \chi)$$

and therefore, from (17) using the fact that $w = 2$ for $f > 1$, we get

$$\frac{1}{\sqrt{p}} \left\{ \frac{\eta\left(\frac{-1+i\sqrt{f^2/p}}{2}\right)}{\eta\left(\frac{-1+i\sqrt{f^2 p}}{2}\right)} \right\}^2 = \begin{cases} (\epsilon(f)^{h(-pf) \cdot h(f)}, & f \equiv 1 (\text{mod } 4) \\ (\epsilon(pf)^{h(pf) \cdot h(-f) \cdot 2/w_0}, & f \equiv -1 (\text{mod } 4) \end{cases}$$

where $h(-pf), \ldots$ are class numbers, $\epsilon(f)$ and $\epsilon(pf)$ are the fundamental units in the real quadratic fields $Q(\sqrt{f})$ and $Q(\sqrt{pf})$ respectively and w_0 the number of roots of unity in $Q(\sqrt{-f})$.

From the definition of λ_n and formula (**??**), we see that we can evaluate λ_n if $p = 3$ and the conditions (23) are satisfied.

They are indeed satisfied in cases

$$df^2 = -3 \cdot 5^2, -3 \cdot 7^2$$

as seen from the tables in [1]. In case $p = 3$, $f = 5$, we have

$$\epsilon(f) = (\sqrt{5}+1)/2 \quad \text{and} \quad h(-15) = 2.$$

We therefore have

$$\lambda_{25} = \frac{1}{3\sqrt{3}}, \left\{ \frac{\eta\left(\frac{-1+i\sqrt{25/3}}{2}\right)}{\eta\left(\frac{-1+i\sqrt{75}}{2}\right)} \right\}^6 = \left(\frac{\sqrt{5}+1}{2}\right)^6 = (2+\sqrt{5})^2 \qquad (5)_R$$

In a similar way, if $p = 3$, $f = 7$, $h(21) = 1$, $\epsilon(21) = (5+\sqrt{21})/2$. This gives since $w_0 = 2$,

$$\lambda_{49} = \left(\frac{5+\sqrt{21}}{2}\right)^3 = 55 + 12\sqrt{21}. \qquad (5)_R$$

The value of λ_{25} enables us to prove Ramanujan's statements (**??**) and (**??**). It is known, by taking $\tau = i\sqrt{3}$ in ([4, p. 700]) that

$$[S(i\sqrt{3})]^{-1} + 1 - S(i\sqrt{3}) = \frac{\eta(i\sqrt{3}/5)}{\eta(i5\sqrt{3})} \cdot \frac{f(i\sqrt{3}/5)}{f(i5\sqrt{3})} \qquad (26)$$

where $f(\tau)$ is Schlefli's modular function

$$f(\tau) = e^{-\pi i/24} \cdot \frac{\eta((1+\eta)/2)}{\eta(\eta)} = f(-1/\tau) \qquad (27)$$

If we use the formula

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \qquad (28)$$

Then

$$\frac{\eta(i\sqrt{3}/5)f(i\sqrt{3}/5)}{\eta(i5\sqrt{3})f(i5\sqrt{3})} = \left(\frac{5}{\sqrt{3}}\right)^{1/2} \frac{\eta[(1+i\sqrt{25/3})/2]}{\eta(1+i\sqrt{75}/2)}$$

so that, by definition of λ_{25},

$$[S(i\sqrt{3})]^{-1} + 1 - S(i\sqrt{3}) = \sqrt{5} - \lambda_{25}^{1/6} = \sqrt{5}(\sqrt{5}+1)/2$$

Solving the above quadratic equation for $S(i\sqrt{3})$ and using the fact that $S(i\sqrt{3}) > 0$, we get

$$S(i\sqrt{3}) = \frac{-(3+\sqrt{5}) + \sqrt{6(5+\sqrt{5})}}{4} \qquad (3)_R$$

The value of $S(i/\sqrt{3})$ can be obtained by again using (27) and (28) or by using the formula

$$\left(S(\tau) + \frac{\sqrt{5}-1}{2}\right)\left(S(-1/\tau) + \frac{\sqrt{5}-1}{2}\right) = \sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right) \qquad (29)$$

which was stated by Ramanujan in his Notebooks. It was first proved 158
by Watson. (See also [4]).

If we use the formulae (29) and

$$\begin{aligned}\left((S(\tau))^5 + \left(\frac{\sqrt{5}-1}{2}\right)^5\right) + \left((S(-1/5\tau)^5 + \left(\tfrac{\sqrt{5}-1}{2}\right)^5\right)\\ = 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5\end{aligned} \qquad (30)$$

proved by us, one can obtain the values of $S[i5^k(\sqrt{3})^1]$ where k is any rational integer and $1 = \pm 1$.

4 We shall prove now the other statements of Ramanujan in (**??**).

In the first place,

$$\lambda_1 = \frac{1}{3\sqrt{3}}\left(\frac{\eta(i/\sqrt{3})}{\eta(i\sqrt{3})} \cdot \frac{f(i/\sqrt{3})}{f(i\sqrt{3})}\right)^6.$$

If we now use the formulae (27) and (28) we get

$$\lambda_1 = 1$$

Consider now λ_9. By definition,

$$\lambda_9 = \frac{1}{3\sqrt{3}}\left(\frac{\eta((1+i\sqrt{3})/2)}{\eta((1+i3\sqrt{3})/2)}\right)^6$$

If we use the product expansion of the η-function, then

$$\lambda_9 = \frac{-i}{3\sqrt{3}}\left(\frac{\eta(\omega)}{\eta(3\omega)}\right)^6, \ \omega = \frac{-1+i\sqrt{3}}{2}. \tag{31}$$

On the other hand,

$$\alpha = 27\left(\frac{\eta(3\omega)}{\eta(\omega)}\right)^6$$

is a root of the equation

$$x^4 + 18x^2 + \lambda_3(\omega)\chi - 27 = 0$$

where

$$\lambda_3(\omega) = \sqrt{j(\omega) - 1728}$$

and $j(\omega)$ is the well-known Klein's invariant ([9, p. 504]). Weber has shown that

$$\lambda_3((-1+i\sqrt{3})/2) = i.24\sqrt{3}$$

159 and therefore $\lambda = \lambda_9$ is a root, positive, of

$$x^4 - 8x^3 + 18x^2 - 27 = 0.$$

This however equals

$$(x+1)(x-3)^3$$

which shows that

$$\lambda_9 = 3 \tag*{(5)$_R$}$$

Consider now $\omega = (1+i\sqrt{11})/2$. Then

$$\lambda_{33} = \frac{-i}{3\sqrt{3}}\left(\frac{\eta(\omega)}{\eta(3\omega)}\right)^6 = \frac{3\sqrt{3}i}{\alpha}$$

where

$$= 27\cdot\left(\frac{\eta(3(-1+i\sqrt{11})/2)}{\eta((-1+i\sqrt{11})/2)}\right)^6$$

From Weber [9, p. 504],

$$\lambda_3\left(\frac{-1+i\sqrt{11}}{2}\right) = 56i\sqrt{11}$$

and hence λ is the positive root of

$$9x^4 - 56\sqrt{33}x^3 + 18\cdot 3^2\cdot x^2 - 3^5 = 0 \tag{32}$$

This quartic equation can be solved by the classical methods of the theory of algebraic equations. One obtains

$$\lambda = \lambda_{33} = 3(6+\sqrt{33}).$$

In fact, Weber (loc. cit) has given the values of $\lambda_3((-1+i\sqrt{n})/2)$ for $n = 19, 43, 67$ and 163 and thus λ_{3n} is a root of a quadratic equation like (32) from which λ_{3n} can be evaluated.

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THE ADJOINT HECKE OPERATOR II

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1 Introduction Progress on the theory of modular forms and associated Euler products can be divided roughly into three stages. At the first fundamental stage there is the work of Hecke [3], who introduced the linear operators T_n now associated with his name. The second stage comprises the work of Petersson [8], who observed that the space M of cusp forms of given level, weight and character is a finite-dimensional Hilbert space, and showed that the adjoint Hecke operator T_n^* is a scalar multiple of T_n, provided that n is a prime to the level N of M. The foundations of the third stage were laid by Atkin and Lehner [1], who separated off from M the subspace M^- consisting essentially of forms of lower level, and concentrated their attention on its orthogonal complement M^+, showing by delicate methods that M^+ has an orthogonal basis of forms that are eigenforms for all the operators T_n and not only for those with n prime to N.

The present paper arose from an effort to simplify the arguments of the third stage, by investigating the properties of the adjoint operator T_n^* for all n, and showing, if possible, that it commutes with T_n on the subspace M^+. We recall that, for any forms f and g in M, T_n^* is defined by

$$(f|T_n, g) = (f, g|T_n^*). \tag{1.1}$$

Petersson proved that $T_n^* = \overline{\chi}(n)T_n$ if $(N, n) = 1$, where χ is the associated Dirichlet character. For this he provided two proofs. Of these one [8] was fairly direct, but had a combinatorial part in which a common left and right transversal of a certain group was shown to exist (Hilfssatz 2), and did not seem applicable to other values of n. On the other hand, his other earlier proof [7] (p. 68), although only valid when the weight k of the space exceeds 2, seemed more promising, although technically somewhat complicated, as it involved the evaluation of $G|T_n$ for an arbitrary Poincaré series G in M.

This is the method developed in my first paper under the same title [10], where is yielded an apparently previously unknown explicit definition of T_n^* for $(n, N) \neq 1$. The case when N is a prime number was then investigated in detail using properties of Poincaré series. However, for composite N this method becomes decidedly more complicated, because of increased number of incongruent cusps of verying cusp widths
162 and parameters. In the present paper the general case is considered in a relatively simple way without the use of Poincaré series, and the explicit definition of the adjoint operator, found in [10], is proved by a different method.

2 Groups, matrices and characters

As is customary, we write

$$\Gamma(1) := \mathrm{SL}(2, \mathbb{Z}) \tag{2.1}$$

for the modular group and, for any positive real number m, we denote by Ω_m the set of all matrices

$$T := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.2}$$

belonging to $\mathrm{GL}(2, \mathbb{R})$ and having determinant m. We shall be particularly concerned with the group

$$\Gamma_0(N) = \{T \in \Gamma(1) : c \equiv 0 (\mathrm{mod}\ N)\}, \tag{2.3}$$

where N is a positive integer, and require the following special matrices in $\Gamma(1)$:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \tag{2.4}$$

Write also

$$\Gamma(N) = \{T \in \Gamma(1) : T \equiv I (\mathrm{mod}\ N)\}. \tag{2.5}$$

For various positive rational values of m we write

$$J_m = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}. \tag{2.6}$$

Throughout k will be a positive integer and, for typographical reasons we write

$$K = \frac{1}{2}k - 1. \tag{2.7}$$

Let

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}, \tag{2.8}$$

and put, as customary,

$$e(z) = \exp(2\pi i z) \quad (z \in \mathbb{C}). \tag{2.9}$$

For $T \in \Omega_m$, we define

$$Tz := \frac{az + b}{cz + d}, \quad T : z = cz + d. \tag{2.10}$$

For any function $f : \mathbb{H} \to \mathbb{C}$ and $T \in \Omega_m (m > 0)$ we define

$$f(z)|T := (T : z)^{-k} (\det T)^{k/2} f(Tz). \tag{2.11}$$

This depends of course on k, which is fixed. Note that

$$f(z)|J_m = m^{-k/2} f(z/m), \ f(z)|J_m^{-1} = m^{k/2} f(mz). \tag{2.12}$$

The letters p and q will always denote prime numbers, and we write

$$P = \{0, 1, \ldots, p-1\}, \ P^* = \{1, 2, \ldots, q-1\}, \tag{2.13}$$

$$Q = \{0, 1, \ldots, q-1\}, \ Q^* = \{1, 2, \ldots, q-1\}. \tag{2.14}$$

Throughout, χ denotes a character modulo N such that **163**

$$\chi(-1) = (-1)^k; \tag{2.15}$$

it follows that, when k is odd, $N \geq 3$. We denote by $N(\chi)$ the conductor of χ and put

$$n(\chi) = N/N(\chi). \tag{2.16}$$

Note that, for any positive integer r,

$$r | n(\chi) \Leftrightarrow N(\chi) | (N/r). \tag{2.17}$$

The principal character modulo m is denoted by ϵ_m.
Accordingly, χ may be written as

$$\chi = \chi^* \epsilon_N, \tag{2.18}$$

where χ^* is a primitive character modulo $N(\chi)$. When (2.17) holds

$$\chi_r := \chi^* \epsilon_{N/r} \tag{2.19}$$

is a character modulo N/r with conductor χ^*.

3 $M(N, k, \chi)$ and its subspaces

A standard notation for the vector space of cusp forms belonging to a group Γ and having weight k and multiplier system v is

$$\{\Gamma, k, v\}_0 \tag{3.1}$$

and we shall write

$$M = M(N, k, \chi) = \{\Gamma_0(N), k, \chi\}_0. \tag{3.2}$$

Thus M is a space of level N, weight k and character χ.

For any positive integers r and s satisfying

$$r | n(\chi), \quad s | r \tag{3.3}$$

we define

$$C(r, s, \chi_r) := M(N/r, k, \chi_r) | J_s^{-1}, \tag{3.4}$$

where χ_r is defined by (2.19). Note that

$$C(1, 1, \chi_1) = M. \tag{3.5}$$

It is easy to see that

$$C(r, s, \chi_r) \subseteq M(Ns/r, k, \chi_{r/s}) \subseteq M. \tag{3.6}$$

Whenever $r > 1$ and $s < r$ the level of $C(r, s, \chi_r)$ is less than N. When $r = s > 1$ the level is N, but the space is isomorphic to $M(N/r, k, \chi_r)$ and so can be regarded as a space of essentially lower level. For this reason we define

$$M^- = \bigoplus_{r,s} C(r, s, \chi_r)(r > 1, r|n(x), s|r). \tag{3.7}$$

Then M^- is a subspace of M of essentially lower level and any member of M^- is called an *oldform*. Now M is a finite-dimensional Hilbert space, and we define M^+ to be the orthogonal complement of M^- in M, so that

$$M = M^- \oplus M^+. \tag{3.8}$$

The definition of M^- can be simplified, as the following Theorem 164
shows.

Theorem 3.1. *We have*

$$M^- = \bigoplus_{p|n(\chi)} C(p, \chi_p), \tag{3.9}$$

where

$$C(p, \chi) = C(p, 1, \chi_p) \oplus C(p, p, \chi_p). \tag{3.10}$$

Proof. We observe that

$$C(rs, t, \chi_{rs}) \subseteq C(r, t, \chi_r)(t|r, rs|n(\chi)) \tag{3.11}$$

and

$$C(rs, rt, \chi_{rs}) \subseteq C(s, t, \chi_s)(t|s, rs|n(\chi)) \tag{3.12}$$

It is clear that (3.11) holds. To prove (3.12) take any

$$F \in C(rs, rt, \chi_{rs}),$$

so that we can put

$$F = g|J_t^{-1} = f|J_{rt}^{-1},$$

where

$$f \in M(N/rs, k, \chi_{rs}).$$

Now take any $T \in \Gamma_0(N/s)$, so that

$$T_1 := J_r^{-1} T J_r \in \Gamma_0(N/(rs)).$$

Then

$$g|T = f|J_r^{-1}T = f|T_1 J_r^{-1} = \chi_{rs}(T_1) f|J_r^{-1}$$
$$= \chi_{rs}(T_1) g = \chi_s(T) g,$$

so that $g \in M(N/s, k, \chi_s)$, and this proves (3.12).

By successive applications of (3.11) and (3.12) we complete the proof of the theorem. □

4 The Fricke involution H_r

For any $r \in \mathbb{N}$ define

$$H_r = J_r V = \begin{bmatrix} 0 & -1 \\ r & 0 \end{bmatrix} \tag{4.1}$$

so that $H_r^2 = -rI$ and $H_r^{-1} = -r^{-1} H_r$. It is easily verified that

$$H_r^{-1} \Gamma_0(r) H_r = H_r \Gamma_0(r) H_r^{-1} = \Gamma_0(r). \tag{4.2}$$

Lemma 4.1. *Let $N = rs$ where $r | n(\chi)$. Then*

$$M(N/r, k, \chi_r) | H_s = M(N/r, k, \overline{\chi}_r). \tag{4.3}$$

In particular

$$M(N, k, \chi) | H_N = M(N, k, \overline{\chi}). \tag{4.4}$$

Further, if $N = pt$, where $p | n(\chi)$, then

$$C(p, p, \chi_p) | H_N = C(p, 1, \overline{\chi}_p) \tag{4.5}$$

and

$$C(p, 1, \chi_p) | H_N = C(p, p, \overline{\chi}_p). \tag{4.6}$$

165 *Proof.* Let $f \in M(N/r, k, \chi_r)|H_s$ so that $f = g|H_s$, where $g \in M(N/r, k, \chi_r)$. Take any $T \in \Gamma_0(N/r)$ so that

$$H_s^{-1} T H_s \in \Gamma_0(s).$$

Then

$$\begin{aligned} f|T &= g|TH_s = g|H_s H_s^{-1} T H_s = \chi_r(H_s^{-1} T H_s) g|H_s \\ &= \chi_r(H_s^{-1} T H_s) f = \overline{\chi}_r(T) f. \end{aligned}$$

Moreover

$$\begin{aligned} C(p, p, \chi_p)|H_N &= C(p, 1, \chi_p)|J_p^{-1} H_N = C(p, 1, \chi_p)|H_t \\ &= C(p, 1, \chi_p), \end{aligned}$$

by (4.4) with N replaced by t. This gives (4.5) and (4.6) follows by replacing χ by $\overline{\chi}$ and operating again on the right with H_N. □

Lemma 4.2. *If $N = pt$, then*

$$H_N J_p U^u H_N^{-1} = p W^{-ut} J_p^{-1}. \tag{4.7}$$

Proof. Straightforward. □

5 The Hecke operators T_n

For any $n \in \mathbb{N}$ and any $f \in M(N, k, \chi)$ define the operator $T_n(N, \chi) = T_n$ by

$$f|T_n = n^k \sum_{ad=n} \sum_{u=1}^{d} \chi(a) f|J_d U_u J_a^{-1}, \tag{5.1}$$

where J is given by (2.7), and observe that, for any prime p we have, in particular,

$$f|T_p = p^K \left(\sum_{u \in P} f|J_p U^u + \chi(p) f|J_p^{-1} \right); \tag{5.2}$$

see (2.12). It is clear that, in (5.2), u can run through any complete set of residues modulo p.

If

$$N = pt, \ p \nmid t \quad \text{and} \quad f \in C(p, 1, \chi_p), \tag{5.3}$$

if follows from (5.2) that

$$f|T_p(t, \chi_p) = f|T_p(N, \chi) + p^K \chi_p(p) f|J_p^{-1} \tag{5.4}$$

and we note that $\chi_p(p) \neq 0$ in this case.

We now summarize some of the known properties of the operators. For any $f \in M$, let

$$f(z) = \sum_{r=1}^{\infty} a(r)e(rz). \tag{5.5}$$

166 Then

$$f(z)|T_n = \sum_{r=1}^{\infty} a_n(r)e(rz), \tag{5.6}$$

where

$$a_n(r) = \sum_{d|(n,r)} d^{k-1}\chi(d)a(nr/d^2). \tag{5.7}$$

Moreover, we have

$$M|T_n \subset M \quad (n \in \mathbb{N}) \tag{5.8}$$

and

$$(f|T_m)|T_n = \sum_{d|(m,n)} d^{k-1}\chi(d) f|T_{mn/d^2} (m \in \mathbb{N}, n \in \mathbb{N}). \tag{5.9}$$

It follows that the operators commute and that T_n is completely determined when T_p is known for each prime $p|n$.

Moreover, as shown in Petersson [8], if f and g belong to M, then

$$(f|T_n, g) = \chi(n)(f, g|T_n) \quad \text{for} \quad (n, N) = 1. \tag{5.10}$$

Here the inner product is defined for cusp forms f and g of weight k on a subgroup Γ of finite index h in $\Gamma(1)$ by

$$(f, g) = (f, g; \Gamma) = \frac{1}{h} \iint_{\mathscr{F}} f(z)g(z)y^{k-2}dxdy. \tag{5.11}$$

where $x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$ and $\mathscr{F}$ is any fundamental region in $\mathbb{H}$ for Γ. In §6 we shall require this definition for various subgroups Γ contained in $\Gamma(1)$. It follows from (5.10) that T_n^*, the adjoint operator, is given by

$$T_n^* = \overline{\chi}(n) T_n \quad \text{for} \quad (n, N) = 1. \tag{5.12}$$

6 The adjoint operator T_p^* for $p|N$

For any prime $p|N$ and $f \in M$ define the operator $T_p^* = T_p^*(N, \chi)$ by

$$\begin{aligned} f|T_p^* := f|H_N T_p H_N^{-1} &= f|H_N^{-1} T_p H_N \qquad (6.1)\\ &= p^K \sum_{u \in P} f|H_N J_p U^u H_N^{-1} \\ &= p^K \sum_{u \in P} f|W^{-ut} J_p^{-1} \qquad (6.2) \end{aligned}$$

by Lemma 4.2.

Since $T_p(N, \chi) = T_p(N, \overline{\chi})$ it follows from (4.4) that

$$M|T_p^* \subseteq M. \tag{6.3}$$

Theorem 6.1. *For any prime $p|N$, T_p^* is the adjoint operator to T_p; i.e.*

$$(f|T_p^*, g) = (f, g|T_p) \quad \text{for} \quad f \quad \text{and} \quad g \quad \text{in} \quad M. \tag{6.4}$$

For the proof we require the following Lemma, which we quote from Theorem 5.2.1 of [9].

Lemma 6.2. (i) *If Γ_1 and Γ_2 are subgroups of $\Gamma(1)$ of finite index in $\Gamma(1)$* 167
and $\Gamma_1 \subseteq \Gamma_2$, then

$$(f, g : \Gamma_1) = (f, g : \Gamma_2)$$

whenever f and g both belong to $\{\Gamma_2, k, v\}_0$.

(ii) *Let* Γ *be a congruence subgroup of* $\Gamma(1)$ *of finite index and let, for any prime* p,

$$\Gamma_p = \Gamma \cap \Gamma(p). \tag{6.5}$$

Suppose that f *and* g *belong to* $\{\Gamma, k, v\}_0$ *and let* $L \in \Omega_p$. *Then*

$$(f, g; \Gamma) = (f|L, g|L; L^{-1}\Gamma_p L). \tag{6.6}$$

Proof of Theorem. Take any f and g in M and write

$$F = f|T_p^*,$$

so that $f \in M$ by (6.3). Note that, if $S \in \Gamma(pN)$, then

$$f|W^{-ut}J_p^{-1}S = f|S'W^{-ut}J_p^{-1},$$

where $S' \in \Gamma(N)$, so that $\chi(S') = 1$. Hence, for any $u \in \mathbb{Z}$,

$$f|W^{-ut}J_p^{-1} \in \{\Gamma(pN), k, 1\}_0,$$

and so, by Lemma 6.2(i) and (6.2),

$$\begin{aligned}(F, g : \Gamma_0(N)) = (F, g; \Gamma(pn)) &= p^K \sum_{u \in P} (f|W^{-ut}J_P^{-1}, g; \Gamma(pN)) \\ &= p^K \sum_{u \in P} (f|U^{-u}W^{-ut}J_P^{-1}, g; \Gamma(pN)).\end{aligned}$$

Write

$$A_u = J_p W^{ut} U^u = W^{uN} J_p U^u \in \Omega_p$$

and note that, when $\Gamma = \Gamma(pN)$,

$$\Gamma_p = \Gamma(pN),$$

by (6.5), so that

$$A_u^{-1}\Gamma_p A_u = J_p^{-1}\Gamma(pN)J_p \supseteq \Gamma(pN^2).$$

Taking $L = A_u$ in (6.6), we get

$$
\begin{aligned}
(F, g; \Gamma_0(N)) &= p^K \sum_{u\in p} (f, g|A_u; \Gamma(pN^2)) \\
&= p^K \left(f, \sum_{u\in P} g|W^{uN} J_p U^u; \Gamma(pN^2) \right) \\
&= p^K \left(f, \sum_{u\in P} g|J_p U^u;\ \Gamma(pN^2) \right) \\
&= (f, g|T_p;\ \Gamma(pN^2)) \\
&= (f, g|T_p;\ \Gamma_0(N)).
\end{aligned}
$$

This completes the proof of the theorem.

Theorem 6.3. *Let m and n be positive integers. Then the following pairs of operators on M commute :* 168

(i) T_m, T_n; (ii) T_m^*, T_n^*; (iii) T_m, T_n^* *provided that* $(m, n, N) = 1$.

Proof. (i) follows from (5.9) and this yields (ii), since

$$(f|T_m^* T_n^*, g) = (f, g|T_n T_m).$$

By (5.12) and (i) we need only prove that T_p^* and T_q commute when p and q are different primes dividing N.

Write $N = pqs$ and define $S_{u,w}$ by

$$S_{u,w} J_q U^{up} W^{-wqs} J_p^{-1} = W^{-wq^2 s} J_p^{-1} J_q U^u$$

for $(u, w) \in Q \times P$. Then it is easy to see that $S_{u,w} \in \Gamma_0(N)$ and that $\chi(S_{u,w}) = 1$. Now, if $f \in M$, since up and wq run through complete sets of residues modulo p and modulo q respectively, we have

$$
\begin{aligned}
f|T_p^* T_q &= (pq)^K \sum_{u\in Q} \sum_{w\in P} f|W^{-wq^2 s} J_p^{-1} J_q U^u \\
&= (pq)^K \sum_{w\in P} \sum_{u\in Q} f|J_q U^{up} W^{-wqs} J_p^{-1} \\
&= f|T_q T_p^*.
\end{aligned}
$$

□

7 The action of the operators on M

Theorem 7.1. *For all $n \in \mathbb{N}$*

$$M^{-}|T_n \subseteq M^{-} \cdot M^{-}|T_n^* \subset M^{-}. \tag{7.1}$$

In particular, if p is any prime dividing N and $N = pt$, we have :

(i) *For $(n, N) = 1$ and $p|n(\chi)$*

$$C(p, 1, \chi_p)|T_n \subseteq C(p, 1, \chi_p), C(p, p, \chi_p)|T_n \subseteq C(p, p, \chi_p). \tag{7.2}$$

(ii) *If $p|n(\chi)$,*

$$C(p, p, \chi_p)|T_p \subseteq C(p, 1, \chi_p), C(p, 1, \chi_p)|T_p^* \subseteq C(p, p, \chi_p). \tag{7.3}$$

(iii) *If p and q are different primes dividing $n(\chi)$,*

$$C(q, 1, \chi_q)|T_p \subseteq C(q, 1, \chi_q), C(q, 1, \chi_q)|T_p^* \subseteq C(q, 1, \chi_q), \tag{7.4}$$
$$C(q, q, \chi_q)|T_p \subseteq C(q, q, \chi_q), C(q, q, \chi_q)|T_p^* \subseteq C(q, q, \chi_q). \tag{7.5}$$

(iv) *If $p|n(\chi)$ and $p^2|N$,*

$$C(p, 1, \chi_p)|T_p \subseteq C(p, 1, \chi_p), C(p, p, \chi_p)|T_p^* \subseteq C(p, p, \chi_p) \tag{7.6}$$

(v) *If $p|n(\chi_p)$ and $p^2 \nmid N$.*

$$C(p, 1, \chi_p)|T_p \subseteq C(p, \chi_p), C(p, p, \chi_p)|T_p^* \subset C(p, \chi_p). \tag{7.7}$$

Proof. In view of Theorem 3.1, (7.1) will follow if we prove parts (i)-(v) of the theorem. For the proof of (i) see pp. 321-322 of [9]. By (6.1), (4.5) and (4.6) it is only necessary to prove those parts of (7.3)-(7.7) that involve the operator T_p.

169 For (7.3) we note that

$$C(p, p, \chi_p)|J_p U^u = C(p, 1, \chi_p)|U^u = C(p, 1, \chi_p).$$

For (7.4) note that the operator $T_p(N,\chi)$ is the same as the operator $T_p(N/q,\chi_q)$ since p divides N/q and the latter operator maps the space $M(N/q,k,\chi_q)$ into itself. Also, if $N = pqs$ and $f = g|J_q^{-1} \in C(q,q,\chi_q)$ then $g \in C(q,1,\chi_q)$ and

$$f|T_p = p^K \sum_{u\in P} g|J_q^{-1} J_p U^u = p^K \sum_{u=P} g|J_p U^{uq} J_q^{-1}$$
$$= g|T_p J_q^{-1} \subseteq C(p,1,\chi_q)|J_q^{-1} = C(q,q,\chi_q).$$

which proves (7.5).

(7.6) follows for the same reason as (7.4), since p divides N/p and the operators $T_p(N,\chi)$ and $T_p(N/p,\chi_p)$ are identical.

Finally, assume that $p|n(\chi)$ but $p^2 \nmid N$. We assume (5.3) and deduce that

$$f|T_p(N,\chi) \in C(p,1,\chi_p) \oplus C(p,p,\chi_p) = C(p,\chi_p),$$

from (5.4). □

Theorem 7.2. *For all $n \in \mathbb{N}$*

$$M^+|T_n \subseteq M^+ \quad \text{and} \quad M^+|T_n^* \subseteq M^+.$$

Proof. Take any $f \in M^+$ and $g \in M^-$. Then

$$(f|T_n, g) = (f, g|T_n^*) = 0$$

by Theorem 7.1. Hence $f|T_n \in M^+$. The proof of the second part is similar. □

Theorem 7.3. *Let $M_H = M|H_N$. Then $(M_H)^- = M^-|H_N$.*

Proof. By (4.4)

$$M_H = M(N,k,\overline{\chi}),$$

so that $(M_H)^-$ is a vector sum of the spaces $C(p,1,\overline{\chi}_p)$ and $C(p,p,\overline{\chi}_p)$; the result follows. □

8 The operators $T_p^*T_p$ and $T_pT_p^*(p|N)$

Lemma 8.1. *Let* $\mathfrak{R}$ *be a right transversal of* $\Gamma_0(N)$ *in* $\Gamma_0(t)$*, where* $N = tp$*, and put*

$$\mathfrak{R}_0 = \bigcup_{w \in P} W^{-wt}. \tag{8.1}$$

Then we may take

(i) $\mathfrak{R} = \mathfrak{R}_0$, *when* $p \nmid t$, *and*

(ii) $\mathfrak{R} = \mathfrak{R}_0 \cup R^*$, *when* $p \nmid t$, *where*

$$R^* = p^{-1} J_p U^s W^t J_p = \begin{pmatrix} (1+st)/p & s \\ t & p \end{pmatrix} \tag{8.2}$$

and s *is chosen so that* $s \in P$ *and* $st \equiv -1 (\bmod\ p)$.

170 This is straightforward : note that $R^* \in \Gamma(1)$.

Lemma 8.2. *Let* $f \in M(N, k, \chi)$*, where* $p | n(\chi)$*. Then*

$$F := \sum_{R \in \mathfrak{R}} \overline{\chi}(R) f | R \in M(t, k, \chi_p),$$

so that $F \in M^-$.

Proof. Let the members of $\mathfrak{R}$ be $R_r (r = 1, 2, \ldots, h)$ where $h = [\Gamma_0(t) : \Gamma_0(N)]$, and take any $S \in \Gamma_0(t)$. Then $\mathfrak{R}S$ is also a right transversal of $\Gamma_0(N)$ in $\Gamma_0(t)$ and so

$$R_r S = S_r R'_r,$$

where $R'_r \in \mathfrak{R}$ and $S_r \in \Gamma_0(N)$. Note that

$$\chi(R_r)\chi(S) = \chi(S_r)\chi(R'_r).$$

Then

$$F|S = \sum_{r=1}^{h} \overline{\chi}(R_r) f | R_r S = \sum_{r=1}^{h} \overline{\chi}(R_r) f | S_r R'_r$$

$$= \sum_{r=1}^{h} \overline{X}(R_r)\chi(S_r)f|R'_r = \sum_{r=1}^{h} \chi(S)\overline{\chi}(R'_r)f|R'_r$$

$$= \chi(S)F.$$

It follows that $F \in M(t, k, \chi_p)$ and this proves the lemma. □

Lemma 8.3. *Suppose that $N = pt$, where $p|t$, and that χ is a character modulo N. Then, for some integer $m \in P$,*

$$\chi(1 + rt) = e(mr/p)(r \in \mathbb{Z}). \tag{8.3}$$

Moreover

$$m = 0 \text{ if and only if } p|n(\chi). \tag{8.4}$$

Proof. Since

$$(1 + t)^p \equiv 1(\text{mod } N),$$

$\chi(1 + t) = e(m/p)$ for some $m \in P$ and (8.3) follows since

$$1 + rt \equiv (1 + t)^r(\text{mod } p).$$

If $p|n(\chi)$, then $N(\chi)|t$ and so $m = 0$, Conversely, if $N(\chi) \nmid t$, then

$$\chi(n) \neq \chi(n + t)$$

for some n prime to N and so, taking $rn \equiv 1(\text{mod } N)$.

$$1 \neq \chi(1 + rt),$$

from which it follows that $m \neq 0$, by (8.3).

We now define

$$\delta(\chi) = 0 \text{ if } p \nmid n(\chi); \ \delta(\chi) = 1 \text{ if } p|n(\chi). \tag{8.5}$$

Further, for any prime p dividing N we put $N = pt$, as usual, and define

$$\alpha(p) = \begin{cases} 0 & \text{if } p|t, p|n(\chi), \\ p^{k-2} & \text{if } p \nmid t, p|n(\chi), \\ p^{k-1} & \text{if } p \nmid n(\chi). \end{cases} \tag{8.6}$$

Then we have □

Theorem 8.4. *Let $f \in M(N, k, \chi)$ and suppose that $p|N$. Then*

$$f|T_p^*T_p - \alpha(p)f \in M^-, \tag{8.7}$$

and

$$f|T_pT_p^* - \alpha(p)f \in M^-. \tag{8.8}$$

Accordingly, if $f \in M^+$, then

$$f|T_pT_p^* = f|T_p^*T_p = \alpha(p)f. \tag{8.9}$$

Proof. We write $N = pt$ and, in the first instance, assume that $p \nmid t$. Then we can write

$$\chi = \psi_p\psi_t,$$

where ψ_p and ψ_t are characters modulo p and modulo t, respectively. For any integers u, v, w we write

$$S(u, v, w) := W^{-wt}U^uW^{-vt} \tag{8.10}$$

$$= \begin{bmatrix} 1 - uvt & u \\ t(-v - w + uvwt) & 1 - uwt \end{bmatrix} \tag{8.11}$$

and, for any $n \in P^*$ we define $n' = P^*$ by $nn' \equiv 1(\text{mod } p)$.

The finite set of ordered pairs $P \times P$ can be written as

$$P \times P = A^* \cup \bigcup_{v \in P} A_v \tag{8.12}$$

where

$$A^* = \{(w, u) \in P^2 : w \neq 0,\ u = (wt)'\} \tag{8.13}$$

$$A_0 = \{(w, u) \in P^2 : w = 0\} \tag{8.14}$$

and, for $v \in P^*$

$$A_v = \{(w, u) \in P^2 : u \neq (vt)',\ w = (ut - v')'\}. \tag{8.15}$$

It is easily checked that these $p + 1$ sets are disjoint and that their union is P^2. Note also that, for $(w, u) \in A_v (v \neq 0)$, we have $S(u, v, w) \in \Gamma_0(N)$.

For any $f \in M(N, k, \chi)$ we write

$$s^* = \sum_{(w,u)\in A^*} f|W^{-wt}U^u, \;\; s_v = \sum_{(w,u)\in A_v} f|W^{-wt}U^u (v \in P) \tag{8.16}$$

so that

$$f|T_p^* T_p = p^{k-2}\{s^* + \sum_{v\in P} s_v\}.$$

For $(w, u) \in A^*$, we put

$$S_w = W^{-wt}U^u (R^*)^{-1}, \tag{8.17}$$

where R^* is defined by (8.2). Then $S_w \in \Gamma_0(N)$ and

$$\begin{aligned}\chi(S_w) &= \chi\{wst + (1 - wut)(1 + st)/p\} \\ &= \psi_p(-w)\overline{\psi}_t(p)\end{aligned}$$

and so, by (8.16), 172

$$\begin{aligned}s^* = \sum_{w\in P^*} f|S_w R^* &= \overline{\psi}_t(p) \sum_{w\in P^*} \psi_p(-w) f|R^* \\ &= (p-1)\delta(\chi)\overline{\psi}_t(p) f|R^*,\end{aligned} \tag{8.18}$$

by (8.5), since $p|n(\chi)$ if and only if ψ_p is the principal character.

Clearly

$$s_0 = \sum_{w\in P^*} f|U^u = pf \tag{8.19}$$

and, for $v \neq 0$,

$$\begin{aligned}s_v &= \sum_{(w,u)\in A_v} f|S(u, v, w)W^{vt} \\ &= \sum_{(w,u)\in A_v} \chi(1 - uvt) f|W^{vt} \\ &= \sum_{u\in P} \chi(1 - uvt) f|W^{vt} = \sum_{u\in P} \psi_p(1 - uvt) f|W^{vt}\end{aligned}$$

$$= \delta(\chi)(p-1)f|W^{vt} \tag{8.20}$$

Accordingly, by (8.18, 19, 20) we have

$$f\,T_p^*T_p = p^{k-2}\left((pf + \delta(\chi)(p-1)\left\{\sum_{v\in P^*} f|W^{vt} + \overline{\psi}_t(p)f|R^*\right\}\right). \tag{8.21}$$

This gives

$$f|T_p^*T_p = \alpha(p)f \quad \text{for} \quad p\nmid t, p\nmid n(\chi). \tag{8.22}$$

If, however, $p|n(\chi)$, then $\psi_t = \chi_t$ and

$$f|T_p^*T_p = \alpha(p)f + p^{k-2}(p-1)F, \tag{8.23}$$

where

$$F = \sum_{v\in P} f|W^{vt} + \overline{\psi}_t(p)f|R^*$$

$$= \sum_{R\in\Re} \overline{\chi}_t(R)f|R. \tag{8.24}$$

It follows from Lemma 8.2 that $F \in M^-$.

It remains to consider the case when $p|t$. In this case $S(w,w,u) \in \Gamma_0(N)$ and we write

$$P\times P = \bigcup_{w\in P} B_w, \tag{8.25}$$

where

$$B_w = \{(w,u) : u \in P\}. \tag{8.26}$$

173 Then

$$f|T_p^*T_p = p^{k-2}\sum_{w\in P} t_w, \tag{8.27}$$

where

$$t_w = \sum_{(w,u)\in B_w} f|W^{-wt}U^u$$

$$= \sum_{u\in P} f|S(u,w,w)W^{wt}$$

$$= \sum_{u\in P} \chi(1 - wut) f|W^{wt} \tag{8.28}$$

Hence

$$t_0 = pf. \tag{8.29}$$

When $w \in P^*$, we have by Lemma 8.3 that

$$\chi(1 - wut) = e(-mwu/p)(m \in P),$$

where $m = 0$ if and only if $p|n(\chi)$. Hence, by (8.5),

$$t_w = p\delta(\chi) f|W^{wt}$$

and so

$$f|T_p^* T_p = p^{k-1}\{f + \delta(\chi) \sum_{w\in P^*} f|W^{wt}\}. \tag{8.30}$$

Accordingly, if $p|n(\chi)$, then

$$f|T_p^* T_p = p^{k-1} \sum_{R\in\Re_0} f|R \in M^- \tag{8.31}$$

while, when $p \nmid n(\chi)$,

$$f|T_p^* T_p = p^{k-1} f. \tag{8.32}$$

Accordingly, (8.7) holds in either case.

To prove (8.8) we write $f = g|H_N$, where $g \in M(N, k, \overline{\chi})$. Then

$$\begin{aligned} f|T_p T_p^* &= g|H_N T_p T_p^* = g|H_N T_p H_N^{-1} \cdot H_N T_P^* \\ &= g|T_p^* H_N T_p^* = g|T_p^* T_p H_N \\ &= \{\alpha(p) g + h\}|H_N = \alpha(p) f + h|H_N \\ &= \alpha(p) f + h', \end{aligned}$$

say, where $h \in M^-(\overline{\chi})$ and therefore, by Theorem 7.3. $h' \in M^-$.

It remains to prove (8.9). If $f \in M^+$, then $f|T_p^* T_p \in M^+$ and so $f|T_p^* T_p - \alpha(p) f \in M^+$. It follows from (8.7) that

$$f|T_p^* T_p = \alpha(p) f.$$

The proof that $f|T_p T_p^* = \alpha(p) f$ is similar. Theorem 8.4 is proved. □

Corollary 8.5. *Suppose that $f \in M^+$ and that, for some $p|N$ we have $p|n(\chi)$. Then, in the notation of Lemma 8.2,* 174

$$\sum_{T \in \mathfrak{R}} \overline{\chi}(T) f|T = 0.$$

This follows from (8.24) and (8.31).

9 The action of the operators on M^+

In this section we are concerned solely with the space M^+.

We recall that a linear operator on a Hilbert space is said to be *normal* if it commutes with its adjoint. Let $\mathscr{F}$ be the family of all the operators T_n and $T_n^*(n \in \mathbb{N})$. Then it follows from Theorems 6.3 and 8.4 that $\mathscr{F}$ is a family of normal operators acting on M^+ and that any two members of $\mathscr{F}$ commute. Now M^+ is a finite-dimensional Hilbert space and so, from a standard theorem on operators on such spaces (see pp. 267 and 291 of [2]), we deduce

Theorem 9.1. *M^+ has an orthogonal basis of forms, each of which is an eigenvector for all the operators in $\mathscr{F}$. Moreover, if f is such a basis element, with Fourier expansion* (5.5)*, we may assume that f is primitive, i.e. that $a(1) = 1$, and then*

$$f|T_n = a(n)f, \ \ f|T_n^* = \overline{a(n)} f (n \in \mathbb{N}). \tag{9.1}$$

Further,

$$\overline{a(n)} = \overline{\chi}(n) a(n) \quad \text{for} \quad (n, N) = 1, \tag{9.2}$$

and, for any prime p dividing N,

$$|a(p)|^2 = \alpha(p). \tag{9.3}$$

Proof. If f is an eigenvector of all the operators $T_n(n \in \mathbb{N})$ with eigenvalue $\lambda(n)$, then we have, taking $r = 1$ in (5.7)

$$\lambda(n) a(1) = a_n(1) = a(n)(n \in \mathbb{N}),$$

which shows that $a(1) \neq 0$; by division, we may assume that $a(1) = 1$ and then we have $\lambda(n) = a(n)$. Since $T_n^* = \overline{\chi}(n) T_n$ for $(n, N) = 1$, (9.2) follows and (8.9) gives (9.3).

Each basis element is called a *newform* and M^+ is the *newform space.*

It may be noted that, from (8.6) and (9.3), the absolute value of the eigenvalue $a(p)$, where p divides N, emerges naturally from the proof of Theorem 8.4. In certain cases one can determine $a(p)$ rather than $|a(p)|$; see [1], [4], [6], [9]. In this connexion I take the opportunity to correct an error in the statement of Theorem 9.4.8 (iii) of [9], where condition (d) should be replaced by

$$p \nmid t_1, \ p \nmid (N/N_\chi).$$

In conclusion, it may be noted that, although the paper [5] is not 175
concerned with the determination of adjoint Hecke operators, the linear operator C_q there introduced has points of similarity with the operator $T_q + T_q^*$, which is clearly normal on each of the subspaces M, M^- and M^+. □

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ON ZETA FUNCTIONS ASSOCIATED WITH SELF-DUAL HOMOGENEOUS CONES

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Let $\mathscr{C}$ be and (irreducible) self-dual homogeneous cone is a vec- 177
tor space V with a $\mathbb{Q}$-structure such that the automorphism group $G = \mathrm{Aut}(V, \mathscr{C})^\circ$ is defined over $\mathbb{O}$. Let M be a lattice in V and let $\Gamma = \{g \in G | gM = M\}$. Then by definition the zeta function associated with $\mathscr{C}$ is given by

$$Z_{\mathscr{C}}(M; s) = \sum_{x:\Gamma\backslash\mathscr{C}\cap M} |\Gamma_x|^{-1} N(x)^{-s} \quad (s \in \mathbb{C}), \tag{1}$$

where $\Gamma_x = \{\lambda \in \Gamma | \lambda x = x\}$ and $N(x)$ is the "norm" of x (see 1).

The purpose of this note is to supplement our previous report [SO] in the following points. First, in 2, we will show that, except for the case $\mathscr{C} = \mathscr{P}_r(\mathbb{R})$ and G is $\mathbb{O}$-split (treated in [**?**]), the fundamental assumption (2.6) in [SS] is satisfied, so that we can apply the general results of Sato-Shintani on the zeta functions of prehomogeneous vector spaces to our case. In §5, we will determine that poles and the residues of the zeta functions, and, in 6, the functional equations (cf. [SO], Th. 2.3.1, 2.3.3). These will be done including the case d is odd, which was excluded in [SF] and [SO]. In particular, we will show that the matrix $U^{(r)}(x)$ giving the functional equations is always diagonalizable.

1 Let V be a real vector space of dimension n endowed with a positive definite inner product $\langle\ \rangle$. Let $\mathscr{C}$ be a self-dual homogeneous cone in V, i.e. an open convex cone with vertex at 0 satisfying the following two conditions :

(i) $\mathscr{C}$ is "self-dual", *i.e.* one has

$$\mathscr{C} = \mathscr{C}^* = \{x \in V | \langle x, y \rangle \rangle 0 \ \text{ for all } \ y \in \overline{\mathscr{C}} - \{0\}\}.$$

(ii) The automorphism group of $\mathscr{C}$,

$$G = \mathrm{Aut}(V, \mathscr{C})^\circ = \{g \in \mathrm{GL}(V) | g\mathscr{C} = \mathscr{C}\}^\circ,$$

is transitive on $\mathscr{C}$ ($^\circ$ denotes the connected component of the identity.)

In what follows, we assume for simplicity that $\mathscr{C}$ is irreducible and exclude the trivial case $\mathscr{C} = \mathbb{R}_+$ (the half-line of positive numbers). Then G is (the identity connected component of) a reductive algebraic group defined over $\mathbb{R}$ with $\mathbb{R}$-rank $r \geq 2$ and one has $G = G^s \times \mathbb{R}_+$, where G^s is $\mathbb{R}$-simple. (For the treatment of the reducible case, see [SO].) For any
178 $c_0 \in \mathscr{C}$, the stabilizer $K = G_{c_0}$ is a maximal compact subgroup of G and one has $G/K \cong \mathscr{C}$. We can (hence will) assume that the base point c_0 and the inner product $\langle\ \rangle$ are so chosen that for $g \in G$ one has $gc_0 = c_0$ if and only if ${}^t g^{-1} = g$. We further normalize $\langle\ \rangle$ by $\langle c_0, c_0 \rangle = r$.

We set $\mathbf{g} = \mathrm{Lie}\, G$, $\mathbf{k} = \mathrm{Lie}\, K$, $\mathbf{k} = \mathrm{Lie}\, K$ and let $\mathbf{g} = \mathbf{k} + \mathbf{p}$ be the corresponding Cartan decomposition. As is well-known (see *e.g.* [S1]), there exists a unique structure of Jordan algebra on V with the unit element c_0 such that, denoting by $T_x (x \in V)$ the Jordan multiplication $y \mapsto xy (y \in V)$, one has $\mathbf{p} = \{T_x (x \in V)\}$. We denote by $N(x)$ the reduced norm of this Jordan algebra. Then $N : V \to \mathbb{R}$ is a polynomial function of degree r defined over $\mathbb{R}$ satisfying the following conditions :

$$N(c_0) = 1, N(gx) = \det(g)^{r/n} N(x) \quad (g \in G, x \in V). \tag{2}$$

It is then clear that $\chi(g) = \det(g)^{r/n}$ is a rational character of the algebraic group G.

One can find a system of mutually orthogonal primitive idempotents $\{e_i (1 \leq i \leq r)\}$ such that

$$c_0 = \sum_{i=1}^{r} e_i, \qquad e_i e_j = \delta_{ij} e_i,$$

which we call a "primitive decomposition" of c_0. Then $\mathbf{a} = \{T_{e_i}(1 \leq i \leq r)\}_{\mathbb{R}}$ is a maximal (abelian) subalgebra in $\mathbf{p}$. It is known that the system of $\mathbb{R}$-roots (relative to $\mathbf{a}$) is of type (A_r) and all the $\mathbb{R}$-roots have the same multiplicity d. One has a direct sum decomposition

$$V = \bigoplus_{k \leq l} V_{kl},$$

where

$$V_{kl} = \begin{cases} \{x \in V | e_k x = x\} & (k = l), \\ \{x \in V | e_k x = e_l x = \frac{1}{2} x\} & (k < l), \end{cases}$$

and one has $\dim V_{kk} = 1$ and $\dim V_{kl} = d (k < l)$. Hence one has the relation

$$\frac{n}{r} = 1 + \frac{d}{2}(r - 1). \tag{3}$$

We assume that there is given a $\mathbb{O}$-structure on V (*i.e.* a $\mathbb{O}$-vector space $V_{\mathbb{O}}$ in V with $V = V_{\mathbb{O}} \otimes_{\mathbb{O}} \mathbb{R}$) such that G is defined over $\mathbb{O}$ and $c_0 \in V_{\mathbb{O}}$. Then, clearly, K, $\langle\ \rangle$, N, χ are all defined over $\mathbb{O}$. We denote by r_0 the $\mathbb{O}$-rank of G. Then it can be shown that r_0 is a divisor of r. So we set $\delta = r/r_0$. The possible values of δ are as listed below.

$\mathscr{C}$	r	d	δ
$\mathscr{P}_r(\mathbb{R})$	≥ 2	1	1 or 2 (r even)
$\mathscr{P}_r(\mathbb{C})$	≥ 2	2	$\delta \vert r$
$\mathscr{P}_r(\mathbb{H})$	≥ 3	4	1
$\mathscr{P}_3(\mathbb{O})$	3	8	1
$\mathscr{P}(1, n-1)$	2	≥ 3	1

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2 To define the zeta function, we fix a lattice M in V compatible with the given $\mathbb{O}$-structure and let Γ be the stabilizer of M in G. Then Γ is an arithmetic subgroup of G acting properly discontinuously on $\mathscr{C}$. For $x \in V$, we denote by G_x and Γ_x the stabilizers of x in G and Γ.

Let S denote the singular set $\{x \in V | N(x) = 0\}$ and put $V^{\times} = V - S$. Let $V_i^{\times}$ denote the set of all $x \in V^{\times}$ with "signature" $(r - i, i)$ (see §3).

Then one has an (open G-orbit decomposition

$$V^\times = \coprod_{i=0}^{r} V_i^\times. \tag{4}$$

Clearly one has $V_i^\times = -V_{r-i}^\times$, $V_0^\times = \mathscr{C}$.

For $x \in V_{\mathbb{O}}^\times$, G_x is a reductive subgroup defined over $\mathbb{O}$ and Γ_x is an arithmetic subgroup. We denote by $\mu(x)$ the volume of $\Gamma_x \backslash G_x$ with respect to a suitably normalized Haar measure on G_x. In particular, if $x \in \mathscr{C}$, then G_x is compact, Γ_x is finite, and one has $\mu(x) = |\Gamma_x|^{-1} < \infty$. For all $x \in V_{i\mathbb{Q}}^\times (1 \leq i \leq r-1)$, one has $\mu(x) < \infty$ except for the case $r = 2, d = \delta = 1$. In what follows, we exclude this case, which is treated in [Si] and [**?**]. For $0 \leq i \leq r$ we define a zeta function associated with the G-orbit $V_i^\times$ by

$$\xi_i(M; s) = \sum_{x \in \Gamma \backslash M \cap V_i^\times} \mu(x) |N(x)|^{-s}, \tag{5}$$

where the summation is taken over a complete set of representatives of the Γ-orbits in $M \cap V_i^\times$. Clearly one has $\xi_i = \xi_{r-i}$ and $\xi_0(M; s)$ is the zeta function $Z_{\mathscr{C}}(M; s)$ associated with the self-dual homogeneous cone $\mathscr{C}$.

To discuss the convergence of these zeta functions, we need

Lemma 1. *Let* $G^1 = \{g \in G | \det(g) = 1\}$ *and for* $f \in \mathscr{S}(V)$ *(the Schwartz space) set*

$$I(f, M) = \int_{G^1/\Gamma \cap G^1} \left(\sum_{x \in M} f(gx) \right) \mathrm{d}^1 g.$$

180 *where* $\mathrm{d}^1 g$ *is a (suitably normalized) Haar measure on* G^1. *Then, if* $d\delta \geq 2$, *the integral on the right hand side is absolutely convergent and the map* $f \mapsto I(f, M)$ *is a tempered distribution on* V.

This is proved by applying Weil's criterion ([W], p. 90, Lem. 5). For $c > 0$ put

$$A_c = \{\mathrm{diag}(t_1, \ldots, t_{r_0}) | t_i \in \mathbb{R}_+, \prod_{i=1}^{r_0} t_i = 1, t_i/t_{i+1} \geq c$$

$$(1 \leq i \leq r_0 - 1)\}.$$

Then, since every $\mathbb{O}$-root has the multiplicity $d\delta^2$, it is enough to show that

$$\int_{A_c} \left\{ \prod_{i=1}^{r_0} \mathrm{Sup}(1, t_i^{-2})^{\delta(1+(d/2)(\delta-1))} \times \right.$$
$$\left. \times \prod_{1 \leq i < j \leq r_0} \mathrm{Sup}(1, t_i^{-1} t_j^{-1})^{d\delta^2} (t_i t_j^{-1})^{-d\delta^2} \right\}^{1/2} \prod_{i=1}^{r_0 - 1} t_i^{-1} \mathrm{d}t_i < \infty.$$

(See [SS], p. 166, Lem. 4.3.) Putting $\tau_i = (t_i t_{i+1}^{-1})^{1/r_0}$, one has for some $c_1 > 0$

$$\mathrm{Sup}(1, t_i^{-2}) \leq c_1 \prod_{k=1}^{i-1} \tau_k^{2k},$$
$$\mathrm{Sup}(1, t_t^{-1} t_j^{-1}) \leq c_1 \prod_{k=1}^{i-1} \tau_k^{2k} \prod_{\substack{i \leq k < j \\ k \geq r_0/2}} \tau_k^{2k - r_0} (i < j),$$
$$\prod_{i<j} (t_i t_j^{-1})^{-1} = \prod_{i=1}^{r_0 - 1} \tau_i^{-i(r_0 - i) r_0}.$$

In view of these estimates, one sees that the above integral is

$$\leq c_2 \int_{A_c} \left(\prod_{i=1}^{r_0 - 1} \tau_i^{i(r_0 - i)\delta v_i} \right)^{1/2} \prod_{i=1}^{r_0 - 1} t_i^{-1} \mathrm{d}t_i$$

for some $c_2 > 0$, where

$$v_i = \begin{cases} 2 - d(\delta i + 1) & \text{for } 1 \leq i \leq [r_0/2], \\ 2 - d(\delta(r_0 - i) + 1) & \text{for } [r_0/2] + 1 \leq i \leq r_0 - 1. \end{cases}$$

If $d\delta \geq 2$, one has $v_i < 0$ for all $1 \leq i \leq r_0 - 1$, which proves our assertion.

In what follows, we assume that $d\delta \geq 2$. Then Lemma 1 assures that the fundamental assumption (2.6) in [SS] is satisfies, so that we can 181
apply the general results obtained there. (As we shall see in §3, the condition (2.13) in [SS] is also satisfied). In particular, by [SS], Theorem 2, (i), the Dirichlet series on the right hand side of (5) converges absolutely for Re $s > n/r$ and the function $\xi_i(M; s)$ thus defined can be continued to a meromorphic function on the whole complex plane. It is known that, even in the case $d = \delta = 1$, the Dirichlet series defining $Z_{\mathscr{C}}(M; s)$ has the same property ([**?**]).

3 We now consider the G-orbit decomposition of the singular set $S = V - V^{\times}$. Every element x in V can be expressed in the form

$$x = k\left(\sum_{\nu=1}^{r} \alpha_\nu e_\nu\right) \quad \text{with} \quad k \in K, \alpha_\nu \in \mathbb{R}, \tag{6}$$

where $(\alpha_1, \ldots, \alpha_r)$ is uniquely determined up to the order (independently of the choice of the primitive decomposition $\{e_\nu\}$) ([S3], Prop. 3). We say that x is of *rank* ρ and of *signature* $(\rho - i, i)$ if, in a suitable order of (α_ν), one has $\alpha_1, \ldots, \alpha_i < 0$, $\alpha_{i+1}, \ldots, \alpha_p > 0$, $\alpha_{\rho+1} = \ldots = \alpha_r = 0$. For $0 \leq \rho \leq r-1$ and $0 \leq i \leq \rho$, we set

$$S^{(\rho)} = \{x \in V | \operatorname{rank} x = \rho\},$$
$$S_i^{(\rho)} = \{x \in V | \operatorname{sign} x = (\rho - i, i)\}.$$

Then it is easy to see that the G-orbit decomposition of S is given by

$$S = \coprod_{\substack{0 \leq \rho \leq r-1 \\ 0 \leq i \leq \rho}} S_i^{(\rho)}. \tag{7}$$

Since G^1 is transitive on each $S_i^{(\rho)}$, (7) is also the G^1-orbit decomposition of S. Thus the condition (2.13) in [SS] is certainly satisfied.

By [SS], Lemmas 2.7 and 2.8, (i), there exists a G^1-invariant measure $\mathrm{d}\nu^{(\rho)}(v)$ on $S^{(\rho)}$ satisfying the relation

$$\mathrm{d}\nu^{(\rho)}(gv) = \chi(g)^{s_{\rho,i}} \mathrm{d}\nu^{(\rho)}(v) \qquad (g \in G, v \in S_i^{(\rho)}) \tag{8}$$

for some $s_{\rho,i} \in \mathbb{R}$. To describe the measure $dv^{(\rho)}$ explicitly, we use the following parametrization of $S^{(\rho)}$.

Set

$$e = \sum_{\nu=1}^{r-\rho} e_\nu, \quad e' = c_0 - e = \sum_{\nu=r-\rho+1}^{r} e_\nu$$

and $V_\lambda = V_\lambda(e) = \{x \in V | ex = \lambda x\}$. Writing $v \in V$ in the form

$$v = v_1 + v_{1/2} + v_0, \ \ v_\lambda \in V_\lambda(e),$$

we set 182

$$S^{(\rho)}(e') = \{v \in S^{(\rho)} | N_0(v_0) \neq 0\}, \tag{9}$$

where N_0 denotes the norm of the Jordan subalgebra $V_0(e)$. Then $S^{(\rho)}(e')$ is a Zariski open set in $S^{(\rho)}$ and by [S3], Lemma 1 every element v in $S^{(\rho)}(e')$ can be written uniquely in the form

$$v = \exp(e\Box y)v_0 \quad \text{with} \quad y \in V_{1/2}, v_0 \in V_0, \tag{10}$$

where in general $x\Box y = T_{xy} + [T_x, T_y]$ (Koecher's notation).

By a well-known identity in the Jordan algebra one has for any y, y' in $V_{1/2}(e)$

$$[[T_y, T_{y'}], T_e] = T_{y(y'e)-y'(ye)} = 0.$$

Hence one has $[e\Box y, e\Box y'] = 0$ (by [S3], (4)) and

$$\exp(e\Box y) \cdot \exp(e\Box y') = \exp(e\Box(y + y')).$$

Therefore $S^{(\rho)}(e')$ can be viewed as a principal bundle of the additive group $V_{1/2}(e)$ with base space $V_0 = V_0(e)$ by the action

$$v \mapsto \exp(e\Box y)v(y \in V_{1/2}(e), v \in S^{(\rho)}(e')).$$

It follows, that, if one puts

$$d\mu(v) = dy \cdot dv_0 \quad \text{for} \quad v = \exp(e\Box y)v_0, y \in V_{1/2}, v_0 \in V_0,$$

then there exists a continuous function $\varphi = \varphi_{\rho,i} : V_0(e) \to \mathbb{R}$ such that $dv^{(\rho)}(v) = \varphi(v_0)^{-1} d\mu(v)$. Putting $g = \lambda 1$ in (8), one sees that φ is homogeneous of degree $\rho(1 + \frac{d}{2}(\rho - 1)) - rs_{\rho,i}$.

Not let G_0 be the subgroup of G generated by $\exp T_x (x \in V_0(e))$. Then the $V_\lambda(e)'$s are G_0-invariant. For $g_0 \in G_0$, one has $g_0|V_1(e) = \mathrm{id}.$ and $N_0(g_0 v_0) = \chi_0(g_0) N_0(v_0)$ for $v_0 \in V_0(e)$, where χ_0 is a rational character of G_0 satisfying the relation

$$\det(g_0|V_0(e)) = \chi_0(g_0)^{1+(d/2)(\rho-1)}. \tag{11}$$

Lemma 2. *For $g_0 \in G_0$, one has*

$$\det(g_0|V_{1/2}) = \chi_0(g_0)^{(d/2)(r-\rho)}, \tag{12}$$

$$\chi(g_0) = \chi_0(g_0). \tag{13}$$

Proof. Since $g_0 e = e$, one has for $y \in V_{1/2}$

$$g_0(e \Box y) v_0 = (e \Box {}^t g_0^{-1} y) g_0 v_0.$$

Hence $g_0(y v_0) = ({}^t g_0^{-1} y) \cdot (g_0 v_0)$, or

$$g_0 T_{v_0} {}^t g_0 = T_{g_0 v_0} \quad \text{on} \quad V_{1/2}(e). \tag{*}$$

By [SF], Lemma 2, (i), one has $\det(2T_{v_0}|V_{1/2}) = N_0(v_0)^{d(r-\rho)}$. Taking the determinant of both sides of (*), one has

$$\det(g_0|V_{1/2})^2 N_0(v_0)^{d(r-\rho)} = N_0(g_0 v_0)^{d(r-\rho)},$$

183 whence follows (12). Since

$$\chi(g_0)^{1+(d/2)(r-1)} = \det(g_0) = \det(g_0|V_{1/2}) \cdot \det(g_0|V_0),$$

(13) follows from (11) and (12). □

By (11) and (12) one has

$$\begin{aligned} \mathrm{d}\mu(g_0 v) &= d({}^t g_0^{-1} y) \cdot \mathrm{d}(g_0 v_0) \\ &= \chi_0(g_0)^{-(d/2)(r-\rho)+1+(d/2)(\rho-1)} \mathrm{d}\mu(v). \end{aligned}$$

Hence by (8) and (13) one has

$$\varphi(g_0 v_0) = \chi_0(g_0)^{1-(d/2)(r-2\rho+1)-s_{\rho,i}} \varphi(v_0).$$

In particular, φ is homogeneous of degree $\rho(1-\frac{d}{2}(r-2\rho+1)-s_{\rho,i})$. Combining this with what we mentioned above, we see that $s_{\rho,i}=\frac{d}{2}\rho$, which is independent of $0\le i\le\rho$, and that $\varphi(v_0)$ is given by $cN(v_0)^{1-(d/2)(r-\rho+1)}$ for some $c>0$. We normalize $\mathrm{d}v^{(\rho)}(v)$ by putting $c=2^{-d\rho(r-\rho)}$.

Summing up, we have

Lemma 3. *In the expression* (10)*, a* G^1*-invariant measure on* $S^{(\rho)}$ *is given by*

$$\mathrm{d}v^{(\rho)}(v)=2^{-d\rho(r-\rho)}N_0(v_0)^{(d/2)(r-\rho+1)-1}\mathrm{d}y\,\mathrm{d}v_0. \tag{14}$$

and one has

$$\mathrm{d}v^{(\rho)}(gv)=\chi(g)^{(d/2)\rho}\mathrm{d}v^{(\rho)}(v)\qquad(g\in G, v\in S^{(\rho)}). \tag{15}$$

4 We set $\mathscr{V}_k=\oplus_{j<k}V_{jk}(2\le k\le r)$. Then every element v in $S^{(\rho)}(e')$ can be written uniquely in the form

$$v=\sum_{k=r-\rho+1}^{r}\epsilon_k t_k\left(e_k+\frac{1}{2}x'_k+\frac{1}{4}x'^2_k(1-e_k)\right), \tag{16}$$
$$x'_k\in\mathscr{V}_k, t_k\in\mathbb{R}_+, \epsilon_k=\pm1(r-\rho+1\le k\le r)$$

(see [S3], Prop. 1). In this expression, it is easy to see that

$$\mathrm{d}v^{(\rho)}(v)=2^{-(d/2)\rho(2r-\rho-1)}\prod_{k=r-\rho+1}^{r}(t_k^{(d/2)(2k-r+\rho-1)-1}\mathrm{d}t_k)\mathrm{d}x', \tag{17}$$

where $\mathrm{d}x'=\Pi\mathrm{d}x'_k$ is the Euclidean measure on $\oplus_{k=r-\rho+1}^{r}\mathscr{V}_k$.

For $0\le\rho\le r-1$ and $0\le i\le\rho$, let $\mathscr{E}_{\rho,i}^{(r)}$ denote the set of all r-tuples $\epsilon=(\epsilon_\nu)\in\{0,1,-1\}^r$ such that

$$\epsilon_\nu=0(0\le\nu\le r-\rho),=\pm1(r-\rho+1\le\nu\le r),$$
$$\text{and }\ \sharp\{\nu|\epsilon_\nu=-1\}=i.$$

For $\epsilon=(\epsilon_\nu)\in\mathscr{E}_{\rho,i}^{(r)}$, let $S_\epsilon^{(\rho)}$ denote the set of all v in $S^{(\rho)}(e')$ of the form **184**

(16) with the given (ϵ_v). Then clearly one has

$$S^{(\rho)}(e') \cap S_i^{(\rho)} = \coprod_{\epsilon \in \mathscr{E}_{\rho,i}^{(r)}} S_\epsilon^{(\rho)}.$$

Let $\mathscr{E}^{(r)} = \{\pm 1\}^r$, and for $\eta = (\eta_k) \in \mathscr{E}^{(r)}$ let $V_\eta^\times$ denote the AN-orbit of $\sum_{k=1}^r \eta_k e_k$, where $A = \exp \mathbf{a}$ and N is the unipotent subgroup of G generated by $\exp(\sum_{k=2}^r e_k \Box x_k)(x_k \in \mathscr{V}_k)$. Then $\coprod_{\eta \in \mathscr{E}^{(r)}} V_\eta^\times$ is a Zariski open subset of $V^\times$.

Proposition 1. *Let* $0 \leq \rho \leq r-1$ *and* $0 \leq i \leq \rho$. *Then, for* $f \in C_0^\infty(\coprod V_\eta^\times)$, *one has*

$$\int_{S_i^{(\rho)}} \widehat{f}(v) \mathrm{d}v^{(\rho)}(v) = \prod_{k=1}^{\rho} [(2\pi)^{-(d/2)k} \Gamma\left(\frac{d}{2}k\right) \times$$

$$\times \sum_{\epsilon,\eta} \mathbf{e}\left(\frac{d}{8} N_{\epsilon\eta}^{(\rho)}\right) \int_{V_\eta^\times} f(u) |N(u)|^{-(d/2)\rho} \mathrm{d}u,$$

where the summation is taken over all $\epsilon \in \mathscr{E}_{\rho,i}^{(r)} \eta \in \mathscr{E}^{(r)}$, $\mathbf{e}(\cdot)$ *stands for* $\exp(2\pi\sqrt{-1}\cdot)$ *and*

$$N_{\epsilon\eta}^{(\rho)} = \sum_{k=r-\rho+1}^{r} \epsilon_k \left(\sum_{l=1}^{k-1} \eta_1 + \eta_k(k-r+\rho) \right).$$

The proof is similar to that of [SF], (21), and we use basically the same notation as in [SF]. Write $u \in V$ in the form

$$u = \sum_{k=1}^{r} \xi_k e_k + \sum_{k=2}^{r} u_k,$$

with

$$\xi_k \in \mathbb{R}^\times, \ u_k \in \mathscr{V}_k$$

and for $1 \le k \le r$ set

$$u^{(k)} = \sum_{j=1}^{k} \xi_j e_j + \sum_{j=2}^{k} u_j.$$

For a self-adjoint linear operator T on $\mathscr{V}_k$ and $x_k \in \mathscr{V}_k$, we write

$$T[x_k] = \langle x_k, T x_k \rangle.$$

In particular, **185**

$$u^{(k-1)}[x_k] = T_{u^{(k-1)}}[x_k] = \langle x_k, u^{(k-1)} x_k \rangle.$$

Now, writing $v \in S_{\epsilon}^{(\rho)}$ in the form (16), one has

$$\langle u, v \rangle = \sum_{k=r-\rho+1}^{r} \epsilon_k t_k \left(\xi_k + \frac{1}{2}\langle u_k, x'_k \rangle + \frac{1}{4} u^{(k-1)}[x'_k] \right)$$

(see [SF], p. 476). For $\lambda > 0$ and $r - \rho + 1 \le k \le r$, put

$$Q_k = Q_k(\lambda, \epsilon_k, u^{(k-1)}) = \lambda 1_{\mathscr{V}_k} - \frac{\sqrt{-1}}{2} \epsilon_k T_{u^{(k-1)}} | \mathscr{V}_k.$$

Then, for $u \in \coprod V_{\eta}^{\times}$, one has

$$\langle u, v \rangle = \frac{\sqrt{-1}}{2} \lim_{\lambda \to 0} \sum_{k=r-\rho+1}^{r} t_k (\lambda - 2\sqrt{-1}\epsilon_k \xi_k + \\ + Q_k \left[x'_k - \frac{\sqrt{-1}}{2} \epsilon_k Q_k^{-1} u_k \right] + \frac{1}{4} Q_k^{-1}[u_k])$$

Hence, putting $q_k(\lambda) = \lambda - 2\sqrt{-1}\epsilon_k \xi_k + \frac{1}{4} Q_k^{-1}[u_k]$, one has

$$I_{\epsilon} = \int_{S_{\epsilon}}^{(\rho)} \widehat{f}(v) \mathrm{d}v^{(\rho)}(v)$$

$$= \int_{S_\epsilon^{(\rho)}} \left(\int_V f(u)\mathbf{e}(\langle u, v\rangle)\mathrm{d}u \right) \mathrm{d}v^{(\rho)}(v)$$

$$= 2^{-(d/2)\rho(2r-\rho-1)} \sum_{\eta \in \mathscr{E}^{(r)}} \int_{V_\eta^\times} f(u)\mathrm{d}u \times$$

$$+ \lim_{\lambda\to 0} \sum_{k=r-\rho+1}^{r} \int_0^\infty t_k^{(d/2)(2k-r+\rho-1)-1} \mathbf{e}\left(\frac{\sqrt{-1}}{2} t_k q_k(\lambda) \right) \mathrm{d}t_k \times$$

$$\times \int_{???????????} \mathbf{e}\left(\frac{\sqrt{-1}}{2} t_k Q_k \left[x_k' - \frac{\sqrt{-1}}{2} \epsilon_k Q_k^{-1} u_k \right] \right) \mathrm{d}x_k'.$$

186 For $u \in V_\eta^\times$, one has, in the notation of [SF], sign $\chi_k(u) = \eta_k (1 \le k \le r)$. Hence

$$\int_{\mathscr{V}_k} \mathbf{e}\left(\frac{\sqrt{-1}}{2} t_k Q_k \left[x_k' - \frac{\sqrt{-1}}{2} \epsilon_k Q_k^{-1} u_k \right] \right) \mathrm{d}x_k' = t_k^{-(d/2)(k-1)} \det(Q_k)^{-1/2}$$

$$\to 2^{d(k-1)} t_k^{-(d/2)(k-1)} \mathbf{e}\left(\frac{d}{8} \epsilon_k \sum_{l=1}^{k-1} \eta_l \right) |N^{(k-1)}(u^{(k-1)})|^{-d/2}$$

$$(\lambda \to 0),$$

$$\int_0^\infty t^{(d/2)(k-r+\rho)-1} \mathbf{e}\left(\frac{\sqrt{-1}}{2} t_k q_k(\lambda) \right) \mathrm{d}t_k$$

$$= \Gamma\left(\frac{d}{2}(k-r+\rho) \right) (\pi q_k(\lambda))^{-(d/2)(k-r+\rho)}$$

$$\to \Gamma\left(\frac{d}{2}(k-r+\rho) \right) \mathbf{e}\left(\frac{d}{8} \epsilon_k \eta_k (k-r-\rho) \right) (2|\chi_k(u)|)^{-(d/2)(k-r+\rho)}$$

$$(\lambda \to 0).$$

Moreover, one has

$$\prod_{k=r-\rho+1}^{r} N^{(k-1)}(u^{(k-1)}) |\chi_k(u)|^{k-r+\rho} = N(u)^\rho.$$

Therefore, putting $k' = k - r + \rho(r - \rho + 1 \le k \le r)$, one has

$$I_\epsilon = 2^{(d/2)\rho(2r-\rho-1)} \sum_{\eta\in\mathscr{E}^{(r)}} \int_{V_\eta^\times} f(u)\mathrm{d}u \times$$
$$\times \lim_{\lambda\to 0}\left(\sum_{k=r-\rho+1}^{r} \det(Q_k)^{-1/2}\Gamma\left(\frac{d}{2}k'\right)(\pi q_k(\lambda))^{-(d/2)k'}\right)$$
$$= \prod_{k'=1}^{\rho}\left(2\pi)^{-(d/2)k'}\Gamma((d/2)k')\right]$$
$$\times \sum_{\eta} \mathbf{e}\left(\frac{d}{8}\sum_{k=r-\rho+1}^{r} \epsilon_k\left(\left(\sum_{l<k}\eta_l\right) - \eta_k(k-r+\rho)\right)\right)\int_{V_\eta^\times} f(u)|N(u)|^{-(d/2)\rho}\mathrm{d}u,$$

which proves the Proposition.

5 We put

$$\lambda_\rho = \prod_{k=1}^{\rho}\left[(2\pi)^{-(d/2)k}\Gamma\left(\frac{d}{2}k\right)\right]$$

Also, putting $n(\eta) = \sharp\{k|1 \le k \le r, \eta_k = -1\}$ for $\eta \in \mathscr{E}^{(r)}$, we set 187
$\mathscr{E}_j^{(r)} = \{\eta \in \mathscr{E}^{(r)}|n(\eta) = j\}$. Then for $\eta \in \mathscr{E}_j^{(r)}$ an easy computation shows that

$$N_{\epsilon\eta}^{(\rho)} = (r-2j)\rho - 2i(r-\rho-2p-1) +$$
$$+ \sum_{k=r-\rho+1}^{r}(1-\epsilon_k)\left\{\sum_{l=r-\rho+1}^{k-1}(1-\eta_l) - (1+\eta_k)(k-r+\rho)\right\},$$

where $p = \sharp\{k|1 \le k \le r-\rho, \eta_k = -1\}$ and Σ' indicates that the summation is taken over k (or l) $\ge r - \rho + 1$.

We claim that $\sum_{\epsilon\in\mathscr{E}_{\rho,i}^{(r)}} \mathbf{e}\left(\frac{d}{8}N_{\epsilon\eta}^{(\rho)}\right)$ depends only on i and $j = n(\eta)$ and is independent of the choice of $\eta \in \mathscr{E}_j^{(r)}$. Then the formula in Proposition

1 can be written as

$$\int_{S_i^{(\rho)}} f(v)\mathrm{d}v^{(\rho)}(v) = \sum_{j=0}^{r} r_{ij} \int_{V_j^{\times}} f(u)|N(u)|^{-(d/2)\rho}\mathrm{d}u, \tag{19}$$

where

$$r_{ij} = \lambda_\rho \sum_{\epsilon\in\mathscr{E}_{\rho,i}^{(r)}} \mathbf{e}\left(\frac{d}{8}N_{\epsilon\eta}^{(\rho)}\right) \quad (\eta \in \mathscr{E}_j^{(r)}). \tag{20}$$

When d is even, the above claim is obvious, since

$$\mathbf{e}\left(\frac{d}{8}N_{\epsilon\eta}^{(\rho)}\right) = \sqrt{-1}^{(d/2)r\rho}(-1)^{(d/2)((r-\rho-1)i+\rho j)}.$$

In this case, one has

$$r_{ij} = \sqrt{-1}^{(d/2)r\rho}\lambda_\rho \cdot (-1)^{(d/2)(r-\rho-1)i}\binom{\rho}{i} \cdot (-1)^{(d/2)\rho j}. \tag{21}$$

Introducing a new variable y, one has the relation

$$\sum_{i=0}^{\rho} r_{ij}y^i = \sqrt{-1}^{(d/2)r\rho}\lambda_\rho \cdot (-1)^{(d/2)\rho j}(1 + (-1)^{(d/2)(r-\rho-1)}y)^\rho. \tag{22}$$

When d is odd, we set

$$\omega = (-1)^\rho(-\sqrt{-1})^{d(r-\rho-1)}$$

Then for $\eta \in \mathscr{E}_j^{(r)}$ one has

$$\sum_{i=0}^{\rho}\left(\sum_{\epsilon\in\mathscr{E}_{\rho,i}^{(r)}} \mathbf{e}\left(\frac{d}{8}N_{\epsilon\eta}^{(\rho)}\right)\right)y^i = \zeta'^{\,dr\rho}_{8}(-\sqrt{-1})^{d\rho j} \times$$

$$\times \sum_{i,\epsilon}(-1)^{\sum_k (1-\epsilon_k)/2\{\sum_{l<k}(1-\eta_l)/2+(1+\eta_k)(k-r+\rho)/2\}}(\omega y)^i$$

$$= \zeta_8^{dr\rho}(-\sqrt{-1})^{d\rho j}\prod_{k'=1}^{\rho}(1+(-1)^{\sum'_{l<k}(1-\eta_l)/2+(1+\eta_k)k/2}\omega y),$$

188 where the summation $\sum\limits_{i,\epsilon}$ is taken over all $0 \le i \le \rho$, $\epsilon \in \mathscr{E}_{\rho,i}^{(r)}$. It can be shown that the last expression depends only on i and j. Actually, defining r_{ij} by (20), one obtains the relation

$$\sum_{i=0}^{\rho} r_{ij}y^i = \zeta_8^{dr\rho}\lambda_\rho(-\sqrt{-1})^{d\rho j} \tag{22'}$$

$$\times\begin{cases}(1-\omega^2y^2)^{\rho/2} & (\rho \text{ even}),\\ (1-\omega^2y^2)^{(\rho-1)/2} & (1-(-1)^j(-\sqrt{-1})^{d(r-\rho-1)}y)(\rho \text{ odd}).\end{cases}$$

Now, for the given $\mathbb{Q}$-structure one has $S_i^{(\rho)} \cap V_{\mathbb{Q}} \neq \emptyset$ if and only if ρ is a multiple of $\delta = r/r_0$. By the general theory of Sato-Shintani, we know that for $x \in S_i^{(\rho)} \cap V_{\mathbb{Q}}$ the stabilizer G_x^1 of x in G^1 is unimodular and for a normalized Haar measure $\mathrm{d}v_x$ on G_x^1 one has

$$\mu^1(x) = \int_{G_x^1/\Gamma_x} \mathrm{d}v_x < \infty,$$

the normalization being made by the relation

$$\int_{G^1/\Gamma_x} f(gx)\mathrm{d}^1g = \mu^1(x)\int_{S_i^{(\rho)}} f(v)\mathrm{d}v^{(\rho)}(v) \qquad (f \in C_0^\infty(S_i^{(\rho)})).$$

It is known furthermore that one has

$$\kappa_i^{(\rho)}(M) = \sum_{x:\Gamma\backslash M\cap S_i^{(\rho)}} \mu^1(x) < \infty \tag{23}$$

([SS], Proof of Lemma 2.7). It can be shown that $\kappa_i^{(\rho)}(M)$ is a finite sum of certain special values of the zeta functions associated with the "rational boundary components" in $S_i^{(\rho)}$.

By [SS], Theorem 2, (*ii*), one obtains the follows

Proposition 2. *Assume that* $d\delta \geq 2$. *Then the zeta functions* $\xi_j(M; s)(0 \leq j \leq r)$ *are holomorphic except for possible simple poles at* $s = \frac{n}{r} - \frac{d}{2}\rho$ *with* $0 \leq \rho \leq r-1$, $\delta|\rho$ *and*

$$\mathrm{Res}_{S=\frac{n}{r}-\frac{d}{2}\rho}\, \xi_j = \mathrm{vol}(V/M^*) \sum_{i=0}^{\rho} \kappa_i^{(\rho)}(M^*) r_{ij}, \tag{24}$$

189 *where* M^* *is the dual lattice of* M *and* $\kappa_i^{(\rho)}(M^*)$ *and* r_{ij} *are given by* (23) *(for* M^**) and* (22), ((22′)).

In the case d is even, we set

$$R^{(\rho)} = \mathrm{vol}(V/M^*)\lambda_\rho \sqrt{-1}^{(d/2)r\rho} \cdot \sum_{i=0}^{\rho} (-1)^{(d/2)(r-\rho+1)i} \binom{\rho}{i} \kappa_i^{(\rho)}(M^*). \tag{25}$$

In particular, for $\rho = 0$, one has $R^{(0)} = \mathrm{vol}(V/M^*) \cdot \mu^1(0) > 0$.

Lemma 4. *When* $d \equiv 0(4)$ *or* $\rho \equiv r-1(2)$, *one has* $R^{(\rho)} \neq 0$. *When* $d \equiv 2$ *(4) and* $\rho \equiv r \equiv 1(2)$, *one has* $R^{(\rho)} = 0$.

This follows immediately from the fact that $\kappa_i^{(\rho)}(M^*) > 0$ and

$$\kappa_i^{(\rho)}(M^*) = \kappa_{\rho-i}^{(\rho)}(M^*).$$

By Proposition 2, the residue of ξ_j at $s = \frac{n}{r} - \frac{d}{2}\rho(0 \leq \rho \leq r-1, \delta|\rho)$ is given by $(-1)^{(d/2)\rho j} R^{(\rho)}$. Hence, in particular, every $\xi_j(0 \leq J \leq r)$ has a pole at $s = n/r$. It follows from Lemma 4 that, when $d \equiv 0(4)$, every ξ_j has exactly r_0 poles at $s = \frac{n}{r} - \frac{d}{2}\delta k(0 \leq k \leq r_0 - 1)$. When $d \equiv 2(4)$ and r is odd, every ξ_j has exactly $(r_0+1)/2$ poles at the above s with even k. When $d \equiv 2(4)$ and r is even, one can only say that, if δ is odd, every ξ_j has at least $[(r_0+1)/2]$ poles.

6 By the general theory of Sato-Shintani we know that the zeta functions satisfy functional equations of the following form

$$\xi_j\left(M^*; \frac{n}{r} - s\right) = \mathrm{vol}(V/M) \prod_{k=1}^{r} \left[(2\pi)^{-s+(d/2)(k-1)} \Gamma\left(s - \frac{d}{2}(k-1)\right)\right] \times \tag{26}$$

$$\times\, \mathbf{e}\left(\frac{r}{4}s\right)\sum_{i=0}^{r}\xi_i(M;s)u_{ij}(s)$$

$$(0 \le j =\le r),$$

where $u_{ij}(s)$ is a polynomial in $\mathbf{e}(-\frac{s}{2})$ determined by the following relation

$$\int_{V_i^\times}\widehat{f}(u)|N(u)|^{s-(n/r)}\mathrm{d}u = \sum_{k=1}^{r}\left[(2\pi)^{-s+(d/2)(k-1)}\Gamma\left(s-\frac{d}{2}(k-1)\right)\right]\times \quad (27)$$

$$\times\, \mathbf{e}\left(\frac{r}{4}s\right)\sum_{j=0}^{r}u_{ij}(s)\int_{V_j^\times}f(u)|N(u)|^{-s}\mathrm{d}u$$

$$(0 \le i \le r, f \in \mathscr{S}(V)).$$

By computing the left hand side of (27), it was shown in [SF] that **190**

$$u_{ij}(s) = \sum_{\epsilon\in\mathscr{E}_i^{(r)}}u_{\epsilon\eta}(s) \qquad (n \in \mathscr{E}_j^{(r)}), \qquad (\sharp)$$

where

$$u_{\epsilon\eta}(s) = \mathbf{e}\left(\frac{d}{8}N_{\epsilon\eta}^{(r)} + \frac{1}{4}\sum_{k=1}^{r}\left(\epsilon_k\eta_k\left(s-\frac{d}{2}r\right)-s\right)\right).$$

(The fact that the right hand side of ($\sharp$) depends only on $j = n(\eta)$ is shown in the following lines. This is not *a priori* clear as stated in [SF], p. 477.)

First, one can write

$$u_{\epsilon\eta}(s) = \mathbf{e}\left(\frac{d}{2}\sum_{k=1}^{r}\frac{1-\epsilon_k}{2}\left(\sum_{l=1}^{k-1}\frac{1-\eta_l}{2}-\frac{1+\eta_k}{2}k\right)+\frac{d}{4}(i-rj)-\right.$$

$$\left.-\frac{1}{2}\sum_{k=1}^{r}\left(\frac{1+\epsilon_k}{2}\frac{1-\eta_k}{2}+\frac{1-\epsilon_k}{2}\frac{1+\eta_k}{2}\right)\left(s-\frac{d}{2}r\right)\right)$$

Put

$$x = \mathbf{e}(-\frac{d}{2}), \quad \zeta = \sqrt{-1}^{d(r+1)},$$

$$\beta_k = \mathbf{e}\left(\frac{d}{2}\sum_{l=1}^{k-1}\frac{1-\eta_l}{2}\right) = \begin{cases} 1 & (d \text{ even}), \\ (-1)^{\#\{l|1\le l\le k-1,\eta_l=-1\}} & (d \text{ odd}), \end{cases}$$

$$u_\eta(x,y) = \sum_{i,\epsilon} u_{\epsilon\eta}(s)y^i,$$

where the summation Σ_ϵ is taken over all $0 \le i \le r$, $\epsilon \in \mathscr{E}_i^{(r)}$. Then by an easy computation one obtains

$$\begin{aligned} u_\eta(x,y) &= (-\sqrt{-1})^{drj}\prod_{k=1}^{r}\left(\mathbf{e}\left(-\frac{1}{2}\frac{1-\eta_k}{2}\left(s-\frac{d}{2}r\right)\right) + \right. \\ &\quad \left. + \mathbf{e}\left(-\frac{1}{2}\left(\frac{1+\eta_k}{2}\left(s-\frac{d}{2}r+dk\right) - d\sum_{l=1}^{k-1}\frac{1-\eta_l}{2}\right)\sqrt{-1}^d y\right) \\ &= \prod_{\eta_k=1}(1+(-1)^{dk}\zeta\beta_k xy)\prod_{\eta_k=-1}(x+(-1)^d\zeta^{-1}\beta_k y). \end{aligned}$$

When d is even, one has $\zeta = (-1)^{(d/2)(r+1)}, \beta_k = 1$ and

$$u_\eta(x,y) = (1+\zeta xy)^{r-j}(x+\zeta y)^j. \tag{28}$$

When d is odd, one has $\zeta^2 = (-1)^{r+1}$ and

$$\begin{aligned} u_\eta(x,y) &= (1+\zeta xy)^{[(r-j)/2]}(1-\zeta xy)^{r-j-[(r-j)/1]} \times \\ &\quad \times (x+\zeta^{-1}y)^{[j/2]}(x-\zeta^{-1}y)^{j-[j/2]}. \end{aligned} \tag{29}$$

191 Thus, in either case, one sees that $u_\eta(x,y)$ depends only on $j = n(\eta)$. Hence one writes $u_j(x,y)$ for it. [(28) and (29) are the same as (25) in [SF].]

The semisimplicity of the matrix $U^{(r)}(x) = (u_{ij}(x))$ was shown in [SF] except for the case $d = 1$. Hence, in the rest of the paper, we assume that $d = 1$. By our assumption $d\delta \ge 2$, one then has $\delta = 2$ and so r is even. However, we include also the case r is odd.

First, suppose that r is odd. Then $\zeta = (-1)^{(r+1)/2}$ and one has

$$u_j(x,y) = \begin{cases} (1-x^2y^2)^{(r-j-1)/2}(x^2-y^2)^{j/2}(1-\zeta xy) & (j \text{ even}), \\ (1-x^2y^2)^{(r-j)/2}(x^2-y^2)^{(j-1)/2}(x-\zeta y) & (j \text{ odd}). \end{cases} \tag{30}$$

Dividing the set of indices in two blocks by their parity, we write

$$U^{(r)}(x) = \begin{pmatrix} U_{++} & U_{+-} \\ U_{-+} & U_{--} \end{pmatrix},$$

where $U_{++,\ldots}$ are square matrices of size $\frac{r+1}{2}$ consisting of $u_{ij}(x)$ with (i, j) of the given parity (*e.g.* U_{+-} consisting of $u_{ij}(x)$ with i even and j odd). Then (30) gives

$$U_{++} = \rho_{(r-1)/2}\left(\begin{pmatrix} 1 & x^2 \\ -x^2 & -1 \end{pmatrix}\right), \quad U_{+-} = xU_{++},$$
$$U_{-+} = -\zeta x U_{++}, \quad U_{--} = -\zeta U_{++},$$

or more symbolically

$$U^{(r)}(x) = \rho_{(r-1)/2}\left(\begin{pmatrix} 1 & x^2 \\ -x^2 & -1 \end{pmatrix}\right) \otimes \begin{pmatrix} 1 & x \\ -\zeta x & -\zeta \end{pmatrix}, \tag{31}$$

where $\rho_{(r-1)/2}$ denotes the symmetric tensor representation of degree $\frac{r-1}{2}$. Thus we see the $U^{(r)}(x)$ is diagonalizable and similar to

$$\rho_{(r-1)/2}\left(\begin{pmatrix} 0 & 1-x^2 \\ 1+x^2 & 0 \end{pmatrix}\right) \otimes \begin{pmatrix} 0 & 1-x \\ 1+x & 0 \end{pmatrix} (\zeta = 1),$$

or

$$\rho_{(r-1)/2}\left(\begin{pmatrix} 0 & 1-x^2 \\ 1+x^2 & 0 \end{pmatrix}\right) \otimes \begin{pmatrix} 1+x & 0 \\ 0 & 1-x \end{pmatrix} (\zeta = -1).$$

Next, suppose that r is even. Then $\zeta = (-1)^{r/2}\sqrt{-1}$ and

$$u_j(x, y) = \begin{cases} (1 + x^2y^2)^{(r-j)/2}(x^2 + y^2)^{j/2} & (j \text{ even}), \\ (1 + x^2y^2)^{(r-j-1)/2}(x^2 + y^2)^{(j-1)/2}(1 - \zeta xy)(x + \zeta y) & (j \text{ odd}). \end{cases} \tag{32}$$

Hence, in the notation similar to the above, one has 192

$$U_{++} = \rho_{r/2}\left(\begin{pmatrix} 1 & x^2 \\ x^2 & 1 \end{pmatrix}\right),$$

$$
\begin{aligned}
U_{+-} &= x(\delta_{ij} + \delta_{i,j+1})\rho_{r/2-1}\left(\begin{pmatrix} 1 & x^2 \\ x^2 & 1 \end{pmatrix}\right), \\
U_{-+} &= 0, \\
U_{--} &= \zeta(1 - x^2)\rho_{r/2-1}\left(\begin{pmatrix} 1 & x^2 \\ x^2 & 1 \end{pmatrix}\right).
\end{aligned}
$$

Thus $U^{(r)}(x)$ is again diagonalizable and similar to

$$
\rho_{r/2}\left(\begin{pmatrix} 1 + x^2 & 0 \\ 0 & 1 - x^2 \end{pmatrix}\right) \oplus \zeta(1 - x^2)\rho_{r/2-1}\left(\begin{pmatrix} 1 + x^2 & 0 \\ 0 & 1 - x^2 \end{pmatrix}\right).
$$

These results imply that one can simplify the functional equations, introducing certain L-functions, and obtain some information about the special values of the zeta functions, as was done in [SO] in the case d is even.

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THE NUMBER OF RATIONAL APPROXIMATIONS TO ALGEBRAIC NUMBERS AND THE NUMBER OF SOLUTIONS OF NORM FORM EQUATIONS

By Wolfgang M. Schmidt

195 Recently, after I had lectured on Norm Form equations [11], Schinzel said : "But now can it be, how can it be in number theory, that one could possibly prove the finiteness of a set of natural numbers, but obtain no estimate of its cardinality ?" The next day, he himself provided the following answer : Suppose we can prove for a set S of natural numbers that for any x, x' in S we have $x' \leq 2x$. Then S is finite, but unless we can find a particular $x \in S$, we cannot estimate the cardinality of S, and even less can be provide a bound for the size of elements in S.

More generally, for $C > 1$, call a set S of positive integers a C-set, if $x' \leq Cx$ for any $x, x' \in S$. What we said above applies more generally to any C-set. On the other hand, we define a λ-set where $\lambda > 1$ to be a set S with the following Gap Principle : When x, x' are in S with $x' > x$, then $x' \geq \lambda x$. A (C, λ)-set is both a C-set and a λ-set. Let $x_0 < x_1 < \ldots < x_v$ be elements of a (C, λ)-set S. Then $x_v \leq Cx_0$ and $x_i \geq \lambda x_{i-1} (i = 1, \ldots, v)$, so that $x_v \geq \lambda^v x_0$. Therefore $\lambda^v \leq C$, and $v \leq (\log C)/(\log \lambda)$, so that S has cardinality

$$|S| \leq 1 + (\log C)/(\log \lambda).$$

In this argument, we did not need to assume that S consists of integers.

A situation very much like this occurs in the Thue-Siegel-Roth Theorem. Let me begin with Thue's Theorem. It asserts that when α is algebraic of degree $r \geq 3$, and if $\mu > (r/2) + 1$, then there are only finitely many rationals x/y with $|\alpha - (x/y)| < |y|^{-\mu}$. Let Y be the set of positive y such that there is a reduced fraction x/y which satisfies this inequality; then Y is finite according to Thue. It is well known that Thue's Theorem (and subsequent theorems of Siegel, Roth, Schmidt) are "ineffective" in the sense that they do not provide an upper bound for the (size of) elements of Y. An analysis of Thue's proof shows that it yields an explicit constant $B = B(\alpha, \mu)$, such that if S_1 is the set of $y \in Y$ with $y < B(\alpha, \mu)$, the so-called "small solutions", and if S_2 is the set of $y \in Y$ with $y \geq B(\alpha, \mu)$, the so-called "large solutions", then the following holds. First, it is trivial that the cardinality of S_1 does not exceed the explicit bound $B(\alpha, \mu)$. Second, there is an explicit $C = C(\alpha, \mu)$ such that S_2 is an *exponential C-set* in the sense that

$$y' \leq y^C$$

for any y, y' in S_2. This shows that S_2 (and hence Y) is finite, but 196
gives no information on the cardinality. However, it turns out that there is also an exponential Gap Principle. Suppose $y' > y$ lie in S_2 and $|\alpha - (x/y)| < y^{-\mu}$, $|\alpha - (x'/y')| < y'^{-\mu}$. Since Y was defined in terms of reduced fractions, we have $x/y \neq x'/y'$ and

$$\frac{1}{yy'} \leq \left|\frac{x}{y} - \frac{x'}{y'}\right| \leq \left|\alpha - \frac{x}{y}\right| + \left|\alpha - \frac{x'}{y'}\right| < y^{-\mu} + y'^{-\mu} < 2y^{-\mu},$$

so that $y' > \frac{1}{2}y^{\mu-1}$. Now if $1 < \lambda < \mu - 1$, say if $\lambda = \mu/2$ and if $y \geq B(\alpha, \mu)$ where $B(\alpha, \mu)$ was chosen large enough, then we have

$$y' > y^{\lambda},$$

i.e., an exponential Gap Principle. Thus S_2 is an exponential (C, λ)-set, and its cardinality may be explicitly bounded by $1 + (\log C)/(\log \lambda)$. But note that even though we can estimate the cardinality of a (C, λ)-set, we cannot estimate the size of its elements, and hence Thue's Theorem remains ineffective.

The situation is similar for Siegel's Theorem, where the condition $\mu > (r/2) + 1$ is relaxed to $\mu > 2\sqrt{r}$, and for the Dyson-Gelfond Theorem, with the condition $\mu > \sqrt{2r}$.

Roth finally relaxed the condition to $\mu > 2$. His Theorem says that for α as above and $\delta > 0$, there are only finitely many rationals x/y with $|\alpha - (x/y)| < |y|^{-2-\delta}$. Again we can form the set Y of denominators and distinguish small solutions $y < B(\alpha, \delta)$ and large solutions $y \geq B(\alpha, \delta)$. This time we cannot assert that the large solutions form an exponential C-set. But there are explicit $C = C(\alpha, \delta)$ and $m = m(\alpha, \delta)$ where m is an integer, such that the large solutions are the union of m exponential C-sets, hence (C, λ)-sets. This again gives a bound for the cardinality.

Explicit bounds for the number of solutions of $|\alpha - (x/y)| < y^{-2-\delta}$ with $y > 0$ were given by Davenport and Roth [3]. More recently, Bombieri and Van der Poorten [2] came up with better bounds by using a new theorem of Esnault and Viehweg [4] in place of "Roth's Lemma". They considered the slightly stronger inequality.

$$\left|\alpha - \frac{x}{y}\right| < \frac{1}{64y^{2+\delta}}$$

and showed that the number of solutions x/y in reduced form with $y > 0$ is

$$\leq \frac{\log\log 4H}{\log(1+\delta)} + 3000\frac{(\log r)^2 \log(50\delta^{-2}\log r)}{\delta^5}, \tag{1}$$

197 provided that $0 < \delta < \delta_0$ with some absolute δ_0. Here $H = H(\alpha)$ is the height of α (related to the naive height, which is the maximum modulus of the coefficients of the minimal defining polynomial of α over $\mathbb{Z}$).

The first summand in (1) comes essentially from the small solutions, the second summand from the large solutions. Although initially the small solutions appeared to be more tractable, it turns out that they are responsible for the dependency of H in (1). In fact, the first summand in (1) is best possible (See, e.g. [8].)

Now let us turn to simultaneous approximation. Some years ago, I proved the following [9] : Suppose $\alpha_1, \ldots, \alpha_n$ are algebraic, with $1, \alpha_1, \ldots, \alpha_n$ linearly independent over $\mathbb{Q}$. Then there are only finitely many rational points $(x_1/y, \ldots, x_n/y)$ with $y > 0$ and

$$|\alpha_i - (x_i/y)| < y^{-1-(1/n)-\delta} (i = 1, \ldots, n) \tag{2}$$

for given $\delta > 0$. Here, when $n > 1$, we cannot at present estimate the number of solutions.

Following the method in [11], we can try to find explicit B, C, m depending only on $\alpha_1, \ldots, \alpha_n, \delta$, such that the numbers $y > B$ occurring in solutions of (2) (the "large solutions") constitute not more than m exponential C-sets. The real difficulty is with the Gap Principle.

Write $\alpha = (\alpha_1, \ldots, \alpha_n)$, $x = (x_1/y, \ldots, x_n/y)$, and write (2) as

$$\overline{abc} < y^{-1-(1/n)-\delta}, \tag{3}$$

where $\overline{}$ denotes the maximum norm. Now let $x_0, \ldots, x_n$ be solutions of (3) with $y_0 \leq y_1 \leq \ldots \leq y_n$ where $y_i = y(x_i)(i = 0, \ldots, n)$. Writing $x_i = (x_{i1}/y_i, \ldots, x_{in}/y_i)$, and assuming the determinant is not zero, we have

$$\frac{1}{y_0 y_1 \ldots y_n} \leq \left\| \begin{matrix} 1 \frac{x_{01}}{y_0} \ldots \frac{x_{0n}}{y_0} \\ \ldots \\ 1 \frac{x_{n1}}{y_n} \ldots \frac{x_{nn}}{y_n} \end{matrix} \right\| = \left\| \begin{matrix} 1 \frac{x_{01}}{y_0} - \alpha_1 \ldots \frac{x_{0n}}{y_0} - \alpha_n \\ \ldots \\ 1 \frac{x_{n1}}{y_n} - \alpha_1 \ldots \frac{x_{nn}}{y_n} - \alpha_n \end{matrix} \right\|$$
$$\leq (n+1)!(y_0 y_1 \ldots y_{n-1})^{-1-(1/n)-\delta}.$$

Therefore

$$y_n > \frac{1}{(n+1)!}(y_0 y_1 \ldots y_{n-1})^{(1/n)+\delta} \geq \frac{1}{(n+1)!} y_0^{1+n\delta}$$

With $\lambda = 1 + (n\delta/2)$ and $y_0 \geq B$ large, this leads to $y_n > y_0^{\lambda}$. Although this 198
would give a Gap Principle involving only every n^{th} term in a sequence x_0, $x_1, \ldots$ of approximations, it would be very useful in estimating the number of such approximations.

Where is the catch ? The catch lies in the assumption that the above determinant is not zero. The determinant will be zero precisely when $x_0, x_1, \ldots, x_n$ (with are in $\mathbb{R}^n$) lie in an $(n-1)$-dimensional linear submanifold of $\mathbb{R}^n$. Henceforth we shall call such a submanifold a *hyperplane*. E.g., when $n = 2$, the determinant will be zero when x_0, x_1, x_2 lie on a line.

So we cannot really estimate the number of solutions x of (3). What we can do is the following (see [12]). We can give an explicit $t = t(\alpha, \delta)$, such that the solutions x of (3) lie in a collection of t hyperplanes.

The following argument shows why it is unlikely that we will soon be able to estimate the number of solution x of (3) when $n > 1$. Given $Q > 1$, the inequalities

$$|u| \leq Q, |\alpha_i u - v_i| \leq Q^{-1/(n-1)} \qquad (i = 1, \ldots, n-1)$$

define a parallelepiped Π of volume 2^n in the space of vectors $u = (v, u_1, \ldots u_{n-1})$. By Minkowski, there is a nonzero integer point u in Π. In fact, for given $\epsilon > 0$ and for large Q, the n^{th} minimum $\mu_n = \mu_n(Q)$ of Π has $\mu_n < Q^\epsilon$ (See [9]. But we don't know how large Q has to be). For such Q there are n independent integer points $\mathbf{u}_1, \ldots, \mathbf{u}_n$ in the blown up parallelepiped $Q^\epsilon\Pi$. The linear combinations $c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$ with $|c_1| + \cdots + |c_n| \leq Q^\epsilon$ lie in $Q^{2\epsilon}\Pi$. Thus for large Q, there are $\geq c(n)Q^{n\epsilon} > Q^\epsilon$ nonproportional integer points in $Q^{2\epsilon}\Pi$.

Suppose now that

$$0 < \delta < \frac{1}{n(n-1)}$$

and let ϵ have $\delta + 7\epsilon < 1/(n(n+1))$. Suppose we have a rational hyperplane $\mathfrak{H}$ which comes very close to $\alpha = (\alpha_1, \ldots, \alpha_n)$. Say $\mathfrak{H}$ is given by

$$a_0 + a_1 X_1 + \cdots + \cdots + a_n X_n = 0,$$

with coprime integers $a_0, a_1, \ldots, a_n$. Suppose $|a_n| = \max(|a_1|, \ldots, |a_n|) = a$, say and $\mathfrak{H}$ is so close to α that

$$|a_0 + a_1\alpha_1 + \cdots + a_n\alpha_n| < a^{-2/\epsilon}. \tag{4}$$

Set $Q = a^{1/\epsilon}$, and let $\mathbf{u}$ be in $Q^{2\epsilon}\Pi(Q)$, so that

$$|u| \leq Q^{1+2\epsilon}, |\alpha_i u - v_i| \leq Q^{-(1/(n-1))+2\epsilon} \quad (i = 1, \ldots, n-1). \tag{5}$$

199 Then with $y = a_n u$, $x_i = a_n v_i (i = 1, \ldots, n-1)$ we have

$$|\alpha_i y - x_i| \leq aQ^{-(1/n-1)+2\epsilon} = Q^{-(1/(n-1))+3\epsilon} \quad (i = 1, \ldots, n-1).$$

On the other hand, if we set $x_n = -(a_0 u + a_1 v_1 + \cdots + a_{n-1} v_{n-1})$, then (4) yields

$$|\alpha_n y - x_n| < naQ^{-(1/(n-1))+2\epsilon} + |u|a^{-2/\epsilon} \leq nQ^{-(1/(n-1))+3\epsilon} + Q^{1+2\epsilon}Q^{-2} < Q^{-(1/(n-1))+4\epsilon}$$

if a, and hence Q, is large. When $\mathbf{u} \neq 0$, then $u \neq 0$ by (5), so let us say that $u > 0$. Then $y = au \leq Q^{1+3\epsilon}$; and since

$$\left(\frac{1}{n-1} - 4\epsilon\right)\Big/(1 + 3\epsilon) \geq \frac{1}{n-1} - 7\epsilon > \frac{1}{n} + \delta,$$

we have

$$\left|\alpha_i - \frac{x_i}{y}\right| < y^{-1-(1/n)-\delta}(i = 1, \ldots, n).$$

This holds for $\mathbf{u} \neq 0$ in $Q^{2\epsilon}\Pi(Q)$. By what we said above, there will in general be $\geq Q^{\epsilon}$ non-proportional such u, and hence there will be $\geq Q^{\epsilon}$ solutions to (2) or (3).

This will happen if there is a *single* hyperplane $\mathfrak{H}$ with (4) and with large a. As is well known, the linear form inequality (4) is dual to simultaneous approximations. Thus in order to bound the number of solutions of (2), we would have to give a bound for the *size a* of solutions of the dual inequality (4). Thus we would have to have an "effective" result on the linear forms inequality (4). But such an effective result is unknown even in the case $n = 1$, since the Thue-Siegel-Roth Theorem is ineffetive.

Now let us turn to Thue equations and Norm Form equations. A *Thue* equation is an equation

$$F(x, y) = m, \tag{6}$$

where F is a binary form of degree $r \geq 3$ with rational integer coefficients which is irreducible over the rationals. Over $\mathbb{C}$ it factors as $F = \alpha(x - \alpha_1 y) \ldots (x - \alpha_r y)$ where $a \in \mathbb{Z}$, and $\{\alpha_1, \ldots, \alpha_r\}$ is a set of conjugate algebraic numbers. A *Norm Form* equation is an equation

$$F(x_1, \ldots, x_n) = m \tag{7}$$

where F is a norm form, i.e.

$$F = a\prod_{i=1}^{r}(\alpha_1^{(i)}x_1 + \cdots + \alpha_n^{(i)}x_n),$$

200 where $\alpha_1, \ldots, \alpha_n$ lie in an algebraic number field K of degree r, and where $\alpha \mapsto \alpha^{(1)}, \ldots, \alpha \mapsto \alpha^{(r)}$ are the embeddings of K into $\mathbb{C}$.

Using his result on approximation to algebraic numbers (applied to $\alpha_1, \ldots, \alpha_r$), Thus showed that (6) has only finitely many solutions. Bounds involving m, r and the size of the coefficients of F were given by Lewis and Mahler [6]. Siegel [14] had conjectured that there were bounds depending only on m and r. The first such bounds were derived by Evertse [5]. Later, Bombieri and Schmidt [1] established the bound

$$cr^{1+\omega}$$

for the number of coprime solutions x, y, where c is an absolute constant and $\omega = \omega(m)$ the number of distinct prime factors of m. Notice the contrast with diophantine approximation, where the dependency of (1) on the height H cannot be eliminated! Siegel also had conjectured that there should be a bound which depends only on the number of nonzero coefficients. It turns out that there is no bound independent of m. However, Mueller and Schmidt [7] proved Siegel's conjecture in the modified form that there is a bound which depends only on m and the number of nonzero coefficients of F (but which is independent of the degree r).

Some years age ([10]), I proved that when F is "non-degenerate", then the equation (7) has only finitely many solutions. I now can give an explicit bound for the number of solutions which depends only on n, r, m, but which is independent of the coefficients of F. In fact, the proof by induction on the number n of variables would break down if at some stage we had dependency on the coefficients. It will be convenient to formulate the result in terms of *primitive* solutions with g.c.d. $(x_1, \ldots, x_n) = 1$. I can prove (see [13]) that the number of primitive solutions of a non-degenerate Norm Form equation with coefficients in $\mathbb{Z}$ is

$$\leq C_1 C_2$$

where

$$C_1 = \min(r^{2^{30n} r^2}, r^E) \quad \text{with} \quad E = (2n)^{n^{2^{n+4}}},$$
$$C_2 = \binom{r}{n-1}^{\omega} d_{n-1}(m^r),$$

with $d_{n-1}(x)$ denoting the number of factorizations $x = x_1 \dots x_{n-1}$ with positive factors.

In the proof, I write the equation as $aL_1 \dots L_r = m$ where $L_i = \alpha_1^{(i)} x_1 + \dots + \alpha_n^{(i)} x_n (i = 1, \dots, r)$, and I deduce the existence of $i_1, \dots, i_n$
such that $L_{i_1}, \dots, L_{i_n}$ are independent and 201

$$|L_{i_1}(\mathbf{x}) \dots L_{i_n}(\mathbf{x})| < |\mathbf{x}|^{-\delta} |\det(L_{i_1}, \dots, L_{i_n})| \tag{8}$$

with suitable $\delta > 0$, where $\mathbf{x} = (x_1, \dots, x_n)$, and $|\mathbf{x}|$ is its norm. Inequalities (8) are derived in various ways, depending on whether $\mathbf{x}$ is "small" or "large". Then (8) is dealt with by a semieffective version of the Subspace Theorem [11].

It is to be hoped that these results will lead to bounds for the number of solutions of S-unit equations, and the multiplicities of linear recursive sequences.

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LINEAR OPERATORS AND AUTOMORPHIC FORMS

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1 We consider a bounded symmetric complex domain B in the sense 203
of Elie Cartan[1] and denote the group of analytic mappings of B onto itself by G, points in B by z and the elements of G by $g : z \to gz$. By a multiplier (or automorphy factor) $\rho_g(z)$, we understand a function defined on $G \times B$ which is analytic (holomorphic) in z and differentiable in g such that

$$\rho_{g_1 g_2}(z) = \rho_{g_1}(g_2 z)\rho_{g_2}(z). \tag{1.1}$$

Any such multiplier defines a kernel function $k_\rho(z, \overline{z})$ which transforms in the way

$$k_\rho(gz, \overline{gz}) = \rho_g(z)\overline{\rho_g(z)}k_p(z, \overline{z}). \tag{1.2}$$

We need only to write, for some fixed z_0 in B,

$$k_\rho(z, \overline{z}) = |\rho_g(z_0)|^2$$

where g is a solution of $z = gz_0$ and it is clear that this does not depend on the particular g chosen, but only on the point z.

From (1.2), we get that

$$ds^2 = \sum_{i,j} \frac{\partial^2 \log k_p(z, \overline{z})}{\partial z_i \partial \overline{z}_j} \cdot dz_i d\overline{z}_j$$

is an invariant metric on B under the action of the group G. Thus, if B is irreducible, we get that this metric can differ only by a constant

[1]See for instance Siegel [5], Chapter XI.

factor from the Bergmann metric. If B is reducible, it must be a linear combination of the Bergmann metrics of the irreducible factors of B. For irreducible B, one easily derives that, up to a factor of the form $cf(z)\overline{f(z)}$ where $f(z)$ is analytic, $k_\rho(z,\overline{z})$ coincides with a real power of the Bergmann kernel function and $\rho_g(z)$ is, apart from a 'trivial' multiplier of the form $f(gz)/f(z)$, equal to a power of the jacobian $j_g(z)$ of the mapping g. Similarly, if B is reducible, $\rho_g(z)$ is, apart from a trivial factor $f(gz)/f(z)$, equal to a product of powers of the jacobians of the mapping g with respect to the various irreducible factors of B.

204 We may mention that essentially the same conclusion could be drawn from the weaker premise that instead of (1.1), $\rho_g(z)$ satisfies the relation

$$|\rho_{g_1g_2}(z)| = |\rho_{g_1}(g_2z)||\rho_{g_2}(z)|, \tag{1.3}$$

then, apart from a factor of the form $\epsilon_g f(gz)/f(z)$ where f is analytic and $|\epsilon_g| = 1$, $\rho_g(z)$ is equal to a product of powers of the jacobians of the mapping $z \to gz$ with respect to the irreducible factors of B.

We shall study linear operators on functions defined on B, which have the property of transforming with a multiplier on each side under the mappings of the group G. Call the operator $L = L_z$ and define L_{gz} through the relation

$$L_{gz}F(z) = [L_zF(g^{-1}z)]_{z\to gz};$$

then L should transform according to the rule

$$L_{gz} = \rho_g(z)L_2\sigma_g^{-1}(z), \tag{1.4}$$

where ρ and σ are two multipliers.[2]

We ask the question : for which B and which choices of multipliers ρ and σ do such operators exist? And when they exist, determine their form as explicitly as possible.

It is easily seen that we may restrict ourselves to the case that B is irreducible and then derive the results for the general case from those obtained for the irreducible factors of B in case B is irreducible.

[2]From now on, we disregard trivial multipliers and consider only products of powers of the jacobians for the irreducible factors of B. This is only an apparent restriction.

As is known[3] there are six types of irreducible bounded symmetric domains. If we denote a matrix with m rows and n columns by $Z^{(m,n)}$ and the (n, n) unit matrix E or $E^{(n)}$, there are the four main types :

(I) $Z = Z^{(m,n)},\ E - \overline{Z}'Z > 0,$

(II) $Z = Z^{(n,n)},\ Z' = -Z,\ E - \overline{Z}'Z > 0.$

(III) $Z = Z^{(n,n)},\ Z' = Z,\ E - \overline{Z}'Z > 0,$ and

(IV) $Z = Z^{(n,1)},\ \overline{Z}'Z < \frac{1}{2}(1 + |Z'Z|^2) < 1.$

Here Z' denotes the transposed matrix. In addition, there are the types V and VI — the two exceptional bounded symmetric domains of complex dimension 16 and 27 respectively; we shall not give a definition here.

It should also be noted that for $n = 2$, the domain IV is reducible **205**
and that there is also some overlapping between the four types for low dimension; the unit circle $|z| < 1$ in one complex variable is, for instance, a special case of all the four types.

2 The question of linear operators that transform according to the rule (1.4) can be split in two :

(*a*) Operators that conserve the multiplier, i.e. when (1.4) holds, but with $\rho_g(z) \equiv \sigma_g(z)$.

We shall refer to such operators as invariant (though, strictly speaking, they are so only if $\rho_g(z) \equiv 1$ identically).

It is well-known that invariant operators exist for all the bounded symmetric domains B and for all multipliers $\rho_g(z)$; for given ρ the differential operators form a finitely generated ring, where the number of independent generators equals the rank of the group G (or of the symmetric space). In this ring, all elements, except the constant, contain

[3] See Siegel [5], Chapter XI for instance.

differentiations both with respect to z and $\overline{z}$. The form of integral operators is easily given explicitly for B irreducible; if $\rho_g(z) = (j_g(z))^{-r}$ where $j_g(z)$ denotes the jacobian of the mapping g, then

$$Lf = \int_B x(z,\zeta)\left(\frac{k(z,\overline{\eta})}{k(\zeta,\overline{\zeta})}\right)^r f(\zeta)d\omega_\zeta \tag{2.1}$$

where $d\omega_\zeta$ is the invariant volume element, $k(z,\overline{\zeta})$ is the Bergmann kernel function and $x(z,\zeta)$ is a "point pair invariant" satisfying $x(gz, g\zeta) = x(z,\zeta)$ for all z and ζ in B and g in G.

In particular, for analytic functions $f(z)$, we have the reproducing operator

$$f(z) = c_r \int_B \left(\frac{k(z,\overline{\zeta})}{k(\zeta,\overline{\zeta})}\right)^r f(\zeta)d\omega_\zeta \tag{2.2}$$

where c_r is a certain polynomial in r. (2.2) is valid for a certain hilbert-space of analytic functions if $r > r_0$, the largest zero of the polynomial c_r.[4]

(b) Operators that change the multiplier, i.e. which transform in the way (1.4) but with $\rho_g(z) \not\equiv \sigma_g(z)$.

The question (b) is more complex than (a), but it is not difficult to establish that linear operators that change the multiplier do not exist for all the irreducible domains, but only for a certain subclass.

To see this, we may look at the compact subgroup of G which leaves
206 some point z_0 in B fixed, the so-called stability group or isotropy group of z_0. It is simplest to choose the point O where all the coordinates are zero and the compact subgroup K_0 which keeps O fixed. For all the six types of bounded symmetric domains, the way they are usually defined, the elements of K_0 are linear transformations, and K_0 is essentially (sometimes, a slight change of variables being necessary as in type III where we would put a factor $1/\sqrt{2}$ in the elements of the symmetric matrix which are off the main diagonal) a subgroup of the unitary group

[4]See Selberg [4]

$U(N)$ where N is the complex dimension of B and $j_k(z) = j_k(0)$, where $k \in K_0$, is a one-dimensional representation of K_0.

It is clear that if there exists a linear operator satisfying (1.4), then, in particular, (1.4) must hold for g restricted to K_0 and we consider the functional $\mathscr{L}$ that L_z represents at $z = 0$.

On the other hand, it is not hard to show that if we have a linear functional $\mathscr{L}$ which has the required property (1.4) for g in K_0, then it can be extended to a linear operator L_z by means of the relation (1.4) with $z = gO$, but the general form of this operator seems awkward to obtain in this way, particularly if it is a differential operator.

It is easily seen that an integral operator

$$L_z f = \int_B h(z,\zeta) f(\zeta) d\omega_\zeta$$

where $h(z,\zeta)$ is a short form for $h(z,\bar{z},\zeta,\bar{\zeta})$, must in order to satisfy (1.4), have a kernel $h(z,\zeta)$ which satisfies

$$h(gz, g\zeta) = \rho_g(z)\sigma_g^{-1}(\zeta)h(z,\zeta) \tag{2.3}$$

and, in particular, for $g = k \in K_0$, if we put $\zeta = 0$, we get

$$h(kz, 0) = \rho_k(z)\sigma_k^{-1}(0)h(z,0)$$

or since $\rho_k(z) = \rho_k(0)$,

$$h(kz, 0) = \rho_k(0)\sigma_k^{-1}(0)h(z,0). \tag{2.4}$$

Since we may assume that $h(z,\zeta)$ is analytic in z and z[5], it is clear that the expansion of $h(z,0)$ in terms of powers of z and $\bar{z}$ for z near 0, must start with a homogeneous polynomial $\rho(z,\bar{z})$ which also transforms by the factor $\rho_k(0)\sigma_k^{-1}(0)$ when we replace z by kz.

Similarly, if D_z is a differential operator which obeys the transfor- **207**

[5]If our original $h(z,\zeta)$ is not so, we may form the convolution of L_z with a suitable operator of the form (2.1) on the left which preserves the multiplier $\rho_g(z)$.

mation rule (1.4), at $z = 0$ it takes the form of a polynomial in $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \overline{z}}$:

$$D_0 = P\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}\right).$$

When z and so dz undergoes a unitary transformation from K_0, $\dfrac{\partial}{\partial z}$ undergoes the contragredient transformation; so we are again led to a polynomial (which we may assume to be homogeneous, otherwise taking the homogeneous part of lowest degree that is not identically zero) which transforms in the way (2.4) when the variables undergo the contragredient transformation to k (Actually, if we interchange $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \overline{z}}$, the vector $\left(\dfrac{\partial}{\partial \overline{z}}, \dfrac{\partial}{\partial z}\right)$ undergoes the same transformation as $(z, \overline{z})$).

It is now easy to see for the various types of B whether such polynomials exist when $\rho_g(z) \not\equiv \sigma_g(z)$.

We find that for type I, they exist only if $m = n$ and are then of the form

$$|z|^r P(z, \overline{z}) \quad \text{or} \quad |\overline{z}|^r P(z, \overline{z})$$

where $|z|$ is the determinant of z, r some positive integer and $P(z, \overline{z})$ some homogeneous polynomial which is invariant under K_0.

For type II, they exist only if n is even and are then of the form

$$P_f^r(z)P(z, \overline{z} \quad \text{or} \quad P_f^r(\overline{z})P(z, \overline{z})$$

where $P_f(z)$ is the polynomial called the Pfaffian of z (actually $|z|^{1/2}$, since the determinant is a square in this case), r again is a positive integer and P, a homogeneous polynomial which is invariant under K_0.

For type III, they exist for all n and are of the form

$$|z|^r P(z, \overline{z}) \quad \text{or} \quad |\overline{z}|^r P(z, \overline{z})$$

where r is a positive integer and $P(z, \overline{z})$ homogeneous and invariant under K_0.

For type IV, they again exist and are of the form

$$(z'\, z)^r P(z, \overline{z}) \quad \text{or} \quad (\overline{z}'\, \overline{z})^r P(z, \overline{z})$$

with r a positive integer and $P(z, \overline{z})$ *again homogeneous and invariant under* K_0.

For the types V and VI which (for good reason!) we have not exhibited explicitly, we find they do not exist for type V but, *for type* VI, *they exist and are given by the form* 208

$$p_3(z)^r P(z, \overline{z}) \quad \textit{or} \quad p_3(\overline{z})^r P(z, \overline{z})$$

where r and P are as before and p_3 *is a certain cubic polynomial in* 27 *variables.*

3 In order to derive more explicitly the form of the linear operators that transform according to (1.4) in the cases where we have seen they can exist, we note that the cases we have listed in the previous section are precisely the cases when the bounded domain B, by a suitable analytic mapping, becomes a so-called "positive half-space"[6], and when the group G by this mapping, becomes a real group (by which we mean that in this new unbounded version of our domain, we have $\overline{gz} = g\overline{z}$).

By a positive half-space, we understand a domain of $z = x + iy$, where the column vector x is unrestricted, while the vector y is required to lie in a homogeneous positivity-domain Y in the sense of Koecher[7]. As before, we shall use N to denote the complex dimension.

We recall some of the properties of a homogeneous positivity domain Y. It is a cone such that, for any two vectors $y^{(1)}$ and $y^{(2)}$ in Y, we have always

$$y^{(1)} y^{(2)} > 0^8 \tag{3.1}$$

[6] M. Koecher [2] writes "half-space"; I prefer "positive half-space" since it indicates the connection with a positivity-domain.

[7] M. Koecher [1]

[8] Koecher's definition is more general; he has (3.1) in the form $y^{(1)} S\, y^{(2)} > 0$, where S is a nonsingular symmetric real matrix, but (3.1) covers the cases we consider.

and so that if, for some vector $y^{(1)}$, (3.1) holds for all $y^{(2)}$ in Y, then $y^{(1)}$ also lies in Y.

There also exists a group G_Y of real matrices A such that $y \to Ay$ maps Y onto itself; this group is transitive on Y. In particular, for any scalar $\lambda > 0$, we have $\lambda y \in Y$ for $y \in Y$, so that Y is a cone. It is seen from (3.1) that if A is in G_Y, then $y \to A'^{-1}y$ also maps Y onto itself; so, we may, without restriction, assume that with A, always A'^{-1} also lies in G_Y.

There exists a homogeneous polynomial $Q(y)$, which we choose to be of minimal degree $q > 0$ such that $Q(y)$ is positive in Y and

$$Q(Ay) = |A|^{q/N} Q(y)^9 \tag{3.2}$$

209 If we define for $i = 1, \ldots, N$,[9]

$$y_i^* = \frac{\partial \log Q(y)}{\partial y_i}, \tag{3.3}$$

then $y \to y^*$ is an involution which carries Y into itself. We have[9]

$$Q(y^*)Q(y) = \text{constant}, \tag{3.4}$$

and, by a suitable choice of Q (which by (3.2) is only determined up to a constant factor) we get

$$Q(y^*)Q(y) = 1 \tag{3.4'}$$

Also, (3.2) gives $y^{*'} y = q$ and $(A\,y)^* = A'^{-1} y^*$.

On Y, we have an invariant volume element

$$dV_y = (Q(y)^{-N/q} dy \tag{3.5}$$

where we have written dy for the euclidean volume element. We also have an invariant metric

$$ds^2 = -\sum_{1 \le i,j \le N} \frac{\partial^2}{\partial y_i \partial y_j} \log Q(y) dy_i dy_j. \tag{3.6}$$

[9]Our $Q(y) = (N(y))^{q/N}$, where $N(.)$ is Koecher's "Norm-function".

The involution $y \to y^*$ (which is actually a symmetry) has a fixed point e and we have $Q(e) = 1$.

Now consider the positive half-space of $z = x + iy$ where x is unrestricted and y is in Y and the group generated by translations of the form $z \to z + a$ where a is a real vector, $z \to Az$ for A in G_Y and $z \to z^*$ where

$$z_i^* = -\frac{\partial}{\partial z_i} \log Q(z), \quad \text{for} \quad i = 1, \ldots, N.$$

We call this group G.

If we write

$$D_z = Q\left(\frac{\partial}{\partial z}\right), \tag{3.7}$$

we shall show that for g in G,

$$D_{gz}^r = (j_g(z))^{-(1/2)(rq/N+1)} D_z^r (j_g(z))^{-(1/2)(rq/N-1)} \tag{3.8}$$

where r is any positive integer. If $gz = z + a$, (3.8) is obvious and also for $gz = Az$ with A in G_Y so we really need to prove (3.8) only for $gz = z^*$.

To do this, we first look at the Y space. Actually Y is a symmetric space; for any two points $y^{(1)}$ and $y^{(2)}$ in Y, there exists an A in G_Y such that $Ay^{(1)} = y^{(2)*}$, $Ay^{(2)} = y^{(1)*}$. Thus G_Y and the $*$ operation satisfy the **210**
conditions for G and μ in Selberg [3].[10]

Also, we see that if r is a positive integer, then

$$L_y = Q^r(y) Q^r\left(\frac{\partial}{\partial y}\right) \tag{3.9}$$

is an operator invariant under the group G_Y.

It follows from a general result[11] that under the $*$ operation the operator L given by (3.9) goes into the formal adjoint L^* with respect to the invariant measure dV_y or otherwise expressed

$$L_{y^*} = L_y^*$$

[10] See Selberg [10], p. 51.

[11] See Selberg [3], top of p. 53. In the context given there, the proof is obvious.

Thus for two suitable functions f and g, we have

$$\int_Y f(y) L_y g(y)(Q(y))^{-N/q} dy = \int_Y g(y) L_y^* f(y)(Q(y))^{-N/q} dy.$$

Inserting the expression for L, we see that it is easy to find the formal adjoint L^*, since the formal adjoint of $Q^r\left(\frac{\partial}{\partial y}\right)$ with respect to the euclidean measure is $Q^r\left(-\frac{\partial}{\partial y}\right) = (-1)^{rq} Q^r\left(\frac{\partial}{\partial y}\right)$. We get

$$\begin{aligned}
\int_Y f(y) L g(y)(Q(y))^{-N/q} dy &= \int_Y f(y)(Q(y)^{r-N/q} Q^r\left(\frac{\partial}{\partial y}\right) g(y) dy \\
&= \int_Y g(y) Q^r\left(-\frac{\partial}{\partial y}\right)(Q(y))^{r-N/q} f(y) dy \\
&= \int_Y g(y)\left(Q(y)^{N/q} Q^r\left(-\frac{\partial}{\partial y}\right) Q(y)^{r-N/q} f(y)\right) \times \\
&\quad \times \frac{dy}{(Q(y))^{N/q}}.
\end{aligned}$$

Thus

$$L^* = Q^{N/q}(y) Q^r\left(-\frac{\partial}{\partial y}\right) Q^{r-N/q}(y).$$

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$$\begin{aligned}
L^* &= Q^r(y^*) Q^r\left(\frac{\partial}{\partial y^*}\right) \\
&= Q^{-r}(y) Q^r\left(\frac{\partial}{\partial y^*}\right).
\end{aligned}$$

Comparing these two expressions for L^*, we get

$$Q^r\left(\frac{\partial}{\partial y^*}\right) = Q^{r+N.q}(y) Q^r\left(-\frac{\partial}{\partial y}\right) Q^{r-N/q}(y) \qquad (3.10)$$

$$= (-1)^{rq} Q^{r+N/q}(y) Q^r\left(\frac{\partial}{\partial y}\right) Q^{r-N/q}(y).$$

But, from (3.10), it follows immediately that

$$Q^r\left(\frac{\partial}{\partial z^*}\right) = Q^{r+N/q}(z) Q^r\left(\frac{\partial}{\partial z}\right) Q^{r-N/q}(z). \tag{3.11}$$

It remains to determine the jacobian of the mapping $z \to z^*$ or $j_*(z)$. We have

$$\frac{\partial z_i^*}{\partial z_j} = -\frac{\partial^2}{\partial z_i \partial z_j} \log Q(z)$$

so that

$$j_*(z) = \left|-\frac{\partial^2}{\partial z_i\, \partial z_j} \log Q(z)\right|.$$

If, as before, dy denotes the euclidean volume element, we have, for the invariant volume element in Y,

$$(Q(y^*))^{-N/q} dy^* = (Q(y))^{-N/q} dy$$

or using (3.4′),

$$dy^* = (Q(y))^{-2N/q} dy.$$

Since the symmetry $y \to y^*$ preserves orientation or not according as N is even or odd, we get

$$\left|\frac{\partial y_i^*}{\partial y_j}\right| = (-1)^N (Q(y))^{-2N/q},$$

$$\text{or} \quad \left|\frac{\partial^2 \log Q(y)}{\partial y_i\, \partial y_j}\right| = (-1)^N (Q(y))^{-2N/q}.$$

It is therefore obvious that 212

$$j_*(z) = \left|-\frac{\partial^2 \log Q(z)}{\partial z_i\, \partial z_j}\right| = (Q(z))^{-2N/q}.$$

Combining this with (3.11), we get that (3.8) holds also for $gz = z^*$; thus (3.8) holds for all g in G.

It is however clear that (3.8), which is really an algebraic identity, holds in a much larger group than G. Let us define $\widetilde{G}_Y$ as the group of complex matrices whose entries satisfy the algebraic relations which define G_Y and consider the group $\widetilde{G}$ generated by translations $z \to z + a$ where a may now be a complex vector, $z \to Az$ for A in $\widetilde{G}_Y$ and $z \to z^*$. *Clearly* (3.8) *as an algebraic identity holds for any transformation g in* G.

The transformations of $\widetilde{G}$ do not, in general, map the positive half-space onto itself. $\widetilde{G}$ is actually large enough to map the positive half-space back into a bounded symmetric domain-in most cases, the original one (this being, for instance, true for the first three types listed at the end of §2) — or one may have to add a final unitary transformation which does not lie in K_0 (this being the case for type IV where the transformation $z_1 \to z_1$, $z_j \to iz_j$ for $1 < j \le n$ would be needed at the end; for type IV, the positivity domain can be defined as $y_1 > 0$, $y_1^2 - y_2^2 - \cdots - y_n^2 > 0$ and we have $Q(y) = \frac{1}{2}(y_1^2 - y_2^2 - \cdots - y_n^2)$; so the last transformation is needed to transform $z_1^2 - z_2^2 - \cdots - z_n^2$ into $z'z = z_1^2 + \cdots + z_n^2$). At any rate, we get, in each case, the form of the differential operator and its transformation formula for the original bounded domain.

In the case of type I with $m = n$, we get, writing $\left|\frac{\partial}{\partial Z}\right|$ for $\left|\frac{\partial}{\partial z_{ij}}\right|$, that *if*

$$gZ = (AZ + B)(CZ + D)^{-1}$$

where A, B, C, D are complex (n, n) *matrices such that* $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$, *then*

$$\left|\frac{\partial}{\partial gZ}\right|^r = |CZ + D|^{r+n} \left|\frac{\partial}{\partial Z}\right|^r |CZ + D|^{r-n}. \tag{3.12}$$

213 In the case of type II, for $Z = Z^{(2n,2n)}$ and $Z' = -Z$, *let g be the transformation*

$$gZ = (AZ + B)(CZ + D)^{-1}$$

where A, B, C and D are $(2n, 2n)$ *complex matrices with the property that for* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $J = \begin{pmatrix} O & E \\ E & O \end{pmatrix}$, *we have*

$$M'JM = J$$

and $|M| = 1$. *Then writing* $P_f\left(\frac{\partial}{\partial Z}\right)$ *for* $P_f\left(\frac{\partial}{\partial z_{ij}}\right)$ *where* P_f *is the Pfaffian, we have*

$$\left(P_f\left(\frac{\partial}{\partial gZ}\right)\right)^r = |CZ+D|^{(r+2n-1)/2}\left(P_f\left(\frac{\partial}{\partial Z}\right)\right)^r |CZ+D|^{(r-2n+1)/2}. \quad (3.13)$$

For type III, if we define

$$\left|\frac{\partial}{\partial Z}\right| = \left|\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial z_{ij}}\right|$$

where δ_{ij} *is the Kronecker symbol (*1 *on the main diagonal,* 0 *off it) and* $gZ = (AZ+B)(CZ+D)^{-1}$ *where, for* $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$, $I = \left(\begin{smallmatrix} 0 & -E \\ E & 0\end{smallmatrix}\right)$, *we have* $M'IM = I$ *and* $|M| = 1$, *then again*

$$\left|\frac{\partial}{\partial gZ}\right|^r = |CZ+D|^{r+(n+1)/2}\left|\frac{\partial}{\partial Z}\right|^r |CZ+D|^{r-(n+1)/2}. \quad (3.14)$$

In the case of type IV, we will confine ourselves to stating the form for the original bounded domain without defining the more general group $\widetilde{G}$ or giving the explicit forms of g or the jacobian $j_g(z)$.

If we define $D_z = \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2}$, *then*

$$D_{gz}^r = (j_g(z))^{-(r/n+1/2)} D_z^r (j_g(z))^{-(r/n-1/2)}. \quad (3.15)$$

For the type VI, we do not give explicit formulas.

Since these differential operators only contain differentiations with respect to z and not $\overline{z}$, we see that if, for real α, we put in a factor $(k(z,\overline{z}))^{-\alpha}$ on the left side and a factor $(k(z,\overline{z}))^{\alpha}$ on the right (k being again the Bergmann kernel function), we again get an operator which satisfies (1.4) but the two multipliers ρ_g and σ_g have each been multiplied by $(j_g(z))^{\alpha}$.

Besides the operators D so constructed, we may, of course, also **214**
consider their complex conjugates $\overline{D}$. These do not satisfy (1.4), since

we required our multipliers to be analytic in z. We note that $\overline{D}$ would transform in the way

$$\overline{D}_{gz} = \overline{(j_g(z))}^{-\alpha} \overline{D}_z \overline{(j_g(z))}^{\beta},$$

where α and β depend on D. If we now define

$$\widetilde{D}_z = (k(z,\overline{z}))^{-\alpha} \overline{D}_z (k(z,\overline{z}))^{\beta},$$

we see that

$$\widetilde{D}_{gz} = (j_g(z))^{\alpha} \widetilde{D}_z (j_g(z))^{-\beta}$$

and so this operator has the required behaviour. Here, since the differentiations in $\widetilde{D}_z$ are with respect to $\overline{z}$ and not z, we can clearly replace the pair (α, β) by any other pair of real numbers (α', β') as long as

$$\alpha' - \beta' = \alpha - \beta.$$

It can be shown that all differential operators which satisfy (1.4) can be obtained by combining the operators D or $\widetilde{D}$ with suitable invariant differential operators of the kind mentioned under (a) at the beginning of §2.

4 To find the general form of integral operators that transform in the required way, we may again look at the representation of the domain B as a positive half-space where the analytic mappings gz are real, which is to say : $g\overline{z} = \overline{gz}$. We have, of course, also that $j_g(\overline{z}) = \overline{j_g(z)}$.

Considering the Bergmann kernel function of this half-space, we get thus

$$\begin{aligned} k(gz, g\overline{\zeta}) &= k(gz, \overline{g\zeta}) \\ &= (j_g(z)\overline{j_g(\zeta)})^{-1} k(z, \overline{\zeta}) \\ &= (j_g(z) j_g(\overline{\zeta}))^{-1} k(z, \overline{\zeta}). \end{aligned}$$

If we now write ζ instead of $\overline{\zeta}$, this becomes

$$k(gz, g\zeta) = (j_g(z) j_g(\zeta))^{-1} k(z, \overline{\zeta}).$$

From this, we see that if we put

$$h_{a,b}(z,\zeta) = \frac{(k(z,\overline{\zeta}))^{(a+b)/2}(k(z,\zeta))^{(a-b)/2}}{(k(\zeta,\overline{\zeta}))^{(a+b)/2}}, \tag{4.1}$$

where $b > a$ and $b - a$ is such that $(k(z,\zeta))^{(a-b)/2}$ is single-valued for z 215
and ζ in the positive half-space[12], then (4.1) transforms in the way given by (2.3), with $\rho_g(z) = (j_g(z))^{-a}$, $\sigma_g(\zeta) = (j_g(\zeta))^{-b}$.

If $b < a$, we write

$$h_{a,b}(z,\zeta) = \frac{k^a(z,\overline{z})}{k^b(\zeta,\overline{z})}\overline{h_{b,a}(\zeta,z)}. \tag{4.1$'$}$$

The most general form of a kernel which transforms in the way given by (2.3) is of the form

$$h(z,\zeta) = x(z,\zeta)h_{a,b}(z,\zeta) \tag{4.2}$$

where x is an invariant of the point pair z and ζ while $h_{a,b}(z,\zeta)$ is given by (4.1) or (4.1′) according to the sign of $b - a$.

For the bounded domains, the form of the kernels is more complicated than for the positive half-spaces.

5 For the reducible bounded symmetric domains, these same questions can be answered by using our results for the irreducible factors.

It is possible to generalize the problem we considered and ask similar questions for, say, bilinear operators operating on two functions are more generally, q-linear operators acting on q functions; for instance, to be able to produce from two automorphic forms a new one which depends linearly on these two, but whose multiplier is not the product of the multipliers of these two forms. Again, one would begin by looking at the stability group of the point O in B. Thus, for instance, it is easy to show that such bilinear operators exist for type I of $Z^{(m,n)} = Z^{(2n,n)}$,

[12]This is true if $b - a$ is an integral multiple of $\frac{a}{N}$, since it is not hard to show that apart from a constant factor $k(z,\zeta)$ is equal to $(Q(\frac{z-\overline{\zeta}}{2i}))^{-2N/q}$.

whereas for $Z^{(n,1)}$ there are no such q-linear operators for $q < n$.[13] Whether such multi-linear operators are of much interest is doubtful.

I originally determined the explicit transformation formulas for the differential operators considered in §3, in the year 1960. My first aim was to construct operators that effected the shift in automorphy factors in the same way as the operators $y^{\alpha}\dfrac{d^k}{dz^k}y^{-\alpha}$ and $y^{k+1}\dfrac{d^k}{d\bar{z}^k}y^{k-1}$ do in the case of the upper half-plane.

Later, I used them to effect analytic continuation of Dirichlet se-
216 ries associated with the Fourier expansions of modular forms in positive half-spaces where the Fourier expansion contains singular terms, and also to get the analytic continuation for the Dirichlet series associated with two such modular forms in the case when singular terms are present.

I lectured off and on, on these matters, the first time in Hamburg in the summer of 1961, later at various conferences, at Copenhagen (1964), Jyväskylä (1970), Bar Ilan (1981) and other places abroad.

In the sixties, some of the applications were privately communicated to Hans Maass, Howard Resnikoff and Audrey Terras, all of whom (with my permission) utilised some of this material in their publications.

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SOME EXPONENTIAL DIOPHANTINE EQUATIONS (II)

By T. N. Shorey

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1

Ramanujan [19] observed in 1913 that

$$\begin{cases} 1^2+7=2^3, 3^2+7=2^4, 5^2+7=2^5 \\ 11^2+7=2^7, (181)^2+7=2^{15}. \end{cases} \tag{1}$$

We are looking for the solutions of

$$x^2+7=2^m \qquad \text{in integers } x>0, m>0. \tag{2}$$

Ramanujan [19] conjectured in 1913 that all the solutions of (2) are given by (1). Nagell [17] confirmed this conjecture in 1948. Equation (2) is known as Ramanujan-Nagell equation.

Ratet [20] observed in 1916 that

$$31=\frac{2^5-1}{2-1}=\frac{5^3-1}{5-1}. \tag{3}$$

Thus 31 has all the digits equal to one with respect to the base 2 as well as the base 5. The next year, Goormaghtigh [11] found an other integer satisfying a similar property :

$$8191=\frac{2^{13}-1}{2-1}=\frac{90^3-1}{90-1}. \tag{4}$$

The letter N denotes an integer greater than two and we write $\omega(N-1)$ for the number of distinct prime factors of $N-1$. Let us consider the following equation which gives (3) and (4) :

$$N=\frac{2^n-1}{2-1}=\frac{y^3-1}{y-1} \qquad \text{in integers } n>0, y>2. \tag{5}$$

By completing square on the right hand side of (5), we obtain

$$4N + 4 = (2y + 1)^2 + 7 = 2^{n+2}.$$

Now, we apply the above mentioned result of Nagell to derive that (5) has no solution other than the ones given by (3) and (4).

More general than equation (5) is

$$N = \frac{y_1^{n_1} - 1}{y_1 - 1} = \frac{y_2^{n_2} - 1}{y_2 - 1} \quad \text{in integers } y_1 > 1, y_2 > 1, n_1 > 2, n_2 > 2.$$

Thus N has all the digits equal to one with respect to the base y_1 as well as the base y_2. Let $S(N)$ denote the set of all integers y with $1 < y < N-1$ such that N has all the digits equal to one with respect to the base y. 218
Further, we put

$$s(N) = |S(N)|.$$

Thus

$$s(31) = s(8191) = 2.$$

A conjecture, due to Ratat and Goormaghtigh, states that

$$s(N) \leq 1, \quad N \neq 31 \quad \text{and} \quad N \neq 8191. \tag{6}$$

Goormaghtigh [11] checked this conjecture for $N < (10)^4$. For $y \in S(N)$, we have

$$N = \frac{y^n - 1}{y - 1}, \quad n \geq 3. \tag{7}$$

We put

$$n = l(N; y). \tag{8}$$

For an integer $v > 1$, we denote by $P(v)$ the greatest prime factor of v and $\omega(v)$ the number of distinct prime divisors of v. Further, we write $P(1) = 1$ and $\omega(1) = 0$. Then, the author [29] proved the following result.

Theorem 1. *Let*

$$N \neq 31, \quad N \neq 8191 \quad \text{and} \quad \omega(N - 1) \leq 5. \tag{9}$$

There is at most one $y \in S(N)$ such that $l(N; y)$ is odd.

If N is a prime number, then we see from (7) and (8) that $l(N; y)$ is an odd prime and hence, we derive

Corollary 1. *For a prime N satisfying* (9)*, we have*

$$s(N) \leq 1.$$

Thus, the conjecture (6) is valid for all primes N satisfying $\omega(N - 1) \leq 5$. By sieve methods, it is known that the number of primes $N \leq Z$ with $\omega(N - 1) \leq 5$ is at least constant times $Z(\log Z)^{-2}$. The equation

$$\frac{y_1^{n_1} - 1}{y_1 - 1} = \frac{y_2^{n_2} - 1}{y_2 - 1} \quad \text{in integers} \quad y_1 > 1, y_2 > 1, n_1 > 2, n_2 > 2 \qquad (10)$$

has been considered by several authors. The first results are due to Makowski and Schinzel [14]. Further, Davenport, Lewis and Schinzel [6] applied a theorem of Siegel on integer points on curves to show that equation (10) with fixed n_1 and n_2 has only finitely many solutions in integers $y_1 > 1$ and $y_2 > 1$. It follows from Baker's effective version [2] of Thue's theorem [36] that equation (10) implies that max (n_1, n_2) is bounded by an effectively computable number C depending only on y_1 and y_2, The author [26] applied a theorem of Baker [1] on
219 the approximations of certain algebraic numbers by rationals proved by hyper-geometric method that there are at most 17 pairs (n_1, n_2) satisfying (10). Balasubramanian and the author [4] applied the theory of linear forms in logarithms to show that the number C, as above, depends only on the greatest prime factor of $y_1 y_2$. See also [23] and [27]. Finally, it follows from a theorem of Schinzel and Tijdeman [22] that equation (10) implies that mas (n_1, y_2) is bounded by an effectively computable number depending only on y_1 and n_2. Hence, equation (10) has only finitely many solutions if any two out of the four variables y_1, y_2, n_1, n_2 are fixed.

A weaker conjecture than (6) states that

$$s(N) < C_1, \quad N = 3, 4, \ldots$$

where $C_1 > 0$ is an effectively computable absolute constant. See Loxton [13] where he derived from the theory of linear forms in logarithms that

$$s(N) = O_\epsilon((\log N)^{(1/2)+\epsilon}), \epsilon > 0. \tag{11}$$

The author [29] proved that

$$s(N) \leq \begin{cases} \max(2\omega(N-1)-3, 0) & \text{if} \quad \omega(N-1) \leq 4 \\ 2\omega(N-1)-4 & \text{if} \quad \omega(N-1) > 4 \end{cases} \tag{12}$$

in an elementary way. If $\omega(N-1)$ is small, we observe that (12) is more precise than (11). Finally, it is easy to observe that $s(N) < \omega(N-1)$ whenever N is prime.

In this paragraph, we suppose that N is a perfect power. Then, we can combine theorem 1 with [32, theorem 5(iv)] to derive that $s(N) \leq 1$ for every N exceeding certain effectively computable absolute constant C_2 and satisfying $\omega(N-1) \leq 5$. In fact, it has been conjectured that $s(N) = 0$ for $N > C_2$ and we refer to [27] for an account of results proved in this direction.

Suppose that $N-1$ is a perfect power, say a q-th perfect power. Then, we subtract one on both the sides of (7) to derive that

$$N - 1 = y\frac{y^{n-1}-1}{y-1}, \quad y \in S(N).$$

Consequently, we see that both y as well as $(y^{n-1}-1)/(y-1)$ are q-th perfect powers. Now, we apply [25, theorem 3] to obtain the following result.

Theorem 2. *There exists an effectively computable absolute constant C_3* **220**
such that $s(N) = 0$ whenever $N - 1$ is a perfect power and $N > C_3$.

In other words, the equation

$$z^q + 1 = \frac{y^n - 1}{y - 1} \quad \text{in integers } z > 1, q > 1, y > 1, n > 2.$$

has only finitely many solutions. Furthermore, this assertion is effective.

2 Let us put

$$u_0 = 1, u_1 = 9, u_m = 3u_{m-1} - 2u_{m-2} (m \geq 2).$$

It is easy to check that

$$u_m = 2^{m+3} - 7 (m \geq 0).$$

Ramanujan-Nagell equation (2) asks for squares in this binary recursive sequence. By Nagell, there are only five squares in this sequence. Several authors have worked on finding perfect powers in binary recursive sequences. For example, in the Fibonacci sequence

$$u_0 = 0, u_1 = 1, u_m = u_{m-1} + u_{m-2} (m \geq 2),$$

Cohn [5] and Wyler [37], independently, proved that

$$u_0 = 0, u_1 = 1, u_2 = 1, u_{12} = 144$$

are the only squares and London and Finkelstein [12] showed that

$$u_0 = 0, u_1 = 1, u_2 = 1, u_6 = 8$$

are the only cubes. It has been derived from the theory of linear forms in logarithms that there are only finitely many perfect powers in a non-degenerate binary recursive sequence. See Pethö [18] and Shorey and Stewart [30]; the latter paper and [31] contain also applications of this and related results to certain Diophantine equations.

Now, we turn to define a non-degenerate binary recursive sequence. Let $r, s \in \mathbb{Z}$ with $s \neq 0$ and $r^2 + 4s \neq 0$. Let $u_0, u_1 \in \mathbb{Z}$ and

$$u_m = ru_{m-1} + su_{m-2} (m \geq 2).$$

Let α and β be roots of $X^2 - rX - s$. Observe that $\alpha\beta \neq 0$ and $\alpha \neq \beta$. Further

$$u_m = a\alpha^m + b\beta^m \quad (m \geq 0) \tag{13}$$

where

$$a = \frac{u_0\beta - u_1}{\beta - \alpha}, \quad b = \frac{u_1 - u_0\alpha}{\beta - \alpha}. \tag{14}$$

The sequence $\{u_m\}_{m=0}^{\infty}$ is called non-degenerate if $ab \neq 0$ and α/β is not
221 a root of unity. Ramanujan's τ-function satisfies the following recursive relation :

$$u_0 = 0, u_1 = 1, u_m = \tau(p)u_{m-1} - p^{11}u_{m-2}(m \geq 2, p \ \text{ prime})$$

and

$$u_m = \tau(p^{m-1}) \quad (m \geq 1).$$

Let α_p and β_p be roots of $X^2 - \tau(p)X + p^{11}$. Then, by (13) and (14), we have

$$u_{m+1} = \tau(p^m) = \frac{\alpha_p^{m+1} - \beta_p^{m+1}}{\alpha_p - \beta_p} \tag{15}$$

This sequence is non-degenerate whenever $\tau(p) \neq 0$. Further, by Deligne,

$$|\alpha_p| = |\beta_p| = p^{11/2}.$$

It is well-known that Thue-Siegel-Roth-Schmidt method and Gel'-fond-Baker theory of linear forms in logarithms are powerful tools in studying recursive sequences. In particular, these methods can be applied to obtain some results on Ramanujan's τ-function. For example, an estimate of Baker [3] applied to (15) gives

$$|\tau(p^m)| \geq p^{(11m/2) - C_4 \log(m+1)} \quad \text{if} \quad \tau(p^m) \neq 0.$$

Here $C_4 > 0$ is an effectively computable absolute constant. The author [28] applied this estimate together with [33, Corollary 7.1] to obtain the following result.

Theorem 3. *Let p be a prime number such that $\tau(p) \neq 0$. Then*

$$\tau(p^m) = \tau(p^n) \quad (m \neq n)$$

implies that

$$\max(m, n, p) \leq C_5$$

where $C_5 > 0$ is an effectively computable absolute constant.

We refer to [16] and [28] for an account of applications of the theory of linear forms in logarithms to Ramanujan's τ-function.

3 This section is a continuation of §1 of [27]. Erdös [7] and Rigge [21], independently, proved that the product of two or more consecutive positive integers is never a square. Erdös and Selfridge [9], by developing on an elementary method of Erdös [8], confirmed an old conjecture by proving that the product of two or more consecutive positive integers is never a power. In this section, we consider the corresponding problem for consecutive members of an arithmetical progression.

First, we introduce some notation. Let $b > 0$, $d > 0$, $m > 0$, $y > 0$,
222 $k > 2$ and $l \geq 2$ be integers such that $P(b) \leq k$ and $(m, d) = 1$. Let d_1 be the maximal divisor of d such that all the prime divisors of d_1 are $\not\equiv 1 (\mathrm{mod}\ l)$ and we write $d_2 = d/d_1$. We consider the equation

$$m(m+d)\dots(m+(k-1)d) = by^l. \tag{16}$$

We shall follow this notation, without reference, in this section. As already stated, equation (16) with $d = b = 1$ is not possible. Also, due to Fermat and Euler, equation (16) with $b = 1$, $l = 2$ and $k = 4$ is not possible.

Erdös conjectured that equation (16) with $b = 1$ implies that k is bounded by an effectively computable absolute constant. Under certain restrictions, we wish to confirm this conjecture for equation (16). For this, it is natural to exclude the case that

$$P(m(m+d)\dots(m+(k-1)d)) \leq k. \tag{17}$$

Then, we refer to [34] to observe that (17) implies that either $d = 1$ or $m = 2$, $d = 7$, $k = 3$. Therefore, we always suppose, without reference, that

$$P(m(m+d)\dots(m+(k-1)d)) > k \quad \text{if } d = 1 \quad \text{and} \quad b > 1. \tag{18}$$

By a well-known theorem of Sylvester, the assumption (18) is certainly satisfied whenever $m > k$.

Marszalek [15] confirmed the conjecture of Erdös for a fixed d.

More precisely, he proved that equation (16) with $b = 1$ implies that

$$\begin{cases} k \leq \exp(C_6 d^{3/2}) & \text{if } \ l = 2, \\ k \leq \exp(C_7 d^{7/3}) & \text{if } \ l = 3, \\ k \leq C_8 d^{5/2} & \text{if } \ l = 4, \\ k \leq C_9 d & \text{if } \ l \geq 5, \end{cases} \tag{19}$$

where C_6, C_7, C_8 and C_9 are explicitly given absolute constants.

The author [27] proved that equation (16) with $l \geq 3$ implies that k is bounded by an effectively computable number depending only on the greatest prime factor of d. Also, the author [27] confirmed the conjecture of Erdös whenever $d_2 = 1$ and $l \geq 3$.

Suppose the equation (16) is satisfied. Then, Shorey and Tijdeman [35][1] proved that k is bounded by an effectively computable number depending only on l and $\omega(d)$. More precisely, they proved that

$$l^{\omega(d)} > C_{10} k / \log k \tag{20}$$

where C_{10} and the subsequent letters $C_{11}, \ldots, C_{28}$ are effectively com- 223
putable absolute constants. Further, in [35], they sharpened the above-mentioned result of the author by proving that

$$(d \geq) d_2 > C_{11} k^{l-2}. \tag{21}$$

We combine (20) and (21) to conclude that

$$d \geq k^{C_{12} (\log \log k)/(\log \log \log k)}, \quad k \geq C_{13}.$$

This improves considerably the estimates (19).

For $\epsilon > 0$, Shorey and Tijdeman [35] proved that equation (16) with $k \geq C'_{14} = C'_{14}(\epsilon)$ implies that

$$m \leq d^2 k^{1+\epsilon} \quad \text{if } \ l = 2$$

[1] We refer to [35] for more general and more precise versions of the results stated here from [35].

and

$$m + (k-1)d \le C_{15} k d_2^{l/(l-2)} \quad \text{if } \ l \ge 3. \tag{22}$$

We show that the estimate (22) is quite precise for sufficiently large l.

Theorem 4. *Suppose that equation* (16) *is satisfied. There exist effectively computable absolute constants* $C_{16} > 0$ *and* $C_{17} > 0$ *such that for* $k > C_{16}$*, we have*

$$m \ge d^{1-C_{17}\Delta_l} \tag{23}$$

where

$$\Delta_l = l^{-1}(\log l)^2(\log\log(l+1)). \tag{24}$$

We combine (23) and (21) to derive the following result.

Corollary 2. *There exists an effectively computable absolute constant* $C_{18} > 0$ *such that equation* (16) *with* $l \ge C_{18}$ *implies that* k *is bounded by an effectively computable number depending only on* m*.*

We may combine Corollary 2 and (20) to conclude that equation (16) implies that k is bounded by an effectively computable number depending only on m and $\omega(d)$. The proof of theorem 4 depends on the estimates of Baker [3] and the author [24, lemma 2] on linear forms in logarithms.

Proof of theorem 4. We denote by $C_{19}, \ldots, C_{28}$ effectively computable absolute positive constants. We may assume that $k \ge C_{19}$ with C_{19} sufficiently large. We may also suppose that $l \ge C_{19}$, otherwise (23) follows immediately. We suppose that

$$m < d^{1-\Delta_l} \tag{25}$$

and we shall arrive at a contradiction.

224 By (16), we have

$$m + id = a_i x_i^l \quad (0 \le i < k) \tag{26}$$

where a_i and x_i are positive integers satisfying

$$P(a_i) \le k, \quad \left(x_i, \prod_{p \le k} p\right) = 1.$$

We put

$$S = \{a_0, \ldots, a_{k-1}\}.$$

In view of the theorem of Erdös and Selfridge mentioned in the beginning of this section, we may assume that $b > 1$ whenever $d = 1$. Then, we derive from (18) and [34] that the left hand side of (16) is divisible by a prime $> k$. Now, it follows from (16) that

$$m + (k-1)d \geq (k+1)^l$$

which, by (25), implies that

$$m + d \geq k^{l-1}, \quad d \geq k^{l-1}/2. \tag{27}$$

We denote by S_1 the set of all a_i with $1 \leq i < k$ such that $x_i = 1$. Observe, by (26), that the elements of S_1 are distinct. Now, we apply an argument of Erdös (see [10], lemma 2.1) to derive from (27) that

$$|S_1| \leq k/2 + \pi(k).$$

We denote by T the set of all i with $1 \leq i < k$ such that $a_i \notin S_1$ and we write S_2 for the set of all $a_i \in S$ with $i \in T$. Then

$$|T| \geq k/4. \tag{28}$$

We put

$$b_i = a_i/i \quad (i \in T)$$

and let S' be the set of all b_i with $i \in T$. Let $i \in T$, $j \in T$ with $i > j$ and $b_i = b_j$. Then, by (26),

$$ja_i(x_i^l - x_j^l) = (j-i)m. \tag{29}$$

Now, we observe that the left hand side of (29) exceeds

$$(j-i)(a_i x_i^l)^{(l-1)/l} \geq (j-i)(m+d)^{(l-1)/l}. \tag{30}$$

Therefore, by (29) and (30),

$$d^{(l-1)/l} < (m+d)^{(l-1)/l} < m$$

which implies (23). Thus, we may suppose that the elements of S' are distinct.

For every prime $p \le k$, we choose an $f = f(p) \in T$ such that

$$\mathrm{ord}_p(a_f) \ge \max_{i \in T} \mathrm{ord}_p(a_i).$$

225 We write T_1 for the set obtained by deleting from T all $f(p)$ with $p \le k$. Then, by (28), we see that

$$|T_1| \ge k/8.$$

Further, it is easy to see from (26) and $(m, d) = 1$ that

$$\prod_{i \in T_1} a_i \le k^k.$$

Then, there exists a subset T_2 of T_1 such that

$$t_2 := |T_2| \ge k/16 \tag{31}$$

and

$$a_i \le k^{32}, \quad i \in T_2. \tag{32}$$

We re-arrange the elements b_i with $i \in T_2$ as

$$b_{i_1} < b_{i_2} \ldots < b_{i_{t_2}}.$$

For simplicity of notation, we write

$$B_\nu = b_{i_\nu},\ 1 \le \nu \le t_2.$$

Then, we see from (32) that

$$\sum_{\nu=1}^{t_2 - 1} \log\left(\frac{B_{\nu+1}}{B_\nu}\right) \le 33 \log k.$$

Now, by (31), we see that there exists μ with $1 \le \mu \le t_2$ such that

$$\log\left(\frac{B_{\mu+1}}{B_\mu}\right) \le C_{20}(\log k)/k \le k^{-1/2}. \tag{33}$$

By (26), we have

$$B_{\mu+1}X_{\mu+1}^{l} - B_{\mu}X_{\mu}^{l} = Fm$$

where

$$X_{\mu+1} = x_{i_{\mu+1}},\ X_{\mu} = x_{i_{\mu}},\ F = \left(\frac{i_{\mu} - i_{\mu+1}}{i_{\mu}i_{\mu+1}}\right).$$

Therefore, by (25),

$$\left|\log\left(\frac{B_{\mu+1}}{B_{\mu}}\right) + l\log\left(\frac{X_{\mu+1}}{X_{\mu}}\right)\right| \leq 2md^{-1} < 2d^{-\Delta_l}. \tag{34}$$

Thus, by (34), (33) and (21),

$$\left|\log\left(\frac{X_{\mu+1}}{X_{\mu}}\right)\right| < 2d^{-\Delta_l} + C_{20}(\log k)/k < k^{-1/2}.$$

We apply an estimate of Baker [3] on linear forms in logarithms to derive that the left hand side of (34) exceeds

$$\exp(-C_{21}l^{-1}\log l \quad \log d \log k \log\log k). \tag{35}$$

We combine (34) and (35) to obtain 226

$$\Delta_l \leq C_{22}l^{-1}\log l \log k \quad \log\log k$$

which, by (24), implies that

$$l < k^{C_{23}}. \tag{36}$$

Now, we are ready to apply lemma 2 of [24] in the same way as in the proof of lemma 6 of [24]. We apply this lemma with $n = 2$, $A_1 = k^{33}$, $A = (m + (k-1)d)^{1/t}$, $\tau_1 = C_{24}$, $\tau_2 = C_{25}$ and $u = C_{23}$ to the linear form (34). Then, we observe that the right hand side of inequality (5) of this lemma exceeds

$$d^{-C_{26}l^{-1}} > 2d^{-\Delta_l}$$

Therefore, the lemma implies that

$$\left(\frac{B_{\mu+1}}{B_{\mu}}\right)^{J_1}\left(\frac{X_{\mu+1}}{X_{\mu}}\right) = \eta^{p}$$

where $p \leq C_{27}$ is a prime number and $0 \leq J_1 < p$ is an integer. Then, since each of $X_\mu > 1$ and $X_{\mu+1} > 1$ has no prime factor $\leq k$, we see that X_μ and $X_{\mu+1}$ are p-th perfect powers and

$$m + (k-1)d > k^{lp}.$$

As in the proof of lemma 6 of [24], we carry out $[(\log\ \log k)^2] + 1$ inductive steps to conclude that X_μ and $X_{\mu+1}$ are lL-th perfect powers for some integer L satisfying

$$C_{28}^{(\log\ \log k)^2} \geq L \geq 2^{(\log\ \log k)^2}. \tag{37}$$

Then, we apply an estimate of Baker [3] on linear forms in logarithms, (36) and (37) to derive that the left hand side of (34) exceeds

$$\exp(-C_{28}(lL)^{-1} \log d (\log k)^2 \log\ \log k) > d^{-t^{-1}} > 2d^{-\Delta_l}$$

which contradicts (34). This completes the proof of theorem 4.

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THE DILOGARITHM FUNCTION IN GEOMETRY AND NUMBER THEORY[1]

By D. Zagier

231 The dilograrithm Function is the function defined by the power series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad \text{for } |z| < 1.$$

The definition and the name, of course, come from the analogy with the Taylor series of the ordinary logarithm around 1,

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for } |z| < 1,$$

which leads similarly to the definition of the *polylogarithm*

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \quad \text{for } |z| < 1,\ m = 1, 2, \ldots$$

The relation

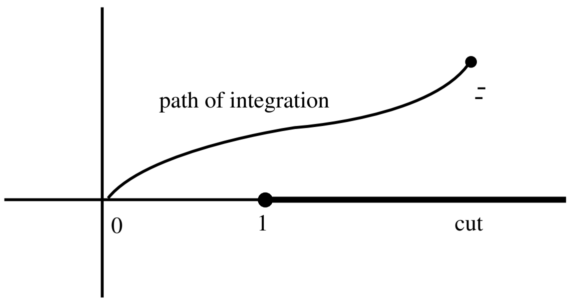
$$\frac{d}{dz}\,\mathrm{Li}_m(z) = \frac{1}{z}\,\mathrm{Li}_{m-1}(z) \quad (m \geq 2)$$

is obvious and leads by induction to the extension of the domain of definition of Li_m to the cut plane $\mathbb{C} - (1, \infty)$; in particular, the analytic

[1]This paper is a revised version of a lecture given in Bonn on the occasion of F. Hirzebruch's 60th birthday, (October 1987) and has also appeared under the title "The remarkable dilogarithm" in the Journal of Mathematical and Physical Sciences, 22(1988).

continuation of the dilogarithm is given by

$$\mathrm{Li}_1(z) = -\int_0^z \log(1-u)\frac{du}{u} \quad \text{for } z \in \mathbb{C} - (1, \infty).$$



Thus the dilogarithm is one of the simplest non-elementary functions one can imagine. It is also one of the strangest. It occurs not quite often enough, and in not quite an important enough way, to be included in the Valhalla of the great transcendental functions — the gamma function, Bessel and Legendre functions, hypergeometric series, or Riemann's zeta function. And yet it occurs too often, and in far too varied contexts, to be dismissed as a mere curiosity. First defined by Euler, it has been studied by some of the great mathematicians of the past — Abel, Lobachevsky, Kummer, and Ramanujan, to name just a few — and there is a whole book devoted to it [4]. Almost all of its appearances in mathematics, and almost all the formulas relating to it, have something of the fantastical in them, as if this function alone among all others possessed a sense of humor. In this paper we wish to discuss some of these appearances and some of these formulas, to give at least an idea of this remarkable and too little-known function. **232**

1 Special values Let us start with the question of special values. Most functions have either on exactly computable special values (Bessel functions, for instance) or else a countable, easily describable set of them; thus, for the gamma function

$$\Gamma(n) = (n-1)!,\ \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{2^n n!}\sqrt{\pi},$$

and for the Riemann zeta function

$$\zeta(2) = \frac{\pi^2}{6},\quad \zeta(4) = \frac{\pi^4}{90},\quad \zeta(6) = \frac{\pi^6}{945},\ldots,$$
$$\zeta(0) = -\frac{1}{2},\quad \zeta(-2) = 0,\quad \zeta(-4) = 0,\ldots,$$
$$\zeta(-1) = -\frac{1}{12},\quad \zeta(-3) = \frac{1}{120},\quad \zeta(-5) = -\frac{1}{252},\ldots.$$

Now so the dilogarithm. As far as anyone knows, there are exactly eight values of z for which z and $\mathrm{Li}_2(z)$ can both be given in closed form :

$$\mathrm{Li}_2(0) = 0$$
$$\mathrm{Li}_2(1) = \frac{\pi^2}{6},$$
$$\mathrm{Li}_2(-1) = -\frac{\pi^2}{12},$$
$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2),$$
$$\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right)\frac{\pi^2}{15} - \log^2\left(\frac{1+\sqrt{5}}{2}\right),$$
$$\mathrm{Li}_2\left(\frac{-1+\sqrt{5}}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{1+\sqrt{5}}{2}\right),$$
$$\mathrm{Li}_2\left(\frac{1-\sqrt{5}}{2}\right) = -\frac{\pi^2}{15} + \frac{1}{2}\log^2\left(\frac{1+\sqrt{5}}{2}\right),$$
$$\mathrm{Li}_2\left(\frac{-1-\sqrt{5}}{2}\right) = -\frac{\pi^2}{10} + \frac{1}{2}\log^2\left(\frac{1+\sqrt{5}}{2}\right).$$

Let me describe a recent experience where these special values figures, and which admirably illustrates what I said about the bizarreness of the occurrences of the dilogarithm in mathematics. From Bruce Berndt via Henri Cohen I learned of a still unproved assertion in the Notebooks of Srinivasa Ramanujan (Vol. 2, p. 289, formula (4)) : Ramanujan says that, for q and x between 0 and 1,

$$\cfrac{q}{x+\cfrac{q^4}{x+\cfrac{q^8}{x+\cfrac{q^{12}}{x+\ddots}}}} = 1 - \cfrac{q^x}{1+\cfrac{q^2}{1-\cfrac{q^3x}{1+\cfrac{q^4}{1-\cfrac{q^5x}{1+\ddots}}}}}$$

"very nearly." He does not explain what this means, but a little experimentation shows that what is meant is that the two expressions are numerically very close when q is near 1; thus for $q = 0.9$ and $x = 0.5$ one has

$$\text{LHS} = 0.7767340194\ldots, \qquad \text{RHS} = 0.7767340180....$$

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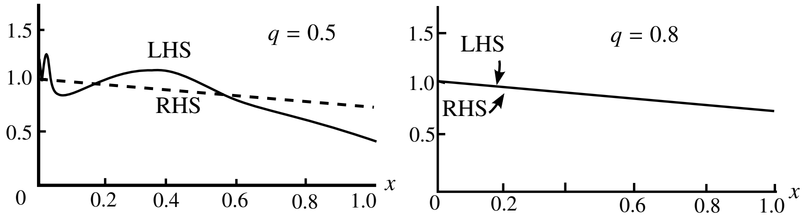
A graphical illustration of this is also shown.



The quantitative interpretation turned out as follows [9] : The difference between the left and right sides of Ramanujan's equation is

$O\dfrac{\pi^2/5}{(e^{\log}q)}$ for $x = 1$, $q \to 1$ (the proof of this used the identities

$$1+\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+}}}} = \prod_{n=1}^{\infty}(1-q^n)^{\left(\frac{n}{5}\right)} = \frac{\Sigma(-1)^r q^{(5r^2+3r)/2}}{\Sigma(-1)^r q^{(5r^2+r)/2}}$$

which are consequences of the Rogers-Ramanujan identities and are surely among the most beautiful formulas in mathematics). For $x \to 0$ and $q \to 1$ the difference is question is $O(e^{(\pi^2/4)/\log q})$, and for $0 < x < 1$ and $q \to 1$ it is $O(e^{c(x)/\log q})$ where $c'(x) = 0(1/x)$ arcsinh $(x/2) = -\frac{1}{x}\log(\sqrt{1+x^2/4}+x/2)$. For these three formulas to be compatible, one needs

$$\int_0^1 \frac{1}{x}\log(\sqrt{1+x^2/4}+x/2)dx = c(0) = c(1) = \frac{\pi^2}{4} - \frac{\pi^2}{5} = \frac{\pi^2}{20}.$$

Using integration by parts and formula A.3.1 (6) of (1) one finds

$$\int \frac{1}{x}\log(\sqrt{1+x^2/4}+x/2)dx = -\frac{1}{2}\operatorname{Li}_2\left((\sqrt{1+x^2/4}-x/2)^2\right) -$$
$$-\frac{1}{2}\log^2\left(\sqrt{1+x^2/4}+x/2\right) + (\log x)\log\left(\sqrt{1+x^2/4}+x/2\right) + C,$$

235 so

$$\int_0^1 \frac{1}{x}\log\left(\sqrt{1+x^2/4}+x/2\right)dx$$
$$= \frac{1}{2}\operatorname{Li}_2(1) - \frac{1}{2}\left(\operatorname{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) + \log^2\left(\frac{1+\sqrt{5}}{2}\right)\right)$$
$$= \frac{\pi^2}{12} - \frac{\pi^2}{30} = \frac{\pi^2}{20}!$$

2 Functional equations In contrast to the paucity of special values, the dilogarithm function satisfies a plethora of functional equations. To begin with, there are the two reflection properties

$$\mathrm{Li}_2(1/z) = -\,\mathrm{Li}_2(z) - (\pi^2/6) - (1/2)\log^2(-z)$$
$$\mathrm{Li}_2(1-z) = -\,\mathrm{Li}_2(z) + (\pi^2/6) - \log(z)\log(1-z).$$

Together they say that the six functions

$$\mathrm{Li}_2(z), \mathrm{Li}_2\left(\frac{1}{1-z}\right), \mathrm{Li}_2\left(\frac{z-1}{z}\right), -\,\mathrm{Li}_2\left(\frac{1}{z}\right), -\,\mathrm{Li}_2(1-z), -\,\mathrm{Li}_2\left(\frac{z}{z-1}\right)$$

are equal modulo elementary functions. Then there is the duplication formula

$$\mathrm{Li}_2(z^2), \mathrm{Li}_2\left(\frac{1}{1-z}\right), \mathrm{Li}_2\left(\frac{z-1}{z}\right), -\,\mathrm{Li}_2\left(\frac{1}{z}\right), -\,\mathrm{Li}_2(1-z), -\,\mathrm{Li}_2\left(\frac{z}{z-1}\right)$$

are equal modulo elementary functions. Then there is the duplication formula

$$\mathrm{Li}_2(z^2) = 2(\mathrm{Li}_2(z) + \mathrm{Li}_2(-z))$$

and more generally the "distribution property"

$$\mathrm{Li}_2(x) = n \sum_{z^n = x} \mathrm{Li}_2(z) \quad (n = 1, 2, 3, \ldots).$$

Next, there is the two-variable, five-term relation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{1-x}{1-xy}\right) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\left(\frac{1-y}{1-xy}\right)$$
$$= \frac{\pi^2}{2} - \log(x)\log(1-x) - \log(y)\log(1-y) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right)$$

which (in this or one of the many equivalent forms obtained by applying the symmetry properties given above) was discovered and rediscovered by Spence (1809). Abel (1827), Hill (1828), Kummer (1840), Schaeffer (1846), and doubtless others. (Despite appearances, this relation is **236** symmetric in the five arguments : if these are numbered cyclically as z_n with $n \in \mathbb{Z}/5\mathbb{Z}$, then $1 - z_n = \dfrac{z_{n-1}}{1 - z_{n-1}} \dfrac{z_{n+1}}{1 - z_{n+1}} = z_{n-2}z_{n+2}$.) There is also the six-term relation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \Rightarrow \mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2(z)$$
$$= \frac{1}{2}\left[\mathrm{Li}_2\left(-\frac{xy}{z}\right) + \mathrm{Li}_2\left(-\frac{yz}{x}\right) + \mathrm{Li}_2\left(-\frac{zx}{y}\right)\right]$$

discovered by Kummer (1840) and Newman (1892). Finally, there is the strange many-variable equation

$$\mathrm{Li}_2(z) = \sum_{\substack{f(x)=z \\ f(z)=1}} \mathrm{Li}_2\left(\frac{x}{a}\right) + C(f), \qquad (1)$$

where $f(x)$ is any polynomial without constant term and $C(f)$ a (complicated) constant depending on f. For f quadratic, this reduces to the five-term relation, while for f of degree n it involves $n^2 + 1$ values of the dilogarithm.

All of the functional equations of Li_2 are easily proved by differentiation, while the special values given in the previous section are obtained by combining suitable functional equations. See [4].

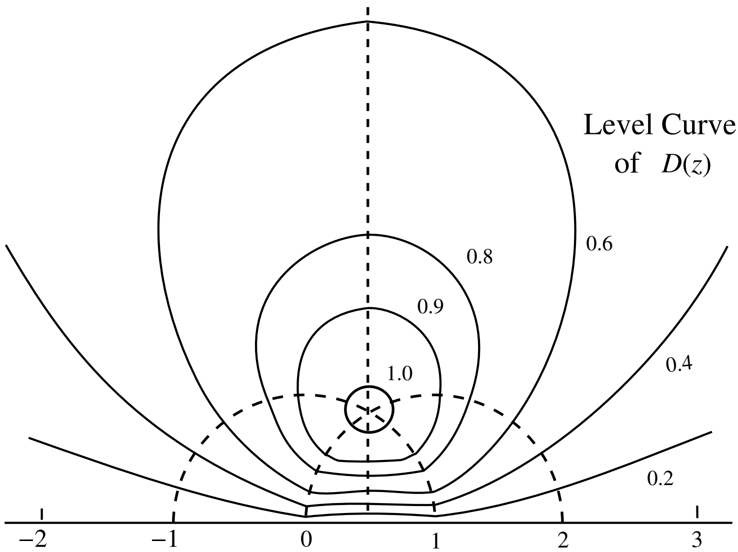
3 The Block-Wigner function $D(z)$ and its generalization

The function $\mathrm{Li}_2(z)$, extended as above to $\mathbb{C} - (1, \infty)$, jumps by $2\pi i \log|z|$ as z crosses the cut. Thus the function $\mathrm{Li}_2(z) + i \arg(1 - z) \log|z|$, where arg denotes the branch of the argument lying between $-\pi$ and π, is continuous. Surprisingly, its imaginary part

$$D(z) = \Im(\mathrm{Li}_2(z)) + \mathrm{art}(1 - z) \log|z|$$

is not only continuous, but satisfies

(I) $D(z)$ is real analytic on $\mathbb{C}$ except at the two points 0 and 1, where it is continuous but not differentiable (it has singularities of type $r \log r$ there.)



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The above graph shows the behaviour of $D(z)$. (We have plotted the level curves $D(z) = 0, .2, .4, .6, .8, .9, 1.0$ in the upper half-plane. The values in the lower half-plane are obtained from $D(\bar{z}) = -D(z)$. The maximum of D is $1.0149\ldots$, attained at the point $(1 + i\sqrt{3})/2$.)

The function $D(z)$, which was discovered by D. Wigner and S. Bloch (cf. [1]), has many other beautiful properties. In particular :

(II) $D(z)$, which is a real-valued function of $\mathbb{C}$, can be expressed in terms of a function of a single real variable, namely

$$D(z) = \frac{1}{2}\left[D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1-1/z}{1-1/\bar{z}}\right) + D\left(\frac{1/(1-z)}{1/(1-\bar{z})}\right)\right] \qquad (2)$$

which expresses $D(z)$ for arbitrary complex z in terms of the function

$$D(e^{i\theta}) = \Im[\mathrm{Li}_2(e^{i\theta})] = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}.$$

Note that the real part of Li_2 on the unit circle is elementary : 238

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = \frac{\pi^2}{6} - \frac{\theta(2\pi-\theta)}{4} \quad \text{for} \quad 0 \le \theta \le 2\pi.)$$

Formula (2) is due to Kummer.

(III) All of the functional equations satisfied by $\mathrm{Li}_2(z)$ lose the elementary correction terms (constants and products of logarithms) when expressed in terms of $D(z)$. In particular, one has the 6-fold symmetry

$$D(z) = D\left(1 - \frac{1}{z}\right) = D\left(\frac{1}{1-z}\right)$$
$$= -D\left(\frac{1}{z}\right) = -D(1-z) = -D\left(\frac{z}{z-1}\right) \quad (3)$$

and the five-term relation

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0, \quad (4)$$

while replacing Li_2 by D in the many-term relation (1) makes the constant $C(f)$ disappear.

The functional equations become even cleaner if we think of D as being a function not of a single complex number but of the cross-ratio of four such numbers, i.e. if we define

$$D(z_0, z_1, z_2, z_3) = D\left(\frac{z_0 - z_2}{z_0 - z_3}\frac{z_1 - z_3}{z_1 - z_2}\right) (z_0, z_1, z_2, z_3 \in \mathbb{C}). \quad (5)$$

Then the symmetry properties (3) say that $\widetilde{D}$ is invariant under even, anti-invariant under odd permutations of its four variables, the five-term relation (4) takes on the attractive form

$$\sum_{i=0}^{4} (-1)^i \widetilde{D}(z_0, \ldots, \widehat{z_i}, \ldots, z_4) = 0 \quad (z_0, \ldots, z_4 \in \mathbb{P}^1(\mathbb{C})). \quad (6)$$

(we will see the geometric interpretation of this later), and the multivariable formula (1) generalizes to the following beautiful formula :

$$\sum_{\substack{z_1 \in f^{-1}(a_1) \\ z_2 \in f^{-1}(a_2) \\ z_3 \in f^{-1}(a_3)}} \widehat{D}(z_0, z_1, z_2, z_3) = n\widetilde{D}(a_0, a_1, a_2, a_3) \quad (z_0, a_1, a_2, a_3 \in \mathbb{P}^1)$$

239 where $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a function of degree n and $a_0 = f(z_0)$. (Equation (1) is the special case when f is a polynomial, so $f^{-1}(\infty)$ is ∞ with multiplicity n.)

Finally, we mention that a real-analytic function on $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ built up out of the polylogarithms in the same way as $D(z)$ was constructed from the dilogarithm, has been defined by Ramakrishnan [6]. His function (slightly modified) is given by

$$D_m(z) = \mathscr{R}\left(i^{m+1} \left[\sum_{k=1}^{m} \frac{(-\log|z|)^{m-k}}{(m-k)!} \operatorname{Li}_k(z) - \frac{(\log|z|)^m}{2m!} \right] \right)$$

(so $D_1(z) = \log|z^{1/2} - z^{-1/2}|$, $D_2(z) = D(z)$) and satisfies

$$D_m\left(\frac{1}{z}\right) = (-1)^{m-1} D_m(z),$$

$$\frac{\partial}{\partial z} D_m(z) = \frac{i}{2z}\left(D_{m-1}(z) + \frac{i}{2} \frac{(-i\log|z|)^{m-1}}{(m-1)!} \frac{1+z}{1-z} \right).$$

However, it does not seem to have analogues of the properties (II) and (III) : for example, it is apparently impossible to express $D_3(z)$ for arbitrary complex z in terms of only the function $D_3(e^{i\theta}) = \sum_{n=1}^{\infty} (\cos n\theta)/n^3$, and passing from Li_3 to D_3 removes many but not all of the numerious lower-order terms in the various functional equations of the trilogarithm, e.g. :

$$\begin{aligned} D_3(x) + D_3(1-x) + D_3\left(\frac{x}{x-1}\right) \\ = D_3(1) + \frac{1}{12} \log|x(1-x)| \log\left|\frac{x}{(1-x)^2}\right| \log\left|\frac{x^2}{1-x}\right|, \end{aligned}$$

$$\begin{aligned} D_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) + D_3(xy) + D_3\left(\frac{x}{y}\right) - 2D_3\left(\frac{x}{y}\frac{1-y}{1-x}\right) \\ - 2D_3\left(\frac{x(1-y)}{x-1}\right) - 2D_3\left(\frac{y(1-x)}{y-1}\right) - 2D_3\left(\frac{1-y}{1-x}\right) \\ - 2D_3(x) - 2D_3(y) = 2D_3(1) - \frac{1}{4} \log|xy| \log\left|\frac{x}{y}\right| \log\left|\frac{x}{y}\frac{(1-y)^2}{(1-x)^2}\right|. \end{aligned}$$

Nevertheless, these higher Bloch-Wigner functions do occur. In studying the so-called "Heegner points" on modular curves, B. Gross and I had to study for $n = 2, 3, \ldots$ "higher weight Green's functions" for $\mathfrak{H}/\Gamma$ ($\mathfrak{H}$ = complex upper half-plane, $\Gamma = SL_2(\mathbb{Z})$ or a congruence subgroup). These are functions $G_n(z_1, z_2) = G_n^{\mathfrak{H}/\Gamma}(z_1, z_2)$ defined on $\mathfrak{H}/\Gamma \times \mathfrak{H}/\Gamma$, real-analytic in both variables except for a logarithmic singularity along the diagonal $z_1 = z_2$, and satisfying $\Delta_{z_1} G_n = \Delta_{z_2} G_n = n(n-1)G_n$, where $\Delta_z = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ is the hyperbolic Laplace operator with respect to $z = x + iy \in \mathfrak{H}$. They are obtained as 240

$$G_n^{\mathfrak{S}/\Gamma}(z_1, z_2) = \sum_{\gamma \in \Gamma} G_n^{\mathfrak{H}}(z_1, \gamma z_2)$$

where $G_n^{\mathfrak{H}}$ is defined analogously to $G_n^{\mathfrak{S}/\Gamma}$ but with $\mathfrak{H}/\Gamma$ replaced by $\mathfrak{H}$. The functions $G_n^{\mathfrak{H}} (n = 2, 3, \ldots)$ are elementary, e.g.,

$$G_2^{\mathfrak{H}}(z_1, z_2) = \left(1 + \frac{|z_1 - z_2|^2}{2y_1 y_2}\right) \log \frac{|z_1 - z_2|^2}{|z_1 - \bar{z}_2|^2} + 2.$$

In between $G_n^{\mathfrak{H}}$ and $G_n^{\mathfrak{H}/\Gamma}$ are the functions $G_n^{\mathfrak{H}/\Gamma} = \Sigma_{r \in Z} G_n^{\mathfrak{H}}(z_1, z_2 + r)$. It turns out [10] that these are expressible in terms of the $D_m (m = 1, 3, \ldots, 2n-1)$, e.g.,

$$\begin{aligned} G_2^{\mathfrak{H}/Z}(z_1, z_2) = {} & \frac{1}{4\pi^2 y_1 y_2} (D_3(e^{2\pi i(z_1 - z_2)}) + D_3(e^{2\pi i(z_1 - \bar{z}_2)}) \\ & + \frac{y_1^2 + y_2^2}{2 y_1 y_2} (D_1(e^{2\pi i(z_1 - z_2)}) + D_1(e^{2\pi i(z_1 - \bar{z}_2)})) \end{aligned}$$

I do not know the reasons for this connection.

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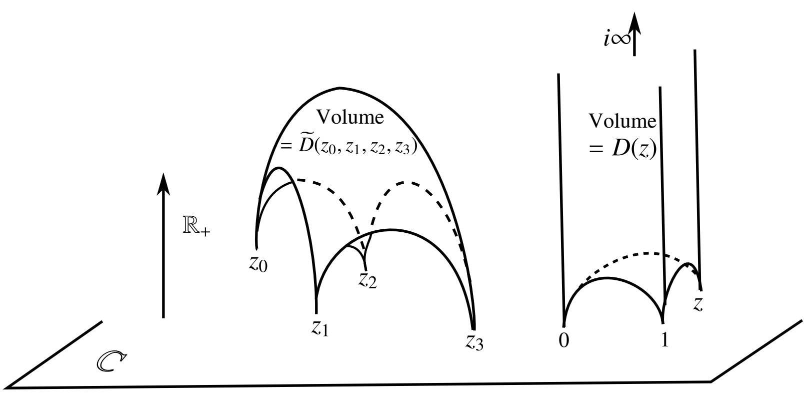
4 Volumes of Hyperbolic 3-manifolds...

The dilogarithm occurs in connection with measurement of volumes in Euclidean, spherical, and hyperbolic geometry. We will be concerned with the last of these. Let $\mathfrak{H}_3$ be the Lobachevsky space (space of non- Euclidean solid geometry). We will use the half-space model, in which $\mathfrak{H}_3$ is represented by $\mathbb{C} \times R_+$ with the standard hyperbolic metric in which the geodesics are either vertical lines or semicircles in vertical planes with endpoints in $\mathbb{C}\times\{0\}$ and the geodesic planes are either vertical planes or else hemispheres with boundary in $\mathbb{C}\times\{0\}$. *An ideal tetrahedron* is a tetrahedron whose vertices are all in $\partial\mathfrak{H}_3=\mathbb{C}\cup\{\infty\}=\mathbb{P}^1(\mathbb{C})$. Let Δ be such a tetrahedron. Although the vertices are at infinity, the (hyperbolic) volume is finite. It is given by

$$\mathrm{Vol}(\Delta)=\widetilde{D}(z_0,z_1,z_2,z_3), \tag{7}$$

where $z_0,\dots,z_3\in\mathbb{C}$ are the vertices of Δ and $\widetilde{D}$ is the function defined in (5). In the special case that three of the vertices of Δ are ∞, 0, and 1, equation (7) reduces to the formula (due essentially to Lobachevsky)

$$\mathrm{Vol}(\Delta)=D(z). \tag{8}$$



In fact, equations (7) and (8) are equivalent since any 4-tuple of points $z_0,\dots,z_3$ can be brought into the form $\{\infty,0,1,z\}$ by the action of some element of $SL_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$, and the group $SL_2(\mathbb{C})$ acts on $\mathfrak{H}_3$ by isometries.

The (anti-) symmetry properties of $\widetilde{D}$ under permutations of the z_i **242**
are obvious from the geometric interpretation (7), since renumbering the vertices leaves Δ unchanged but may reverse its orientation. Formula (6) is also an immediate consequence of (7), since the five tetrahedra

spanned by four at a time of $z_0, \ldots, z_4 \in \mathbb{P}^1(\mathbb{C})$, counted positively or negatively as in (6), add up algebraically to the zero 3-cycle.

The reason that we are interested in hyperbolic tetrahedra is that these are the building blocks of hyperbolic 3-manifolds, which in turn (according to Thurston) are the key objects for understanding three-dimensional geometry and topology. A hyperbolic 3-manifold is a 3-dimensional riemannian manifold M which is locally modelled on (i.e., isometric to portions of) hyperbolic 3-space $\mathfrak{H}_3$; equivalently, it has constant negative curvature -1. We are interested in complete oriented hyperbolic 3-manifolds which have finite volume (they are then either compact or have finitely many "cusps" diffeomorphic to $S^1 \times S^1 \times \mathbb{R}_+$). Such a manifold can obviously be triangulated into small geodesic simplices which will be hyperbolic tetrahedra. Less obvious is that (possibly after removing from M a finite number of closed geodesics) there is always a triangulation into *ideal* tetrahedra (the part of such a tetrahedron going out towards a vertex at infinity will then either tend to a cusp of M or else spiral in around one of the deleted curves). Let these tetrahedra be numbered $\Delta_1, \ldots, \Delta_n$ and assume (after an isometry of $\mathfrak{H}_3$ if necessary) that the vertices of Δ_ν are at ∞, 0, 1 and z_ν. Then

$$\mathrm{Vol}(M) = \sum_{\nu=1}^{n} D(z_\nu). \tag{9}$$

Of course, the numbers z_ν are not uniquely determined by Δ_ν since they depend on the order in which the vertices were sent to $\{\infty, 0, 1, z_\nu\}$, but the non-uniqueness consists (since everything is oriented) only in replacing z_ν by $1 - 1/z_\nu$ or $1/(1 - z_\nu)$ and hence does not affect the value of $D(z_\nu)$.

One of the objects of interest in the study of hyperbolic 3-manifolds is the "volume spectrum"

$$\mathbf{Vol} = \{\mathrm{Vol}(M) | M \text{ hyperbolic 3-manifold}\} \subset \mathbb{R}_+.$$

From the work of Jørgensen and Thurston one knows the **Vol** is a countable and well-ordered subset of $\mathbb{R}_+$ (i.e. every subset has a smallest

element), and its exact nature is of considerable interest both in topology and number theory. Equation (9) as it stands says nothing about this set since any real number can be written as a finite sum of values 243
$D(z)$, $z \in \mathbb{C}$. However, the parameters z_ν of the tetrahedra triangulating a complete hyperbolic 3-manifold satisfy an extra relation, namely

$$\sum_{\nu=1}^{n} z_\nu \wedge (1 - z_\nu) = 0, \tag{10}$$

where the sum is taken in the abelian group $\wedge^2\mathbb{C}^\times$ (the set of all formal linear combinations $x \wedge y$, $x, y \in \mathbb{C}^\times$, subject to the relations $x \wedge x = 0$ and $(x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y$). (This follows from assertions in [3] or form Corollary 2.4 of [5] applied to suitable x and y). Now (9) *does* give information about **Vol** because the set of numbers $\Sigma_{\nu=1}^{n} D(z_\nu)$ with z_ν satisfying (10) is countable. This fact was proved by Bloch [1]. To make a more precise statement, we introduce the *Bloch group*. Consider the abelian group of formal sums $[z_1] + \cdots + [z_n]$ with $z_1, \ldots, z_n \in \mathbb{C}^\times - \{1\}$ satisfying (10). As one easily checks, it contains the elements

$$[x] + \left[\frac{1}{x}\right],\ [x] + [1-x],\ [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right] \tag{11}$$

for all x and y in $\mathbb{C}^\times - \{1\}$ with $xy \neq 1$, corresponding to the symmetry properties and 5-term relation satisfied by $D(\cdot)$. The Block group is defined as

$$\mathscr{B}_{\mathbb{C}} = \{[z_1] + \cdots + [z_n] \text{ satisfying (10)}\}/(\text{subgroup generated by the elements (11)}) \tag{12}$$

(this is slightly different from the usual definitions). The definition of the Block group in terms of the relations satisfied by $D(\cdot)$ makes it obvious that D extends to a linear map $D : \mathscr{B}_{\mathbb{C}} + \mathbb{R}$ by $[z_1] + \cdots + [z_n] \mapsto D(z_1) + \cdots + D(z_n)$, and Bloch's result (related to Mostow rigidity) says that the set $D(\mathscr{B}_{\mathbb{C}})$ coincides with $D(\mathscr{B}_{\overline{\mathbb{Q}}})$ (where $\mathscr{B}_{\overline{\mathbb{Q}}}$ is defined by (12) but with the z_ν lying in $\overline{\mathbb{Q}}^\times - \{1\}$). Thus $D(\mathscr{B}_{\mathbb{C}})$ is countable, and (9) and (10) imply that **Vol** is contained in this countable set. The structure of $\mathscr{B}_{\overline{\mathbb{Q}}}$ which is very subtle, will be discussed below.

We give an example of a non-trivial element of the Bloch group. For convenience, set $\alpha = \dfrac{1-\sqrt{-7}}{2}$, $\beta = \dfrac{-1-\sqrt{-7}}{2}$. Then

$$2\left(\frac{1+\sqrt{-7}}{2}\right) \wedge \left(\frac{1-\sqrt{-7}}{2}\right) + \left(\frac{-1+\sqrt{-7}}{4}\right) \wedge \left(\frac{5-\sqrt{-7}}{4}\right)$$

$$2(-\beta) \wedge \alpha + \left(\frac{1}{\beta}\right) \wedge \left(\frac{\alpha^2}{\beta}\right) = \beta^2 \wedge \alpha - \beta \wedge \alpha^2 = 2 \cdot \beta \wedge \alpha - 2 - \cdot\beta \wedge \alpha = 0,$$

244 so

$$2\left[\frac{1+\sqrt{-7}}{2}\right] + \left[\frac{-1+\sqrt{-7}}{4}\right] \in \mathscr{B}_{\mathbb{C}}. \tag{13}$$

This example should make it clear why non-trivial elements of $\mathscr{B}_{\mathbb{C}}$ can only arise from algebraic numbers — the key relations $1+\beta = \alpha$ and $1-\beta^{-1} = \alpha^2/\beta$ above forced α and β to be algebraic.

5 ...and values of Dedekind zeta functions

Let F be an algebraic number field, say of degree N over $\mathbb{O}$. Among its most important invariants are the discriminant d, the numbers r_1 and r_2 of real and imaginary archimedean valuations, and the Dedekind zeta-function $\zeta_F(s)$. For the non-number-theorist we recall the (approximate) definitions. The field F can be represented as $\mathbb{Q}(\alpha)$ where α is a root of an irreducible monic polynomial $f \in \mathbb{Z}[x]$ of degree N. The discriminant of f is an integer d_f and d is given by $c^{-2}d_f$ for some natural number c with $c^2|d_f$. The polynomial f, which is irreducible over $\mathbb{Q}$, in general becomes reducible over $\mathbb{R}$, where it splits into r_1 linear and r_2 quadratic factors (thus $r_1 \geq 0$, $r_2 \geq 0$, $r_1 + 2r_2 = N$). It also in general becomes reducible when it is reduced modulo a prime p, but if $p \nmid d_f$ then its irreducible factors modulo p are all distinct, say $r_{1,p}$ linear factors, $r_{2,p}$ quadratic ones, etc. (so $r_{1,p} + 2r_{2,p} + \cdots = N$). Then $\zeta_F(s)$ is the Dirichlet series given by an Euler product $\Pi_p Z_p(p^{-s})^{-1}$ where $Z_p(t)$ for $p \nmid d_f$ is the monic polynomial $(1-t)^{r_{1,p}}(1-t^2)^{r_{2,p}} \ldots$ of degree N and $Z_P(t)$ for $p|d_f$ is a certain monic polynomial of degree $\leq N$. Thus (r_1, r_2) and $\zeta_F(s)$ encode the information about the behaviour of f (and hence F) over the real and p-adic numbers, respectively.

As an example, let F be an imaginary quadratic field $\mathbb{Q}(\sqrt{-a})$ with $a \geq 1$ squarefree. Here $N = 2$, $d = -a$ or $-4a$, $r_1 = 0$, $r_2 = 1$. The Dedekind zeta function has the form $\sum_{n\geq 1} r(n)n^{-s}$ where $r(n)$ counts representations of n by certain quadratic forms of discriminant d; it can also be represented as the product of the Riemann zeta function $\zeta(s) =$
$\zeta_{\mathbb{Q}}^{(s)}$ with an L-series $L(s) = \sum_{n\geq 1} \left(\frac{d}{n}\right) n^{-s}$ where $\left(\frac{d}{n}\right)$ is a symbol taking the 245
values ± 1 or 0 and which is periodic of period $|d|$ in n. Thus for $a = 7$

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(s) = \frac{1}{2} \sum_{(x,y)\neq(0,0)} \frac{1}{(x^2 + xy + 2y^2)^s}$$
$$= \left(\sum_{n=1}^{\infty} n^{-s}\right)\left(\sum_{n=1}^{\infty} \left(\frac{-7}{n}\right) n^{-s}\right)$$

where $\left(\frac{-7}{n}\right)$ is $+1$ for $n \equiv 1, 2, 4 (\mathrm{mod}\ 7)$, -1 for $n \equiv 3, 5, 6 (\mathrm{mod}\ 7)$, and 0 for $n \equiv 0 (\mathrm{mod}\ 7)$.

One of the questions of interest is the evaluation of the Dedekind zeta function at suitable integer arguments. For the Riemann zeta function we have the special values cited at the beginning of this paper. More generally, if F is totally real (i.e., $r_1 = N$, $r_2 = 0$), then a theorem of Siegel and Klingen implies that $\zeta_F(m)$ for $m = 2, 4, \ldots$ equals $\pi^{mN}/\sqrt{d}$ times a rational number. If $r_2 > 0$, then no such simple result holds. However, in the case $F = \mathbb{Q}(\sqrt{-a})$, then using the representation $\zeta_F(s) = \zeta(s)L(s)$ and the formula $\zeta(2) = \pi^2/6$ and writing the periodic function (d/n) as a finite linear combination of terms $e^{2\pi in\ \ d}$, we obtain

$$\zeta_F(2) = \frac{\pi^2}{6\sqrt{|d|}} \sum_{n=1}^{|d|-1} \left(\frac{d}{n}\right) D(e^{2\pi in\ \ d}) \quad (F\ \text{ imaginary quadratic}),$$

e.g.,

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{\pi^2}{3\sqrt{7}} \left(D(e^{2\pi i/7}) + D(e^{4\pi i/7}) - D(e^{6\pi i/7})\right)$$

Thus the values of $\zeta_F(2)$ for imaginary quadratic fields can be expressed in closed form in terms of values of the Bloch-Wigner function $D(z)$ at algebraic arguments z.

By using the ideas of the last section we can prove a much stronger statement. Let $\mathscr{O}$ denote the ring of integers of F (this is the $\mathbb{Z}$-lattice in $\mathbb{C}$ spanned by 1 and $\sqrt{-a}$ or $(1+\sqrt{-a})/2$, depending whether $d=-4a$ or $d=-a$). Then the group $\Gamma = SL_2(\mathscr{O})$ is a discrete subgroup of $SL_2(\mathbb{C})$ and therefore acts on hyperbolic space $\mathfrak{H}_3$ by isometries. A classical result of Humbert gives the volume of the quotient space $\mathfrak{H}_3/\Gamma$
246 as $|d|^{3/2} \times \zeta_F(2)/4\pi^2$. On the other hand, $\mathfrak{H}_3/\Gamma$ (or, more precisely, a certain covering of it of low degree) can be triangulated into ideal tetrahedra with vertices belonging to $\mathbb{P}^1(F) \subset \mathbb{P}^1(C)$, and this leads to a representation

$$\zeta_F(2) = \frac{\pi^2}{3|d|^{3/2}} \sum_v n_v D(z_v)$$

with n_v in $\mathbb{Z}$ and z_v *in F itself* rather than in the much larger field $\mathbb{Q}(e^{2\pi i\ d})$([8], Theorem 3). For instance, in our example $F=\mathbb{Q}(\sqrt{-7})$ we find

$$\zeta_F(2) = \frac{4\pi^2}{21\sqrt{7}}\left(2D\left(\frac{1+\sqrt{-7}}{2}\right) + D\left(\frac{-1+\sqrt{-7}}{4}\right)\right).$$

This equation together with the fact that $\zeta_F(2) = 1.89484144897\ldots \neq 0$ implies that the element (13) has infinite order in $\mathscr{B}_{\mathbb{C}}$.

In [8], it was pointed out that the same kind of argument works for all number fields, not just imaginary quadratic ones. If $r_2 = 1$ but $N > 2$ then one can again associate to F (in many different ways) a discrete subgroup $\Gamma \subset SL_2(\mathbb{C})$ such that $\mathrm{Vol}(\mathfrak{H}_3/\Gamma)$ is a rational multiple of $d|^{1/2}\zeta_F(2) \times \pi^{2(1-N)}$. This manifold $\mathfrak{H}_3/\Gamma$ is now compact, so the decomposition into ideal tetrahedra is a little less obvious than in the case of imaginary quadratic F, but by decomposing into non-ideal tetrahedra (tetrahedra with vertices in the interior of $\mathfrak{H}_3$) and writing these as differences of ideal ones, it was shown that the volume is an integral linear combination of values of $D(z)$ with z of degree at most 4 over F. For F completely arbitrary there is still a similar statement, except that now one gets discrete groups Γ acting on $\mathfrak{H}_3^{r_2}$; the final result ([8], Theorem 1) is that $|d|^{1/2} \times \zeta_F(2)/\pi^{2(r_1+r_2)}$ is a rational linear combination of r_2-fold products $D(z^{(1)})\ldots D(z^{(r_2)})$ with each $z^{(i)}$ of degree ≤ 4 over F (more precisely, over the i^{th} complex embedding $F^{(i)}$ of F, i.e. over the subfield $\mathbb{Q}(\alpha^{(i)})$ of $\mathbb{C}$ where $\alpha^{(i)}$ is one of the two roots of the i^{th} quadratic factor of $f(x)$ over $\mathbb{R}$).

But in fact much more is true : the $z^{(i)}$ can be chosen in $F^{(i)}$ itself (rather than of degree 4 over this field), and the phrase "rational linear combination of r_2-fold products" can be replaced by "rational multiple of an $r_2 \times r_2$ determinant." We will not attempt to give more than a very sketchy account of why this is true, lumping together work of Wigner, Bloch, Dupont, Sah, Levine, Merkuriev, Suslin, ... for the purpose (references are [1], [3], and the survey paper [7]). This work connects the Bloch group defined in the last section with the algebraic K-theory of **247** the underlying field; specifically, the group[1] $\mathscr{B}_F$ is equal, at least after tensoring it with $\mathbb{O}$, to a certain quotient $K_3^{\text{ind}}(F)$ of $K_3(F)$. The exact definition of $K_3^{\text{ind}}(F)$ is not relevant here. What is relevant is that this group has been studied by Borel [2], who showed that it is isomorphic (modulo torsion) to $\mathbb{Z}^{r_2}$ and that there is a canonical homomorphism, the "regulator mapping," from it into $\mathbb{R}^{r_2}$ such that the co-volume of the image in a non-zero rational multiple of $|d|^{a/2}\zeta_F(2)/\pi^{2r_1+2r_2}$; moreover, it is known that under the identification of $K_3^{\text{ind}}(F)$ with $\mathscr{B}_F$ this mapping corresponds to the composition $\mathscr{B}_F \to (\mathscr{B}_{\mathbb{C}})^{r_2} \xrightarrow{D} \mathbb{R}^{r_2}$, where the first arrow comes from using the r_2 embeddings $F \subset \mathbb{C}(\alpha \to \alpha^{(i)})$. Putting all this together gives the following beautiful picture : The group $\mathscr{B}_F/\{\text{torsion}\}$, is isomorphic to $\mathbb{Z}^{r_2}$. Let $\zeta_1, \ldots, \zeta_{r_2}$ by any r_2 linearly independent elements of it, and form the matrix with entires $D(\zeta_j^{(i)})$, $(i, j = 1, \ldots, r_2)$. Then the determinant of this matrix is a non-zero rational multiple of $|d|^{1/2}\zeta_F(2)/\pi^{2r_1+2r_2}$. If instead of taking any r_2 linearly independent elements we choose the ζ_j to be a basis of $\mathscr{B}_F/\{\text{torsion}\}$, then this rational multiple (chosen positively) is an invariant of F, independent of the choice of ζ_j. This rational multiple is then conjecturally related to the quotient of the order of $K_3(F)_{\text{torsion}}$ by the order of the

[1]It should be mentioned that the definition of $\mathscr{B}_F$ which we gave for $F = \mathbb{C}$ or $\overline{\mathbb{Q}}$ must be modified slightly when F is a number field because $F^\times$ is no longer divisible; however, this is a minor point, affecting only the torsion in the Bloch group, and will be ignored here.

finite group $K_2(\mathscr{O}_F)$ where $\mathscr{O}_F$ denotes the ring of integers of F (Lichtenbaum conjectures).

This all sounds very abstract, but it is fact not. There is a reasonably efficient algorithm to produce many elements of $\mathscr{B}_F$ for any number field F. If we do this, for instance, for F an imaginary quadratic field, and compute $D(\zeta)$ for each element $\zeta \in \mathscr{B}_F$ which we find, then after a while we are at least morally certain of having identified the lattice $D(\mathscr{B}_F) \subset \mathbb{R}$ exactly (after finding k elements at random, we have only about one chance in 2^k of having landed in the same non-trivial sublattice each time). By the results just quoted, this lattice is generated by a number of the form $\kappa|d|^{3/2}\zeta_F(2)/\pi^2$ with κ rational, and the conjecture referred to above says that κ should have the form $\frac{3}{2T}$ where T is the order of the finite group $K_2(\mathscr{O}_F)$, at least for $d < -4$ (in this case the order of $K_3(F)_{\text{torsion}}$ is always 24). Calculations done by H. Gangl in Bonn for several hundred imaginary quadratic fields support this; the κ he found
248 all have the form $\frac{3}{2T}$ for some integer T and this integer agrees with the order of $K_2(\mathscr{O}_F)$ in the few cases where the latter is known. Here is a small excerpt from his tables :

$\lvert d\rvert$	7	8	11	15	19	20	23	24	31	35	39	40	...	303	472	479	491	555	583
T	2	1	1	2	1	1	2	1	2	2	6	1	...	22	5	14	13	28	34

(the omitted values contain only the primes 2 and 3; 3 occurs whenever $d \equiv 3 (\text{mod } 9)$ and there is also some regularity in the powers of 2 occurring). Thus one of the many virtues of the mysterious dilogarithm is that it gives, at least conjecturally, an effective way of calculating the orders of certain groups in algebraic K-theory!

To conclude, we mention that Borel's work connects not only $K_3^{\text{ind}}(F)$ and $\zeta_F(2)$ but more generally $K_{2m-1}^{\text{ind}}(F)$ and $\zeta_F(m)$ for any integer $m > 1$. No elementary description of the higher K-groups analogous to the description of K_3 in terms of B is known, but one can at least speculate that these groups and their regulator mappings may be related to the higher polylogarithms and that, more specifically, the value of $\zeta_F(m)$ is always a simple multiple of a determinant ($r_2 \times r_2$ or $(r_1+r_2)\times(r_1+r_2)$ depending whether m is even or odd) whose entries are linear combinations of values of the Bloch-Wigner-Ramakrishnan function $D_m(z)$ with arguments $z \in F$. As the simplest case, one can guess that for a *real* quadratic field F the value of $\zeta_F(3)/\zeta(3) = L(3)$, where $L(s)$ is a Dirichlet L-Function of a real quadratic character of period d) is equal to $d^{-5/2}$ times a simple rational linear combination of differences $D_3(x) - D_3(x')$ with $x \in F$, where x' denotes the conjugate of x over $\mathbb{Q}$. Here is one (numerical) example of this :

$$2^{-5}5^{5/2}\zeta_{\mathbb{Q}(\sqrt{5})}(3)/\zeta(3) = D_3\left(\frac{1+\sqrt{5}}{2}\right) - D_3\left(\frac{1-\sqrt{5}}{2}\right) - \\ - \frac{1}{3}[D_3(2+\sqrt{5}) - D_3(2-\sqrt{5})]$$

(both sides are equal approximately to 1.4933174117785447 26). I have found many other examples, but the general picture is not yet clear.

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