RADON MEASURES ON ARBITRARY TOPOLOGICAL SPACES AND CYLINDRICAL MEASURES

RADON MEASURES ON ARBITRARY TOPOLOGICAL SPACES AND CYLINDRICAL MEASURES

TATA INSTITUTE OF FUNDAMENTAL RESEARCH STUDIES IN MATHEMATICS

General Editor: M. S. Narasimhan

1. M. Hervé : Several Complex Variables
2. M. F. Atiyah and others : Differential Analysis
3. B. Malgrange : Ideals of Differentiable Functions
4. S. S. Abhayankar and others : Algebraic Geometry
5. D. Mumford : Abelian Varieties
6. L. Schwartz : Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures

RADON MEASURES ON ARBITRARY TOPOLOGICAL SPACES AND CYLINDRICAL MEASURES

LAURENT SCHWARTZ

Published for the
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
OXFORD UNIVERSITY PRESS
1973

Oxford University Press, Ely House, London W 1

GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON
CAPE TOWN IBADAN NAIROBI DAR ES SALAAM LUSAKA ADDIS ABABA
DELHI BOMBAY CALCUTTA MADRAS KARACHI LAHORE DACCA
KUALA LUMPUR SINGAPORE HONG KONG TOKYO

Oxford House, Apollo Bunder, Bombay 400 001.

Laurent SCHWARTZ, 1915

© Tata Institute of Fundamental Research, 1973

PRINTED IN INDIA

INTRODUCTION

There are essentially two methods of presenting the theory of measure: the abstract theory and the theory of Radon measures on locally compact spaces. The justification for the abstract theory is essentially that the topology has, *a priori*, nothing to do with the problem. The justification for the Radon measures is that, in fact, every set in analysis occurs with a topology or several topologies.

The defects of the abstract theory are (i) the catastrophe of image measures (see § 5), (ii) the product of Borel σ-algebras of Hausdorff spaces is not in general the Borel σ-algebra of the product, (iii) an abstract measure even on the Borel σ-algebra has in general no support, etc. The fundamental defect of the usual theory of Radon measures is that the spaces of functions which occur in the theory of probability are not locally compact.

The present theory of Radon measures on arbitrary (for the most part) Hausdorff topological spaces combines all the advantages of both methods; the Radon measures studied have all the good properties of the Radon measures studied in Bourbaki. One finds in the literature numerous theorems where additional hypotheses are put on abstract measures in order to give them some of the properties of Radon measures on locally compact spaces. (See § 8, page 58, for references); however the results never seem to have adequate cohesion. The notion of Radon measure presented here gives a unification.

We define a Radon measure to be a measure on the Borel σ-algebra which is locally finite and inner regular; in § 8 we discuss other methods which yield the same measures.

Part I of the book studies these new Radon measures on arbitrary (Hausdorff) topological spaces and the special properties, relative to these measures of polish, Lusin and Suslin spaces.

Part II studies cylindrical probabilities and radonifying maps. It has, for its point of departure, the results of the Soviet School and in particular the theorems of Sazonov and Minlos. The Radon

measures of Part I are particularly well adapted for treating these problems.

During the preparation of this book, other works have appeared on the same subject: Chapter IX of Bourbaki's Intégration, and numerous articles on radonifying maps appear for the most part in the Seminaire Schwartz 1969-70 de l' École Polytechnique. Practically, all the results of Bourbaki are contained in the present book. On the contrary it has not been possible to incorporate those in the Seminaire de l' École Polytechnique without over-loading this book and delaying its publication.

REMERCIEMENT

Je tiens à remercier le Tata Institute of Fundamental Research de s'être chargé de l'édition délicate de ce livre.

Je me dois également de remercier tout particulierement d'une part les professeurs J. Choksi et K. N. Gowrisankaran, d'autre part le professeur J. Horvath, qui ont bien voulu en rédiger respectivement la premiere et la seconde partie. Ils y ont fortement engagé leur personnalité scientifique et leur style. Qu'ils trouvent ici toute ma reconnaissance.

Laurent SCHWARTZ

CONTENTS

INTRODUCTION v

REMERCIEMENTS vii

PART I

SUMMARY OF CHAPTER I 3

CHAPTER I. RADON MEASURES: MAIN PROPERTIES AND OPERATIONS

1. Preliminaries 7
2. Definition of Radon measure on general topological spaces 12
3. Induced measures 20
4. Product of a measure by a locally integrable function 23
5. Measurable mappings and image measures . . 25
6. Other properties of Radon measures . . . 40
7. Real and complex measures 53
8. Different ways of defining Radon measures . . 58
9. Tensor product of measures 63
10. Projective limit of measures 74
11. Non-Hausdorff spaces 82

SUMMARY OF CHAPTER II 89

CHAPTER II. POLISH, LUSIN, SUSLIN AND RADON SPACES

1. Definitions and topological properties . . . 92
2. Examples of Lusin spaces 112
3. Radon spaces 117
4. Lifting of L^∞ to $\mathscr{L}^\infty$ 130
5. Spaces of regulated functions 136

TERMINOLOGICAL INDEX TO PART I 144

REFERENCES 146

PART II

INTRODUCTION TO PART II 151

SUMMARY OF CHAPTER I 152

CHAPTER I. EQUIVALENT TOPOLOGIES

1. Equivalent measures 153
2. Equivalent topologies 155
3. A Borel-graph theorem 160
4. Equivalence of the strong and weak topologies . 161
5. Equivalence of the strong topology with coarser topologies 166

SUMMARY OF CHAPTER II 171

CHAPTER II. CYLINDRICAL MEASURES

1. Definition of cylindrical measures . . . 172
2. Operations with cylindrical measures . . . 180
3. Concentrations 188

SUMMARY OF CHAPTER III 206

CHAPTER III. CYLINDRICAL MEASURES AND HILBERT-SCHMIDT OPERATORS

1. Hilbert-Schmidt operators 207
2. The main lemma 212
3. The theorem of Sazonov 214

SUMMARY OF CHAPTER IV 218

CHAPTER IV. CYLINDRICAL MEASURES AND NUCLEAR SPACES

1. Nuclear operators 219
2. Nuclear spaces 226
3. The theorem of Minlos 233
4. The theorem of Sazonov-Badrikian . . . 237

SUMMARY OF CHAPTER V 242

CHAPTER V. CYLINDRICAL MEASURES AND PROBABILITY THEORY

1. Basic concepts of probability theory . . . 244
2. Randon functions 254
3. Examples and applications 269
4. An analogue of Prokhorov's theorem for random variables with values in a Suslin space . . . 288
5. Categories of random functions 305

SUMMARY OF CHAPTER VI 324

CHAPTER VI. GAUSS MEASURES

1. The definition of Gauss measures . . . 325
2. The Hilbert subspaces of a locally convex space . 330
3. The converse of the theorems of Sazonov and Minlos 341
4. The analogue of Sazonov's theorem for l^p spaces $(0 < p \leqslant +\infty)$ and applications . . . 347
5. An application to Brownian motion . . . 350

SUMMARY OF APPENDIX 368

APPENDIX. COMPLEMENTS ON NARROW CONVERGENCE

1. Narrow convergence on an arbitrary topological space 369
2. Riemann-integrable functions 374

3. A compactness criterion for the narrow topology . 379
4. A theorem of Paul Lévy 382
5. The space of finite measures on a Suslin space . 385

TERMINOLOGICAL INDEX TO PART II 390

REFERENCES 392

RADON MEASURES ON ARBITRARY TOPOLOGICAL SPACES AND CYLINDRICAL MEASURES

Part I

RADON MEASURES ON ARBITRARY TOPOLOGICAL SPACES

SUMMARY OF CHAPTER I

§ 1 recalls the main classical properties of abstract measures on a set X equipped with a σ-algebra τ : measurability, integrability, Lebesgue dominated convergence theorem, Fatou property.

§ 2 defines Radon measures on an arbitrary Hausdorff topological space X. We notice that, on a locally compact space X, a Radon measure μ defines two abstract measures m and M with special properties on the Borel σ-algebra and, conversely, every system of two Borel measures m and M with these properties defines a Radon measure. Therefore it becomes possible to define a Radon measure μ on a general space X by a pair of Borel measures m and M having these special properties 1-5 of Definition R_1, page 12. Instead of starting from m and M, one can start only from M, Definition R_2, or from m, Definition R_3; finally, this third definition will appear, in practice, as the most convenient, and will be adopted in the rest of the book. Then one defines immediately upper integrals, and integrable functions. Definition 5, page 17, gives the measurable subsets, Definition 6 the negligible subsets, Definition 7 the Borel measurable maps. This demonstrates the classical elementary properties for such Radon measures. We point out that the notions of integrability and essential integrability, used by Bourbaki, are called here respectively strict integrability and integrability; the shorter name, integrability, is given to the more useful notion, which happens to be Bourbaki's essential integrability.

§ 3 deals with induced measures. X is a topological space, μ a Radon measure on X, Y is a μ-measurable subspace, one defines the induced measure μ_Y by μ on Y, page 21. The main property is Theorem 1; Theorem 2, page 23, gives the transitivity of induced measures.

§ 4 deals with the product of a Radon measure μ by a non-negative, locally integrable function p, Definition: page 23. The main property is Theorem 3, page 24; Theorem 4, page 24, gives the transitivity of products; Proposition 3 the compatibility of

induced measures with multiplication. §§ 3 and 4 are very similar in form; the essential fact is that integrability with respect to μ_Y or $p\mu$ can be immediately deduced from integrability with respect to μ itself.

§ 5, among the most important ones, studies measurable mappings and image measures. Its importance comes from the fact that, with respect to the mappings, Radon measures have a much better behaviour than abstract measures (which may have the counter-property of the image measure catastrophe). Lusin measurable maps are defined, Definition 9; this is specific to Radon measures. Measurability is a local property: Proposition 4. Theorem 5, page 26, gives the relationship between Borel and Lusin measurability; this will be improved later on with Suslin spaces (Chap. II, Theorem 14, page 129). Theorem 6, page 28, is that of Egoroff. Theorem 7, page 28, gives the relationship between measurability and integrability: f is μ-integrable if and only if it is μ-measurable and $\mu^*(\|f\|) < +\infty$; here is a great advantage of Lusin measurability over Borel measurability (Lusin measurability exists only for Radon measures). Then, Lusin measurability must be used for the image; if $H : X \to Y$ is μ-proper (Definition 10), one can define the image measure $H(\mu)$ which is always Radon. Moreover Theorem 8, page 32, holds without analogue in abstract measure; hence its Corollary 1, and Theorem 9, Corollaries 1 and 2 which are very important, permit one in all cases to deduce the measurability and integrability with respect to $H(\mu)$ from the corresponding properties with respect to μ itself. All these things do not hold for abstract measures. With Theorem 10, page 37, we start a new problem: if $H : X \to Y$ is injective, then $\mu \to H(\mu)$ is also injective; Theorems 11 and 12 give conditions for surjectivity.

§ 6 consists of miscellaneous properties of Radon measures. Definition 12, page 40, defines the support; Proposition 5, page 42, gives the limit theorem for an increasing directed family of lower semi-continuous functions; Proposition 6, page 43, expresses $\mu^*(f)$ for $f \geqslant 0$, as the infimum of the $\mu^*(g)$, for g lower semi-continuous

$> f$; (c), page 43, studies moderated measures (a stronger condition than σ-finiteness); (d), page 44, glueing of pieces (the sheaf of measures) (e); page 45, concassage, Theorem 13; (f) Lebesgue-Nikodym theorem, Theorem 14. After that, we give extension lemmas: if an abstract measure is defined on a sub-σ-algebra of the Borel σ-algebra, when is it continuable as a Radon measure on the Borel σ-algebra? Theorem 15, page 48, is due to Choksi, Theorem 16, page 51, to Henry. These two theorems will be used later on, several times, in particular, for tensor products and projective limits.

The measures we studied before were always intended to be ≥ 0. § 7 studies real and complex measures. First of all we give other properties of non-negative measures with respect to the order relation $\mu \leq \nu$, Proposition 7 and Definition 14. This allows us to define the real and complex measures, page 57, and to give their main property: the space of real measures is a complete lattice.

§ 8 shows various possible definitions of Radon measures. We adopted a definition in § 2, but many other equivalent definitions are possible. 1) defines them by using immersions of X in locally compact spaces, if X is completely regular. 2) is more or less the compactology of André Weil; this definition has been given by P.A. Meyer. A Radon measure on X is a coherent system of Radon measures on the compact subspaces. Choquet gave definition 3): a Radon measure on X being a function on the collection of the compact subsets of X.

§ 9 deals with tensor products of Radon measures. X and Y are topological spaces, μ and ν are Radon measures respectively on X and Y, we define $\mu \otimes \nu$ as the unique Radon measure λ such that $\lambda^*(A \times B) = \mu^*(A)\, \nu^*(B)$, for A and B Borel. We give several proofs of existence and uniqueness. Theorem 18, page 65, is Lebesgue-Fubini's theorem for positive functions; we give also several proofs. Theorem 20, page 72, is the same for Banach-valued functions. This section is rather long, and more complicated than one would expect; but finally, the properties are exactly those one expects.

§ 10 introduces the projective limit of a projective system of probabilities. The fundamental theorem is Prokhorov's Theorem 21, page 74, for which we give several proofs: the first one uses the notion of a decreasing projective system; the second one is almost trivial, but assumes the spaces to be completely regular; the third one uses Choksi's extension lemma. Theorem 22, page 81, is a generalization of Prokhorov's theorem which is still more useful in the applications. It will be the main theorem in the theory of cylindrical measures; the study of the latter is the object of Part II.

§ 11 studies the case of the topological non-Hausdorff spaces. The definition of Radon measures is the same, but for replacing the compact subsets by the quasi-compact closed subsets. Most of the previous results hold with only slight modifications. We indicate only what is to be modified, and what does not hold. We have no idea whether this section is useful or not because only very few applications are known.

CHAPTER I

RADON MEASURES: MAIN PROPERTIES AND OPERATIONS

1. Preliminaries. We begin by briefly sketching the development and some of the essential aspects of the abstract and (on locally compact spaces) the Radon theories of measure and integration, and then proceed to give the definition of Radon measures on general Hausdorff topological spaces. The standard reference for the abstract theory of measure is P. R. Halmos [1], (see also the older book of S. Saks [1]) and for the Radon theory on locally compact spaces is N. Bourbaki [1, 2, 3]. The bibliographical note in Halmos and the note historique in Bourbaki [2] give references to earlier work. We will have, by convention, $0 . \infty = 0$ and measures and non-negative valued functions possibly taking the value $+ \infty$.

Let (X, τ, m) be an abstract measure space where τ is a σ-algebra and m a measure. We recall that a σ-algebra in X is a non-void collection of subsets of X, closed for complementation and countable unions (and intersections) and containing the empty set. A *measure* on a σ-algebra is a non-negative valued function (possibly taking the value $+ \infty$), equal to 0 on the empty set (monotone increasing), and countably additive. Let m^* be the outer measure induced by m on the collection of all subsets of X, where $m^*(A) = \inf m(B)$ for all $B \supset A$ and $B \in \tau$. Then m^* is a countably subadditive function with the Fatou property, viz. if $A_n \uparrow A$, then $m^*(A_n) \uparrow m^*(A)$.

DEFINITION 1. *A step function on X is a function which takes only countably many values such that each value is taken on a set belonging to the σ-algebra τ.*

A non-negative real valued step function g is, hence, a countable linear combination with non-negative coefficients (including $+ \infty$) of

characteristic functions of sets in τ, i.e. $g = \sum_{i=1}^{\infty} \alpha_i \chi_{E_i}$. Put $m(g) = \sum_{i=1}^{\infty} \alpha_i m(E_i)$. For a non-negative (possibly infinite) valued function f on X, define the *upper integral*:

$$m^*(f) = \inf \{m(g): g \text{ step function}, g \geqslant f\}.$$

m^* is countably subadditive and has the Fatou property, i.e. if f is the limit of an increasing sequence of non-negative functions f_n, then $m^*(f) = \lim_{n \to \infty} m^*(f_n)$. The function f is said to be *negligible* if $m^*(f) = 0$. A set is *negligible* if its characteristic function is negligible; and we say that a property holds *m-almost everywhere* if it holds except on a negligible set.

Let $\vec{f}$ be a function on X with values in a Banach space $\vec{F}$. Then, $\vec{f}$ is said to be *m-integrable*, if, for every $\epsilon > 0$, there exists an elementary step function $\vec{g}$ with values in $\vec{F}$ (i.e. a step function which takes only finitely many values in $\vec{F}$ and all of them, except perhaps $\vec{0}$, on a set in τ of finite measure), such that, $m^*(\|\vec{f} - \vec{g}\|) \leqslant \epsilon$. This leads to the definition of *m-integral* for *m*-integrable functions and

LEBESGUE'S DOMINATED CONVERGENCE THEOREM. *Let $\{\vec{f}_n\}$ be a sequence of Banach-valued integrable functions such that* (i) *$\vec{f}_n$ converges to $\vec{f}$ m-almost everywhere on X and* (ii) *$\|\vec{f}_n\| < h$ where h is an m-integrable function, for all n. Then $\vec{f}$ is m-integrable and the m-integrals of $\vec{f}_n$ converge to the m-integral of $\vec{f}$.*

The *essential outer measure m^* associated to m* is defined by

$$\bar{m^*}(A) \equiv \sup \{m^*(C) : C \subset A, m^*(C) < +\infty\}.$$

Then, m^* is an outer measure, viz. it is countably subadditive and satisfies the Fatou property. The *essential upper integral*† of any $f \geq 0$ is defined by setting

$$\bar{m^*}(f) \equiv \sup \{m^*(g) : g \leq f,\ m^*(g) < +\infty\}.$$

It can be checked that $\bar{m^*}(f)$ also equals $\sup m^*(f\chi_C)$, χ_C the characteristic function of C with $m^*(C) < +\infty$, C arbitrary or $C \in \tau$. Using the elementary step functions as before, but $\bar{m^*}(\| \vec{f} - \vec{g} \|)$ instead of $m^*(\|\vec{f} - \vec{g}\|)$, we have now the notion of an *m-essentially integrable Banach-valued function* f and its *essential integral.* Clearly, integrability implies essential integrability and then the two integrals coincide. With the notion of m-essentially negligible functions or sets, and m-essentially almost everywhere, we still have Lebesgue's dominated convergence theorem for the essential integral.

The measure m is said to be *σ-finite* if X is a countable union of sets of τ of finite measures. Then $m^* = \bar{m^*}$ and the essential integral coincides with the integral.

DEFINITION 2. *A set $A \subset X$ is said to be m-integrable (resp. m-essentially integrable) if its characteristic function χ_A is m-integrable (resp. m-essentially integrable).*

If $A \in \tau$, it is integrable iff $m(A) < +\infty$; it is essentially integrable iff $\bar{m^*}(A) < +\infty$, or iff

$$\sup \{m(B):\ B \in \tau,\ B \subset A,\ m(B) < +\infty\} < +\infty.$$

The set A is integrable iff there exist $B', B'' \in \tau, B' \subset A \subset B''$ with $m(B') = m(B'') < +\infty$; also iff there exists $B \in \tau$ such that the symmetric difference between A and B is negligible and $m(B) < +\infty$.

The set A is essentially integrable iff there exists $B \in \tau$, such that the symmetric difference between A and B is essentially negligible and $\bar{m^*}(B) < +\infty$; we can also say $m(B) < +\infty$ since every $B \in \tau$

† We shall use a different terminology while discussing the Radon measures on general topological spaces. We shall call this the upper integral (and denote it by $\mu^*(f)$) and the other integral the *strict upper integral* and denote it by $\mu^\bullet(f)$.

such that $m^{\overline{*}}(B) < +\infty$ is the union $B' \cup N$ with $B' \in \tau$, $m(B') < +\infty$, and $m^{\overline{*}}(N) = 0$. But note that the first characterisation of integrable sets probably does not extend to an arbitrary essentially integrable set, i.e. if A is essentially integrable, it is unlikely that there exist $B', B'' \in \tau$ with $B' \subset A \subset B''$ and $m^{\overline{*}}(B') = m^{\overline{*}}(A) = m^{\overline{*}}(B'') < \infty$. Similarly $m^{\overline{*}}(A) = 0$ probably has no implication analagous to the statement that $m^*(A) = 0$ implies there exists $B \in \tau$ with $B \supset A$ and $m(B) = 0$†.

DEFINITION 3. *A set A is said to be m-measurable if its intersection with every m-integrable set C is m-integrable.*

In the above definition we may restrict C to be in τ. The definition of an *m-essentially measurable* set is similar. A set A is m-measurable iff it is m-essentially measurable, i.e. iff for every m-essentially integrable set C, $A \cap C$ is m-essentially integrable. For, assume that A is m-measurable and C is m-essentially integrable. Then there exists a $B \in \tau$, $m(B) < +\infty$, such that the symmetric difference N between B and C is m-essentially negligible; then $A \cap C$ differs from $A \cap B$, which is integrable, by a set contained in $A \cap N$. Now $A \cap N$ is essentially negligible and hence $A \cap C$ is essentially integrable. Conversely, assume that A is m-essentially measurable. If C is integrable, $A \cap C$ is essentially integrable and is contained in C which is integrable and therefore $A \cap C$ is integrable.

The m-measurable sets form a σ-algebra τ_m containing τ and m^* is a measure on τ_m extending m. (Note that this is the familiar Lebesgue completion of m on (X, τ)). Moreover, if we do the same process of extension for m^* on (X, τ_m), the resulting σ-algebra is τ_m itself. $m^{\overline{*}}$ is also a measure on τ_m.

DEFINITION 4. *If $(Y, \mathscr{S})$ is a measure space and f a mapping $f: X \to Y$, then f is said to be $(m, \mathscr{S})$-measurable, if $f^{-1}(B)$ belongs to τ_m, for every $B \in \mathscr{S}$‡.*

†In other words, $m^{\overline{*}}$ is perhaps not the Lebesgue continuation of its restriction to τ.

‡ We shall say usually m-measurable (or even measurable) when there is no confusion possible about the underlying σ-algebra $\mathscr{S}$ (or the measure m).

Suppose $\vec{f} : X \to \vec{F}$, where $\vec{F}$ is a separable Banach space. Then, $\vec{f}$ is m-integrable (resp. m-essentially integrable) if and only if $\vec{f}$ is measurable and $m^*(\|\vec{f}\|) < +\infty$ (resp. $\overline{m^*}(\|\vec{f}\|) < +\infty$).

Let, now, X be a locally compact Hausdorff space and μ a positive linear functional on the space $\mathscr{K}$ of continuous functions with compact support. Then μ induces an abstract measure μ^* on τ_β, the Borel algebra of X. First, we define, for any non-negative lower semi-continuous function ψ on X,

$$\mu^*(\psi) = \sup \{\mu(f) : f \in \mathscr{K},\ 0 \leqslant f \leqslant \psi\}.$$

Then, for any non-negative valued function ϕ, we set,

$$\mu^*(\phi) = \inf \{\mu^*(\psi) : \psi \geqslant \phi,\ \psi \text{ l.s.c.}\}.$$

The function μ^* defined on the set of all subsets of X (by setting for any $A \subset X$, $\mu^*(A) = \mu^*(\chi_A)$) is an outer measure. Now, the μ^*-integrable functions and their integrals are defined, more or less as in the abstract case, except that the elementary step functions are replaced by continuous functions with compact support.

Let $\overline{\mu^*}$ be the essential measure associated to μ^*. It is well known (see Bourbaki [1,2]) that the following five properties are verified by $\overline{\mu^*}$ and μ^* (on τ_β).

1) $\overline{\mu^*}$ is the essential measure associated to μ^*.

2) μ^* is outer regular, i.e. for any $B \in \tau_\beta$,

$$\mu^*(B) = \inf \{\mu^*(V) : V \text{ open},\ V \supset B\}.$$

3) $\overline{\mu^*}$ is inner regular, i.e. for any $B \in \tau_\beta$,

$$\overline{\mu^*}(B) = \sup \{\mu^*(K) : K \text{ compact},\ K \subset B\}.$$

4) μ^* is locally finite, i.e. every point of X has a neighbourhood which has a finite μ^*-measure.

5) $\mu^*(B) = \overline{\mu^*}(B)$ if B is open or if $\mu^*(B) < +\infty$.

Conversely, if M and m are two measures on τ_β satisfying the above five properties, then there exists a unique Radon measure

μ on X such that $\mu^*=M$ and $\mu^{*-}=m$. This is seen as follows. Consider the integral with respect to M. In view of the condition 4, the integral is finite for every $f\in\mathscr{K}$. Hence, we have a positive linear functional μ on $\mathscr{K}$, defined by $\mu(f)=M(f)$; i.e. μ is a Radon measure. We shall show that $\mu^*=M$. For this it is enough to show that μ^* and M coincide on all open sets. Let V be any open set. Then

$$\begin{aligned}\mu^*(V) &= \sup\{\mu(f): f\in\mathscr{K},\ 0\leqslant f\leqslant\chi_V\}\\ &= \sup\{M(f): f\in\mathscr{K},\ 0\leqslant f\leqslant\chi_V\}.\\ &\leqslant M(V).\end{aligned}$$

But
$$\begin{aligned}M(V) &= m(V)\\ &= \sup\{m(K): K\subset V,\ K \text{ compact}\}\\ &\leqslant \sup\{M(f): f\in\mathscr{K},\ 0\leqslant f\leqslant\chi_V\}\\ &= \mu^*(V),\end{aligned}$$

since, for every compact set $K\subset V$, there exists an $f\in\mathscr{K}$ satisfying $0\leqslant\chi_K\leqslant f\leqslant\chi_V$. The uniqueness of μ is evident; for, the value of μ is known on all Borel sets, therefore on step functions and hence, by passage through uniform limits, on all continuous functions with compact support.

2. Definition of Radon measure on general topological spaces. Let us now go on to define Radon measures on general Hausdorff topological spaces. Let X be a Hausdorff topological space and τ_β its Borel algebra.

DEFINITION R_1. *A Radon measure μ on X is a pair of measures M and m on τ_β satisfying the following five properties:*

1) *m is the essential measure associated with M;*

2) *M is outer regular on τ_β, i.e. for every $B\in\tau_\beta$,*

$$M(B)=\inf\{M(V): V\supset B,\ V \text{ open}\};$$

3) *M is locally finite, i.e. every point has a neighbourhood which has a finite M-measure. Consequently every compact set has a neighbourhood of finite M-measure, and therefore has itself finite M-measure;*

(4) *m is inner regular on τ_β, i.e. for every $B \in \tau_\beta$,*

$$m(B) = \sup \{m(K): K \subset B, K \text{ compact}\};$$

and (5) *$m(B) = M(B)$ whenever B is open or $M(B) < +\infty$.*

This definition is equivalent to either of the following two definitions.

DEFINITION R_2. *A Radon measure on X is a measure M on τ_β satisfying the following three conditions:*

(1) *M is locally finite;*

(2) *M is outer regular (as in (2) above);*

and (3) *for every open set V,*

$$M(V) = \sup \{M(K): K \subset V, K \text{ compact}\}.$$

DEFINITION R_3. *A Radon measure on X is a measure m on τ_β satisfying the conditions:*

(1) *m is locally finite;*

and (2) *m is inner regular (as in (4) of R_1).*

EQUIVALENCE OF R_1, R_2 AND R_3.

It is clear that if μ is a Radon measure on X, according to the definition R_1, then M (resp. m) is a Radon measure according to R_2 (resp. R_3).

$R_3 \Rightarrow R_1$. Let m be a measure satisfying R_3. Define, for every $B \in \tau_\beta$,

$$M(B) = \inf \{m(V): V \text{ open}, V \supset B\}.$$

Then M is a measure on τ_β and the pair (M, m) satisfies R_1.

PROOF. (i) It is trivial to see that M is monotone increasing and countably subadditive and $M \geqslant m$.

(ii) $M(V) = m(V)$ for all open sets (obvious). Also, for any $B \in \tau_\beta$ with $M(B) < +\infty$, $M(B) = m(B)$. For, let V be an open set containing B such that $m(V) < +\infty$. From the inner regularity of m and the fact that $m(V - B) < +\infty$, for every $\epsilon > 0$, we can find a compact set $K_\epsilon \subset V - B$, such that, $m(V - B) < m(K_\epsilon) + \epsilon$. Now, $V'_\epsilon = V - K_\epsilon$ is open and contains B; further,

$$M(B) < m(V'_\epsilon) = m(B) + m(V - B) - m(K_\epsilon)$$
$$< m(B) + \epsilon.$$

From this it follows that $M(B) = m(B)$, if $M(B) < +\infty$.

(iii) Let $B_n \in \tau_\beta$ and $\{B_n\}$ be a mutually disjoint sequence. Let $B = \bigcup_{n=1}^{\infty} B_n$. If $M(B) = +\infty$, then the countable subadditivity shows that $M(B) = \sum_{n=1}^{\infty} M(B_n)$. On the other hand if $M(B) < +\infty$, so is $M(B_n)$, for every n, and we get, by (iii),

$$M(B) = m(B) = \sum_{n=1}^{\infty} m(B_n) = \sum_{n=1}^{\infty} M(B_n).$$

Hence M is a measure on τ_β. It is outer regular and locally finite.

(iv) For every $B \in \tau_\beta$,

$$\begin{aligned} m(B) &= \sup \{m(K) : K \text{ compact}, K \subset B\} \\ &= \sup \{M(K) : K \text{ compact}, K \subset B\}^\dagger \\ &= \sup \{M(B') : B' \subset B, M(B') < +\infty\}. \\ &= \overline{M^*}(B). \end{aligned}$$

That is, m is the essential measure associated to M.

Hence, the pair (M, m) is a Radon measure (according to R_1) on X. The proof is complete.

Note that the measure M defined in this way is unique, as it is known on all open sets and hence, by outer regularity, on all Borel sets.

$R_2 \Rightarrow R_1$. Let M be a measure on τ_β satisfying the conditions of R_2. Then, M and the associated essential measure m constitute a pair satisfying R_1.

PROOF. Let B be a Borel set with $M(B) < +\infty$. Then, $M(B) = \sup \{M(K) : K \subset B$, and K compact$\}$. For, by the outer regularity of M, there exists an open set $W \supset B$ for which

† Because every compact K is contained in an open set of finite measure, and therefore, by (ii), $m(K) = M(K)$.

$M(W) < +\infty$. But, from (3), there exists a sequence of compact sets K_n such that $K_n \subset W$, for every n, and $\lim_{n\to\infty} M(K_n) = M(W)$.

Hence, $M(W - K_n) \to 0$, as $M(W) < +\infty$. Hence,

$$M(B) = \lim_{n\to\infty} M(B \cap K_n).$$

We get

$$M(B) = \sup \{M(B') : B' \in \tau_\beta,\ B' \subset B, B' \text{ relatively compact}\}. \quad \text{(I)}$$

Now, let A be any relatively compact set such that $A \subset B$ and $A \in \tau_\beta$. If $K = \bar{A}$, since M is outer regular, we have

$$\begin{aligned} M(K - A) &= \inf \{M(V) : V \text{ open},\ V \supset K - A\} \\ &= \inf \{M(V \cap K) : V \text{ open},\ V \supset K - A\}. \end{aligned}$$

That is,

$$M(K - A) = \inf \{M(W') : W' \supset K - A,\ W' \text{ relatively open in } K\}.$$

Since $M(K) < +\infty$, K being a compact set, we deduce immediately that

$$M(A) = \sup \{M(K') : K' \text{ relatively closed in } K, \text{ i.e. } K' \text{ compact and } K' \subset A\}. \quad \text{(II)}$$

From (I) and (II), we deduce the required inner regularity for m in the case $M(B) < +\infty$; if $M(B) = +\infty$, it is trivial because $m(B) = \sup \{M(B'),\ B' \subset B,\ M(B') < +\infty\}$. Moreover we get immediately that $M = m$ on open sets and sets with M-measure finite. The proof is complete.

The three definitions being equivalent, we may use any one of them to define Radon measures on general topological spaces. The definition by m is the simplest one. But, we note the fact that, if we start with m (satisfying R_3), it is necessary to construct M in order to develop the theory of integration. For, it is M^* (resp. $\overline{M^*}$) which decides the integrability (resp. essential integrability) of functions and sets. We shall give later on (p. 45) an example of a measure on the Borel σ-algebra τ_β of a topological (in fact, compact) space X which is finite but which is not inner regular. This measure, therefore, fails to satisfy all the conditions of R_3

and cannot be used to define a Radon measure. It follows that on this space X, the set of measures on τ_β which satisfy the conditions of R_3 is a strict subset of the set of locally finite measures on τ_β.

In the sequel, it will be the essential upper integral which will play the main role, instead of the upper integral; for this reason, *we shall adopt the shorter name for the more frequently used concept and make a change from the usual terminology and notation.* We shall call the *upper integral* of a positive function f, with respect to a Radon measure μ, what is usually called the essential upper integral, that is $M^{\bar{*}}(f)$; we shall denote it by $\mu^\cdot(f)$. And we shall call *strict upper integral* what is usually called the upper integral, that is $M^*(f)$, and denote it by $\mu^*(f)$. A function $\vec{f}$ having values in a Banach space will be said to be *μ-integrable* (called in Bourbaki μ-essentially integrable), if it is M-essentially integrable, and its *μ-integral* will be its M-essential integral: it is *μ-strictly integrable* (in Bourbaki μ-integrable) if it is M-integrable and its *μ-strict integral* is its M-integral.

A Radon measure is said to be *finite* if M (or m) is a finite measure, i.e. if $M(X) < +\infty$ (or $m(X) < +\infty$); it follows that m (or M) is also finite and $M = m$ since X is open and so $M(X) = m(X)$. μ is said to be *σ-finite* if m is σ-finite, i.e. if there exist sets X_n, $n = 1, 2, 3, \ldots$, such that $X = \bigcup_{n=1}^{\infty} X_n$ and $m(X_n) < +\infty$ for each n. μ is said to be *moderated* if M is σ-finite, equivalently if there exist open sets $X_n, n = 1, 2, 3 \ldots$, such that $X = \bigcup_{n=1}^{\infty} X_n$ and $M(X_n)\,(= m(X_n)) < +\infty$.

We always have for any set

$$M^{\bar{*}} \leq m^* \leq M^*.$$

The second inequality is trivial. To prove the first, if B is any Borel set such that $M^*(B) < \infty$, then $m^*(B) = M^*(B)$ and so $M^*(A \cap B) = m^*(A \cap B) \leq m^*(A)$ for any set A. Hence

$$\begin{aligned} M^{\bar{*}}(A) &= \sup \{M^*(A \cap B) : B \text{ Borel}, M^*(B) < \infty\} \\ &\leq m^*(A). \end{aligned}$$

If μ is σ-finite then $M^{*}=m^{*}$ for any set. For in that case $X=(\bigcup_{n=1}^{\infty} K_n)\cup N$ where K_n is an increasing sequence of compact sets and N is Borel with $m(N)=0$. If A is any set then

$$m^{*}(A)=\lim_{n\to\infty} m^{*}(A\cap K_n)=\lim_{n\to\infty} M^{*}(A\cap K_n)\ (\leq \text{ hence})=M^{*}(A).$$

We do not know of examples for which $m^{*}\neq M^{*}$, but we should expect them to exist.

If μ is moderated then $M^{*}=m^{*}=M^{*}$ for any set and $\mu^{*}=\mu^{\cdot}$, the theory then becomes much less complicated. This occurs in many interesting cases.

An example where $m^{*}\neq M^{*}$ (and in fact where μ is σ-finite but not moderated) is given in Bourbaki [1] page 234, Exercises of 1, No. 5.

It is easy to give examples of Radon measures which are not σ-finite, e.g. the measure induced on the topological sum of uncountably many copies of the line by taking Lebesgue measure on each.

DEFINITION 5. *A set A is said to be μ-measurable if it is either M-measurable or m-measurable.*

(We recall that we have already shown that the measurable and essentially measurable sets coincide for an abstract measure and so the M-measurable and m-measurable sets coincide).

Note that if μ is σ-finite, then a set A is μ-measurable if and only if there exist Borel sets $B'\subset A\subset B''$ such that $\mu^{\cdot}(B''-B')=0$.

The μ-measurable sets form a σ-algebra τ_{μ} on which μ^{*} and $\mu^{\cdot}$ are still measures, $\mu^{\cdot}$ is the essential measure associated with μ^{*}. Clearly, μ^{*} is outer regular. It is not quite so obvious that $\mu^{\cdot}$ is inner regular on τ_{μ}; we show this in Proposition 1.

LEMMA 1. *For an arbitrary subset A of X,*

$$\mu^{\cdot}(A)=\sup\{\mu^{\cdot}(A\cap K) : K\subset X,\ K \text{ compact}\}.$$

PROOF. We observe first that $\mu^{\cdot}(A)=\sup\{\mu^{\cdot}(A\cap B) : B\in\tau_{\beta},\ \mu^{*}(B)<+\infty\}$. For all the subsets of B, $\mu^{*}=M^{*}$ and

$\mu^\bullet = \overline{M^*}$ coincide. Now, B is a union of an increasing sequence $\{K_n\}$ of compact sets and N, a μ^*-negligible set, by the inner regularity of $\mu^\bullet = \mu^*$ on B. Then $\mu^\bullet(A \cap B) = \sup_n \mu^\bullet(A \cap K_n)$ and this proves the lemma.

PROPOSITION 1. *The measure $\mu^\bullet$ is inner regular on the σ-algebra τ_μ of μ-measurable sets:*

$$\mu^\bullet(A) = \sup \{\mu^\bullet(K) : K \text{ compact}, K \subset A\}.$$

PROOF. By the above lemma, we need only prove the proposition when A is relatively compact and measurable. But then A is integrable; hence, $A = B' \cup N$ where $B' \in \tau_\beta$ and $\mu^\bullet(N) = 0$. We observe that $\mu^\bullet$ is inner regular for B' and this completes the proof.

REMARK 1. $A \subset X$ is μ-measurable iff for every compact set $K \subset X$, $A \cap K$ is measurable.

For, assume that $A \cap K$ is measurable for every compact set $K \subset X$. Let $B \in \tau_\beta$ with $\mu^*(B) < +\infty$. Then B is the union of an increasing sequence of compact sets K_n and a set N such that $\mu^*(N) = 0$. All the $A \cap K_n$ and $A \cap N$ are μ-strictly integrable (i.e. M-integrable), therefore, so is $A \cap B$. Hence A is μ-measurable.

REMARK 2. Measurability is a local property. More precisely, $A \subset X$ is μ-measurable iff, for every $x \in X$, there is an open neighbourhood V_x such that $A \cap V_x$ is μ-measurable.

Suppose $A \subset X$ is such that, for every $x \in X$, there is an open neighbourhood V_x with $A \cap V_x$ μ-measurable. Then, for every compact set $K \subset X$, K is covered by a finite number of the V_x: therefore, $A \cap K$ is a finite union of measurable subsets of the form $A \cap K \cap V_x$. Hence $A \cap K$ is μ-measurable. It follows by Remark 1, that A is μ-measurable.

DEFINITION 6. *A set $N \subset X$ is μ-negligible (in Bourbaki μ-locally negligible) if $\mu^\bullet(A) = 0$. It is μ-strictly negligible (in Bourbaki μ-negligible) if $\mu^*(N) = 0$.*

Note that

(1) N is μ-negligible iff, for every compact set $K \subset X$, $N \cap K$ is μ-negligible. (Trivial consequence of Lemma 1).

(2) N is μ-negligible iff, for every $x \in X$, there exists an open neighbourhood V_x of x such that $N \cap V_x$ is μ-negligible.

The proof is as for Remark 2 above. Thus μ-negligibility is a local property.

Since μ^* is the more frequently used concept it is desirable to be able to construct the theory of integration from m alone, without having to go through M. If a function $f \geqslant 0$ is carried by a compact set (i.e. f is zero outside this compact set),

$$\mu^*(f) = \bar{M^*}(f) = M^*(f) = m^*(f) = \inf \{m(g) : g \text{ step function}, g \geqslant f\}$$

And, it can be seen as in Lemma 1, that for any $f \geqslant 0$

$$\mu^*(f) = \bar{M^*}(f) = \sup \{\mu^*(f \chi_K) : K \text{ compact}\}.$$

Finally, a function $\vec{f}$ with values in a Banach space $\vec{F}$ is μ-integrable (we recall that 'μ-integrable' here is the same as 'μ-essentially integrable' according to Bourbaki), if, for every $\epsilon > 0$, there exists a step function $\vec{g}$, which takes only finitely many values and all of them except for $\vec{0}$ on Borel sets of finite measure (or even compact sets) such that $\mu^*(\| \vec{f} - \vec{g} \|) \leqslant \epsilon$ and the integral of $\vec{f}$ is then defined in the obvious way. A subset A of X is μ-measurable if it is m-measurable, i.e. for every compact set $K \subset X$, $K \cap A$ is m-integrable.

DEFINITION 7. *A mapping $f : X \to Y$, where $\mathscr{S}$ is the Borel σ-algebra of Y, is said to be Borel μ-measurable, if the inverse image of every set in $\mathscr{S}$ is μ-measurable.*

DEFINITION 8. *A hereditary family $\mathscr{F}$ of compact sets of X, closed under finite unions, is said to be μ-dense if every compact set of X is the union of a μ-negligible set and an element of $\mathscr{F}_\sigma$, where $\mathscr{F}_\sigma$ is the family of all countable unions of elements of $\mathscr{F}$.*

The following proposition is proved easily.

PROPOSITION 2. *For every positive function* f, $\mu^*(f) = \sup\{\mu^*(f\chi_K), K \in \mathscr{F}\}$, *where* $\mathscr{F}$ *is a* μ*-dense family.*

3. **Induced measures.** Let us first consider the case of an abstract measure (X, τ, m). Let $Y \subset X$ be an m-measurable set. Let τ^Y be the induced σ-algebra on Y, i.e. $\tau^Y = \{B \cap Y : B \in \tau\}$. Then, τ^Y is precisely the Borel σ-algebra of Y. The measure m induces a measure m_Y on (Y, τ^Y); m_Y is defined by setting

$$m_Y(B) = m(B), \text{ for all } B \in \tau^Y.$$

(Note that any $B \in \tau^Y$ is necessarily m-measurable). Then, the following properties can be verified : for any $A \subset Y$,

$$\left.\begin{aligned} (m_Y)^*(A) &= m^*(A), \\ (m_Y)^{\bar{*}}(A) &= m^{\bar{*}}(A). \end{aligned}\right\} \qquad (\alpha)$$

And, for any $f \geqslant 0$ and carried by Y (i.e. zero outside Y),

$$\left.\begin{aligned} (m_Y)^*(f) &= m^*(f), \\ (m_Y)^{\bar{*}}(f) &= m^{\bar{*}}(f). \end{aligned}\right\} \qquad (\beta)$$

The requirement that Y be m-measurable is actually superfluous. For *any* $Y \subset X$, τ^Y defined as above is the Borel σ-algebra of Y. For $B \in \tau^Y$ the function m_Y defined by

$$m_Y(B) = m^*(B)$$

is countably additive, and so defines an abstract measure on Y. m_* is also countably additive on τ^Y, this defines another induced abstract measure on Y. We use these measures later on in the proof of Theorem 16. But the situation is slightly different in the case of Radon measures.

Let us now take a Radon measure $\mu[= (M, m)]$ on X. The measure M_Y induced by M need not satisfy the conditions of R_2. For, if this were true, we should get in general,

$$\begin{aligned} M^*(Y) = (M_Y)^*(Y) &= \sup\{M_Y^*(K) : K \subset Y, K \text{ compact}\} \\ &= \sup\{M^*(K) : K \subset Y, K \text{ compact}\} = M^{\bar{*}}(Y). \end{aligned}$$

On the other hand, the measure induced on Y by m may not satisfy the inner regularity condition stipulated in R_3, as this will lead to the conclusion that $m^*(Y) = \sup\{m(K): K \text{ compact } \subset Y\} = \bar{M}^*(Y)$, which is in general not true. Hence, we shall give for Radon measures a special definition based on $\mu^\cdot$; *and the measure will not be directly induced by either of M and m.*

Let Y be a μ-measurable set; we define,

$$(\mu_Y)^\cdot(B) = \mu^\cdot(B)$$

for every Borel set B of Y. This function $B \to (\mu_Y)^\cdot(B)$, on the Borel algebra of Y, is the one induced by $\bar{M}^*$ on the σ-algebra of all μ-measurable sets of X; and is hence, locally finite and inner regular. It follows that $(\mu_Y)^\cdot$ defines a Radon measure on Y. We shall now prove the properties similar to (α) and (β) above, *but only for $\mu^\cdot$, not for μ^*.*

LEMMA 2. *Let $\mu^*(Y) < +\infty$. Then, for any $A \subset Y$, and any $f \geq 0$ and carried by Y, the following hold good:*

$$(\mu_Y)^*(A) = \mu^*(A), \quad (\mu_Y)^*(f) = \mu^*(f)$$

and

$$(\mu_Y)^\cdot(A) = \mu^\cdot(A), \quad (\mu_Y)^\cdot(f) = \mu^\cdot(f).$$

PROOF. This is true because $\mu^\cdot$ and $(\mu_Y)^\cdot$ coincide with M^* and M_Y^* respectively and the rest follows as in the case of abstract measures.

LEMMA 3. *Let Y and Z be subsets of X such that* (i) *Y is a μ-measurable set and* (ii) *$Z \subset Y$ and Z is Borel in Y. Then $(\mu_Y)_Z = \mu_Z$.*

PROOF. If B is any Borel subset of Z, then B is also Borel in Y and hence,

$$(\mu_Z)^\cdot(B) = \mu^\cdot(B) = (\mu_Y)^\cdot(B) = ((\mu_Y)_Z)^\cdot(B).$$

Since this is true for every Borel subset B of Z, the assertion of the lemma follows.

THEOREM 1. *Let Y be a μ-measurable set. Then for any $A \subset Y$ and any $f \geq 0$ carried by Y, the following hold good:*

$$(\mu_Y)^\cdot(A) = \mu^\cdot(A) \text{ and } (\mu_Y)^\cdot(f) = \mu^\cdot(f).$$

Further, for any Banach-valued function $\vec{f}$ on X, which is $\vec{0}$ outside Y, μ_Y-integrability and μ-integrability are equivalent; and in the case of such an integrable function, the two integrals are equal.

PROOF. Let $f > 0$ be a function on X, carried by Y. Then

$$\begin{aligned}(\mu_Y)^\bullet(f) &= \sup\{(\mu_Y)^\bullet(f\chi_K): K \subset Y, K \text{ compact}\}\\ &= \sup\{[(\mu_Y)_K]^\bullet(f\chi_K): K \subset Y, K \text{ compact}\} \quad \text{(Lemma 2)}\\ &= \sup\{(\mu_K)^\bullet(f\chi_K): K \subset Y, K \text{ compact}\} \quad \text{(Lemma 3)}\\ &= \sup\{\mu^\bullet(f\chi_K): K \subset Y, K \text{ compact}\} \quad \text{(Lemma 2)}\\ &\leq \mu^\bullet(f).\end{aligned}$$

On the other hand, let V be an open set such that $\mu^*(V) < +\infty$. Then obviously $\mu^*(V \cap Y) < +\infty$, but, also, $(\mu_Y)^*(V \cap Y) < +\infty$. For, $V \cap Y$ is open in Y and

$$(\mu_Y)^*(V \cap Y) = (\mu_Y)^\bullet(V \cap Y) = \mu^\bullet(V \cap Y) \leq \mu^*(V) < +\infty.$$

Now, since $\mu^\bullet(f) = \sup\{\mu^\bullet(f\chi_K): K \text{ compact}\}$ and $\sup\{\mu^\bullet(f\chi_V): V$ open with $\mu^*(V) < +\infty\}$ lies between $\mu^\bullet(f)$ and $\sup\{\mu^\bullet(f\chi_K): K$ compact$\}$, we get

$$\begin{aligned}\mu^\bullet(f) &= \sup\{\mu^\bullet(f\chi_V): V \text{ open with } \mu^*(V) < +\infty\}\\ &= \sup\{(\mu_{V\cap Y})^\bullet(f\chi_V): V \text{ open with } \mu^*(V) < +\infty\} \quad \text{(Lemma 2)}\\ &= \sup\{[(\mu_Y)_{V\cap Y}]^\bullet(f\chi_V): V \text{ open with } \mu^*(V) < +\infty\} \quad \text{(Lemma 3)}\\ &= \sup\{\mu^\bullet{}_Y(f\chi_V): V \text{ open with } \mu^*(V) < +\infty\} \quad \text{(Lemma 2)}\\ &\leq (\mu_Y)^\bullet(f).\end{aligned}$$

This shows that $(\mu_Y)^\bullet(f) = \mu^\bullet(f)$, for any such f. The rest of the proof is completed easily.

The following statement can be easily verified and we omit the proof.

A real-valued function f, taking a unique value outside Y, is μ-measurable iff it is μ_Y-measurable.

The following theorem completes the statement of Lemma 3.

THEOREM 2. (Transitivity of induced measures). *For any two measurable sets Y and Z such that $X \supset Y \supset Z$, the induced measures satisfy $(\mu_Y)_Z = \mu_Z$.*

PROOF. Let B be any Borel subset of Z. Then, we have by Theorem 1

$$[(\mu_Y)_Z]^*(B) = (\mu_Y)^*(B) = \mu^*(B) = (\mu_Z)^*(B).$$

It follows that $\mu_Z = (\mu_Y)_Z$, completing the proof.

4. Product of a measure by a locally integrable function. Let τ be a σ-algebra on a set X and m an abstract measure on (X, τ). Let $p > 0$ be a measurable function. Let us define the measure pm on τ by setting

$$(pm)^*(B) = m^*(p\chi_B).$$

Then this measure satisfies, for every non-negative valued function f on X,

$$(pm)^*(f) = m^*(pf).$$

However, in general, $(pm)^{\overline{*}}(f) \neq m^*(pf)$. This is certainly true, if we assume, moreover, that the function p is finite m-essentially almost everywhere. We shall take for granted these results of the abstract case.

Let us now consider a Radon measure μ on a topological space X. Let $p > 0$ be a measurable function which is locally integrable. We remark, firstly, that the couple (pM, pm) of measures does not verify the definition R_1. The measure pM does not satisfy R_2; as this will lead to the conclusion, $M^*(p) = \sup\{M(p\chi_K)\colon K \text{ compact } \subset X\} = M^*(p)$, which is not true in general. On the other hand, the measure pm is not, in general, inner regular, as that will make us conclude that $m^*(p) = M^*(p)$. Once again we give for Radon measures a special definition by taking μ^*, viz.

$$(p\mu)^*(B) = \mu^*(p\chi_B) \; (= M^{\overline{*}}(p\chi_B)).$$

This measure is locally finite (since p is locally integrable) and inner regular since

$$\mu^*(p\chi_B) = \sup \{\mu^*(\chi_K\, p\chi_B) : K \text{ compact}\}$$

Thus $p\mu$ comes neither from pM nor from pm.

We have the following two results as in the case of induced measures (*only for* μ^*, *not for* μ^* !).

THEOREM 3. *For any set* A *and any non-negative valued function* f, (i) $(p\mu)^*(A) = \mu^*(p\chi_A)$ *and* (ii) $(p\mu)^*(f) = \mu^*(pf)$. *Further, a function* $\vec{f}$ *with values in a Banach space* $\vec{B}$ *is* $p\mu$*-integrable if and only if* $p\vec{f}$ *is* μ*-integrable; further, in case of integrability, the two integrals are equal. The same result holds for the measurability of a real-valued function* f.

PROOF.. Let $f > 0$ be any function on X. The required relation, $(p\mu)^*(f) = \mu^*(pf)$, is a result of the abstract case, when the space X is compact, as in such a situation $p\mu$ will be nothing but pM.

Suppose Y is any Borel set of X. Then $p\mu_Y = (p\mu)_Y$. For, if $B \subset Y$ is any Borel set, then,

$$\begin{aligned}(p\mu_Y)^*(B) &= (\mu_Y)^*(p\chi_B)\\ &= \mu^*(p\chi_B) = (p\mu)^*(\chi_B) \qquad \text{(Theorem 1)}\\ &= [(p\mu)_Y]^*(B).\end{aligned}$$

Hence, if the function considered is carried by a compact set of X, then the required equality follows by considering the induced measures. In the general case,

$$\begin{aligned}(p\mu)^*(f) &= \sup \{(p\mu)^*(f\chi_K) : K \text{ compact}\}\\ &= \sup \{\mu^*(pf\chi_K) \;\; : K \text{ compact}\}\\ &= \mu^*(pf).\end{aligned}$$

The rest of the proof is completed easily.

THEOREM 4. (Transitivity of products). *Let* q *be a locally* $p\mu$*-integrable function (and this is equivalent to local* μ*-integrability of* pq). *Then the two measures* $q(p\mu)$ *and* $(pq)\mu$ *are identical.*

PROOF. This follows since, for any Borel set B,

$$[q(p\mu)]^*(B) = (p\mu)^*(q\chi_B) \qquad \text{(definition)}$$
$$= \mu^*(pq\chi_B) \qquad \text{(Theorem 3)}$$
$$= [(pq)(\mu)]^*(B) \qquad \text{(definition).}$$

This completes the proof.

PROPOSITION 3. (Compatibility between product and induction). *If Y is a μ-measurable set, then for any locally μ-integrable function p, Y is $p\mu$-measurable, and*

$$(p\mu)_Y = p\mu_Y.$$

This is proved easily by considering the values of either side on any set.

5. Measurable mappings and image measures. Let μ be a Radon measure on X. Let $H: X \to Y$ where Y is a Hausdorff topological space.

DEFINITION 9. *The mapping H is said to be Lusin μ-measurable if, for every compact set $K \subset X$ and every $\delta > 0$, there exists a compact set $K_\delta \subset K$ with $\mu(K - K_\delta) < \delta$ and such that H restricted to K_δ is continuous.*

We recall here the weaker notion of Borel μ-measurable functions: H is Borel μ-measurable if, the inverse image under H of every Borel set in Y, is a μ-measurable set.

A mapping H is Lusin μ-measurable iff for every compact set $K \subset X$ the restriction $H|K$ is Lusin μ_K-measurable. Therefore, a function whose restriction to every compact set is continuous, is Lusin measurable for every Radon measure.

PROPOSITION 4. (Measurability is a local property) *The function $H : X \to Z$ is Lusin μ-measurable if for every $x \in X$, there exists an open neighbourhood V_x of x such that the restriction $H|V_x$ is Lusin measurable for the induced measure μ_{V_x}.*

PROOF. First of all if H is Lusin μ-measurable, and if Y is a μ-measurable subset, then the restriction $H|Y$ is trivially Lusin μ_Y-measurable; therefore the Lusin μ-measurability of H implies

the Lusin μ_V-measurability of $H|V$, for all the open neighbourhoods V of x.

Assume now that the local property holds and $\delta > 0$. Let K be a compact subset. There is a finite number of elements, say $x_1, x_2, \ldots, x_n$, such that $\bigcup_{i=1}^{n} V_{x_i} \supset K$. The restriction of H to $V_{x_i} \cap K$ is Lusin $\mu_{V_{x_i} \cap K}$-measurable, therefore there exists a compact $K_i \subset V_{x_i} \cap K$, such that $\mu(V_{x_i} \cap K - K_i) \leqslant \delta/n$, and $H \mid K_i$ is continuous. If we put $K_\delta = \bigcup_{i=1}^{n} K_i$ we have $\mu(K - K_\delta) \leqslant \delta$. Further, the space K_δ is the finite union of compact subsets K_i; the restriction of H to each K_i is continuous; hence, the restriction of H to K_δ is continuous. It follows that H is Lusin μ-measurable, completing the proof.

REMARK 1. By inner regularity, H is Lusin μ-measurable iff for every μ-measurable subset A with $\mu^*(A) < +\infty$, and $\delta > 0$ there exists a compact set $A_\delta \subset A$ such that $\mu^*(A - A_\delta) \leqslant \delta$ and $H \mid A_\delta$ is continuous.

REMARK 2. H is Lusin μ-measurable iff the family $\mathscr{F}$ of the compact sets $K \subset X$ such that $H \mid K$ is continuous is μ-dense (cf. Definition 8).

THEOREM 5. *Let $H : X \to Y$. If H is Lusin μ-measurable, then H is Borel μ-measurable, and conversely, if Y is metrizable and separable, then every Borel μ-measurable function is also Lusin μ-measurable.*

PROOF. Let $H : X \to Y$ be Lusin μ-measurable. It is enough to show that the inverse image $H^{-1}(F)$ of every closed set $F \subset Y$ is μ-measurable. For this it is enough if we show that, for every compact set $K \subset X$, the set $K \cap H^{-1}(F)$ is μ-measurable. Now, the Lusin μ-measurability of H gives us that K is the union $\bigcup_{n=1}^{\infty} K_n \cup N$, where K_n are compact sets and N is a μ-negligible set such that, $H \mid K_n$, for each n, is continuous. Hence, $H^{-1}(F) \cap K =$

$(H^{-1}(F) \cap N) \cup (\bigcup_{n=1}^{\infty} K_n \cap H^{-1}(F))$; and, for each n, $H^{-1}(F) \cap K_n$ is compact. It follows that $H^{-1}(F) \cap K$ is μ-measurable. Hence, H is Borel μ-measurable.

Conversely, let H be a Borel μ-measurable mapping of $X \to Y$, where Y is a separable and metrizable space. The following steps prove the converse.

(1) If Y is separable and metrizable then, every Borel μ-measurable function with values in Y is the uniform limit of a sequence of step-functions with values in Y. For, given any $\epsilon > 0$, Y is the union of a countable number of balls $\{B_n\}_{n=0}^{\infty}$ of radii ϵ. Let, for every n, $A_n = B_n \cap \mathsf{C}(\bigcup_{i<n} B_i)$. Let $E_n = H^{-1}(A_n)$. Then, the step functions f_ϵ on X, $f_\epsilon(x) = y_n$ where y_n is an arbitrary element in A_n for every n, approximate to H uniformly on X.

(2) If Y is metrizable then the uniform limit of a sequence of Lusin μ-measurable functions is Lusin μ-measurable.

The proof of the converse is completed by the following

LEMMA 4. *Suppose Y is any Hausdorff topological space and $f: X \to Y$ is a step function. Then f is Lusin μ-measurable.*

PROOF. Let $\{b_n\}$ be the sequence of distinct values that the function f takes in Y. Let $E_n = f^{-1}(\{b_n\})$. Let K be any compact subset of X and $\epsilon > 0$. We can find a positive integer N such that

$$\sum_{N=1}^{\infty} \mu^*(E_n \cap K) = \mu^*\left(\bigcup_{N=1}^{\infty} E_n \cap K\right) < \epsilon/2.$$

Now, for each $n < N$, choose a compact set $K_n \subset E_n \cap K$ such that $\mu^*(E_n \cap K) < \mu^*(K_n) + \epsilon/2N$. Now, $\bigcup_{n=1}^{N} K_n = K' \subset K$ and,

$$\mu(K') \geqslant \mu(K) - \epsilon.$$

Moreover, f restricted to K' is continuous, since $K_n \subset K$ are compact and mutually disjoint and $K' = \bigcup_{n=1}^{N} K_n$. This shows that f is Lusin μ-measurable, proving the lemma.

COROLLARY. *A lower-semi-continuous function f on X, having values in $\overline{\mathbf{R}}$, is Lusin-measurable for every Radon measure μ.*

THEOREM 6. (Egoroff). *Let μ be a finite Radon measure on X. Let Y be a metrizable space. Let $\{f_n\}$ be a sequence of Borel μ-measurable functions from X to Y, converging μ-almost everywhere to a function $f: X \to Y$. Then, given $\delta > 0$, there exists a μ-measurable set of measure less than δ such that f_n converges uniformly to f in the complement of this set, and f is Borel μ-measurable.*

The proof is exactly as in Halmos [1, p. 88] and we omit it.

We have the following result in the case $Y = \vec{B}$, a Banach space which is not necessarily separable.

THEOREM 7. *A function $\vec{f}: X \to \vec{B}$ is strictly μ-integrable (resp. μ-integrable) if and only if $\mu^*(\|\vec{f}\|) < +\infty$ (resp. $\mu^*(\|\vec{f}\|) < +\infty$) and $\vec{f}$ is Lusin μ-measurable.*

PROOF. Let us suppose that $\vec{f}$ is a strictly μ-integrable function. Then, by taking a sequence of step functions $\{\vec{g}_n\}$ such that $\mu^*(\|\vec{f} - \vec{g}_n\|) < \frac{1}{2^n}$, we see immediately that $\vec{f}$ equals, μ-almost everywhere, the pointwise limit of a sequence of Lusin μ-measurable functions $\vec{g}_n$. Now, given a compact set K and $\epsilon > 0$, by Egoroff's Theorem, we can find a compact K_ϵ such that, $\mu(K - K_\epsilon) < \epsilon$ and all the functions $\vec{g}_n$ are continuous on K_ϵ and moreover $\vec{g}_n$ converges uniformly to $\vec{f}$ on K_ϵ. Hence, $\vec{f}$ is continuous on K_ϵ and this shows that f is Lusin μ-measurable. Moreover, it is trivial to see that $\mu^*(\|\vec{f}\|) < +\infty$.

Conversely, let us suppose that $\vec{f}$ satisfies the two conditions and we shall show that $\vec{f}$ is strictly μ-integrable. Let us choose a step function h such that $h \geq \|\vec{f}\|$ and $\mu^*(h) < +\infty$. Let $h = \sum_{n=0}^{\infty} \alpha_n \chi_{A_n}$.

We can find compact sets $K_n \subset A_n$ such that $K_n = \emptyset$ for all but a finite number of indices and $\sum_{n=0}^{\infty} \alpha_n \mu(A_n - K_n) < \epsilon$. Let $K = \bigcup_n K_n$. Then K is compact and $\mu^*(\|\vec{f}\chi_{CK}\|) < \epsilon$. On K, $\|\vec{f}\|$ is bounded by some positive constant. Hence, $\vec{f}$ would be strictly μ-integrable if $\vec{f}\chi_K$ were strictly μ-integrable or equivalently, $\vec{f}\chi_K$ were strictly μ_K-integrable. Hence, we have to consider the following situation: K compact Hausdorff, μ a Radon measure and $\vec{f}$ a Lusin μ-measurable function such that $\|\vec{f}\| < M$.

Now, given $\epsilon > 0$, let K_ϵ be a compact set such that $\mu(K - K_\epsilon) < \epsilon$ and $\vec{f}$ restricted to K_ϵ is continuous. The function $\vec{f}$ is uniformly continuous on K_ϵ. We can cover K_ϵ by a finite number of open sets such that the oscillation of f in any one of them is $\leqslant \delta$, where δ is any real number > 0. We can replace this finite union of open sets by a finite disjoint union of Borel sets and get a step function $\vec{g}$ such that $\|\vec{f} - \vec{g}\| < \delta$ in K_ϵ and $\vec{g} = 0$ on $K - K_\epsilon$. Then

$$\mu^*(\|\vec{f} - \vec{g}\|) < \delta\, \mu(K) + M\epsilon$$

and from this the strict μ-integrability of $\vec{f}$ follows.

The other part of the proof, for the essential integrability case, is on the same lines and we omit it.

Let us now consider the image of a measure, in the abstract case. Let $H : (X, \tau) \to (Y, \mathscr{S})$ be a measurable mapping and μ a measure on (X, τ). Consider the image measure $H\mu$ on $(Y, \mathscr{S})$, defined by $(H\mu)(B) = \mu[H^{-1}(B)]$ for every $B \in \mathscr{S}$. Then, for any set $A \subset Y$,

$$\begin{aligned}(H\mu)^*(A) &= \inf\{(H\mu^*)(B) : B \supset A,\ B \in \mathscr{S}\} \\ &= \inf\{\mu^*(H^{-1}(B)) : B \supset A,\ B \in \mathscr{S}\} \\ &> \mu^*[H^{-1}(A)].\end{aligned}$$

Similarly, we can show that, for any positive real valued function f on Y, $(H\mu)^*(f) \geqslant \mu^*(f \circ H)$. Hence, if a function $\vec{f}$ on Y with values in a Banach space $\vec{B}$ is $H\mu$-integrable then, $\vec{f} \circ H$ is μ-integrable. But we cannot say anything in the reverse direction, nor can we assert anything about the essential integrals.

In particular, with the abstract measures, we have the so-called "image measure catastrophe". It may so happen, that for a set $A \subset Y$, $H^{-1}(A)$ is m-measurable but A is not Hm-measurable. This is not in itself abnormal or catastrophic, if say the only elements of $\mathscr{S}$ are Y and $\varnothing$; then it is quite natural. However what is abnormal and catastrophic is that Hm need not be concentrated on $H(X)$, even for topological spaces and Borel σ-algebras. $H^{-1}(H(X)) = X$ is m-measurable, but this does not imply that $H(X)$ is Hm-measurable. $H^{-1}[\complement H(X)] = \varnothing$, but $\complement H(X)$ has only inner measure 0, and need not be Hm-measurable. Consider the following example. $Y = [0, 1]$, with Lebesgue measure n; X a non-measurable subset of Y, with $n^*(X) = 1$, or $n_*(Y - X) = 0$ (where n_* is the inner measure). Let $H : X \to Y$ be the natural injection. The pre-image of a subset of Y is simply its intersection with X. Let τ and $\mathscr{S}$ be the Borel σ-algebras of Y and X respectively; the Borel subsets of X are simply the intersections of the Borel subsets of Y with X. Define the measure m on X in the following way. For B, a Borel set in X, put $m(B) = n(B')$, for any Borel subset B' of Y such that $B' \cap X = B$; this is independent of the choice of B', because the symmetric difference between two such B' is a Borel subset of $Y - X$, therefore has measure zero. It is trivial to see that m is a measure on the Borel σ-algebra of X. The canonical injection H being continuous, is $(\mathscr{S}, m)$-measurable; and the image measure Hm is exactly the Lebesgue measure n on Y. However, $H(X) = X$ is not n-measurable and $Y - X$, whose pre-image is void, is not n-measurable either. We see that the image measure catastrophe may occur even in the simplest cases: m is a probability ($m(X) = 1$), H the canonical injection of a subspace. We also observe that this catastrophe has great inconveniences. If $A \subset Y$

such that $H^{-1}(A)$ is m-measurable, we should like to be able to say that A is Hm-measurable and that $m[H^{-1}(A)] = Hm(A)$, since it is m that we know and not Hm. The previous example gives also a measure m on the Borel σ-algebra of a topological space X, which is finite and hence locally finite, but not Radon, since it is not inner regular. For, the equality

$$\begin{aligned} n^*(X) = m(X) &= \sup \{m(K) : K \text{ compact } \subset X\} \\ &= \sup \{n(K) : K \text{ compact } \subset X\} \end{aligned}$$

would mean that X is n-measurable, which is not true.

Various authors have considered abstract measures with some restrictions which enable them to have some of the properties of Radon measures. See Alexandrov[1], Halmos and von Neumann[1], Marczewski[1], Ryll-Nardzewski[1], Gnedenko and Kolmogorov[1], and Blackwell[1]. They succeed in avoiding the image measure catastrophe when Y is a complete separable metric space.

Let us now consider the situation when μ is a Radon measure on the space X.

DEFINITION 10. *A mapping $H: X \to Y$ is said to be μ-proper if it is Lusin μ-measurable and every point of Y has a neighbourhood whose inverse image is μ-integrable.*

Let us assume that H is μ-proper. As in the other cases (viz. induced measure) the measures HM and Hm do not satisfy the conditions of R_2 or R_3. Let us define the image (Radon) measure H in a special way, as follows. For every Borel set $B \subset Y$, let

$$(H\mu)^*(B) \equiv \mu^*(H^{-1}(B));$$

we shall show that $(H\mu)^*$ satisfies the conditions of R_3. Now, $(H\mu)^*$ is a measure on $\mathscr{S}_\beta$, the Borel σ-algebra of Y and the assumption that H is μ-proper gives us the fact that $(H\mu)^*$ is locally finite. It is enough to verify the inner regularity of $(H\mu)^*$.

Let $B \in \mathscr{S}_\beta$. For any $\alpha < \mu^*[H^{-1}(B)]$, we can find a compact set $K \subset H^{-1}(B)$, such that $\mu(K) > \alpha$. Again, by the Lusin μ-measurability of H, we can find a compact set $K_\epsilon \subset K$ such that $\mu(K_\epsilon) \geqslant \alpha$

and H restricted to K_ϵ is continuous. Let $L = H(K_\epsilon)$. Then L is compact and is contained in B. Now, $H^{-1}(B)$ is a μ-measurable set and contains K_ϵ. Hence,

$$(H\mu)^*(L) = \mu^*[H^{-1}(L)] > \mu(K_\epsilon) > \alpha.$$

This proves the inner regularity of $(H\mu)^*$. Hence $(H\mu)^*$ defines a Radon measure on Y.

We shall now prove the main theorem about the image measure.

THEOREM 8. *For every set $A \subset Y$, $\mu^*[H^{-1}(A)] = (H\mu)^*(A)$.*

PROOF. CASE 1. Suppose μ is a finite Radon measure on X. In this case, $\mu^* = \mu^*$, $(H\mu)^* = (H\mu)^*$ and $H\mu^* = (H\mu)^*$. In this case of finite measure, we have, for any $A \subset Y$,

$$(H\mu)^*(A) > \mu^*[H^{-1}(A)]$$

and

$$(H\mu)_*(A) < \mu_*[H^{-1}(A)].^\dagger$$

Hence every set A which is $H\mu$-measurable is such that $H^{-1}(A)$ is μ-measurable. On the other hand, let A be such that $H^{-1}(A)$ is μ-measurable. Given $\epsilon > 0$, we can find compact sets K and K_ϵ such that (i) $K_\epsilon \subset K \subset H^{-1}(A)$ (ii) $H \mid K_\epsilon$ is continuous and (iii) $\mu(K_\epsilon) > \mu(K) - \epsilon/2 > \mu_*[H^{-1}(A)] - \epsilon$. Let $L = H(K_\epsilon)$. Then, the compact set $L \subset A$ has the property

$$(H\mu)^*(L) = \mu^*[H^{-1}(L)] > \mu(K_\epsilon) > \mu_*[H^{-1}(A)] - \epsilon.$$

Hence, we get

$$(H\mu)_*(A) = \mu_*[H^{-1}(A)].$$

From this we get

$$(H\mu)^*(A) = \mu^*[H^{-1}(A)]$$

by using $(H\mu)_*[\complement A] = \mu_*[H^{-1}(\complement A)]$. In particular it follows that a set A is $H\mu$-measurable iff $H^{-1}(A)$ is μ-measurable.

Before going to the other cases, let us prove the following lemma.

† The inner measure $\mu_*(E)$ of any set E is defined to be the supremum of the measures of measurable sets F contained in E.

LEMMA 5. *Let Z be a Borel subset of Y. Then*

$$(H\mu)_Z = H(\mu_{H^{-1}(Z)}).$$

PROOF. Let B be any Borel subset of Z (then B is also Borel in Y). Then we have

$$\begin{aligned}[(H\mu)_Z]^*(B) &= (H\mu)^*(B) \\ &= \mu^*[H^{-1}(B)] \\ &= [\mu_{H^{-1}(Z)}]^*(H^{-1}(B)) \\ &= (H[\mu_{H^{-1}(Z)}])^*(B).\end{aligned}$$

This proves the lemma.

Let us now consider the Case 2 in the proof of the theorem.

CASE 2. A is any relatively compact set in Y. Then we have,

$$\begin{aligned}(H\mu)^*(A) &= ((H\mu)_{\overline{A}})^*(A) && \text{(Theorem 1)} \\ &= (H(\mu_{H^{-1}(\overline{A})}))^*(A) && \text{(Lemma 5)} \\ &= (\mu_{H^{-1}(\overline{A})})^*(H^{-1}(A)) \\ &\quad \text{(Case 1: since } \mu_{H^{-1}(\overline{A})} \text{ is a finite measure)} \\ &= \mu^*(H^{-1}(A)). && \text{(Theorem 1)}\end{aligned}$$

GENERAL CASE. Let μ be an arbitrary Radon measure on X and A any subset of Y. It is easy to see that

$$(H\mu)^*(A) < \mu^*(H^{-1}(A)).$$

For,

$$\begin{aligned}(H\mu)^*(A) &= \sup_K (H\mu)^*(A \cap K) \\ &= \sup_K \mu^*[H^{-1}(A \cap K)] && \text{(Case 2)} \\ &< \mu^*[H^{-1}(A)].\end{aligned}$$

To prove the opposite inequality, suppose $\alpha < \mu^*(H^{-1}(A))$. We can find a compact set $K \subset X$ with $\mu^*[H^{-1}(A) \cap K] > \alpha$. Given $\epsilon > 0$, choose a compact set $K_\epsilon \subset K$, satisfying the following conditions: (i) $\mu(K_\epsilon) > \mu(K) - \epsilon$ and (ii) H restricted to K_ϵ is continuous. Let $L_\epsilon = H[K_\epsilon \cap H^{-1}(A)]$. Then, L_ϵ is relatively compact in Y and we have,

$$(H\mu)^*(L_\epsilon) = \mu^*[H^{-1}(L_\epsilon)]. \qquad \text{(Case 2)}$$

Hence,

$$\begin{aligned}(H\mu)^*(A) &\geqslant (H\mu)^*(L_\epsilon)\\ &= \mu^*[H^{-1}(L_\epsilon)]\\ &\geqslant \mu^*[K_\epsilon \cap H^{-1}(A)]\\ &\geqslant \mu^*[K \cap H^{-1}(A)] - \mu(K - K_\epsilon).\\ &\geqslant \mu^*[K \cap H^{-1}(A)] - \epsilon\\ &\geqslant \alpha - \epsilon.\end{aligned}$$

It follows that $(H\mu)^*(A) \geqslant \mu^*[H^{-1}(A)]$ and this completes the proof of the theorem.

COROLLARY. *For any non-negative valued function f, possibly taking the value $+\infty$, on Y,*

$$(H\mu)^*(f) = \mu^*(f \circ H).$$

PROOF. The proof of the corollary is done in three steps as in the proof of the theorem.

First, let us consider a function $f \geqslant 0$ and bounded on Y, say $|f| < M$, and let the measure be finite. Then, using the inner measures we see easily that $(H\mu)_*(M - f) \geqslant \mu_*[(M - f) \circ H]$ and from this we get the required result in this case. For a finite measure μ and any $f > 0$, by considering the sequence $f_n = \inf(f, n)$ and applying Fatou's property we get the result.

Now, for any Radon measure, but for functions carried by a compact set, the result is established, using the same technique as in the second step of the proof of the theorem.

Finally, consider the general case. The inequality $(H\mu)^*(f) \leqslant \mu^*(f \circ H)$ is easily proved. To prove the opposite inequality, let us first note that the family $\mathcal{F}$ of the compact sets of X, for which the restriction of H to each of its elements is continuous, is μ-dense. Hence,

$$\begin{aligned}\mu^*(f \circ H) &= \sup_{K \in \mathcal{F}} \mu^*((f \circ H)\chi_K)\\ &< \sup_{K \in \mathcal{F}} \mu^*[(f \circ H)\chi_{H^{-1}[H(K)]}]\end{aligned}$$

$$\leq \sup_{K \in \mathcal{K}} (H\mu)^*[f\chi_{H(K)}]$$
$$\leq (H\mu)^*(f).$$

This proves the corollary.

THEOREM 9. *Let X, Y, Z be Hausdorff topological spaces. Let μ be a Radon measure on X. Let $H: X \to Y$ be a Lusin μ-measurable and μ-proper mapping. Let $f: Y \to Z$. Then f is Lusin $H\mu$-measurable on Y iff $f \circ H$ is Lusin μ-measurable on X.*

PROOF. Suppose that f is Lusin $H\mu$-measurable. Let $K \subset X$ be any compact set. Given $\epsilon > 0$, we can find a compact set $K_\epsilon \subset K$ such that (i) $\mu(K_\epsilon) \geq \mu(K) - \epsilon/2$ and H restricted to K_ϵ is continuous. Let $L'_\epsilon = H(K_\epsilon)$. Then, we can find a compact set $L_\epsilon \subset L'_\epsilon$ satisfying $(H\mu)(L'_\epsilon) \leq (H\mu)(L_\epsilon) + \epsilon/2$ and f restricted to L_ϵ is continuous. Clearly, $f \circ H$ restricted to $H^{-1}(L_\epsilon) \cap K_\epsilon$ is continuous. Further, $H^{-1}(L_\epsilon) \cap K_\epsilon$ being a closed subset of K_ϵ is compact; and also, $\mu[H^{-1}(L_\epsilon) \cap K_\epsilon] \geq \mu(K) - \epsilon$. This being true for all $\epsilon > 0$ and in turn for all compact sets of K, we get that $f \circ H$ is Lusin μ-measurable.

Conversely, let us suppose that $f \circ H$ is Lusin μ-measurable. Let $C \subset Y$ be a compact set. Given $\delta > 0$, we can find compact sets $K_\delta \subset K \subset H^{-1}(C)$ satisfying

(i) $\mu(K_\delta) \geq \mu(K) - \delta/2 \geq \mu^*[H^{-1}(C)] - \delta$,

(ii) $f \circ H$ restricted to K is continuous, and

(iii) H restricted to K_δ is also continuous.

Let $H(K_\delta) = C_\delta$. Then C_δ is compact, is contained in C and moreover, $(H\mu)(C_\delta) \geq (H\mu)(C) - \delta$. Now, we assert that f restricted to C is a continuous function. For, if F is any closed set in Z then, $(f \circ H)^{-1}(F) \cap K_\delta$ being closed is compact. Also, H is continuous on K_δ and hence, $H[(f \circ H)^{-1}(F) \cap K_\delta]$ is compact. But this set is precisely $f^{-1}(F) \cap H(K_\delta) = f^{-1}(F) \cap C_\delta$. Therefore f is Lusin $H\mu$-measurable. This completes the proof of the theorem.

COROLLARY 1. *Let H be a μ-proper mapping of $X \to Y$. Then, a Banach-valued function $\vec{f}$ on Y is $H\mu$-integrable iff $\vec{f} \circ H$ is μ-integrable; moreover, in the case of integrable functions, the two integrals are equal.*

PROOF. From the above theorem we have that $\vec{f} \circ H$ is Lusin μ-measurable if and only if $\vec{f}$ is Lusin $H\mu$-measurable. From Theorem 7, we deduce that $\mu \cdot (\|\vec{f} \circ H\|) = (H\mu) \cdot (\|\vec{f}\|)$. Hence Theorem 6 gives us that $\vec{f}$ is $H\mu$-integrable iff $\vec{f} \circ H$ is μ-integrable. Then the equality of the integrals is trivial to see by taking limits of elementary step-functions. The proof is complete.

COROLLARY 2. (Transitivity). *In addition to the hypotheses of the theorem, if $f \circ H$ is μ-proper or, equivalently, f is $H\mu$-proper, then $f(H\mu) = (f \circ H)\,\mu$.*

The following transitivity relations are easily seen to be true from the properties we have.

(1) (Compatibility of image and induction). If $H: X \to Y$ is μ-proper, then for any $H\mu$-measurable set $Z \subset Y$,

$$(H\mu)_Z = H[\mu_{H^{-1}(Z)}].$$

(2) (Compatibility of image and multiplication). Under the same assumptions regarding H, if $p > 0$ is a locally $H\mu$-integrable function on Y (or, equivalently, a function on Y such that $p \circ H$ is locally μ-integrable), then

$$p(H\mu) = H[(p \circ H)\,\mu].$$

Finally,

(3) If $Z \subset X$ is a μ-measurable set and j is the canonical injection then $j(\mu_Z) = \chi_Z\,\mu$.

Let $R_1(X)$ denote the set of all finite Radon measures on any Hausdorff topological space X; it is a subset of the set of all finite Borel measures (i.e. measures on the Borel σ-algebra). Suppose f is a continuous function $f: X \to Y$, where Y is also a Hausdorff space. Then, clearly, for every $\mu \in R_1(X)$, f is μ-proper and the

measure $f\mu \in R_1(Y)$. That is, f induces a mapping $\tilde{f}: R_1(X) \to R_1(Y)$. We shall see below the conditions under which $\tilde{f}$ can be an injection or a surjection.

THEOREM 10. *If f is an injection, then so also is $\tilde{f}$.*

PROOF. Suppose $f\mu_1 = f\mu_2$ for $\mu_1, \mu_2 \in R_1(X)$. Then, for every compact set $K \subset X$,

$$\begin{aligned} \mu_1(K) &= (f\mu_1)[f(K)] \qquad (f \text{ is an injection}) \\ &= (f\mu_2)[f(K)] \\ &= \mu_2(K). \end{aligned}$$

The inner regularity gives us immediately that $\mu_1 = \mu_2$. The proof is complete.†

THEOREM 11. *Let f be an injection. Let $\nu \in R_1(Y)$. Then a necessary and sufficient condition that ν may be the image of a measure on X, viz. $\nu = f\mu$ where $\mu \in R_1(X)$, is the simultaneous fulfilment of the following two conditions:*

(1) $\nu[\complement f(X)] = 0$;

and (2) f^{-1} (which is defined ν almost everywhere on Y) is Lusin ν-measurable.

PROOF. Let $\nu = f\mu$, for some $\mu \in R_1(X)$. Then, obviously (1) is true. Now, f^{-1} is defined ν almost everywhere on Y. Moreover, in

$$X \xrightarrow{f} Y \xrightarrow{f^{-1}} X$$

f is Lusin μ-measurable and $f^{-1} \circ f = I_X$ is Lusin measurable. Hence, by Theorem 9, f^{-1} is Lusin ν-measurable (since $f\mu = \nu$).

Conversely, suppose, for the measure ν, the two conditions are verified. Then, in $Y \xrightarrow{f^{-1}} X \xrightarrow{f} Y$, f^{-1} is Lusin ν-measurable. Let $\mu = f^{-1}\nu$. Then, by Corollary 2 to Theorem 9, we have $f\mu = \nu$. This completes the proof.

An important application of the above two theorems is the possible comparison between $R_1(X')$ and $R_1(X'')$, where X' and X''

† The same result does not hold for Borel measures.

are two Hausdorff topological spaces with the same underlying set X, and comparable topologies. It follows that there is an injection $R_1(X') \to R_1(X'')$, where X'' is the space with the weaker topology.

DEFINITION 11. *A mapping $f: X \to Y$, where X and Y are Hausdorff topological spaces, is said to be universally measurable if, for every $\mu \in R_1(X)$, the mapping f is Lusin μ-measurable.*

If the identity mapping $i: X'' \to X'$ is universally measurable, then, $R_1(X')$ and $R_2(X'')$ are identical. We shall say that the two topologies on the same underlying set are *Radon equivalent* if the sets of all finite Radon measures for either of the topologies are identical. A result due to Phillips and Grothendieck asserts that the weak and the strong topologies on a locally convex metrisable topological vector space are Radon equivalent. The result will be proved in Part II.

Suppose, now, that T_1 and T_2 are two not necessarily comparable Hausdorff topologies on the set X. Let the respective Borel σ-algebras be τ_1 and τ_2. Let $\mu_1 \in R_1(X, \tau_1)$ and $\mu_2 \in R_1(X, \tau_2)$. We shall say that μ_1 is equivalent to μ_2 if $i: (X, \tau_1) \to (X, \tau_2)$ is Lusin μ_1-measurable and $i\mu_1 = \mu_2$. Then μ_2 is also equivalent to μ_1. The integrable functions, the upper integrals and the Lusin measurable functions are the same for both the measures. This is then an equivalence relation of the set of such couples (τ, μ). An equivalence class is called a *Radon measure* on X, without any reference to the topology.

LEMMA 6. *Let X be a compact Hausdorff space and $\mathscr{C}(X)$ the class of all real-valued continuous functions on X. Any bounded linear functional L on $\mathscr{C}(X)$ such that $L(1) = \| L \|$ is a positive linear functional on $\mathscr{C}(X)$.*

PROOF. Let $g \in \mathscr{C}(X)$ such that $g \geqslant 0$ and the lower bound of g is zero. If $f = g - \| g \|$, then it is clear that $\| g \| = \| f \|$, where $\| g \| = \sup |g|$. Hence, $L(g) = L(f + \| g \|) = L(f) + \| L \| \| g \|$, as $L(1) = \| L \|$, and this is $\geqslant 0$. If $g \in \mathscr{C}^+(X)$ and $\alpha > 0$ is the greatest

lower bound of g, then $L(g-\alpha) \geq 0$ and hence $L(g) = L(g-\alpha) + L(\alpha) \geq 0$. This proves the lemma.

THEOREM 12. *Let X and Y be Hausdorff topological spaces and $H : X \to Y$ a continuous map. Let $\nu \in R_1(Y)$. A necessary and sufficient condition that $\nu = H\mu$, for some $\mu \in R_1(X)$, is that ν must be concentrated in a countable union of images of compact sets in X.*

PROOF. The necessity of the condition is easily proved by using the inner regularity of the elements of $R_1(X)$ and Theorem 7. We shall prove the sufficiency in two steps.

CASE 1. Assume that X and Y are compact spaces and H a surjection. We may assume $\nu(Y) = 1$, then ν is a positive linear functional on $\mathscr{C}(Y)$ with norm 1. Let $\mathscr{C}' = \{\phi \circ H\}_{\phi \in \mathscr{C}(Y)}$. Then $\mathscr{C}' \subset \mathscr{C}(X)$ and $\phi \circ H \to \nu(\phi)$ defines a linear functional μ' on $\mathscr{C}'$. This linear functional can be extended to a continuous linear functional μ on $\mathscr{C}(X)$, without changing the norm (Hahn-Banach theorem). Now, $\mu'(1) = 1$, i.e. $\mu'(1) = \mu(1) = 1 = \|\mu\|$. It follows by the previous lemma that $\mu \geq 0$. It is clear that $\nu = H\mu$ and this completes the proof in this particular case.

CASE 2. (General case).

Suppose $\{K_n\}_0^\infty$ is a sequence of compact sets of X such that ν is concentrated on $\bigcup_{n=0}^{\infty} H(K_n)$. We can assume that $L_n = H(K_n)$ increases with n. Let $B_n = L_n - L_{n-1}$ for $n > 1$ and $B_0 = L_0$. Let ν_n be the image in L_n of the restriction of ν to B_n. Then, for each n, there exists a Radon measure $\tilde{\mu}_n$ on K_n such that $H\tilde{\mu}_n = \nu_n$. Let μ_n be the image of this measure on X, for each n. Then, the measure $\mu = \sum_{n=0}^{\infty} \mu_n$, defined by $(\sum_n \mu_n)^*(B) = \sum_n \mu_n^*(B)$ for B Borel, satisfies the required condition $H\mu = \nu$. To prove this, let us first note that the measures ν_n are carried by B_n and hence both $\tilde{\mu}_n$ and μ_n are carried by $H^{-1}(B_n)$. Now, for any Borel set $B \subset Y$,

$$\nu^*(B) = \sum_{n=0}^{\infty} \nu^*(B \cap B_n) = \sum_{n=0}^{\infty} \nu_{B_n}{}^*(B \cap B_n)$$

$$= \sum_{n=0}^{\infty} \nu_n^*(B \cap B_n)$$

$$= \sum_{n=0}^{\infty} \tilde{\mu}_n^*[H^{-1}(B \cap B_n)]$$

$$= \sum_{n=0}^{\infty} \mu_n^*[H^{-1}(B \cap B_n)]$$

$$= \sum_{n=0}^{\infty} \mu_n^*[H^{-1}(B)]$$

$$= \mu^*[H^{-1}(B)]$$

$$= (H\mu)^*(B).$$

This completes the proof of the theorem.

REMARK. If $f: X \to Y$ is surjective, we just saw that, if X is compact, $\tilde{f}$ is surjective too. This does not hold if X is an arbitrary Hausdorff space. However we shall prove it later on if X is a Suslin space.

6. Other properties of Radon measures. Let μ be a Radon measure on a Hausdorff topological space X. We shall prove a number of properties of μ.

(a) *Support of a measure. If $\{V_i\}_{i \in I}$ is a family of open sets of X such that $\mu(V_i) = 0$, for every $i \in I$, then $\mu(\bigcup_{i \in I} V_i) = 0$.*

PROOF. Let $K \subset \bigcup_{i \in I} V_i$ be any compact set. Then K is covered by a finite number of V_i and hence, $\mu(K) = 0$. The inner regularity gives us immediately that $\mu(\bigcup_{i \in I} V_i) = 0$. The proof is complete.

DEFINITION 12. *The support of the measure μ is the complement of the largest open set $V \subset X$ such that $\mu(V) = 0$. The support of a measure has the usual properties*†.

(b) *Lower semi-continuous functions. For every non-negative valued lower semi-continuous function f on X,*

$$\mu^*(f) = \mu^*(f).$$

† One can give an example of a finite *Borel* measure μ on a compact space X which is 0 in the neighbourhood of every point, but has the total mass 1. See p. 45.

PROOF. Every function which is strictly μ-integrable is also μ-integrable and the two integrals are equal. Hence, the statement will be proved if we show that $\mu^*(f) = +\infty$ implies that $\mu^\bullet(f) = +\infty$. If f vanishes outside a countable increasing union of open sets O_n of finite measure then $\mu^*(f) = \sup \mu^*(\chi_{O_n} f) = \sup \mu^\bullet(\chi_{O_n} f) = \mu^\bullet(f)$. If this is not the case, then there exists at least one $\epsilon > 0$ such that the open set $A_\epsilon = \{x : f(x) > \epsilon\}$ has $\mu(A_\epsilon) = +\infty$. Hence, $\mu^\bullet(f) \geqslant \epsilon \mu^\bullet(A_\epsilon) = +\infty$. This completes the proof.

LEMMA 7. (Dini). *Let $\{f_i\}_{i \in I}$ be an increasing directed family of lower semi-continuous functions on X and let f be the limit of $\{f_i\}_{i \in I}$. If g is an upper semi-continuous function on X, with $g < f$ then corresponding to every compact set $K \subset X$, there exists at least one $i_K \in I$, such that $g(x) < f_{i_K}(x)$, for all $x \in K$.*

PROOF. For each point $x \in K$, there exists at least one index $i_x \in I$ such that $f_{i_x}(x) > g(x)$. Hence, in a neighbourhood N_x of x, $f_{i_x} > g$. Since K is compact, we can choose a finite number of indices $i_1, \ldots i_n$, such that, for each point $y \in K$, there is at least one of these indices i_k satisfying $f_{i_k}(y) > g(y)$. The family f_i being increasing directed, the proof can be completed easily.

LEMMA 8. *Let $f > 0$ and measurable. Then*

$$\mu^\bullet(f) = \sup \left\{ \mu(g) : \begin{array}{l} g \text{ upper semi-continuous on } X, \\ g \text{ with compact support and } 0 < g < f \end{array} \right\}$$

PROOF. Since every non-negative measurable function is the increasing limit of a sequence of non-negative step functions taking only a finite number of values, it is enough to prove the result for the latter functions. Again, it is enough if we prove the same result for characteristic functions of measurable sets. For measurable sets this is nothing but the inner regularity. This completes the proof†.

LEMMA 9. *Let $\{f_i\}_{i \in I}$ be an increasing directed family of non-negative valued functions and let $f = \lim_i f_i$. If there exists a μ-dense*

† One cannot replace the upper semi-continuous functions g by continuous functions.

family $\mathcal{F}$ of compact sets such that on each $K \in \mathcal{F}$, the above convergence is uniform, then $\mu^*(f) = \lim_i \mu^*(f_i)$.

PROOF. By Proposition 2, $\mu^*(f)$ can be approached by $\mu^*(f\chi_K)$ for $K \in \mathcal{F}$. Since $\{f_i\}$ converges uniformly on K, we have $\mu^*((f - f_i)\chi_K) \to 0$ following $i \in I$. But,

$$\mu^*(f\chi_K) \leq \mu^*(f_i\chi_K) + \mu^*((f - f_i)\chi_K)$$

and the result of the lemma follows.

PROPOSITION 5. *Let $f \geq 0$ be any function on X. Let $\{g_i\}_{i \in I}$ be an increasing directed family of non-negative lower semi-continuous functions with the limit g. Then* $\mu^*(fg) = \lim_{i \in I} \mu^*(fg_i)$.

PROOF. Let us suppose first of all that f and g are bounded functions. The function g, being lower semi-continuous, is μ-measurable. Now, by the Theorem 5, g is Lusin measurable and it follows that the family $\mathcal{F}$ of compact sets, restricted to which g is continuous, is μ-dense. But, on each compact set belonging to $\mathcal{F}$, g being bounded and continuous, the family $\{g_i\}$ converges uniformly to g on this set, by Lemma 7. Hence, fg_i converges uniformly to fg on K, for every K. Hence, in this case, $\mu^*(fg) = \lim_i \mu^*(fg_i)$ by Lemma 9.

Let us now consider the case when f and g are not necessarily bounded. Define, for each positive integer n and all $i \in I$, $g_{i,n} = \inf(g_i, n)$; $f_n = \inf(f, n)$ and $g_n = \inf(g, n)$. Now, by the Fatou property, $\mu^*(fg) = \lim_{n \to \infty} \mu^*(f_n g_n)$. But $\mu^*(f_n g_n) = \lim_i \mu^*(f_n g_{i,n})$. Now, the required result follows immediately, completing the proof.

COROLLARY. *Let $f \geq 0$ be any function on X. Let $\{V_i\}_{i \in I}$ be an increasing directed family of open sets of X. Then,*

$$\mu^*(f\chi_V) = \lim_i \mu^*(f\chi_{V_i})$$

where $V = \bigcup_{i \in I} V_i$. By complementation with respect to an open set of finite measure: if $(K_i)_{i \in I}$ is a decreasing directed family of compact subsets, $\mu(\bigcap_{i \in I} K_i) = \lim_i \mu(K_i)$.

PROPOSITION 6. *For every non-negative valued function f,*

$$\mu^*(f) = \inf \{\mu^*(g) ; g \text{ lower semi-continuous} \geqslant f\}.$$

PROOF. It is enough to prove the result for non-negative step functions. This is deduced easily from the outer regularity of the measure, completing the proof.

(c) *Moderated functions and measures.* A set $A \subset X$ is called *μ-moderated* if it is contained in a countable union of open subsets of finite μ-measure. A function f, with values in $\mathbf{R}$ or a vector space, is called *moderated* if it is carried by a moderated subset. The measure μ is said to be moderated if X is μ-moderated; in this case $\mu^* = \mu^\bullet$. The converse is not true; one can have $\mu^* = \mu^\bullet$ but μ not moderated, e.g. if X is uncountable and discrete and μ has mass 1 at each point of X. If however μ is *diffuse* (i.e. if μ has no point masses) and if $\mu^* = \mu^\bullet$ then μ is moderated. Let $X = \bigcup_{i \in I} K_i \cup N$ be a μ-concassage such that the K_i are also μ-full (see (e) below for the definition and proof of existence). Let A be a set containing exactly one point of each K_i, then $\mu^\bullet(A) = 0$ since μ is diffuse. But then since the K_i are μ-full, any open set containing A has infinite measure unless I is countable, hence I must be countable else $0 = \mu^\bullet(A) < \mu^*(A) = +\infty$. Finally $0 = \mu^\bullet(N) = \mu^*(N)$ and so N is contained in an open set of finite measure and so μ is moderated.

Let $f \geqslant 0$, then $\mu^*(f) = \mu^\bullet(f)$ if f is μ-moderated, $\mu^*(f) = +\infty$, if f is not μ-moderated. Assume f is moderated; it is carried by the union of an increasing sequence of open sets V_n with $\mu^*(V_n) < +\infty$. Then

$$\begin{aligned} \mu^*(f) &= \lim_{n \to \infty} \mu^*(f \chi_{V_n}) \\ &= \lim_{n \to \infty} \mu^\bullet(f \chi_{V_n}) \\ &= \mu^\bullet(f). \end{aligned}$$

Assume now that f is not μ-moderated. Let $g \geqslant f$, g a lower semi-continuous function. There exists an integer n such that the set $\{x \in X : f(x) > 1/n\}$ is not moderated. Then the set

$\Omega_n = \{x \in X : g(x) > 1/n\}$ is open, of infinite measure, $\mu^*(g) \geqslant (1/n)\mu^*(\Omega_n) = +\infty$, and hence $\mu^*(f) = +\infty$ from Proposition 6.

We see easily (by Theorem 7) that a Banach valued function $\vec{f}$ is μ-strictly integrable iff it is μ-integrable and μ-moderated.

(d) *Glueing of pieces.* Suppose ν_n is an abstract measure on a Borel set X_n of a topological space X, for every positive integer n. If the sequence of measures satisfies the condition that $\nu_n = \nu_m$ on $X_n \cap X_m$, then there exists evidently a *unique* abstract measure ν on $Y = \bigcup_n X_n$ such that $\nu | X_n = \nu_n$ for every n†. However, if we consider an arbitrary family $\{\nu_i\}_{i \in I}$ of such (abstract) measures (on Borel sets X_i) the situation is different. In this case a measure ν satisfying $\nu | X_i = \nu_i$ exists; in fact, there exists a smallest measure having this property. [This is so, because, by taking first unions for all finite subsets of indices in I, we can assume that X_i is increasing directed. The measure ν on $Y = \bigcup_{i \in I} X_i$ defined by setting $\nu(B) = \lim_i \nu(B \cap X_i)$ is clearly the smallest measure meeting the requirement $\nu | X_i = \nu_i$]. But the uniqueness of such a measure is not guaranteed, even if the X_i are all open sets.

However, if the sets X_i are all open and ν_i are Radon measures, then the smallest measure on the union $\bigcup_{i \in I} X_i$ is also Radon. The local finiteness is trivial and the inner regularity is seen as follows.

Consider first of all a finite union $X_J = \bigcup_{i \in J} X_i$, J finite. Let $B \subset X_J$ and let $B = \bigcup_{i \in J} B_i$ where B_i are disjoint Borel sets with $B_i \subset X_i$, for every $i \in J$. Clearly, the inner regularity of ν_i on X_i implies the inner regularity of ν_J on X_J. Let now B be a Borel subset of $Y = \bigcup_{i \in I} X_i$. Then

†One simply replaces the X_n by a trivial family of disjoint subsets, X_0, $X_1 - X_0$ $X_2 - X_0 - X_1, \ldots$

$$\begin{aligned}\nu(B) &= \lim_J \nu_J(B \cap X_J) \\ &= \lim_J (\sup \{\nu_J(K) : K \text{ compact}, K \subset B \cap X_J\}) \\ &= \sup \{\nu(K) : K \text{ compact}, K \subset B\},\end{aligned}$$

since any compact subset of Y is contained in an X_J for some J. For this same reason all measures ν satisfying $\nu | X_i = \nu_i$ can have but one value on each compact set; the inner regularity shows that there is only one Radon measure ν on Y which can satisfy the condition $\nu | X_i = \nu_i$.

This property of glueing of pieces is equivalent to saying that the pre-sheaf of Radon measures on X is a sheaf.

Consider the following example due to Dieudonné [1]. Let Ω be the first uncountable ordinal. Let $\overline{X}$ be the set of all ordinals less than or equal to Ω. Let $X = \overline{X} - \{\Omega\}$. The class of all intervals of the form $\{0 < \alpha < x < \beta\}$ together with the set $\{0\}$ forms a base for a topology on $\overline{X}$; provided with this topology $\overline{X}$ is a compact Hausdorff space. The class of all unbounded (i.e. uncountable) closed subsets of X is closed under the formation of countable intersections. The set of subsets A of $\overline{X}$ such that either A or $\complement A$ contains an unbounded, closed subset of X, forms a σ-algebra, so every Borel subset of $\overline{X}$ belongs to this σ-algebra. Hence if A_n are Borel sets each not containing an unbounded closed subset of X, so is $\bigcup_n A_n$. The set function μ defined on the σ-algebra of Borel subsets of $\overline{X}$ by setting $\mu(E) = 1$ or 0 according as E does or does not contain an unbounded closed subset of X, is thus countably additive and so is a Borel measure. But μ is not a Radon measure. This is seen from the fact that $\mu(\{\Omega\}) = 0$ but the μ-measure of every open set containing Ω is 1. This measure shows that on X the uniqueness of the glueing property is not valid either. For instance, every point of the open set X has an open neighbourhood which is of measure zero; but the measure of X is 1. Thus the measure does not have a support.

(e) *'Concassage'.*

DEFINITION 13. *Let μ be a Radon measure on a Hausdorff topological space X. A μ-concassage of X is a partition of X into $N \cup (\bigcup_{i \in I} K_i)$ where N is a μ-negligible set and K_i are compact sets such that the family $(K_i)_{i \in I}$ is locally countable and mutually disjoint, i.e. $K_i \cap K_j = \emptyset$ if $i \neq j$ and every point has a neighbourhood which meets at most a countable number of K_i.*

The following theorem due to R. Godement and N. Bourbaki establishes the existence of a μ-concassage.

THEOREM 13. *Given a Radon measure μ on X, there exists a μ-concassage $\{(K_i)_{i \in I}, N\}$ of μ. Further these compact sets K_i can be chosen to be μ-full, i.e. the support of μ_{K_i} is K_i.*

PROOF. Consider the set whose elements are the families of compact disjoint μ-full subsets of X. This set is non-empty, ordered by inclusion and inductive. So, by Zorn's lemma, there exists a maximal element. Suppose this maximal element is the family $\{K_i\}_{i \in I}$ of compact disjoint μ-full subsets of X. Let μ_i be the measure induced on K_i by μ, for every $i \in I$. To prove the theorem it is enough to show that the family $\{K_i\}_{i \in I}$ is locally countable and that $X - \bigcup_{i \in I} K_i = N$ is locally μ-negligible.

Now, let $x \in X$ and let V be an open set of finite measure such that $x \in V$. We have, $\mu(V \cap K_i) > 0$ for at most a countable number of $i \in I$. Since each K_i is μ_i-full, if $\mu(V \cap K_i) = 0$, then $V \cap K_i = \emptyset$. Hence $V \cap K_i$ is not void for at most a countable number of $i \in I$, i.e. $\{K_i\}_{i \in I}$ is locally countable. It follows that $\bigcup_{i \in I} V \cap K_i$ is a Borel set in X and hence, $V \cap (X - \bigcup_{i \in I} K_i)$ is also a Borel set. We get, therefore, that $N = X - \bigcup_{i \in I} K_i$ is locally Borel and hence this set is μ-measurable. Now, N contains no compact set of strictly positive μ-measure (else, it contains a μ-full compact set, and so $\{K_i\}_{i \in I}$ is not maximal). It follows from the inner regularity for measurable subsets that $\mu^*(N) = 0$, i.e. N is μ-negligible. This completes the proof of the theorem.

(f) *Lebesgue-Nikodym Theorem.*

THEOREM 14. *Let μ and ν be Radon measures on X such that every μ-negligible set is also ν-negligible. Then there exists a positive real-valued μ-measurable and locally μ-integrable function p on X (unique up to a μ-negligible set) such that $\nu = p\mu$. p may be chosen locally Borel.*

PROOF. Let $\{(K_i)_{i\in I}, N\}$ be a μ-concassage. Since every μ-negligible set is also ν-negligible, we get that $\{(K_i)_{i\in I}, N\}$ is also a ν-concassage. Now, by the Lebesgue-Nikodym theorem for compact spaces, there exists p_i which is μ_{K_i}-integrable such that $\nu_{K_i} = p_i\mu_{K_i}$, for every $i \in I$. Let us define the function p on X by setting $p = p_i$ on K_i, for each $i \in I$ and $p = 0$ on N. For every compact set $K \subset X$, because of the local countability of I, we have,

$$\begin{aligned}\nu(K) = \sum_{i\in I} \nu(K \cap K_i) &= \sum_{i\in I} \nu_{K_i}(K \cap K_i) = \sum_{i\in I} \mu^*_{K_i}(p_i \chi_{K_i \cap K}) \\ &= \sum_{i\in I} \mu^*(p\,\chi_{K_i \cap K}) \\ &= \mu^*(p\,\chi_K) \\ &= (p\mu)(K).\end{aligned}$$

Observe that p may be chosen locally Borel.

If $p_1\mu = p_2\mu$, for two functions p_1 and p_2, then $p_1 = p_2$ μ-almost everywhere; for, the points x for which $p_1(x) \neq p_2(x)$, on every compact set $K \subset X$, form a set of μ_K-measure zero. This completes the proof of the theorem.

COROLLARY. *Let μ and ν be two Radon measures on X. Define the sum $\mu + \nu$ by $(\mu + \nu)^*(B) = \mu^*(B) + \nu^*(B)$ for $B \in \tau_\beta$. Then $\mu + \nu$ is a Radon measure, and for every $f > 0$, $(\mu + \nu)^*(f) = \mu^*(f) + \nu^*(f)$.*

PROOF. Trivially, $\mu^* + \nu^*$ is a measure on τ_β and it is locally finite. For any $\alpha < \mu^*(B)$, $\beta < \nu^*(B)$, there are compact subsets H and K of B such that $\mu(H) > \alpha$ and $\nu(K) > \beta$. Hence, $(\mu + \nu)(H \cup K) > \mu(H) + \nu(K) > \alpha + \beta$. Hence, $\mu + \nu$ is inner regular and $\mu + \nu$, as defined above, is a Radon measure. Now

by the Lebesgue-Nikodym Theorem there exist functions p and q such that $\mu = p(\mu + \nu)$, $\nu = q(\mu + \nu)$ and $p + q = 1$, $(\mu+\nu)$-almost everywhere. Finally, $\mu^*(f) = [p(\mu + \nu)]^*(f) = (\mu + \nu)^*(pf)$ and similarly $\nu^*(f) = (\mu + \nu)^*(qf)$, hence $\mu^*(f) + \nu^*(f) = (\mu + \nu)^*(f)$.

(g) *Extension Lemma.*

THEOREM 15. *Let X be a Hausdorff space and S a σ-subring† of the σ-algebra τ_β of all Borel subsets of X such that S contains a base for the topology of X. Let μ be a measure on S such that*

(a) *μ is locally finite;*

(b) *μ is inner regular with respect to the closed sets, i.e. for every $E \in S$, $\mu(E) = \sup \{\mu(F) : F \text{ closed}, F \in S \text{ and } F \subset E\}$;*

(c) *for every $E \in S$ and $\alpha < \mu(E)$, there exists a compact set K such that, for all $F \in S$ with $K \subset F \subset E$, the inequality $\alpha < \mu(F)$ is satisfied.*

For all compact sets $K \subset X$, put

$$\overline{\mu}(K) = \inf \{\mu(E) : E \in S \text{ and } E \supset K\},$$

and

$$\overline{\mu}(B) = \sup \{\overline{\mu}(K) : K \subset B \text{ and } K \text{ compact}\}$$

for every $B \in \tau_\beta$.

Then, the set function μ on τ_β defines a Radon measure on X such that $\overline{\mu} = \mu$ on S.

If further μ is a totally finite measure (i.e. if $X \in S$ and $\mu(X) < \infty$) then the condition (c) *can be replaced by:*

(c′) *given $\epsilon > 0$, there exists a compact set K such that, for every $E \in S$ with $K \subset E \subset X$, the inequality $\mu(E) \geqslant \mu(X) - \epsilon$ is verified.*

PROOF. We first show that (c′) implies (c) when the measure is totally finite and the other conditions of the hypothesis hold good. We assume that $\mu(X) = 1$. By (c′), there exist increasing compact sets C_n such that $\mu^*(C_n) \geqslant 1 - 1/n$, where μ^* is the outer

† A σ-subring of τ_β is a non-void subfamily of τ_β closed under countable unions and intersections and differences.

measure defined by μ. Let $D = \bigcup_{n=1}^{\infty} C_n$. Then $\mu^*(D) = 1$. Given $B \in S$ and an $\epsilon > 0$, there exists a closed set F belonging to S and $F \subset B$ such that $\mu(B-F) < \epsilon/2$ (by (b)). Now, $\mu^*(F \cap D) \leq \mu^*(F)$. Further, if H is such that $F \cap D \subset H \subset F$, $H \in S$ and $\mu(H) = \mu^*(F \cap D)$, then $H \cup (X - F) \supset D$ and $\mu(H) + \mu(X - F) = 1$. So, $\mu(F) = \mu(H) = \mu^*(F \cap D)$. Finally, $\mu^*(F \cap C_n) \uparrow \mu^*(F \cap D) = \mu(F)$. And hence, there exists n such that $\mu^*(F \cap C_n) \geq \mu(F) - \epsilon/2 \geq \mu(B) - \epsilon$. The set $F \cap C_n$ is compact and (c) follows.

We now come to the proof of the theorem. We first show that for any compact set K,

$$\mu^*(K) = \inf \{\mu(V) : V \in S,\ V \text{ open} \supset K\}. \qquad (\alpha)$$

Now since μ is locally finite and every open covering of K has a finite subcovering, there is an open set $U \supset K$ which is an element of S (as S contains a base for the topology) such that $\mu(U) < +\infty$. Let $H \in S$, $K \subset H \subset U$ and $\mu^*(K) = \mu(H)$. Then, $U - H \in S$, and by (b), there exists a closed set F, $F \in S$, $F \subset U - H$ such that $\mu(F) \geq \mu(U - H) - \epsilon$. Now, $U' = U - F$ is open, $U' \in S$ and $U' \supset H$. So,

$$\mu(U) = \mu(H) + \mu(U - H) = \mu(U') + \mu(F) \geq \mu(U') + \mu(U - H) - \epsilon.$$

Hence, $\mu(U') \leq \mu(H) + \epsilon = \mu^*(K) + \epsilon$, proving ($\alpha$). It follows in particular that $\mu^*(K) < +\infty$.

We now show that μ^* on the class of all compact sets of X is a *regular content*, in the sense of Choksi [2], i.e. μ^* is monotone increasing, (i) strongly additive and (ii) regular.

It is obvious that μ^* is monotone increasing. Suppose, now, that K and K' are disjoint compact subsets of X. If V is any open set, $V \in S$ and $V \supset K \cup K'$, then there exist disjoint open sets V_1 and V_2, $V_1 \in S$, $V_2 \in S$ with $K \subset V_1 \subset V$ and $K' \subset V_2 \subset V$, as S contains a base for the topology. It follows from (α) that

$$\mu^*(K \cup K') \geq \mu^*(K) + \mu^*(K').$$

μ^* being an outer measure is subadditive and we deduce that μ^* is additive on the class of all compact sets.

Let, now, K_1 and K_2 be any two compact sets. If V_1 and V_2 are open sets of finite measure in S containing respectively K_1 and K_2, then $K_1 \cup K_2 \subset V_1 \cup V_2$ and $K_1 \cap K_2 \subset V_1 \cap V_2$, so from ($\alpha$) it follows that

$$\mu^*(K_1) + \mu^*(K_2) \geqslant \mu^*(K_1 \cup K_2) + \mu^*(K_1 \cap K_2).$$

Now, let $U \supset K_1 \cup K_2$ and $U' \supset K_1 \cap K_2$, where U and U' are open sets of finite measure in S and $U \supset U'$. Let $D = \overline{U} - U'$, D is closed. Let $D_1 = D \cap K_1$ and $D_2 = D \cap K_2$. Then D_1 and D_2 are disjoint compact sets. Hence, there exist disjoint open sets V_1 and V_2 in S such that $D_1 \subset V_1 \subset U$ and $D_2 \subset V_2 \subset U$.

Let $W_1 = V_1 \cup U'$ and $W_2 = V_2 \cup U'$. Then $W_1, W_2 \in S$ and are open sets of finite measure and further $W_1 \cap W_2 = U'$ and $W_1 \cup W_2 \subset U$. So,

$$\mu(W_1) + \mu(W_2) < \mu(U) + \mu(U').$$

But $W_1 \supset K_1$ and $W_2 \supset K_2$ and hence

$$\mu^*(K_1) + \mu^*(K_2) < \mu(U) + \mu(U').$$

So we get from (α),

$$\mu^*(K_1) + \mu^*(K_2) < \mu^*(K_1 \cup K_2) + \mu^*(K_1 \cap K_2),$$

proving the strong additivity.

It remains to verify the regularity property, viz. for two compact sets C_1 and C_2 such that $C_1 \subset C_2$ and for any given $\epsilon > 0$, there exists a compact set $K \subset C_2 - C_1$ satisfying

$$\mu^*(C_1) + \mu^*(K) \geqslant \mu^*(C_2) - \epsilon.$$

We can assume that $\mu^*(C_1) < \mu^*(C_2)$, else the regularity is trivial. There exist sets H_1, H_2 in S, such that $C_1 \subset H_1$, $C_2 \subset H_2$, $H_1 \subset H_2$ and $\mu(H_1) = \mu^*(C_1)$, $\mu(H_2) = \mu^*(C_2)$. By (c), there exists a compact set $K' \subset H_2 - H_1$ with

$$\mu^*(K') \geqslant \mu(H_2 - H_1) - \epsilon = \mu(H_2) - \mu(H_1) - \epsilon.$$

Let $K = K' \cap C_2$. By strong additivity of μ^* on compact sets,

$$\mu^*(K') + \mu^*(C_2) = \mu^*(K' \cup C_2) + \mu^*(K).$$

But $K' \cup C_2 \subset H_2$, so $\mu^*(C_2) = \mu(H_2) = \mu^*(K' \cup C_2)$ and $\mu^*(K) = \mu^*(K')$. Hence, we get $K \subset C_2 - C_1$ and

$$\mu^*(K) + \mu^*(C_1) \geqslant \mu^*(C_2) - \epsilon.$$

This shows that μ^* is a regular content. By a theorem of Choksi ([2] page 390), μ^* can be extended to an inner regular measure $\tilde{\mu}$ on S_1, the least σ-ring containing the compact sets. If B is any Borel set, then $B \cap K$ belongs to S_1, for every compact set K. It is well known that, the set function $\bar{\mu}$ defined by setting

$$\bar{\mu}(B) = \sup \{\tilde{\mu}(B \cap K) : K \text{ compact}\}$$

is a measure on τ_β and moreover, $\bar{\mu}$ is inner regular with respect to the compact sets. The condition (c) ensures that $\bar{\mu}$ coincides with μ on S. Since $\bar{\mu}$ is locally finite (condition (a)) we conclude that $\bar{\mu}$ is a Radon measure on X (by R_3). The proof of the theorem is complete.

(h) *Henry's extension theorem.* We give below an extension theorem due to Henry [1] which is extremely useful in cases where the Hahn-Banach theorem is not available.

THEOREM 16. *Let X be a Hausdorff topological space, τ an algebra of Borel subsets of X and μ a finite 'Radon measure' on τ, i.e. μ is* (i) *non-negative,* (ii) *finitely additive and* (iii) *inner regular, i.e. for any $B \in \tau$*

$$\mu(B) = \sup \{\mu(K): K \text{ compact}, K \subset B, K \in \tau\}.$$

Then μ can be extended to a Radon measure $\bar{\mu}$ on the Borel σ-algebra τ_β of X.

PROOF. (a) Condition (iii) ensures that μ is countably additive (which justifies the name 'Radon measure'). It is enough to show that if B_n is a decreasing sequence of sets in τ such that $\bigcap_{n=1}^{\infty} B_n = \emptyset$, then $\mu(B_n) \to 0$. Let $\epsilon > 0$ and for each n choose K_n compact contained in B_n such that $\mu(B_n - K_n) < \frac{\epsilon}{2^{n+1}}$. Since $\bigcap_{i=1}^{\infty} K_i = \emptyset$, for n sufficiently large $\bigcap_{i=1}^{n} K_i = \emptyset$. But

$$B_n = B_n - \bigcap_{i=1}^{n} K_i \subset \bigcup_{i=1}^{n} (B_i - K_i),$$

so $\mu(B_n) < \epsilon$. Clearly classical theorems permit us to extend μ to a finite Radon measure on the σ-algebra generated by τ.

(b) The set of all pairs (τ', μ') where τ' is a sub-algebra of τ_β containing τ, μ' is a finite Radon measure on τ' extending μ, is partially ordered by the relation $(\tau_1', \mu_1') < (\tau_2', \mu_2')$ if $\tau_1' \subset \tau_2'$ and μ_2' is an extension of μ_1'. If $\{(\tau_i, \mu_i)\}$ is a chain in this ordering, put $\tau' = \bigcup_i \tau_i$, and define μ' on τ' by putting $\mu'(E) = \mu_i(E)$ if $E \in \tau_i$. τ' is an algebra containing τ and μ' is clearly well-defined, finitely additive and inner regular, i.e. a Radon measure on τ', extending μ; so (τ', μ') is an upper bound for $\{(\tau_i, \mu_i)\}$. By Zorn's lemma there exists a maximal element $(\bar{\tau}, \bar{\mu})$ extending (τ, μ). $\bar{\tau}$ must be a σ-algebra, else (since $\bar{\mu}$ is countably additive as in (a)) we could extend $\bar{\mu}$ to the σ-algebra generated by $\bar{\tau}$ which would contradict the maximality of $(\bar{\tau}, \bar{\mu})$.

(c) Suppose there exists a closed set $A \subset X$, such that $A \notin \bar{\tau}$. The algebra $\tilde{\tau}$ generated by $\bar{\tau}$ and A consists of sets of the form $(E \cap A) \cup (F \cap \complement A)$ where E and $F \in \bar{\tau}$. [For clearly any algebra containing $\bar{\tau}$ and A must contain all sets of this form, and this class of sets is an algebra.] We extend $\bar{\mu}$ to a set function $\tilde{\mu}$ on $\tilde{\tau}$ as follows. Put

$$\tilde{\mu}(B) = \bar{\mu}^*(B), \text{ for } B \subset A,\ B \in \tilde{\tau};$$

$$\tilde{\mu}(B) = \bar{\mu}_*(B), \text{ for } B \subset \complement A,\ B \in \tilde{\tau};$$

$$\tilde{\mu}(B) = \tilde{\mu}(B \cap A) + \tilde{\mu}(B \cap \complement A), \text{ for arbitrary } B \in \tilde{\tau}.$$

Recall that as shown on page 20 in §3, $\bar{\mu}^*$ and $\bar{\mu}_*$ actually define (countably) additive set functions for the intersections of measurable sets with a fixed non-measurable set; $\tilde{\mu}(B)$ is then defined for all sets B in $\tilde{\tau}$ and is clearly finite, non-negative and additive. It is further inner regular. It is sufficient to show this separately for sets $B \subset A$ and sets $B \subset \complement A$. For $B \subset \complement A$ it is obvious since

$$\tilde{\bar{\mu}}(B) = \bar{\mu}_*(B) = \sup \{\bar{\mu}(B') : B' \in \bar{\tau},\ B' \subset B\}$$

and

$$\bar{\mu}(B') = \sup \{\bar{\mu}(K) : K \text{ compact},\ K \in \bar{\tau},\ K \subset B'\}.$$

Now if $B \subset A$, suppose $B = A \cap E$, $E \in \bar{\tau}$. There exists an increasing sequence of compact sets $K_n \subset E$, $K_n \in \bar{\tau}$ such that $\bar{\mu}(E - K_n) \to 0$, so $\bar{\mu}^*(B - (A \cap K_n)) \to 0$, but $A \cap K_n$ is compact, since A is closed. Hence inner regularity follows.

Thus $(\tilde{\bar{\tau}}, \tilde{\bar{\mu}})$ is an extension of $(\bar{\tau}, \bar{\mu})$, contradicting the maximality of $(\bar{\tau}, \bar{\mu})$. Hence every closed set A must be in $\bar{\tau}$. Since $\bar{\tau}$ is also a σ-algebra it follows that $\bar{\tau} = \tau_\beta$, the σ-algebra of all Borel subsets of X, completing the proof of the theorem.

The extension in the above theorem is clearly not unique, but we have the

COROLLARY. *If τ contains a base for the topology of X, then the extension is unique.*

The assumption that τ is an algebra cannot be relaxed significantly. If, say, we assume merely that τ is a ring of sets then the theorem is false as the following example shows. Let X be an uncountable, non-discrete space, let τ be the ring of finite subsets of X and μ the counting measure on τ. Then μ is finite, countably additive and inner regular, but cannot be extended to a Radon measure on X.

Henry's theorem may be used in place of the Hahn-Banach theorem in the proof of Theorem 12. We shall see later, in § 11, that Henry's theorem and its proof generalise almost unchanged to non-Hausdorff spaces. This will enable us to prove Theorem 12 in such spaces.

7. Real and complex measures. Let m and n be two abstract measures on a σ-algebra τ. The sum $m + n$ of the two measures is defined trivially, so is an order relation $m < n$ compatible with addition. For the two measures, there is a supremum defined by

$$[\sup\ (m, n)]\ (B) = \sup\ [m(B') + n(B'')]$$

for every $B \in \tau$, where the supremum on the right side is taken for all the partitions of B into a union $B' \cup B''$ belonging to τ. It is easily verified that this is indeed a measure and that it is the smallest measure majorising both m and n. Hence, the supremum on the right side can be taken for any subsets B' and B'' of B with $B' \cap B'' = \emptyset$ (without the condition $B' \cup B'' = B$).

On the other hand, if $(m_i)_{i \in I}$ is an increasing directed family of measures on τ, then there is an upper bound for this family which is defined by setting, $[\sup_{i \in I} (m_i)]\,(B) = \sup_{i \in I} [m_i(B)]$, for every $B \in \tau$. Combining the above results, we see that every family of measures has an upper bound.

Let us now consider the lower bounds. Any collection $\mathcal{M}$ of measures has a lower bound, viz. the upper bound of all the measures that minorise every measure in $\mathcal{M}$. But it is not possible to express it by a simple formula. For two measures m and n, $\inf(m, n)$ is again defined by

$$[\inf(m, n)]\,(B) = \inf [m(B') + n(B'')].$$

But the equality

$$\sup(m, n) + \inf(m, n) = m + n$$

is not necessarily true. This is certainly true for every set $B \in \tau$ such that both $m(B)$ and $n(B)$ are finite. To verify this, we can assume that both m and n are finite measures. Then, if B' and B'' are disjoint subsets of B, $B' \cup B'' = B$ and $B', B'' \in \tau$ then we have

$$(m + n)(B) = [m(B') + n(B'')] + [m(B'') + n(B')] < + \infty.$$

This couple B', B'' which maximises $m(B') + n(B'')$ also minimises $m(B'') + n(B')$, so that

$$[\inf(m, n)](B) = m(B) + n(B) - [\sup(m, n)](B).$$

But, if $(m_i)_{i \in I}$ is a decreasing directed family of measures, then it does not necessarily follow that $(\inf_{i \in I} m_i)(B) = \inf_{i \in I} m_i(B)$. Consider the measures m_i on the σ-algebra of all subsets of $\mathbf{N}$, defined as follows :

$m_i(E) = \frac{n(E)}{i}$ if E is a finite set (where $n(E)$ is the number of elements in E),

$= +\infty$ if E is an infinite set.

Then, $m_i(\mathbf{N}) = +\infty$ for all i but $\inf(m_i) = 0$. Evidently the relation would be true if $B \in \tau$ is such that $\inf m_i(B)$ is finite.

Let us now go to the case of Radon measures on a topological space X. Let $\mu = (m, M)$ and $\nu = (n, N)$ be two Radon measures. Then, $\sup(m, n)$ is locally finite and inner regular as is seen by the following formula :

$$\begin{aligned}[\sup(m, n)](B) &= \sup[m(B') + n(B'')] \\ &= \sup\{[m(K') + n(K'')] : K' \subset B', K'' \subset B'', K', K'' \text{ compact}\}, \\ &= \sup\{[\sup(m, n)](K) : K \subset B, K \text{ compact}\}.\end{aligned}$$

Hence, $\sup(m, n)$ defines a Radon measure which is evidently the smallest Radon measure majorising both m and n. Hence, $\sup(\mu, \nu)$ exists in the class of Radon measures and moreover,

$$\sup(\mu^\cdot, \nu^\cdot) = [\sup(\mu, \nu)]^\cdot$$

on τ_β. (Perhaps $\sup(\mu^*, \nu^*) \neq [\sup(\mu, \nu)]^*$. Hence, if $(\mu_i)_{i \in I}$ is an increasing directed family of Radon measures bounded by a Radon measure, then $\sup m_i$ is a locally finite and inner regular measure, i.e. a Radon measure. Hence $\sup \mu_i$ exists and also, $\sup \mu_i^\cdot = (\sup \mu_i)^\cdot$. Combining the above results we get that in the family of Radon measures on X, every collection which is bounded above admits an upper bound. We get also that every collection of Radon measures has a lower bound but there is no simple explicit formula for this lower bound.

However,

$$[\inf(\mu, \nu)]^\cdot(B) = [\inf(\mu^\cdot, \nu^\cdot)](B)$$

for every Borel set B for which both $\mu^\cdot$ and $\nu^\cdot$ are finite; also for a decreasing directed family $(\mu_i)_{i \in I}$,

$$\inf \mu_i^\cdot(B) = (\inf \mu_i)^\cdot(B)$$

whenever the left side is finite. But in the case of Radon measures we always have $\inf(\mu, \nu) + \sup(\mu, \nu) = \mu + \nu$, as can be seen easily by considering the values on the compact sets.

PROPOSITION 7. *An increasing directed family $(\mu_i)_{i\in I}$ of Radon measures is bounded above, if every $x \in X$ has a neighbourhood V_x such that the real numbers $\mu_i^*(V_x)$ are bounded. In this case, $\mu^*(B) = \sup_{i\in I} \mu_i^*(B)$ for $B \in \tau_\beta$ defines a Radon measure μ on X; μ is exactly the upper bound of the family $(\mu_i)_{i\in I}$. Moreover, for every $f > 0$, $\mu^*(f) = \sup_{i\in I} \mu_i(f)$.*

PROOF. We already saw that μ^*, as defined above, is locally finite and inner regular and hence μ is a Radon measure. Evidently μ is the upper bound for the given family of Radon measures.

Now, assume first that f is bounded, say $f < M$, and carried by a compact set K. Then $\mu^*(f) = \mu_i^*(f) + (\mu - \mu_i)^*(f)$; but the second term on the right is bounded by $M(\mu - \mu_i)^*(K)$, which tends to zero following the directed set I, and the required result follows in this case. In the general case,

$$\begin{aligned}\mu^*(f) &= \sup\{\mu^*[\inf(f, n)\,\chi_K] : K \text{ compact} \subset X, n \in \mathbf{N}\} \\ &= \sup_{i,K,n} \mu_i^*[\inf(f, n)\,\chi_K] \\ &= \sup_{i\in I} \mu_i^*(f).\end{aligned}$$

This completes the proof.

DEFINITION 14. *Let $(\mu_i)_{i\in I}$ be a family of Radon measures. The family is said to be summable if every point $x \in X$ has a neighbourhood V_x such that $\sum_{i\in I} \mu_i^*(V_x) < +\infty$. In this case, if we set $\mu^*(B) = \sum_{i\in I} \mu_i^*(B)$, for $B \in \tau_\beta$, then μ^* defines a Radon measure which is called the sum of the family $(\mu_i)_{i\in I}$.*

If μ is the sum of a summable family of Radon measures $\{\mu_i : i \in I\}$ then for every $f > 0$, $\mu^*(f) = \sum_{i\in I} \mu_i^*(f)$. This is true for a finite sum (Corollary to Theorem 14) and for the upper bound of a directed (increasing) family, and we have

$$\sum_{i \in I} \mu_i = \sup [(\sum_{i \in J} \mu_i) : J \text{ finite} \subset I].$$

If $\{(\mu_i)_{i \in I}, N\}$ is a μ-concassage, one has exactly $\mu = \sum_{i \in I} j_i \mu_{K_i}$ where μ_{K_i} is the induced measure and j_i the canonical injection $K_i \to X$: or $\mu = \sum_{i \in I} \chi_{K_i} \mu$. For, by the local countability of I, these measures coincide on the compact subsets of X and therefore on τ_β.

The ordered semi-group of abstract measures on τ_β is not regular, i.e. the relation $m + n = m' + n$ or $< m' + n$ does not necessarily imply $m = m'$ or $m < m'$. (Take, for example, the measure $n = +\infty$ for any non-void set.) However the semi-group of Radon measures is regular. For, such an inequality will imply $\mu(K) < \mu'(K)$ for every compact set and we get immediately $\mu < \mu'$, by the inner regularity. Hence the semi-group of Radon measure generates a group, in fact a vector space over the reals. We shall call an element of this vector space a *real-valued Radon measure*. Moreover this space is a complete lattice. If $\{\mu_i\}_{i \in I}$ is a family of real-valued Radon measures bounded by a similar measure λ, the supremum is defined by going back to positive measures ; thus,

$$\sup_{i \in I} \mu_i = \lambda - \inf_{i \in I} (\lambda - \mu_i).$$

The relations $[\sup(\mu, \nu)](B) = \sup[\mu(B') + \nu(B'')]$ etc. for any relatively compact set B are preserved. We have always, $\sup(\mu, \nu) + \inf(\mu, \nu) = \mu + \nu$. For a real-valued Radon measure μ, we define $\mu^+ = \sup(\mu, 0)$, $\mu^- = \sup(-\mu, 0)$ and $|\mu| = \sup(\mu, -\mu) = \mu^+ + \mu^-$. We have, for any Borel set B, the formula

$$(\mu^+)^*(B) = \sup \{\mu(K) : K \text{ compact} \subset B\}.$$

By a $|\mu|$-concassage, we have a partition $X = X^+ \cup X^-$ such that μ^+ is carried by X^+ and μ^- by X^- ; X^+ and X^- can be chosen to be locally Borel. Thus we have the theory of integration with real-valued Radon measures. Complex-valued Radon measures are easily defined and they have the usual properties.

We note that a real or complex-valued Radon measure μ is not a function on the Borel σ-algebra. The function μ is countably additive

but we can only assert that $\mu(B)$ is defined for every relatively compact Borel set B. A real or complex-valued Radon measure can be defined directly, without first defining positive Radon measures in the following way. It will be a real or complex-valued function defined on the class of all relatively compact Borel sets satisfying (i) countable additivity, (ii) inner regularity in the sense: every relatively compact Borel set B is the union of a countable number of compact sets and a subset N of B where N is such that $\mu(N')=0$ for every Borel set $N' \subset N$; and (iii) local finiteness, i.e. for every point $x \in X$, there is a neighbourhood V_x and an $H>0$ such that $|\mu(B)| \leqslant H$ for every relatively compact Borel set $B \subset V_x$. If μ is a real-valued Radon measure in the above sense and also $\mu \geqslant 0$, we find that μ can be extended uniquely to a Radon measure in the original sense. In any case, the theory of integration has first to be done for positive Radon measures.

8. Different ways of defining Radon measures†. A Radon measure on a general topological space can be introduced in many ways. What we have seen already is one method. But similar set functions have been introduced by several authors and we shall describe some of them here.

(1) Let X and $\tilde{X}$ be two Hausdorff topological spaces and $X \subset \tilde{X}$. To every measure $\tilde{\mu}$ on an open neighbourhood Ω of X, concentrated on X, (i.e. $\mu^*(\Omega - X)=0$), one can associate the induced measure $\mu = \tilde{\mu}_X$ on X. Suppose $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are two measures, defined on open neighbourhoods Ω_1 and Ω_2 of X and concentrated on X; if $\tilde{\mu}_1$ and $\tilde{\mu}_2$ induce the same measure on $\Omega_1 \cap \Omega_2$, or more generally on some open neighbourhood of X (in $\tilde{X}$) contained in $\Omega_1 \cap \Omega_2$, then they induce the same measure on X. Conversely, if $\tilde{\mu}_1$ and $\tilde{\mu}_2$ induce the same measure on X, then they induce the same measure on $\Omega_1 \cap \Omega_2$. For, if $B \subset \Omega_1 \cap \Omega_2$ is a Borel set, then for $i=1, 2$, one has

† See also p. 31.

$$\tilde{\mu}_i^*(B) = \tilde{\mu}_i^*(B \cap X) + \tilde{\mu}_i^*[B \cap (\Omega_i - X)]$$
$$= \tilde{\mu}_i^*(B \cap X),$$

and this last number is the same for $i = 1, 2$.

Let us now show that if μ is a Radon measure on X, then there exists a neighbourhood Ω of X in $\tilde{X}$ and a measure $\tilde{\mu}$ on $\tilde{X}$ concentrated on X such that $\mu = \tilde{\mu}_X$. Consider all the open sets V such that $\mu^*(X \cap V) < +\infty$. For any such open set V, the canonical injection of $X \cap V$ into V is $\mu_{X \cap V}$-proper, because of the condition $\mu^*(X \cap V) < +\infty$. Hence there is a measure $\tilde{\mu}_V$ on V for each such open set. If V_1 and V_2 are two such open sets, then $\tilde{\mu}_{V_1}$ and $\tilde{\mu}_{V_2}$ coincide on $V_1 \cap V_2$. For, if B is any Borel subset of $V_1 \cap V_2$, then

$$\tilde{\mu}_{V_1}^*(B) = \mu^*(B \cap X) = \tilde{\mu}_{V_2}^*(B).$$

Let Ω be the union of these open sets. Then, by using the glueing of pieces of measures, we see immediately that there exists a unique Radon measure $\tilde{\mu}$ on Ω inducing on each open set V (of the above family) the measure $\tilde{\mu}_V$. All the measures $\tilde{\mu}_V$ are carried by X, therefore $\tilde{\mu}$ too. Further, since these open sets are increasing directed, for every Borel set $B \subset X$,

$$\tilde{\mu}^*(B) = \sup_V \mu^*(B \cap V) \leqslant \mu^*(B).$$

On the other hand,

$$\tilde{\mu}^*(B) \geqslant \sup \{\mu^*(B \cap K) : K \subset X,\ K \text{ compact}\},$$

for every compact set K of X is contained in some V of the family and hence, $\tilde{\mu}^*(B) \geqslant \mu^*(B)$ and we have $\tilde{\mu}^*(B) = \mu^*(B)$, i.e. $\tilde{\mu}$ induces on X the measure μ. This particular $\tilde{\mu}$ corresponds to the greatest possible Ω; more precisely, to the largest open neighbourhood Ω of X in $\tilde{X}$ such that the canonical injection of X in Ω is μ-proper. Of course, all this presupposes a knowledge of measures on $\tilde{X}$. Let us call a 'pre-measure' on X any Radon measure $\tilde{\mu}$ on an open neighbourhood Ω of X in $\tilde{X}$ concentrated on X. Two such pre-measures $\tilde{\mu}_i$ $(i = 1, 2)$ on Ω_i will be called equivalent if $\tilde{\mu}_i$ induce the same measure on $\Omega_1 \cap \Omega_2$ or on some open set of the intersection

containing X. A Radon measure on X can then be defined as an equivalence class of pre-measures. We have just seen that the mapping which takes each class of pre-measures $\tilde{\mu}$ to the associated measure on X is bijective.

In particular the Radon theory of measures is known on locally compact Hausdorff topological spaces and the above procedure permits the introduction of Radon measures on *completely regular* spaces. For, a completely regular space X can be embedded in a compact space $\tilde{X}$ and any open set of $\tilde{X}$ is locally compact. It is by this method that we had given (in a course of lectures given in Paris in 1964-65) a theory of Radon measures on completely regular spaces. By considering the real-valued measures $\tilde{\mu}$ on the open subsets of $\tilde{X}$, we get directly the real Radon measures on X.

Let us now consider the particular case of finite Radon measures. Then the largest open set Ω containing X such that the canonical injection of X in Ω is μ-proper is evidently the whole space $\tilde{X}$. Hence, there exists a bijective correspondence between the (finite) Radon measures on X and the Radon measures on $\tilde{X}$, concentrated on X. This one-one correspondence is given by: $\tilde{\mu}$ is the image of the measure μ by the canonical injection and μ is the measure induced on X by $\tilde{\mu}$. Hence, the theory of Radon measures on locally compact spaces (resp. compact spaces) permits us to have Radon measures (resp. finite Radon measures) on completely regular spaces. This approach is justifiable since most of the spaces dealt with in practice are completely regular. Nevertheless, this method is applicable only to the restrictive class of completely regular spaces and further it presupposes a complete knowledge of a theory of measures on locally compact spaces, only to declare finally that local compactness is a superfluous hypotheses.

However, we note that the choice of $\tilde{X}$ above is arbitrary. For a completely regular space X, we can take the Stone-Čech compactification $\check{X}$, as also any other compactification. Let us now take the case of finite Radon measures on X. We know that the

bounded continuous functions on X can be identified with the continuous functions on $\check{X}$. A finite Radon measure on X is a measure $\check{\mu}$ on $\check{X}$ concentrated on X. That is the same as saying that it is a positive linear form on $\mathscr{B}(X)$, the class of all bounded continuous functions on X, verifying the following condition, which we shall call the (ϵ, K) condition:

Given $\epsilon > 0$, there exists a compact set $K \subset X$ such that, for every $\phi \in \mathscr{B}(X)$ satisfying $0 \leq \phi \leq 1$, and $\phi = 0$ on K, the inequality $\mu(\phi) < \epsilon$ holds good.

A finite complex Radon measure will be a continuous linear form μ on $\mathscr{B}(X)$ satisfying the (ϵ, K)-condition, viz. for every $\epsilon > 0$ there exists a compact set $K \subset X$ such that whenever $\phi \in \mathscr{B}(X)$, $|\phi| \leq 1$ and $\phi = 0$ on K, the inequality $|\mu(\phi)| < \epsilon$ is true.

From this definition of finite Radon measures on a completely regular space we can construct the complete theory of finite Radon measure on this space as in locally compact spaces. This is what we had done in the earlier course mentioned above. *cf.* Khelifa Zizi [1]. For earlier work on measures on completely regular spaces, see Alexandrov [1] and Varadarajan [1].

(2) Starting from the notion of '*compactology*' introduced by A. Weil, P.A. Meyer introduced a notion of Radon measures on general Hausdorff topological spaces.

DEFINITION 15. *Let, for every compact set K of a topological (Hausdorff) space X, μ_K be a Radon measure on K. This family of measures is called a compactology if* (i) *the measures μ_K are mutually compatible, i.e. if $B \subset K_1 \cap K_2$, then $\mu_{K_1}(B) = \mu_{K_2}(B)$, where B is a Borel set and* (ii) *for every $x \in X$, there exists a neighbourhood V_x of x such that for all $K \subset V_x$, $\mu_K(K)$ are uniformly bounded.*

If we start with a Radon measure according to our definition then it is trivial to see that μ induced on every compact set K gives a compactology. Conversely, given a compactology it is easy to see that it gives rise to the Radon measure defined by setting for any Borel set B, $\mu^*(B) = \sup_{K \subset X} \mu_K(B \cap K)$. This measure μ is

exactly the upper bound of the image measures $j_K(\mu_K)$ where j_K is the injection of K into X. Meyer's method of studying the measures on general spaces starting from this definition is to feed back information from the study on compact spaces. Once again in this method one starts with measures on compact Hausdorff spaces finally to say that compactness is unnecessary. But perhaps this is not a great defect, since in our proofs of the earlier sections (of course starting from the general definition) we have often used the intermediary stage of passing through compact sets.

In Meyer's method, a complex-valued Radon measure would be a family of complex measures μ_K with the compatibility condition and local finiteness (i.e. for every x, there is a neighbourhood V_x and a number $M > 0$ such that for all K, $|\mu_K|^*(V_x \cap K) < M$).

(3) G. Choquet [1] has defined Radon measures by yet another method. Let μ be a Radon measure on a locally compact space. Then the function defined on the class of all compact sets: $K \longmapsto \mu(K)$, is non-negative, finite-valued, increasing, strongly additive [i.e. $\mu(K_1)+\mu(K_2)=\mu(K_1 \cup K_2)+\mu(K_1 \cap K_2)$] and continuous on the right [i.e. given $\epsilon>0$ and a compact set K, there exists an open set $V \supset K$ such that, for every compact set K' with $K \subset K' \subset V$, $\mu(K) \geq \mu(K')-\epsilon$]. Conversely, Choquet remarked that every set function on the class of compact sets, satisfying the above conditions (viz. positive finite-valued, increasing, strongly additive and right continuous) comes from a unique Radon measure. Hence he defined a Radon measure on any general space as a set function of the above type on compact sets. This definition is equivalent to the others. For a Radon measure in our sense trivially defines such a set function. Conversely, let L be such a set function. For every compact set K, consider the Radon measure μ_K defined on K by the restriction of L to the compact subsets of K. Then these measures μ_K form a compactology, the local finiteness being a consequence of the right continuity of L. This method of Choquet gives a short definition. Moreover, the definition is essentially the same for complex Radon measures; $K \longmapsto \mu(K)$ will be strongly additive and continuous on the right (an analogous definition with the exception

that $|\mu(K)-\mu(K')|<\epsilon$). But starting from this definition one has still to develop a theory of measure and integration and consequently it is difficult to judge the advantages and disadvantages of this approach. However, it looks unlikely that this method will avoid the defect of having to consider first measures on compact spaces and this will bring us back more or less to the method of Meyer.

9. Tensor product of measures. Let $(X, \mathscr{X}, m)$, $(Y, \mathscr{Y}, n)$ be two measure spaces. Let τ be the σ-algebra on $X \times Y$, generated by the sets $A \times B$, $A \in \mathscr{X}$, $B \in \mathscr{Y}$. One proves that there exists a unique measure on τ, called the *tensor product* and denoted by $m \otimes n$, such that $(m \otimes n)\ (A \times B) = m(A)n(B)$ (always with the usual convention $0 \times \infty = 0$). This tensor product satisfies the well-known Lebesgue-Fubini theorem.

Assume that X and Y are Hausdorff topological spaces, and $\mathscr{X}, \mathscr{Y}$ are the Borel σ-algebras. The σ-algebra τ is generally smaller than the Borel σ-algebra of $X \times Y$; for an open set of $X \times Y$ is an arbitrary (not necessarily countable) union of sets $A \times B$, A and B open. Hence the tensor product of two Borel measures on X, Y, need not be a Borel measure. Let us examine the case of Radon measures.

Let μ and ν be Radon measures on the spaces X and Y respectively. We shall now define the tensor product $\mu \otimes \nu$ of these two measures on the product space $X \times Y$. We shall first give a proof using the extension lemma and then an alternative proof using the existence of product measures on a product of compact spaces.

THEOREM 17. *There exists a unique Radon measure λ on $X \times Y$ such that*

$$\lambda^*(A \times B) = \mu^*(A)\, \nu^*(B), \tag{1}$$

where A and B are Borel subsets of X and Y respectively.

This Radon measure λ is called the *tensor product* of μ and ν and is denoted by $\mu \otimes \nu$.

PROOF. Let us call a set of the form $C \times D$ in $X \times Y$ a rectangle; and a Borel rectangle if C and D are Borel subsets of the respective

spaces. There exists at the most one Radon measure on $X \times Y$ with prescribed values on Borel rectangles. For, a finite union of Borel rectangles is a finite union of mutually disjoint Borel rectangles. The values of any such Radon measure are fixed on all finite unions of Borel rectangles and in particular on the finite unions of open rectangles. Hence the values of the measure are known on every open set (since every open set of the product is the increasing limit of an increasing directed family of finite unions of open rectangles). Hence the measure is fixed on all compact sets (by outer regularity) and in turn on all Borel sets of the product (using the inner regularity). It remains to show, hence, the existence of such a measure.

FIRST PROOF OF EXISTENCE AND UNIQUENESS. Let $\mathscr{S}_\mu$ (resp. $\mathscr{S}_\nu$) denote the σ-ring of sets of σ-finite μ(resp. ν) measure. If $\mathscr{S}_\lambda$ denotes the σ-ring generated by sets of the form $A \times B$, $A \in \mathscr{S}_\mu$ and $B \in \mathscr{S}_\nu$, then there exists a unique abstract measure λ on $\mathscr{S}_\lambda$ which is σ-finite and such that $\lambda(A \times B) = \mu(A)\, \nu(B)$, for every $A \in \mathscr{S}_\mu$ and $B \in \mathscr{S}_\nu$.

Since μ and ν are locally finite, it follows that λ is locally finite which is the condition (a) of the extension lemma (Theorem 15). The measure λ is inner regular with respect to the compact sets. This is immediate for rectangles $A \times B$ with sides in $\mathscr{S}_\mu$ and $\mathscr{S}_\nu$ respectively; and hence for the disjoint union of a finite number of any such rectangles. But these form a ring $\mathscr{R}$ of subsets of $X \times Y$ and $\mathscr{S}_\lambda$ is the σ-ring generated by this ring $\mathscr{R}$. Now it can be seen easily that the class of all sets for which λ is inner regular is a monotone class and hence it contains the σ-ring generated by $\mathscr{R}$, i.e. it contains $\mathscr{S}_\lambda$. It follows that for every set in $\mathscr{S}_\lambda$, λ is inner regular with respect to the compact sets and hence in particular the conditions (b) and (c) of the extension lemma are verified. It is easily seen that $\mathscr{S}_\lambda$ contains a base for the topology of $X \times Y$ (namely, open rectangles whose sides are of finite measure). Hence by the extension lemma there exists a unique Radon measure (which we also denote by λ) on $X \times Y$, extending the measure λ on $\mathscr{S}_\lambda$. Now, it can be easily shown by inner regularity that λ is the required

measure, completing the proof of the theorem. Note that in this proof uniqueness is proved at the same time as existence.

ALTERNATIVE PROOF OF EXISTENCE. Let $K_1 \subset X$ and $K_2 \subset Y$ be compact sets. Then the theory of measures on compact Hausdorff spaces gives a unique Radon measure $\lambda_{K_1 K_2}$ on $K_1 \times K_2$ satisfying the fundamental equation (1) on $K_1 \times K_2$. If K is any compact set contained in $X \times Y$, then $K \subset \mathrm{Proj}_X K \times \mathrm{Proj}_Y K$ and taking the induced measure on K we get a Radon measure λ_K on K. It is trivial to verify that these measures (uniquely defined on K) are consistent. Further if $G_1 \subset X$ and $G_2 \subset Y$ are open sets of finite measure then for all compact sets K contained in $G_1 \times G_2$,

$$\lambda_K(K) \leq \mu(G_1)\, \nu(G_2).$$

This shows that λ_K are locally uniformly bounded and this shows that we have a compactology. Hence we have a Radon measure on $X \times Y$. This measure satisfies the required condition (1). For, if $A \times B$ is a Borel rectangle, then for any subset $K \subset A \times B$, $K \subset \mathrm{Proj}_X K \times \mathrm{Proj}_Y K \subset A \times B$ and hence,

$$\lambda^*(A \times B) = \sup\{\lambda(K_1 \times K_2) \colon K_1 \subset A, K_2 \subset B\} = \mu^*(A)\, \nu^*(B).$$

This completes the proof of existence. Uniqueness was shown right at the beginning of the proof of the theorem.

THEOREM 18. (Fubini's Theorem)† (i) If $f \geq 0$ *on* $X \times Y$, *then*

$$\int\int^* f(x, y)\ (\mu \otimes \nu)\ (dx\, dy) \geq \int^* \nu(dy) \int^* f(x, y)\ \mu(dx).$$

(ii) *If f is strictly integrable with respect to the product measure $\mu \otimes \nu$, then for ν-strictly almost all y, $f(x, y)$ is μ-strictly integrable, $y \longmapsto \int^* f(x, y)\ \mu(dx)$ is ν-strictly integrable and*

$$\int\int^* f(x, y)\ (\mu \otimes \nu)\ (dx\, dy) = \int^* \nu(dy) \int^* f(x, y)\ \mu(dx).$$

(iii) *If f is ≥ 0 (not necessarily finite), moderated, and $(\mu \otimes \nu)$-measurable, then for ν-strictly almost all y, $f(x, y)$ is μ-measurable,*

† Observe that Fubini's theorem involves μ^*, ν^* and not $\mu^\cdot$, $\nu^\cdot$.

$y \longmapsto \int^* f(x,y)\mu(dx)$ *is* ν*-measurable and one has the same equality.*

FIRST PROOF. It is enough to prove (i) assuming that the left hand side is finite. If f is the characteristic function of a finite union of open rectangles, the required theorem follows by the abstract Fubini theorem that

$$\int\int f(x,y)\,(\mu\otimes\nu)\,(dx\,dy) = \int \nu(dy)\int f(x,y)\,\mu(dx).$$

If f is the characteristic function of any open set in the product space, then f is the limit of $\{g_i\}_{i\in I}$, where (i) each g_i is the characteristic function of a finite union of open rectangles and (ii) $(g_i)_{i\in I}$ is increasing directed. Hence, we have

$$\int\int f(x,y)\,(\mu\otimes\nu)\,(dx\,dy) = \lim_{i\in I}\int\int g_i(x,y)\,(\mu\otimes\nu)\,(dx\,dy).$$

Also, for each $y\in Y$, $g_i(x,y)$ is the characteristic function of an open set in X and the limit of the increasing directed family $\{g_i(\cdot,y)\}$ of such functions is $f(\cdot,y)$ and so,

$$\int f(x,y)\mu(dx) = \lim_{i\in I}\int g_i(x,y)\,\mu\,(dx).$$

Now, we shall show that $h_i(y) = \int g_i(x,y)\,\mu(dx)$ is a lower semi-continuous function for each $i\in I$. For each $i\in I$,

$$g_i(x,y) = \max\{g_i^1(x,y), \ldots, g_i^k(x,y), \ldots, g_i^{n_i}(x,y)\},$$

where the g_i^k are the characteristic functions of the open rectangles $A_i^k \times B_i^k$ which form the set corresponding to i.

Now, if $y \in \bigcap_{j=1}^{l} B_i^{r_j}$ $(1 \leqslant r_j \leqslant n_i)$ then, $h_i(y) \geqslant \mu(\bigcup_{j=1}^{1} A_i^{r_j})$ and in fact, the equality holds except on the open subset $\bigcup_{p\neq r_1,r_2,\ldots,r_l} [B_i^p \cap \bigcap_{j=1}^{l} B_i^{r_j}]$. If $y \notin B_i^k$ for any k, then $h_i(y) = 0$. Hence, for any $\alpha \geqslant 0$, the set $\{y : h_i(y) > \alpha\}$ is open in Y, this set being the union of all the $\bigcap_{j=1}^{l} B_i^{r_j}$ for which $\alpha < \mu[\bigcup_{j=1}^{l} A_i^{r_j}]$. It follows that h_i is lower semi-continuous on $X \times Y$. Hence,

$$\int \lim_{i \in I} h_i(y)\, \nu\,(dy) = \lim_{i \in I} \int h_i(y)\, \nu\,(dy)$$

and so,

$$\int^* \nu(dy) \int^* f(x, y)\, \mu(dx) = \lim_{i \in I} \int^* \nu(dy) \int^* g_i(x, y)\, \mu\,(dx)$$

$$= \lim_{i \in I} \int\int^* g_i\,(x, y)\, (\mu \otimes \nu)\, (dx\, dy)$$

$$= \int\int^* f\, d(\mu \otimes \nu).$$

It follows that for the characteristic function χ_E of a set with $(\mu \otimes \nu)^*(E) = 0$,

$$\int\int^* \chi_E\, d(\mu \otimes \nu) = \int^* \nu(dy) \int^* \mu(dx)\, \chi_E(x, y) = 0. \qquad (1)$$

Hence the same equality holds if $f \geqslant 0$ is any function carried by a set whose $(\mu \otimes \nu)^*$ -measure is zero. Now, if f is the characteristic function of a measurable set E with $(\mu \otimes \nu)^*(E) < +\infty$, then there exist characteristic functions g_n of open sets of finite measures such that g_n decreases to $f + h$, where h is the characteristic function of a set which is $\mu \otimes \nu$-strictly negligible. Hence, we have

$$\int\int^* f\, d(\mu \otimes \nu) = \int^* \nu(dy) \int^* \mu(dx)\, f(x, y)$$

and for ν-strictly almost all y, f_y is μ-strictly integrable. Hence the equality holds good if f is an arbitrary $\mu \otimes \nu$-strictly integrable step function. Since every $\mu \otimes \nu$-strictly integrable non-negative function f is the limit of an increasing sequence of functions f_n, where f_n (for every n) is a finite positive linear combination of characteristic functions of $(\mu \otimes \nu)$-strictly integrable subsets, the equality is true for all non-negative $\mu \otimes \nu$-strictly integrable functions f (using the monotone convergence theorem and (1) above). In particular, since the right hand side is finite, except for a set which is ν-strictly negligible for all $y \in Y$, $f(\cdot, y)$ is μ-strictly integrable. These results now follow for an arbitrary μ-strictly integrable function, i.e. (ii) is proved. (iii) follows easily (each function $f \geqslant 0$ which is $(\mu \otimes \nu)$-measurable and moderated, is

the limit of an increasing sequence of $(\mu \otimes \nu)$-strictly integrable functions).

It remains to show (i) for any $f \geq 0$ with $\int\int^* f d(\mu \otimes \nu) < +\infty$. Let g be a step function of finite $(\mu \otimes \nu)^*$-integral and $g \geq f$. Then

$$\int\int^* g\, d(\mu \otimes \nu) = \int^* \nu(dy) \int^* g(x, y)\, \mu(dx) \geq \int^* \nu(dy) \int^* f(x, y)\, \mu(dx).$$

So,

$$\int\int^* f\, d(\mu \otimes \nu) = \inf_{g \geq f} \int\int^* g\, d(\mu \otimes \nu) \geq \int^* \nu(dy) \int^* f(x, y)\, \mu(dx),$$

proving (i).

Let us now give an alternative proof of Fubini's Theorem using the properties of product measures on compact spaces.

ALTERNATIVE PROOF. Let $X = \bigcup_{i \in I} X_i \cup N$ and $Y = \bigcup_{j \in J} Y_j \cup M$ be respectively μ- and ν-concassages of X and Y, where the sets X_i (resp. Y_j) are μ (resp. ν) full (Theorem 13). Let $f \geq 0$ be any function on $X \times Y$. There is nothing to prove if $(\mu \otimes \nu)^*(f) = +\infty$ and hence we may assume that this is finite. In other words we can assume that f is carried by a countable union of open sets V_k, each of them of finite $\mu \otimes \nu$-measure. Since $X_i \times Y_j$, for any $i \in I$ and $j \in J$ is $\mu_{X_i} \otimes \nu_{Y_j}$-full, not more than a countable number of them intersect any W_k. Hence, we can find countable subsets I' and J', respectively in I and J such that f is carried by $\bigcup_{i \in I', j \in J'} (X_i \times Y_j) \cup (N \times Y) \cup (X \times M)$. It is clear from the definition of the product measure that $N \times Y$ and $X \times M$ are $(\mu \otimes \nu)$-negligible. In other words $\bigcup_{i \in I, j \in J} (X_i \times Y_j) \cup (N \times Y) \cup (X \times M)$ is a $\mu \otimes \nu$-concassage of $X \times Y$. Now,

$$(\mu \otimes \nu)^*(f) = (\mu \otimes \nu)^\cdot(f) = \sum_{i \in I', j \in J'} (\mu_{X_i} \otimes \nu_{Y_j})^\cdot(f).$$

But now with the knowledge of the tensor product of measures on compact spaces, we get,

$$\sum_{i \in I', j \in J'} (\mu_{X_i} \otimes \nu_{Y_j})^\cdot(f) \geq \sum_{i \in I', j \in J'} \int^\cdot \mu_{X_i} d(x) \int^\cdot \nu_{Y_j}(dy) f(x, y)$$

$$> \sum_{i\in I'} \int^{\bullet} \mu_{X_i}(dx) \sum_{j\in J'} \int^{\bullet} f(x,y)\, \nu_{Y_j}(dy).$$

$$= \sum_{i\in I'} \int^{\bullet} \mu_{X_i}(dx) \sum_{j\in J} \int^{\bullet} f(x,y)\, \nu_{Y_j}(dy)$$

$$= \sum_{i\in I'} \int^{\bullet} \mu_{X_i}(dx) \int^{\bullet} f(x,y)\, \nu(dy).$$

Now replacing $\sum_{i\in I}$ by $\sum_{i\in I'}$, we get

$$\int^{*} f d(\mu\otimes\nu) > \int^{\bullet} \mu(dx) \int^{\bullet} f(x,y)\ \ \nu(dy).^{\dagger}$$

Now we have to replace the integrals by strict integrals on the right side of the above inequality. For this we need the following lemma.

LEMMA 10. *Let Ω be an open set in the product with finite $\mu\otimes\nu$-measure. The function $x \longmapsto \nu^{\bullet}(\Omega_x)$ on X is lower semi-continuous, μ-moderated and finite μ-strictly almost everywhere (where Ω_x is the section through x of Ω).*

PROOF. Let $x \in X$, we know that Ω_x is open. Let $\alpha < \nu^{\bullet}(\Omega_x)$. Let K be a compact set of Y such that $K \subset \Omega_x$ and $\nu(K) > \alpha$. Now, for every $y \in K$, there exist neighbourhoods U_y and V_y of x and y respectively in X and Y such that $U_y \times V_y \subset \Omega$. A finite number of V_y cover K and if U is the intersection of all those U_y's, then $U \times K \subset \Omega$ and hence for every $x' \in U$, the section $\Omega_{x'}$ contains K. It follows that $\nu^{\bullet}(\Omega_{x'}) > \alpha$, for all $x' \in U$. This shows that the function $x \longmapsto \nu^{\bullet}(\Omega_x)$ is lower semi-continuous on X. Now applying to the characteristic function of Ω the result already proved, we get

$$\int^{\bullet} \nu^{\bullet}(\Omega_x)\, \mu(dx) < (\mu\otimes\nu)^{*}(\Omega) < +\infty.$$

† The reader will notice that we interchanged X and Y, μ and ν, in the alternative proof; it has no importance!

But, for the function $\nu^*(\Omega_x)$ which is lower semi-continuous, the integral and strict integral are the same and finite. Hence we deduce that $x \longmapsto \nu^*(\Omega_x)$ is a moderated function and it is finite μ-strictly almost everywhere, completing the proof of the lemma.

Let us now go back to the proof of the theorem. The function f is carried by a countable number of open sets V_k of finite measure. Hence the set of x for which $\nu^*[(V_k)_x] \neq 0$ is μ-moderated and consequently also the set of points for which $\nu^*[V_x] \neq 0$ where $V = \bigcup_k V_k$. Hence, the set of x for which $\int^* f(x, y)\, \nu(dy) \neq 0$ is μ-moderated (as f is carried by V and consequently f_x is carried by V_x). Hence, in the above formula,

$$\int^* \mu(dx) \int^* f(x, y)\, \nu(dy) = \int^* \mu(dx) \int^* f(x, y)\, \nu(dy).$$

On the other hand, for μ-strictly almost every x of X, $\nu^*[(V_k)_x]$ is finite for every k, hence V_x and so f_x is ν-moderated. Hence, for μ-strictly almost every x in X, $\int^* f_x(y)\, \nu(dy) = \int^* f_x(y)\, \nu(dy)$. This leads us to the required inequality

$$\int\int^* f\, d(\mu \otimes \nu) \geq \int^* \mu(dx) \int^* f(x, y)\, \nu(dy).$$

Now to complete the proof of Fubini's theorem, it is enough to note that if $f \geq 0$ is $\mu \otimes \nu$-measurable and $(\mu \otimes \nu)^*(f) < +\infty$, then f is $(\mu \otimes \nu)$-moderated and all the inequalities considered in the proof of the first part are actually equalities.

The above proof gives us immediately the following useful result.

PROPOSITION 8. *If f is a $\mu \otimes \nu$-moderated function then for μ-strictly almost every $x \in X$, the function f_x is ν-moderated and, moreover $x \longmapsto \nu^*(f_x)$ is μ-moderated.*

THEOREM 19. *Let $f \geq 0$ be any lower semi-continuous function on $X \times Y$. Then $x \longmapsto \int^* f(x, y)\, \nu(dy)$ is lower semi-continuous and further,*

$$\int\int^* f\, d(\mu \otimes \nu) = \int^* \mu(dx) \int^* f(x, y)\, \nu(dy).$$

PROOF. The first assertion is true (by Lemma 10) for the characteristic functions of open sets with finite $(\mu \otimes \nu)$-measure. The equality of the integrals has been proved for any $(\mu \otimes \nu)$-moderated lower semi-continuous function f. Utilising Proposition 5 we see that if for an increasing directed family $\{f_i\}_{i \in I}$, $f_i \geqslant 0$, f_i lower semi-continuous, these two properties are true, then the same holds for the limit f of the family. Now every $f \geqslant 0$, lower semi-continuous and $\mu \otimes \nu$-moderated, is the increasing limit of a sequence of linear combinations of functions for which both the properties hold good (for instance, characteristic functions of open sets which are σ-finite). Suppose f is a lower semi-continuous function $\geqslant 0$ and moderated. Then there exists a sequence of simple functions s_n, each one of them > 0, moderated, and such that s_n increases to f. But since f is lower semi-continuous the s_n, for each n, can be constructed as a linear combination of characteristic functions of open sets. For fixed n, call $A_{k,n}$ the set where $\frac{k}{2^n} < f < \frac{k+1}{2^n}$. Put $s_n = \sum_{k=0}^{\infty} \frac{k}{2^n} \chi_{A_{k,n}}$; then $s_n \uparrow f$ for $n \to +\infty$. The set $B_{h,n} = \bigcup_{k \geqslant h} A_{k,n}$ is open since f is lower semi-continuous and we have

$$s_n = \sum_{k=0}^{\infty} \frac{k}{2^n} \left(\chi_{B_{k,n}} - \chi_{B_{k+1,n}} \right)$$

$$= \frac{1}{2^n} \sum_{h=1}^{+\infty} \chi_{B_{h,n}}$$

is lower semi-continuous. Hence the required result is proved for functions of this form.

Now, $\mu \otimes \nu$ being locally finite, the open sets of finite measure form an increasing directed family whose union is the whole space. Hence, any lower semi-continuous $f \geqslant 0$ is the limit of the (increasing directed) family of functions $\{f \chi_V\}$, V any open set of finite measure. This completes the proof.

PROPOSITION 9. *Let $f : X \times Y \to Z$ be a Lusin $\mu \otimes \nu$-measurable function, where Z is any arbitrary topological space. Let f be a*

constant outside a moderated set. Then, for μ-strictly almost every x in X, f_x is ν-Lusin measurable.

PROOF. Let us consider a μ (resp. ν)-concassage $\bigcup_{i\in I} X_i \cup N$ (resp. $\bigcup_{j\in J} Y_j \cup M$) where the compact sets are μ-full (resp. ν-full). Since f is constant outside a moderated set, we can find countable subfamilies I' of I and J' of J such that f is a constant on $X_i \times Y_j$ for $(i, j) \notin I' \times J'$. For μ_{X_i}-almost every x of X_i, the section f_x has a restriction to Y_j which is ν_{Y_j}- Lusin measurable. Hence it is also ν-measurable on $\bigcup_{j\in J'} Y_j$. But since it is a constant on the ν-measurable set $\bigcup_{j\in J'} Y_j$ and M is a ν-negligible set, the sections f_x are ν-measurable on Y for μ_{X_i} almost every x in X_i. Hence the same is true for μ-almost every x in $\bigcup_{i\in I'} X_i$. But since f is a constant for all x in $\bigcup_{i\in I'} X_i$ the ν-measurability of f_x is true for all $x \in X$ but for a set which is μ-negligible. But by assumption f is a constant outside a set A which is $\mu\otimes\nu$-moderated. Hence the set of values of x for which A_x is not ν-negligible is μ-moderated; outside this set f_x is surely ν-measurable. Now, we deduce that f_x is ν-measurable for all $x \in X$ except for a set which is μ-negligible and μ-moderated, hence of μ^*-measure zero.

THEOREM 20. *Let $\vec{f}$ be a function on $X \times Y$ with values in a Banach space $\vec{F}$ and $\vec{f}$ $(\mu\otimes\nu)$-strictly integrable. Then for μ-strictly almost all x, $\vec{f}_x$ is ν-strictly integrable, $x \longmapsto \int^* \vec{f}(x, y)\ \nu(dy)$ is μ-strictly integrable and*

$$\int\int^* \vec{f}\, d(\mu\otimes\nu) = \int^* \mu(dx) \int^* \vec{f}(x, y)\nu(dy).$$

PROOF. This has been proved for $\vec{F} = \mathbf{R}$. It is true for characteristic functions of measurable sets and hence for simple functions with values in $\vec{F}$. Now, if $\vec{f}$ is $(\mu\otimes\nu)$-strictly integrable, then there exists a sequence of simple functions $\vec{f}_n$ such that

$$(\mu\otimes\nu)^*(\|\vec{f}_n - \vec{f}\|) \to 0.$$

Hence, we have,

$$\int\int^* f(x, y)\, d(\mu\otimes\nu) = \lim_{n\to\infty} \int\int \vec{f}_n(x, y)\, d(\mu\otimes\nu).$$

But, $(\mu\otimes\nu)^*(\|\vec{f}_n - \vec{f}\|) \geq \int^* d\mu \int^* \|\vec{f}_n - \vec{f}\|\, d\nu$ and the left side tends to zero. Hence, we can extract a subsequence such that $\int^* \|\vec{f}_n - \vec{f}\|\, d\nu \to 0$ for μ-strictly almost every $x \in X$. Now, since for μ-strictly almost every $x \in X$, $(\vec{f}_n)_x$ is ν-strictly integrable we deduce the same property for $\vec{f}_x$. Hence,

$$\int^* \mu(dx) \left\| \int [\vec{f}(x, y) - \vec{f}_n(x, y)]\ \nu(dy) \right\| \to 0.$$

Hence the function $x \longmapsto \int^* f(x, y)\ \nu(dy)$ is strictly μ-integrable. Further,

$$\int^* \mu(dx) \int^* \vec{f}(x, y)\ \nu(dy) = \lim_{n\to\infty} \int^* \mu(dx) \int^* \vec{f}_n(x, y)\ \nu(dy).$$

Now the proof of the theorem is completed by noting that the required equality is true for all f_n and hence, by taking limits, for f also.

For the convenience of the reader we summarize the results of this section in a single

THEOREM. *Let μ and ν be Radon measures on spaces X and Y respectively. Then*

(i) *there exists a unique Radon measure λ on $X\times Y$ such that*

$$\lambda^*(A\times B) = \mu^*(A)\ \nu^*(B)$$

where A and B are Borel sets of X and Y respectively. This measure λ is called the tensor product of μ and ν and denoted by $\mu\otimes\nu$.

(ii) *If $f \geq 0$ on $X\times Y$, then,*

$$\int\int^* f(x, y)\ (\mu \otimes \nu)\ (dx, dy) \geqslant \int^* \nu(dy) \int^* f(x, y)\ \mu(dx) \qquad (*)$$

(iii) *Equality holds in* (*) *if,*

(a) f *is not necessarily* $\geqslant 0$ *but is* $\mu \otimes \nu$*-strictly integrable* [$f(\cdot, y)$ *is then* μ*-strictly integrable for* ν*-strictly almost all* y *and* $y \longmapsto \int^* f(x, y)\ \mu(dx)$ *is* ν*-strictly integrable*];

(b) $f \geqslant 0$, *moderated and* $\mu \otimes \nu$*-measurable* [$f(\cdot, y)$ *is then* μ*-measurable for* ν*-almost all* y *and* $y \longmapsto \int^* f(x, y)\ \mu(dx)$ *is* ν*-measurable*];

(c) $f \geqslant 0$ *and lower semi-continuous on* $X \times Y$ [$y \longmapsto \int^* f(x, y)\ \mu(dx)$ *is then lower semi-continuous*];

(d) f *is replaced by a Banach-valued* $\mu \otimes \nu$*-strictly integrable function* $\vec{f}$ [*as in* (a), $\vec{f}(\cdot, y)$ *is then* μ*-strictly integrable for* ν*-strictly almost all* y *and* $y \longmapsto \int \vec{f}(x, y)\ \mu(dx)$ *is* ν*-strictly integrable.*

10. Projective limit of measures. Let I be a directed set and let $\{X_i\}_{i \in I}$ be a projective system of Hausdorff spaces, with maps $\pi_{i,j}$: $X_j \to X_i$ when $i \leqslant j$. Let X_I be the projective limit of the X_i and π_i the projection map from X_I to X_i. A *decreasing projective system* of measures is a set of Radon measures μ_i on X_i such that if $i \leqslant j$ then $\pi_{i,j}(\mu_j) \leqslant \mu_i$. An *exact projective system* of measures is a decreasing projective system with the additional condition that for $i \leqslant j$, $\pi_{i,j}(\mu_j) = \mu_i$. The most important special cases of such a system, of frequent occurrence in probability theory, are of the form $\prod_{\lambda \in \Lambda'} X_\lambda$ where Λ' are finite subsets of some index set Λ and the Λ' are ordered by inclusion.

THEOREM 21. (Prokhorov) (a) *Given a decreasing projective system of finite Radon measures* $\{\mu_i\}$ *on* $\{X_i\}$, *there exists a greatest Radon measure* μ *on* X_I *such that* $\pi_i(\mu) \leqslant \mu_i$ *for all* $i \in I$.

(b) *Given an exact projective system of finite Radon measures* $\{\mu_i\}$ *on* $\{X_i\}$ *such that* $\mu_i(X_i) = 1$, *a necessary and sufficient condition that there exists a Radon measure* μ *on* X_I *such that* $\pi_i(\mu) = \mu_i$, *for all* $i \in I$, *is the following:*† *given* $\epsilon > 0$, *there exists a compact set* $K \subset X_I$, *such that for all* $i \in I$, $\mu_i(\pi_i K) \geqslant 1 - \epsilon$.

The measure is called the projective limit of the measures μ_i.

PROOF. The necessity part of (b) is trivial and (b) is known when X_i are all compact spaces. We first prove (a). We then give three different proofs of the sufficiency part of (b). The first proof uses (a). The second, which is the simplest, makes the additional assumption, verified in most cases, that the X_i are completely regular and then uses the Stone-Čech embedding in a compact space. The third proof uses the extension lemma.

PROOF OF (a). We first assume that all X_i are compact. (It follows that X_I is compact). Let Γ denote the subspace of $\mathscr{C} = \mathscr{C}(X_I)$, consisting of functions of the form $\phi_i \circ \pi_i$, for some $i \in I$. Since Γ is a vector space and separates points, it follows at once from the Stone-Weierstrass theorem, that Γ is dense in $\mathscr{C}$, and that any non-negative function in $\mathscr{C}$ is the limit of non-negative functions in Γ, i.e. Γ_+ is dense in $\mathscr{C}_+$.

Let $\phi \in \Gamma_+$, $\phi = \phi_i \circ \pi_i$ where $\phi_i \in \mathscr{C}(X_i)$. Put $\tilde{\mu}_i(\phi) = \mu_i(\phi_i)$. For j such that $j > i$, one has also $\phi = \phi_j \circ \pi_j$ where $\phi_j = \phi_i \circ \pi_{i,j}$; hence,

$$\tilde{\mu}_j(\phi) = \mu_j(\phi_j) = \mu_j(\phi_i \pi_{i,j}) = [\pi_{i,j}(\mu_j)]\,(\phi_i) < \mu_i(\phi_i) = \tilde{\mu}_i(\phi).$$

Hence the set of all $i \in I$ for which ϕ is of the form $\phi_i \circ \pi_i$ is a section of I, and on that section, for $\phi > 0$, $i \longmapsto \tilde{\mu}_i(\phi)$ is ordered and decreasing directed. It has a limit which we call $\tilde{\mu}(\phi)$. Further, if ϕ is continuous and real of the form $\phi_i \circ \pi_i$ for some $i \in I$, it is the difference of two functions in Γ_+ and the $\tilde{\mu}_i(\phi)$ have again a limit $\tilde{\mu}(\phi)$. Thus, $\tilde{\mu} : \phi \longmapsto \tilde{\mu}(\phi)$ is a positive linear form on Γ. It is moreover continuous and can be extended uniquely to a continuous linear form on $\mathscr{C} = \mathscr{C}(X_I)$, i.e. a Radon measure in X_I, which we

† This condition is called the (ϵ, K)-condition.

call μ. As Γ_+ is dense in $\mathscr{C}_+(X_I)$, μ is also positive; it is finite, since $\mu(1) = \lim \mu_i(1)$.

We now show that μ is the required Radon measure. Let $\phi_i \in \mathscr{C}_+(X_i)$; we have,

$$[\pi_i(\mu)]\,(\phi_i) = \mu(\phi_i \circ \pi_i) \leqslant \mu_i(\phi_i \circ \pi_i) = \mu_i(\phi_i),$$

hence $\pi_i(\mu) \leqslant \mu_i$. Conversely, let ν be any positive Radon measure on X_I such that for all $i \in I$, $\pi_i(\nu) \leqslant \mu_i$. Let $\phi \in \Gamma_+$, and i be such that $\phi = \phi_i \circ \pi_i$. Then,

$$\nu(\phi) = \nu(\phi_i \circ \pi_i) = [\pi_i(\nu)]\,(\phi_i) \leqslant \mu_i(\phi_i) = \tilde{\mu}_i(\phi).$$

The same inequality holds good if we replace i by any $j \in I$ such that $i \leqslant j$, so passing to the limit $\nu(\phi) \leqslant \tilde{\mu}(\phi)$. Since Γ_+ is dense in $\mathscr{C}_+(X_I)$ one has $\nu(\phi) \leqslant \mu(\phi)$, for all $\phi \in \mathscr{C}_+$; hence $\nu \leqslant \mu$. This shows that μ is the greatest measure satisfying $\pi_i(\mu) \leqslant \mu_i$ for all $i \in I$.

We now come to the general case. Let K be any compact subset of X_I. We write K_i for $\pi_i(K)$ and π_i^K for the restriction of π_i to K, and $\pi_{i,j}^K$ for the restriction of $\pi_{i,j}$ to K_j. Then $\{K_i, \pi_{i,j}^K\}$ is a projective system of compact spaces. The projective limit of this system is the set of π-coherent families $(x_i)_{i \in I}$ with $x_i \in K_i$; and hence is $\bigcap_{i \in I} \pi_i^{-1}[\pi_i(K)]$. However, since K is compact this intersection coincides with K. [Note that if $x \in X_I$ and $x \notin K$ there exists a finite subset $J \subset I$ and open sets V_j containing x_j for all $j \in J$ such that $\prod_{j \in J} V_j \times \prod_{j \in J} X_j$ does not intersect K. In particular $\pi_j x$ is not in $\pi_j K$ for $j \in J$; if i is an index $> j \in J$, we have $x_i \notin K_i$. Hence the intersection of the sets $\pi_i^{-1}[\pi_i K]$ is K].

We shall now show that $(\mu_i)_{K_i}$ form a decreasing projective system. Let $i \leqslant j$ and B_i be a Borel subset of K_i. Then,

$$\begin{aligned}
\pi_{i,j}^K[(\mu_j)_{K_j}]\,(B_i) &= (\mu_j)_{K_j}[\pi_{i,j}^{-1}(B_i)] \\
&= (\mu_j)[K_j \cap \pi_{i,j}^{-1}(B_i)] \\
&\leqslant \mu_j[\pi_{i,j}^{-1}(B_i)] \\
&= [\pi_{i,j}(\mu_j)]\,(B_i)
\end{aligned}$$

$$\leqslant \mu_i(B_i)$$
$$= (\mu_i)_{K_j}(B_i).$$

Hence, $\pi^K_{i,j}[(\mu_j)_{K_j}] \leqslant (\mu_i)_{K_i}$. [Note that even if the $\{\mu_i\}$ form an exact projective system, because of the first inequality above, the $(\mu_i)_{K_i}$ form in general only a decreasing projective system.]

Thus there exists a μ_K which is the greatest Radon measure on K satisfying $\pi^K_i(\mu_K) \leqslant (\mu_i)_{K_i}$. Replacing the measures $(\mu_i)_{K_i}$ (resp. μ_K) by the measures in X_i (resp. X_I) carried by K_i (resp. K), their images under the injections $K_i \to X_i$ (resp. $K \to X$) (one knows this correspondence to be bijective), one sees that there exists a measure μ^K on X_I carried by K, having the following two properties:

(α) for all $i \in I$, $\pi_i(\mu^K) \leqslant \chi_{K_i}\ \mu_i$;

(β) if ν is a measure on X_I carried by K such that for all $i \in I$, $\pi_i(\nu) \leqslant \chi_{K_i}\ \mu_i$, then $\nu \leqslant \mu^K$.

The positive measures μ^K on X_I form an increasing directed family indexed by the family of all compact subsets of X_I; and the latter set is also increasing directed. For, if $K \subset K'$, then for all i, $K_i \subset K'_i$ and by (α), $\pi_i(\mu^K) \leqslant \chi_{K_i}\ \mu_i \leqslant \chi_{K'_i}\ \mu_i$ and hence by (β) we have $\mu^K \leqslant \mu^{K'}$. The $\mu^K(1)$ are all bounded by $\mu_i(1)$ for any $i \in I$. Hence, there exists a limit Radon measure μ of the μ^K on X. We have for all $i \in I$,

$$\pi_i(\mu) = \lim_K \pi_i(\mu^K) \leqslant \lim_K \chi_{K_i}\ \mu_i = \mu_i,$$

which is the required inequality of the type (α). If, on the other hand, ν is a positive Radon measure on X_I such that $\pi_i(\nu) \leqslant \mu_i$, for all $i \in I$, then $\pi_i(\chi_K \nu) \leqslant \chi_{K_i}\ \mu_i$, since for all Borel sets $B_i \subset K_i$, we have,

$$[\pi_i(\chi_K \nu)](B_i) = (\chi_K \nu)[\pi_i^{-1}(B_i)] = \nu[K \cap \pi_i^{-1}(B_i)]$$
$$\leqslant \nu[\pi_i^{-1}(B_i)]$$
$$= [\pi_i\ \nu](B_i)$$
$$\leqslant \mu_i(B_i) = (\chi_{K_i}\ \mu_i)(B).$$

Since $\chi_K \nu$ is carried by K, we have $\chi_K \nu \leqslant \mu^K \leqslant \mu$. Hence, $\nu = \lim_K \chi_K \nu \leqslant \mu$. This completes the proof of (a) of the theorem.

PROOF OF (b). That the (ϵ, K)-condition is necessary is obvious.

FIRST PROOF OF THE SUFFICIENCY PART OF (b). Assume that the (ϵ, K)-condition is satisfied. Since the measures $\{\mu_i\}$ form a decreasing directed system there exists by (a) a greatest measure μ on X_I such that $\pi_i(\mu) \leqslant \mu_i$ for all $i \in I$. Let K be such that $\mu_i(\pi_i K) \geqslant 1 - \epsilon$, for all $i \in I$. Then

$$(\mu_i)_{K_i}(1) = \mu_i(K_i) \geqslant 1 - \epsilon$$

and hence, $\mu^K(1) \geqslant 1 - \epsilon$ (where μ^K is the measure considered above). Hence, $\mu(1) \geqslant 1 - \epsilon$. Since $\mu(1) \leqslant 1$ and also for arbitrary $\epsilon > 0$, $\mu(1) \geqslant 1 - \epsilon$, we get that $\mu(1) = 1$. So for all $i \in I$, $\pi_i \mu \leqslant \mu_i$ and they have the same total measure 1, hence $\pi_i \mu = \mu_i$. This completes the proof.

SECOND PROOF OF THE SUFFICIENCY PART OF (b). *We assume here the X_i are in addition completely regular.*

As remarked earlier the existence of the projective limit of an exact projective system of measures is known when the X_i are all compact (Bourbaki, [2] page 112; Choksi [1]). In the general case let $\breve{X}_i$ be the Stone-Čech compactification of X_i. Then, for $i \leqslant j$ exists a unique extension of π_{ij} to a map $\breve{\pi}_{i,j}: \breve{X}_j \to \breve{X}_i$; the uniqueness of the extension implies that $\breve{\pi}_{ij} \circ \breve{\pi}_{jk} = \breve{\pi}_{ik}$, if $i \leqslant j$ and $j \leqslant k$. Let $\breve{X}_I$ be the projective limit of $\breve{X}_i$. It follows that the projections $\breve{\pi}_i: \breve{X}_I \to \breve{X}_i$ are extensions of π_i. The injection $f_j: X_j \to \breve{X}_j$ induces a Radon measure $\breve{\mu}_j$ on $\breve{X}_j$. The measures $\breve{\mu}_i$ form an exact system of measures; hence there exists a Radon measure μ on $\breve{X}_I$ such that $\breve{\pi}_i \mu = \breve{\mu}_i$ for all $i \in I$. We now show that μ is concentrated on X_I. For each $\epsilon > 0$, there exists a compact set $K_\epsilon \subset X_I$ such that for each $i \in I$, $\mu_i(\pi_i K_\epsilon) \geqslant 1 - \epsilon$. We show that $\mu(K_\epsilon) \geqslant 1 - \epsilon$. But we have for every $i \in I$, $\mu[\pi_i^{-1}(\pi_i(K_\epsilon))] \geqslant 1 - \epsilon$. Taking the limit and noting that for a decreasing directed family

of compact sets the limit of the measures equals the measure of the limit, we get that $\mu(K_\epsilon) \geqslant 1 - \epsilon$. (As remarked earlier in the first proof, the limit following I of the sets $\pi_i^{-1}(\pi_i K_\epsilon)$ is K_ϵ). From this we deduce that the inner measure of X_I is 1, but the outer measure of X_I is less than or equal to 1. Hence X_I is measurable and $\mu(X_I) = 1$. This completes the proof.

THIRD PROOF OF THE SUFFICIENCY PART OF (b). We first prove the existence of a unique measure $\tilde{\mu}$ on a class of Borel subsets of X_I, which is the projective limit in the sense of Bochner [1]. Let $\mathscr{R}$ denote the class of all sets of the form $\pi_i^{-1}(E_i)$ where E_i is a Borel set in X_i; let $\mathscr{S}$ be the least σ-ring generated by $\mathscr{R}$. Since $\mathscr{R}$ is an algebra of subsets, the equation

$$\tilde{\mu}(\pi_i^{-1}(E_i)) = \mu_i(E_i)$$

consistently defines a finitely additive set function on $\mathscr{R}$. If countably additive it will extend to a unique measure $\tilde{\mu}$ on $\mathscr{S}$ which will be the projective limit measure in the sense of Bochner. Let $\tilde{E}_n$ be a decreasing sequence of sets in $\mathscr{R}$ such that $\tilde{\mu}(\tilde{E}_n) > \epsilon > 0$, for all n. To show that $\tilde{\mu}$ is countably additive it is enough to show that $\bigcap_n \tilde{E}_n \neq \emptyset$.

Since $\tilde{E}_n \in \mathscr{R}$, there exist indices $i_n \in I$ with $i_n < i_{n+1}$ such that $E_n = \pi_{i_n}^{-1}(E_n)$ where E_n is a Borel set in X_{i_n}. There exists, for each n, a compact set $C_n \subset E_n$ such that

$$\mu_{i_n}(C_n) > \epsilon \text{ and } \mu_{i_n}(C_n) > \mu_{i_n}(E_n) - \epsilon/4^n.$$

Let $\tilde{C}_n = \pi_{i_n}^{-1}(C_n)$. Let $\tilde{C}_n' = \bigcap_{r=1}^{n} \tilde{C}_r = \tilde{C}_n - \bigcup_{r=1}^{n-1} (\tilde{E}_r - \tilde{C}_r)$. $\tilde{C}_n$ and $\tilde{C}_n'$ are closed and are in $\mathscr{R}$, $\tilde{C}_n' \supset \tilde{C}_{n+1}'$ and

$$\begin{aligned}\tilde{\mu}(\tilde{C}_n') &> \tilde{\mu}(\tilde{C}_n) - \epsilon \sum_{r=1}^{n-1} \frac{1}{4^r} \\ &> \epsilon - \epsilon \sum_{r=1}^{n-1} \frac{1}{4^r} \\ &> \tfrac{1}{2}\epsilon.\end{aligned}$$

Now, by the (ϵ, K)-condition there exists K compact in X_I such that $\mu_i(\pi_i K) > 1 - \epsilon/2$ for all $i \in I$. Since $\mu_{i_n}(\pi_{i_n} \tilde{C}'_n) > \epsilon/2$, it follows that $\pi_{i_n} K \cap \pi_{i_n} \tilde{C}'_n \neq \emptyset$, for each n, and hence $\tilde{C}'_n \cap K \neq \emptyset$ as $\tilde{C}'_n = \pi_{i_n}^{-1}(\pi_{i_n} \tilde{C}'_n)$. But $\tilde{C}'_n \cap K$ is compact and $\tilde{C}'_n \cap K \supset \tilde{C}'_{n+1} \cap K$. Hence, $\bigcap_n (\tilde{C}'_n \cap K) \neq \emptyset$. It follows that $\bigcap_n \tilde{E}_n \neq \emptyset$. This shows that there exists a measure $\tilde{\mu}$ on $\mathscr{S}$ extending $\tilde{\mu}$ on $\mathscr{R}$.

We next show that the measure $\tilde{\mu}$ on $\mathscr{S}$ satisfies the conditions of the extension lemma. It will then follow that there exists a Radon measure μ on X_I, extending $\tilde{\mu}$ on $\mathscr{S}$ and hence verifying the required condition $\pi_i \mu = \mu_i$. Uniqueness of μ is obvious.

It follows at once from the definition of the topology on the projective limit X_I, that $\mathscr{R}$ and so $\mathscr{S}$ contains a base for the topology of X_I. $\tilde{\mu}$ is finite and hence the condition (a) of the extension lemma is trivial. Every set in $\mathscr{R}$ is inner regular with respect to the closed sets in $\mathscr{R}$. It is now trivial to verify that the family of sets which are inner regular with respect to the closed sets is a monotone class and contains $\mathscr{R}$ and hence it contains $\mathscr{S}$. This is the condition (b) of the extension lemma. It remains only to prove the condition (c′) (since $\tilde{\mu}$ is finite). By the (ϵ, K)-condition, given $\epsilon > 0$, there exists a compact set K in X_I such that $\mu_i(\pi_i K) \geqslant 1 - \epsilon$ for all $i \in I$. We show that $\tilde{\mu}^*(K) \geqslant 1 - \epsilon$. (Note that $\tilde{\mu}(X_I) = 1$). Now, as was shown at the beginning of the extension lemma,

$$\tilde{\mu}^*(K) = \inf \{\tilde{\mu}(G) : G \in \mathscr{S},\ G \text{ open} \supset K\}.$$

If $G \in \mathscr{R}$ and $G \supset K$, then $G = \pi_i^{-1}(G_i)$ for some $i \in I$, and so $\tilde{\mu}(G) = \mu_i(G_i) \geqslant \mu_i(\pi_i K) \geqslant 1 - \epsilon$. If $G \notin \mathscr{R}$, since the open sets in $\mathscr{R}$ form a base, for each $x \in G$ there exists G_x in $\mathscr{R}$, G_x open, such that $x \in G_x$ and $G_x \subset G$. There exist therefore a finite number of such x_i with $K \subset \bigcup_{i=1}^{m} G_{x_i}$. Now, $\bigcup_{i=1}^{m} G_{x_i} \in \mathscr{R}$ and $K \subset \bigcup_{i=1}^{m} G_{x_i} \subset G$, so $\tilde{\mu}(G) \geqslant \tilde{\mu}(\bigcup_{i=1}^{m} G_{x_i}) > 1 - \epsilon$. Hence, $\tilde{\mu}^*(K) > 1 - \epsilon$, proving the condition (c′) of the extension lemma. This completes the proof of the theorem.

COROLLARY. *Given an exact projective system such that I, the indexing set, is countable or admits a countable confinal subset, the (ϵ, K)-condition is always verified.*

PROOF. We can assume that $I = \mathbf{N}$. We can find compact sets, step by step, $K_n \subset X_n$, $K_n \subset \pi_{n-1,n}^{-1}(K_{n-1})$ such that

$$\mu_0(K_0) > 1 - \epsilon/2,$$

$$\mu_n(K_n) \geqslant \mu_n[\pi_{n-1,n}^{-1}(K_{n-1})] - \frac{\epsilon}{2^{n+1}} = \mu_{n-1}(K_{n-1}) - \frac{\epsilon}{2^{n+1}}.$$

The K_n form a projective system (with $\pi_{m,n}$) and let K be the projective limit. This is compact and further $\pi_m(K)$ is the (decreasing) limit of $\pi_{m,n}(K)$ for $n \geqslant m$. Hence, $\mu_m[\pi_m(K)]$ is the limit of $\mu_m[\pi_{m,n}(K_n)]$ for $n \geqslant m$. But we see by recurrence that

$$\begin{aligned}\mu_m[\pi_{m,n}(K_n)] &\geqslant \mu_m(K_m) - \epsilon \sum_{i=m+2}^{n+1} \frac{1}{2^i} \\ &> \mu_0(K_0) - \epsilon/2 > 1 - \epsilon.\end{aligned}$$

This proves the corollary.

THEOREM 22. (Prokhorov). *Let $\{\mu_i\}$ be an exact projective system of finite Radon measures on $\{X_i\}$ such that $\mu_i(X_i) = 1$ for all $i \in I$. Let X be any Hausdorff space and for each $i \in I$, let $\tilde{\pi}_i : X \to X_i$ be such that if $i < j$, $\tilde{\pi}_i = \pi_{ij} \circ \tilde{\pi}_j$. Then in order that there exists a Radon measure μ on X such that $\tilde{\pi}_i\, \mu = \mu_i$, it is necessary and sufficient that for each $\epsilon > 0$, there exists a compact set $K \subset X$ with $\mu_i[\tilde{\pi}_i K] \geqslant 1 - \epsilon$, for all $i \in I$. If the $\tilde{\pi}_i$ separate the points of X, this measure is unique.*

PROOF. The necessity of the condition is obvious. Let X_I denote the projective limit of X_i. The set $\{\tilde{\pi}_i\}$ defines a map $\tilde{\pi} : X \to X_I$. It is immediate that $\tilde{\pi}_i = \pi_i \circ \pi$ and the compact set $\tilde{\pi}\, K \subset X_I$ satisfies the (ϵ, K)-condition. Hence there exists a projective limit measure μ_I on X_I. It follows from Theorem 12 that this is the image of a Radon measure μ on X, which then obviously satisfies $\tilde{\pi}_i \mu = \mu_i$.

If the $\tilde{\pi}_i$ separate the points of X, $\tilde{\pi}$ is injective and according to Theorem 10, μ is unique.

The (ϵ, K)-condition was first introduced by Prokhorov [1]; projective limits of measures on general projective limits (not necessarily product spaces) were first introduced by Bochner [1], see also Choksi [1], Métivier [1].

11. Non-Hausdorff spaces. In this section we consider how far the theory of Radon measures, which we have developed in the preceding sections, is capable of generalisation to spaces which are not necessarily Hausdorff. We recall first of all that a set E in a (not necessarily Hausdorff) space X is said to be *quasi-compact* if every open covering of E has a finite sub-covering. A quasi-compact set in a non-Hausdorff space is not in general closed. A great deal of the theory of Radon measures developed in § 1 to § 10 for Hausdorff spaces, carries over to spaces which are not necessarily Hausdorff if in the definition of Radon measures given in § 2, we replace the word 'compact' by 'closed, quasi-compact'. We also need to change slightly the definition of Lusin measurability. However, those proofs which use the Radon theory for compact spaces do not generalise, and we are forced instead to make greater use of the methods of abstract measure theory. We give below the essential details where the theory differs from that in Hausdorff spaces.

DEFINITION R_1. *A Radon measure μ on a (not necessarily Hausdorff) topological space X, is a pair of measures M and m on the Borel σ-algebra τ_β of X satisfying the following five properties:*

(1) *m is the essential measure associated to M;*

(2) *M is outer regular on τ_β, i.e. for every $B \in \tau_\beta$,*

$$M(B) = \inf \{M(V) : V \supset B \text{ and } V \text{ open}\};$$

(3) *M is locally finite;*

(4) *m is inner regular on τ_β with respect to the closed, quasi-compact sets, i.e. for every $B \in \tau_\beta$,*

$$m(B) = \sup \{m(K): K \subset B, K \text{ closed, quasi-compact}\};$$

(5) *$m(B) = M(B)$ whenever B is open or $M(B) < +\infty$;*

Definitions R_2 and R_3 are similar, inner regularity always being with respect to the closed, quasi-compact sets.

Our first real difficulties occur in § 5 on measurable mappings and image measures. Since the image of a closed, quasi-compact set under a continuous map is quasi-compact, but not necessarily closed, we must modify the definition of Lusin measurability as follows.

DEFINITION 9. *The mapping $H : X \to Y$ is said to be Lusin measurable if for every closed, quasi-compact set $K \subset X$ and every $\delta > 0$, there exists a closed, quasi-compact set $K_\delta \subset K$ with $\mu(K - K_\delta) < \delta$, such that H restricted to K_δ is a continuous and closed (i.e. the image of every closed subset of K_δ is closed in Y) map of K_δ into Y. (It follows that the image of every closed quasi-compact subset of K_δ is closed quasi-compact in Y).*

The additional requirement that $H \mid K_\delta$ is a closed map of K_δ into Y is automatically verified if Y is Hausdorff. (A continuous map on a non-Hausdorff space is not necessarily Lusin-measurable.)

With this definition most of the results on measurable mappings and image measures go through for spaces which are not necessarily Hausdorff, but a little extra care is needed in some of the proofs. We give, for instance the necessary extra steps in the proof that measurability is a local property (Proposition 4). Clearly Lusin measurability implies local Lusin measurability. To prove the converse we proceed as in the proof of Proposition 4, but have to verify, in addition that

$$H \mid K_\delta \left(K_\delta = \bigcup_{i=1}^{n} K_i \right)$$

is a closed map $K_\delta \to Y$. Since $H(K_\delta) = \bigcup_{i=1}^{n} H(K_i)$ and each $H(K_i)$ is closed in Y, so is $H(K_\delta)$.

Also if $F \subset K_\delta$, F is closed in K_δ, then $F \cap K_i$ is closed in K_i, so $H(F \cap K_i)$ is closed in $H(K_i)$ which is closed in Y, so $H(F \cap K_i)$ is

closed in Y. Hence $H(F) = \bigcup_{i=1}^{n} H(F \cap K_i)$ is closed in Y, i.e. $H \mid K_i$ is a closed map.

A Lusin measurable function $H: X \to Y$ is Borel measurable and the converse holds if Y is metrisable and separable (Theorem 5). No change is necessary in the proof, but note that to prove the converse we use Lemma 4 in which we cannot remove the restriction that Y is Hausdorff. Theorem 6 (Egoroff) goes through without change. So does Theorem 7. (Y here is a Banach space, again Hausdorff! Also, a continuous function on a closed quasi-compact set is uniformly continuous).

There are no further problems on image measures until we come to Theorem 12. The proof of this theorem using the Hahn-Banach theorem, of course, breaks down. We note, however that the Theorem of Henry (Theorem 16) and its proof generalise with only the usual modification ('compact' is replaced by 'closed, quasi-compact'). We now use this theorem to prove Theorem 12 as follows.

Case (i). Assume X and Y are quasi-compact and H is a surjection. Let

$$\tau = \{E: E \subset X, E = H^{-1}(B), B \in \text{Borel } \sigma\text{-algebra of } Y\}.$$

For $E \in \tau$, $E = H^{-1}(B)$ say, put $\mu_0(E) = \nu(B)$. Then (τ, μ_0) satisfies the conditions of Theorem 16 and so can be extended to a Radon measure μ on X. Clearly $\nu = H\mu$.

Case (ii). This is essentially unchanged from the earlier proof when X is Hausdorff.

The Lebesgue-Nikodym Theorem (14) goes through with only the usual modification. At first sight this may seem surprising, for the Lebesgue-Nikodym theorem on a compact space is heavily used in the proof. But this theorem, and similarly the corresponding one on a quasi-compact space which is not necessarily Hausdorff, are but special cases of the abstract Lebesgue-Nikodym theorem (see e.g.

Halmos [1] page 128); one makes the additional observation that p can be chosen locally Borel. Hence one merely uses the abstract theorem to get the quasi-compact case, and the μ-concassage to get the general case. The corollary also goes through.

The extension lemma (Theorem 15) is still valid. The *statement* of the theorem requires only the usual change. The *proof* however requires two major modifications. These occur in the proofs that μ^* on the class of closed, quasi-compact subsets of X is (a) additive (b) strongly additive, where the existing proof makes use of the Hausdorff axiom. The proof is unchanged until then. Recall that therefore we have already shown that for any closed, quasi-compact K

$$\mu^*(K) = \inf \{\mu(V) : V \in S,\ V \text{ open}, \supset K\} \ldots (\alpha),$$

and that $\mu^*(K) < \infty$, since the local finiteness of μ implies that $K \subset V$, for some $V \in S$, V open, $\mu(V) < \infty$.

(a) PROOF OF ADDITIVITY: Suppose K and K' are disjoint closed quasi-compact subsets of X. If V is any open set, $V \in S$, $V \supset K \cup K'$, $\mu(V) < \infty$, then since S contains a base for the topology of X and $V - K'$ is open, there exists a set $V_1 \in S$ such that $K \subset V_1 \subset V - K'$. (Every point x of K has a neighbourhood V_x, $V_x \in S$, $V_x \subset V - K'$. These form a covering of K. Since K is quasi-compact, there exists a finite sub-covering, $V_{x_1}, \ldots, V_{x_n}$, $V_1 = \bigcup_{i=1}^{n} V_{x_i} \in S$). Similarly there exists a set $V_2 \in S$ such that $K' \subset V_2 \subset V - K$. Put $B_1 = V_1 - (V_1 \cap V_2)$, $B_2 = V_2 - (V_1 \cap V_2)$. $B_1, B_2 \in S$, $B_1 \cap B_2 = \varnothing$, $K \subset B_1 \subset V$, $K' \subset B_2 \subset V$, so from

$$\mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2)$$

follows

$$\mu^*(K \cup K') \geq \mu^*(K) + \mu^*(K').$$

μ^* being an outer measure is sub-additive, and we deduce that μ^* is additive on the class of all closed, quasi-compact sets.

(b) PROOF OF STRONG ADDITIVITY. Let now K_1 and K_2 be any two closed, quasi-compact sets. If V_1 and V_2 are open sets of finite

measure in S containing respectively K_1 and K_2, then $K_1 \cup K_2 \subset V_1 \cup V_2$ and $K_1 \cap K_2 \subset V_1 \cap V_2$ so from (α) it follows that

$$\mu^*(K_1) + \mu^*(K_2) \geqslant \mu^*(K_1 \cup K_2) + \mu^*(K_1 \cap K_2).$$

Now let U, U' be open sets of finite measure in S such that $U \supset K_1 \cup K_2$, $U' \supset K_1 \cap K_2$ and $U \supset U'$. Let $D = \overline{U} - U'$, then D is closed. Let $D_1 = D \cap K_1$, $D_2 = D \cap K_2$. Then D_1 and D_2 are disjoint, closed quasi-compact sets. So, as in the proof of additivity above, there exist disjoint sets B_1, $B_2 \in S$ such that $D_1 \subset B_1 \subset U$, $D_2 \subset B_2 \subset U$. Let $A_1 = B_1 \cup U'$, $A_2 = B_2 \cup U'$. A_1, $A_2 \in S$, $A_1 \cup A_2 \subset U$, $A_1 \cap A_2 = U'$, $A_1 \supset K_1$, $A_2 \supset K_2$. So

$$\mu(A_1) + \mu(A_2) \leqslant \mu(U) + \mu(U'),$$

whence

$$\mu^*(K_1) + \mu^*(K_2) \leqslant \mu^*(K_1 \cup K_2) + \mu^*(K_1 \cap K_2),$$

proving strong additivity.

The rest of the proof of Theorem 15 is unchanged except for the usual modification.

We have already noted that Theorem 16 and its proof are unchanged. Theorem 17, on the existence and uniqueness of the tensor product of Radon measures still holds in spaces which are not necessarily Hausdorff. However, only the first proof, which uses the extension lemma, goes through without change. The second proof must be modified as follows. If K_1 and K_2 are closed, quasi-compact, so is $K_1 \times K_2$. As in the first proof, there is an abstract measure on the σ-algebra generated by the product sets $\subset K_1 \times K_2$, with the required property; further this measure is finite and inner regular. By Theorem 16 it can be extended to a Radon measure on $K_1 \times K_2$, uniquely because the σ-algebra of product sets contains a base for the topology of $K_1 \times K_2$. The remainder of the proof is unchanged. Fubini's Theorem (18) still holds in spaces which are not necessarily Housdorff, but only the first proof generalises (without change). However Lemma 10 used in the second proof, does generalise, as do Proposition 8 and Theorem 19. For Proposition 9 there are again difficulties coming from the definition of Lusin measurability.

Even for X and Y quasi-compact we do not know whether the result still holds. However, it holds if Z is Hausdorff. It is sufficient to show it for X and Y quasi-compact, and in the general case it holds by concassage. Let $\delta>0$ be given. Then there exists $K_\delta \subset X \times Y$, quasi-compact closed, such that $f|K_\delta$ is continuous, and $(\mu \otimes \nu)(\complement K_\delta) < \delta^2$. For every $x \in X$, the section $K_x = \{y \in Y : (x, y) \in K_\delta\}$ is closed, therefore quasi-compact, the restriction of f to it is continuous, and closed since Z is Hausdorff; and $\nu(K_x) < \delta$ except for a set of x of μ measure $< \delta$. By taking $\delta = (1/n)^{\frac{1}{2}}$, $n \in \mathbf{N}$, one sees easily that, for μ-almost every x, f_x is ν-measurable. Theorem 20 is unchanged.

We must consider the theorem on the existence of a projective limit of Radon measures (Theorem 21). Clearly the first two proofs will not generalise. The third proof uses the extension lemma, and one might hope for a generalisation but here again there are difficulties. In the first place it is obviously necessary to assume that the maps π_{ij} are not only continuous but also closed. However, though this implies that the maps π_j from the projective limit are continuous, it does not seem to imply that they are closed, and without this the proof breaks down.

We conclude this section with some remarks on how the results of Chapter II generalise to spaces which are not necessarily Hausdorff. One can, of course, extend the definition of a Radon space. We give two simple examples of non-Hausdorff spaces, one of which is, and the other is not, Radon. Let $X = \{a, b\}$ where the open sets of X are $\varnothing$, $\{a\}$ and X. The (Dirac) point measure at a is a Borel measure which is not regular, and so X is not Radon. Now let $Y = (\mathbf{R} - \{0\}) \cup \{a, b\}$, where the neighbourhoods of a are of the form

$$\{a\} \cup \{\text{deleted neighbourhood of } 0\}$$

and similarly the neighbourhoods of b are of the form

$$\{b\} \cup \{\text{deleted neighbourhood of } 0\}.$$

a and b cannot be separated by open sets, but $\{a\}$ and $\{b\}$ are closed in Y. Every finite Borel measure μ on Y is Radon. This is immediate if μ has no point mass at a or b. On the other hand point measures

at a and b are also Radon. Every measure is a sum of such measures. Note that a space in which at least one point is Borel, but not closed can never be Radon (the Dirac mass at this point is not a Radon measure !). Finally note that the space X has a refinement which is compact metrisable (and so polish!). Thus it is not easy to see what the definition of a Lusin space should be in the non-Hausdorff case.

SUMMARY OF CHAPTER II

This chapter studies special spaces, the polish, Lusin, Suslin and Radon spaces which have very good properties with respect to Radon measures; moreover, most of the usual spaces in analysis are Lusin. These spaces have been studied by many authors for many years; they took more and more importance in integration and probability.

§ 1 gives the main definitions and properties of polish, Lusin, Suslin and Lindelöf spaces. The polish spaces are defined in Definition 1: a polish space is a separable space whose topology can be defined by a metric which makes it complete. A subset of a polish space is polish if and only if it is a G_δ-set, Theorem 1, page 93. A Hausdorff topological space is Lusin if there exists a finer topology which is polish, Definition 2. The following theorems give the stability of Lusin spaces under the usual operations, which is very great : countable products and sums, countable intersections, countable disjoint unions, Borel subsets; also a countable union of arbitrary Lusin subspaces is Lusin, but that will be proved only later on, Theorem 5, Corollary 2, page 102. A Suslin space is a continuous image of a polish space, Definition 3, page 96. Suslin spaces have the same properties of stability. In Definition 6, page 97, we introduce the subdivisions which will enable us to prove the finer properties of Suslin spaces. A space is Suslin if and only if it has a subdivision, Lemma 9, page 98. The separation property, Theorem 4, page 100, is among the most important ones. As a consequence, a Lusin space is universally Borel, Theorem 5. All the properties of this part are very important. The Lindelöf spaces are introduced in Definition 7, page 103, they have very simple and important properties but the main thing will be that Suslin spaces are Lindelöf, Proposition 3, page 104. Corollary 1 is specially important, page 105. Lemma 11 and the following study the Borel maps between Suslin spaces, they have many applications. We briefly recapitulate the main properties of polish, Lusin and Suslin spaces, page 110.

Theorem 6 gives several equivalent necessary and sufficient conditions for a locally compact space to be polish, Lusin, Suslin.

Most of the usual spaces in analysis are Lusin; § 2 gives the main examples. Theorem 7, page 112, gives the main tool for topological vector spaces. On pages 115 and following, we give examples and applications. All the spaces of distributions $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{O}_M$, $\mathscr{O}'_M$, $\mathscr{O}_c$, $\mathscr{O}'_c$, are Lusin.

§ 3 studies Radon spaces. A space is Radon if every finite Borel measure on it is a Radon measure. A Radon space is universally measurable, Proposition 9, page 119; a subspace of a Radon space is Radon if and only if it is universally measurable, Proposition 8, page 118. The Radon spaces have good properties of stability, page 119. Every Lusin or Suslin space is Radon, Theorems 9 and 10, page 122. A strongly Radon space is a space in which every locally finite Borel measure is Radon, Definition 10, page 124; a Lindelöf Radon space is strongly Radon, in particular a Suslin space is strongly Radon. Lemma 19, Theorem 12, and Corollary 1, give generalizations: the K-analytic spaces; they are of lesser interest. Theorem 13 due to von Neumann, page 127, is fundamental: every surjective Borel map of a Suslin space onto another one has a universally measurable section. Theorem 14 follows: a Borel-measurable map into a Suslin space is Lusin-measurable.

§ 4 deals with the lifting property from L^∞ into $\mathscr{L}^\infty$. The corresponding property for abstract measures has been proved by Dorothy Maharam. We prove it here for Radon measures, Theorem 15, page 131. Then we introduce the local liftings, page 132. Theorem 16, page 134, gives the best which is known on that subject so far: for a Suslin or metrizable space, local liftings always exist. Local liftings are related with the strong liftings of A. and C. Ionescu Tulcea, Theorem 18, page 135.

§ 5 studies the space of regulated functions with values in a polish space. First, we define the D-topology, Theorem 19, page 138,

for which it is Lusin; the corresponding Borel structure is the same for all the countable subsets D of $\mathbf{R}$. The fundamental mapping $\delta : (f, t) \to f(t)$ is Borel, Theorem 20, page 138. From that, we deduce the main property, the measurability of entry times, Theorem 21. The Theorems 22 and 23 give other topologies, again equivalent to the previous ones.

CHAPTER II

POLISH, LUSIN, SUSLIN AND RADON SPACES[1].

1. Definitions and topological properties.

DEFINITION 1. *A topological space (X, τ) is said to be a polish space if it is separable and if there exists a metric on X, for which the topology is τ and X is complete.*

LEMMA 1. *The intersection of any countable family of polish subspaces of any topological space is polish.*

PROOF. To prove the lemma, let us first note that any countable product of polish spaces is polish. Also, any closed subspace of a polish space is polish. Let, now, $\{E_n\}$ be a sequence of polish subspaces of a topological space X. Let $E' = \prod_{n=1}^{\infty} E_n$. Then E' is a polish space. The diagonal Δ of the product $\prod_{n=1}^{\infty} X_n$ (where $X_n = X$, for every n) is a closed set. Hence $\Delta \cap E'$ is closed in E' and hence is a polish subspace. But $\Delta \cap E'$ is homeomorphic to $E = \bigcap_{n=1}^{\infty} E_n$. This proves the lemma.

LEMMA 2. *Every open subset of a polish space is polish.*

PROOF. Let V be an open subset of a polish space (X, τ) (with a metric d for which it is complete).

The function $d(x, \complement V) = \inf_{y \in \complement V} d(x, y)$ has the following properties:

(1) $d(x, \complement V) > 0$, for every $x \in X$ and $d(x, \complement V) > 0$ if and only if $x \in V$;

(2) $d(x, \complement V)$ is uniformly continuous in x; in fact,

$$d(x, y) > |\, d(x, \complement V) - d(y, \complement V)|\,, \text{ for all } x, y \text{ in } X.$$

Let $\rho(x) = 1/d(x, \complement V)$ for all $x \in X$. Define, for x, y in V,

(1) The standard modern reference for polish, Lusin and Suslin spaces is Bourbaki [4] where reference to earlier work can be found.

$$D(x, y) = | \rho(x) - \rho(y) | + d(x, y).$$

Then it is clear that D is a metric on V. Now, suppose $(x_n)_{n\in\mathbf{N}}$ is a sequence in V such that $D(x_n, x) \to 0$, $x \in V$. Then $d(x_n, x) \to 0$, since $D(x, y) \geq d(x, y)$. Conversely, if $(x_n)_{n\in\mathbf{N}} \subset V$ and $x_n \xrightarrow{\tau} x$ in V, then $\rho(x_n) \to \rho(x)$, and we get that $D(x_n, x) \to 0$. This shows that the topology of D on V is precisely the induced topology. Obviously V is separable. Finally let $(x_n)_{n\in\mathbf{N}} \subset V$ be a D-Cauchy sequence. Then the sequence is also d-Cauchy. Since (X, d) is complete, there exists x in X which is the limit of this sequence. This in turn gives that $\rho(x_n)$ converges to $\rho(x)$. But it is clear that $\rho(x_n)$ is bounded and this implies that $\rho(x) < +\infty$ and hence x belongs to V. Hence, V is D-complete. This completes the proof of the lemma.

THEOREM 1. *A subspace of a polish space is polish if and only if it is a G_δ-set.*

PROOF. The sufficiency of the condition is the consequence of the above two lemmas. Conversely, let (X, τ) be a polish space (with a metric d) and $F \subset X$ be a polish subspace (with a metric D). Let $\bar{F}$ be the closure of F in X and define, for every n,

$$F_n = \left\{ x \in \bar{F} : \begin{array}{l} \exists \text{ a neighbourhood } V \text{ of } x \text{ in } X \text{ such that} \\ D(y, z) < 1/n, \text{ for every } y, z \in V \cap F \end{array} \right\}$$

It is clear that F_n, for every n, is a relatively open subset of $\bar{F}$. Let, for each n, V_n be an open set of X such that $V_n \cap \bar{F} = F_n$. Now $F \subset \bigcap_{n=1}^{\infty} F_n$. On the other hand, let $x \in \bigcap_{n=1}^{\infty} F_n$. Then, $x \in \bar{F}$. Take the filter $\mathcal{G}$ which is the trace on F of the filter $\mathcal{F}$ of all neighbourhoods of $x \in X$. By the defining property of the F_n, it is clear that $\mathcal{G}$ is a D-Cauchy filter on F. Hence, there exists a D-limit for $\mathcal{G}$ in F. But the D-limit, which is the same as the τ-limit, is in fact, x. Hence, x belongs to F and we deduce that $F = \bigcap_{n=1}^{\infty} F_n$.

Now, since $\bar{F}$ is closed in X, it is a G_δ. Let $(W_m)_{m\in\mathbf{N}}$ be a sequence of open sets of X such that $\bigcap_{m=1}^{\infty} W_m = \bar{F}$. Then,

$$F = \bigcap_{m,\,n} (W_m \cap V_n).$$

This completes the proof.

COROLLARY. *A topological space is polish if and only if it is homeomorphic to a G_δ set contained in $I^{\mathbf{N}}$ (where $I = [0, 1]$ and $\mathbf{N}$ = natural numbers).*

We note first that $I^{\mathbf{N}}$ being a compact metric space is polish. The condition is evidently sufficient. Conversely, let X be a polish space; since X is metrizable and separable, it is homeomorphic to a subspace of $I^{\mathbf{N}}$. Now the theorem shows that the condition of the corollary is also necessary.

DEFINITION 2[1]. *A Hausdorff topological space, (X, τ) is said to be a Lusin space if, there exists a topology τ' (on X) stronger than τ such that (X, τ') is a polish space. Equivalently, (X, τ) is a Lusin space, if there exists a polish space (Y, σ) and a continuous bijective mapping from Y to X.*

A continuous bijective image of a Lusin space is again Lusin.

LEMMA 3. *Every open (or closed) subspace of a Lusin space is again Lusin.*

PROOF. Let X be a Lusin space. Then there is a polish space Y and a continuous bijection $f: Y \to X$. If A is open (or closed) in X so is $f^{-1}(A)$ and $f^{-1}(A)$ is hence a polish space. The mapping f restricted to $f^{-1}(A)$ is a continuous bijection of $f^{-1}(A) \to A$. This shows that A is Lusin. The proof is complete.

LEMMA 4. *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of Lusin spaces. Then the product $\Pi\, X_n$ is also a Lusin space.*

PROOF. Let, for every n, Y_n be a polish space and f_n a continuous bijection $f_n: Y_n \to X_n$. Let $Y = \Pi\, Y_n$ and $X = \Pi X_n$. Then Y is polish and moreover, $f: Y \to X$ defined by $f[(x_n)] = (f_n(x_n))$ is a continuous bijection. Hence X is a Lusin space, proving the lemma.

(1)Observe that in Bourbaki [4] these spaces are assumed to be metrizable.

LEMMA 5. *Let $(A_n)_{n\in N}$ be a sequence of Lusin spaces. Then, their topological sum A is a Lusin space.*

PROOF. The lemma follows immediately from the fact that the topological sum of a countable family of polish spaces is polish.

COROLLARY. *Every disjoint countable union of Lusin subspaces of a Lusin space is again Lusin.*

LEMMA 6. *Let $(A_n)_{n\in N}$ be a sequence of Lusin subspaces of a Hausdorff space. Then, $\bigcap_n A_n$ is Lusin.*

PROOF. Consider the countable product $Y = \prod_{i=1}^{\infty} X_i$, where $X_i = X$ for all i. The diagonal Δ of the product is a closed subset and is homeomorphic to X. But, $\bigcap_n A_n$ is homeomorphic to $\Delta \cap \prod_n A_n = A'$. Now, A' being a closed subset of $\prod A_n$ is Lusin. This completes the proof.

We shall see only later on that the same result holds with $\bigcup$ instead of $\bigcap$ (Corollary 2 of Theorem 5).

THEOREM 2. *Every Borel subset of a Lusin space is Lusin.*

PROOF. Let X be a Lusin space. Define

$$\mathcal{F} = \{E \subset X : E \text{ and } CE \text{ Lusin subspaces}\}.$$

The family $\mathcal{F}$ contains all open sets (Lemma 3). The theorem will be proved if we show that $\mathcal{F}$ is a σ-algebra. Since $\mathcal{F}$ is closed for complementation, it is enough to show that if $E_n \in \mathcal{F}$ then $E = \bigcup_n E_n$ also belongs to $\mathcal{F}$. Now, $CE = \bigcap_{n=1}^{\infty} CE_n$ and by the Lemma 6, CE is Lusin. Also, if $F_n = E_n \cap \bigcap_{j<n} E_j$, then F_n are Lusin subspaces and $(F_n)_{n\in N}$ is a sequence of mutually disjoint subsets of X. Hence, $\bigcup_n F_n$ is Lusin. (Corollary, Lemma 5). But $E = \bigcup_n F_n$, i.e. E is Lusin. This completes the proof.

DEFINITION 3. *A Hausdorff topological space X is said to be a Suslin space if there exists a polish space Y and a continuous surjection from Y to X; equivalently, if there exists a finer topology on X such that with the new topology, X is homeomorphic to a quotient of a polish space. A continuous image of a Suslin space is again Suslin.*

It is clear that if a space is polish, then it is Lusin and in turn every Lusin space is Suslin. All the stability properties previously given for the Lusin spaces hold good for Suslin spaces also. In particular, we state one of the important properties.

THEOREM 3. *Every Borel subset of a Suslin space is Suslin. Moreover a countable union $\bigcup_n A_n$ of Suslin subspaces A_n of a Hausdorff space X is trivially Suslin, for it is a continuous image of the topological sum of the A_n's.*

Observe that every Suslin space is separable. Moreover, we have

PROPOSITION 0. *Let X be a Suslin space and $A \subset X$. Then A has a countable subset D which is sequentially, dense in A, i.e. every point of A is a limit of a sequence in D.*

PROOF. This is immediate, considering X as a continuous image of a polish space.

We shall now go to prove the converse of Theorem 2. We shall show that every Lusin space is universally Borel.

DEFINITION 4. *A topological space is said to be dispersed if it is polish and moreover, every point has a fundamental system of neighbourhoods consisting of sets which are both open and closed.*

DEFINITION 5. *A Hausdorff topological space is said to be dispersible if it is the continuous image of a dispersed space.*

LEMMA 7. *The product and topological sum of any countable family of dispersible spaces is dispersible. Any countable intersection of dispersible subspaces of a dispersible space is again dispersible. A subspace of a dispersible space is dispersible if and only if it is a G_δ-set.*

The proofs are similar to the proofs of Lemmas 3, 4, 5 and 6.

LEMMA 8. *A Hausdorff topological space is dispersible if it is Suslin; in particular, any Lusin or polish space is dispersible.*

PROOF. Consider the interval $[0,1] = I$. Let Q be the set of all rational numbers. Then $I - (I \cap Q)$ is a G_δ in I and is hence a polish space. It is clear that $I - (I \cap Q)$ is dispersed. Each point of $I \cap Q$ is dispersed. Hence the topological sum of $I \cap Q$ and $I - (I \cap Q)$ is dispersed. Hence, I is dispersible. It follow that $I^{\mathbf{N}}$ is dispersible.

Suppose P is a polish space, then P is a G_δ in $I^{\mathbf{N}}$. (Corollary, Theorem 1). Hence P is dispersible. Now, the result follows for Lusin and Suslin spaces. The lemma is proved.

REMARK. If X is a polish or Lusin space, then there exists a continuous bijection from a dispersed space onto X.

DEFINITION 6. (Subdivision). *Let X be a Hausdorff topological space. Consider the following system. Let $(\mathscr{C}_n, p_n)_{n\geq 0}$ be a sequence such that, for every n, $\mathscr{C}_n$ is a countable set and p_n is a projection of $\mathscr{C}_{n+1}$ onto $\mathscr{C}_n$. Let, for every $n \geq 0$, ϕ_n be a one-one mapping of $\mathscr{C}_n \to \mathscr{P}(X)$ (the set of all subsets of X). Let $\mathscr{F}_n = \phi_n(\mathscr{C}_n) \subset \mathscr{P}(X)$. Such a system is said to constitute a subdivision of X, if the following three conditions are verified. If, in addition, the fourth condition is valid, then the subdivision is said to be a strict subdivision.*

(1) *For every $n > 0$, and every $c \in \mathscr{C}_{n+1}$, $\phi_{n+1}(c) \subset \phi_n[p_n(c)]$.*

(2) *For every $n \geq 0$, and $c \in \mathscr{C}_n$,*

$$\phi_n(c) = \{\cup \phi_{n+1}(d) : d \in \mathscr{C}_{n+1} \text{ and } p_n(d) = c\} \text{ and } X = \bigcup_{c \in \mathscr{C}_0} \phi_0(c).$$

(3) *If $(c_n)_{n\geq 0}$ is a coherent sequence of the subdivision (i.e. $c_{n-1} = p_{n-1}(c_n)$, for all $n > 1$), then the sequence of sets $(\phi_n(c_n))$ converges to an element of X such that the limit is contained in each $\phi_n(c_n)$.*

(4) (Strict subdivision). *For every $n \geq 0$, $\mathscr{F}_n$ (which is contained in $\mathscr{P}(X)$) consists of mutually disjoint subsets of X.*

The condition (3) can be expressed more precisely in the following way. Let $(c_n)_{n\geq 0}$ be a p-coherent sequence (viz. $c_n \in \mathscr{C}_n$ with $p_{n-1}(c_n) = c_{n-1}$ for every $n > 1$). Then $(X_n)_{n\geq 0}$, $X_n = \phi_n(c_n)$, has

the finite intersection property (condition (1)). The condition (3) requires that the filter generated by $(X_n)_{n\geq 0}$ be convergent to $x \in X$ such that $x \in X_n$, for all n. In the sequel we shall refer to such a sequence (X_n) as above [viz. $\phi_n(c_n) = X_n$ for a p-coherent sequence (c_n)] as a p-coherent family of sets from $(\mathscr{F}_n)$.

Suppose that there exists a subdivision of a space X. Using the notation of the definition, let us define mappings $p_{n,m}$ for $m, n \geq 0$. Let $p_{m,m} \equiv$ identity mapping on $\mathscr{C}_m$. Let, for $n > m$,

$$p_{m,n} = p_m \circ p_{m+1} \circ \cdots \circ p_{n-2} \circ p_{n-1}$$

be the mapping from $\mathscr{C}_n \to \mathscr{C}_m$. Then, it is clear that $\{\mathscr{C}_n, p_{n,m}\}$ forms a projective system; let $\mathscr{C}$ be the projective limit.

We assert that there exists a continuous mapping $\lambda: \mathscr{C} \to X$ which is surjective. For, suppose $(c_n)_{n\in\mathbf{N}} \in \mathscr{C}$; then it is easy to see that the family (X_n), $X_n = \phi_n(c_n)$, is p-coherent. Hence this family is convergent to some $x \in X$ (X being Hausdorff, this limit is unique). Let λ be the mapping which assigns to each point of $\mathscr{C}$ the corresponding limit. Moreover, if $x \in X$, choose $c_0 \in \mathscr{C}_0$ such that $x \in \phi(c_0)$. Having chosen $c_0, c_1, \ldots, c_n$, let c_{n+1} belong to $\mathscr{C}_{n+1}$ such that $p_n(c_{n+1}) = c_n$ and $\phi_{n+1}(c_{n+1}) \ni x$. This is possible because of (2). It is clear that $(c_n)_{n\in\mathbf{N}} \in \mathscr{C}$ and that $(\phi_n(c_n))$ converges to x. Hence the mapping λ is surjective.

We note that if, in particular, the subdivision is strict, then at every stage the elements c_n(corresponding to an $x \in X$) are uniquely fixed and hence the mapping λ is a bijection onto X.

LEMMA 9. *A Hausdorff space X admits a subdivision (respectively a strict one) if and only if the space X is Suslin (respectively Lusin).*

PROOF. Suppose X admits a subdivision. Consider the sets $\mathscr{C}_n$ with the discrete topology. Each $\mathscr{C}_n$, being countable, is a dispersed space. Now, the projective limit $\mathscr{C}$, being a closed subset of the product, is itself dispersed. Consider the mapping $\lambda: \mathscr{C} \overset{\text{onto}}{\longmapsto} X$ (as above). This is a continuous mapping and hence X is Suslin.

If, in addition the subdivision is strict, then this mapping is a bijection and it follows that X is Lusin.

Conversely, let us start with a dispersed space D. The set of closed-open neighbourhoods of diameter less than or equal to ϵ at each point forms a covering of D. Hence we can choose a countable number of these sets covering D. Suppose, for every $\epsilon > 0$, $(V_n^\epsilon)_{n\in\mathbf{N}}$ is a sequence of open-closed sets of D, of diameter $\leqslant \epsilon$, covering D. By taking (W_n^ϵ), $W_n^\epsilon = V_n^\epsilon - \bigcup_{m<n} V_m^\epsilon$, we have a disjoint countable covering of D consisting of open-closed sets of diameter less than or equal to ϵ.

Now, choose, for $\epsilon = 1$, a countable covering (as above). Let us denote by $\mathscr{C}_0$ the indices of the covering. Now, each element in this covering is a dispersed space. Hence we can choose for V_{c_0}, $c_0 \in \mathscr{C}_0$, a covering (of V_{c_0}) consisting of non-void open-closed sets of diameter $< \frac{1}{2}$ such that they are mutually disjoint. Let, for every $c_0 \in \mathscr{C}_0$, $I(c_0)$ be the indexing set of the above covering of V_{c_0}. Let $\mathscr{C}_1$ be the sum of sets $I(c_0)$, $c_0 \in \mathscr{C}_0$; then $\mathscr{C}_1$ is countable. Define $p_0 : \mathscr{C}_1 \to \mathscr{C}_0$, by setting $p_0(i) = c_0$, for every $i \in I(c_0)$; this is a surjection on $\mathscr{C}_0$. Proceeding by induction, it is easy to see that we get a system $(\mathscr{C}_n, p_n)_{n\geqslant 0}$ and the corresponding family $\mathscr{F}_n$, satisfying the first two conditions of the subdivision. Further, every element of $\mathscr{F}_n$ is a subset of X of diameter $\leqslant 1/2^n$ for every n. Now, let $(E_n)_{n\geqslant 0}$ be a p-coherent family of sets with $E_n \in \mathscr{F}_n$. By the defining property of E_n, these sets are closed and the diameter of E_n tends to zero as $n \to \infty$. Since D is a complete metric space, (E_n) shrinks to a point, contained in each one of E_n. This shows that we have a subdivision on X. It is clear (from the construction) that this subdivision is strict.

If X is a Suslin (respectively Lusin) space, then there is a dispersed space D and a continuous map (respectively which is a bijection) of D onto X. It is clear from the continuity (respectively and the bijective nature) of this map that the subdivision on D (constructed as above) is carried to a subdivision (respectively a strict subdivision) on X.

The lemma is proved.

REMARK. The above method shows that if X is a Suslin (resp. Lusin) space then for all n, each element of $\mathscr{F}_n$ is a Suslin (resp. Lusin) subspace because it has itself a subdivision (resp. strict subdivision).

THEOREM 4. (*Separation property*). *Let X be a Hausdorff space and $(A_n)_{n\geqslant 0}$ be a mutually disjoint sequence of Suslin subspaces. Then there exists a mutually disjoint sequence of Borel subsets B_n of X such that, for every n, B_n contains A_n.*

PROOF. Suppose A and A' are two disjoint subsets of X and $A = \bigcup_{n=1}^{\infty} E_n$, $A' = \bigcup_{n=1}^{\infty} E'_n$, where E_n and E'_n are arbitrary. If, for every m and n, there is a Borel set $B_{m,n}$ containing E_n and not intersecting E'_m, then $\bigcup_n \bigcap_m B_{m,n} = B_1$ is a Borel set which contains A and the complement of B_1, which is also Borel, contains A'.

Let us now assume that A and A' are Suslin subspaces of X and let $A \cap A' = \varnothing$. By the above lemma, there is a subdivision of each of A and A'. Assume, if possible, that every Borel set of X which contains A intersects A'.

Consider $\mathscr{C}_0$ and $\mathscr{C}'_0$ (corresponding to the subdivisions). Then the sets $E_{c_0} \subset A$ (resp. $E'_{c'_0} \subset A'$) corresponding to the indices in $\mathscr{C}_0$ (resp. $\mathscr{C}'_0$) form a countable covering of A (resp. A'). By what we have observed above, there exists at least one $c_0 \in \mathscr{C}_0$ and $c'_0 \in \mathscr{C}'_0$ such that every Borel set containing E_{c_0} intersects $E'_{c'_0}$. Having chosen c_i and c'_i respectively in $\mathscr{C}_i$ and $\mathscr{C}'_i$ for $i < n$ satisfying (1) every Borel set of X containing E_{c_i} intersects $E'_{c'_i}$ and (2) $p_{i-1}(c_i) = c_{i-1}$ and $p'_{i-1}(c'_i) = c'_{i-1}$; let us consider $E_{c_{n-1}}$ and $E'_{c'_{n-1}}$. The sequences (E_d), $d \in \mathscr{C}_n$, $p_{n-1}(d) = c_{n-1}$ [resp. $(E'_{d'})$, $d' \in \mathscr{C}'_n$, $p'_{n-1}(d') = c'_{n-1}$] form a countable covering of $E_{c_{n-1}}$ (resp. $E'_{c'_{n-1}}$). Further any Borel set of X containing $E_{c_{n-1}}$ intersects $E'_{c'_{n-1}}$. Hence, we can find $c_n \in \mathscr{C}_n$ and $c'_n \in \mathscr{C}'_n$ satisfying (1) $p_{n-1}(c_n) = c_{n-1}$, $p'_{n-1}(c'_n) = c'_{n-1}$ and (2) every Borel set containing E_{c_n} intersects $E'_{c'_n}$.

Now, by the choice, it is clear that $(E_{c_n})_{n\in\mathbf{N}}$ (resp. $(E'_{c'_n})$) is a p (resp. p')-coherent family and hence converges to an element a in A (resp. a' in A'). Since $A \cap A' = \emptyset$, we have $a \neq a'$. Since X is Hausdorff, we can find open neighbourhoods V of a and V' of a' such that $V \cap V'$ is void. Clearly, $E_{c_n} \subset V$ and $E'_{c'_n} \subset V'$ for all n sufficiently large. This is a contradiction.

Thus we have shown that any two disjoint Suslin subspaces of X can be separated by disjoint Borel subsets of X.

Suppose now, $(A_n)_{n\in\mathbf{N}}$ is a mutually disjoint sequence of Suslin subspaces. For every $k \geqslant 0$, there is a Borel subset B'_k containing A_k such that $B'_k \cap A'_k = \emptyset$ where $A'_k = \bigcup_{n\neq k} A_n$. [Note that A'_k is also Suslin and is disjoint from A_k]. Define $B_k = B'_k \cap \bigcap_{i<k} \complement B'_i$. Then $(B_k)_{k\geqslant 0}$ is a mutually disjoint sequence of Borel subsets of X and it is clear that, for every k, B_k contains A_k. This completes the proof.

COROLLARY 1. *If a Hausdorff space X is the union of countably many mutually disjoint Suslin subspaces, then each of them is a Borel subset of X. If two complementary subspaces of X are both Suslin, they are Borel.*

COROLLARY 2. *Let τ_1 and τ_2 be two comparable Suslin topologies on a set X. The Borel σ-algebras of τ_1 and τ_2 coincide.*

PROOF. Assume τ_1 is stronger than τ_2. Then any τ_2-Borel subset is τ_1-Borel. But, if B is τ_1-Borel then it is τ_1-Suslin and hence τ_2-Suslin. Also, B has the same property. It follows, from the first corollary, that B(and $\complement B$) are τ_2-Borel, completing the proof.

REMARK. If τ_1 and τ_2 are two Suslin topologies on a set X, such that inf (τ_1, τ_2) is Hausdorff, then the same conclusion holds, since the Borel σ-algebras of τ_1 and τ_2 coincide with that of inf (τ_1, τ_2).

THEOREM 5. *Let X be a Hausdorff space. Then every Lusin subspace A (with the induced topology) of X is Borel in X.*

PROOF. Since A is Lusin, Lemma 9 assures us of the existence of a strict subdivision of A, all the sets of this subdivision being Lusin.

Let $(\mathscr{C}_n)_{n\geq 0}$ be the sequence of the sets of indices with projections $p_n\colon \mathscr{C}_n \to \mathscr{C}_{n+1}$, giving a strict subdivision of A. For every $c \in \mathscr{C}_0$, there is a Lusin subset $A_c^0 \subset A$ such that $(A_c^0)_{c\in\mathscr{C}_0}$ is a mutually disjoint sequence and $A = \bigcup_{c\in\mathscr{C}_0} A_c^0$. By Theorem 4, we can find Borel subsets B_c^0 of X satisfying (1) $(B_c^0)_{c\in\mathscr{C}_0}$ is a mutually disjoint sequence and for every $c \in \mathscr{C}_0$, $B_c^0 \supset A_c^0$. Moreover, replacing B_c^0 by its intersection with $\bar{A}_c^0$ if necessary, we may assume that $B_c^0 \subset \bar{A}_c^0$. Now, for every $c_0 \in \mathscr{C}_0$, $A_{c_0}^0$ is a Lusin space and is the countable disjoint union of Lusin subspaces $(A_c^1)_{c\in\mathscr{C}_1, p_0(c)=c_0}$. By a similar argument, we can find mutually disjoint Borel subsets $(B_c^1)_{c\in\mathscr{C}_1}$ satisfying $B_c^1 \supset A_c^1$ and $B_c^1 \subset \bar{A}_c^1$ for every $c \in \mathscr{C}_1$. It is clear by induction that for every n, we can find Borel subsets B_c^n, $c \in \mathscr{C}_n$ satisfying (1) $(B_c^n)_{c\in\mathscr{C}_n}$ is a mutually disjoint sequence and (2) $A_c^n \subset B_c^n \subset \bar{A}_c^n$.

Let, for every $n > 0$, $B_n = \bigcup_{c\in\mathscr{C}_n} B_c^n$. Then B_n is a Borel subset of X and $B_n \supset \bigcup_{c\in\mathscr{C}_n} A_c^n = A$. Hence, $B = \bigcap B_n$ is also Borel and $B \supset A$. We shall show that $B = A$ and this will prove the theorem. Suppose $x \in B$. Then for every $n > 0$, x belongs to a $B_{c_n}^n$ for a uniquely determined $c_n \in \mathscr{C}_n$. Hence, $x \in \bar{A}_{c_n}^n$, by construction. Moreover, it is clear that $(A_{c_n}^n)_{n\in\mathbf{N}}$ is p-coherent. Hence, there exists an element $a \in A$ such that $A_{c_n}^n$ converges to a with $a \in A_{c_n}^n$ for all n. If $x \neq a$, we can find open neighbourhoods V of a and V' of x with $V \cap V' = \emptyset$. Then, for all n sufficiently large, $V \supset A_{c_n}^n$ and hence $V' \cap \bar{A}_{c_n}^n = \emptyset$. This contradicts the fact that $x \in \bar{A}_{c_n}^n$ for all n. Hence, we conclude that $x = a$, i.e. $x \in A$. It follows that $A = B$ and hence A is a Borel set. The theorem is proved. The same is not true in general for a Suslin subspace.

The following corollaries are deduced immediately.

COROLLARY 1. *A subspace of a Lusin space is Lusin iff it is a Borel set.*

We remark that this is not true in general for Suslin spaces.

COROLLARY 2. *Let $(A_n)_{n\in\mathbf{N}}$ be a sequence of Lusin subspaces of a Hausdorff space. Then, $\bigcup_n A_n$ is also Lusin.*

COROLLARY 3. *If X is a Lusin space and $f: X \to Y$ is a continuous injection (where Y is any Hausdorff space), then $f(B)$ is Borel in Y for every Borel subset B in X.*

PROOF. For, $A_1, A_2, \ldots, A_{n-1}$ are Borel in X; therefore $B_n = A_n \cap \bigcap_{k=1}^{n-1} CA_k$ is Borel in A_n, hence Lusin and $\bigcup_n A_n = \bigcup_n B_n$, disjoint Lusin subspaces.

REMARK. There is an example of a polish space P with a Suslin subspace S which is not Lusin. In this case $P - S$ is not a Suslin space; for, if it were, we could deduce by the corollary to Theorem 4 that S and $P - S$ are Borel and hence Lusin (being Borel subsets of a polish space). Therefore the complement of a Suslin subspace of a Suslin space need not be Suslin: Corollary 1 does not extend to Suslin spaces, nor Corollary 3, nor Theorem 5 itself. But Corollary 2 trivially holds for Suslin spaces (see Theorem 3).

DEFINITION 7. *A topological space X is said to be strongly Lindelöf if every open cover of any open set of X has a countable subcover.* [*Recall that X is said to be Lindelöf if every open cover of X has a countable subcover*].

LEMMA 10. *Let $(f_i)_{i \in I}$ be a family of lower semi-continuous functions on a Lindelöf space X. There exists a countable subfamily having the same upper envelope $f = \sup f_i$, $i \in I$. (Conversely, this property trivially characterises Lindelöf spaces, since the characteristic function of an open set is lower semi-continuous).*

PROOF. For all rationals $r \in \mathbf{Q}$ and all $i \in I$, let $\mathcal{O}_{i,r} = \{x \in X: f_i(x) > r\}$. $(\mathcal{O}_{i,r})_{i \in I}$ being a family of open sets, there exists a countable subfamily $(\mathcal{O}_{j,r})_{j \in J_r}$, $J_r \subset I$, countable, having the same union. So, if $J = \bigcup_{r \in \mathbf{Q}} J_r$, the family of f_j, $j \in J$, has the required property. In fact, for any $x \in X$ and all $r < f(x)$, we have $x \in \bigcup_{i \in I} \mathcal{O}_{i,r}$, hence $x \in \bigcup_{j \in J_r} \mathcal{O}_{j,r}$, hence $\sup_{j \in J} f_j(x) > r$ and hence $\sup_{j \in J} f_j(x) = f(x)$. This completes the proof.

PROPOSITION 1. *Let X be a regular strongly Lindelöf space. Then, every open subset of X is an F_σ and every closed subset of X is a G_δ.*

PROOF. It is enough to prove one of the properties (since they are equivalent). Let V be an open subset of X. Since the closed neighbourhoods form a fundamental system at every point, we get that V is the union of open sets W such that $W \subset \overline{W} \subset V$. We deduce that there exists a countable number of them covering V, i.e. V is an F_σ; the proof is complete.

PROPOSITION 2. *If X is strongly Lindelöf, every Radon measure μ on X is moderated.*

PROOF. Every point x has an open neighbourhood V_x such that $\mu(V_x) < +\infty$. X is covered by a countable system of such neighbourhoods, hence X is a countable union of open subsets of finite measure.

The following stability properties of strongly Lindelöf spaces are easily verified.

(1) Countable topological sums of strongly Lindelöf spaces are again strongly Lindelöf.

(2) Any subspace of a strongly Lindelöf space is strongly Lindelöf.

(3) Arbitrary intersections and countable unions of strongly Lindelöf spaces are strongly Lindelöf.

(4) If a set X with a topology τ is strongly Lindelöf, then for any topology weaker than τ on X, the space is strongly Lindelöf.

(5) Any continuous image of a strongly Lindelöf space is strongly Lindelöf.

(6) Any space with a countable base $(\omega_n)_{n\in\mathbf{N}}$ for open sets is strongly Lindelöf. Namely, let $\Omega = \bigcup_{i\in I} \Omega_i$. Let $\mathbf{N}'$ be the set of $n \in \mathbf{N}$ such that there exists an $i \in I$ with $\Omega_i \supset \omega_n$; choose such an i, call it $i(n)$. Then $\bigcup_{n\in\mathbf{N}'} \omega_n = \Omega$, therefore $\bigcup_{n\in\mathbf{N}'} \Omega_{i(n)} = \Omega$.

PROPOSITION 3. *Any Suslin space is strongly Lindelöf.*

PROOF. Any polish space has a countable base for its open sets and hence it is strongly Lindelöf. Now property (5) gives us the required result.

REMARK. The cartesian product of two strongly Lindelöf spaces need not be strongly Lindelöf. For instance, let $X = \mathbf{R}^1$ with a basis for a topology consisting of intervals of the form (a, b). X is strongly Lindelöf, but $X \times X$ is not strongly Lindelöf [J. L. Kelley 1, p. 59, Ex. K, L, see also p. 134 and p. 172].

However, if the X_n have countable bases for open sets or are Suslin spaces, then surely $\prod_n X_n$ has the same property, and is therefore strongly Lindelöf. If the spaces X_n are such that $X = \prod_n X_n$ is strongly Lindelöf, then the Borel σ-algebra of X is the product of the Borel σ-algebras of the X_n. For, the open sets of X, which generate the Borel σ-algebra of X, belong to the product σ-algebra. This is true since every open set of X is a union of open rectangles $\prod_{0 \leq i \leq n} V_i \times \prod_{i > n} X_i$ and is hence by the strong Lindelöf property a countable union of such rectangles.

PROPOSITION 4. *Let X be a non-void Hausdorff space such that $X \times X$ is strongly Lindelöf. Let $(f_i)_{i \in I}$ be a family of continuous functions on X with values in Hausdorff spaces Y_i, separating the points of X. Then there is a countable subset $J \subset I$ such that $(f_j)_{j \in J}$ separate the points of X.*

PROOF. Replacing the Y_i by their sum, if necessary, we may assume that the values are taken in the same space Y. Let Δ (resp. Δ') be the diagonal in $X \times X$ (resp. $Y \times Y$). For every $(x, x') \notin \Delta$, there is an f_i in the family such that $(f_i(x), f_i(x')) \notin \Delta'$. This means that the open subsets $(f_i, f_i)^{-1}(\complement\, \Delta')$ form a covering of $\complement\, \Delta$. We can choose a countable subset $J \subset I$ such that $(f_j, f_j)^{-1}(\complement\, \Delta'), j \in J$, is already a cover for $\complement\, \Delta$; the corresponding $f_j, j \in J$, separate the points of X. The proof is complete.

COROLLARY 1. *If X is as in the theorem and the continuous real-valued functions on X separate the points of X, then there is a coarser Hausdorff topology on X which is separable and metrizable.*

The corollary follows by observing that in this case X can be mapped injectively and continuously onto a subspace of $\mathbf{R}^{\mathbf{N}}$.

COROLLARY 2. *If X is a compact space then $X \times X$ is strongly Lindelöf if and only if, X is metrizable. In particular, a compact Suslin space is metrizable.*

LEMMA 11. *Let X and $(Y_n)_{n\geq 0}$ be Hausdorff topological spaces and for every n, $f_n: X \to Y_n$ a Borel[1] map. If, moreover, the Borel σ-algebra of the product $\prod_n Y_n$ is the 'product' of the Borel σ-algebras (this is satisfied, for instance, if all the Y_n are Suslin), then the map $f: X \to \prod_n Y_n$ defined by $f(x) = (f_n(x))_{n\geq 0}$ is Borel.*

PROOF. The lemma follows by noting the fact that the inverse image of every rectangular Borel set of the product space is a Borel set of X.

LEMMA 12. *Let $f: X \to Y$. Let f be a Borel map and let the product σ-algebra (of the Borel σ-algebra of Y with itself) be the Borel σ-algebra of $Y \times Y$. Then, the graph G of f in $X \times Y$ is a Borel set.*

PROOF. We deduce from Lemma 11 that the mapping $f \times I$: $X \times Y \to Y \times Y$ (where I is the identity mapping $Y \to Y$) is Borel. It follows that G, the inverse image of the diagonal under $f \times I$, is a Borel set of $X \times Y$. The lemma is proved.

LEMMA 13. *Let $f: X \to Y$. If the graph G of f is Suslin then the map f is Borel.*

PROOF. Let B be any Borel subset of Y. Let p_1 and p_2 be respectively the projections of $X \times Y$ onto X and Y. Then $f^{-1}(B) = p_1[p_2^{-1}(B) \cap G]$ and $p_2^{-1}(B) \cap G$, being a Borel subset of G, is Suslin and hence $f^{-1}(B)$ is Suslin. But in the same way $f^{-1}(\complement B)$ is also Suslin. Now, the separation property (Theorem 4) gives us that $f^{-1}(B)$ (and also $f^{-1}(\complement B)$) is Borel, as $f^{-1}(B)$ and $f^{-1}(\complement B)$ are disjoint and Suslin and their union is X. This completes the proof of the lemma.

(1) A map $g: X \to Y$ is said to be *Borel* if the inverse image of every Borel set of Y is a Borel set of X.

COROLLARY. *Let X and Y be Suslin spaces. Then a mapping $f: X \to Y$ is Borel iff the graph of f is Suslin or iff it is a Borel subset of $X \times Y$.*

The corollary is an immediate consequence of the last two lemmas.

LEMMA 14. *Let $f: X \to Y$ be an injective map. If the graph G of f is Lusin, then the image $f(B)$ of any Borel set B of X is Borel in Y, and Lusin.*

PROOF. Let p_1 (resp. p_2) be the projection of $X \times Y$ onto X (resp. Y). Let B be a Borel set of X. Then $f(B) = p_2[p_1^{-1}(B) \cap G]$. But $p_1^{-1}(B) \cap G$, being a Borel subset of G, is Lusin and p_2 restricted to G is continuous and injective. It follows that $f(B)$ is Lusin and hence Borel in Y. The proof is complete.

REMARK. The conditions of the lemma are satisfied if, for instance, X and Y are Lusin spaces and f is an injective Borel map.

LEMMA 15. *Let $f: X \to Y$ be such that the graph G is Borel (in $X \times Y$) or G is a Suslin subspace. Then,* (a) *if X is Suslin then the inverse image of every Suslin subspace of Y is Suslin and* (b) *if Y is Suslin then the direct image of every Suslin subspace of X is Suslin.*

PROOF. (a) Let S be a Suslin subspace of Y. Let p_1 and p_2, be the projections of $X \times Y$ onto X and Y respectively. Then, $p_2^{-1}(S) = X \times S$ is Suslin, since X is Suslin. If G is also Suslin, then $G \cap p_2^{-1}(S)$ is Suslin; on the other hand if G is Borel in $X \times Y$ then $G \cap p_2^{-1}(S)$ is Borel in $p_2^{-1}(S)$ and hence again $G \cap p_2^{-1}(S)$ is Suslin. It follows that $f^{-1}(S) = p_1[G \cap p_2^{-1}(S)]$ is also Suslin.

(b) Let $S \subset X$ be a Suslin subspace and Y Suslin. Again, $p_2^{-1}(S) = S \times Y$ is Suslin and with the assumption on the graph G, we see as before that $p_1^{-1}(S) \cap G$ is Suslin. It follows that $f(S) = p_2[G \cap p_1^{-1}(S)]$ is Suslin. The proof is complete.

LEMMA 16. *If X and Y are Suslin spaces and $f: X \to Y$ is Borel, then the direct image and the inverse image of a Suslin subspace are*

Suslin. If, moreover, f is bijective, then the direct image and the inverse image of a Borel subset are Borel.

PROOF. The first part follows from Lemma 15. For the second part, if B is a Borel subset of X, it is Suslin, therefore $f(B)$ is Suslin. But, f being bijective, $f(\complement B) = \complement f(B)$ is Suslin and hence both $f(B)$ and $f(\complement B)$ are Borel. For the inverse image, it is simply the definition of a Borel map. The proof is complete.

LEMMA 17. *If τ_1 and τ_2 are two comparable Suslin topologies on a set X, then they have the same Borel and Suslin subspaces.*

LEMMA 18. [Fernique, 1]. *If $(f_n)_{n\in\mathbf{N}}$ is a sequence of real-valued Borel functions on a Suslin space X separating points, then it generates the Borel σ-algebra of X.*

PROOF. $f = (f_n)_{n\in\mathbf{N}}$ is a map of $X \to \mathbf{R}^{\mathbf{N}}$, each component being Borel. By Lemma 11, f is a Borel map. We deduce from Lemma 16 that $f(X)$ is Suslin and that the Borel bijective map $f : X \to f(X)$ generates a bijective map between the Borel subsets of X and $f(X)$; as the Borel σ-algebra of $f(X)$ is generated by the coordinate functions of $\mathbf{R}^{\mathbf{N}}$, the proof is complete.

COROLLARY. *The Borel σ-algebra of a Suslin space is countably generated.*

PROOF. The characteristic functions of the closures of the sets of a subdivision separate points.

REMARK. The countability of the family of functions is necessary. For example, the characteristic functions of points do not generate the Borel σ-algebra.

PROPOSITION 5. *Let X be a Suslin space and p a Borel mapping of a Suslin space P onto X. A set $B \subset X$ is Borel iff $p^{-1}(B)$ is Borel in P.*

PROOF. The result follows by observing that $B = p[p^{-1}(B)]$ and $\complement B = p[p^{-1}(\complement B)]$ are both Suslin subspaces of X and hence Borel.

COROLLARY 1. *A sequentially closed subset A of a Suslin space X is Borel.*

PROOF. Consider a projection p from a polish space as in the proposition. Then $p^{-1}(A)$ is sequentially closed, but P is polish and hence $p^{-1}(A)$ is closed. It follows that A is Borel, completing the proof.

COROLLARY 2. *If X is Suslin, and Y is an arbitrary topological space, every mapping of X into Y which is sequentially continuous is Borel.*

PROOF. The inverse image of every closed subset of Y is sequentially closed and hence Borel.

COROLLARY 3. *Let X and Y be Hausdorff topological spaces such that every compact subset of Y is metrizable. (In particular, for a Suslin space Y the above holds good). Let Z be any separable metrizable space. Then every map $f: X \times Y \to Z$, continuous in each variable separately is universally Lusin measurable on the product.*

PROOF. We can assume without loss of generality that $Z = \mathbf{R}$. For, a separable metrizable space is embedded in $\mathbf{R}^{\mathbf{N}}$ and to prove the measurability of a mapping into $\mathbf{R}^{\mathbf{N}}$, it is enough to show measurability coordinate-wise. Again, we can assume that X and Y are both compact spaces, in particular Y compact metrizable (hypothesis), since Lusin measurability has to be proved on compact subsets.

If $\mathscr{C}(Y)$ denotes the Banach space of real valued continuous functions on Y, then the mapping f induces $\vec{F}: X \to \mathscr{C}(Y)$ defined by $\vec{F}(x)(y) = f(x, y)$. The fact that f is separately continuous is equivalent to the fact that $\vec{F}$ is continuous when $\mathscr{C}(Y)$ is provided with the pointwise convergence topology. However, since $\mathscr{C}(Y)$ is a polish space, we deduce that $\vec{F}$ is a Borel map, when $\mathscr{C}(Y)$ is provided with the supremum norm topology. It follows that $\vec{F}$ is universally Lusin measurable on X.

Now, suppose μ is a Radon measure on $X \times Y$ of total measure α. Let ν be the projection of this measure onto X. Given $\epsilon > 0$, choose a compact subset K of X such that $\nu(K) > \nu(X) - \epsilon/\alpha$ and the mapping $\vec{F}$ restricted to K is continuous (with values in $\mathscr{C}(Y)$). This implies that the restriction of f to $K \times Y$ is continuous. Moreover, $\mu(K \times Y) = \mu(X \times Y) - \mu((X-K) \times Y) = \mu(X \times Y) - \alpha\nu(X-K) > \alpha - \epsilon$. Since the above holds good for arbitrary $\epsilon > 0$, we deduce that f is μ-Lusin measurable on $X \times Y$. The proof is complete.

For a more general result see Gowrisankaran [1].

To end this article, we shall recall here briefly the important properties (like stability etc.) of the various spaces discussed above.

I. Polish Spaces. The class of polish spaces is closed for countable products and topological sums, countable intersections and projective limit of countable subfamilies[1]. A subspace of a polish space is polish iff it is a G_δ; in particular, closed and open subsets are polish. But even a finite union of polish spaces need not be polish.

A Fréchet space is polish iff it is separable. A compact space is polish iff it is metrizable. (This is true since a compact metrizable space is separable). Also a compact space X is polish iff $\mathscr{C}(X)$, the Banach space of real-valued continuous functions on X, is polish. If X is compact and metrizable, then there is a countable dense subset of $\mathscr{C}(X)$ and hence the latter space is polish. Conversely, if $\mathscr{C}(X)$ has a countable dense set, then there is a countable set separating the points of X. Now, X can be homeomorphically identified with a subspace of a countable product of unit intervals and hence X is metrizable (Corollary 1 of Proposition 4). Finally, a locally compact Hausdorff space is polish iff it has a countable base. This is proved easily by considering the Alexandroff compactification of X.

II. Lusin Spaces. Lusin spaces are closed for (i) countable products, (ii) countable topological sums, (iii) countable unions and

1. A projective limit is a closed subset of the product.

intersections, (iv) complements, (v) Borel subspaces (necessary and sufficient), (vi) countable projective limits, and (vii) for weakening of (Hausdorff) topologies. However, Lusin spaces are unstable for passage to the quotient.

III. SUSLIN SPACES. The properties (i), (ii), (iii), (v), (vi) and (vii) of the class of Lusin spaces are valid for this class also; except that a Suslin subspace of a topological space need not be Borel. This class is further closed for passage to quotient, continuous image, countable inductive limit (an inductive limit is a quotient of the sum). However the class of Suslin spaces is not stable for complementation.

THEOREM 6. *Let X be a locally compact space. Then the following are equivalent*:

(a) *X is polish*;

(b) *X is Lusin*;

(c) *X is Suslin*;

(d) *X has a countable base for open sets*;

(e) *X is separable and metrizable*;

(f) *X is locally metrizable and denumerable at ∞*;

(g) *the Alexandroff compactification $\hat{X}$ has any one of the above properties*;

(h) *the space $\mathscr{C}_{00}(X)$ (of continuous real-valued functions with compact support) endowed with its inductive limit topology is Lusin or Suslin or separable*;

(i) *the Banach space $\mathscr{C}_0(X)$ (of continuous real-valued functions vanishing at ∞) is polish or Lusin or Suslin or separable.*

PROOF. Trivially, any one of the properties (a), (b), (c), (d), (e), (f) for $\hat{X}$ implies the same for X. Any one of the properties (a), (b), (c) for X implies the same for $\hat{X}$. If $\hat{X}$ is Suslin, it is metrizable (Cor. 2, Prop. 4), therefore polish : hence (a), (b), (c) are equivalent

for $\hat{X}$, hence for X. Assume (d) for X; then X is Lindelöf; it is a union of relatively compact open sets, therefore a countable union, therefore denumerable at ∞, and X is metrizable and separable, which implies (d) and (e) for $\hat{\hat{X}}$ and X. Now (e) implies (d) for X, therefore (d) and (e) for $\hat{X}$, and (f) for X and $\hat{X}$. Assume (f); then X is a countable union of compact metrizable Suslin subspaces, therefore X and $\hat{X}$ are Suslin, which is (c), and (f) for $\hat{X}$. It follows that (a) through (g) are equivalent. We shall now show that (a) $\Rightarrow$ (h) $\Rightarrow$ (i). Now X (so $\hat{X}$) being polish, each subspace $\mathscr{C}(K, X)$ (i.e. functions with support in K) of $\mathscr{C}_{00}(X)$ is polish and $\mathscr{C}_{00}(X)$ is a countable inductive limit of polish spaces and hence is Lusin or Suslin, hence separable. If $\mathscr{C}_{00}(X)$ is separable, $\mathscr{C}_0(X)$ in which it is dense is separable and since it is a Banach space, it is polish. Finally, if $\mathscr{C}_0(X)$ is separable, then $\mathscr{C}(\hat{X})$ (i.e. continuous real-valued functions on $\hat{X}$) which is the direct sum of $\mathscr{C}_0(X)$ and $\mathbf{R}$ is separable and is hence polish. It follows that $\hat{X}$ is embeddable in $\mathbf{R}^{\mathbf{N}}$ hence $\hat{X}$ is polish, i. e. (i) $\Rightarrow$ (a). This completes the proof of the theorem.

2. Examples of Lusin spaces. We shall now give some interesting examples of Lusin spaces. We shall prove the following important theorem in this direction.

THEOREM 7. *Let E be the inductive limit of a sequence of separable Fréchet spaces E_n such that* (i) *E_n increases with n, and* (ii) *every compact subset of E is a compact subset of E_n, for some n. Let F be a Hausdorff topological vector space which is the countable union of images (under linear continuous maps) of separable Fréchet spaces.*

Then $\mathscr{L}_c(E, F)$ (the space of continuous linear functionals $E \to F$, provided with the compact open topology) is a Lusin space.

PROOF. CASE 1. E separable Fréchet; F separable Banach.

Define, for a neighbourhood U of 0 in E,

$$\mathscr{L}_U = \{u \in \mathscr{L}(E, F) : u(U) \subset \text{unit ball in } F\}.$$

The set $\mathscr{L}_U$ is closed in $\mathscr{L}_c(E,F)$ and since $\mathscr{L}_c(E,F)$ is complete, so is $\mathscr{L}_U$. On the other hand, $\mathscr{L}_U$ is an equicontinuous set. Hence, by Ascoli's theorem, its uniform structure is identical with the metrizable uniform structure $\mathscr{L}_D(E,F)$ of the simple convergence on a countable dense subset D of E. Hence, $\mathscr{L}_U$ is metrizable and complete.

We shall show that $\mathscr{L}_U$ is separable. Let $\{a_1, a_2, \ldots, a_l\}$ be a finite subset of D. If u is a continuous linear map of $E \to F$, we can without changing the values of u on the vector subspace generated by $\{a_1, a_2, \ldots, a_l\}$, modify u in such a way that it has a finite rank. Hence, we can assume that u is of the form $\Sigma_\nu e'_\nu \otimes f_\nu$. But, each f_ν can be approximated by an element $\tilde{f}_\nu$ situated in a countable dense subset F_0 of F. Also, we know that the dual of a separable Fréchet space is weakly separable. Hence, we can approximate e'_ν by an $\tilde{e}'_\nu$ belonging to a countable set E'_0 of E', weakly dense in E', in such a way that $|\langle e'_\nu - \tilde{e}'_\nu, a_j \rangle|$ is as small as we want (for $j = 1$ to l). Thus u is approximated by $\Sigma (\tilde{e}'_\nu \otimes \tilde{f}_\nu)$ in the topology of simple convergence on $a_1, \ldots, a_l$. This shows that $\mathscr{L}_D(E, F)$ is separable. Now $\mathscr{L}_U$ being the subspace of a separable and metrizable space is itself separable.

Thus, we have shown that $\mathscr{L}_U$ is polish. But, if U varies through a fundamental system of neighbourhoods $(U_n)_{n \in \mathbf{N}}$ of 0 in E, then $\bigcup_n \mathscr{L}_{U_n} = \mathscr{L}(E, F)$. It follows that $\mathscr{L}(E, F)$ (the union of countable Lusin subspaces) is Lusin.

CASE 2. E and F two separable Fréchet spaces.

Let V be a disc neighbourhood of 0 in F and let $\hat{F}_V$ the associated Banach space. Then, $\mathscr{L}_c(E, \hat{F}_V)$ is a Lusin space. But $\mathscr{L}_c(E, F)$ is the projective limit of $\mathscr{L}_c(E, \hat{F}_V)$ and is hence Lusin (being a closed subset of a countable product of Lusin spaces).

CASE 3. E separable Fréchet. $F = \bigcup_n w_n(G_n)$; G_n separable Fréchet spaces and w_n are linear continuous mappings.

Let, for every u, F_n be the image of G_n under w_n. Then, with the canonical factorisation $G_n \to G_n/w_n^{-1}\{0\} \to F_n$, we see that F_n is provided with the structure of a separable Fréchet space in such a way that the canonical injection of $F_n \to F$ is continuous. Now, let $u \in \mathscr{L}(E, F)$. Then, E is the (countable) union of the subspaces $u^{-1}(F_n)$. Hence, by Baire's theorem, we can find at least one m such that $u^{-1}(F_m)$ is not meagre†. Let u_m be the restriction of u to $u^{-1}(F_m)$. We shall show that u_m is defined on the whole of E and that u_m is continuous. We note first that the graph of the map u_m (defined on a subspace of E, with values in F_m) is closed in $E \times F_m$; this graph is nothing but the intersection of $E \times F_m$ and the graph of u in $E \times F$; the graph of u in $E \times F$ is closed and $F_m \to F$ is a continuous map; we get that the graph of u_m is closed in $E \times F_m$. Under these conditions we know [Bourbaki 5, Ch. I § 3, n^0 3, Th. 1] that the domain of definition of u_m is either meagre or the whole space E. Since $u^{-1}(F_m)$, the domain of definition of u_m, is not meagre, we deduce that $u^{-1}(F_m) = E$. Now the closed graph theorem gives us that u_m is in fact continuous. Hence, we have shown that every $u \in \mathscr{L}(E, F)$ belongs to some $\mathscr{L}(E, F_m)$. We get from Case 2 that $\mathscr{L}_c(E, F_m)$ is Lusin, for every m. A fortiori, $\mathscr{L}(E, F_m)$ provided with the topology induced from $\mathscr{L}_c(E, F)$, this topology being less fine than that of $\mathscr{L}_c(E, F_m)$, is necessarily a Lusin space. Now $\mathscr{L}_c(E, F)$ being the countable union of Lusin spaces is itself Lusin.

GENERAL CASE. The space $\mathscr{L}_s(E, F)$, the space of linear continuous functions $E \to F$ provided with the simple convergence topology, is the projective limit of $\mathscr{L}_s(E_n, F)$. But $\mathscr{L}_s(E_n, F)$ are Lusin spaces, the topology being weaker than that of the Lusin space $\mathscr{L}_c(E, F)$. Hence, $\mathscr{L}_s(E, F)$ is a Lusin space. But if $K \subset E$ is a compact set, then K is contained in some E_m; hence $\mathscr{L}_c(E, F)$ is also the projective limit of $\mathscr{L}_c(E_n, F)$. It follows that $\mathscr{L}_c(E, F)$ is Lusin.

† A set is said to be *meagre* if it is contained in a countable union of closed subsets with void interiors.

This completes the proof of the theorem.

COROLLARY 1. *Let E be a space as in the theorem above, then E'_c is Lusin.*

This corollary follows because $E'_c = \mathscr{L}_c(E, \mathbf{C})$.

COROLLARY 2. *Any Hausdorff complex topological vector space which is the countable union of linear continuous images of separable Fréchet spaces is Lusin.*

Applying the theorem with $E = \mathbf{C}$ we get the result.

REMARK. In the above theorem (or the corollaries) the topology of compact convergence cannot be replaced by the topology of bounded convergence. L^∞, the strong dual of L^1, is not separable and hence not Lusin.

EXAMPLES. The following familiar spaces are Lusin.

(A) L^p, L^p_{loc}, $H^{s,p}$, $H^{s,p}_{\text{loc}}$. $\mathscr{E}^m$, $\mathscr{S}$, $p \neq \infty$.

These are separable Fréchet spaces.

(B) L^p_{comp}, $H^{s,p}_{\text{comp}}$, $\mathscr{D}^m$, $\mathscr{D}$, $p \neq \infty$.

These are strict inductive limits of countably many separable Fréchet spaces.

(C) $\mathscr{E}'$, $\mathscr{S}'$, $\mathscr{D}'$.

The spaces $\mathscr{E}$, $\mathscr{S}$, $\mathscr{D}$ are Montel and hence the strong topology on the dual coincides with the topology of compact convergence. It follows that the spaces in (C) are Lusin.

(D) $\mathscr{O}_M$ and $\mathscr{O}'_c$ are the spaces of multiplication and convolution operators of $\mathscr{S}$ in $\mathscr{S}$, respectively; and their topologies are precisely those induced by $\mathscr{L}(\mathscr{S}, \mathscr{S})$. But $\mathscr{L}(\mathscr{S}, \mathscr{S})$ is Lusin and the above spaces being closed, are Lusin.

(E) Let $K \subset \mathbf{C}^n$ be a compact set. Let $\mathscr{H}(K)$ be the space of holomorphic functions on "K"; viz. consider the class of holomorphic functions defined in some open neighbourhood of K. Two functions of this class are said to be equivalent if they coincide

on an open set containing K. $\mathscr{H}(K)$ is the set of these equivalence classes. For every open set Ω containing K, there is a natural linear mapping of $\mathscr{H}(\Omega)$ (the space of holomorphic functions on Ω) into $\mathscr{H}(K)$. The space $\mathscr{H}(\Omega)$ provided with the topology of compact convergence is a separable, Montel and Fréchet space. (Hence it is Lusin). The space $\mathscr{H}(K)$, provided with the finest topology which makes the above mappings continuous is a locally convex topological vector space. We assert that this space and its strong dual are Lusin.

There are countably many open sets Ω_n containing K, forming a fundamental system; we get that $\mathscr{H}(K)$ is the countable union of continuous images of the separable Fréchet spaces $\mathscr{H}(\Omega_n)$ and hence $\mathscr{H}(K)$ is Lusin. Now, it can be proved [Martineau, 1, Lemma A, p. 14] that every bounded set in $\mathscr{H}(K)$ comes from a bounded and hence relatively compact set in $\mathscr{H}(\Omega)$, for some Ω; we deduce that the topologies of bounded convergence and compact convergence coincide on the dual of $\mathscr{H}(K)$. We deduce that the strong dual of $\mathscr{H}(K)$ is Lusin.

The tremendous stability of the Lusin and Suslin spaces under the elementary operations on topological spaces allows one usually to prove directly that a space is Lusin or Suslin, without using Theorem 6. Consider for instance $\mathscr{E}'$. Every equicontinuous set is compact metrizable and is therefore polish; $\mathscr{E}'$ is a countable union of such spaces, and hence $\mathscr{E}'$ is Lusin. (This is exactly a part of the reasoning in the proof of Theorem 6). Now, if K is a compact set of $\mathbf{R}^n$, $(\mathscr{D}_K)'$ is Lusin for the same reason; and $\mathscr{D}'$ is the projective limit of a countable system of $(\mathscr{D}_K)'$; therefore $\mathscr{D}'$ is Lusin. This again is a part of the reasoning in the proof of Theorem 6. Let now, $\mathscr{B}^\circ$ be the Banach space of continuous functions converging to 0 at infinity; it is polish. Let $\mathscr{S}^\circ$ be the space of rapidly decreasing continuous functions: $f \in \mathscr{S}^\circ$ iff, for every polynomial q, qf belongs to $\mathscr{B}^\circ$. Consider the weakest topology on $\mathscr{S}^\circ$ for which all the multiplications by the polynomials are continuous from $\mathscr{S}^\circ$ to $\mathscr{B}^\circ$. It is a countable projective limit of spaces isomorphic to $\mathscr{B}^\circ$, therefore a separable Fréchet space. The dual $\mathscr{O}'_M$ of $\mathscr{O}_M$ is the space of

distributions which are finite sums of derivatives of functions in $\mathscr{B}^\circ$. Every differential operator $P(D)$ with constant coefficients operates continuously: $\mathscr{S}^\circ \to \mathscr{O}'_M$, and $\mathscr{O}'_M$ is exactly the union of the $P(D)\mathscr{S}^\circ$, for a countable infinity of P; $\mathscr{O}'_M$ is a countable union of linear continuous images of Fréchet spaces, therefore $\mathscr{O}'_M$ is Lusin.

Interchange now projective and inductive limits. Call $\mathscr{O}^0_M$ the space of slowly increasing continuous functions, with the natural topology: it is a countable union of images $q\,\mathscr{B}^0$ of $\mathscr{B}^0$ by multiplications by polynomials q, and each of these multiplications is continuous; hence $\mathscr{O}^0_M$ is Lusin. Now $\mathscr{O}_M$ is a countable projective limit of spaces $\mathscr{O}^0_M$ by maps $\mathscr{O}_M \xrightarrow{P(D)} \mathscr{O}^0_M$; therefore $\mathscr{O}_M$ is Lusin. One proves in the same way that $\mathscr{O}_c$ and $\mathscr{O}'_c$ are Lusin (changing the order of multiplications and differential operators). One may write (symbolically!):

$$\mathscr{O}'_M = \varinjlim_{\rightarrow} P(D) \varprojlim_{\leftarrow} (Q)\,\mathscr{B}^0 \qquad \mathscr{O}_M = \varprojlim_{\leftarrow} P(D) \varinjlim_{\rightarrow} (Q)\,\mathscr{B}^0$$

$$\mathscr{O}_c = \varinjlim_{\rightarrow} (Q) \varprojlim_{\leftarrow} P(D)\,\mathscr{B}^0 \qquad \mathscr{O}'_c = \varprojlim_{\leftarrow} (Q) \varinjlim_{\rightarrow} P(D)\mathscr{B}^0$$

OPEN PROBLEMS. If E and F are two Lusin or Suslin locally convex topological vector spaces, what can we say about $E \hat{\otimes}_\pi F$, $E \hat{\otimes}_\varepsilon F$, $L_c(E; F)$?

3. Radon spaces.

DEFINITION 8. *A Hausdorff space X is called a Radon space if every finite Borel measure on X is a Radon measure, i.e. is inner regular.*

PROPOSITION 6. *A compact metrizable space X is Radon.*

PROOF. The subsets B of X for which the inner regularity holds, for a given finite Borel measure m on X, form a family, stable for countable unions and intersections. Therefore the subsets B, such that the inner regularity holds for both B and $\complement B$, form a σ-algebra. But the closed subsets have this property, because every open subset of X is a countable union of closed subsets; therefore all the

Borel subsets have this property: m is inner regular or Radon, which proves the proposition.

PROPOSITION 7. *Let μ be any locally finite Borel measure on a Radon space X and $\tilde{\mu}$ the associated essential measure. Then $\tilde{\mu}$ is a Radon measure.*

PROOF. We shall show that $\tilde{\mu}$ satisfies the conditions of R_3 (Chapter I). By assumption $\tilde{\mu}$ is locally finite and it remains to verify the inner regularity. Let B be a Borel set of finite μ measure. Then $\chi_B \tilde{\mu}$ (which is also equal to $\chi_B \mu$) is a finite Borel measure and is hence Radon. We get

$$\mu(B) = \tilde{\mu}(B) = \sup \{\mu(K): \text{compact } K \subset B\}.$$

If B' is any Borel set, we have

$$\tilde{\mu}(B') = \sup \{\mu(B): \text{Borel } B \subset B' \text{ and } \mu(B) < +\infty\}.$$

The inner regularity of $\tilde{\mu}$ follows, completing the proof.

PROPOSITION 8. *If X is a Radon space and Y a universally measurable subset of X (viz. Y is measurable for every finite Radon measure on X), then Y (with the induced topology) is a Radon space.*

PROOF. Let m be a finite Borel measure on Y and j the canonical injection $Y \to X$. Now, jm is a finite Borel measure on X and is hence Radon. Consider the restriction $(jm)_Y$ of this Radon measure to the measurable set Y. For any Borel subset B of Y,

$$\begin{aligned}(jm)_Y(B) &= (jm)^*(B) \qquad \text{(definition)}\\ &= \inf \{(jm)(B'): B' \text{ Borel in } X, B' \supset B\}\\ &= \inf \{m(B' \cap Y): B' \text{ as above}\}\\ &= m(B) \ (B \text{ is the intersection with } Y \text{ of a Borel set in } X).\end{aligned}$$

We know that $(jm)_Y$ is a Radon measure on Y and $m = (jm)_Y$. This completes the proof.

The following result proves the converse.

PROPOSITION 9. *Let X be a Radon space. Let X be a topological subspace of any Hausdorff space Z. Then X is measurable for every measure on the Borel σ-algebra of Z. (viz., any Radon space is universally measurable).*

PROOF. Let m be a Borel measure on Z. We have to show that for every m-integrable Borel set $B \subset Z$, $X \cap B$ is m-integrable. $X \cap B$ is a Borel subset of X and is hence a Radon space. By considering the measure m_B induced on B and the set $X \cap B$ we are brought to the case when the whole space is of finite measure.

We shall hence assume that m is finite on Z and X is a subspace which is Radon. Let, now, Z_0 be a Borel subset of Z such that (i) $X \subset Z_0$ and (ii) $m(Z_0) = m^*(X)$. If B is a Borel subset of X, then there is a Borel set B' of Z such that $B' \subset Z_0$ and $B' \cap X = B$. Define the set function

$$\tilde{m}(B) = m(B')$$

on the σ-algebra of Borel sets of X. If for any B, B' and B'' are two Borel sets in X contained in Z_0 such that $B' \cap X = B = B'' \cap X$, then using the fact that $m^*(X) = m(Z_0)$ or $m_*(Z_0 - X) = 0$ and $B' \Delta B'' \subset Z_0 - X$, we see that $m(B' \Delta B'') = 0$†. Hence $m(B') = m(B'')$. Now, it is easily verified that m is a Borel measure on X. But since X is a Radon space, this is a Radon measure. Hence,

$$m^*(X) = \tilde{m}(X) = \sup\{\tilde{m}(K) : K \text{ compact } \subset X\}.$$

But $\tilde{m}(K) = m(K)$, for any compact set $K \subset X$, and it follows that X is m-measurable. The proposition is proved.

PROPOSITION 10. *Countable unions and countable intersections of Radon spaces are themselves Radon.*

PROOF. Let $(A_n)_{n \in \mathbf{N}}$ be a sequence of Radon subspaces of a space X. For every finite Radon measure on X, each A_n is measurable and hence $\bigcap_{n=1}^{\infty} A_n$ is measurable. A fortiori, A is

† Δ denotes the symmetric difference.

universally measurable for every finite Radon measure on A_1 (by taking the image measure for the injection $A_1 \to A$). But A_1 is Radon and it follows that A is Radon. Let now, $A' = \bigcup_{n=1}^{\infty} A_n$. A' can be expressed as the countable disjoint union of A'_m, $A'_m = A_m \cap \bigcap_{1 \leqslant i < m} \complement A_i$. Each A_m is universally measurable (for every Borel measure on X) and hence the same is true of A'_m. Now, A'_m being also a subspace of A_m, we deduce as above that A'_m is a Radon space. Let μ be any finite Borel measure on A'. Since A'_m are universally measurable, $\mu_{A'_m}$ is a finite Borel measure on A'_m and is hence Radon. Let j_m be the canonical injection of $A'_m \to A'$. Then $\mu = \sum_m j_m \mu_{A'_m}$. Since the measures $j_m \mu_{A'_m}$ are all Radon, it follows that μ is Radon, completing the proof.

A trivial example of a non-Radon space is a non-Lebesgue measurable subset of [0, 1] (because of Proposition 9); we have actually given in Chap. I, §5 a Borel measure on it, which is not Radon. Such a non-Radon space is metrizable and separable; all its compact subspaces are Radon spaces.

In Chapter I, §6 (d), we have given an example (due to Dieudonné) of a compact Hausdorff space on which there is a Borel measure which is not Radon. Hence a compact space is not necessarily Radon. This space has a basis of cardinal $\aleph_1$ (the first cardinal $> \aleph_0$) and so (by Urysohn's theorem —see Bourbaki, [4] or Kelley, [1]) may be embedded in a cube with $\aleph_1$ sides, i.e. $\prod_{i \in I} [0, 1]$ where the cardinality of I is $\aleph_1$. Thus a cube $[0, 1]^J$ with cardinality of $J \geqslant \aleph_1$ is not a Radon space. However, a discrete space of the cardinality of the continuum is a Radon space, if one assumes the continuum hypothesis. This follows from an old result (S. Banach [1], S. Banach and K. Kuratowski [2], S. M. Ulam [1]) that there is no finite non-zero measure on the algebra of all subsets of the continuum which vanishes for points. It follows, that the one point compactification of a discrete space of the cardinality of the continuum is a Radon space. We have thus a compact Radon

space which does not have a countable base for open sets, and is therefore not metrizable (hence not Suslin, see Theorem 6).

THEOREM 8. *Let $(X_n)_{n\in N}$ be a countable family of Radon spaces such that for every n, each compact set of X_n is metrizable. Then $\prod_n X_n$ is Radon.*

PROOF. If $K_n \subset X_n$ is any compact set, then $\prod_n K_n$ is compact and metrizable (since each K_n is metrizable) and hence $\prod_n K_n$ is a Radon space.

Let $X = \prod_n X_n$ and let m be any finite Borel measure on X. Let p_n be the projection of X onto X_n. Let for every i, m_i be the measure $= p_i(m)$. Each m_i is a finite Borel measure on X_i and so is Radon. Thus there exists a compact set $K_n \subset X_n$ with $m_n(X_n - K_n) < \epsilon/2^n$. Let $K = \prod_n K_n$. Then,

$$\begin{aligned} m(X - K) &< \sum_n m[p_n^{-1}(X_n - K_n)] \\ &= \sum_n m_n(X_n - K_n) \\ &< \epsilon. \end{aligned}$$

Thus, taking $\epsilon = 1, 1/2, \ldots, 1/n, \ldots$, we can construct an increasing sequence of compact sets C_n in X, each set C_n is a product set, such that $m(X - C_n) < \frac{1}{n}$. Now, for any Borel set B in X, $m(B) = \lim_n m(B \cap C_n)$. But each C_n is a Radon space. So m_{C_n} the restriction of m to C_n, is a Radon measure. Now we deduce that $m(B) = \sup\{m(D)$: compact $D \subset B\}$. That is, m is a Radon measure on X, completing the proof.

REMARK 1. The conditions of the theorem are verified if (i) each X_n is a Radon space and (ii) if each X_n has a countable base for open sets, or if it is metrizable, or if it is Suslin.

REMARK 2. The proof shows that any countable product of Radon spaces would also be a Radon space if any countable product

of compact Radon spaces is again Radon. We do not know whether the latter property is true even for a product of two compact Radon spaces. However an uncountable product of (even compact) Radon spaces is generally not Radon (we saw that $[0,1]^I$ is not Radon for card $I > \aleph_1$).

We shall now see that every Suslin space is a Radon space. We first prove

THEOREM 9. *Every Lusin space is a Radon space.*

PROOF. Let (X, τ) be a Lusin space. Then there exists a finer topology τ' on X such that (X, τ') is polish. But the Borel sets of either of the topologies are identical (Cor. 3, Theorem 5). Let m be a finite Borel measure on (X, τ); then it is also a Borel measure on (X, τ'). Assume we know that m is a Radon measure in (X, τ'); then it will be a fortiori one on (X, τ). Hence it suffices to prove that any polish space is Radon. But any polish space is homeomorphic to a G_δ in $I^{\mathbf{N}}$, i.e. it is a universally measurable set of a compact metrizable, therefore Radon, space $I^{\mathbf{N}}$. It follows that any polish space is Radon. This completes the proof of the theorem.

That every polish space is Radon was first proved by J.C. Oxtoby and S. M. Ulam[1].

THEOREM 10. (P.A. Meyer) *Every Suslin space is Radon.*

PROOF. To prove the theorem it is enough to show that for any Suslin space and every finite Borel measure on this space, the measure of the whole space is the supremum of the measures of its compact sets. For, let X be a Suslin space and μ a finite Borel measure on X. Let B be any Borel set of X and consider the finite Borel measure μ_B on B. Since B is in turn a Suslin space it has also the same property and it follows that μ on X is inner regular and hence μ is Radon, i.e. X is a Radon space.

Let μ^* be the outer measure generated by μ on the class of all subsets of X. Then the outer measure has the following properties: (1) it is monotone increasing; (2) for any increasing sequence of

Borel sets $A_n \subset X$, $\lim \mu^*(A_n) = \mu^*(\bigcup A_n)$; and (3) for any decreasing sequence of Borel sets $B_n \subset X$, $\mu^*(\bigcap B_n) = \lim \mu^*(B_n)$. (The third property is true since $\mu^* = \mu$ on the Borel σ-algebra of X).

Now, since X is a Suslin space, we can find a G_δ-set P in $Y' = [0, 1]^{\mathbf{N}}$ and a continuous onto mapping $f: P \to X$. Since Y is compact metrizable, we can find a countable family of compact sets K_n^m satisfying (i) $\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} K_n^m = P$ and (ii) for every m, K_n^m increases with n.

Now, let $\lambda < \mu(X)$. The sequence of sets $f[K_n^1 \cap P]$ increases and the limit is X. Hence, we can find an integer p_1 such that $\mu^*[f(K_{p_1}^1 \cap P)] > \lambda$. Again, the sequence of sets $f[K_n^2 \cap K_{p_1}^1 \cap P]$ increases to $f[K_{p_1}^1 \cap P]$ and we can find an integer p_2 satisfying $\mu^*[f(K_{p_2}^2 \cap K_{p_1}^1 \cap P)] > \lambda$. Proceeding this way, we can choose, by induction, a sequence of integers (p_j) verifying, for every j,

$$\mu^*[f(K_{p_j}^j \cap \ldots \cap K_{p_2}^2 \cap K_{p_1}^1 \cap P)] > \lambda.$$

Let $C_j = \bigcap_{i<j} K_{p_i}^i$. Then $(C_j)_{j>1}$ is a decreasing sequence of compact sets of Y and $C = \bigcap_j C_j \subset P$.

Let us now consider $\overline{f(C_n \cap P)}$ in X. These sets form a decreasing sequence of closed sets of X and hence,

$$\mu^*[\bigcap_n \overline{f(C_n \cap P)}] = \lim_{n \to \infty} \mu^*[\overline{f(C_n \cap P)}] > \lambda.$$

The theorem would be proved if we show that the compact set $f(C)$ is precisely $\bigcap_n f(P \cap C_n)$. It is clear that $f(C) \subset \bigcap_n f(C_n \cap P)$. Now, $f(C)$ being compact, the intersection of all closed neighbourhoods of $f(C)$ is precisely $f(C)$. Hence if $x \in X - f(C)$, then there exists a closed neighbourhood V of $f(C)$ with $x \notin V$. But $f^{-1}(V)$ is a neighbourhood of C in P and hence there is a neighbourhood W of C in Y such that $W \cap P = f^{-1}(V)$. Since $\bigcap C_n = C$, there exists an n_0 such that $C_{n_0} \subset W$; hence, $C_{n_0} \cap P \subset W \cap P = f^{-1}(V)$. It follows that $f(P \cap C_{n_0}) \subset V$. But V is, by assumption, closed and hence

$\overline{f(P \cap C_{n_0})} \subset V$. This proves that $x \notin \bigcap_n \overline{f(P \cap C_n)}$ and hence $f(C) = \bigcap_n \overline{f(P \cap C_n)}$.

The proof is complete.

COROLLARY 1. *Any Suslin space X is universally measurable (for any Borel measure on any space of which X is a subspace).*

COROLLARY 2. *Let (X, τ_1) and (X, τ_2) be two Suslin spaces such that the two topologies τ_1 and τ_2 are comparable. Then, $\mathscr{B}_1$ and $\mathscr{B}_2$, the spaces of finite Radon measures of the two spaces, are identical.*

PROOF. They are just the spaces of finite Borel measures, and we saw that the Borel σ-algebras on τ_1 and τ_2 are the same (Theorem 4, Cor. 2).

REMARK 1. The conclusion of the second corollary is not true if (X, τ_1) and (X, τ_2) are only *Radon* spaces and not *Suslin* spaces. Consider, for example, (X, τ_2) a compact metrizable space (say, $I^{\mathbf{N}}$) and (X, τ_1) the discrete space. (X, τ_1), it has been observed already, is a Radon space. It is trivial to see that the Radon measures on (X, τ_1) and (X, τ_2) are not identical.

REMARK 2. There are Radon spaces which are not Suslin. We saw on p. 120 that the one point-compactification of a discrete space with the power of the continuum is Radon and compact; it is not Suslin since it is not separable. The discrete space itself is Radon and metrizable, and not Suslin. If now we take, in a polish space (say $[0,1]$) a subset X which is the complement of a Suslin subspace but is not itself Suslin, it is surely universally measurable and therefore Radon and even metrizable and separable; however, it is not Suslin.

REMARK 3. Sunyach [1], has given a necessary and sufficient condition for a completely regular topological space to be universally Radon measurable.

DEFINITION 10. *A Hausdorff space is said to be strongly Radon if every locally finite Borel measure on this space is a Radon measure.*

THEOREM 11. *Every Lindelöf Radon space (for instance, every Suslin space) is strongly Radon and every Radon measure on it is moderated.*

PROOF. Let μ be a locally finite Borel measure on a Lindelöf Radon space X. Every $x \in X$ has an open neighbourhood V_x of finite μ-measure. It is possible to choose a countable number of them which cover X. The induced measure V_x is Radon on V_x, for every x. Hence μ is the unique abstract (Borel) measure on X which glues the countably many μ_{V_x} (V_x covering X). But this measure is also Radon. This completes the proof.

Before proceeding to see some examples of Suslin and Radon spaces in analysis, we shall summarise some essential properties of Radon spaces (stability etc.).

The class of Radon spaces is closed for

(1) countable topological sums;

(2) countable unions and intersections, complementation;

(3) universally measurable subspaces (necessary and sufficient);

(4) countable product of spaces such that every compact set in each of them is metrizable.

But the class of Radon spaces is unstable for

(1) weakening of topology;

(2) passage to continuous (even injective) image.

EXAMPLE. A discrete space, having the power of the continuum, is Radon (if we assume the continuum hypothesis). If weakening the topology would preserve Radon spaces, any space having the power of the continuum would be Radon; however a non-Lebesgue-measurable subset of [0, 1] is not Radon.

COMPLEMENTS ABOUT MEYER'S PROOF (THEOREM 10). The thing which has been actually proved is the following: let P be a topological space, which is a $K_{\sigma\delta}$, i.e. a countable intersection of countable unions of compact spaces ($= \bigcap_m \bigcup_n K_n^m$), and let f be a continuous

surjective map of P onto X. Then every finite Borel measure on X is carried by a countable union of images (through f) of compact subsets of P. (For every $\lambda < \mu(X)$, there exists a compact $C \subset P$ such that $\mu(f(C)) > \lambda$).

Let us call every continuous image of a $K_{\sigma\delta}$-space *K-analytic.* Suslin and compact spaces are K-analytic. Now we have

LEMMA 19. *Let g be a continuous surjective map from a K-analytic space U onto another such space V_2; then every finite Borel measure on V is carried by a countable union of images (through g) of compact subsets of U.*

PROOF. Let P be a $K_{\sigma\delta}$ space, p a continuous surjective map from P onto U. Taking $f = g \circ p : P \to V$, one is led back to the previous assertion, which proves the lemma.

Taking now $U = V = X$, a K-analytic space, g the identity map, m a finite Borel measure on X, we get that m has the property of inner regularity for X, and also for all the closed subspaces, which are K-analytic too. But m need not be a Radon measure because the Borel subsets of X need not be K-analytic; and we know very well that every compact space is not Radon. However, if X is Suslin, all its Borel subsets are Suslin too; then every finite Borel measure is inner regular and hence X is a Radon space. That is the key to Meyer's proof. Now we have

THEOREM 12. *Let $f : X \to Y$ be a continuous surjective map, and $\tilde{f}$ the corresponding map for finite Radon measures. If X is K-analytic (in particular, compact or Suslin), $\tilde{f}$ is surjective.*

PROOF. It suffices to apply Theorem 12 of Chapter I (page 39), and Lemma 19 to complete the proof.

REMARK. This is not true without assumptions on X. Take for instance $X = I^{\mathbf{N}}$ discrete and $Y = I^{\mathbf{N}}$ compact.

COROLLARY 1. *Let τ_1 and τ_2 be two comparable topologies on a K-analytic (for τ_1 and τ_2) set X. Then the spaces of finite Radon*

measures on (X, τ_1) *and* (X, τ_2) *coincide; the identity maps* $(X, \tau_1) \to (X, \tau_2)$ *and* $(X, \tau_2) \to (X, \tau_1)$ *are universally Lusin measurable.*

Apply the previous theorem and Theorem 11 of Chapter I (page 37). Then Theorem 11 proves that the identity is Lusin measurable for every finite Radon measure; but for an infinite measure we go back immediately to the finite case, remembering that a map is measurable iff its restriction to every compact subspace is measurable with respect to the induced measure.

COROLLARY 2. *Let X be as in the first corollary. If a map f from a topological space Z equipped with a Radon measure μ is μ-Lusin measurable from Z into (X, τ_1), it is also μ-Lusin measurable from Z into (X, τ_2).*

PROOF. Now, with our hypothesis on f, Z, μ, if $K \subset Z$ is compact, $f|_K$ is μ_K-Lusin measurable and the identity $(X, \tau_1) \to (X, \tau_2)$ is $(f|_K)$ (μ_K)-Lusin measurable, therefore $f|_K$ is μ_K-Lusin measurable from $Z \to (X, \tau_2)$; therefore f is μ-Lusin measurable.

LEMMA 20. *Let C_n, for every $n > 0$, be a countable set and $C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \ldots \leftarrow C_n \leftarrow \ldots$ a system of surjective mappings defining a projective limit C (a closed subset of $\Pi_n C_n$). Suppose each C_n is ordered by a bijective mapping onto* N *(or a subset of* N*). Then, for the lexicographic order on C, every closed subset of C has a smallest element.*

PROOF. Let π_k be the projection $\prod_n C_n \to C_k$. Let A be a closed subset of C. There is a smallest element $c_0 \in C_0$ in the set $\pi_0(A)$. There is a smallest element $c_1 \in C_1$ in the set $\pi_1[\pi_0^{-1}(\{c_0\}) \cap A]$. It is clear that by induction we can choose a coherent sequence, i.e. an element of C, $\{c_0, c_1, \ldots, c_n, \ldots\}$, such that every element of A is $>$ this element. Further, for every n, there is an element of A in $\pi_0^{-1}(c_0) \cap \pi_1^{-1}(c_1) \cap \ldots \cap \pi_n^{-1}(c_n)$. Hence, this element of C is in the closure of A. But A being closed, it is in A. This proves the lemma.

THEOREM 13 (von Neumann). *Let p be a surjective Borel map of a Suslin space X onto a Suslin space Y. Then there is a section $s: Y \to X$, universally Lusin measurable, such that $p \circ s =$ identity.*

PROOF. Let us first prove the theorem assuming that p is a continuous function. We can suppose that the space X is polish and therefore Lusin measurability is equivalent to Borel measurability. Again, by lifting, we can always suppose that the space X is a projective limit of the form $C \subset \prod_n C_n$ as in the lemma above. Define, for every $y \in Y$, $s(y)$ = the smallest element, in the lexicographic order, of the closed set $p^{-1}(\{y\})$ (Lemma 20). It is clear that $p \circ s$ is the identity on Y. We have only to show that s is universally measurable. For this we have to show that, for every Borel set $B \subset C$, $s^{-1}(B)$ is universally measurable. For this it is enough to show that for every rectangular open set, the inverse image by s is universally measurable. Hence B can be taken to be a set of the form, $(\{c_0\} \times \{c_1\} \times \ldots \times \{c_n\} \times \prod_{m>n} C_m) \cap C$. But by the choice of s, it is clear that $s^{-1}(B)$ is precisely the set of all points of $p(B)$ which do not belong to any $p(B')$ where $B' = \{c_0'\} \times \ldots \times \{c_n'\} \times \prod_{m>n} C_m$ where $(c_0', c_1', \ldots, c_n') < (c_0, c_1, \ldots, c_n)$. Now B and every B' are open sets of C and hence $p(B)$ and $p(B')$ are Suslin spaces. Now the $p(B')$ are at most countable and hence $s^{-1}(B) = p(B) - \bigcup p(B')$ is universally measurable. Hence this proves the result in the case when p is continuous.

Suppose now that p is only Borel. Then the graph G of p in $X \times Y$ is Suslin. The projection $G \to Y$ is continuous. Hence there is a universally measurable section $Y \to G$. Now composing it with the canonical projection $G \to X$ we get the required result. The proof is complete.

REMARK. In the proof of the theorem we have shown that $s^{-1}(B)$, for every Borel set of X, is in the σ-algebra generated by the Suslin subspaces of Y.

COROLLARY 1. *Let f be a map of a topological space Z, equipped with a Radon measure μ, into Y. Let f be a μ-measurable map. Then there exists a μ-measurable map g from Z into X such that $f = p \circ g$.*

PROOF. We can take $g = s \circ f$ where s is a universally mesurable section $Y \to X$ of p. If $K \subset Z$ is compact, $f|K$ is μ_K-measurable, s is $(f|K)(\mu_K)$-measurable, therefore $g|K$ is μ_K-measurable. Since K is arbitrary, g is μ-measurable and $p \circ g = p \circ s \circ f = f$, completing the proof.

THEOREM 14. *Let f be a map from a topological space Z equipped with a Radon measure μ into a Suslin space Y. If f is μ-Borel measurable, then it is μ-Lusin measurable.* (cf. Ch. I, Th. 5).

PROOF. Let p be a continuous surjection from a polish space X onto Y and s a universally measurable section $Y \to X$ and $g = s \circ f$. Then $g = s \circ f$ is μ-Borel measurable for Z into X. But X is polish, hence metrizable and separable; therefore g is also μ-Lusin measurable and p being continuous, $f = p \circ g$ is μ-Lusin measurable too. The proof is complete.

EXAMPLE 1. Let E be the set of all real-valued continuous functions on $\mathbf{R}$ provided with the topology of uniform convergence on the compact subsets of $\mathbf{R}$. It is trivial to see that this is a polish space. The mapping $(f,t) \to f(t)$ of $E \times \mathbf{R} \to \mathbf{R}$ is continuous. Let

$$D = \{(f,t) \in E \times \mathbf{R} : \text{the right derivative of } f \text{ at } t \text{ exists}\}.$$

PROPOSITION 11. *The set D defined above is Borel in E.*

PROOF. Given $\eta > 0$, let, for every h, k, between 0 and η,

$$F(t,h,k,f) = \left| \frac{f(t+h) - f(t)}{h} - \frac{f(t+k) - f(t)}{k} \right|.$$

Let $\epsilon > 0$. For every h, k, ϵ fixed, the set $D(\epsilon, h, k)$ of $(f, t) \in E \times \mathbf{R}$ such that $F(t, h, k, f) < \epsilon$ is a closed subset of $E \times \mathbf{R}$. Hence, for every $\epsilon > 0$ and $\eta > 0$,

$$D(\epsilon, \eta) = \bigcup_{\substack{0 \le h \le \eta \\ 0 \le k \le \eta}} D(\epsilon, h, k,)$$

is also closed in $E \times \mathbf{R}$. Now, it is easy to see that

$$D = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} D\left(\frac{1}{m}, \frac{1}{n}\right)$$

and hence D is a Borel set in $E \times \mathbf{R}$, completing the proof.

It follows that D is a Lusin subspace and hence its projection on E is Suslin. But this projection is nothing but the set of all f in E such that the right derivative of f exists at least at one point of $\mathbf{R}$. Hence its complement, viz. the set of real continuous functions f on $\mathbf{R}$ such that the right derivative of f exists at no point, is a Radon space (since it is the complement of a Suslin space in a polish space). Similarly the set of functions which have a left and right derivative at least at one point of $\mathbf{R}$ is a Suslin subspace of E; and hence, the functions which are differentiable (viz., right derivative $=$ left derivative) at least at one point (resp. at no point) is a Suslin (resp. Radon) subspace of E. In a similar way, the set of functions which are not differentiable at least at one point (resp. differentiable everywhere) is a Suslin (resp. Radon) subspace of E. On the other hand, the functions which are of class C^1 (once continuously differentiable) is polish and hence Borel in E; in turn it is a Lusin subspace of E. In particular, all the spaces considered above are universally measurable for all Borel measures on E.

EXAMPLE 2. Consider $\mathbf{R}^{\mathbf{Z}}$, i.e. the space of real sequences. This is clearly a polish space. For $s \in \mathbf{R}^{\mathbf{Z}}$, if $s = (a_n)_{n \in \mathbf{Z}}$ define $T(s)$, the trigonometric series,

$$[T(s)](t) = \sum_{n \in \mathbf{Z}} a_n e^{int}.$$

Then, the set of elements $(s, t) \in \mathbf{R}^{\mathbf{Z}} \times \mathbf{R}$ such that the trigonometric series $T(s)$ converges at t, is a Borel set. Hence, the set of s in $\mathbf{R}^{\mathbf{Z}}$ for which the trigonometric series converges at least at one point (resp. nowhere) is a Suslin subspace (resp. a Radon subspace) of $\mathbf{R}^{\mathbf{Z}}$. Similarly, the set of sequences for which the trigonometric series does not converge at least at one point (resp. converges everywhere) is a Suslin subspace (resp. Radon subspace) of $\mathbf{R}^{\mathbf{Z}}$.

4. Lifting of L^∞ to $\mathscr{L}^\infty$. The problem is the following. Let X be a topological space with a Radon measure $\mu > 0$ on X. Does there exist a mapping ρ of $L^\infty(X, \mu)$ (the space of equivalence classes of bounded measurable functions endowed with its usual norm: if f is bounded measurable, f° its equivalence class, then

$\|f^\circ\|_{L^\infty} = \text{ess. sup}_{x\in X} |f(x)|$) to $\mathscr{L}^\infty(X, \mu)$ (the space of bounded measurable functions endowed with the norm $\|f\|_{\mathscr{L}^\infty} = \sup_{x\in X} |f(x)|$) having the following properties:

(a) ρ is linear, continuous and isometric: $\|\rho f^\circ\|_{\mathscr{L}^\infty} = \|f^\circ\|_{L^\infty}$;

(b) $\rho\, 1^\circ = 1$;

(c) $\rho f^\circ > 0$ if $f^\circ > 0$;

(d) the class of ρf° in L^∞ is f°, i.e. $(\rho f^\circ)^\circ = f^\circ$?

In these relations, f° is an element of L^∞, hence a class of functions and not a function; 1° is the class of the function 1; a class is > 0 if it contains a function > 0, in other words, in (c) $f^\circ > 0$ says that f is μ-almost everywhere > 0, further $\rho f^\circ > 0$ says that ρf° is everywhere > 0. One can if one wishes consider only $\mathscr{L}^\infty$, and seek a mapping u of $\mathscr{L}^\infty$ into itself (which would simply be the composition of $f \to f^\circ$, $\mathscr{L}^\infty \to L^\infty$ and ρ) with the following properties:

(a) u is linear, continuous and $\|uf\|_{\mathscr{L}^\infty} = \inf\{\|f'\|_{\mathscr{L}^\infty} : f' = f, \mu\text{-a.e.}\}$;

(b) $u1 = 1$;

(c) $uf > 0$ if f is > 0 μ-almost everywhere;

(d) f and g are equal μ-almost everywhere $\Leftrightarrow uf = ug$; and uf is equal to f μ-almost everywhere.

A mapping such as ρ or u is called a *lifting*.

In the case of an abstract σ-finite measure not identically zero, the lifting is always possible according to a theorem of Dorothy Maharam [1]; see also A and C. Ionescu Tulcea [1, 2] (if μ is the zero measure, $L^\infty = \{0\}$ and the conditions (a) and (b) are contradictory).

THEOREM 15. *If μ is a Radon measure, $\mu > 0$, $\neq 0$, then the lifting is always possible.*

PROOF. Let $X = \bigcup_{i\in I} X_i \cup N$ be a μ-concassage of X. We suppose that the X_i are μ-full, i.e. that $\overline{\bigcup_{i\in I} X_i} = S$, the support of μ.

For each i, let ρ_i be a relative lifting of the measure μ_{X_i} on the compact set X_i. We define ρ as follows:

(a) if $x \in X_i$, $(\rho f^\circ)(x) = (\rho_i f^\circ)(x)$;

(b) if $x \in N$, we choose an ultrafilter $\mathcal{U}_x$ such that $\bigcup_{i \in I} X_i \in \mathcal{U}_x$, and we put $(\rho f^\circ)(x) = \lim_{\mathcal{U}_x} (\rho' f^\circ)$, where ρ' is the lifting on $\bigcup_{i \in I} X_i$ defined by (a). One can verify that ρ has the required properties. The limit along the ultrafilter exists, since $\rho' f^\circ$ is bounded; the necessity of taking such an ultrafilter arises from the fact that we desire at the same time the linearity of ρ and the condition $\rho\, 1^\circ = 1$.

LIFTING OF LOCAL TYPE. Let us work with $u: \mathscr{L}^\infty \to \mathscr{L}^\infty$ instead of ρ. Is it possible to find a lifting of local type, i.e. one such that for each open set Ω, the value of uf in Ω depends only on f in Ω? This is clearly impossible if the support of μ is not the whole of X. For if Ω is contained in the complement of S, the function equal to $+1$ on S, and to -1 on the complement $\complement S$ of S has $+1$ as its lifting, but since it coincides on Ω with the function -1, it must have a lifting equal to -1 on Ω. It is the condition (b) for a lifting which is troublesome here. We replace it from now on by the following

(b') $\rho 1^\circ$ or $u1$ equals 1 on the support of μ.

In this case one can repeat the proof of the preceding theorem modifying it as follows: if $x \in S$, $\mathcal{U}_x$ must be chosen to be an ultrafilter containing $\bigcup_{i \in I} X_i$ and converging to x; if $x \notin S$, one takes $(\rho f^\circ)(x) = 0$. We remark that the definition then works even for $\mu = 0$. Replacing once and for all the condition (b) by the condition (b'), we shall say that a lifting u is of *local type* if for any open set Ω, if f is μ-almost everywhere > 0 in Ω, then uf is > 0 in Ω.

LEMMA 21. *If X is a Lusin space, it admits a lifting of local type for each finite measure $\mu > 0$.*

PROOF. Consider a strict sub-division of X: we have a sequence of countable sets $C_0, C_1, C_2, \ldots, C_n, \ldots$, of surjections $C_{n+1} \to C_n$, and of mappings $\phi_n: C_n \to \mathscr{P}(X)$, where the images under ϕ_n of two distinct elements of C_n are disjoint Borel subsets of X. Let us call

τ_n the σ-ring of Borel subsets of X generated by the $\phi_n(c)$, $c \in C_n$; it is simply the set of subsets of X which are countable unions of subsets $\phi_n(c)$, $c \in C_n$. Let us call X_n, the union of sets $\phi_n(c)$, $c \in C_n$, of μ-measure > 0; the support S of μ is evidently contained in $\overline{X}_n$. For all n, we define the function f_n as follows: for all $\phi_n(c)$, $c \in C_n$, of μ-measure > 0, it equals the mean of f on $\phi_n(c)$, i.e. $\int_{\phi_n(c)} f\,d\mu / \int_{\phi_n(c)} d\mu$; outside X_n it is null. Then f_n is measurable τ_n, and its integral on each subset of ϕ_n is the same as that of f; it is what is called, in probability theory, a conditional expectation of f relative to μ and to the σ-ring τ_n. The theory of martingales (see e.g. Meyer [1]) tells that f_n then converges to f μ-almost everywhere as $n \to \infty$ (since the σ-ring τ generated by the τ_n being generated on the Lusin space X by a denumerable family of Borel subsets which separate points, is the entire σ-algebra of Borel sets). We now proceed to define uf as follows. We choose once and for all, an ultrafilter $\mathcal{U}$ on N, finer than the filter of complements of finite sets. On $X_\infty = \bigcap_n X_n$ (the sequence X_n is decreasing), we put $uf = \lim_{\mathcal{U}} f_n$. The preceding result on martingales shows us that, whatever be the definition of uf on $\complement X_\infty$, $uf = f$ μ-almost everywhere. Since further if $f = g$ μ-almost everywhere, one has $f_n = g_n$ on X_n, one has $uf = ug$ everywhere on X_∞. If $f \geqslant 0$ μ-almost everywhere one has $f_n \geqslant 0$ on X_n and so $uf \geqslant 0$ on X_∞. If $f = 1$ one has $f_n = 1$ on X_n and so $uf = 1$ on X_∞. It remains therefore to define u on $\complement X_\infty$. For $x \in \complement X_\infty \cap S$, x is adherent to X_∞; we choose an ultrafilter $\mathcal{U}_x$ on X finer than the filter of neighbourhoods of x and containing X_∞, and we put $(uf)(x) = \lim_{\mathcal{U}_x} uf$ (uf being already defined on X_∞). For $x \notin S$, we put $(uf)(x) = 0$. The mapping u is linear, because all the stages are linear. Since $uf = f$ μ-almost everywhere we have certainly $uf \in \mathscr{L}^\infty$. If $f \geqslant 0$, we have $uf \geqslant 0$. If $f = g$ μ-almost everywhere we have $uf = ug$ on X_∞ and hence everywhere. Hence the conditions (c) and (d) are verified. For the condition (a) of the norm, one has $\| f_n \|_{\mathscr{L}^\infty} < \| f^\circ \|_{L^\infty}$ on X_n hence $\| uf \|_{\mathscr{L}^\infty} < \| f^\circ \|_{L^\infty}$ on X_∞, hence everywhere. But $uf = f$ μ-almost everywhere, hence $\| uf \|_{\mathscr{L}^\infty} = \| f \|_{\mathscr{L}^\infty}$.

We now verify condition (b'). If $f = 1$, $uf = 1$ on X_∞, by the preceding construction $uf = 1$ on S and 0 on $\mathsf{C}\, S$, which is (b').

It remains to verify that the above lifting is of local type. Let Ω be an open set and f a function > 0 in Ω. Let $x \in X_\infty \cap \Omega$; by the properties of a sub-division, for n sufficiently large, the unique $\phi_n(c)$, $c \in C_n$, which contains x and which is necessarily of μ-measure > 0, is contained entirely in Ω; hence $f_n(x) > 0$, hence $(uf)(x) > 0$. Hence uf is > 0 everywhere in $X_\infty \cap \Omega$. If now $x \notin X_\infty$, but $x \in S \cap \Omega$, then $(uf)(x)$ is the limit of uf along the ultrafilter $\mathscr{U}_x$, which contains X_∞ and also Ω; hence this ultrafilter contains $X_\infty \cap \Omega$, and since $uf > 0$ on $X_\infty \cap \Omega$ we have $(uf)(x) > 0$, and so $uf > 0$ on $S \cap \Omega$. On $\mathsf{C}\, S \cap \Omega$, $uf = 0$, so $uf > 0$ on Ω, completing the proof.

LEMMA 22. *If X has a μ-concassage $\bigcup_{i \in I} X_i \cup N$, where the X_i are μ-full, and if, for all X_i, μ_{X_i} has the local lifting property on X_i, then the same is true of μ on X.*

PROOF. It suffices to repeat the construction of the proof of Theorem 15, with the modification indicated just after the statement of condition (b'), and one obtains a local lifting. From Lemmas 21 and 22 one deduces immediately

THEOREM 16. *Let X be a topological space whose compact subsets are metrizable (for example, X metrizable or X Suslin). Then one has the property of lifting of local type for every Radon measure $\mu > 0$ on X.*

It is not known yet (works of Ionescu Tulcea, see [2]) if one has the property of lifting of local type for all compact spaces. This is equivalent to having it for all cubes $[0, 1]^I$, since all compact spaces can be embedded in a $[0, 1]^I$; or for all topological spaces by Lemma 22.

THEOREM 17. *Let X be a topological space, μ a Radon measure on X, $\mu > 0$, u a lifting of local type on X. Then u is a strong lifting (in the sense of Ionescu Tulcea): i.e. if $f \in \mathscr{L}^\infty(X, \mu)$ is continuous in an open set Ω, then $uf = f$ on $S \cap \Omega$ (in particular, if $S = X$, then $uf = f$*

in Ω). *In fact for any* $f \in \mathscr{L}^\infty$, *if* $x \in S \cap \Omega$ *is a point of continuity of* f, *then* $(uf)(x) = f(x)$.

PROOF. Let $x \in S \cap \Omega$. If f is continuous at x there exists an open subset ω of Ω, containing x, in which f takes only values lying between $f(x) - \epsilon$ and $f(x) + \epsilon$; hence, the lifting being local, and the lifting of a constant being constant in S, uf also, in $\omega \cap S$ takes only values between $f(x) - \epsilon$ and $f(x) + \epsilon$; since ϵ is arbitrary, $(uf)(x) = f(x)$, completing the proof.

One clearly cannot expect the converse to be true in general, since there may be very few continuous functions on X. However when X is completely regular we have

THEOREM 18. *Let* X *be a completely regular space. Let* μ *be a Radon measure on* X, $\mu \geqslant 0$, *and* u *a lifting on* X *relative to* μ *having the following property: if* f *is continuous and bounded in* X, *then* $uf = f$ *on* S (*i.e.* u *is a strong lifting in the sense of Ionescu Tulcea*). *Then* u *is a lifting of local type.*

PROOF. Let Ω be an open set, f° a function in $\mathscr{L}^\infty$, μ-almost everywhere > 0 in Ω. Let $M = \|f^\circ\|_{L^\infty}$. For each $x \in \Omega$, one can find a continuous function ϕ on X, everywhere < 0, zero at x, and equal to $-M$ on $\complement\Omega$. It follows that $f \geqslant \phi$ μ-almost everywhere on X; hence $uf \geqslant u\phi$ everywhere. But $u\phi = \phi$ since ϕ is continuous on X; hence $(uf)(x) \geqslant 0$, completing the proof.

The existence of an abstract lifting was first proved by von Neumann [2] for the unit interval and by Dorothy Maharam [1] for the general case. A different proof was given by A. and C. Ionescu Tulcea [1]. Most of their results on lifting (and in particular the idea of a strong lifting) are discussed in their book (Ionescu Tulcea [2]). An important result of theirs not in their book is the proof of the existence of a strong lifting commuting with the left translations of an arbitrary locally compact group, (see Ionescu Tulcea [3]).

5. Spaces of regulated functions. Let us recall that a function on $\mathbf{R}$ with values in a topological space is said to be *regulated* if it has at every point a right and a left limit.

LEMMA 23. *Let J be a compact interval in $\mathbf{R}$, D a denumerable, dense subset of $\mathbf{R}$, $\epsilon > 0$, $n \in \mathbf{N}$, and X a complete metric space with distance d. Let $\mathscr{C}(J, D, \epsilon, n, X)$ denote the set of functions on D with values in X, having the following property: for every finite strictly increasing sequence $t_1 < t_2 < \ldots < t_i$ of elements of $D \cap J$ one has $d(f(t_i), f(t_{i+1})) > \epsilon$ for at most n couples (t_i, t_{i+1}). Let f be a function on D with values in X. In order that f be the trace on D, of a regulated function on $\mathbf{R}$, with values in X, it is necessary and sufficient that, for all J and $\epsilon > 0$, there exists $n \in \mathbf{N}$ such that $f \in \mathscr{C}(J, D, \epsilon, n, X)$.*

PROOF. SUFFICIENCY. Suppose f is not the restriction of a regulated function on $\mathbf{R}$. There exists at least one J and a point $c \in J$ such that $f(t)$ does not tend to a limit, when $t \in D \cap J$ tends to c either from the left or from the right. Since X is complete, there must therefore exist an infinite sequence $(t_m)_{m \in \mathbf{N}}$ in $D \cap J$, strictly increasing or strictly decreasing, tending to c, and an $\epsilon > 0$ such that $d(f(t_{2p}), f(t_{2p+1})) > \epsilon$; hence whatever be n, $f \notin \mathscr{C}(J, D, \epsilon, n, X)$.

NECESSITY. Suppose that the condition of the lemma is not verified. There then exist a J and an $\epsilon > 0$ such that for every $n \in \mathbf{N}$, one can find disjoint open intervals (t_1, t_1'), (t_2, t_2'), $\ldots$, (t_n, t_n') with $t_1, t_1', \ldots, t_n, t_n'$ in $D \cap J$, and $d(f(t_i), f(t_i')) > \epsilon$. Fix k points $a_1, a_2, \ldots, a_k \in J$. When $n > k$, at least one of the preceding intervals does not contain any of the a_i, and has length $\leqslant$ (length of J)$/(n - k)$; we choose such an interval and call it (θ_n, θ_n'). One can choose a convergent subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$; the $\{\theta_{n_k}'\}$ have the same limit which we call θ, obviously $\theta \in J$. Suppose first that for at least one system $(a_1, a_2, \ldots, a_k)$ and one choice of θ_n, θ_n' and a convergent subsequence $\{\theta_{n_k}\}$, θ is one of the a_i. Since a_i is never in (θ_n, θ_n'), the function f does not have a limit as t tends to this a_i, either from the left or from the right, which proves that f if

not the restriction to $D \cap J$ of a regulated function on J. Suppose on the other hand that θ is never one of the a_i. Every extension $\bar{f}$ of f to J is necessarily discontinuous at θ with an oscillation $\geq \epsilon$. Hence the set of points of discontinuity of $\bar{f}$, of oscillation $\geq \epsilon$, is necessarily infinite (since whatever be the points $a_1, \ldots, a_k$, this set contains θ different from the a_i) and $\bar{f}$ is not regulated. So f is not the restriction of a regulated function on J and a fortiori is not the restriction of a regulated function on $\mathbf{R}$.

COROLLARY 1. *Let D be a countable dense subset of $\mathbf{R}$, X a complete metric space. The set of mappings of D into X, which are restrictions to D of regulated mappings of $\mathbf{R}$ into X, is Borel in X^D.*

PROOF. For each fixed J, ϵ, and n, and $t_1 < t_2 < \ldots < t_l$, the set of $f \in X^D$ for which $d(f(t_i), f(t_{i+1})) < \epsilon$ for at most n pairs (t_i, t_{i+1}) is closed in X^D. Hence $\mathscr{C}(J, D, \epsilon, n, X)$ is closed (being the intersection of closed sets); hence the set of restrictions of regulated functions which is exactly

$$\bigcap_{p \in \mathbf{N}} \bigcap_{m \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \mathscr{C}\left([-p, +p], D, \frac{1}{m}, n, X\right)$$

is Borel.

COROLLARY 2. *The set of mappings of D into X which are restrictions of regulated, right continuous maps of $\mathbf{R}$ into X, is Borel in X^D.*

PROOF. In order that $f \in X^D$ be the restriction of a regulated, right continuous function on $\mathbf{R}$, it is necessary and sufficient that it be the restriction of a regulated function and that in addition it be right continuous at each point of D; for then one can choose its extension by putting, for each $t \in \mathbf{R}$, $f(t) = \lim_{t' \to t+, t' \in D} f(t')$. It suffices to show that the set of right continuous functions on D is Borel. For $t \in D$, $\epsilon > 0$, $\eta > 0$, denote by $\mathscr{C}(D, t, \epsilon, \eta, X)$ the set of functions $D \to X$ which satisfy $d(f(t), f(t')) < \epsilon$ for $t < t' < t + \eta$. $\mathscr{C}(D, t, \epsilon, \eta, X)$ is the intersection of closed sets and so is closed. The set of right continuous functions on D is $\bigcap_{t \in D} \bigcap_{m \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \mathscr{C}\left(D, t, \frac{1}{m}, \frac{1}{n}, X\right)$, and so is Borel.

THEOREM 19. *Let X be a polish space, D a countable, dense subset of $\mathbf{R}$. The space $\mathscr{R}(\mathbf{R}, X; D)$ of regulated, right continuous mappings of $\mathbf{R}$ into X, endowed with the topology of simple convergence on D, is Lusin. For all choices of D, the Borel subsets of $\mathscr{R}(\mathbf{R}, X; D)$ are the same, as also the Suslin subsets, and the bounded Radon measures.*

PROOF. The mapping $f \to f_D$ which takes each function f to its restriction to D, is an injection of $\mathscr{R}(\mathbf{R}, X)$ in X^D, since D is dense. The topology $\mathscr{R}(\mathbf{R}, X; D)$ of simple convergence on D is, by identification with the image in X^D, the topology induced by X^D. But (as we see by choosing a metric on X which makes it complete) X^D is polish, and the image of $\mathscr{R}(\mathbf{R}, X; D)$ is Borel in X^D, hence this space is Lusin. To compare two dense denumerable sets, one can compare each with their union; now if $D \subset D'$, then $\mathscr{R}(\mathbf{R}, X; D')$ is finer than $\mathscr{R}(\mathbf{R}, X; D)$, since they are both Lusin, they have the same Borel subsets, the same Suslin subsets and the same bounded Radon measures.

From now on, for X polish, the space $\mathscr{R}(\mathbf{R}, X)$ of regulated, right-continuous mappings of $\mathbf{R}$ in X, will be assumed to be endowed with one of the preceding topologies $\mathscr{R}(\mathbf{R}, X; D)$.

It is pointless to endow $\mathscr{R}(\mathbf{R}, X)$ with a topology; we only consider its Borel σ-algebra, since the topologies considered are all Lusin, and consequently, the Borel subsets, the Suslin subsets and the bounded Radon measures are the same for all the topologies, and depend only on the Borel σ-algebra. This Borel σ-algebra is simply that generated by the set of functions $f \longmapsto f(t)$, $t \in \mathbf{R}$, or by any subset of these functions with t ranging over a dense subset of $\mathbf{R}$ as is seen by taking the topology $\mathscr{R}(\mathbf{R}, X; D)$. It follows that a mapping of a topological space in $\mathscr{R}(\mathbf{R}, X)$ is Borel if and only if its composition with each map $f \longmapsto f(t)$ is Borel. On the other hand we also know that every countable family of Borel functions on $\mathscr{R}(\mathbf{R}, X)$ which separates points, generates the Borel σ-algebra.

THEOREM 20. *The mapping $\delta\colon (f, t) \longmapsto f(t)$ of $\mathscr{R}(\mathbf{R}, X)$ in X is Borel.*

PROOF. First note, that for fixed t, $f \longmapsto f(t)$ is a Borel map of $\mathscr{R}(\mathbf{R}, X)$ in X; in fact, it suffices to choose D containing t, it is then continuous.

Let D_n be the set of numbers $k/2^n$, $n \in \mathbf{Z}$. For each $t \in \mathbf{R}$, let t_n be the smallest element of D_n majorising t; the sequence t_n is decreasing and tends to t. Now the map $(f, t) \to f(t_n)$ of $\mathscr{R}(\mathbf{R}, X) \times \mathbf{R}$ in X is Borel. To see this, note that D_n determines a partition of $\mathbf{R}$ into a countable union of Borel subsets, namely the intervals $D_{k,n} = (k/2^n, (k+1)/2^n]$; it suffices therefore to show that the mapping we are considering, restricted to each $\mathscr{R}(\mathbf{R}; X) \times D_{k,n}$ is Borel. Now this mapping is the composition of the mapping $(f, t) \longmapsto f$, which is continuous, and the mapping $f \longmapsto f((k+1)/2^n)$ which, we have seen, is Borel. But our original mapping $(f, t) \longmapsto f(t)$ is the simple limit as $n \to \infty$ of the mappings $(f, t) \longmapsto f(t_n)$, hence it is Borel, since X is metrizable.

THEOREM 21. *Let A be a Suslin subset of X. The set $\mathscr{A}$ of mappings f in $\mathscr{R}(\mathbf{R}; X)$ whose trajectories meet A, i.e. those for which $f(\mathbf{R}) \cap A \neq \emptyset$, is a Suslin subset of $\mathscr{R}(\mathbf{R}, X)$; and if τ denotes the 'entry time' in A, i.e. τ is the function on $\mathscr{R}(\mathbf{R}, X)$ with values in $\overline{\mathbf{R}}$ defined by*

$$\tau(f) = \inf\{t : f(t) \in A\}, \text{ if } f(\mathbf{R}) \cap A \neq \emptyset$$
$$= +\infty \qquad \text{if } f(\mathbf{R}) \cap A = \emptyset,$$

then τ is a universally measurable function.

PROOF. The set $B = \{(f, t) : f(t) \in A\}$ is the inverse image of A under the mapping δ of the previous theorem; hence it is a Suslin subset of $\mathscr{R}(\mathbf{R}, X) \times \mathbf{R}$. The set $\mathscr{A}$ of f whose trajectories meet A is the projection of B on $\mathscr{R}(\mathbf{R}, X)$, so it is also Suslin.

For each $t \in \overline{\mathbf{R}}$, the set $\{f : \tau(f) < t\}$ is also the set of f whose trajectories meet A in the 'time' interval $(-\infty, t)$, thus it is the projection on $\mathscr{R}(\mathbf{R}, X)$ of $B \cap (\mathscr{R}(\mathbf{R}, X) \times (-\infty, t))$ and so by what we have seen above, is Suslin and therefore universally measurable; hence the function τ is universally measurable.

THEOREM 22. *If X is a bounded subset in a Banach space $\vec{F}$ which is polish in the induced topology (in this case for any one of the topologies defined by dense subsets D of $\mathscr{R}$, $\mathscr{R}(\mathbf{R}, \vec{F})$ is a topological vector space), and if μ is a finite, positive Radon measure on $\mathbf{R}$, the functions*

$$\vec{f} \longmapsto \int_{\mathbf{R}} \vec{f}(t)\, d\mu(t) \text{ and } f \longmapsto |||\vec{f}|||_{L^p(\mathbf{R}, \mu; \vec{F})} \ (0 < p \leqslant +\infty)$$

are Borel on $\mathscr{R}(\mathbf{R}, X)$. If the support of μ is the whole line, the topology induced on $\mathscr{R}(\mathbf{R}, X)$ by $L^p(X, \mu; \vec{F})$, $0 < p < \infty$, is (Borel) equivalent to its Lusin topologies.

PROOF. Let J be a compact interval in $\mathbf{R}$. Then $\int_J \vec{f}(t)\, d\mu(t)$ is the limit of Riemann sums since $\tilde{f}$ is regulated; a Riemann sum is of the form

$$\sum_i \mu((t_i, t_{i+1}))\, \vec{f}(\theta_i) + \mu(\{t_i\})\, \vec{f}(t_i), \ \theta_i \in (t_i, t_{i+1}),$$

one can take a sequence of Riemann sums, where the t_i run over the set of all point masses of μ and where the largest of the $t_{i+1} - t_i$ tends to zero. Each Riemann sum is a Borel function of $\vec{f}$, by Theorem 20 above; hence $\vec{f} \longmapsto \int_J \vec{f}\, d\mu$ is Borel, and passing to the limit $\vec{f} \to \int_{\mathbf{R}} \vec{f}\, d\mu$ is Borel.

If $f \in \mathscr{R}(\mathbf{R}; X)$, the function $t \longmapsto \|\vec{f}(t)\|_{\vec{F}}$, which we write $\|\vec{f}\|$ is in $\mathscr{R}(\mathbf{R}, \mathbf{R})$ and the mapping $\vec{f} \longmapsto \|\vec{f}\|$ of $\mathscr{R}(\mathbf{R}, X)$ in $\mathscr{R}(\mathbf{R}, \mathbf{R})$ is Borel from $\mathscr{R}(\mathbf{R}, X)$ to $\mathbf{R}$ for each $t \in \mathbf{R}$ since $f \longmapsto \|f(t)\|$ is Borel. Next, what we have called $|||\vec{f}|||_{L^p(\mathbf{R}, \mu; \vec{F})}$ is nothing but $\left(\int_{\mathbf{R}} \|\vec{f}(t)\|_{\vec{F}}^p\, d\mu(t)\right)^{1/p}$ (this is a norm if $1 \leqslant p < \infty$, but not for $0 < p < 1$), and it is therefore Borel from the previous result on the integral. If now $p = +\infty$, we remark that for a regulated, right-continuous function $\vec{f}$,

$$|||\vec{f}|||_{L^\infty(\mathbf{R},\mu;\vec{F})} = \sup_{t\in D} \|\vec{F}(t)\|_{\vec{F}},$$

where D is a countable, dense subset of $\mathbf{R}$; the function $\vec{f} \longmapsto |||\vec{f}|||_{L^\infty}$ is therefore the upper envelope of a countable family of Borel functions and is therefore Borel.

Suppose now that the support of μ is $\mathbf{R}$. Then the canonical mapping of $\mathscr{R}(\mathbf{R}, X)$ in $L^p(\mathbf{R}, \mu; \vec{F})$ is injective; if we knew that this injection is a Borel map in a Lusin space, it would follow that the Borel σ-algebra of $\mathscr{R}(\mathbf{R}, X)$ is that of the topology induced by $L^p(\mathbf{R}, \mu; \vec{F})$. One can suppose that $\vec{F}$ is separable, because X is separable and so contained in a separable, Banach subspace of $\vec{F}$. Thus $L^p(\mathbf{R}, \mu; \vec{F})$ is itself also separable if $p \neq \infty$, and so Lusin (in fact even polish). Hence its subspace $\mathscr{R}(\mathbf{R}, X)$ is separable, metrizable for the induced topology τ. In other words there exists a sequence of functions $f_n \in \mathscr{R}(\mathbf{R}, X)$ dense in $\mathscr{R}(\mathbf{R}, X)$ for the topology τ induced by $L^p(\mathbf{R}, \mu; \vec{F})$. Therefore this induced topology τ is also that induced by $\mathbf{R}^{\mathbf{N}}$, when one embeds $\mathscr{R}(\mathbf{R}, X)$ in $\mathbf{R}^{\mathbf{N}}$ by the map

$$\vec{f} \longmapsto (|||\vec{f} - \vec{f}_n|||_{L^p(\mathbf{R},\mu;\vec{F})})_{n\in\mathbf{N}};$$

further, a map into $\mathbf{R}^{\mathbf{N}}$ is Borel if and only if each of its components is Borel; but we have seen that $\vec{f} \longmapsto |||\vec{f} - \vec{f}_n|||_{L^p(\mathbf{R},\mu;F)}$ is Borel on $\mathscr{R}(\mathbf{R}; X)$. We have thus shown that the injection $\mathscr{R}(\mathbf{R}; X) \to L^p(\mathbf{R}, \mu; \vec{F})$ is a Borel map in a Lusin space, from which follows the equivalence of the topology τ with the earlier Lusin topologies of $\mathscr{R}(\mathbf{R}; X)$. This is evidently false for L^∞, since $L^\infty(\mathbf{R}, \mu; \vec{F})$ is not separable, and the topology induced by L^∞ is not effectively separable on $\mathscr{R}(\mathbf{R}; X)$.

THEOREM 23. *Let μ be a positive measure on $\mathbf{R}$. Let us consider the space* Mes $(\mathbf{R}, \mu; X)$ *of equivalence classes of μ-measurable functions on $\mathbf{R}$ with values in X; for each distance d on X, one can endow*

this space with the topology of convergence in measure. Then the canonical mapping $\mathscr{R}(\mathbf{R}; X) \to \mathrm{Mes}\,(\mathbf{R}, \mu; X)$ *is Borel; and if* μ *has support* $\mathbf{R}$, *the topology* τ *induced on* $\mathscr{R}(\mathbf{R}; X)$ *by* $\mathrm{Mes}\,(\mathbf{R}, \mu; X)$ *is (Borel) equivalent to the earlier Lusin topologies.*

PROOF. We can restrict ourselves to the case when μ is finite. We recall that the topology of $\mathrm{Mes}\,(\mathbf{R}, \mu; X)$ depends on the uniform structure of X, hence on the distance chosen. One can reason as in the proof of the previous theorem: $\mathrm{Mes}(\mathbf{R}, \mu; X)$ is separable, metrizable, and one can take, for example, for the distance

$$\tilde{d}(f, g) = \sup_{k \in \mathbf{N}} \frac{1}{k}\, \mu \left\{ t \in \mathbf{R} : d(f(t), g(t)) \geqslant \frac{1}{k} \right\}.$$

One can, as before, choose a sequence f_n dense in $\mathscr{R}(\mathbf{R}; X)$ for the topology τ induced by $\mathrm{Mes}\,(\mathbf{R}, \mu; X)$; one must then show that each map $f \longmapsto \tilde{d}\,(f, f_n)$ is Borel from $\mathscr{R}(\mathbf{R}, X)$ to $\mathbf{R}$. Since the function $f \longmapsto d(f,g)$ of $\mathscr{R}(\mathbf{R}, X)$ to $\mathscr{R}(\mathbf{R}, \mathbf{R})$ is Borel, one has only to show that the function $f \longmapsto \mu\{t \in \mathbf{R} : f(t) > \alpha\}$ is Borel from $\mathscr{R}(\mathbf{R}, \mathbf{R})$ to $\mathbf{R}$. Now $f \longmapsto \inf\,(f,\alpha)/\alpha$ is Borel from $\mathscr{R}(\mathbf{R}, \mathbf{R})$ to $\mathscr{R}(\mathbf{R}, [0,1])$, and one has only to show that $f \longmapsto \mu\{t \in \mathbf{R} : f(t) = 1\}$ is Borel from $\mathscr{R}(\mathbf{R}; [0,1])$ to $\mathbf{R}$. But by the monotone convergence theorem of Lebesgue, if f has values in $[0,1]$, $\mu\{t \in \mathbf{R} : f(t) = 1\}$ is the limit of $\int_{\mathbf{R}} (f(t))^q\, d\mu(t)$ as q tends to infinity and so it is Borel by the previous theorem. If μ has support $\mathbf{R}$, $\mathscr{R}(\mathbf{R}; X) \to \mathrm{Mes}(\mathbf{R}, \mu; X)$ is injective, and the same reasoning as in the previous theorem shows that the topology τ is equivalent to the earlier Lusin topologies.

REMARK. Let us now denote the space of regulated right-continuous functions by $\mathscr{R}^+(\mathbf{R}; X)$ instead of just $\mathscr{R}(\mathbf{R}; X)$. We now denote by $\overline{\mathscr{R}}(\mathbf{R}; X)$ the equivalence classes of regulated functions regarding as equivalent, 2 regulated functions which have the same left-hand limit and the same right-hand limit at every point; for regulated functions this coincides with Lebesgue equivalence (equality almost everywhere) for any non-atomic measure whose support is $\mathbf{R}$. Now one can define an obvious bijection from $\overline{\mathscr{R}}(\mathbf{R}; X)$

to $\mathscr{R}^+(\mathbf{R};X)$, from which one has a family of Lusin topologies for $\overline{\mathscr{R}}(\mathbf{R};X)$; but one can define also a bijection of $\overline{\mathscr{R}}(\mathbf{R};X)$ to the space $\mathscr{R}^-(\mathbf{R};X)$ of regulated, left-continuous functions, from which one has another family of Lusin topologies for $\overline{\mathscr{R}}(\mathbf{R};X)$. These topologies are all Borel equivalent, as is also the coarsest topology finer than any two of them; they define the same Borel σ-algebra on $\overline{\mathscr{R}}(\mathbf{R};X)$ generated by the maps $f \longmapsto f(t+0)$ or by the maps $f \longmapsto f(t-0)$, of $\mathscr{R}^-(\mathbf{R};X)$ in X. To see this, it suffices to remark that on $\mathscr{R}^+(\mathbf{R};X)$, the function $f \longmapsto f(t-0)$ is Borel, since it is the limit of the sequence of Borel functions $f \longmapsto f(t-1/n)$.

TERMINOLOGICAL INDEX

PART I

1. Compactology, p. 61
2. Dispersed Hausdorff space, p. 96
   Dispersible Hausdorff space, p. 96
3. Extension lemma, p. 48
4. Glueing of pieces, p. 44
5. Image of a measure, p. 29
6. Image measure catastrophe, p. 30
7. Induced measure, p. 20
8. Lifting of L^∞ to $\mathscr{L}^\infty$, p. 130
9. Lusin measurable maps, p. 25
10. Lusin space, p. 94
11. Measure, p. 7
12. m-almost everywhere, p. 8
    m-integrable function, p. 8
    m-integrable set, p. 9
    m-integral, p. 8
    $(m, \mathscr{S})$-measurable map, p. 10
    m-measurable set, p. 10
13. m-essential outer measure, p. 8
    m-essentially integrable function, p. 9
    m-essentially-integrable set, p. 9
    m-essential integral, p. 9
    m-negligible function, p. 8
    m-negligible set, p. 8
    m-upper integral, p. 8
14. Moderated function, p. 43
    Moderated set, p. 43

15. μ-Borel measurable, p. 19
μ-concassage, p. 46
μ-dense family, p. 19
μ-integrable function, p. 16
μ-strictly integrable function, p. 16
μ-measurable set, p. 17
μ^*, $\mu^\bullet$, p. 16
μ-negligible, p. 18
μ-proper, p. 31

16. Polish space, p. 92

17. Projective (decreasing) system of measures, p. 74
Projective (exact) system of measures, p. 74
Projective limit of measures, p. 75
Prokhorov's theorem on projective limits, p. 74

18. Radon measure, p. 12
Diffuse Radon measure, p. 43
Moderated Radon measure, p. 16
Real and complex Radon measures, p. 53
Summable family of Radon measures, p. 56
Tensor product of Radon measure, p. 63

19. Radon equivalent topologies, p. 38

20. Randon space, p. 117
strongly Radon space, p. 124

21. Subdivision, p. 97
Strict sub-division, p. 97

22. Strongly Lindelöf space, p. 103

23. Suslin space, p. 96

24. Universally measurable maps, p. 38

REFERENCES

A. D. ALEXANDROV: [1] Additive set-functions in abstract spaces, *Mat. Sb.* (N.S.) 8(50), 307-348; 9(51), 563-628; 13(55), 169-238;

S. BANACH: [1] Über additive Massfunktionen in abstrakten Mengen, *Fund. Math.* 15 (1930), 97-101.

S. BANACH and K. KURATOWSKI: [1] Sur une généralisation du problème de la mesure, *Fund. Math.* 14 (1929), 127-131.

D. BLACKWELL: [1] On a class of probability spaces, *Proc. Third Berkeley Symposium*, Vol. II (1955), 1-6.

S. BOCHNER: [1] *Harmonic analysis and the theory of probability*, Berkeley (1955).

N. BOURBAKI: [1] *Intégration*, Chapitres I-IV, 2ième ed. Hermann, Paris (1965).

[2] *Intégration*, Chapitre V, Hermann, Paris (1956).

[3] *Intégration*, Chapitre VI, Hermann, Paris (1959).

[4] *Topologie générale*: Chapitre IX, 2ième ed., Hermann, Paris (1959).

[5] *Espaces vectoriels topologiques*, Chapitres I & II, Hermann, Paris (1952).

[6] *Intégration*, Chapitre IX, Hermann, Paris (1969).

J. R. CHOKSI: [1] Inverse limits of measure spaces, *Proc. London Math. Soc.* (3) 8 (1958), 321-342.

[2] On compact contents, *J. London Math. Soc.*, 33 (1958), 387-398.

G. CHOQUET: [1] Theory of capacities, *Ann. Inst. Fourier*, V (1955), 131-295.

J. DIEUDONNÉ: [1] Un exemple d'espace normal non susceptible d'une structure uniforme d'espace complet, *C.R. Acad. Sci. Paris*, 209 (1939), 145-147.

X. FERNIQUE: [1] Processus linéaires, processus généralisés, *Ann. Inst. Fourier* XVII (1967), 1-92.

B. V. GNEDENKO and A. N. KOLMOGOROV: [1] *Limit distributions for sums of independent random variables*, English translation by K. L. Chung, Addison-Wesley, Cambridge, Mass. (1954).

K. N. GOWRISANKARAN : [1] Measurability of functions in product spaces, *Proc. American Math. Soc.*, 31(1972), 485-488.

P. R. HALMOS : [1] *Measure theory*, Van Nostrand, New York, (1950).

P. R. HALMOS and J. VON NEUMANN : [1] Operator methods in classical mechanics II, *Ann. of Math.* (2) 43 (1942), 332-350.

O. HAUPT and C. Y. PAUC : [1] Bemerkungen über Inhalte und Masse in lokal bikompakten Raumen, *Akad. Wiss. Lit. Mainz Abh. Mat.-Nat. Kl.* (1955) No. 7, 187-218, and later papers in the same journal.

J. P. HENRY : [1] Prolongement des mesures de Radon, *Ann. Inst. Fourier*, XIX (1969), 237-238.

A. and C. IONESCU TULCEA : [1] On the lifting property, *J. Math. Anal. and Appl.* 3 (1961), 534-546.

[2] *Topics in the theory of lifting*, Springer, Berlin (1969).

[3] On the existence of a lifting commuting with the left translations of an arbitrary locally compact group, *Proc. Fifth Berkeley Symposium*, Vol. II, Part I (1965), 63-97.

J. L. KELLEY : [1] *General Topology*, Van Nostrand, Princeton (1965).

D. MAHARAM : [1] On a theorem of von Neumann, *Proc. American Math. Soc.* 9 (1958), 987-994.

E. MARCZEWSKI : [1] On compact measures, *Fund. Math.* 40 (1953), 113-124.

A. MARTINEAU : [1] Fonctionelles analytiques et transformations de Fourier-Borel, *J. d'Analyse*, 11 (1963), 1-164.

M. MÉTIVIER : [1] Limites projectives de mesures, Martingales, Applications, *Annali di Mat.* (4) 63 (1963), 225-352.

P. A. MEYER : [1] *Probability and Potentials*. Blaisdell, Waltham, Mass., 1966.

J. VON NEUMANN : [1] Algebraische Repräsentanten der Funktionen bis auf eine Menge vom Masse Null, *J. Reine Angew-Math.* 165 (1931), 109-115.

J. C. OXTOBY and S. M. ULAM : [1] On the existence of a measure invariant under a transformation, *Ann. of Math.* 40 (1939), 560-566.

U. V. PROKHOROV : [1] Convergence of random processes and limit theorems in probability theory, *Theory of probability and its applications* I (1956), 157-214.

U. V. PROKHOROV : [2] The method of characteristic functionals, *Fourth Berkeley Symposium*, Vol. II (1961), 403-419.

C. RYLL-NARDZEWSKI : [1] On quasi-compact measures, *Fund. Math.* 40 (1953), 125-130.

S. SAKS : [1] *Theory of the integral*, Warsaw (1937).

SUNYACH : [1] Une caractèrisation des espaces universellement Radon-mesurables, *Note C. R. Ac. Sc., Paris*, 268 (1969), 864-866.

S. M. ULAM : [1] Zur Masstheorie in der allgemeinen Mengenlehre, *Fund. Math.* 16 (1931), 140-150.

V. S. VARADARAJAN : [1] Measures on topological spaces, *Mat. Sb.* (*N.S.*) 55 (97) (1961), 35-100. English translation in A.M.S. Translation Series 2, 48, 161-228.

KHÉLIFA ZIZI : [1] *Measures de Radon bornées sur les espaces complètement réguliers*, Inst. Henri Poincaré.

Part II

CYLINDRICAL MEASURES

In this Part all topological spaces will be Hausdorff and all Radon measures will be finite and positive unless the contrary is explicitly stated. In particular, a positive measure with total mass 1 *will be called a probability.*

INTRODUCTION TO PART II

Probabilities on Banach spaces or topological vector spaces have been studied extensively; they form an important part of probability theory. We shall here study especially the cylindrical measures on topological vector spaces, and their relations with random functions; Prokhorov's theorem is the key for relations between cylindrical measures and Radon measures, see Part I, Chapter 1, Th. 20. The first important application has been given by Sazonov [19] and Minlos [20]; the theorem of Sazonov characterizes the 0-radonifying maps in Hilbert spaces as the Hilbert-Schmidt maps, while the theorem of Minlos says that a cylindrical measure on a conuclear space, scalarly concentrated on the family of convex balanced compact subsets, is a Radon measure. Developments on the same topics will be found in Gelfand-Vilenkin [6]. All the papers of Gross and Segal (see for instance [21] with a long bibliography) give many improvements to the theory of Gauss measures on Banach spaces.

All along the present book, and in most of the works cited in the bibliographical index, Hilbert-space techniques are constantly used and also the Fourier transform. However, these methods are no longer available for the study of p-radonifying maps, which are based on completely different techniques (see Schwartz [17], which gives the largest part of the results known in 1970 on the subject).

SUMMARY OF CHAPTER I

This chapter defines equivalent topologies (with respect to Radon measures): Definition 2, page 155. The main theorem about equivalent topologies in this chapter is: in a locally convex metrizable vector space, the strong and the weak topology are equivalent, Theorem 3, page 162; after the proof, this result is compared with another one of Part I, the equivalence of a polish topology (for instance that of a complete metrizable separable vector space) with every coarser topology, linear or not. In another direction, Theorem 1, page 157, has important applications; in particular, it proves the Borel graph theorem, Theorem 2, page 160.

I

EQUIVALENT TOPOLOGIES

1. Equivalent measures. Let X be a set and let τ_1 and τ_2 be two topologies on X. We assume that τ_1 is finer than τ_2. If we denote by X_k $(k = 1, 2)$ the set X equipped with the topology τ_k, then the identity map $i: X_1 \to X_2$ is continuous. Let μ_1 be a Radon measure on X_1 and $\mu_2 = i(\mu_1)$ its image on X_2. We know that the measurable sets are the same and have the same measure for μ_1 and μ_2 (Part I, Th. 8 and 9). It follows that for any subset of X the outer measures μ_1^* and μ_2^* are the same and that for any Banach space F the integrable maps $X \to F$ are the same for μ_1 and μ_2 and have the same integrals. Furthermore for any topological space Y the Lusin-measurable maps $X \to Y$ are the same for both μ_1 and μ_2. This leads us to the following definition.

DEFINITION 1. *Let X be a set and let τ_1 and τ_2 be two topologies on X. Denote by X_k the set X equipped with the topology τ_k $(k = 1, 2)$. A Radon measure μ_1 on X_1 and a Radon measure μ_2 on X_2 will be said to be equivalent if the following two conditions are satisfied.*

(1) *The measurable sets are the same for both μ_1 and μ_2, and any measurable set has the same measure for these two measures.*

(2) *For any topological space Y the Lusin-measurable maps $X \to Y$ are the same with respect to μ_1 and μ_2.*

If two Radon measures μ_1 and μ_2 on a set X are equivalent, then it follows from condition (1) that the outer measure of any subset of X is the same for μ_1^* and for μ_2^*. Also if F is a Banach space, the integrable maps $X \to F$ are the same and have the same integrals with respect to the measures μ_1 and μ_2 (Part I, Ch. I, Th. 7). Finally, condition (1) of Def. 1 implies condition (2) for any Suslin space Y, since for such a space a map $X \to Y$ is Lusin-measurable if and only if it is Borel-measurable, i.e. the inverse image of any Borel set is measurable (Part I, Ch. II, Th 14). For a general topological space, however, Douady has given an example of

two measures which satisfy condition (1) but do not satisfy condition (2).

It is clear that the equivalence of two Radon measures as defined in Def. 1 is indeed an equivalence relation. An equivalence class of Radon measures on the set X will be called an *abstract Radon measure* on X.

Let X be a set, τ_1 and τ_2 two topologies on X and X_k the set X equipped with the topology τ_k $(k = 1, 2)$. Let μ_1 be a Radon measure on X_1 and μ_2 a Radon measure on X_2. Assume that there exists a Hausdorff topology τ_3 on X which is coarser than both τ_1 and τ_2, and let X_3 be X equipped with τ_3. If the image of μ_1 under the identical bijection $X_1 \to X_3$ is the same as that of μ_2 under the identical bijection $X_2 \to X_3$, then it follows from the considerations preceding Def. 1 that μ_1 and μ_2 are equivalent.

PROPOSITION 1. *Let τ_1 and τ_2 be two topologies on the set X. Denote by X_k the set X equipped with the topology $\tau_k (k = 1, 2)$. Let μ_1 and μ_2 be two Radon measures on the spaces X_1 and X_2, respectively. The measures μ_1 and μ_2 are equivalent if and only if the identity map $i: X_1 \to X_2$ is μ_1-Lusin-measurable and $\mu_2 = i(\mu_1)$.*

PROOF. (a) Suppose that i is μ_1-Lusin-measurable and that $\mu_2 = i(\mu_1)$. Then a set $A \subset X$ is μ_2-measurable if and only if $i^{-1}(A) = A$ is μ_1-measurable. Furthermore, $\mu_2(A) = \mu_1(i^{-1}(A)) = \mu_1(A)$. Also if Y is any topological space, and $f: X_2 \to Y$ a map, then f is μ_2-Lusin-measurable if and only if $f \circ i = f: X_1 \to Y$ is μ_1-Lusin-measurable. Thus μ_1 and μ_2 are equivalent.

(b) Suppose that μ_1 and μ_2 are equivalent. Since the identity map $i: X_2 \to X_2$ is certainly μ_2-Lusin-measurable, applying condition (2) of Def. 1 to $Y = X_2$ we obtain that $i: X_1 \to X_2$ is μ_1-Lusin-measurable. Let μ_2' be the image of the measure μ_1 with respect to the map $i: X_1 \to X_2$. It follows from the definition of an image measure and from condition (1) of Def. 1 that a set $A \subset X$ is measurable for either μ_2 or μ_2' if and only if it is μ_1-measurable, and $\mu_2(A) = \mu_2'(A) = \mu_1(A)$. Thus $\mu_2 = \mu_2'$.

Let us now list some operations that can be performed on abstract Radon measures.

1. Let X be a set, μ an abstract Radon measure on X; let Y be a topological space and $f: X \to Y$ a map. By condition (2) of Def. 1 it is meaningful to say that f is μ-Lusin-measurable if it is Lusin-measurable for any of the Radon measures which define μ. If f is μ-Lusin-measurable, then $f(\mu)$ is a well-defined Radon measure on Y.

2. If μ is an abstract Radon measure on a set X and p a positive μ-integrable numerical function, then the product $p\mu$ is an abstract Radon measure on X.

3. If μ is an abstract Radon measure on a set X and A is a μ-measurable subset of X, then μ induces an abstract Radon measure on A.

4. If X and Y are two sets, and μ and ν are abstract Radon measures on them, then the abstract Radon measure $\mu \otimes \nu$ on $X \times Y$ is well-defined. In fact this follows from Prop. 1 and from the remark that if μ_1 and ν_1 are Radon measures on the topological spaces X_1 and Y_1 respectively, $f: X_1 \to X_2$ is μ_1-Lusin-measurable, and $g: Y_1 \to Y_2$ is ν_1-Lusin-measurable, then

$$f \times g: X_1 \times Y_1 \to X_2 \times Y_2$$

is $\mu_1 \otimes \nu_1$-Lusin-measurable and its image is $f(\mu_1) \otimes g(\nu_1)$.

On the other hand, there is no possibility to define the sum of two abstract Radon measures.

2. Equivalent topologies.

DEFINITION 2. *Let X be a set, τ_1 and τ_2 two topologies on X. Denote by X_k the set X equipped with the topology τ_k $(k = 1, 2)$. The topologies τ_1 and τ_2 are said to be Radon-equivalent if every Radon measure for one of these topologies is equivalent* (Def. 1) *to a Radon measure for the other.*

Observe that if μ_1 is a Radon measure on X_1, then μ_1 can be equivalent to at most one Radon measure on X_2. Indeed, if μ_1 is equivalent to both μ_2 and μ_2', then μ_2 and μ_2' are equivalent Radon measures for the same topology τ_2 and thus $\mu_2 = \mu_2'$.

By Prop. 1 the topologies τ_1 and τ_2 are Radon-equivalent if and only if for any Radon measure μ_1 on X_1 the identity map $X_1 \to X_2$ is μ_1-Lusin-measurable, and for every Radon measure μ_2 on X_2 the identity map $X_2 \to X_1$ is μ_2-Lusin-measurable. This leads us to the more general

DEFINITION 3. *The topology τ_1 on X is said to be Radon-finer than τ_2 if for every Radon measure μ_1 on X_1 the identity map $X_1 \to X_2$ is μ_1-Lusin-measurable.*

Thus τ_1 and τ_2 are Radon-equivalent if each is Radon-finer than the other. Clearly, if τ_1 is finer than τ_2, then τ_1 is also Radon-finer than τ_2, but the converse is false. In particular, we shall see in a moment that there exist Radon-equivalent topologies τ_1 and τ_2 such that τ_1 is strictly finer than τ_2.

Let $f\colon X \to Y$ be a map from the topological space X into the topological space Y. We shall say that f is *universally Lusin-measurable* if f is μ-Lusin-measurable for every Radon measure μ on X. A continuous map is clearly universally Lusin-measurable, but the converse is again false in general.

With this terminology we can say that τ_1 is Radon-finer than τ_2 if the identity map $i\colon X_1 \to X_2$ is universally Lusin-measurable.

Examples of two topologies τ_1 and τ_2 which are Radon equivalent but τ_1 is strictly finer than τ_2 can be obtained from

PROPOSITION 2. *Let X be a set equipped with two topologies τ_1 and τ_2. Assume that X is the union of a sequence (A_n) of pairwise disjoint sets which are Borel sets for both topologies, and that on each A_n both τ_1 and τ_2 induce the same topology. Then τ_1 and τ_2 are Radon-equivalent.*

PROOF. Let X_k be the set X equipped with the topology $\tau_k (k = 1, 2)$. By symmetry it is sufficient to prove that for any Radon measure μ_1 on X_1 the identity map $i\colon X_1 \to X_2$ is μ_1-Lusin measurable. Since $\sum_{n=1}^{\infty} \mu_1(A_n) = \mu_1\left(\bigcup_{n=1}^{\infty} A_n\right) < \infty$, given $\delta > 0$, there exists an integer $m \geqslant 1$ such that $\sum_{n=m+1}^{\infty} \mu_1(A_n) < \frac{1}{2}\delta$.

On the other hand, for each index n such that $1 \leqslant n \leqslant m$ there exists a compact subset K_n of A_n such that $\mu_1(A_n - K_n) \leqslant \frac{\delta}{2m}$. Since each set K_n is both closed and open in $K = \bigcup_{n=1}^{m} K_n$, the restriction of i to K is continuous by hypothesis. But $\mu_1(X_1 - K) \leqslant \delta$, hence i is indeed μ_1-Lusin-measurable.

Now consider, for instance, the real line $\mathbf{R}$ equipped with its usual topology, which shall be our τ_2. The disjoint union (topological sum) of the sets $\mathbf{Q}$ and $\mathscr{C}_{\mathbf{R}}\mathbf{Q}$, each equipped with the topology induced by $\mathbf{R}$, defines on $\mathbf{R}$ a topology τ_1 which is strictly finer than τ_2 but which, by Prop. 2, is Radon-equivalent to τ_2.

Cor. 2 of Th. 14 of Part I, Ch. II also yields many examples of Radon-equivalent topologies: if X equipped with the topology τ is a Suslin space, then any Hausdorff topology coarser than τ (for which X will of course still be a Suslin space) is Radon-equivalent to τ.

In view of these examples, the following result (proved by Douady as a modernization of classical results of Banach) is of particular interest.

THEOREM 1. *Let E be an ultrabornological topological vector space and F a locally convex Hausdorff topological vector space. If the linear map $u: E \to F$ is scalarly universally measurable, then it is continuous.*

Let us recall that a locally convex Hausdorff space E is *ultrabornological* if there exists a family (E_ι) of Banach spaces and a family of linear maps $f_\iota: E_\iota \to E$ such that the topology of E is the finest locally convex topology for which the f_ι are continuous. In particular any quasi-complete bornological Hausdorff space is ultrabornological.

Let us further recall that a map $f: X \to Y$ from a topological space X into a topological space Y is *universally Borel-measurable* if for every Radon measure μ on X and every Borel set $B \subset Y$ the inverse image $f^{-1}(B)$ is μ-measurable. (We know that a universally Lusin-measurable map is universally Borel-measurable but that the

converse is not necessarily true. If, however, Y is a Suslin space, then the two notions of universal measurability coincide). The scalar measurability means: for every $f \in F'$, $f \circ u$ is universally measurable from E into $\mathbf{R}$.

To prove Theorem 1 we shall need an auxiliary result. Let B be the vector space (without topology) of all bounded sequences of real numbers. Denote by K the product $I^{\mathbf{N}}$, where I is the interval $[0, 1]$ of $\mathbf{R}$. We consider K imbedded into B. Equip K with the product topology; it is then a compact space. Let μ be the Radon measure on K which is the product of the Lebesgue measures on the factor spaces. Let us denote by e_n the n-th unit vector in B, i.e. $e_n = (t_j)$, where $t_n = 1$ and $t_j = 0$ for $j \neq n$.

LEMMA 1. *If g is a linear form on B such that $|g(e_n)| \geq n$ for $n \in \mathbf{N}$, then the restriction of g to K is not μ-Borel-measurable.*

PROOF. For any $M > 0$ denote by A_M the subset of K where $|g| < M$. If g were μ-Borel-measurable, then each set A_M would be μ-measurable. Since K is the union of the sets A_M, we could choose M so that $\mu(A_M) > \frac{1}{2}$.

Let $M > 0$ be arbitrary and choose $n \in \mathbf{N}$ so that $n \geq 4M$. We can represent K as the product $X \times I$, where $X = I^{\mathbf{N}-\{n\}}$. For each $x \in X$ the diameter of the set $(\{x\} \times I) \cap A_M$ is less than $\frac{1}{2}$. Indeed, if u and v are two elements of I such that $(x, u) \in A_M$ and $(x, v) \in A_M$, then on the one hand

$$|g(x, u) - g(x, v)| < |g(x, u)| + |g(x, v)| < 2M,$$

and on the other

$$|g(x, u) - g(x, v)| = |g(0, u - v)| = |g((u - v)e_n)| \geq |u - v|\, n \geq |u - v|\, 4M.$$

Thus

$$|u - v| < \frac{2M}{4M} = \frac{1}{2}.$$

It follows by Fubini's theorem that $\mu(A_M) < \frac{1}{2}$ for any $M > 0$. But we have seen that if g were μ-Borel-measurable then we would

have $\mu(A_M) > \frac{1}{2}$ for sufficiently large M. This shows that g cannot be μ-Borel-measurable and concludes the proof.

PROOF OF THEOREM 1. (a) It is sufficient to prove the theorem in the case when E is a Banach space. Indeed, let (E_ι) be a family of Banach spaces and $f_\iota : E_\iota \to E$ a family of linear maps such that the topology of E is the finest locally convex topology for which the f_ι are continuous. If $u: E \to F$ is scalarly universally measurable, then every map $u \circ f_\iota$ is scalarly universally measurable. Hence, if we assume that the theorem is true for Banach spaces, each map $u \circ f_\iota$ is continuous and thus u is continuous [3, Chap. II, 2nd ed, § 4, n° 4, Prop. 5].

(b) It is sufficient to prove the theorem for linear forms, i.e. for the case when F is the field of scalars. Indeed, let $u: E \to F$ be a scalarly universally measurable linear map from the Banach space E into the locally convex Hausdorff space F, then for every continuous linear form $f \in F'$ the linear form $f \circ u = {}^t u(f)$ on E is universally measurable.

If we assume that the theorem holds for linear forms, then ${}^t u(f)$ is continuous, i.e. ${}^t u$ maps F' into E' and therefore u is continuous for the weakened topologies $\sigma(E, E')$ and $\sigma(F, F')$ [3, Chap. II, 2nd ed., §6, n° 4, Prop. 5]. Thus u maps weakly bounded subsets of E into weakly bounded subsets of F, and in particular the image in F of the unit ball of E is weakly bounded and therefore also bounded for the original topology of F [3, Chap. IV, § 2, n° 4, corol. of. Th. 1]. It follows that u is continuous [3, Chap. IV, §5, n° 3, Prop. 7].

(c) Let u be a linear form on the Banach space E and assume that u is not continuous. There exists a sequence (x_n) of elements of E such that $\sum_{n=0}^{\infty} \|x_n\| < \infty$ and $|u(x_n)| \geqslant n$ for all $n \in \mathbf{N}$. Since E is complete, setting $\xi: (t_n) \longmapsto \sum_{n=0}^{\infty} t_n x_n$ we obtain a linear map from the vector space B of Lemma 1 into E. The restriction of ξ to K is continuous. Indeed, given $\epsilon > 0$ there exists an integer $m > 0$ such that $\sum_{n=m}^{\infty} \|x_n\| < \frac{1}{2}\epsilon$. Setting $a = \sum_{n=0}^{m-1} \|x_n\|$ we see that

if $(s_n) \in K$ and $(t_n) \in K$ are such that $|s_n - t_n| < \epsilon/2a$ for $0 < n < m$, then

$$\|\xi((s_n)) - \xi((t_n))\| = \|\sum_{n=0}^{\infty} (s_n - t_n)\, x_n\|$$

$$< (\epsilon/2a) \sum_{n=0}^{m-1} \|x_n\| + 2 \sum_{n=m}^{\infty} \|x_n\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Now $\xi(e_n) = x_n$ and $|(u \circ \xi)(e_n)| = |u(x_n)| > n$ for every $n \in \mathbf{N}$. It follows from Lemma 1 that the restriction of $g = u \circ \xi$ to K is not μ-Borel-measurable, i.e. u is not Borel-measurable with respect to the Radon-measure $\xi(\mu)$ on E. Thus u is not universally Borel-measurable.

3. A Borel-graph theorem. Theorem 1, combined with a result proved in Part I, has the following consequence.

THEOREM 2 [13]. *Let E be an ultrabornological space and F a Suslin locally convex space. If $u: E \to F$ is a linear map whose graph is a Borel set in $E \times F$, then u is continuous.*

PROOF. (a) It is sufficient to prove the theorem in the case when E is a Banach space. Indeed, let (E_ι) be a family of Banach spaces and $f_\iota: E_\iota \to E$ a family of linear maps such that the topology of E is the finest locally convex topology for which the f_ι are continuous. Let G be the graph of u in $E \times F$ and for each ι let G_ι be the graph of $u \circ f_\iota$ in $E_\iota \times F$. If G is a Borel set, then G_ι is a Borel set since it is the inverse image of G with respect to the continuous map $f_\iota \times 1_F$ [2, Chap. IX, 2nd ed. §6, Prop. 10]. Hence, if the theorem is true for Banach spaces, each map $u \circ f_\iota$ is continuous, i.e. u is continuous [3, Chap. II, 2nd ed., § 4, n° 4, Prop. 5].

(b) We may also assume that the Banach space E is separable (i.e. contains as everywhere dense countable subset). Indeed, to show that $u: E \to F$ is continuous it is sufficient to show that it maps every sequence (x_n) converging to 0 into a bounded set [3, Chap. III, § 2, exerc. 11 b)]. Hence we can replace E by the closed subspace of E generated by the sequence (x_n).

(c) Let u be a linear map from the separable Banach space E into the Suslin locally convex space F whose graph is a Borel subset of $E \times F$. Since E is a Suslin space u is a Borel map, i.e. for every Borel set $B \subset F$ the inverse image $u^{-1}(B)$ is a Borel subset of E (Part I, Ch. II, Lemma 13). In particular u is universally measurable, hence it follows from Theorem 1 that u is continuous.

Theorem 2 applies to cases which are not covered by the classical forms of the closed-graph theorem. Thus for F we can take any of the spaces $\mathscr{D}, \mathscr{D}', \mathscr{E}, \mathscr{E}', \mathscr{S}, \mathscr{S}', \mathcal{O}_M, \mathcal{O}_M', \mathcal{O}_C, \mathcal{O}_C', \mathscr{L}(\mathscr{D}', \mathscr{D}'), \ldots$ which are all Suslin spaces. On the other hand all Suslin spaces are separable, therefore Theorem 2 does not contain even the classical theorem when E and F are Banach spaces. Let us observe that Raikov has proved a closed-graph theorem for linear maps $\mathscr{D}' \to \mathscr{D}'$. Other proofs of Theorem 2 were given by Martineau and by De Wilde whose results are probably the best ones for all the applications; one can also see a bibliography in De Wilde [5].

COROLLARY 1. *Let F be a Suslin locally convex space, E an ultrabornological space and $v: F \to E$ a continous surjective linear map. Then v is a strict morphism.*

PROOF. Let $N = v^{-1}(0)$ be the kernel of v. The injection $v: F/N \to E$ associated with v is continuous, hence its graph is closed in $(F/N) \times E$. It follows that the graph of $\bar{v}^{-1}$ is closed in $E \times (F/N)$. Now the quotient of a Suslin space is a Suslin space, hence by virtue of Theorem 2 the map $\bar{v}^{-1}$ is continous. This means that $\bar{v}$ is an isomorphism, i.e. that v is a strict morphism.

COROLLARY 2. *Let E be an ultrabornological Suslin locally convex space. If M and N are two closed subspaces of E which are algebraic supplements of each other, then they are also topological supplements.*

PROOF. The product space $M \times N$ is a Suslin space, and the map $(x, y) \longmapsto x + y$ is continuous, bijective and linear from $M \times N$ onto E. Hence by Corollary 1 the map is an isomorphism.

4. Equivalence of the strong and weak topologies. In this section we prove the following result which is a variant of theorems due

to Phillips, Dunford-Pettis and Grothendieck [8, Chap. I, § 4, $n°$ 1, Th. 4, p. 104].

THEOREM 3. *Let E be a locally convex metrizable topological vector space. The original ("strong") topology on E and the weak topology $\sigma(E, E')$ are Radon-equivalent.*

The proof proper of the theorem will follow from a number of auxiliary results among which Theorem 4 is of independent interest.

Let $f: X \to E$ be a map from the topological space X into the Hausdorff locally convex space E, and let μ be a Radon measure on X. We shall say that f is *scalarly μ-measurable* if for every $e' \in E'$ the numerical function $x \longmapsto \langle f(x), e' \rangle$ is μ-measurable.

LEMMA 2. *Let X be a topological space and μ a Radon measure on X. If the map $f: X \to \mathbf{C}^{\mathbf{N}}$, is scalarly μ-measurable, then it is μ-measurable.*

PROOF. Each projection $\pi_n: \mathbf{C}^{\mathbf{N}} \to \mathbf{C}$ $(n \in \mathbf{N})$ is a continuous linear form on the vector space $\mathbf{C}^{\mathbf{N}}$, hence by our hypothesis the maps $\pi_n \circ f$ are μ-measurable. But a map into a product of a countable family of topological spaces is Lusin-measurable if and only if the corresponding maps into the factor spaces are Lusin-measurable [4, Chap. IV, 2nd ed., § 5, $n°$ 3, Th. 1].

Using the fact that if f is a map from X into the subspace A of a topological space Y, and if f is Lusin-measurable as a map into Y, then it is also Lusin-measurable as a map into A, we obtain

LEMMA 3. *Let X be a topological space, μ a Radon measure on X and E a topological vector space which is a linear subspace of $\mathbf{C}^{\mathbf{N}}$ with the induced topology. If the map $f: X \to E$ is scalary μ-measurable, then it is μ-measurable.*

LEMMA 4. *Let E be a Suslin locally convex Hausdorff space and E' its dual. There exists a countable subset of E' which is dense for the *-weak topology $\sigma(E', E)$.*

PROOF. The elements of E', i.e. the continuous linear forms on E, separate the points of E [3, Chap. II, 2nd ed., §4, $n°$ 2, Cor. of Prop. 2], and thus by a property common to all Suslin spaces (Part I, Ch. II, Prop. 4) there exists a countable subfamily D' of E' whose elements still separate the points of E. It follows from the Hahn-Banach theorem that D' is total in E' for the weak topology $\sigma(E', E)$ [3, Chap. II, 2nd ed., §6, $n°$ 2, Cor. 1 of Prop. 3], and therefore the set of all finite linear combinations of elements of D' with rational (complex) coefficients is countable and dense in E' for the topology $\sigma(E', E)$.

THEOREM 4. *Let X be a topological space, μ a Radon measure on X and E a linear subspace of a Suslin locally convex Hausdorff space with the induced topology. If the map $f: X \to E$ is scalarly μ-measurable, then it is μ-Lusin-measurable.*

PROOF. The fact noted before Lemma 3 implies that we may assume that E itself is a Suslin locally convex Hausdorff space. By Lemma 4 there exists a countable subset D' of the dual E' of E which is dense for $\sigma(E', E)$. Denote by E_0 the vector space E equipped with the topology of simple convergence on D'. The map $e \longmapsto (\langle e, e' \rangle)_{e' \in D'}$ is an injection of E into $\mathbf{C}^{D'} \simeq \mathbf{C}^{\mathbf{N}}$ and the topology of E_0 is the inverse image of the topology of $\mathbf{C}^{\mathbf{N}}$ with respect to this injection. It follows from Lemma 3 that f is a μ-Lusin-measurable map from X into E_0. Now the topology of E_0 is coarser than that of E but still Hausdorff, hence E_0 and E are Radon-equivalent (Part I, Ch. II, Cor. 2 of Th. 10) and therefore f is also μ-Lusin-measurable as a map from X into E.

We are now able to prove Theorem 3 under the additional hypothesis that E is separable. The identity map $E_{\text{strong}} \to E_{\text{weak}}$ is continuous and therefore universally Lusin-measurable. Therefore we have to prove only that the identity map $E_{\text{weak}} \to E_{\text{strong}}$ is universally Lusin-measurable. Let us choose $X = E_{\text{weak}}$ and $E = E_{\text{strong}}$. For every $e' \in E'$ the function $e \longmapsto \langle e, e' \rangle$ on X is continuous by the definition of the weak topology. Thus in particular it is μ-measurable for every Radon measure μ on X. Since the completion of E

is a Suslin space, it follows therefore from Theorem 4 that the identity map $e \longmapsto e$ from X onto E is μ-Lusin-mesurable for every Radon measure μ on X, i.e. that the identity map $E_{\text{weak}} \to E_{\text{strong}}$ is universally Lusin-mesurable.

To get rid of the hypothesis that E is separable, we have to make use of some further results. In the first place, we quote without proof the following result of Grothendieck [8, Th. 1, p. 139].

LEMMA 5. *Let F be a locally convex Hausdorff space, and μ an arbitrary Radon measure on a locally compact space X. If a continuous linear map $L^1(X, \mu) = L^1(\mu) \to F$ maps the unit ball of $L^1(\mu)$ onto a weakly relatively compact subset of F, then it maps every weakly relatively compact subset of $L^1(\mu)$ onto a strongly relatively compact subset of F.*

If the Radon measure μ is finite, we have $L^\infty(\mu) \subset L^1(\mu)$. The unit ball B of $L^\infty(\mu)$ is weakly compact in $L^1(\mu)$. Indeed, since $L^\infty(\mu)$ is the dual of $L^1(\mu)$, the ball B is compact for $\sigma(L^\infty(\mu), L^1(\mu))$ [3, Chap. IV, §5, $n°$ 1, Prop. 1]. A fortiori B is compact for the coarser topology $\sigma(L^\infty(\mu), L^\infty(\mu))$ induced by $\sigma(L^1(\mu), L^\infty(\mu))$ on $L^\infty(\mu)$. Hence we obtain from Lemma 5 the following

LEMMA 6. *Let F be a locally convex Hausdorff space, and μ a Radon measure on a space X. If a linear map $L^1(\mu) \to F$ sends the unit ball of $L^1(X, \mu) = L^1(\mu)$ into a weakly relatively compact subset of F, then it sends the unit ball of $L^\infty(\mu)$ into a strongly relatively compact subset of F.*

Our last lemma is the Deus ex machina which allows to lift the hypothesis that E be separable.

LEMMA 7. *Let E be an arbitrary Fréchet space, K a weakly compact subset of E, and μ a Radon measure on K equipped with the weak topology. Then the support of μ is separable for the strong topology.*

PROOF. Every $f \in L^1(\mu)$ yields a measure $f d\mu$ supported by K. Let us associate with each $f \in L^1(\mu)$ the point $\int x f(x)\, d\mu(x)$ of E (which, upto the factor $(\int f d\mu)^{-1}$ is the barycenter of $\int d\mu$ [4,

Chap. IV, 2nd ed., §7, n° 1, Def. 1]). We shall show that the linear map $f \longmapsto \int x f(x) d\mu(x)$ sends the unit ball of $L^1(K, \mu)$ into a weakly relatively compact subset of E.

Indeed if $\|f\| \leqslant 1$ in $L^1(K, \mu)$, then its image is contained in $(\int |f| d\mu) \circ \hat{K} \subset \hat{K}$, where $\hat{K}$ denotes the balanced, closed, convex hull of K in the weak completion of E [4, Chap. III, 2nd ed. §3, n°2, Prop. 5]. We now invoke a theorem due to Krein and Šmulian [9, Chap. I, p. 45; 11, 17.12, p.159] according to which in a strongly quasi-complete space the balanced, closed, convex hull of a weakly compact set is weakly compact. It follows that $\hat{K}$ is a weakly compact subset of E.

By virtue of Lemma 6, the unit ball of $L^\infty(K, \mu)$ is mapped onto a strongly relatively compact subset A of E. Since E is metrizable, A is separable. It follows that the image of $L^\infty(K, \mu)$ in E is itself separable. But $L^\infty(K, \mu)$ is dense in $L^1(K, \mu)$ and therefore the image of $L^1(K, \mu)$ in E is contained in the closure of the image of $L^\infty(K, \mu)$ in E. Thus the image of $L^1(K, \mu)$ in E is separable for the strong topology.

Let a be a point belonging to the support of μ. For every weak neighbourhood V of a we have $\mu(V) \neq 0$ and we can consider the functions $f_V = \chi_V / \mu(V)$, where χ_V is the characteristic function of V. If for instance V is open or closed, then f_V is measurable, $f_V \in L^1(K, \mu)$, and $\int f_V d\mu = 1$. The map $f \longmapsto \int x f(x) d\mu(x)$ sends f_V into the point $\int x \frac{\chi_V(x)}{\mu(V)} d\mu(x)$ of $\hat{V}$. Since $\hat{V}$ is an arbitrarily small weak neighbourhood of a, we see that Supp μ is contained in the weak closure of the image of $L^1(K, \mu)$ in E.

Now the image of $L^1(K, \mu)$ is a linear subspace of E and therefore its weak closure coincides with its strong closure. Hence Supp μ is contained in the strong closure of the image of $L^1(K, \mu)$ and consequently also in the strong closure of a countable subset of E.

Proof of Theorem 3. We have to show that for every Radon measure μ on E_{weak}, the identity map $E_{\text{weak}} \to E_{\text{strong}}$ is μ-Lusin-measurable.

Given $\epsilon > 0$, there exists a compact subset K of E_{weak} such that μ is concentrated up to $\frac{1}{2}\epsilon$ on K, i.e. that $\mu(E - K) \leqslant \frac{1}{2}\epsilon$. The completion $\hat{E}$ of E_{strong} is a Fréchet space, and K is a weakly compact subset of $\hat{E}$. If μ_K is the measure induced on K by μ, then by Lemma 7 the support of μ_K is contained in a separable closed subspace F of $\hat{E}$. By the particular case of the theorem we have already proved following Theorem 4, the identity map $F_{\text{weak}} \to F_{\text{strong}}$ is μ_K-Lusin-measurable. Thus there exists a weakly compact set $H \subset K$ such that $\mu_K(K - H) < \frac{1}{2}\epsilon$ and the restriction of the identity map to H is continuous. But then $\mu(E - H) \leqslant \epsilon$ and the restriction of the identity map $E_{\text{weak}} \to E_{\text{strong}}$ to H is continuous. This proves that $E_{\text{weak}} \to E_{\text{strong}}$ is μ-Lusin-measurable.

5. Equivalence of the strong topology with coarser topologies. It would have been sufficient to prove Theorem 3 for Fréchet spaces, since it is clear, a priori, that if its assertion holds for all Fréchet spaces, then it holds for all metrizable locally convex spaces. Indeed, let E be a metrizable locally convex space and $\hat{E}$ its completion (for the strong topology, of course). We have the commutative diagram

$$\begin{array}{ccc} E_{\text{weak}} & \xrightarrow{i} & E_{\text{strong}} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ (\hat{E})_{\text{weak}} & \xrightarrow{\hat{i}} & (\hat{E})_{\text{strong}} \end{array}$$

where i and $\hat{i}$ are identity maps, j and k are the canonical injections. Now if $\hat{i}$ is universally measurable, then so is $\hat{i} \circ j = k \circ i$, since j is continuous (cf. Prop. 3 below). It follows from the observation made before Lemma 3 that i is universally measurable.

It follows from Theorem 3 that if E is a locally convex metrizable space, then its topology is Radon-equivalent to any topology which is coarser than the strong topology and finer than the weak topology. This is a consequence of the following result.

PROPOSITION 3. *Let X, Y, and Z be three topological spaces and $f: X \to Y$, $g: Y \to Z$ two universally measurable maps. Then $g \circ f$ is universally measurable.*

PROOF. If μ is a Radon measure on X then $f(\mu)$ is a Radon measure on Y and g is $f(\mu)$-measurable. Thus $g \circ f$ is μ-measurable.

COROLLARY. *Let τ_1, τ_2, and τ_3 be three topologies on a set X and assume that τ_1 is finer than τ_2 and τ_2 is finer than τ_3. If τ_1 and τ_3 are Radon-equivalent then τ_2 is Radon-equivalent to both τ_1 and τ_3.*

PROOF. Let X_k be the set X equipped with the topology τ_k ($k=1, 2, 3$). Since the identity map $X_2 \to X_3$ is continuous, it is sufficient to prove that the identity map $X_3 \to X_2$ is universally measurable. Now the map $X_3 \to X_2$ is composed of the maps $X_3 \to X_1$ and $X_1 \to X_2$, of which the first is universally measurable and the second is continuous.

It follows from results proved in Part I (Ch. II, Cor. 2 of Th. 10) that if E is a separable Fréchet space, then its topology is Radon equivalent to *any* coarser topology, whether or not it is compatible with the vector space structure of E and whether or not it is finer than the weak topology. Here the passage to a noncomplete space indicated at the beginning of the section, is not feasible since a topology on E, coarser than its original topology, is not necessarily induced by a topology on $\hat{E}$, coarser than its original topology. We shall show in fact, with the help of examples, that neither the separability condition nor the completeness condition can be removed without destroying the validity of the assertion.

EXAMPLE 1. Let E be the strong dual $\mathscr{C}'[0, 1]$ of the Banach space $\mathscr{C}[0, 1]$ of all continuous functions on the closed interval $[0, 1]$. The Banach space E is not separable. The space $\mathscr{C} = \mathscr{C}[0, 1]$ is imbedded into its bidual $\mathscr{C}''$, and the weak* topology $\sigma(\mathscr{C}', \mathscr{C})$ on $\mathscr{C}' = \mathscr{C}'[0, 1]$ is strictly coarser than the weak topology $\sigma(\mathscr{C}', \mathscr{C}'')$. Let us denote by $\mathscr{C}'_*$ the space $\mathscr{C}'$ equipped with the

topology $\sigma(\mathscr{C}', \mathscr{C})$. We shall show that the identity map $i: \mathscr{C}'_* \to E$ is not universally measurable.

The map $f: a \to \delta_a$ from $[0, 1]$ into $\mathscr{C}'_*$ is a homeomorphism onto the image of $[0, 1]$ in $\mathscr{C}'_*$. If i were universally measurable, then $i \circ f: [0, 1] \to E$ would also be so. For any two points $a \neq b$ of $[0, 1]$ we always have $\| \delta_a - \delta_b \| = 2$, thus the only sets in $[0, 1]$, on which the restriction of $i \circ f$ is continuous, are finite sets. On the other hand, if $i \circ f$ were measurable for the Lebesgue measure on $[0, 1]$, then there would exist compact sets K whose measure is arbitrarily close to 1 and such that the restriction of $i \circ f$ to K is continuous.

One can construct a similar example with the space L^∞, dual of the space L^1, and the weak* topology $\sigma(L^\infty, L^1)$. In this case the map $a \longmapsto \tau_a Y = Y_a$ from $[0, 1]$ into L^∞_* is a homeomorphism (where Y is the Heaviside function) but $\| Y_a - Y_b \| = 1$ for $a \neq b$.

EXAMPLE 2. We consider the Banach space l^∞ of all bounded sequences $c = (c_n)_{n \in \mathbf{N}}$ equipped with the norm $\| c \| = \sup | c_n |$. We know that l^∞ is the dual of the Banach space l^1 of all sequence $\lambda = (\lambda_n)$ such that $\Sigma | \lambda_n | < \infty$, and that l^∞ is not separable. We want to show that the topology $\sigma(l^\infty, l^1)$ on l^∞ is not Radon-equivalent to the topology defined by the norm on l^∞.

Define the map $f: [0, 1] \to l^\infty$ by setting $f(t) = (e^{2\pi i n t})_{n \in \mathbf{N}}$. This map is continuous if we equip l^∞ with the topology $\sigma(l^\infty, l^1)$. Indeed, for any $\lambda \in l^1$ the function $t \longmapsto \sum_{n=0}^{\infty} \lambda_n e^{2\pi i n t}$ is continuous. On the other hand, the map f is not measurable for the normed topology on l^∞ and the Lebesgue measure on $[0, 1]$. Indeed, if t_0 and t are such that $0 < | t - t_0 | \leqslant \frac{1}{4}$, then there exists $n \in \mathbf{N}$ such that

$$| n | t - t_0 | - \tfrac{1}{2} | < \tfrac{1}{4}$$

and therefore

$$| e^{2\pi i n t} - e^{2\pi i n t_0} | = | e^{2\pi i n | t - t_0 |} - 1 | \geqslant \sqrt{2}.$$

Consequently if t converges to t_0 the sequence $(e^{2\pi i n t})$ does not converge to $(e^{2\pi i n t_0})$ with respect to the norm of l^∞. It follows that the only subsets of $[0, 1]$, on which the restriction of f is continuous, are

the finite ones, whereas, if f were measurable for the Lebesgue measure, one could find compact sets K with measure arbitrarily close to 1 such that the restriction of f to K is continuous. We see exactly as in the preceding example that the identity map $l^\infty \to l^\infty$ from l^∞ equipped with $\sigma(l^\infty, l^1)$ onto l^∞ equipped with the topology defined by the norm is not universally measurable.

EXAMPLE 3. Let E be a separable Banach space and f a non-continuous linear form on E. By Theorem 1, the form f is not universally measurable. Set $H = \mathrm{Ker}(f)$, and let $e \in E$ be such that $f(e) = 1$. The hyperplane H is dense in E and therefore it is not complete if we equip it with the topology induced by E. The map $\chi : H \oplus \mathbf{R} \to E$ defined by $\chi(x, t) = x + te$ $(x \in H,\ t \in \mathbf{R})$ is a continuous bijection. The inverse of the map χ is given by

$$y \longmapsto (y - ef(y),\ f(y))$$

and this map is not universally measurable because f is not. Thus on the separable non-complete normed space $H \oplus \mathbf{R}$ the inverse image with respect to χ of the topology of E is a coarser topology on $H \oplus \mathbf{R}$ which is not Radon-equivalent to its own topology.

Let X be a topological space, μ a Radon measure on X, and E a Fréchet space. We consider the following kinds of maps $f: X \to E$:

(1) strongly μ-measurable;

(2) weakly μ-measurable;

(3) scalarly μ-measurable.

Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and by Theorem 3 we have (1) $\Leftrightarrow$ (2). If E is separable, then by Theorem 4 we have (3) $\Rightarrow$ (1), i.e., in this case the three conditions are equivalent. We show by an example if E is not separable, then (3) does not imply (1).

EXAMPLE 4. Let $\mathscr{H}$ be a Hilbert space having a Hilbert basis $(e_t)_{t \in I}$, where $I = [0,1]$, which has the power of the continuum. Consider the map $f: t \longmapsto e_t$ from I into $\mathscr{H}$. This map is scalarly measurable for the Lebesgue measure μ on I. Indeed, $\mathscr{H}$ can be identified with its dual, and each element $x \in \mathscr{H}$ can be written in the form $x = \sum_{t \in I} x_t e_t$, where at most countably many coefficients

x_t are different from zero. Thus for each $x \in \mathscr{H}$ the function $t \longmapsto \langle f(t), x \rangle = \langle e_t, x \rangle = x_t$ is zero except for countably many values $t \in I$, and in particular it is μ-measurable.

On the other hand $\| e_t - e_{t'} \| = \sqrt{2}$ for $t \neq t'$, and therefore f is continuous only on the finite subsets of I. Thus f is not strongly μ-measurable. Let us observe that we can see directly that f is not weakly μ-measurable either. Indeed, if t tends *to* t_0 through values different from t_0, then e_t does not tend to e_{t_0} weakly, since $\langle e_t, e_{t_0} \rangle = 0$ but $\langle e_{t_0}, e_{t_0} \rangle = 1$. Thus also for the weak topology on $\mathscr{H}$ the function f is continuous only on finite subsets of I.

Let us recapitulate our results.

If E is a locally convex metrizable space, then the strong and weak topologies on E are Radon-equivalent.

If E is a locally convex Suslin space (in particular if E is a separable Fréchet space) then the strong topology of E is Radon-equivalent to any coarser topology on E, not necessarily compatible with the vector space structure of E.

If E is a linear subspace of a locally convex Suslin space (in particular if E is a separable metrizable space) then every scalarly measurable map $X \to E$ is measurable.

SUMMARY OF CHAPTER II

Cylindrical measures (or better, probabilities) will be the main subject of this whole part; this chapter gives their definition and main properties. The definition is given on page 172 (a cylindrical measure is a coherent system of Radon measures on the finite codimensional Hausdorff quotient spaces). A Radon measure is a cylindrical measure (page 173), the converse is not true (Prokhorov's theorem, page 174). An equivalent definition of cylindrical measures is given, page 175; it is very important. § 2 gives the usual operations on cylindrical measures: tensor product (page 180), image by a weakly continuous linear map (page 181), convolution (page 183), Fourier transform (page 184). The Fourier transform will play a fundamental role in this book, and plays a fundamental role in probability theory (although, in the most recent developments on p-radonifying maps, this role is much decreasing). The characterization of the Fourier transform of a cylindrical measure is Proposition 2 (page 187). § 3 defines the concentrations, scalar and cylindrical. It is essential; the definitions are given, page 188, and Definition 1, page 192. The main theorem is Theorem 1, page 193; it gives the correspondence between scalar concentration of a cylindrical measure and continuity of its Fourier transform. Also the scalar concentration on weakly compact sets gives the key for the injectivity, for cylindrical measures, by continuous linear injective maps between vector spaces. Theorem 3, page 198; Proposition 6, page 200; Theorem 4, page 203; and their corollaries, are very useful tools (but one may skip them at first reading). Proposition 7, page 204, is very useful in applications.

II

CYLINDRICAL MEASURES

1. Definition of cylindrical measures. Let E be a locally convex Hausdorff space over the field $\mathbf{R}$. For an arbitrary closed subspace of finite codimension $F \subset E$ we denote by $\pi_{E/F}$ the canonical surjection $E \to E/F$. The spaces E/F form a projective system since if F and G are two closed subspaces of E, of finite codimension, such that $F \supset G$, then we have a canonical map $\pi_{E/F,E/G}$: $E/G \to E/F$, and furthermore for three subspaces we have the obvious composition rule. For $E \supset F \supset G$ we have also $\pi_{E/F,E/G} \circ \pi_{E/G} = \pi_{E/F}$.

Suppose now that on each finite-dimensional space E/F we are given a probability $\mu_{E/F}$ and that for $F \supset G$ we have

$$\mu_{E/F} = \pi_{E/F,E/G}(\mu_{E/G}).$$

Such a projective system of probabilities $(\mu_{E/F})$ will be called a *cylindrical measure* (or cylindrical probability) *on* E†.

Let us explain the origin of the name "cylindrical measure". A subset A of E is said to be a *cylindrical set* if there exists a closed subspace of finite codimension $F \subset E$, and a subset $B \subset E/F$, such that $A = \pi^{-1}_{E/F}(B)$. We say that B is a base and F a generator subspace of the cylindrical set A. Let G be another closed subspace of E of finite codimension, contained in F. If we set $B_0 = \pi^{-1}_{E/F,\,E/G}(B)$, then B_0 is also a base and G a generator subspace of A, since obviously

$$A = \pi^{-1}_{E/F}(B) = \pi^{-1}_{E/G}(\pi^{-1}_{E/F,\,E/G}(B)) = \pi^{-1}_{E/G}(B_0).$$

If now $(\mu_{E/F})$ is a cylindrical measure on E, and A a cylindrical set with measurable base B and generator subspace F, then we can define the "measure" of A by setting $\mu(A) = \mu_{E/F}(B)$, but let us

† Cylindrical probability is the only correct word. Nevertheless, like many people we shall always say cylindrical measure! For the sake of simplicity, we always assume the space E to be locally convex. It would be sufficient, in most of the theorems, to assume that E is a topological vector space, separated by its dual.

emphasize immediately that $A \longmapsto \mu(A)$ is not at all a measure in the usual sense of the word. To show that the definition of $\mu(A)$ is independent of the particular choice of F and B, let us first observe that with the above notation we have

$$\mu(A) = \mu_{E/F}(B) = \pi_{E/F,\,E/G}(\mu_{E/G})(B)$$
$$= \mu_{E/G}(\pi^{-1}_{E/F,\,E/G}(B)) = \mu_{E/G}(B_0).$$

Let now F_1 and F_2 be two generator subspaces of A, then $G = F_1 \cap F_2$ is a closed subspace of finite codimension of E, contained in both F_1 and F_2. If B_1 is the basis of A in E/F_1 and B_2 in E/F_2, then clearly

$$B_0 = \pi^{-1}_{E/F_1,E/G}(B_1) = \pi^{-1}_{E/F_2,E/G}(B_2)$$

and therefore, by what we have already seen,

$$\mu_{E/F_1}(B_1) = \mu_{E/G}(B_0) = \mu_{E/F_2}(B_2).$$

On the cylindrical sets which have a fixed subspace F as generator subspace this "measure" μ is countably additive. However, if we let F vary, then μ will be finitely additive, since given two cylindrical sets A_1 and A_2 we can assume that $A_1 = \pi^{-1}_{E/F}(B_1)$ and $A_2 = \pi^{-1}_{E/F}(B_2)$ for the same subspace F, but it will not be countably additive any more. For a counterexample, see Chap. 6, § 1, p. 328.

It is important to observe that a cylindrical measure μ does not in general define a Radon measure on E. We know, however (Part I, Ch. I, Th. 20), that it defines a probability $\check{\mu}$ on the product $\prod_F (E/F)^\vee$ of the Čech compactifications $(E/F)^\vee$ of the quotient spaces E/F. We have a canonical injection $E \to \prod_F (E/F)^\vee$ but, as we shall see, it can happen that E has measure zero for the measure defined on the product. The map $\mu \longmapsto \check{\mu}$, from the set of cylindrical measures on E, into the set of measures on $\prod_F (E/F)^\vee$, is injective.

If μ is a Radon measure on E, then the measures $\mu_{E/F} = \pi_{E/F}(\mu)$ form a cylindrical measure on E (and therefore define a Radon measure $\check{\mu}$ on $\prod_F (E/F)^\vee$): if j is the canonical injection $E \to \prod_F (E/F)^\vee$, then $j(\mu) = \check{\mu}$, and we say that the cylindrical measure "comes"

from the Radon measure μ; but $\mu \longmapsto j(\mu)$ is injective, therefore μ is entirely determined by the corresponding cylindrical measure and we can say that the cylindrical measure "is" the Radon measure μ: *we identify the Radon measures on E with a subset of the set of the cylindrical measures on E.*

By Prokhorov's theorem (Part I, Chap. I, § 9, Th. 20) the measure $\check{\mu}$ on $\prod_F (E/F)^{\check{}}$ defined by the cylindrical measure $(\mu_{E/F})$ comes from a Radon measure μ on E if and only if for every $\epsilon > 0$ there exists a compact subset K of E such that for every F we have $\mu_{E/F}(\pi_{E/F}(K)) \geqslant 1 - \epsilon$.

An equivalent condition is the following: there exists a sequence (K_n) of compact subsets of E such that $\check{\mu}(j(\bigcup K_n)) = 1$.

This condition cannot be replaced by any of the following weaker conditions:

(a) there exists a sequence (K_n) of compact subsets of E such that for every F we have $\mu_{E/F}(\pi_{E/F}(\bigcup_n K_n)) = 1$;

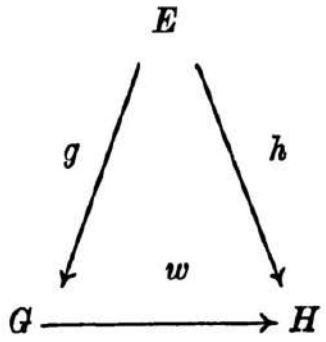
(b) for every $\epsilon > 0$ and every F there exists a compact subset K of E such that $\mu_{E/F}(\pi_{E/F}(K)) \geqslant 1 - \epsilon$.

As an example let us consider a Hilbert space $\mathscr{H}$ equipped with its weak topology. We shall see in Chap. 6 that there exist cylindrical measures on $\mathscr{H}$ which are not Radon measures on $\mathscr{H}$. Nevertheless $\mathscr{H}$ is the countable union of balls, which are compact sets, and therefore condition (a) is satisfied by every cylindrical measure on $\mathscr{H}$.

On the other hand, condition (b) is always satisfied. Indeed, since $\mu_{E/F}$ is a Radon measure on E/F, there exists a compact subset L of E/F such that $\mu_{E/F}(L) \geqslant 1 - \epsilon$. Now F has a topological supplement M [3, Chap. I, § 2, n° 3, Prop. 3] and E/F is isomorphic to M. This isomorphism transforms L into a compact set $K \subset M \subset E$ such that $\pi_{E/F}(K) = L$.

Since $\Pi_F(E/F)$ induces on E the weak topology $\sigma(E, E')$, it is clear that $\check{\mu}$ comes from a Radon measure on E equipped with $\sigma(E, E')$ if and only if $\check{\mu}(E) = 1$, i.e. if $\check{\mu}$ is concentrated on E.

There is another way of looking at cylindrical measures. We associate with every finite-dimensional vector space G (equipped with its unique Hausdorff topology) and with every continuous linear map $g: E \to G$ a probability μ_g on G. We assume furthermore that if H is another such space, $h: E \to H$ a continuous linear map, and $w: G \to H$ a linear map such that the diagram



is commutative, then $\mu_h = w(\mu_g)$. Such a family (μ_g) defines a cylindrical measure on E. Indeed, since the spaces E/F are finite-dimensional, given a family (μ_g), we have a family $(\mu_{E/F})$ which satisfies the required compatibility conditions. Conversely, assume that we are given a cylindrical measure $(\mu_{E/F})$ and let $g: E \to G$ be a continuous linear map into the finite-dimensional vector space G. If F is a closed subspace of finite codimension of E contained in the kernal of g, then g factorizes into

$$E \xrightarrow{\pi_{E/F}} E/F \xrightarrow{\bar{g}} G.$$

We see that $\mu_g = \bar{g}(\mu_{E/F})$ is independent of the particular choice of F and that we obtain a family (μ_g) with the property explained above.

REMARK 1. Let (F_ι) be a family of closed subspaces with finite codimension of the locally convex Hausdorff space E, and assume that (F_ι) is cofinal with the set of all closed subspaces with finite codimension, i.e. that for any closed subspace F with finite

codimension of E, there exists an F_ι contained in F. Assume further that on each quotient space E/F_ι we are given a probability μ_ι, and if $F_\iota \supset F_\kappa$ and $\pi_{\iota\kappa}: E/F_\kappa \to E/F_\iota$ is the corresponding canonical surjection, then $\mu_\iota = \pi_{\iota\kappa}(\mu_\kappa)$. Then the family of probabilities (μ_ι) gives rise to a cylindrical measure on E. Indeed, for any closed subspace with finite codimension F of E let $F_\iota \subset F$ and denote by π_{E/F_ι} the canonical surjection $E/F_\iota \to E/F$. We can check in a straightforward manner that the probability $\mu_{E/F} = \pi_{E/F_\iota}(\mu_\iota)$ is independent of the particular choice of the subspace $F_\iota \subset F$ and the family of probabilities $(\mu_{E/F})$ so obtained satisfies the coherence condition which is required in order to yield a cylindrical measure on E.

Similarly, let $(G_\iota)_{\iota \in I}$ be a family of finite-dimensional Hausdorff vector spaces, (g_ι) a family of continuous linear maps $g_\iota : E \to G_\iota$ and on each space G_ι let μ_ι be a probability. Assume that the families (G_ι), (g_ι) and (μ_ι) satisfy the following conditions.

(1) The family $(G_\iota)_{\iota \in I}$ is directed, i.e. I is an increasingly directed ordered set, and for $\kappa \geqslant \iota$ we have a linear map $w_{\iota\kappa} : G_\kappa \to G_\iota$ such that $w_{\iota\iota}$ is the identity map and $w_{\iota\kappa} \circ w_{\kappa\lambda} = w_{\iota\lambda}$ if $\lambda \geqslant \kappa \geqslant \iota$.

(2) For $\kappa \geqslant \iota$ we have $g_\iota = w_{\iota\kappa} \circ g_\kappa$.

(3) For $\kappa \geqslant \iota$ we have $\mu_\iota = w_{\iota\kappa}(\mu_\kappa)$.

(4) The family (g_ι) is coinitial with the set of continuous linear maps into finite-dimensional vector spaces. In other words, given a continuous linear map $g: E \to G$ into a finite-dimensional Hausdorff vector space G, there exists an index $\iota \in I$ such that g factors into $g = w_{g\iota} \circ g_\iota$, where $w_{g\iota}$ is a linear map $G_\iota \to G$.

For every continuous linear map $g: E \to G$ we can define a probability μ_g on G by setting $\mu_g = w_{g\iota}(\mu_{g\iota})$. It can be checked immediately that μ_g is independent of the particular choice of the index $\iota \in I$. Also if $h: E \to H$ is a second map and $h = w_{hg} \circ g$, where $w_{hg}: G \to H$, then $w_{hg}(\mu_g) = \mu_h$. Thus the triple family $(G_\iota, g_\iota, \mu_\iota)_{\iota \in I}$ defines a cylindrical measure on E.

If E is finite-dimensional, then $\{0\}$ is a subspace of finite codimension, and a cylindrical measure is completely described if a

probability $\mu_{E/\{0\}} = \mu_E$ on E is given. In particular the probability on $\Pi(E/F)^\vee$ defined by this cylindrical measure comes from the Radon measure μ_E on E.

PROPOSITION 1. *Let E be a locally convex Hausdorff space and let $\mathscr{F}$ be the directed set of all closed subspaces of finite codimension of E. If $F \in \mathscr{F}$, $G \in \mathscr{F}$ and $G \subset F$, denote by $\pi_{E/F,\, E/G}$ the canonical map $E/G \to E/F$. The spaces E/F and maps $\pi_{E/F,\, E/G}$ from a projective system and the projective limit of this system equipped with the projective limit topology can be identified canonically with the algebraic dual E'^* of the dual E' of E equipped with the weak topology $\sigma(E'^*, E')$.*

PROOF. Let $(x_{E/F})_{F \in \mathscr{F}}$ be a point of the projective limit $\varprojlim E/F$, i.e. a family of points $x_{E/F} \in E/F$ such that for $F \in \mathscr{F}$, $G \in \mathscr{F}$ and $G \subset F$ we have $x_{E/F} = \pi_{E/F,\, E/G}(x_{E/G})$. Now each $x_{E/F} \in E/F$ defines a linear form on the subspace $(E/F)' = F^\perp$ of E' which is given by

$$y' \longmapsto \langle x_{E/F}, y' \rangle = \langle x, y' \rangle$$

for $y' \in F^\perp$, where $x \in E$ is an arbitrary representative of the coset $x_{E/F}$. If $F \supset G$, then $F^\perp \subset G^\perp$ and if x_1 is a representative of $x_{E/G}$ in E, we have $x - x_1 \in F$ since

$$\begin{aligned} \pi_{E/F}(x - x_1) &= \pi_{E/F}(x) - \pi_{E/F,\, E/G}(\pi_{E/G}(x_1)) \\ &= x_{E/F} - \pi_{E/F,\, E/G}(x_{E/G}) = 0. \end{aligned}$$

Thus for $y' \in F^\perp$ we have

$$\langle x_{E/F}, y' \rangle = \langle x, y' \rangle = \langle x_1, y' \rangle = \langle x_{E/G}, y' \rangle.$$

From here it follows that if $F_1^\perp$ and $F_2^\perp$ are any two finite-dimensional subspaces of E', then the linear forms defined by $(x_{E/F})$ on these two subspaces coincide on their intersection. Since E' is the union of subspaces of finite dimension, we see that $(x_{E/F})$ defines a linear form x on E'. It is clear that the map $(x_{E/F}) \longmapsto x$ from $\varprojlim E/F$ into E'^* is linear.

Now let $x \in E'^*$ be a linear form on E'. Each finite-dimensional subspace of E' is of the form $F^\perp$, where F is a closed subspace of

finite codimension of E. The restriction of x to $F^\perp$ is a linear form on $F^\perp$, i.e. an element $x_{E/F}$ of $(F^\perp)' = E/F$. If $F \supset G$, i.e. $F^\perp \subset G^\perp$, then $\pi_{E/F,\,E/G}: (G^\perp)' \to (F^\perp)'$ is the transpose of the canonical injection $F^\perp \to G^\perp$, and we have $x_{E/F} = \pi_{E/F,\,E/G}(x_{E/G})$. Thus $(x_{E/F})$ is an element of $\varprojlim E/F$, and it follows that the map $(x_{E/F}) \longmapsto x$ is bijective.

Finally, let us prove that the weak topology $\sigma(E'^*, E')$ on $E'^* = \varprojlim E/F$ coincides with the projective limit topology. The element $x = (x_{E/F})$ converges to 0 for the projective limit topology if and only if each $x_{E/F}$ tends to zero for each $F \in \mathscr{F}$. Now this means that $\langle x, y' \rangle$ tends to zero for each $y' \in E'$ since each $y' \in E'$ is contained in some subspace $F^\perp$ and every $F^\perp$ is spanned by finitely many $y' \in E'$.

It follows in particular from Prop. 1 that besides the canonical injection $E \to E'^*$ we also have the canonical injection $E'^* \to \Pi(E/F)^\vee$. To round out the picture let us recall that E'^* is the completion of E for the topology $\sigma(E, E')$ [3, Chap. II, 2nd ed., §6, n°7, Prop. 9].

If F is any closed subspace of finite codimension of E we have $F \subset F^{\perp\perp} \subset E'^*$ and therefore the canonical injection $E \to E'^*$ yields an injective map $E/F \to E'^*/F^{\perp\perp}$. Since the dimensions of both E/F and $E'^*/F^{\perp\perp}$ are finite and equal, namely equal to $\dim F^\perp$, the map $E/F \to E'^*/F^{\perp\perp}$ is in fact an isomorphism. Also every closed subspace of finite codimension of E'^* is of the form $F^{\perp\perp}$ with $F = F^{\perp\perp} \cap E$, therefore we can identify the family $(E/F)_F$ with the family $(E'^*/F^{\perp\perp})_F$ and in particular there is a bijective correspondence between the cylindrical measures on E and the cylindrical measures on E'^*. One can even say more:

If E_1 and E_2 are two locally convex topological vector spaces having the same weak completion, $E_1'^* = E_2'^*$, even if there is no inclusion relation between them, there exists a bijective correspondence between the cylindrical measures on E_1 and E_2.

It is now very easy to give an example of a cylindrical measure μ on E such that for the associated Radon measure $\check{\mu}$ on $\Pi(E/F)^\vee$

the subset E has measure zero. Let a be a point of E'^* which does not belong to E. To a there corresponds in each quotient space $E/F \simeq E'^* F^{\perp\perp}$ a point $a_{E/F}$, and the Dirac measures $\delta_{a_{E/F}}$, form a projective system., i.e. define a cylindrical measure μ on E. The associated Radon measure $\check{\mu}$ on $\prod_F (E/F)^\vee$ is the Dirac measure δ_a so that we have indeed $\check{\mu}(E) = 0$.

In the preceding example we have $\check{\mu}(E'^*) = 1$, i.e. $\check{\mu}$ is concentrated on E'^*. We shall now exhibit an example of a cylindrical measure μ on E for which we even have $\check{\mu}(E'^*) = 0$.

EXAMPLE 1. Let I be an uncountable index set and $E = \mathbf{R}^I$. The dual of E is the direct sum $E' = \mathbf{R}^{(I)}$, the product topology on E coincides with the weak topology $\sigma(E, E')$ [3, Chap. IV, § 2, n^0 8, Prop. 10], and E is complete for this topology, i.e. $E'^* = E$. For each $\iota \in I$ let μ_ι be a probability on $\mathbf{R}$ such that the support of μ_ι is all of $\mathbf{R}$ (e.g. let each μ_ι be the Gauss measure $\gamma = e^{-\pi t^2}\, dt$, cf. Chap. 6, § 1). For each finite subset J of I let μ_J be the probability $\bigotimes_{\iota \in J} \mu_\iota$ on $\mathbf{R}^J$. If $L \subset J$ and π denotes the canonical map $\mathbf{R}^J \to \mathbf{R}^L$, then $\pi(\mu_J) = \mu_L$ and therefore by Remark 1 we obtain a cylindrical measure μ on $\mathbf{R}^I$.

Let K be a compact subset of $\mathbf{R}^I$ and for every $\iota \in I$ denote by K_ι the projection of K in $\mathbf{R}^{\{\iota\}} \simeq \mathbf{R}$. If J is a finite subset of I, and π_J the projection from $\mathbf{R}^I$ onto $\mathbf{R}^J$, then we have $\mu_J(\pi_J(K)) \leqslant \prod_{j \in J} \mu_j(K_j)$. Since K is compact we have $K = \bigcap_J \pi_J^{-1}(\pi_J(K))$, and therefore $\check{\mu}(K) = \inf_J \mu(\pi_J^{-1}(\pi_J(K))) = \inf_J \mu_J(\pi_J(K))$, and this last quantity is zero since $\mu_\iota(K_\iota) < 1$ for each $\iota \in I$ and I is uncountable. Thus $\check{\mu}(K) = 0$ for every compact subset K of $E = \mathbf{R}^I$, and therefore $\check{\mu}(E) = \check{\mu}(E'^*) = 0$.

From here we can infer that if E is an arbitrary locally convex Hausdorff space whose dual E' has an uncountable algebraic basis, then there exists on E a cylindrical measure μ for which $\check{\mu}(E'^*) = 0$. Indeed, if the set I is equipotent with an algebraic basis of E', then E'^* is isomorphic to $\mathbf{R}^I$, hence there exists by the above

considerations a cylindrical measure on E'^* which has the required property. But, as we already remarked, a cylindrical measure on E'^* is the same thing as a cylindrical measure on E.

Let us observe that if E' has a countable basis then every cylindrical measure on E'^* is a Radon measure on E'^* since in this case $E'^* \simeq \mathbf{R}^{\mathbf{N}}$ and countable projective limits always exist (Part I, Chap. I, §9, Corol. of Prop. 20).

2. Operations with cylindrical measures. As a general rule, one can perform the same operations with cylindrical measures as with Radon measures except that one cannot integrate a function with respect to a cylindrical measure, and one cannot speak of the measure of a set with respect to a cylindrical measure.

(a) TENSOR PRODUCT. Let E_1 and E_2 be two locally convex Hausdorff spaces and F a closed subspace of finite codimension in their product $E = E_1 \times E_2$. Let $F^\perp$ be the subspace orthogonal to F in the dual $E' = E'_1 \times E'_2$ of E. Then $F^\perp$ is finite-dimensional and so its projections $F_1^\perp$ and $F_2^\perp$ in the factor spaces E'_1 and E'_2 are both finite-dimensional. Thus $F_1^\perp \times F_2^\perp$ is a finite-dimensional subspace of E' such that $F^\perp \subset F_1^\perp \times F_2^\perp$. If we set $F_1 = F_1^{\perp\perp} \subset E_1$ and $F_2 = F_2^{\perp\perp} \subset E_2$, then $F_1 \times F_2$ is a closed subspace of $E_1 \times E_2$, and it is of finite codimension since the dual of $(E_1 \times E_2)/(F_1 \times F_2)$ can be identified with $F_1^\perp \times F_2^\perp$. Furthermore $F \supset F_1 \times F_2$. Thus we have proved that the family of closed subspaces of finite codimension of the form $F_1 \times F_2$ is cofinal with the set of all closed subspaces of finite codimension of $E_1 \times E_2$.

Now let μ_1 be a cylindrical measure on E_1 and μ_2 a cylindrical measure on E_2. For any closed subspace with finite codimension F_j of E_j $(j = 1, 2)$ we have a probability $(\mu_1)_{E_1/F_1} \otimes (\mu_2)_{E_2/F_2}$ on the space $(E_1/F_1) \times (E_2/F_2)$, canonically isomorphic to the quotient $(E_1 \times E_2)/(F_1 \times F_2)$. It is easy to check that these probabilities satisfy the usual coherence condition and therefore it follows from Remark 1 of §1 and the observation made at the beginning that they form a cylindrical measure on $E_1 \times E_2$ which we shall call the

tensor product of the cylindrical measures μ_1 and μ_2, and denote by $\mu_1 \otimes \mu_2$.

If μ_1 and μ_2 are Radon measures, they define cylindrical measures, and the Radon measure $\mu_1 \otimes \mu_2$ defines a cylindrical measure which is the tensor product of the cylindrical measures defined by μ_1 and μ_2. One can define exactly in the same way an infinite tensor product $\bigotimes_{i \in I} \mu_i$ on an infinite product $\prod_{i \in I} E_i$ of locally convex vector spaces E_i. In this case one uses only the subspaces of the form $\prod_{i \in I} F_i$, where F_i is a finite-codimensional closed vector subspace of E_i, all but finitely many of them being the whole space.

(b) IMAGE WITH RESPECT TO A WEAKLY CONTINUOUS LINEAR MAP. Let E_1 and E_2 be two locally convex Hausdorff spaces, $u: E_1 \to E_2$ a weakly continuous linear map, and μ a cylindrical measure on E_1. We suppose that μ is defined with the help of a family (μ_{g_1}) of Radon measures attached to weakly continuous linear maps $g_1: E_1 \to G$ from E_1 into finite-dimensional vector spaces G (p. 175). If $g_2: E_2 \to G$ is a weakly continuous linear map, then $g_1 = g_2 \circ u$ is a weakly continuous linear map $E_1 \to G$. Setting $\nu_{g_2} = \mu_{g_1}$ for each g_2, we obtain a family (ν_{g_2}) of Radon measures, which defines a cylindrical measure $\nu = u(\mu)$ on E_2, called the image of μ with respect to u. If μ is a Radon measure on E_1, $u(\mu)$ coincides with the image of this measure by u, which is a Radon measure on $\sigma(E_2, E'_2)$ or E_2 itself according as u is weakly continuous or continuous for the initial topologies.

Let E be a locally convex Hausdorff space, μ a cylindrical measure on E and F a closed subspace of finite codimension of E. The measure $\mu_{E/F}$ on E/F is the image of μ with respect to the canonical surjection $\pi_{E/F}: E \to E/F$ in the sense just defined. We know (p. 176) that we can consider $\mu_{E/F}$ as a cylindrical measure since E/F is finite-dimensional.

If E_1, E_2, F_1, F_2 are four locally convex Hausdorff spaces, $u_1: E_1 \to F_1$, $u_2: E_2 \to F_2$ continuous linear maps, μ_1 a cylindrical measure on E_1 and μ_2 a cylindrical measure on E_2, then clearly

$$(u_1 \times u_2)(\mu_1 \otimes \mu_2) = u_1(\mu_1) \otimes u_2(\mu_2). \tag{1}$$

PROPOSITION 1 BIS. *Let μ be a cylindrical measure on a countable product $E = \prod_{n \in \mathbf{N}} E_n$ of locally convex vector spaces. It is a Radon measure on E if and only if its projection $p_n\mu$ on every E_n is a Radon measure on E_n.*

PROOF. The condition is trivially necessary; we shall prove that it is sufficient. We shall use a simple lemma.

LEMMA. *Let J be a finite set, $X = \prod_{n \in J} X_n$ a finite product of topological spaces. Let $A_n \subset X_n$, $A = \prod_{n \in J} A_n$ and ν a Radon probability on X. Then the inequalities $\nu_n(A_n) \geqslant 1 - \epsilon_n$, $n \in J$, where ν_n is the image of ν by the projection $X \to X_n$, imply $\nu(A) \geqslant 1 - \epsilon$, where $\epsilon = \sum_{n \in J} \epsilon_n$.*

The proof is trivial: the complement $\complement A$ is the union

$$\bigcup_{n \in J} (\complement A_n \times \prod_{\substack{m \in J \\ m \neq n}} X_m),$$

hence its measure is less than

$$\sum_{n \in J} \nu (\complement A_n \times \prod_{\substack{m \in J \\ m \neq n}} X_m) = \sum_{n \in J} \nu_n (\complement A_n) < \sum_{n \in J} \epsilon_n.$$

Let us prove the proposition. Call p_J the projection of E on $E_J = \prod_{n \in J} E_n$, for $J \subset \mathbf{N}$. For every n, $p_n \mu$ is assumed to be Radon. Hence it verifies Prokhorov's condition. For $\epsilon > 0$, there exists a compact subspace K_n of E_n such that if F_n is a closed finite codimensional vector subspace of E_n, we have $(\pi_n p_n \mu)(\pi_n K_n) \geqslant 1 - \epsilon/2^{n+1}$, where π_n is the canonical projection $E_n \to E_n/F_n$. Put $K = \prod_{n \in \mathbf{N}} K_n$. Let J be a finite subset of $\mathbf{N}$, and for every $n \in J$ let F_n be a finite codimensional closed subspace of E_n. Call π_J the canonical projection $\prod_{n \in J} E_n \to \prod_{n \in J} E_n/F_n$. Let us compute $(\pi_J p_J \mu)(\pi_J p_J K)$. We have

$$(\pi_n p_J \mu)(\pi_n p_J K) = (\pi_n p_n \mu)(\pi_n K_n) \geqslant 1 - \epsilon/2^{n+1};$$

according to the lemma applied to $X_n = E_n/F_n$, $A_n = \pi_n K_n$, $\nu = \pi_J p_J \mu$, we get $(\pi_J p_J \mu)(\pi_J p_J K) \geqslant 1 - \epsilon$, $\epsilon = \sum_{n \in \mathbf{N}} (\epsilon/2^{n+1})$.

But we know that the spaces of the form $\prod_{n\in J} F_n \times \prod_{m\in J} E_m$ form a cofinal set to the set of closed, finite-codimensional vector subspaces of E. Therefore X satisfies Prokhorov's condition for ϵ.

COROLLARY. *Let E, F be two locally convex vector spaces, u and v weakly continuous linear maps $E \to F$. Let μ be a cyclindrical measure on E. Assume $u(\mu)$ and $v(\mu)$ are Radon measures; then $(u+v)(\mu)$ is a Radon measure too.*

PROOF. The map $u+v : E \to F$ can be factorized as follows: $E \xrightarrow{w} F \times F \to F$, where $E \xrightarrow{w} F \times F$ is the map $x \longmapsto (u(x), v(x))$, and $F \times F \to F$ is the addition $(x, y) \longmapsto x + y$. Therefore $(u+v)(\mu)$ is the image with respect to addition of the cylindrical measure $w(\mu)$ on $F \times F$. But $w(\mu)$ is a cylindrical measure on $F \times F$ whose projections are Radon measures $u(\mu)$, $v(\mu)$; according to the proposition $w(\mu)$ is a Radon measure, which proves the corollary.

(c) CONVOLUTION. Let E be a locally convex Hausdorff space and consider first two Radon measures μ and ν on E. If σ is the map $(x,y) \longmapsto x + y$ from $E \times E$ into E, then we define the convolution $\mu * \nu$ of μ and ν as the image of $\mu \otimes \nu$ with respect to σ. If u is a continuous linear map from E into a second locally convex Hausdorff space F, then the diagram

$$\begin{array}{ccc} E \times E & \xrightarrow{u \times u} & F \times F \\ \Big\downarrow{\scriptstyle \sigma_E} & & \Big\downarrow{\scriptstyle \sigma_F} \\ E & \xrightarrow{u} & F \end{array}$$

is commutative and therefore we have

$$u(\mu * \nu) = u(\mu) * u(\nu). \tag{2}$$

Let now μ and ν be cylindrical measures on E rather than Radon measures. We can still define their convolution $\mu * \nu$ as the image of

$\mu \otimes \nu$ with respect to the map σ. Explicitly $\mu * \nu$ is defined by the family of probabilities $\mu_{E/F} * \nu_{E/F}$ on the finite-dimensional quotient spaces E/F. Indeed, by what we have just said, for $F \supset G$ we have

$$\mu_{E/F} * \nu_{E/F} = \pi_{E/F, E/G}(\mu_{E/G}) * \pi_{E/F, E/G}(\nu_{E/G})$$

$$= \pi_{E/F, E/G}(\mu_{E/G} * \nu_{E/G}),$$

i.e. the probabilities $\mu_{E/F} * \nu_{E/F}$ form a cylindrical measure. On the other hand, by (1) we have

$$(\mu * \nu)_{E/F} = \pi_{E/F}(\mu * \nu) = \pi_{E/F}(\sigma_E(\mu \otimes \nu))$$

$$= \sigma_{E/F}(\pi_{E/F}(\mu) \otimes \pi_{E/F}(\nu)) = \mu_{E/F} * \nu_{E/F}$$

i.e. the cylindrical measure defined by the $\mu_{E/F} * \nu_{E/F}$ is indeed $\mu * \nu$. Relation (2) holds also for cylindrical measures μ and ν.

(d) FOURIER TRANSFORM. Let E be a locally convex Hausdorff space and consider first again a finite Radon measure μ on E. For $x \in E$ and $\xi \in E'$ let $(x, \xi) \longmapsto \langle x, \xi \rangle$ denote the canonical bilinear form on $E \times E'$. The Fourier transform $\phi = \mathscr{F}(\mu)$ of μ is the function defined on E' given by the formula

$$\phi(\xi) = \int_E e^{-2\pi i \langle x, \xi \rangle} \, d\mu(x)$$

for $\xi \in E'$.

Let now $\mu = (\mu_{E/F})$ be a cylindrical measure on E. If F is a closed subspace of finite codimension of E then for $\xi \in (E/F)' = F^\perp$, the Fourier transform of $\mu_{E/F}$ is defined by

$$\phi_F(\xi) = \int_{E/F} e^{-2\pi i \langle x, \xi \rangle} \, d\mu_{E/F}(x).$$

Let now G be another closed subspace of finite codimension of E contained in F, then $F^\perp \subset G^\perp$ and the injection map $j: F^\perp \to G^\perp$ is the transpose of the canonical map $\pi = \pi_{E/F, E/G}: E/G \to E/F$. Thus for $\xi \in F^\perp$ we have

$$\phi_G(j(\xi)) = \int_{E/G} e^{-2\pi i\langle x,\, j(\xi)\rangle}\, d\mu_{E/G}(x)$$

$$= \int_{E/G} e^{-2\pi i\langle \pi(x),\, \xi\rangle}\, d\mu_{E/G}(x)$$

$$= \int_{E/F} e^{-2\pi i\langle x,\, \xi\rangle}\, d\pi(\mu_{E/G})(x)$$

$$= \int_{E/F} e^{-2\pi i\langle x,\, \xi\rangle}\, d\mu_{E/F}(x) = \phi_F(\xi).$$

It follows that if F_1 and F_2 are two closed subspaces of finite codimension of E, then ϕ_{F_1} and ϕ_{F_2} coincide on $F_1^\perp \cap F_2^\perp$. Since every $\xi \in E'$ is contained in some subspace $F^\perp$, the functions ϕ_F define a unique function ϕ on E' which will be, by definition, the Fourier transform $\mathscr{F}(\mu)$ of the cylindrical measure $\mu = (\mu_{E/F})$.

The Fourier transform $\phi = \mathscr{F}(\mu)$ of the cylindrical measure $\mu = (\mu_{E/F})$ can be expressed with the help of a useful formula. Consider an element $\xi \in E'$, i.e. a continuous linear map $E \to \mathbf{R}$. The image $\xi(\mu)$ of μ with respect to ξ is a Radon measure on $\mathbf{R}$. If F is a closed subspace of finite codimension of E and $\xi \in F^\perp$, then we have clearly $\xi(\mu) = \xi(\mu_{E/F})$. The Fourier transform $\mathscr{F}(\xi(\mu))$ of $\xi(\mu)$ is of course defined by

$$\mathscr{F}(\xi(\mu))\,(t) = \int_{\mathbf{R}} e^{-2\pi i s t}\, d(\xi(\mu))(s)$$

for $t \in \mathbf{R}$. Now if $\xi \in F^\perp$ and $t \in \mathbf{R}$, we have

$$\phi(t\xi) = \phi_F(t\xi) = \int_{E/F} e^{-2\pi i\langle x,\, t\xi\rangle}\, d\mu_{E/F}(x)$$

$$= \int_{E/F} e^{-2\pi i t\xi(x)} d\mu_{E/F}(x)$$

$$= \int_{\mathbf{R}} e^{-2\pi i t s} d(\xi(\mu))(s) = \mathscr{F}(\xi(\mu))(t)$$

which gives the required formula

$$\phi(\xi) = \mathscr{F}(\xi(\mu))(1). \tag{3}$$

This formula holds, of course, a fortiori if μ is a Radon measure on E, and could have served to *define* the Fourier transform $\phi = \mathscr{F}(\mu)$ of the cylindrical measure μ.

Let E_1 and E_2 be two locally convex Hausdorff spaces, μ_1 a cylindrical measure on E_1 and μ_2 a cylindrical measure on E_2; then we have

$$\mathscr{F}(\mu_1 \otimes \mu_2) = \mathscr{F}(\mu_1) \otimes \mathscr{F}(\mu_2),$$

as can be seen easily from the definitions.

If μ and ν are two cylindrical measures on the same locally convex Hausdorff space E, then

$$\mathscr{F}(\mu * \nu) = \mathscr{F}(\mu).\mathscr{F}(\nu).$$

Indeed, for any $\xi \in E'$ formula (2) yields $\xi(\mu * \nu) = \xi(\mu) * \xi(\nu)$ and by the classical properties of the Fourier transform on $\mathbf{R}$ we have $\mathscr{F}(\xi(\mu) * \xi(\nu)) = \mathscr{F}(\xi(\mu)).\mathscr{F}(\xi(\nu))$. The assertion follows now from (3).

Let E_1 and E_2 be two locally convex Hausdorff spaces, $u : E_1 \to E_2$ a continuous linear map and μ a cylindrical measure on E_1. We have

$$\mathscr{F}(u(\mu)) = \mathscr{F}(\mu) \circ {}^t u, \tag{4}$$

where ${}^t u : E_2' \to E_1'$ denotes the transpose of u. Indeed, if we set $\phi = \mathscr{F}(\mu)$, by virtue of (3) we have, for $\eta \in E_2'$,

$$\mathscr{F}(u(\mu))(\eta) = \mathscr{F}(\eta(u(\mu)))(1) = \mathscr{F}({}^t u(\eta)\,(\mu))(1)$$
$$= \mathscr{F}(\mu)\,({}^t u(\eta)) = (\phi \circ {}^t u)(\eta),$$

since ${}^t u(\eta) = \eta \circ u$, and therefore $\eta(u(\mu)) = {}^t u(\eta)\,(\mu)$.

Any map $f : X \to Y$ defines a map f^* from the set of all functions on Y into the set of all functions on X if we define $f^*(g) = g \circ f$ for any function g on Y. With this convention (4) can also be written as

$$\mathscr{F}(u(\mu)) = {}^t u^*(\mathscr{F}(\mu)). \tag{5}$$

Let us consider in particular a linear form $\xi \in E'$ as a linear map from E into $\mathbf{R}$. Its transpose is the map ${}^t\xi : \tau \to \tau\xi$ from $\mathbf{R}$ into E' and (4) yields the relation

$$\mathscr{F}(\xi(\mu))\,(\tau) = \mathscr{F}(\mu)\,({}^t\xi(\tau)) = \mathscr{F}(\mu)\,(\tau\xi),$$

which is already familiar.

A function $\phi: \xi \longmapsto \phi(\xi)$ defined on an additive group is said to be *of positive type* if for arbitrary elements $\xi_1, \dots, \xi_n$ of the group and arbitrary complex numbers $z_1, \dots, z_n$ we always have

$$\sum_{i,j=1}^{n} z_i \bar{z}_j \varphi(\xi_i - \xi_j) \geqslant 0.$$

PROPOSITION 2. *A function φ defined on the dual E' of a locally convex Hausdorff space E is the Fourier transform of a cylindrical measure μ on E if and only if it is of positive type, its restriction to every finite-dimensional subspace of E' is continuous and $\varphi(0) = 1$.*

PROOF. Let $\phi = \mathscr{F}(\mu)$, where $\mu = (\mu_{E/F})$ is a cylindrical measure. If $F^\perp$ is any finite-dimensional subspace of E', then for $\xi \in F^\perp$ we have

$$\mathscr{F}(\mu)\,(\xi) = \mathscr{F}(\mu_{E/F})\,(\xi).$$

By a classical result of Bochner the restriction of ϕ to $F^\perp$ is a continuous function of positive type. Now only finitely many vectors $\xi_1, \dots, \xi_n$ enter at the same time into the definition of positive type, hence φ is of positive type throughout E'. Finally $\mathscr{F}(\mu)(0) = \text{total mass of any } \mu_{E/F} = 1$.

Conversely, let φ be a function of positive type on E', whose restriction to every $F^\perp$ is continuous and such that $\varphi(0) = 1$. For every closed subspace of finite codimension F of E there exists by Bochner's theorem a probability $\mu_{E/F}$ on E/F such that $\mathscr{F}(\mu_{E/F})\,(\xi) = \phi(\xi)$ for $\xi \in F^\perp$. Furthermore, if $F^\perp \subset G^\perp$, then $\mathscr{F}(\mu_{E/F})\,(\xi) = \mathscr{F}(\mu_{E/G})\,(\xi) = \phi(\xi)$ for $\xi \in F^\perp$. On the other hand, the canonical map $\pi = \pi_{E/F,E/G}: E/G \to E/F$ is the transpose of the injection $j: F^\perp \to G^\perp$ and thus for $\xi \in F^\perp$ we have

$$\mathscr{F}(\mu_{E/G})\,(j(\xi)) = \mathscr{F}(\pi(\mu_{E/G}))\,(\xi).$$

Comparing with the above formula, we see that $\mu_{E/F} = \pi(\mu_{E/G})$. Thus the probabilities $\mu_{E/F}$ form a cylindrical measure on E whose Fourier transform is ϕ.

3. **Concentrations.** Let μ be a probability on the topological space E. In agreement with the usual terminology [4, Chap. V, §5, n° 7, Def. 4] we say that μ is concentrated on the subset A of E, or that A carries μ, if $\mu(A) = 1$, i.e., if $\mu(\complement A) = 0$. Given $0 < \delta < 1$, we say that μ is *concentrated up to* δ on the subset A of E if $\mu_*(A) \geq 1 - \delta$, i.e. if $\mu^*(\complement A) \leq \delta$, where μ_* is the inner measure and μ^* the outer measure associated with μ.

Let now $\mu = (\mu_{E|F})$ be a cylindrical measure on the locally convex Hausdorff space E and let $0 \leq \delta \leq 1$. We say that μ is *cylindrically concentrated up to* δ on the subset A of E if for every closed subspace of finite codimension $F \subset E$ we have $(\mu_{E|F})_*(\pi_{E|F}(A)) \geq 1 - \delta$. This can be expressed equivalently by saying that for every finite-dimensional Hausdorff vector space G and for every continuous linear map $g : E \to G$, we have $(g(\mu))_*(g(A)) \geq 1 - \delta$. In the case $\delta = 0$ we shall sometimes say simply that μ is cylindrically concentrated on A. Clearly if μ is cylindrically concentrated on A up to $\delta + \epsilon$ for any $\epsilon > 0$, then μ is cylindrically concentrated up to δ on A.

We shall say that $\mu = (\mu_{E|F})$ is *scalarly concentrated up to* δ on $A \subset E$ if for every closed hyperplane $H \subset E$ we have $(\mu_{E|H})_*(\pi_{E|H}(A)) > 1 - \delta$. This amounts to saying that for every continuous linear form $\xi \in E'$ we have $(\xi(\mu))_*(\xi(A)) > 1 - \delta$. In the case $\delta = 0$ we shall sometimes say simply that μ is scalarly concentrated on A. Again it is clear that if μ is scalarly concentrated up to $\delta + \epsilon$ on A for every $\epsilon > 0$, then μ is scalarly concentrated up to δ on A. It is also obvious that if μ is cylindrically concentrated up to δ on A, then μ is scalarly concentrated up to δ on A.

With the terminology just introduced the condition given on p. 174 can be restated as follows. A cylindrical measure μ on E defines a probability on E if and only if for every $\epsilon > 0$ there exists a compact subset K of E such that μ is concentrated cylindrically up to ϵ on K.

Let E_1 and E_2 be two locally convex Hausdorff spaces and u: $E_1 \to E_2$ a continuous linear map. If μ is a Radon measure on E_1, concentrated up to δ on the set $A \subset E_1$, then $u(\mu)$ is concentrated

up to δ on $u(A)$. Similarly, if μ is a cylindrical measure on E_1, concentrated cylindrically (resp. scalarly) up to δ on the set $A \subset E_1$, then $u(\mu)$ is concentrated cylindrically (resp. scalarly) up to δ on $u(A)$.

PROPOSITION 3. *Let μ be a probability on the locally convex Hausdorff space E. If μ is concentrated cylindrically up to δ on the weakly closed subset A of E, then μ is concentrated up to δ on A.*

PROOF. Let F be a closed subspace of finite codimension of E. Then $\pi_{E/F}(\mu) = \mu_{E/F}$ is concentrated up to δ on $\overline{\pi_{E/F}(A)}$, hence we have a fortiori $\mu(\pi_{E/F}^{-1}(\overline{\pi_{E/F}(A)})) \geqslant 1 - \delta$. Let $\tilde{A}$ be the image of A in $\Pi(E/F)^\vee$. Then $\tilde{\bar{A}} = \bigcap \pi_{E/F}^{-1}(\overline{\pi_{E/F}(A)})$, and therefore $\mu(\tilde{\bar{A}}) \geqslant 1 - \delta$ since the sets $\pi_{E/F}^{-1}(\overline{\pi_{E/F}(A)})$ form a decreasingly directed system of closed subsets of $\Pi(E/F)^\vee$. On the other hand $\tilde{\bar{A}} \subset E'^*$, and since the topology $\sigma(E'^*, E')$ induces $\sigma(E, E')$ on E, we have $\tilde{\bar{A}} \cap E = \bar{A}$. Now μ is concentrated on E, hence $\mu(\bar{A}) = \mu(\tilde{\bar{A}})$. By hypothesis $A = \bar{A}$, and thus we have finally $\mu(A) = \mu(\tilde{\bar{A}}) \geqslant 1 - \delta$.

This proposition can be applied when A is a closed, convex subset of E, and more particularly if A is a closed ball in a Banach space.

Cylindrical concentration up to δ is much stronger than scalar concentration up to δ, as can be seen from the following consideration. Let $E = \mathbf{R}^n$ and let μ be a Radon measure on E which is concentrated scalarly up to δ on the unit ball $\sum_{i=1}^{n} x_i^2 \leqslant 1$. This means that for every $\xi \in E'$ such that $|\xi| = 1$ the measure μ is concentrated up to δ in the strip $|\langle x, \xi \rangle| \leqslant 1$ bounded by the two hyperplanes $\langle x, \xi \rangle = \pm 1$ tangent to the unit sphere. The unit cube K of $\mathbf{R}^n$, given by $\max_{1 \leqslant j \leqslant n} |x_j| \leqslant 1$, is the intersection of the strips $|x_j| \leqslant 1$ and therefore $u(\complement K) \leqslant n\delta$, i.e. μ is concentrated upto $n\delta$ on K. The smallest ball containing K has radius $\sqrt{n}$ and the best we can say with this method is that μ is concentrated up to $n\delta$ on the ball with radius $\sqrt{n}$. This result is weaker the larger n becomes.

PROPOSITION 4. *Let μ be a cylindrical measure on the locally convex Hausdorff space E which is scalarly concentrated on any closed homogeneous hyperplane of E. Then μ is the Dirac measure $\delta_{(0)}$.*

PROOF. Let $g: E \to G$ be a continuous linear map into the n-dimensional Hausdorff space G. The measure $g(\mu)$ is then concentrated up to $\delta = 0$ on any hyperplane through the origin of G and therefore $g(\mu)$ is concentrated upto $n\delta = 0$ on $\{0\}$. Thus $g(\mu) = \delta_{(0)}$ for every g, hence μ is the Dirac measure on E.

PROPOSITION 5. *Let μ be a cylindrical measure on the locally convex Hausdorff space E which is scalarly concentrated on the convex weakly compact set $A \subset E$. Then μ is a Radon measure on E equipped with the weak topology, and it is concentrated on A.*

PROOF. Let $g : E \to G$ be a continuous linear map into the finite-dimensional Hausdorff space G. Then $g(\mu)$ is scalarly concentrated on the convex set $g(A)$ and in particular $g(\mu)$ is concentrated on any closed half-space containing $g(A)$ and therefore also on any finite intersection of such half-spaces. Now the compact, convex set $g(A)$ is the intersection of the decreasingly directed collection of closed sets consisting of finite intersections of closed half-spaces containing $g(A)$, hence $g(\mu)$ is concentrated on $g(A)$. Thus μ is cylindrically concentrated on the weakly compact set A and therefore by Prokhorov's theorem μ is a Radon measure on E equipped with the weak topology. It follows finally from Prop. 3 that μ is concentrated on A.

Let $(A_n)_{n\geq 0}$ be a decreasing sequence of subsets of the locally convex Hausdorff space E, and let μ be a cylindrical measure on E which is cylindrically concentrated up to δ on each set A_n. We shall show by an example that μ is not necessarily concentrated scalarly up to δ on the set $\bigcap_{n=0}^{\infty} A_n$.

EXAMPLE 2. The cylindrical measure constructed in this example will be used repeatedly later in this section. Let F be the Banach space l^1 of all absolutely summable sequences $c = (c_n)_{n<0}$ equipped

with the norm $\|c\| = \sum_{n=0}^{\infty} |c_n|$. It is well known that l^1 can be identified with the dual of the Banach space c_0 of all sequences which tend to zero. Denote by E the space l^1 equipped with the weak* topology $\sigma(l^1, c_0)$. The dual of E is the Banach space l^∞ of all bounded sequences, and the dual of E is c_0.

Let $\mathfrak{U}$ be an ultrafilter on N finer than the Fréchet filter and let $\xi = (\xi_n)_{n>0}$ be an arbitrary element of l^∞. Since $n \longmapsto \xi_n$ maps N into the compact interval $[-\|\xi\|, \|\xi\|]$, the limit $a(\xi) = \lim_{\mathfrak{U}} \xi_n$ exists. The function $\xi \longmapsto a(\xi)$ is linear in ξ, we have $|a(\xi)| \leqslant \|\xi\|$, and if $\lim_{n\to\infty} \xi_n$ exists, then $a(\xi) = \lim_{n\to\infty} \xi_n$. Thus a is a continuous linear form on $F' = l^\infty$, with norm 1, which vanishes on the subspace c_0.

Let μ be the cylindrical measure on F corresponding to the Radon measure δ_a on $F'' = (l^\infty)'$. Since $a \notin F$, the measure δ_a is concentrated on the complement of F, and therefore μ is not a Radon measure on F.

Let us show that μ is concentrated cylindrically on any ball with radius > 1 of F. Let g be a continuous linear map from F *onto* a finite-dimensional Hausdorff vector space G. The map $g : F \to G$ can be extended to a map $\bar{g} : (l^\infty)' \to G$ and we have $g(\mu) = \delta_{\bar{g}(a)}$. Now a belongs to the unit ball of F' and is therefore adherent with respect to the topology $\sigma((l^\infty)', l^\infty)$ to the unit ball B of F [3, Chap. IV, §5, $n°2$, Prop. 5], hence $\bar{g}(a)$ is adherent to $g(B)$. But since g is surjective, it is an open map and thus for any $\epsilon > 0$ the set $g(\epsilon B)$ is a neighbourhood of 0 in G. It follows that the set $g((1+\epsilon)B) = g(B) + g(\epsilon B)$ contains the closure of $g(B)$ and in particular $\bar{g}(a) \in g((1+\epsilon)B)$ which proves that $g(\mu)$ is carried by $g((1+\epsilon)B)$.

Finally let us show that μ is not concentrated scalarly on the unit ball B of F. Let $\xi = (\xi_n)_{n>0}$ be an element of $l^\infty = F'$ such that $0 < \xi_n < 1$ and $\lim_{n\to\infty} \xi_n = 1$. As a continuous linear form on F, the map $\xi : F \to \mathbf{R}$ maps B onto the open interval $[-1, 1]$. On the other

hand $a(\xi) = 1$, and therefore $\xi(\mu) = \delta_{\tilde{\xi}(a)} = \delta_1$ which is concentrated on the set $\{1\}$ not contained in $]-1, 1[$.

If, however, $(A_\iota)_{\iota \in I}$ is a decreasingly directed family of *weakly compact* subsets of the locally convex Hausdorff space E, and if μ is a cylindrical measure on E which is concentrated cylindrically (resp. scalarly) up to δ on each set A_ι, then μ is concentrated cylindrically (resp. scalarly) up to δ on $A = \bigcap_\iota A_\iota$. This follows immediately from the well-known [2, § 3, Chap. IX 2nd ed., § 6, $n°9$, Prop. 15] fact that for any continuous map $g: E \to G$ we have $g(A) = \bigcap_{\iota \in I} g(A_\iota)$.

DEFINITION 1. *Let $\mathfrak{S}$ be a collection of subsets of the locally convex Hausdorff space E. A cylindrical measure μ on E is concentrated cylindrically (resp. scalarly) on the sets of $\mathfrak{S}$ if for every $\delta > 0$ there exists a set $A \in \mathfrak{S}$ such that μ is concentrated cylindrically (resp. scalarly) up to δ on A.*

If μ is concentrated cylindrically (resp. scalarly) on the sets of $\mathfrak{S}$, we shall sometimes say briefly that μ is cylindrically (resp. scalarly) $\mathfrak{S}$-concentrated.

Using the terminology of Definition 1 we can say that a cylindrical measure on E defines a probability on E if and only if it is concentrated cylindrically on the collection of compact subsets of E (cf. p. 188).

It is well known that if the sets belonging to $\mathfrak{S}$ are bounded, then the topology of uniform convergence on sets belonging to $\mathfrak{S}$, also called the $\mathfrak{S}$-topology, is compatible with the vector space structure of the dual E' of E [3, Chap. III, § 3, $n°1$, Prop. 1]. A fundamental system of neighborhoods of 0 in E' for the $\mathfrak{S}$-topology is given by the finite intersections of sets of the form λA°, where $A^\circ = \{\xi \mid |\langle x, \xi \rangle| < 1 \text{ for all } x \in A\}$ is the polar of the set $A \in \mathfrak{S}$ and $\lambda \neq 0$. *We shall denote by $E'_{\mathfrak{S}}$ the space E' equipped with the $\mathfrak{S}$-topology.*(1)

(1) We do not assume $\bigcup_{A \in \mathfrak{S}} A$ to be dense in E, therefore $E'_{\mathfrak{S}}$ need not be Hausdorff.

We shall say that $\mathfrak{S}$ *is saturated, if it is invariant by homothecies, increasingly directed* (i.e. *for* $A \in \mathfrak{S}$, $B \in \mathfrak{S}$ *there exists* $C \in \mathfrak{S}$ *with* $(A \cup B) \subset C$), *and if all the sets* $A \in \mathfrak{S}$ *are bounded and balanced.* In this case, the A° form a fundamental system of neighborhoods of the origin in $E'_{\mathfrak{S}}$.

More generally, if F is a topological vector space and $\mathscr{L}(E,F)$ the vector space of all continuous linear maps from E into F, then the topology of uniform convergence on sets belonging to $\mathfrak{S}$ (more briefly: the $\mathfrak{S}$-topology) is compatible with the vector space structure of $\mathscr{L}(E, F)$ [3, Chap. III, §3, n° 1, Corol. of Prop. 1]. A fundamental system of neighborhoods of 0 in $\mathscr{L}(E, F)$ for the $\mathfrak{S}$-topology is given by the finite intersections of sets of the form $W(A, U)$, where $A \in \mathfrak{S}$, U belongs to a fundamental system of neighborhoods of 0 in F, and $W(A, U)$ is the set of all those maps $u \in \mathscr{L}(E, F)$ which map A into U. The vector space $\mathscr{L}(E, F)$ equipped with the $\mathfrak{S}$-topology will be denoted by $\mathscr{L}_{\mathfrak{S}}(E, F)$.

THEOREM 1. *Let* $\mathfrak{S}$ *be a saturated collection of subsets of a locally convex Hausdorff space* E.

Let μ *be a cylindrical measure on* E. *The following conditions are equivalent*:

(i) μ *is scalarly* $\mathfrak{S}$*-concentrated*;

(ii) $\phi = \mathscr{F}(\mu)$ *is continuous on* E' *for the* $\mathfrak{S}$*-topology*;

(ii') $\phi = \mathscr{F}(\mu)$ *is continuous at the origin of* E' *for the* $\mathfrak{S}$ *topology*.

LEMMA 1. *If* μ *is concentrated scalarly up to* δ *on* A, *then* $\phi = \mathscr{F}(\mu)$ *satisfies the inequality*

$$|1 - \varphi(\xi)| \leqslant 2\delta + 2\pi N(\xi),$$

where $N(\xi) = \inf \{\lambda \mid \lambda \geqslant 0, \xi \in \lambda A^\circ\}$ *is the gauge of* A°.

PROOF. For any $\xi \in E'$ the Radon measure $\nu = \xi(\mu)$ on $\mathbf{R}$ is concentrated up to δ on $\xi(A)$. By the definition of $N(\xi)$ the set $\xi(A)$ is contained in the interval $[-N(\xi), N(\xi)]$. Indeed, let $\lambda > N(\xi)$.

Then $\xi \in \lambda A^\circ$, i.e. $|\langle \xi, x \rangle| \leq \lambda$ for every $x \in A$, which proves that $\xi(A) \subset [-\lambda, \lambda]$.

By formula (3) we have

$$\phi(\xi) = \mathscr{F}(\nu)(1) = \int_R e^{-2\pi i t} d\nu(t),$$

hence

$$|1 - \phi(\xi)| \leq \int_R |1 - e^{-2\pi i t}| d\nu(t)$$

$$= \int_{|t| \leq N(\xi)} |1 - e^{-2\pi i t}| d\nu(t) + \int_{|t| \geq N(\xi)} |1 - e^{-2\pi i t}| d\nu(t).$$

By virtue of the inequality $|1 - e^{iu}| \leq |u|$, the first integral is less than $2\pi N(\xi)$, and the second is less than 2δ.

LEMMA 2. *Under the hypotheses of Lemma* 1 *we have* $|1 - \phi(\xi)| \leq k\delta$ *for* $\xi \in \delta A^\circ$, *where* k *is a universal constant.*

PROOF. If $\xi \in \delta A^\circ$, then $N(\xi) \leq \delta$, hence by Lemma 1 we have $|1 - \phi(\xi)| \leq 2(1 + \pi)\delta$.

We can now prove the implication (i) $\Rightarrow$ (ii') in Theorem 1. Let us show that given $\epsilon > 0$ the set $\{\xi \mid |1 - \phi(\xi)| \leq \epsilon\}$ is a neighborhood of the origin in E'. Since $\phi(0) = 1$, this will prove the continuity of ϕ at the origin. Let us choose $\delta = \epsilon/k$. By assumption there exists $A \in \mathfrak{S}$ such that μ is concentrated scalarly up to δ on A. By Lemma 2 we have $|1 - \phi(\xi)| \leq \epsilon$ for $\xi \in \delta A^\circ$, and δA° is a neighborhood of 0 for the $\mathfrak{S}$-topology. Therefore (i) $\Rightarrow$ (ii').

Next we prove that (ii') implies (ii). Given $\epsilon > 0$, there exists a neighborhood V of 0 in E' such that $|1 - \mathrm{Re}\,\phi(\eta)| \leq \epsilon^2/2$ for $\eta \in V$. Let ξ_1 and ξ_2 be two points of E' such that $\xi_2 - \xi_1 \in V$. The subspace L of E' generated by ξ_1 and ξ_2 is locally compact, and by Prop. 2(§2) the restriction of ϕ to L is a continuous function of positive type. We have therefore the classical inequality, valid for any continuous function of positive type on a locally compact group:

$$|\phi(\xi_1) - \phi(\xi_2)|^2 \leq 2\phi(0)\{\phi(0) - \mathrm{Re}\,\phi(\xi_2 - \xi_1)\} \qquad (6)$$

[7, p. 23], from which we obtain $|\phi(\xi_1) - \phi(\xi_2)| < \epsilon$.

It is obvious that (ii) implies (ii'). To prove the implication (ii') $\Rightarrow$ (i) we need two more lemmas.

LEMMA 3. *Let ν be a probability on the real line* $\mathbf{R}$ *and* let $\psi = \mathcal{F}(\nu)$ *be its Fourier transform. If* $|1 - \psi(\tau)| < \delta$ *for* τ *in the interval* $]-1/L,\ 1/L[$, *then* ν *is concentrated up to* $k\delta$ *on the interval* $]-L/\delta,\ L/\delta[$, *where* k *is a universal constant.*

PROOF. We shall use the fact that the Fourier transform of the function $e^{-\pi t^2}$ is the function $e^{-\pi\tau^2}$ and thus the Fourier transform of $e^{-\pi\delta^2 t^2/L^2}$ is $\frac{L}{\delta} e^{-\pi L^2\tau^2/\delta^2}$. By Parseval's theorem we have

$$\int_{\mathbf{R}} e^{-\pi\delta^2 t^2/L^2}\, d\nu(t) = \int_{\mathbf{R}} e^{-\pi L^2\tau^2/\delta^2}\, \frac{L}{\delta}\, \psi(\tau)\, d\tau.$$

Now

$$1 = \int_{\mathbf{R}} d\nu(t) = \int_{\mathbf{R}} e^{-\pi L^2\tau^2/\delta^2}\, \frac{L}{\delta}\, d\tau. \tag{7}$$

If we subtract from this equation the previous one we get

$$\int_{\mathbf{R}} (1 - e^{-\pi\delta^2 t^2/L^2})\, d\nu(t) = \int_{\mathbf{R}} (1 - \psi(\tau))\, e^{-\pi L^2\tau^2/\delta^2}\, \frac{L}{\delta}\, d\tau. \tag{8}$$

The left-hand side of (8) is greater than

$$\int_{|t| \geq L/\delta} (1 - e^{-\pi})\, d\nu(t) = (1 - e^{-\pi})\, \nu(\mathbf{C}]-L/\delta,\ L/\delta[).$$

The right-hand side of (8) is smaller than

$$\int_{|\tau| < 1/L} |1 - \psi(\tau)|\, e^{-\pi L^2\tau^2/\delta^2}\, \frac{L}{\delta}\, d\tau + \int_{|\tau| \geq 1/L} |1 - \psi(\tau)|\, e^{-\pi L^2\tau^2/\delta^2}\, \frac{L}{\delta}\, d\tau$$

$$< \delta + 2 \int_{|\tau| \geq 1/L} e^{-\pi L^2\tau^2/\delta^2}\, \frac{L}{\delta}\, d\tau,$$

where we used the hypothesis, the relation (7) and the inequality $|1-\psi(\tau)| \leqslant 1+|\psi(\tau)| \leqslant 2$. Now there exists a universal constant k_1 such that

$$\int_{|s| \geqslant 1/\delta} e^{-\pi s^2}\, ds \leqslant k_1 \delta$$

for every $\delta > 0$, and therefore the last expression is smaller than $\delta + 2k_1\delta = (1+2k_1)\delta$. Collecting our results we obtain

$$\nu(\mathbf{C}] - L/\delta, L/\delta[) \leqslant \frac{2k_1+1}{1-e^{-\pi}}\,\delta,$$

which proves the lemma.

LEMMA 4. *Let μ be a cylindrical measure on E, let $\phi = \mathscr{F}(\mu)$ and A a balanced subset of E. If $|1-\phi(\xi)| \leqslant \delta$ for $\xi \in A^0$, then μ is concentrated scalarly up to $k\delta$ on $\frac{1}{\delta}A$.*

PROOF. The conclusion of the Lemma means that for each $\xi \in E'$ the Radon measure $\xi(\mu)$ on $\mathbf{R}$ is concentrated up to $k\delta$ on $(1/\delta)\,\xi(A)$. We know that if we set $\psi = \mathscr{F}(\xi(\mu))$ we have $\psi(\tau) = \phi(\tau\xi)$ (p. 187) and thus by assumption

$$|1-\psi(\tau)| \leqslant \delta \text{ for } \tau\xi \in A^0.$$

Now $\tau\xi \in A^0$ is equivalent to $|\tau|\,N(\xi) \leqslant 1$, where N is the gauge of A^0. It follows from Lemma 3 that $\xi(\mu)$ is concentrated up to $k\delta$ on the interval $]-N(\xi)/\delta, N(\xi)/\delta[$.

By the definition of the gauge, $N(\xi)$ is the greatest lower bound of all $\lambda > 0$ such that $\xi(A) \subset [-\lambda, \lambda]$. Since A is balanced, $\xi(A)$ is an interval of $\mathbf{R}$, symmetric with respect to the origin and so

$$]-N(\xi), N(\xi)[\;\subset\; \xi(A) \;\subset\; [-N(\xi), N(\xi)].$$

Thus $\xi(\mu)$ is a fortiori concentrated up to $k\delta$ on $\frac{1}{\delta}\xi(A)$.

We can now conclude the proof of Theorem 1 by proving (ii') $\Rightarrow$ (i). Assume that ϕ is continuous at 0 on E' equipped with the $\mathfrak{S}$-topology. Given $\epsilon > 0$, let $\delta = \epsilon/k$, where k is the universal constant of Lemmas 3 and 4. Since $\mathfrak{S}$ is increasingly directed and

invariant with respect to homothecies, a fundamental system of neighbourhoods of 0 in E' is given by the sets A^0, where A varies in $\mathfrak{S}$. Thus there exists $A \in \mathfrak{S}$ such that $|1 - \phi(\xi)| \leqslant \delta$ for $\xi \in A^0$. But then by Lemma 4 the cylindrical measure μ is concentrated scalarly up to $k\delta = \epsilon$ on $(1/\delta)A$. Since $(1/\delta)A \in \mathfrak{S}$, the theorem is proved.

THEOREM 2. *Let μ be a cylindrical measure on the locally convex Hausdorff space E, and let $\mathfrak{S}$ be a saturated collection of subsets of E.*

If μ is scalarly $\mathfrak{S}$-concentrated, then for every finite-dimensional Hausdorff topological vector space G the map $u \longmapsto u(\mu)$ is continuous from $\mathscr{L}_{\mathfrak{S}}(E, G)$ into $\mathscr{M}^1(G)$ always equipped with the vague topology, (i.e. the topology of pointwise convergence on the space of continuous functions with compact support [4, Chap. III, 2nd ed., §I, n° 9]). *Conversely, if the map $\xi \longmapsto \xi(\mu)$ from $E'_{\mathfrak{S}}$ into $\mathscr{M}^1(\mathbf{R})$ is continuous at the origin, then μ is scalarly $\mathfrak{S}$-concentrated.*

COROLLARY. *The three following properties are equivalent*:

(i) *μ is scalarly $\mathfrak{S}$-concentrated;*

(ii) *$\xi \longmapsto \xi(\mu)$ is continuous from $E'_{\mathfrak{S}}$ into $\mathscr{M}^1(\mathbf{R})$;*

(ii') *$\xi \longmapsto \xi(\mu)$ is continuous at the origin.*

The corollary is an immediate consequence of the theorem.

PROOF. We shall invoke a classical theorem due to Paul Lévy according to which a probability ν on a locally compact abelian group converges vaguely to the probability ν_0 if and only if $\mathscr{F}(\nu)$ converges to $\mathscr{F}(\nu_0)$ uniformly on every compact set (cf. Appendix, § 4)

Assume that μ is scalarly $\mathfrak{S}$-concentrated, and let $e_1, \ldots, e_n$ be a basis of G, which identifies it with $\mathbf{R}^n$. Then every $u \in \mathscr{L}(E, G)$ is determined according to the formula $u(x) = \sum_{j=1}^{n} \xi_j(x) e_j$ by n linear forms $\xi_1, \ldots, \xi_n \in E'$. Suppose that the linear maps u_k converge to u in $\mathscr{L}_{\mathfrak{S}}(E, G)$, and that the u_k are determined by $u_k(x) = \sum_{j=1}^{n} \xi_{kj}(x) e_j$,

where $\xi_{kj} \in E'$. Then for each j $(1 < j < n)$ ξ_{kj} converges to ξ_j in $E'_{\mathfrak{S}}$, and therefore $t_1 \xi_{k1} + t_2 \xi_{k2} + \ldots + t_n \xi_{kn}$ converges to $t_1 \xi_1 + t_2 \xi_2 + \ldots + t_n \xi_n$ in $E'_{\mathfrak{S}}$, uniformly if the t_j remain bounded. It follows from our assumption, from Theorem 1 and from formula (6) that $\mathscr{F}(\mu)$ is uniformly continuous on $E'_{\mathfrak{S}}$, hence $\mathscr{F}(\mu)$ $(t_1 \xi_{k_1} + t_2 \xi_{k_2} + \ldots + t_n \xi_{k_n})$ converges to $\mathscr{F}(\mu)$ $(t_1 \xi_1 + t_2 \xi_2 + \ldots + t_n \xi_n)$, uniformly if the t_j remain bounded. Now observe that the map $(t_1, \ldots, t_n) \longmapsto \sum_{j=1}^{n} t_j \xi_j$ from $\mathbf{R}^n$ into E' is the transpose ${}^t u$ of the map $u : x \longmapsto \Sigma\, \xi_j(x)\, e_j$, hence by formula (4) we have

$$\mathscr{F}(\mu)\,(t_1 \xi_1 + t_2 \xi_2 + \ldots + t_n \xi_n) = \mathscr{F}(u(\mu))\,(t_1, t_2, \ldots, t_n),$$

and similarly

$$\mathscr{F}(\mu)\,(t_1 \xi_{k_1} + t_2 \xi_{k_2} + \ldots + t_n \xi_{k_n}) = \mathscr{F}(u_k(\mu))\,(t_1, t_2, \ldots, t_n).$$

Thus $\mathscr{F}(u_k(\mu))$ converges to $\mathscr{F}(u(\mu))$, uniformly on every compact subset of $\mathbf{R}^n$, hence by Paul Lévy's theorem $u_k(\mu)$ converges to $u(\mu)$ vaguely on G.

Assume now that the map $\xi \longmapsto \xi(\mu)$ from $E'_{\mathfrak{S}}$ into $\mathscr{M}^1(\mathbf{R})$ is continuous at the origin. Then, if ξ converges to 0 in $E'_{\mathfrak{S}}$, the probability $\xi(\mu)$ converges to δ vaguely on $\mathbf{R}$, and therefore by Paul Lévy's theorem $\mathscr{F}(\mu)\,(\xi) = \mathscr{F}(\xi(\mu))\,(1)$ converges to 1, i.e. the function $\mathscr{F}(\mu)$ is continuous on $E'_{\mathfrak{S}}$ at the origin. It follows from Theorem 1 that μ is scalarly $\mathfrak{S}$-concentrated.

In Chap. 5, §2, Theorem 2, we shall see a further necessary and sufficient condition for a cylindrical measure to be scalarly $\mathfrak{S}$-concentrated, expressed in terms of the continuity of a certain random function.

The following consequence of Theorem 2 will be used later.

THEOREM 3. *Let E and F be two locally convex Hausdorff spaces, $j : F \to E$ a weakly continuous, injective, linear map, and $\mathfrak{S}$ a saturated collection of weakly compact not necessarily convex subsets of F. We denote by ${}^t j : E' \to F'$ the transpose of j, and assume that ${}^t j(E')$ is dense in $F'_{\mathfrak{S}}$. Let μ be a scalarly $\mathfrak{S}$-concentrated cylindrical measure on F. Assume that for $D \in \mathfrak{S}$ and $\epsilon > 0$ the measure $j(\mu)$ is concentrated*

cylindrically upto ϵ *on* $j(D)$. *Then* μ *is concentrated cylindrically upto* ϵ *on* D.

PROOF. Let G be a finite-dimensional Hausdorff vector space and $g : F \to G$ a continuous linear map. We have to show that $g(\mu)(g(D)) \geq 1 - \epsilon$. Let us observe that if j is a strict morphism, i.e. if we can consider F as a subspace of E with the induced topology, then the conclusion is trivial. Indeed, in this case by the Hahn-Banach theorem there exists a continuous linear map $g' : E \to G$ such that $g = g' \circ j$. Since by hypothesis $g'(j(\mu))\ (g'(j(D))) \geq 1 - \epsilon$, and on the other hand $g'(j(\mu))\ (g'(j(D))) = g(\mu)(g(D))$, the conclusion follows.

In the general case we use the fact that ${}^t j(E')$ consists exactly of those continuous linear forms ξ on F for which there exists a continuous linear form ξ' on E such that $\xi = \xi' \circ j = {}^t j(\xi')$. Since by our assumption ${}^t j(E')$ is dense in $F'_{\mathfrak{S}}$ and since G is finite-dimensional, g can be approximated, uniformly on every set belonging to $\mathfrak{S}$, by continuous linear maps $g_k : F \to G$ for which there exist corresponding continuous linear maps $g'_k : E \to G$ such that $g_k = g'_k \circ j$. The observation made at the beginning of the proof shows that $g_k(\mu)\ (g_k(D)) \geq 1 - \epsilon$ for every such g_k.

Now D is weakly compact and g continuous, hence also weakly continuous, hence $g(D)$ is a compact subset of G. Let U and V be two relatively compact open neighbourhoods of $g(D)$ such that $V \supset \overline{U} \supset g(D)$, and let φ be a continuous function on G such that $\operatorname{Supp} \varphi \subset V$, $\varphi = 1$ on U and $0 \leq \varphi \leq 1$ everywhere. Since g_k tends to g uniformly on D, we have ultimately $g_k(D) \subset U$ and therefore

$$g_k(\mu)(\varphi) \geq g_k(\mu)\ (U) \geq g_k(\mu)\ (g_k(D)) \geq 1 - \epsilon.$$

By Theorem 2 the measures $g_k(\mu)$ converge to $g(\mu)$ for the vague topology on $\mathscr{M}^1(G)$, hence we have also $g(\mu)\ (\varphi) \geq 1 - \epsilon$, and a fortiori $g(\mu)\ (V) \geq 1 - \epsilon$.

The compact set $g(D)$ is the intersection of countably many of its relatively compact open neighbourhoods V, and for each such neighbourhood V there exists an open neighbourhood U of $g(D)$ such

that $V \supset \bar{U} \supset g(D)$. As we have just seen, we have $g(\mu)(V) \geqslant 1 - \epsilon$ for every such V, hence $g(\mu)(g(D)) \geqslant 1 - \epsilon$.

COROLLARY 1. *Let E and F be two locally convex Hausdorff spaces, $j : F \to E$ a continuous, injective, linear map, and $\mathfrak{S}$ a saturated collection of weakly compact subsets of F. We assume that ${}^t j(E')$ is dense in $F'_{\mathfrak{S}}$. Let μ be a scalarly $\mathfrak{S}$-concentrated cylindrical measure on F. If $j(\mu)$ is concentrated cylindrically on the sets of $j(\mathfrak{S})$, then μ is concentrated cylindrically on the sets of $\mathfrak{S}$.*

Now we assume the sets of $\mathfrak{S}$ to be convex.

COROLLARY 2. *Let E and F be two locally convex Hausdorff spaces, $j: F \to E$ a continuous, injective, linear map and μ a cylindrical measure on F which is scalarly concentrated on the balanced, convex, weakly compact subsets of F. Let D be a balanced, convex, weakly compact subset of F and $\epsilon > 0$. If $j(\mu)$ is concentrated cylindrically up to ϵ on $j(D)$, then μ is concentrated cylindrically up to ϵ on D.*

PROOF. Denote by $\mathfrak{S}$ the collection of all balanced, convex, weakly compact subsets of F. The $\mathfrak{S}$-topology on F' is then the Mackey topology $\tau(F', F)$ [3, Chap. IV, § 2, $n°$ 3, Corol. of Th. 2]. Since j is injective, ${}^t j(E')$ is dense in F' for the weak* topology $\sigma(F', F)$ [3, Chap. II, 2nd ed., § 6, $n°$ 4, Corol. 2 of Prop. 6] and therefore also for the Mackey topology [3, Chap. IV § 2, $n°$ 3, Prop. 4]. The corollary follows therefore from Theorem 3.

COROLLARY 3. *Let E and F be two locally convex Hausdorff spaces, $j: F \to E$ a continuous, injective, linear map, μ a cylindrical measure on F which is scalarly concentrated on the balanced, convex, weakly compact subsets of F, and $\mathfrak{S}$ a collection of balanced, convex weakly compact subsets of F. If $j(\mu)$ is concentrated cylindrically on the sets of $j(\mathfrak{S})$, then μ is concentrated cylindrically on the sets of $\mathfrak{S}$.*

Another consequence of Theorem 2 is the following.

PROPOSITION 6. *Let E and F be two locally convex Hausdorff spaces, $j: F \to E$ a continuous, injective, linear map and $\mathfrak{S}$ a saturated collection of subsets of F. Assume that ${}^t j(E')$ is dense in $F'_{\mathfrak{S}}$. Let*

μ and ν be two cylindrical measures on F which are both scalarly concentrated on the sets of $\mathfrak{S}$, and assume that $j(\mu) = j(\nu)$. Then $\mu = \nu$.

PROOF. Let G be a finite-dimensional Hausdorff space and $g: F \to G$ a continuous linear map. It follows from the assumption that ${}^t j(E')$ is dense in $F'_{\mathfrak{S}}$, as in the proof of Theorem 3, that g can be approximated, uniformly on every set belonging to $\mathfrak{S}$, by continuous linear maps $g_k: F \to G$ for which there exist corresponding continuous linear maps $g'_k: E \to G$ such that $g_k = g'_k \circ j$. On the other hand it follows from theorem 2 that $g_k(\mu)$ tends vaguely to $g(\mu)$ and $g_k(\nu)$ tends vaguely to $g(\nu)$. Since by hypothesis $g_k(\mu) = g'_k(j(\mu)) = g'_k(j(\nu)) = g_k(\nu)$, we have $g(\mu) = g(\nu)$, and, since the map g is arbitrary, we obtain finally $\mu = \nu$.

Exactly the same way as we deduced from Theorem 3 its Corollary 2, we obtain from Proposition 6 the

COROLLARY. *Let E and F be two locally convex Hausdorff spaces and $j: F \to E$ a continuous, injective, linear map. If μ and ν are two cylindrical measures on F, both scalarly concentrated on the balanced, convex, weakly compact subsets of F, and if $j(\mu) = j(\nu)$, then $\mu = \nu$.*

EXAMPLE 3. Let us show that in Corollary 2 of Theorem 3 and in the Corollary of Prop. 6 the condition that the cylindrical measures involved are scalarly concentrated on the balanced convex, weakly compact subsets, is not superfluous. Let F be a locally convex Hausdorff space and E the underlying vector space of F equipped with a coarser topology so that the dual of E is smaller than that of F. We can take for instance for F the space l^1 equipped with the normed topology, and for E the same vector space equipped with the topology $\sigma(l^1, c_0)$ (*cf.* example 2).

Let a be a non-zero linear form on F' which vanishes on E'. Then a is an element of F'^* whose image under the canonical map $F'^* \to E'^*$ is zero. The Dirac measure δ_a on F'^* defines a cylindrical measure μ on F, and since the image measure of δ_a on E'^* is δ_0, also the image of μ on E is δ_0. Thus we have our counterexamples:

(a) δ_0 is concentrated cylindrically on the subset $\{0\}$ of E, while μ is not;

(b) both cylindrical measures δ_0 and μ on F have the same image δ_0 on E but they are not equal.

REMARK. There are, however, other cases in which $j(\mu)=j(\nu)$ implies $\mu=\nu$, *without hypotheses of concentration. It is always true if j is a strict morphism for the weakened topologies.* Namely, in this case, every continuous linear map from F into a finite-dimensional vector space G can be factorized through j and a continuous linear map from E into G, according to Hahn-Banach theorem, which gives the result.

A consequence of Prop. 6 is

THEOREM 4. *Let F be a locally convex Hausdorff space, $\mathfrak{S}$ a saturated collection of subsets of F, and μ,ν two cylindrical measures on F, both scalarly concentrated on the sets of $\mathfrak{S}$. Let I be an ordered, increasingly directed set, and $(G_\iota)_{\iota\in I}$ a family of finite-dimensional Hausdorff vector spaces. Assume that for $\kappa \geqslant \iota$ we have a linear map $w_{\iota\kappa}: G_\kappa \to G_\iota$ such that $w_{\kappa\kappa}$ is the identity map, and $w_{\iota\kappa}\circ w_{\kappa\lambda} = w_{\iota\lambda}$ for $\lambda \geqslant \kappa \geqslant \iota$. For each $\iota\in I$ we suppose given a continuous linear map $g_\iota: F \to G_\iota$ such that $g_\iota = w_{\iota\kappa}\circ g_\kappa$ for $\kappa \geqslant \iota$. We make furthermore the following hypotheses:*

(i) *the maps g_ι separate the points of F, i.e. for any $x\neq 0$ in F there exists an $\iota\in I$ such that $g_\iota(x)\neq 0$;*

(ii) *the union of the sets ${}^t g_\iota(G_\iota')$ is total in $F'_{\mathfrak{S}}$.*

If $g_\iota(\mu) = g_\iota(\nu)$ for every $\iota\in I$, then $\mu=\nu$.

PROOF. Denote by E the underlying vector space of F equipped with the coarsest topology for which all the maps g_ι are continuous. The space E is obviously locally convex and its topology is Hausdorff by (i). The canonical bijection j: $F\to E$ is continuous since the maps $g_\iota: F\to G_\iota$ are continuous. Since the set I is increasingly directed, a linear form ξ on E is continuous if and only if there exists $\iota\in I$ and a linear form η on G_ι such that $\xi = \eta\circ g_\iota = {}^t g_\iota(\eta)$ [3, Chap. IV,

§2, n° 8, Corol. of Prop. 10]. It follows therefore from (ii) that ${}^t j(E')$ is dense in $F'_{\mathfrak{S}}$.

We have to prove that $j(\mu)=j(\nu)$ since it will then follow from Prop. 6 that $\mu=\nu$. Let g be a continuous linear map from E into a finite-dimensional Hausdorff vector space G, and denote by B the unit ball of G. Using again the fact that the G_ι form a directed system we see that there exists, by the definition of the topology of E, an index $\lambda\in I$ and a ball B_λ in G_λ such that $g_\lambda^{-1}(B_\lambda)\subset g^{-1}(B)$. For any $x\in E$ we set $w(g_\lambda(x))=g(x)$ and show that w is a well-defined map from $g_\lambda(E)$ into G. Indeed, if $g_\lambda(x)=g_\lambda(y)$, i.e. $g_\lambda(x-y)=0$, then $x-y\in g_\lambda^{-1}(aB_\lambda)\subset g^{-1}(aB)$ and thus $g(x)-g(y)\in aB$ for every $a>0$, which shows that $g(x)=g(y)$.

We now extend w to all of G_λ so that $w\colon G_\lambda\to G$ shall be a linear map which satisfies $g=w\circ g_\lambda$. Thus we have $g(\mu)=w(g_\lambda(\mu))$ $=w(g_\lambda(\nu))=g(\nu)$. Since this is true for any continuous linear map g from E into a finite-dimensional Hausdorff space G, we have $j(\mu)=j(\nu)$.

COROLLARY. *Let F be a locally convex Hausdorff space, and μ, ν two cylindrical measures on F, both scalarly concentrated on the balanced, convex, weakly compact subsets of F. Let I be an ordered, increasingly directed set, and $(G_\iota)_{\iota\in I}$ a family of finite-dimensional Hausdorff vector spaces. Assume that for $\kappa\geq\iota$ we have a linear map $w_{\iota\kappa}\colon G_\kappa\to G_\iota$ such that $w_{\kappa\kappa}$ is the identity map, and $w_{\iota\kappa}\circ w_{\kappa\lambda}=w_{\iota\lambda}$ for $\lambda\geq\kappa\geq\iota$. For each $\iota\in I$ we suppose given a continuous linear map $g_\iota\colon F\to G_\iota$ such that $g_\iota=w_{\iota\kappa}\circ g_\kappa$ for $\kappa\geq\iota$ and we assume furthermore that the maps g_ι separate the points of F (condition* (i) *of Theorem* 4).

If $g_\iota(\mu)=g_\iota(\nu)$ for every $\iota\in I$, then $\mu=\nu$.

PROOF. Condition (ii) of Theorem 4 was used only to prove that ${}^t j(E')$ is dense in $F'_{\mathfrak{S}}$. Now we observe, as in the proof of Corollary 2 of Theorem 3, that if $\mathfrak{S}$ is the collection of all balanced, convex, weakly compact subsets of F, then the $\mathfrak{S}$-topology on F' is the Mackey topology $\tau(F', F)$. Since j is injective, ${}^t j(E')$ is dense in F' for $\sigma(F', F)$ and therefore also for $\tau(F', F)$.

REMARK 2. It follows from the preceding corollary that if μ is concentrated scalarly on the balanced, convex, weakly compact subsets of F, then it is uniquely determined by its images $g_\iota(\mu)$ with respect to a directed family of continuous linear maps $g_\iota : F \to G_\iota$ separating the points of F, of which we do not have to assume that it is coinitial with the set of all continuous linear maps into finite-dimensional Hausdorff vector spaces (cf. Remark 1, §1).

The following result, which is based on Theorem 3 of Chap. 1, §4, will be used in Chap. 3, §3.

PROPOSITION 7. *Let μ be a cylindrical measure on a reflexive Banach space E, concentrated cylindrically on the balls of E. Then μ is a Radon measure on E, and, in particular, it is concentrated on the strongly compact subsets of E.*

PROOF. Since every ball in E is weakly compact [3, Chap. IV, §5, n° 2, Prop. 6], by Prokhorov's theorem μ is a Radon measure on E equipped with the weak topology. But by Theorem 3 of Chap. 1, §4, the weak topology on E is Radon-equivalent to the strong topology, therefore μ is also a Radon measure on E for the original normed topology.

EXAMPLE 4. The following *erroneous* analogue to Prop. 7 shows that cylindrical measures must be handled with extreme caution.

(*)"A cylindrical measure μ on l^1, which is cylindrically concentrated on the balls of l^1, is a Radon measure on l^1".

The faulty argument runs as follows: denote by F the space l^1 equipped with the normed topology and by E the space l^1 equipped with the weak* topology $\sigma(l^1, c_0)$. By the Bourbaki-Alaoglu theorem [3, Chap. IV, §5, n° 1, Prop. 5] the balls of F are compact in E hence by Prokhorov's theorem μ is a Radon measure on E. But F is a Polish space, hence its topology is Radon-equivalent to every coarser topology, in particular to $\sigma(l^1, c_0)$. Thus μ is also a Radon measure on F.

In view of Example 3 the flaw in the argument is easy to detect. Denote by j the identical bijection $F \to E$. Then $\mu_1 = j(\mu)$ is a

Radon measure on E, hence there exists a Radon measure ν on F such that $\mu_1 = j(\nu)$. But we may not assert that μ is equal to ν.

An example which shows that (*) is false is given by the cylindrical measure μ constructed in Example 2, which is not a Radon measure but is cylindrically concentrated on every ball with radius > 1 of l^1.

If we assume a priori that μ is scalarly concentrated on the weakly compact subsets of l^1, then the above argument becomes correct since by virtue of the corollary of Prop. 6 we can conclude that $\mu = \nu$. Taking into account that by a theorem of Orlicz the weakly compact subsets of l^1 are the compact subsets [3, Chap. IV, §5, exerc. 4 *b*)] we obtain

PROPOSITION 8. *A cylindrical measure on l^1, which is scalarly concentrated on the compact subsets of l^1 and cylindrically concentrated on the balls of l^1, is a Radon measure on l^1.*

SUMMARY OF CHAPTER III

This chapter develops the relationship between Hilbert-Schmidt maps and radonifying maps. Hilbert-Schmidt operators are not assumed to be known, they are defined in § 1, Definition 1, page 208, and their main properties are given; they are essential in what follows. Sazonov's theorem says that a map between Hilbert spaces is radonifying if and only if it is Hilbert-Schmidt; the part "if" is Theorem 1, page 209, the part "only if" will be proved later, page 273. The lemmas used for the proof are useful by themselves. We use here Fourier transform, as it has been the case in all the previous proof of Sazonov's theorem; in the developments of p-radonifying maps, after the writing of this book, there are completely different proofs.

III

CYLINDRICAL MEASURES AND HILBERT-SCHMIDT OPERATORS

1. Hilbert-Schmidt operators. Let E and F be two Hilbert spaces and $u: E \to F$ a continuous linear map. The theorem of Sazonov states that the image $u(\mu)$ of every cylindrical measure μ concentrated scalarly on the balls of E is a Radon measure on F if and only if u is a Hilbert-Schmidt operator. In this chapter we shall prove that the condition is sufficient and we postpone to Chapters V and VI the proof of the necessity. We start, however, with a survey of Hilbert-Schmidt operators.

In what follows, it will be convenient to introduce the anti-space $\overline{E}$ of a vector space E over $\mathbf{C}$. The underlying additive group of $\overline{E}$ is the same as that of E. If we denote by $x \mapsto \bar{x}$ the identical bijection $E \to \overline{E}$, then we define the product of $\alpha \in \mathbf{C}$ and $\bar{x} \in \overline{E}$ by $\alpha . \bar{x} = (\bar{\alpha}\, x)^{-}$. Thus $\overline{E}$ is a vector space over $\mathbf{C}$, and $\overline{\alpha x} = \bar{\alpha} . \bar{x}$, i.e. the map $x \mapsto \bar{x}$ is semi-linear.

Let now E and F be two Hilbert spaces, and let $E \otimes F$ be the tensor product of the underlying vector spaces. If $(e_1 \mid e_2)_E$ is the scalar product on E and $(f_1 \mid f_2)_F$ the scalar product on F, then

$$(e_1, f_1, \bar{e}_2, \bar{f}_2) \longmapsto (e_1 \mid e_2)_E (f_1 \mid f_2)_F$$

is a quadrilinear form on $E \times F \times \overline{E} \times \overline{F}$ and defines a sesquilinear form on $(E \otimes F) \times (E \otimes F)$ for which

$$(e_1 \otimes f_1 \mid e_2 \otimes f_2) = (e_1 \mid e_2)_E (f_1 \mid f_2)_F .$$

If (e_i) and (f_j) are orthonormal bases of E and F, then every element of $E \otimes F$ can be written in the form $\Sigma \lambda_{ij} . e_i \otimes f_j$ and $(e_i \otimes f_j)$ is an orthonormal family in $E \otimes F$. From this it follows immediately that the sesquilinear form on $(E \otimes F) \times (E \otimes F)$, which we have defined, is positive hermitian and non-degenerate, and thus defines a separated prehilbert structure on $E \otimes F$. We shall denote by $E \hat{\otimes}_2 F$ the Hilbert space obtained by completing the prehilbert

space $E \otimes F$, and $E \hat{\otimes}_2 F$ will be called the Hilbert-Schmidt tensor product of E and F.

In a general way, if E and F are for instance two normed spaces, then the elements of $E' \otimes F$ define operators of finite rank from E into F if we associate with the element $\Sigma x_i' \otimes y_i$ the linear map $x \longmapsto \Sigma \langle x, x_i' \rangle y_i$. Now if E is a Hilbert space, then $\bar{E}$ can be identified with E' [3, Chap. V, §1, n^0 6, Th. 3] and thus we have a canonical injection $\bar{E} \otimes F \to \mathscr{L}(E, F)$. This injection will be continuous if we equip $\bar{E} \otimes F$ with the Hilbert-Schmidt norm and the space $\mathscr{L}(E, F)$ of all continuous linear maps $E \to F$ with its usual norm. Indeed, if $u = \Sigma \bar{x}_i \otimes y_i$ is an element of $\bar{E} \otimes F$, which we identify with its image in $\mathscr{L}(E, F)$, then for $\bar{e} \otimes f \in \bar{E} \otimes F$ we have

$$\begin{aligned}(u \mid \bar{e} \otimes f) &= \Sigma (\bar{x}_i \mid \bar{e})_{\bar{E}} (y_i \mid f)_{\hat{F}} \\ &= \Sigma (e \mid x_i)_E (y_i \mid f)_F = (u(e) \mid f)_F. \end{aligned} \tag{1}$$

Now

$$\begin{aligned}\| u \|_{\mathscr{L}(E, F)} &= \sup_{\|e\| \leqslant 1} \| u(e) \| = \sup_{\|e\| \leqslant 1, \|f\| \leqslant 1} | (u(e) \mid f)_F | \\ &= \sup_{\|e\| \leqslant 1, \|f\| \leqslant 1} (u \mid e \otimes f) \leqslant \| u \|_{\bar{E} \otimes F},\end{aligned}$$

since $\| e \|_E \leqslant 1$ and $\| f \|_F \leqslant 1$ imply $\| \bar{e} \otimes f \|_{\bar{E} \otimes F} \leqslant 1$. The injection $\bar{E} \otimes F \to \mathscr{L}(E, F)$ can be extended by continuity into a map $\bar{E} \hat{\otimes}_2 F \to \mathscr{L}(E, F)$ and by a miracle this map is also injective. Indeed, formula (1) can be extended by continuity, and we have

$$(u \mid \bar{e} \otimes f)_{\bar{E} \hat{\otimes}_2 F} = (\tilde{u}(e) \mid f)_F$$

for $u \in \bar{E} \hat{\otimes}_2 F$, where $\tilde{u}$ denotes the image of u in $\mathscr{L}(E, F)$. If $\tilde{u} = 0$, then $(u \mid \bar{e} \otimes f)_{\bar{E} \hat{\otimes}_2 F} = 0$ for every $\bar{e} \otimes f \in \bar{E} \otimes F$, and since the $\bar{e} \otimes f$ are total in $\bar{E} \hat{\otimes}_2 F$, we have $u = 0$.

DEFINITION 1. *An operator $u \in \mathscr{L}(E, F)$ is of Hilbert-Schmidt type if it belongs to the image of $\bar{E} \hat{\otimes}_2 F$ in $\mathscr{L}(E, F)$.*

Every Hilbert-Schmidt operator is compact since it is the limit in $\mathscr{L}(E, F)$ of operators of finite rank. The converse is false.

THEOREM 1. *Let $(e_i)_{i\in I}$ be an orthonormal basis of the Hilbert space E and let $u\in\mathscr{L}(E,F)$. The value $\sum_{i\in I}\|u(e_i)\|^2$ is independent of the choice of the orthonormal basis (e_i) and it is finite if and only if u is Hilbert-Schmidt. If this is the case, then for the Hilbert-Schmidt norm $\|u\|_2=\|u\|_{\bar{E}\hat{\otimes}_2 F}$ we have*

$$\|u\|_2^2=\sum_{i\in I}\|u(e_i)\|^2.$$

PROOF. Let us first assume that u is a Hilbert-Schmidt operator. Let $(e_i)_{i\in I}$ be an orthonormal basis of E and $(f_j)_{j\in J}$ an orthonormal basis of F. Then $(\bar{e}_i\otimes f_j)$ is an orthonormal basis of $\bar{E}\hat{\otimes}_2 F$ and we have

$$\|u\|_2^2=\sum_{i,j}|(u\,|\,\bar{e}_i\otimes f_j)|^2=\sum_{i,j}|(u(e_i)\,|\,f_j)_F|^2=\sum_i\|u(e_i)\|^2. \tag{2}$$

In particular, this last expression is finite and it is independent of the particular choice of the orthonormal basis (e_i).

Conversely it is sufficient to prove that if the sum $\sum_i\|u(e_i)\|^2$ is finite for *one* orthonormal basis, then u is Hilbert-Schmidt, since then, by the first part of the proof, it will have the same value for any other orthonormal basis. If $\sum_i\|u(e_i)\|^2<\infty$, then there exists a countable subset $N\subset I$ such that $u(e_i)=0$ for $i\notin N$. Let us identify N with $\mathbf{N}$, and let u_n be the operator defined by $u_n(e_i)=u(e_i)$ if $i\in N$ and $i<n$, $u_n(e_i)=0$ if $i\in N$, $i>n$ or $i\notin N$. Each operator u_n is of finite rank and a fortiori Hilbert-Schmidt. (u_n) is a Cauchy sequence in $\bar{E}\otimes F$ since for $n<m$ we have

$$\|u_m-u_n\|_2^2=\sum_{i=n+1}^{m}\|u(e_i)\|^2$$

and by hypothesis this sum is arbitrarily small for sufficiently large n. Thus u_n converges to some $v\in\bar{E}\hat{\otimes}_2 F$ in $\bar{E}\hat{\otimes}_2 F$ and therefore also in $\mathscr{L}(E,F)$. But $u(e_i)=v(e_i)$ for each $i\in I$, hence $u=v$ and thus u is Hilbert-Schmidt.

PROPOSITION 1. *If $u\in\mathscr{L}(E,F)$ is a Hilbert-Schmidt operator, then its adjoint $u^*\in\mathscr{L}(F,E)$ is also a Hilbert Schmidt operator and*

we have $\|u^*\|_2 = \|u\|_2$. *If* u *is defined by the element* $\Sigma \bar{x}_k \otimes y_k$ *of* $\overline{E} \hat{\otimes}_2 F$, *then* u^* *is defined by the element* $\Sigma \bar{y}_k \otimes x_k$ *of* $\overline{F} \hat{\otimes}_2 E$.

PROOF. If $\bar{u} \in \mathscr{L}(E,F)$ is the Hilbert-Schmidt operator defined by $\Sigma \bar{y}_k \otimes x_k \in \overline{F} \hat{\otimes}_2 E$, then for $e \in E$ and $f \in F$ we have

$$(e \mid \bar{u}(f))_E = \overline{(\bar{u}(f) \mid e)_E} = \overline{\sum_k (f \mid y_k)_F (x_k \mid e)_E}$$

$$= \Sigma (e \mid x_k)_E (y_k \mid f)_F = (u(e) \mid f)_F.$$

Thus $\bar{u} = u^*$ and u^* is a Hilbert-Schmidt operator.

If $(e_i)_{i \in I}$ is an orthonormal basis in E and $(f_j)_{j \in J}$ an orthonormal basis in F, then we have (*cf.* (2))

$$\| u^* \|_2^2 = \sum_{i,j} | (u^*(f_j) \mid e_i)_E |^2 = \sum_{i,j} | (u(e_i) \mid f_j)_F |^2 = \| u \|_2^2.$$

PROPOSITION 2. *Let* u *be a Hilbert-Schmidt operator,* v *and* w *two continuous linear maps. Then* $v \circ u \circ w$ *is a Hilbert-Schmidt operator and*

$$\| v \circ u \circ w \|_2 \leqslant \| v \| \cdot \| u \|_2 \cdot \| w \|.$$

PROOF. Let (e_i) be a family of vectors such that $\Sigma \| u(e_i) \|^2$ converges. Then

$$\Sigma \| v(u(e_i)) \|^2 \leqslant \| v \|^2 \Sigma \| u(e_i) \|^2,$$

hence it follows from Theorem 1 that $v \circ u$ is Hilbert-Schmidt and $\| v \circ u \|_2 \leqslant \| v \| \cdot \| u \|_2$.

By Prop. 1 the operator u^* is Hilbert-Schmidt and $\| u^* \|_2 = \| u \|_2$. By the first part of the proof $w^* \circ u^*$ is a Hilbert-Schmidt operator and $\| w^* \circ u^* \|_2 \leqslant \| w^* \| \cdot \| u^* \|_2 = \| u \|_2 \cdot \| w \|$. But $(w^* \circ u^*)^* = (u \circ w)^{**} = u \circ w$ and thus, again by Prop. 1, the operator $u \circ w$ is Hilbert-Schmidt and $\| u \circ w \|_2 = \| w^* \circ u^* \|_2$.

We have already mentioned that a Hilbert-Schmidt operator is compact. It is well known that a *normal* operator $u \in \mathscr{L}(E) = \mathscr{L}(E, E)$ is compact if and only if E is the Hilbert sum of a sequence of eigenspaces (E_n) corresponding to a sequence (λ_n) of eigenvalues which tends to zero, and the subspaces E_n, except

possibly that which corresponds to the eigenvalue zero, are finite dimensional.

PROPOSITION 3. *A normal operator $u \in \mathscr{L}(E)$ is Hilbert-Schmidt if and only if E is the Hilbert sum of eigenspaces E_n, and the eigenvalues λ_n belonging to the E_n are such that $\sum \dim E_n \cdot |\lambda_n|^2$ converges. In particular the E_n, with the possible exception of the eigenspace belonging to the eigenvalue zero, are finite-dimensional. If u is a Hilbert-Schmidt operator, then*

$$\|u\|_2 = (\sum \dim E_n \cdot |\lambda_n|^2)^{1/2}. \tag{3}$$

PROOF. Assume that $\sum \dim E_n \cdot |\lambda_n|^2 < \infty$ and choose in each E_n an orthonormal basis (e_i) of E such that $u(e_i) = \lambda_i e_i$, and (λ_i) is the sequence of all eigenvalues, each counted as often as its multiplicity $\dim E_n$ indicates. The sum

$$\sum \|u(e_i)\|^2 = \sum \|\lambda_i e_i\|^2 = \sum |\lambda_i|^2 \tag{4}$$

is finite, hence by Theorem 1 the operator u is Hilbert-Schmidt and (3) holds.

Conversely, if u is a Hilbert-Schmidt operator, then in particular it is compact and therefore E is the Hilbert sum of eigenspaces E_n belonging to eigenvalues λ_n. Choosing an orthonormal basis (e_i) as in the first part of the proof, we obtain from (4) that $\sum |\lambda_i|^2$ converges.

PROPOSITION 4. *Let $u: E \to F$ be a Hilbert-Schmidt operator and assume that $u(E)$ is contained in a closed subspace F_1 of F. Then u, considered as a map from E into F_1, is a Hilbert-Schmidt operator.*

Indeed, $u: E \to F_1$ can be factored into

$$E \xrightarrow{u} F \xrightarrow{p} F_1,$$

where p is an orthogonal projector and $\|p\| \leqslant 1$. The proposition follows from Prop. 2. One could also consider the sum $\|u\|_2^2 = \sum \|u(e_i)\|^2$, where (e_i) is an orthonormal basis of E, and apply Theorem 1.

It is obvious that if $u: E \to F_1$ is a Hilbert-Schmidt operator, and $F_1 \subset F$, then u is a Hilbert-Schmidt operator from E into F.

PROPOSITION 5. *Let $u:E \to F$ be a Hilbert-Schmidt operator and let N be the kernel of u. Then the associated injection $\bar{u} : E/N \to F$ is a Hilbert-Schmidt operator and $\| \bar{u} \|_2 = \| u \|_2$.*

PROOF. The map $\bar{u}: E/N \to F$ can be factored into $E/N \to N^\perp \to E \to F$, where the first two maps are isometries. By Prop. 2 the map $\bar{u}$ is a Hilbert-Schmidt operator and $\| \bar{u} \|_2 \leqslant \| u \|_2$. On the other hand, u can be factored into $E \xrightarrow{\varphi} E/N \xrightarrow{\bar{u}} F$. Thus $\| u \|_2 \leqslant \| \bar{u} \|_2$ since for the canonical surjection φ we have $\| \varphi \| \leqslant 1$.

2 The main lemma. The proof of Sazonov's theorem in the next section will depend essentially on the following elementary lemma whose proof is similar to that of Lemma 3 of Chap. 2, § 3.

LEMMA 1. *Let μ be a probability on $\mathbf{R}^n$, scalarly concentrated up to $\eta > 0$ on the unit ball $B_1 = \{x \mid \sum x_i^2 \leqslant 1\}$. Then μ is concentrated up to ϵ on $B_2 = \{x \mid \sum \alpha_i^2 x_i^2 < 1\}$, where $\sum \alpha_i^2 = S^2$, $\epsilon = C \max(\eta, S)$, and C is universal constant.*

PROOF. The Fourier transform of $e^{-\pi \sum \alpha_i^2 x_i^2}$ is the function

$$\frac{1}{\alpha_1 \alpha_2 \dots \alpha_n} e^{-\pi \sum \xi_i^2/\alpha_i^2}$$

(p. 195). By Parseval's theorem we have

$$\int e^{-\pi \sum \alpha_i^2 x_i^2} d\mu(x) = \int e^{-\pi \sum \xi_i^2/\alpha_i^2} \varphi(\xi) \frac{d\xi}{\alpha_1 \alpha_2 \dots \alpha_n}, \tag{5}$$

where $\varphi = \mathscr{F}(\mu)$. Now $\int d\mu(x) = 1$ and

$$\int e^{-\pi \sum \xi_i^2/\alpha_i^2} \frac{d\xi}{\alpha_1 \alpha_2 \dots \alpha_n} = \prod_{i=1}^{n} \int e^{-\pi \xi_i^2/\alpha_i^2} \frac{d\xi_i}{\alpha_i} = 1, \tag{6}$$

therefore if we subtract both sides of (5) from 1, we get

$$\int (1 - e^{-\pi \sum \alpha_i^2 x_i^2}) \, d\mu(x) = \int (1 - \varphi(\xi)) \, e^{-\pi \sum \xi_i^2/\alpha_i^2} \frac{d\xi}{\alpha_1 \alpha_2 \dots \alpha_n}. \tag{7}$$

The left-hand side of (7) is greater than

$$\int_{\mathbf{C}B_2} (1 - e^{-\pi \sum \alpha_i^2 x_i^2}) \, d\mu(x) \geqslant (1 - e^{-\pi}) \, \mu(\mathbf{C}\, B_2). \tag{8}$$

By Lemma 1 of Chap. 2, § 3 we have

$$|1 - \varphi(\xi)| \leq 2\eta + 2\pi N(\xi), \tag{9}$$

where N is the gauge of the unit ball B_1^0, i.e. $N(\xi) = (\Sigma \xi_i^2)^{1/2}$. It follows from the inequality between the arithmetic and the geometric mean that

$$\eta N(\xi) \leq \tfrac{1}{2}(\eta^2 + N(\xi)^2),$$

or

$$N(\xi) \leq \tfrac{1}{2}\left(\eta + \frac{N(\xi)^2}{\eta}\right),$$

and therefore (9) yields

$$|1 - \varphi(\xi)| \leq C_1\left(\eta + \frac{\Sigma \xi_i^2}{\eta}\right),$$

where we can take for instance $C_1 = 2 + \pi$. The right-hand side of (7) is thus smaller than

$$C_1\eta \int e^{-\pi\Sigma\xi_i^2/\alpha_i^2} \frac{d\xi}{\alpha_1 \dots \alpha_n} + \frac{C_1}{\eta}\int \sum_{i=1}^{n} \xi_i^2 e^{-\pi\Sigma\xi_i^2/\alpha_i^2} \frac{d\xi}{\alpha_1 \dots \alpha_n}$$

$$\leq C_2\left(\eta + \frac{\Sigma \alpha_i^2}{\eta}\right) = C_2\left(\eta + \frac{S^2}{\eta}\right),$$

where we have used (6) and

$$\int \sum_{i=1}^{n} \xi_i^2 e^{-\pi\Sigma\xi_i^2/\alpha_i^2} \frac{d\xi}{\alpha_1\alpha_2 \dots \alpha_n}$$

$$= \sum_{i=1}^{n} \alpha_i^2 \left(\int \frac{\xi_i^2}{\alpha_i^2} e^{-\pi\xi_i^2/\alpha_i^2} \frac{d\xi_i}{\alpha_i}\right)\left(\prod_{j\neq i} \int e^{-\pi\xi_j^2/\alpha_j^2} \frac{d\xi_j}{\alpha_j}\right)$$

$$= \sum_{i=1}^{n} \alpha_i^2 \int e^{-\pi\tau^2}\tau^2 d\tau = C_3 \sum_{i=1}^{n} \alpha_i^2.$$

Collecting our results we obtain

$$\mu(\mathbf{C}B_2) \leq \frac{C_2}{1 - e^{-\pi}}\left(\eta + \frac{S^2}{\eta}\right).$$

If $\eta > S$, then $\frac{S^2}{\eta} < S$, and $\eta + \frac{S^2}{\eta} < 2\eta$. If $\eta \leq S$, then μ is a fortiori scalarly concentrated up to S on the unit ball B_1, and we have the

previous inequality for $\mu(\mathbf{C}B_2)$, with $\eta + \frac{S^2}{\eta}$ replaced by $S + \frac{S^2}{S} = 2S$.

In all cases, we can replace $\eta + \frac{S^2}{\eta}$ by $2 \max(\eta, S)$, which proves the lemma.

3. The theorem of Sazonov. Consider in the space $\mathbf{R}^n$ the unit ball $B_1 = \{x \mid \sum x_i^2 \leqslant 1\}$ and the ellipsoid $B_2 = \{x \mid \sum \alpha_i^2 x_i^2 < 1\}$. Denote by $\mathbf{R}^n_{B_1}$ the space $\mathbf{R}^n$ equipped with its usual norm $\|x\|_{B_1} = (\sum x_i^2)^{1/2}$, which is the gauge of the set B_1, and denote by $\mathbf{R}^n_{B_2}$ the space $\mathbf{R}^n$ with the gauge $\|x\|_{B_2} = (\sum \alpha_i^2 x_i^2)^{1/2}$ of B_2 as norm. Then the Hilbert-Schmidt norm of the identity map $I: x \longmapsto x$ from $\mathbf{R}^n_{B_1}$ onto $\mathbf{R}^n_{B_2}$ is $S = (\sum \alpha_i^2)^{1/2}$. Indeed, if $e_1, \ldots, e_n$ is the canonical basis of $\mathbf{R}^n_{B_1}$, we have

$$\sum_{i=1}^{n} \|I(e_i)\|^2_{B_2} = \sum_{i=1}^{n} \alpha_i^2 = S^2.$$

These considerations and Lemma 1 give the following, more intrinsic result.

LEMMA 2. *Let E and F be two Hilbert spaces having the same finite dimension and let $u: E \to F$ be a linear bijection. If μ is a probability on E which is concentrated scalarly up to $\eta > 0$ on the unit ball of E, then $u(\mu)$ is concentrated up to $\epsilon = C \max(\eta, S)$ on the unit ball of F, where $S = \|u\|_2$ and C is a universal constant.*

PROOF. Let us denote by $\tilde{F}$ the Hilbert space obtained by transporting the Hilbert space structure of E onto F with the help of the bijection u. Then u can be factored into

$$E \xrightarrow{v} \tilde{F} \xrightarrow{I} F,$$

where v is a Hilbert space isomorphism and I is the identity bijection $x \longmapsto x$. There exists an orthonormal basis (e_i) of $\tilde{F}$ such that the vectors e_i are orthogonal for the scalar product in F [1, Chap. IX, §7, n°3, Prop. 6 *a*]. If we set $\alpha_i = \|e_i\|_F$ and $x = \sum x_i e_i$, then we have, of course, $\|x\|^2_{\tilde{F}} = \sum x_i^2$ and

$$\|x\|_F^2 = \left(\sum x_i e_i \,\middle|\, \sum x_j e_j\right)_F = \sum x_i^2 \|e_i\|_F^2 = \sum \alpha_i^2 x_i^2.$$

Thus the basis (e_i) defines Hilbert space isomorphisms between the spaces $\tilde{F}$ and $\mathbf{R}^n_{B_1}$ and the spaces F and $\mathbf{R}^n_{B_2}$. The assertion now follows immediately from Lemma 1 and the considerations at the beginning of this section.

LEMMA 3. *Let E and F be two Hilbert spaces, $u: E \to F$ a Hilbert-Schmidt operator, $S = \|u\|_2$ and μ a cylindrical measure on E, concentrated scalarly up to η on the unit ball of E. Then $u(\mu)$ is concentrated cylindrically up to $\epsilon = C \max(\eta, S)$ on the unit ball of F.*

PROOF. Let G be a finite-dimensional vector space equipped with the Hausdorff topology and $g: F \to G$ a continuous linear map. We have to prove (cf. p. 188) that $g(u(\mu))\,(g(B_2)) > 1 - \epsilon$, where B_2 is the unit ball of F.

Let E_1 be the orthogonal complement of the kernel of $g \circ u: E \to G$. The map $g \circ u$ can be factored into

$$E \xrightarrow{p} E_1 \xrightarrow{u/E_1} F \xrightarrow{g} G,$$

where p is the orthogonal projector onto E_1. Now $\|p\| = 1$ and the probability $p(\mu)$ is concentrated scalarly up to η on the unit ball of E_1. On the other hand $u \mid E_1$ maps E_1 bijectively onto the image $u(E) = F_2 \subset F$. It follows from Propositions 4 and 5 that $u \mid E_1: E_1 \to F_2$ is a Hilbert-Schmidt operator and $\|u \mid E_1\|_2 = \|u\|_2 = S$. Lemma 2 shows that the image of $p(\mu)$ under $u \mid E_1$ is concentrated up to ϵ on the unit ball of F_2, and this gives the desired conclusion since $g(B_2)$ clearly contains the image of the unit ball of F_2.

THEOREM 2 (Sazonov)†. *Let E and F be two Hilbert spaces, $u: E \to F$ a Hilbert-Schmidt operator and μ a cylindrical measure on E concentrated scalarly on the balls of E. Under these conditions $u(\mu)$ is a Radon measure on F.*

PROOF. Let $\epsilon > 0$ and set $\eta = S = \epsilon/C$. Then the expression $C \max(\eta, S)$ of the preceding lemmas is equal to ϵ. There exists a

† See bibliographical index (21).

ball B_1 in E such that μ is scalarly concentrated up to η on B_1. Denote by E_{B_1} the vector space E equipped with the gauge of B_1 as norm. On the other hand let F_{B_2} be the vector space F equipped with the gauge of the ball $B_2 \subset F$ as norm, where we choose B_2 so that the Hilbert-Schmidt norm of $u: E_{B_1} \to F_{B_2}$ shall be $\leqslant S$. This can be achieved by taking the radius of B_2 sufficiently large. It follows then from Lemma 3 that the cylindrical measure $u(\mu)$ is concentrated cylindrically up to ϵ on B_2.

Thus $u(\mu)$ is concentrated cylindrically on the balls of F, and therefore by Prop. 7 of Chap. 2, §3 it is a Radon measure on F.

REMARK. For a fixed S the expression $C\max(\eta, S)$ which figures in the estimates of the preceding lemmas and which will figure again in Chap. 4, §3 tends to CS if η tends to 0. Now this estimate, so useful to prove Theorem 2, is in some sense quite weak. To see this, consider a cylindrical measure μ on a Hilbert space E which is concentrated scalarly on the unit ball B_1 of E, and let $u: E \to F$ be a Hilbert-Schmidt map into a Hilbert space F with Hilbert-Schmidt norm $\|u\|_2 = S$. Then μ is concentrated scalarly up to any $\eta > 0$ on B_1 but the only conclusion Lemma 3 lets us draw from this is that $u(\mu)$ is concentrated cylindrically up to CS on the unit ball B_2 of F and the bound CS cannot be lowered. If we renorm F by taking the ball $R.B_2$ with radius R as the new unit ball, then the Hilbert-Schmidt norm of u becomes S/R and we obtain that $u(\mu)$ is cylindrically concentrated up to CS/R on $R.B_2$. But our method does not let us prove that $u(\mu)$ is concentrated up to 0 on *any* ball of F.

This is a bad result since, as a matter of fact $u(\mu)$, which by Theorem 2 is a Radon measure, is carried by the closed ball with radius S of F. Indeed, $\|u\| \leqslant \|u\|_2 = S$, and therefore $u(B_1)$ is contained in $S.B_2$. On the other hand $u(\mu)$ is scalarly concentrated on the compact, convex set $u(B_1)$, hence by Prop. 5 of Chap. 2, §3, it is concentrated on $u(B_1)$.

Theorem 2 can be proved without using Prop. 7 of Chap. 2, §3, and therefore the deep Theorem 3 of Chap. 1, § 4 if one applies instead the following result.

PROPOSITION 6. *Every Hilbert-Schmidt operator $u: E \to F$ can be factored into*

$$E \xrightarrow{v} H \xrightarrow{w} F,$$

where v is a Hilbert-Schmidt operator and w is a compact operator.

Let us first show how one can deduce Theorem 2 with the help of Prop. 6. If μ is a cylindrical measure on E scalarly concentrated on the balls of E, then the first half of the proof of Theorem 2 shows that $v(\mu)$ is concentrated cylindrically on the balls of H. But w transforms the balls of H into compact sets of F, hence $u(\mu) = w(v(\mu))$ is concentrated cylindrically on the compact sets of F. Hence by Prokhorov's theorem (cf. p. 192) $u(\mu)$ is a Radon measure on F.

PROOF OF PROPOSITION 6. By going over to the adjoint maps (Prop. 1, § 1) it is sufficient to prove that $u: E \to F$ can be factored into

$$E \xrightarrow{\alpha} H \xrightarrow{\beta} F,$$

where α is compact and β Hilbert-Schmidt. Let (e_i) be an orthonormal basis of E. We have (Theorem 1, § 1) $\sum \| u(e_i) \|^2 < \infty$. In particular we have $u(e_i) \neq 0$ only for indices i belonging to a countable subset N of I. By duBois-Reymond's theorem there exists a sequence $(\lambda_i)_{i \in N}$ of positive numbers which converges to $+\infty$ and is such that $\sum \lambda_i^2 \| u(e_i) \|^2 < \infty$. Let us choose $H = E$ and define α by $\alpha(e_i) = (1/\lambda_i) e_i$ for $i \in N$ and $\alpha(e_i) = 0$ for $i \notin N$. Then α is a compact operator since its eigenvalues $1/\lambda_i$ tend to zero. Let us define $\beta: H \to F$ by setting $\beta(e_i) = \lambda_i u(e_i)$. Clearly $\beta \circ \alpha = u$ and β is a Hilbert-Schmidt operator since

$$\sum \| \beta(e_i) \|^2 = \sum \lambda_i^2 \| u(e_i) \|^2 < \infty.$$

It follows from the preceding proof that in general $\| v \|_2 > \| u \|_2$ but $\| v \|_2$ can be chosen arbitrarily close to $\| u \|_2$.

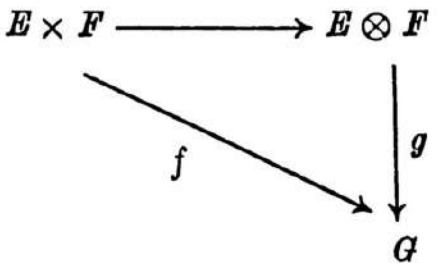
SUMMARY OF CHAPTER IV

This chapter gives the fundamental theorem of Minlos, the identity of cylindrical measures, scalarly concentrated on the compact subsets, with the Radon measures, in a quasi-complete conuclear space. The nuclear operators and spaces are not assumed to be known. Nuclear operators are defined in § 1, Definition 1, page 222 and their main properties are given. An operator between Hilbert spaces is nuclear iff it is a product of two Hilbert-Schmidt operators, Proposition 3, page 224. The nuclear spaces are defined in § 2, Definition 2, page 226, with an alternative definition using Hilbert-Schmidt operators, Proposition 5, page 229. Conuclear spaces are defined in definition 3, page 227, with an alternative definition using Hilbert-Schmidt operators, Proposition 6, page 230. The relationships, through duality, between nuclear and conuclear spaces is given by Theorem 1, page 231. The § 3 gives Minlos' theorem, Theorem 2, page 233, the essential result of the whole theory. The § 4 gives complement of less importance.

IV

CYLINDRICAL MEASURES AND NUCLEAR SPACES

1 Nuclear operators. Let E and F be two vector spaces over the same field. It is well known [1, Chap. II, 3rd ed., §3, $n°$ 1, Def. 1 and Prop. 1] that their tensor product $E \otimes F$ and the canonical bilinear map$(x, y) \longmapsto x \otimes y$ are characterized by the following universal property: given a vector space G and a bilinear map $f: E \times F \to G$, there exists a unique linear map $g: E \otimes F \to G$ such that $f(x, y) = g(x \otimes y)$ for every $x \in E$ and $y \in F$.



PROPOSITION 1. *Let E and F be two Banach spaces. For $z \in E \otimes F$ let us define $\| z \|_1$ by*

$$\| z \|_1 = \inf \sum_i \| x_i \| \cdot \| y_i \|,$$

where the greatest lower bound is taken with respect to all decompositions $z = \Sigma x_i \otimes y_i$. The function $z \longmapsto \| z \|_1$ is a norm on $E \otimes F$ and has the following property:

Let G be a normed space. Denote by $B(E,F;G)$ the normed space of all continuous bilinear maps $E \times F \to G$ and by $\mathscr{L}(E \otimes F, G)$ the normed space of all continuous linear maps $E \otimes F \to G$. Then the linear map $f \longmapsto g$ referred to above maps $B(E, F; G)$ isometrically onto $\mathscr{L}(E \otimes F, G)$.

PROOF. It is clear that $z \longmapsto \|z\|_1$ is a semi-norm. If $f(x, y) = g(x \otimes y)$ for every $x \in E$ and $y \in F$, then for every decomposition $z = \sum_i x_i \otimes y_i$ we have

$$\|g(z)\| \leqslant \sum_i \|g(x_i \otimes y_i)\| = \sum_i \|f(x_i, y_i)\| \leqslant \|f\| \cdot \sum_i \|x_i\| \cdot \|y_i\|.$$

Thus $\|g(z)\| \leqslant \|f\| \cdot \|z\|_1$ and therefore $\|g\| \leqslant \|f\|$.

Next we prove that for elements of the form $z_0 = x_0 \otimes y_0$ we have $\|z_0\|_1 = \|x_0\| \cdot \|y_0\|$. Clearly $\|z_0\|_1 \leqslant \|x_0\| \cdot \|y_0\|$. By the Hahn-Banach theorem there exist $x' \in E'$, $y' \in F'$ such that $x'(x_0) = \|x_0\|$, $y'(y_0) = \|y_0\|$, $\|x'\| = 1$, $\|y'\| = 1$. The bilinear form

$$f\colon (x, y) \longmapsto \langle x, x' \rangle \langle y, y' \rangle$$

has norm 1 and

$$\|x_0\| \cdot \|y_0\| = f(x_0, y_0) = g(x_0 \otimes y_0) \leqslant \|f\| \cdot \|x_0 \otimes y_0\|_1,$$

i.e. $\|x_0\| \cdot \|y_0\| \leqslant \|z_0\|_1$.

Now we can prove that $\|f\| \leqslant \|g\|$. Indeed, for $x \in E$ and $y \in F$ we have

$$\|f(x, y)\| = \|g(x \otimes y)\| \leqslant \|g\| \cdot \|x \otimes y\|_1 = \|g\| \cdot \|x\| \cdot \|y\|.$$

Finally, $z \longmapsto \|z\|_1$ is a norm. Indeed, let $z = \sum_{i=1}^{n} x_i \otimes y_i \neq 0$. We may assume that the family (x_i) is free in E and the family (y_i) is free in F. By the Hahn-Banach theorem there exist $x' \in E'$ and $y' \in F'$ such that $x'(x_1) = 1$, $x'(x_i) = 0$ for $2 \leqslant i \leqslant n$ and $y'(y_1) = 1$, $y'(y_i) = 0$ for $2 \leqslant i \leqslant n$. Then $z'\colon \Sigma u_i \otimes v_i \longmapsto \Sigma \langle u_i, x' \rangle \langle v_i, y' \rangle$ is a well-defined linear form on $E \otimes F$, and

$$|z'(\Sigma u_i \otimes v_i)| \leqslant \|x'\| \cdot \|y'\| \cdot \Sigma \|u_i\| \cdot \|v_i\|,$$

i.e. $|z'(\Sigma u_i \otimes v_i)| \leqslant \|x'\| \cdot \|y'\| \cdot \|\Sigma u_i \otimes v_i\|_1$. Now $z'(z) = z'(\Sigma x_i \otimes y_i) = \langle x_1, x' \rangle \langle y_1, y' \rangle = 1$ and therefore $1 \leqslant \|x'\| \cdot \|y'\| \cdot \|z\|_1$, which proves that $\|z\|_1 \neq 0$.

The topology π defined by the norm $z \longmapsto \|z\|_1$ will be called the projective tensor product of the topologies of E and F. The space $E \otimes F$ equipped with the norm $z \longmapsto \|z\|_1$ will be denoted by $E \otimes_1 F$ or $E \otimes_\pi F$ and called the projective tensor product of E and F. Its completion will be denoted by $E \hat{\otimes}_1 F$ or by $E \hat{\otimes}_\pi F$.

It can be proved [9, Chap. I, § 2, n°1, Theorem 1.3, p. 51] that every element $z \in E \hat{\otimes}_1 F$ can be written in the form $\sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$

where $\|x_i\| \leqslant 1$, $\|y_i\| \leqslant 1$ and $\sum_{i=1}^{\infty} |\lambda_i| < \infty$. Furthermore the norm in $E \hat{\otimes}_1 F$ is given by $\|z\|_1 = \inf \sum_{i=1}^{\infty} |\lambda_i|$, where the greatest lower bound is taken with respect to all such decompositions.

Contrarily to what happens for the Hilbert-Schmidt tensor product, if E_1 is a closed subspace of E and F_1 is a closed subspace of F, then the topology π on $E \otimes F$ does not induce necessarily the topology π on $E_1 \otimes F_1$.

If $u: E_1 \to E_2$ and $v: F_1 \to F_2$ are two continuous linear maps, then we can associate with them a continuous linear map $u \otimes v$: $E_1 \otimes_\pi F_1 \to E_2 \otimes_\pi F_2$ such that $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$. This follows immediately from Prop. 1 and from the fact that the bilinear map $(x, y) \longmapsto u(x) \otimes v(y)$ from $E_1 \times F_1$ into $E_2 \otimes_\pi F_2$ is continuous since

$$\|u(x) \otimes v(y)\|_1 = \|u(x)\| \cdot \|v(y)\| \leqslant \|u\| \cdot \|v\| \cdot \|x\| \cdot \|y\|.$$

Furthermore this association is functorial, i.e. the following two conditions are satisfied:

(1) if $u: E \to E$ and $v: F \to F$ are identity maps, then $u \otimes v$ is the identity map $E \otimes F \to E \otimes F$;

(2) *if* $u: E_1 \to E_2$, $u_1: E_2 \to E_3$, $v: F_1 \to F_2$, $v_1: F_2 \to F_3$, then

$$(u_1 \circ u) \otimes (v_1 \circ v) = (u_1 \otimes v_1) \circ (u \otimes v)$$

[9, Chap. I, § 1, n°2, p. 37].

Let now E' be the dual of the Banach space E, and let us consider the injection $E' \otimes F \to \mathscr{L}(E, F)$ which associates with the element $e' \otimes f$ the linear map $u: e \longmapsto \langle e, e' \rangle f$ (p. 208) If we equip $\mathscr{L}(E, F)$ with its usual norm, then the bilinear map

$$(e', f) \longmapsto u$$

from $E' \times F$ into $\mathscr{L}(E, F)$ is continuous since $\|u(e)\| = |\langle e, e' \rangle| \cdot \|f\| \leqslant \|e\| \cdot \|e'\| \cdot \|f\|$ and thus $\|u\| \leqslant \|e'\| \cdot \|f\|$. It follows that the injection $E' \otimes F \to \mathscr{L}(E, F)$ is continuous if we equip $E' \otimes F$ with the topology π, and therefore we can extend it to a continuous

linear map $E' \hat{\otimes}_1 F \to \mathscr{L}(E, F)$. Contrarily to what happens for the map $E' \hat{\otimes}_2 F \to \mathscr{L}(E, F)$ in the case of Hilbert spaces (p. 208), we do not know whether the map $E' \hat{\otimes}_1 F \to \mathscr{L}(E, F)$ is always injective.

DEFINITION 1. *An operator $u \in \mathscr{L}(E, F)$ is nuclear if it belongs to the image of $E' \hat{\otimes}_1 F$ in $\mathscr{L}(E, F)$. The nuclear norm or trace norm of a nuclear operator u will be its quotient norm in $E' \hat{\otimes}_1 F/N$, where N is the kernel of $E' \hat{\otimes}_1 F \to \mathscr{L}(E, F)$, and it will still be denoted by $\|u\|_1$.*

We have $\|u\| \leqslant \|u\|_1$. Indeed the bilinear map $(e', f) \longmapsto u$ from $E' \times F$ into $\mathscr{L}(E, F)$ has norm $\leqslant 1$ as we have seen above. It follows from Prop. 1 that the corresponding linear map from $E' \otimes F$ into $\mathscr{L}(E, F)$ has norm $\leqslant 1$. Thus for any $u \in E' \hat{\otimes}_1 F$ the quotient norm $\|u\|_1$ is $\geqslant \|u\|$. It will be convenient to set $\|u\|_1 = \infty$ for an operator $u \in \mathscr{L}(E, F)$ which is not nuclear.

By what has been said earlier, the nuclear operators $u \in \mathscr{L}(E, F)$ are the ones which can be written in the form $u = \sum_{i=1}^{\infty} \lambda_i (e_i' \otimes f_i)$, where $\|e_i'\| \leqslant 1$, $\|f_i\| \leqslant 1$, $\Sigma |\lambda_i| < \infty$, and we have $\|u\|_1 = \inf \Sigma |\lambda_i|$.

If u is a nuclear operator and v, w two continuous linear maps, then $v \circ u \circ w$ is a nuclear operator and $\|v \circ u \circ w\|_1 \leqslant \|v\| \cdot \|u\|_1 \cdot \|w\|$. Indeed, if $u = \Sigma \lambda_i (e_i' \otimes f_i)$, then

$$v \circ u \circ w = \Sigma \lambda_i ({}^t w(e_i') \otimes v(f_i)) = \Sigma \|{}^t w\| \cdot \lambda_i \cdot \|v\| \left(\frac{{}^t w(e_i')}{\|{}^t w\|} \otimes \frac{v(f_i)}{\|v\|} \right),$$

from which the assertion immediately follows in view of $\|{}^t w\| = \|w\|$.

Every nuclear operator is compact since it is the limit in $\mathscr{L}(E, F)$ of operators of finite rank $\sum_{i=1}^{n} \lambda_i (e_i' \otimes f_i)$.

We shall now study nuclear maps from a Hilbert space into another Hilbert space, and we begin by considering normal operators $u \in \mathscr{L}(E)$ which are nuclear.

PROPOSITION 2. *Let $u \in \mathscr{L}(E)$ be a normal operator on a Hilbert space E. The operator u is nuclear if and only if E is the Hilbert sum*

of eigenspaces E_n, and the eigenvalues λ_n belonging to the E_n are such that $\Sigma \dim E_n . |\lambda_n| < \infty$. *In particular the E_n, with the possible exception of the eigenspace belonging to the eigenvalue zero, are finite-dimensional. If u is nuclear, we have*

$$\|u\|_1 = \Sigma \dim E_n . |\lambda_n| .$$

PROOF. If u is a compact normal operator, then it is of the form $u = \Sigma \lambda_i (\bar{e}_i \otimes e_i)$, where in the sequence (λ_i) each eigenvalue occurs as often as its multiplicity indicates, and (e_i) is an orthonormal basis of E. Hence if $\Sigma |\lambda_i| < \infty$, then u is nuclear and $\|u\|_1 \leqslant \Sigma |\lambda_i|$.

It remains to show that if u is nuclear, then $\Sigma |\lambda_i| \leqslant \|u\|_1$. Let us consider for a moment an arbitrary Banach space F and its dual F'. The canonical bilinear form $(f', f) \longmapsto \langle f', f \rangle$ on $F' \times F$ is continuous and its norm is $\leqslant 1$. Thus there corresponds to it a continuous linear form on $F' \hat{\otimes}_1 F$, whose norm is $\leqslant 1$, called the trace form and denoted by $u \longmapsto \mathrm{Tr}(u)$. We have by definition $\mathrm{Tr}(f' \otimes f) = \langle f', f \rangle$ for $f' \in F'$ and $f \in F$. For any $u \in F' \hat{\otimes}_1 F$ we have $|\mathrm{Tr}(u)| \leqslant \|u\|_1$.

Suppose now that E is a finite-dimensional Hilbert space. Then $\bar{E} \otimes_1 E = \bar{E} \hat{\otimes}_1 E$ can be identified with $\mathscr{L}(E)$, and if $u = \Sigma \lambda_i \bar{e}_i \otimes e_i$ is a normal operator then $|\mathrm{Tr}(u)| = |\Sigma \lambda_i| \leqslant \|u\|_1$. We can find a linear map $v \in \mathscr{L}(E)$, whose matrix is diagonal with elements of absolute value 1 along the diagonal, such that $\|v\| \leqslant 1$ and $vu = \Sigma |\lambda_i| \bar{e}_i \otimes e_i$. Hence $\Sigma |\lambda_i| = |\mathrm{Tr}(vu)| \leqslant \|vu\|_1 \leqslant \|v\| . \|u\|_1 \leqslant \|u\|_1$.

We can now conclude the proof. Assume that u is a normal nuclear map. Then u is compact, E is the Hilbert sum of the eigenspaces E_n of u and the eigenvalues of u tend to zero. Arrange the non-zero eigenvalues in decreasing order, and let $E^{(k)}$ be the sum of the spaces E_n belonging to the first k non-zero eigenvalues. The space $E^{(k)}$ is finite-dimensional. Denote by $i_k : E^{(k)} \to E$ the canonical injection and by $p_k : E \to E^{(k)}$ the orthogonal projection. The operator $p_k \circ u \circ i_k : E^{(k)} \to E^{(k)}$ is of the form $\sum_{i \leqslant k} \lambda_i \bar{e}_i \otimes e_i$ and thus by the preceding argument $\sum_{i \leqslant k} |\lambda_i| \leqslant \|p_k \circ u \circ i_k\|_1 \leqslant \|p_k\| . \|u\|_1 . \|i_k\| \leqslant \|u\|_1$. Letting $k \to \infty$ we obtain $\Sigma |\lambda_i| \leqslant \|u\|_1$.

It follows that if u is a normal nuclear map then it is Hilbert-Schmidt since

$$\sum \dim E_n \cdot |\lambda_n|^2 \leqslant (\max |\lambda_m|) \sum \dim E_n \cdot |\lambda_n| \leqslant (\sum \dim E_n \cdot |\lambda_n|)^2$$

and it suffices to apply Prop. 3 of Chap. 3, § 1. More precisely we have

PROPOSITION 3. *Let u be a continuous linear map from a Hilbert space E into a Hilbert space F. The map u is nuclear if and only if it can be factored into*

$$E \xrightarrow{A} G \xrightarrow{B} F,$$

where G is a Hilbert space, A and B are Hilbert-Schmidt maps.

PROOF. If $E = F$ and u is normal, then the proof is trivial. Indeed we have $\sum |\lambda_i| < \infty$ if and only if there exist two sequences (μ_i) and (ν_i) such that $\lambda_i = \mu_i \nu_i$, $\sum |\mu_i|^2 < \infty$, $\sum |\nu_i|^2 < \infty$.

In the general case we shall use the polar decomposition of the linear operator $u : E \to F$. We recall that u can be written as $u = U\sqrt{u^*u}$, where $U : E \to F$ is a partially isometric operator and $\sqrt{u^*u}$ is a positive hermitian endomorphism of E. Indeed, u^*u is a positive hermitian operator, hence it possesses a unique positive square root $\sqrt{u^*u} : E \to E$. Furthermore

$$\|\sqrt{u^*u}\, x\|^2 = (\sqrt{u^*u}\, x \mid \sqrt{u^*u}\, x) = (u^*ux \mid x) = (ux \mid ux) = \|ux\|^2.$$

If $E_1 \subset E$ is the image of $\sqrt{u^*u}$, then we define an isometry from E_1 into F by setting $\sqrt{u^*u}\, x \longmapsto ux$. This isometry is then extended by continuity to the closure of E_1 and defining it as the zero map on the orthogonal complement of E_1 in E we obtain the partial isometry $U : E \longmapsto F$ which satisfies $u = U\sqrt{u^*u}$. In a similar way, we can write $\sqrt{u^*u} = Vu$, where $V : F \to E$ is a partially isometric map. It follows that u is a nuclear map if and only if $\sqrt{u^*u}$ is a nuclear map, and $\|\sqrt{u^*u}\|_1 = \|u\|_1$; also u is a Hilbert-Schmidt map if and only if $\sqrt{u^*u}$ is a Hilbert-Schmidt map and $\|\sqrt{u^*u}\|_2 = \|u\|_2$.

Assume now that u is nuclear. Then $\sqrt{u^*u}$ is a positive hermitian nuclear map and has therefore the form $\sum \lambda_i \bar{e}_i \otimes e_i$ with $\lambda_i \geqslant 0$ and

$\Sigma\lambda_i < \infty$. If we write $A = \Sigma\sqrt{\lambda_i}(\bar{e}_i \otimes e_i)$ and $B = U \circ \Sigma_i \sqrt{\lambda_i}(\bar{e}_i \otimes e_i)$, then $A: E \to E$ and $B: E \to F$ are Hilbert-Schmidt operators and $B \circ A = U \circ \sqrt{u^* u} = u$. Furthermore, we have obviously $\| A \|_2 = \| B \|_2 = \| u \|_1^{1/2}$.

Conversely suppose that $u = B \circ A$, where A and B are Hilbert-Schmidt maps. If $A = UT$ is the polar decomposition of A, with $T = \sqrt{A^* A}$, then T is a positive hermitian Hilbert-Schmidt operator and therefore there exists an orthonormal basis (e_i) of E such that the vectors Te_i are orthogonal. Since U is a partial isometry, the vectors $UTe_i = Ae_i$ are orthogonal in G and therefore the vectors $Ae_i / \| Ae_i \|$ for which $Ae_i \neq 0$ form an orthonormal system in G which can be completed to become an orthonormal basis of G. Since B is a Hilbert-Schmidt operator, we have

$$\sum \frac{\| BAe_i \|^2}{\| Ae_i \|^2} \leqslant \| B \|_2^2$$

and thus by the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum \| BAe_i \| &= \sum \frac{\| BAe_i \|}{\| Ae_i \|} \| Ae_i \| \\ &< \left(\sum \frac{\| BAe_i \|^2}{\| Ae_i \|^2} \right)^{1/2} (\Sigma \| Ae_i \|^2)^{1/2} \\ &< \| B \|_2 . \| A \|_2 . \end{aligned}$$

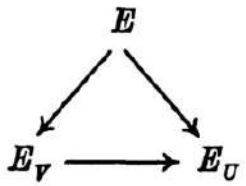
In particular $\Sigma \| u(e_i) \| < \infty$ and since

$$u = \Sigma \bar{e}_i \otimes u(e_i) = \Sigma \| u(e_i) \| \left(\bar{e}_i \otimes \frac{u(e_i)}{\| u(e_i) \|} \right),$$

we see that u is nuclear and $\| u \|_1 \leqslant \| A \|_2 \| B \|_2$.

Let now E and F be again two Banach spaces. If the linear map $u: E \to F$ is nuclear, then its transpose $u: F' \to E'$ is also nuclear and we have $\| {}^t u \|_1 \leqslant \| u \|_1$. Indeed, if $u = \Sigma\lambda_i(e_i' \otimes f_i)$, then ${}^t u = \Sigma\lambda_i(f_i \otimes e_i')$. On the other hand ${}^t u$ can be nuclear without u being nuclear. The reason for this is that if a map $u: E \to F$ is nuclear and sends E into a subspace F_1 of F, then the map $u: E \to F_1$ is not necessarily nuclear [9, Chap. I, § 3, n°2, Remark 9, pp. 87-88]. Thus if ${}^t u$ is nuclear, then the map ${}^{tt} u: E \to F''$ is nuclear, but may fail to be so as a map from E into F [loc. cit.].

2. Nuclear spaces. Let E be a locally convex topological vector space. If U is a balanced, convex, open neighborhood of 0 in E and q_U is the gauge of U, then we denote by E_U the Hausdorff space associated with the space E equipped with the semi-norm q_U. In other words, if $N = \bigcap_{\lambda>0} \lambda U = \{x \mid q_U(x) = 0\}$ is the closure of $\{0\}$ for the topology defined on E by the semi-norm q_U, then E_U is the normed space E/N equipped with the quotient of the semi-norm q_U. For each such U we have a surjective continuous linear map $E \to E_U$ and if V is a second balanced, convex, open neighborhood of 0 in E such that $V \subset U$, then we have a canonical surjection $E_V \to E_U$ such



that the diagram is commutative.

DEFINITION 2. *A locally convex Hausdorff space E is said to be nuclear if for every balanced, convex, open neighborhood U of 0 in E there exists a balanced, convex, open neighborhood V of 0 in E, contained in U and such that the canonical map $\hat{E}_V \to \hat{E}_U$ is nuclear.*

If a Banach space E is nuclear, then it is finite-dimensional since its unit ball is then compact.

Now let A be a balanced, convex, bounded and closed subset of the locally convex space E. We denote by E_A the subspace of E generated by A and equipped with the gauge of A as semi-norm. If E is a Hausdorff space, then so is E_A, i.e. E_A is a normed space whose closed unit ball is A. If furthermore A is complete, then E_A is complete, i.e. a Banach space [3, Chap. III, §3, $n°$ 4, Lemma 1]. If B is a second balanced, closed, bounded, convex set such that $A \subset B$, then we have a continuous linear map $E_A \to E_B$ with norm $\leqslant 1$. Thus if U and V are two balanced, convex, open neighborhoods of zero such that $V \subset U$, we have the sequence of continuous linear maps

$$E_A \to E_B \to E \to E_V \to E_U.$$

DEFINITION 3. *Let $\mathfrak{S}$ be a saturated collection of balanced, closed, convex subsets of the locally convex Hausdorff space E. We say that E is $\mathfrak{S}$-conuclear if for every $A \in \mathfrak{S}$ there exists $B \in \mathfrak{S}$, $B \supset A$ such that the canonical map $\hat{E}_A \to \hat{E}_B$ is nuclear.*

If E is nuclear, then its dual E' is ϵ-conuclear, where ϵ refers to the collection of all balanced, convex, closed, equicontinuous subsets of E'. We shall see later (Theorem 1) that the converse is also true. To prove the assertion, let A be a balanced, closed, convex, equicontinuous subset of E'. Then $A = U^\circ$, where U is a balanced, convex, open neighbourhood of 0 in E. Since E is nuclear, there exists another such neighbourhood $V \subset U$, such that $\hat{E}_V \to \hat{E}_U$ is nuclear. But the dual of E_U or of $\hat{E}_U$ is the space E'_A, hence if we set $B = V^\circ$, the transposed map $E'_A \to E'_B$ will be nuclear.

The spaces $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{E}$, $\mathscr{E}'$, $\mathcal{O}_M$, $\mathcal{O}'_M$, $\mathcal{O}_C$, $\mathcal{O}'_C$, $\mathscr{S}$ and $\mathscr{S}'$ are nuclear, [9, Chap. II, §2, n° 3, Theorem 10, p. 55]. The countable direct product $\mathbf{R}^{\mathbf{N}}$ and its dual, the countable direct sum $\mathbf{R}^{(\mathbf{N})}$, are nuclear spaces.

We shall now give a characterization of nuclear spaces which does not use the concept of nuclear maps. For this we shall need

PROPOSITION 4. *If u is a nuclear map from a Banach space E into a Banach space F, then u can be factored into*

$$E \xrightarrow{\alpha} H \xrightarrow{\beta} F,$$

where H is a Hilbert space, α and β are continuous linear maps.

PROOF. Suppose that $u = \sum_{i \in I} \lambda_i (e'_i \otimes f_i)$ with $\|e'_i\| \leqslant 1$, $\|f_i\| \leqslant 1$ and $\Sigma |\lambda_i| < \infty$. For H we shall take the space l^2_I of all families (t_i) such that $\Sigma |t_i|^2 < \infty$. The map $\alpha: E \to H$ is defined by $x \longmapsto (\langle x, e'_i \rangle \sqrt{|\lambda_i|})$. Since

$$\Sigma |\langle x, e'_i \rangle| \sqrt{|\lambda_i|^2} \leqslant \|x\|^2 \Sigma |\lambda_i|,$$

α maps E into H and $\|\alpha\| \leqslant \|u\|_1^{1/2}$. Next we define $\beta: H \to F$ by

$$(t_i) \longmapsto \sum_{i \in I} t_i \frac{\lambda_i}{\sqrt{|\lambda_i|}} f_i.$$

By virtue of the inequality

$$\Sigma \| t_i \frac{\lambda_i}{\sqrt{|\lambda_i|}} f_i \| < (\Sigma |t_i|^2)^{1/2} (\Sigma |\lambda_i|)^{1/2}$$

this map is also well-defined and $\|\beta\| < \|u\|_1^{1/2}$. Clearly $u = \beta \circ \alpha$.

DEFINITION 4. [12, §1, p. 124]. *Let F be a locally convex Hausdorff space. We shall say that $\mathscr{H}$ is a Hilbert subspace of F, if $\mathscr{H}$ is a linear subspace of the vector space F, the vector space $\mathscr{H}$ is equipped with a Hilbert space structure, and the canonical injection map $\mathscr{H} \to F$ is continuous.*

Let us emphasize that according to this definition, if F is a Hilbert space, there is no relation between the Hilbert space structure of a Hilbert subspace $\mathscr{H}$ of F and the Hilbert space structure of F.

Suppose now that we have a continuous linear map u from a Hilbert space H into F. The kernel N of u is a closed subspace of H and u yields a continuous, injective, linear map u from the Hilbert space H/N, isomorphic to $N^\perp$, into F. The Hilbert space structure of H/N can be transported with the help of $\bar{u}$ onto $\mathscr{H} = u(H)$ and thus we obtained a Hilbert subspace $\mathscr{H}$ of F as defined in Def. 4.

It follows from Prop. 4 that if $u: E \to F$ is a nuclear map, then it can be factored into two continuous linear maps $E \to \mathscr{H} \to F$, where $\mathscr{H}$ is a Hilbert subspace of F and $\mathscr{H} \to F$ is the canonical injection.

Let E be a nuclear space. Given a balanced, convex, open neighbourhood U of 0 in E, there exists by definition another such neighbourhood $V \subset U$ so that the canonical map $\hat{E}_V \to \hat{E}_U$ is nuclear and therefore factors into two continuous linear maps $\hat{E}_V \to \mathscr{H} \to \hat{E}_U$, where $\mathscr{H}$ is a Hilbert subspace of E_U. Let W be the inverse image in E of the open unit ball of $\mathscr{H}$ with respect to the composite map $E \to \hat{E}_V \to \mathscr{H}$. Then W is a balanced, convex,

open neighbourhood of 0 in E which is absorbed by U. Furthermore E_W has a pre-Hilbert structure obtained by transporting onto E_W the pre-Hilbert structure of its image in $\mathscr{H}$.

In general we shall call a balanced, convex, open neighbourhood W of 0 in a locally convex space E a *pre-Hilbert neighborhood* if the normed space E_W is equipped with a pre-Hilbert structure, i.e. its norm can be deduced from a scalar product. In this case the completion $\hat{E}_W$ of E_W is of course a Hilbert space.

With this terminology it follows from the preceding considerations that every nuclear space E possesses a fundamental system of neighbourhoods of the origin which are pre-Hilbert. Furthermore E can be imbedded into the product ΠE_W, where W runs through all pre-Hilbert neighbourhoods of 0 in E. Thus every nuclear space is the projective limit of a family of pre-Hilbert spaces.

PROPOSITION 5. *A locally convex Hausdorff space E is nuclear if and only if the following two conditions are satisfied*:

(1) *E possesses a fundamental system of neighbourhoods of 0 formed by pre-Hilbert neighbourhoods*;

(2) *given a pre-Hilbert neighbourhood U of 0, there exists a pre-Hilbert neighbourhood $V \subset U$ of 0 such that the canonical map $\hat{E}_V \to \hat{E}_U$ is Hilbert-Schmidt.*

PROOF. We have just seen that if E is nuclear, then condition (1) is satisfied. Furthermore if U is a pre-Hilbert neighbourhood of 0, there exists a neighbourhood of zero $V \subset U$ which we may assume to be pre-Hilbert, such that the canonical map $\hat{E}_V \to \hat{E}_U$ is nuclear, hence by Prop. 3 (§1) also Hilbert-Schmidt.

Conversely assume that the two conditions are satisfied. Let U be a neighbourhood of 0 which by condition (1) may be taken to be pre-Hilbert. By condition (2) there exists a pre-Hilbert neighbourhood $V \subset U$ such that $\hat{E}_V \to \hat{E}_U$ is a Hilbert-Schmidt operator, and a pre-Hilbert neighbourhood $W \subset V$ such that $\hat{E}_W \to \hat{E}_V$ is a Hilbert-Schmidt

operator. By Prop. 3 the composite map $\hat{E}_W \to \hat{E}_V \to \hat{E}_U$ is nuclear and E is a nuclear space.

We want to give now a similar characterization of $\mathfrak{S}$-conuclear spaces (Def. 3) in the case when the sets belonging to $\mathfrak{S}$ are completing.

DEFINITION 5. *A balanced, convex, bounded subset A of a locally convex Hausdorff space E will be called a Hilbert set if the Banach space E_A has a Hilbert space structure, i.e. the norm of E_A can be deduced from a scalar product, and E_A is complete.*

DEFINITION 6. *A subset A of a locally convex Hausdorff space E is said to be completing, if it is contained in the image of the unit ball of a Banach space by a continuous map of this Banach space into E or, equivalently, if there exists a bounded convex balanced subset B in E containing A such that E_B is complete.*

If A is bounded, convex, balanced and closed, it is completing if and only if E_A is complete. Indeed, if $A \subset B$, B bounded, convex and balanced, E_B complete, then A is a bounded, convex and balanced subset of E_B, which is closed in E_B therefore complete, hence E_A is complete [3, Chap. III, §3, $n°4$, Lemma 1].

A completing subset is bounded; a bounded, convex, balanced, complete subset of E is completing (previous reference).

PROPOSITION 6. *Let E be a locally convex Hausdorff space and let $\mathfrak{S}$ be a saturated collection of balanced, convex, completing subsets of E, and such that if A is a balanced, convex, completing subset of E contained in some $B \in \mathfrak{S}$ then $A \in \mathfrak{S}$. The space E is $\mathfrak{S}$-conuclear if and only if the following two conditions are satisfied;*

(1) *there exists in $\mathfrak{S}$ a fundamental system of Hilbert sets, i.e. every $A \in \mathfrak{S}$ is contained in a Hilbert set $B \in \mathfrak{S}$;*

(2) *given a Hilbert set $A \in \mathfrak{S}$, there exists a Hilbert set, $B \in \mathfrak{S}$ containing A such that the canonical map $E_A \to E_B$ is Hilbert-Schmidt.*

PROOF. Assume that E is $\mathfrak{S}$-conuclear and let $A \in \mathfrak{S}$. Since the sets belonging to $\mathfrak{S}$ are completing there exists $B \in \mathfrak{S}$, $B \subset A$ such

that $E_A \to E_B$ is nuclear. By Prop. 4 this map can be factored into $E_A \to H \to E_B$, where H is a Hilbert space. The image C of the unit ball of H in E_B is a balanced, convex, weakly (and therefore also strongly) closed subset of E_B, because the unit ball is weakly compact. Since C is absorbed by B it is complete and $C \in \mathfrak{S}$. On the other hand A is absorbed by C, and since each set λC belongs to $\mathfrak{S}$ and is Hilbert, we see that condition (1) is satisfied.

If $A \in \mathfrak{S}$ is a Hilbert set, there exists a set $B \in \mathfrak{S}$, $B \supset A$ such that $E_A \to E_B$ is a nuclear map. By the first part of the proof we may assume that B is a Hilbert set, and then the map $E_A \to E_B$ is a fortiori Hilbert-Schmidt (Prop. 3, § 1).

Conversely assume that the two conditions are satisfied. Let A be a set belonging to $\mathfrak{S}$ which by condition (1) may be taken to be a Hilbert set. By condition (2) there exists a Hilbert set $B \in \mathfrak{S}$ containing A such that $E_A \to E_A$ is a Hilbert-Schmidt operator, and a Hilbert set $C \in \mathfrak{S}$ containing B such that $E_B \to E_C$ is a Hilbert-Schmidt operator. By Prop. 3 the composite map $E_A \to E_B \to E_C$ is nuclear and the space E is $\mathfrak{S}$-conuclear.

Proposition 6 can be applied in the following cases:

(a) $\mathfrak{S}$ is the collection of *all* balanced, bounded, convex, complete subsets of E.

(b) $\mathfrak{S}$ is the collection of all balanced, convex, compact subsets of E.

(c) $\mathfrak{S}$ is the collection ϵ of all balanced, convex, closed equicontinuous (and thus weak*compact) subsets of the dual E' of a locally convex Hausdorff space E.

We are now in the position to prove a result announced earlier (p. 227).

THEOREM 1. (a) *A locally convex Hausdorff space E is nuclear if and only if its dual E' is ϵ-conuclear.*

(b) *If $\mathfrak{S}$ is a collection of subsets of E having the properties listed in Prop. 6 such that $E_0 = \bigcup_{A \in \mathfrak{S}} A$ is dense in E, then E is $\mathfrak{S}$-conuclear if and only if $E'_{\mathfrak{S}}$ is nuclear.*

PROOF. (a) We know already that if E is nuclear then E' is ϵ-conuclear.

Suppose that E' is ϵ-conuclear and let U be a balanced, convex, open neighbourhood of 0 in E. Then U° is a balanced, closed, convex, equicontinuous subset of E', hence there exists by Prop. 6, condition (1), an equicontinuous Hilbert set A containing U°. The interior V of A° is a balanced, convex, open neighbourhood of 0 in E and $V \subset U$. On the other hand $(E_V)' = E'_A$ is equipped with a Hilbert space structure and therefore E_V is equipped with a pre-Hilbert structure. Thus condition (l) of Prop. 5 is satisfied.

Let U be a pre-Hilbert neighbourhood of 0 in E. Then $A = U^\circ$ is an equicontinuous Hilbert set in E', hence by condition (2) of Prop. 6 there exists an equicontinuous Hilbert set $B \supset A$ such that the canonical map $E'_A \to E'_B$ is Hilbert-Schmidt. If V is the interior of B°, then V is a pre-Hilbert neighbourhood of 0 in E and the map $\hat{E}_V \to \hat{E}_U$, which is the transpose of the preceding one, is a Hilbert-Schmidt operator. Thus condition (2) of Prop. 5 is also satisfied and E is a nuclear space.

(b) We have $E'_{\mathfrak{S}} = (E_0)'_{\mathfrak{S}}$, and E is $\mathfrak{S}$-conuclear if and only if E_0 is $\mathfrak{S}$-conuclear.

Therefore we may assume, for the sake of simplicity, $E_0 = E$. In the dual space $(E'_{\mathfrak{S}})' \subset E'^*$, the equicontinuous subsets are exactly the subsets of the $*$-weak-closures A'' of the sets $A \in \mathfrak{S}$. Therefore, it results from (a) that $E'_{\mathfrak{S}}$ is nuclear if and only if $(E'_{\mathfrak{S}})'$ is $\mathfrak{S}''$-conuclear.

It is now sufficient to prove that $\mathfrak{S}$ possesses a fundamental system of weakly compact subsets, so that for them $A'' = A$. If we assume E to be $\mathfrak{S}$-conuclear, then $\mathfrak{S}$ has a fundamental system of Hilbert subsets, and this is true. If we assume $E'_{\mathfrak{S}}$ is nuclear, then $(E'_{\mathfrak{S}})'$ is $\mathfrak{S}''$-conuclear; for every $A \in \mathfrak{S}$, there will exist a Hilbert set $\tilde{B}$ of E'^*, and a set $C \in \mathfrak{S}$, with $A'' \subset \tilde{B} \subset C''$. Put $B = \tilde{B} \cap E$; $\tilde{B}$ is a convex balanced weakly compact subset of E'^*, therefore B a closed convex balanced subset of E contained in C, therefore completing. Therefore E_B is complete (see after Definition 6); but

it is a subspace of the Hilbert space $(E'^*)_{\widetilde{B}}$ therefore E_B is Hilbert, therefore B is a Hilbert set, therefore weakly compact. Now $A \subset B \subset C$, therefore $B \in \mathfrak{S}$, which achieves the proof.

EXAMPLES. Let us denote by E'_{cc} the dual of the locally convex Hausdorff space E equipped with the topology of uniform convergence on the collection $\mathfrak{S} = cc$ of all balanced, convex, compact subsets of E. Then E'_{cc} is nuclear if and only if E is cc-conuclear. Indeed, it follows from Mackey's theorem [3, Chap. IV, §2, $n°$ 3, Corol. of Th. 2] that the dual of E'_{cc} is E and the equicontinuous subsets of the dual of E'_{cc} are the sets contained in some balanced, convex, compact subset of E.

In particular the spaces $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{O}_M$, $\mathscr{O}'_M$, $\mathscr{O}_C$, $\mathscr{O}'_C$, $\mathscr{S}$, $\mathscr{S}'$, the product $\mathbf{R}^N$ and the direct sum $\mathbf{R}^{(N)}$ are cc-conuclear.

3. The theorem of Minlos.

THEOREM 2. (Minlos [20]). *Let E be a locally convex Hausdorff space. Let $\mathfrak{S}$ be a saturated collection of convex, completing subsets of E, such that if A is a balanced, convex, completing subset of E contained in some $B \in \mathfrak{S}$, then $A \in \mathfrak{S}$. Assume that E is $\mathfrak{S}$-conuclear* (§2, Def. 3). *If μ is a cylindrical measure on E concentrated scalarly on the sets of $\mathfrak{S}$* (Chap. 2, §3, Def. 1) *then μ is a Radon measure on E, concentrated on the sets of $\mathfrak{S}$.*

LEMMA 1. *Let n and N be two positive integers, $n < N$, and let us identify $\mathbf{R}^n$ with the subspace of $\mathbf{R}^N$ formed by all vectors $x = (x_1, \ldots, x_N)$ such that $x_i = 0$ for $n < i < N$. Let μ be a probability on $\mathbf{R}^N$ scalarly concentrated up to $\eta > 0$ on the unit ball $B_1 = \{x \mid x \in \mathbf{R}^n, \sum x_i^2 < 1\}$ of $\mathbf{R}^n$. Then μ is concentrated up to ϵ on $B_2 = \{x \mid x \in \mathbf{R}^n, \sum \alpha_i^2 x_i^2 < 1\} \subset \mathbf{R}^n$, where $\epsilon = C \max(\eta, S)$, $S^2 = \sum \alpha_i^2$ and C is a universal constant.*

PROOF. The lemma will be a consequence of Lemma 1 of Chap. 3, §2. Under the hypothesis, for any $\lambda \in \mathbf{R}$ the probability μ is a fortiori concentrated scalarly up to η on the subset

$$\widetilde{B}_1 = \left\{ x \mid x \in \mathbf{R}^N, \sum_{i=1}^{n} x_i^2 + \lambda^2 \sum_{i=n+1}^{N} x_i^2 < 1 \right\}$$

of $\mathbf{R}^N$. Denote by E the space $\mathbf{R}^N$ with the balanced convex set $\widetilde{B}_1$ considered as unit ball and by F the space $\mathbf{R}^N$ where the unit ball is the set

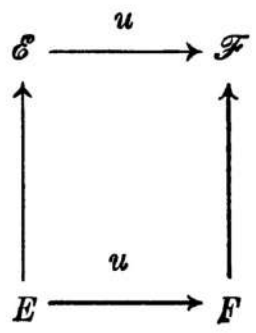
$$\widetilde{B}_2 = \left\{ x \mid x \in \mathbf{R}^N, \sum_{i=1}^{n} \alpha_i^2 x_i^2 + \alpha^2 \lambda^2 \sum_{i=n+1}^{N} x_i^2 \leqslant 1 \right\}.$$

The square of the Hilbert-Schmidt norm of the identity map $I: E \to F$ is then

$$\| I \|_2^2 = \sum_{i=1}^{n} \alpha_i^2 + (N - n)\, \alpha^2$$

(cf. p. 214). It follows from Lemma 2 of Chap. 3, §3 that μ is concentrated up to $\epsilon = C \max(\eta, \| I_2 \|_2^2)$ on $\widetilde{B}_2$. If we let λ tend to $+\infty$, we see that μ is concentrated up to ϵ on B_2. Then if we let α tend to zero, $\| I \|_2$ will tend to S, and we see that we can choose $\epsilon = C \max(\eta, S)$.

LEMMA 2. *Let $\mathscr{E}$ and $\mathscr{F}$ be two locally convex Hausdorff spaces and $u: \mathscr{E} \to \mathscr{F}$ a continuous linear map. Let E be a Hilbert subspace of $\mathscr{E}$* (§2, Def. 4) *and F a Hilbert subspace of $\mathscr{F}$. Let us suppose that u maps E into F and that $u: E \to F$ is a Hilbert-Schmidt map with norm $\| u \|_2 \leqslant S$. Let μ be a cylindrical measure on $\mathscr{E}$ concentrated scalarly up to η on the unit ball of E. Then the cylindrical measure $u(\mu)$ on $\mathscr{F}$ is concentrated cylindrically up to $\epsilon = C \max(\eta, S)$ on the unit ball of F. Furthermore, for every $\epsilon' > \epsilon$ it is concentrated cylindrically up to ϵ' on a compact subset of F.*



PROOF. Let G be a finite-dimensional Hausdorff topological vector space and $g: \mathscr{F} \to G$ a continuous linear map. The kernel $\mathscr{N} = (g \circ u)^{-1}(0)$ of $g \circ u$ is a closed subspace of finite codimension of $\mathscr{E}$ and $N = \mathscr{N} \cap E$ is a closed subspace of finite codimension of E. Let $E_1 = N^\perp$ be the orthogonal complement of N in E and $\mathscr{E}_1$ a supplement of $\mathscr{N}$ containing E_1 in $\mathscr{E}$. Let p_1 be the continuous projector from $\mathscr{E}$ onto the finite-dimensional subspace $\mathscr{E}_1$ parallel to $\mathscr{N}$. The restriction of p_1 to E is the orthogonal projector onto E_1.

If $\mathscr{F}_2$ is the image of $\mathscr{E}_1$ in $\mathscr{E}$ with respect to u, we have the maps

$$\mathscr{E} \xrightarrow{p_1} \mathscr{E}_1 \xrightarrow{u/\mathscr{E}_1} \mathscr{F}_2 \xrightarrow{g} G$$

and since $g \circ u: \mathscr{E}_1 \to G$ is injective, $u|\mathscr{E}_1 : \mathscr{E}_1 \to \mathscr{F}_2$ is bijective. Furthermore the map u from E_1 onto its image $F_2 \subset F$ is also bijective and we have the commutative diagram

$$\begin{array}{ccccccc} \mathscr{E} & \xrightarrow{p_1} & \mathscr{E}_1 & \xrightarrow{u} & \mathscr{F}_2 & \xrightarrow{g} & G \\ \uparrow & & \uparrow & & \uparrow & & \\ E & \xrightarrow{p_1} & E_1 & \xrightarrow{u} & F_2 & & \end{array}$$

Since $\mathscr{E}_1$ is finite-dimensional, $p_1(\mu)$ is a Radon measure on $\mathscr{E}_1$. Furthermore $p_1(\mu)$ is concentrated scalarly up to η on the unit ball of E_1. Let us denote by $\widetilde{F}_2$ the Hilbert space whose underlying vector space is that of F_2 and whose Hilbert structure is obtained from that of E_1 with the help of the bijection $u \mid E_1$. Then $(u \circ p_1)(\mu)$ is concentrated scalarly up to η on the unit ball of $\widetilde{F}_2 \subset \mathscr{F}_2$. We see exactly as in the proof of Lemma 2 of Chap. 3, § 3, that we can identify $\mathscr{F}_2$ with $\mathbf{R}^N$, the Hilbert space $\widetilde{F}_2$ with $\mathbf{R}^n_{B_1}$ and the Hilbert space F_2 with $\mathbf{R}^N_{B_2}$, and therefore Lemma 1 implies that $(u \circ p_1)(\mu)$ is concentrated up to $\epsilon = C \max(\eta, S)$ on the unit ball of F_2.

It follows that $(g \circ u \circ p_1)(\mu) = (g \circ u)(\mu)$ is concentrated up to ϵ on the image by g of the unit ball of F_2 and a fortiori on the image of the

unit ball of F. By definition this means that $u(\mu)$ is concentrated cylindrically up to ϵ on the unit ball of F.

Finally, it follows from Prop. 6 of Chap. 3, §3 and the remark made after its proof, that the map $u: E \to F$ can be factored into $E \xrightarrow{v} \mathscr{H} \xrightarrow{w} F$, where $\mathscr{H}$ is a Hilbert subspace of F, w is a compact operator and v is a Hilbert-Schmidt operator whose norm $\|v\|_2 > \|u\|_2$ can be taken arbitrarily close to $\|u\|_2$. Choose $\|v\|_2$ so that $\epsilon = C \max(\eta, S) \leqslant C \max(\eta, \|v\|_2) < \epsilon'$. Then, by the first part of the proof, $v(\mu)$ is cylindrically concentrated up to ϵ' on the unit ball of $\mathscr{H}$, hence $u(\mu) = w(v(\mu))$ is cylindrically concentrated up to ϵ' on a compact subset of F.

PROOF OF THEOREM 2. Given $\epsilon > 0$ choose $\eta = S = \epsilon/C$. Then $C \max(\eta, S) = \epsilon$. There exists $A \in \mathfrak{S}$ such that μ is concentrated scalarly up to η on A and by Prop. 6 (§2) we may assume that A is a Hilbert set. Again by Prop. 6 there exists a Hilbert set $B \in \mathfrak{S}$ containing A such that the canonical map $E_A \to E_B$ is Hilbert-Schmidt. Since $\mathfrak{S}$ is invariant with respect to homothecies, we can choose B so large that the Hilbert-Schmidt norm of this map is $< S$. Then by Lemma 2 the cylindrical measure μ is concentrated cylindrically up to ϵ on B.

Again, by Lemma 2, for any $\epsilon' > \epsilon$ the cylindrical measure μ is concentrated cylindrically up to ϵ' on a compact subset of E_B, hence a fortiori on a compact subset of E. Thus μ is a Radon measure (p. 188).

COROLLARY 1. *If E is cc-conuclear* (p. 233) *then every cylindrical measure scalarly concentrated on the balanced, convex, compact subsets of E is a Radon measure.*

This applies in particular to the spaces $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{O}_M$, $\mathscr{O}'_M$, $\mathscr{O}_C$, $\mathscr{O}'_C$, $\mathbf{R}^N$, $\mathbf{R}^{(N)}$ whose duals equipped with the cc-topology (equal to the strong topology) are nuclear.

COROLLARY 2. *If E is cc-conuclear, then every continuous function of positive type on E'_{cc} is the Fourier transform of a Radon measure on E.*

Indeed, by Theorem 1 of Chap. 2, § 3 such a function is the Fourier transform of a cylindrical measure concentrated scalarly on the balanced, convex, compact subsets of E.

COROLLARY 3. *Let F be a nuclear space. Every scalarly ϵ-concentrated cylindrical measure on $\sigma(F', F)$ is an ϵ-concentrated Radon measure on $\sigma(F', F)$ and F'_{cc}. Every continuous function of positive type on F is the Fourier transform of a Radon measure on $\sigma(F', F)$ or F'_{cc}.*

For $\sigma(F', F)$, it is a particular case of the theorem for $E = \sigma(F', F)$. For F'_{cc}, we simply observe that the equicontinuous subsets of F' are relatively compact for F'_{cc}, therefore the Radon measure μ is also a Radon measure on F'_{cc}.

REMARK. It is also true that the *Radon* measure μ is a Radon measure on F'_c, since the equicontinuous subsets of F' are relatively compact in F'_c. But, if F is not quasicomplete, then the dual of F' is no longer F, and a *cylindrical* measure on $\sigma(F', F)$ *is not* a cylindrical measure on F'_c, *it is therefore wrong to say that the cylindrical measure μ is a Radon measure on F'.*

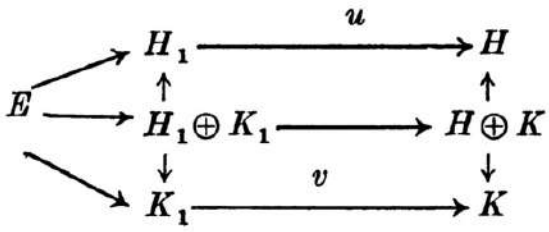
4. The theorem of Sazonov-Badrikian. We shall now generalize Corollary 2 of Theorem 2 by giving a necessary and sufficient condition for a function of positive type to be the Fourier transform of a Radon measure.

Let E be a locally convex Hausdorff space and H a Hilbert space. In this section we shall call a continuous linear map $f: E \to H$ a *Hilbert-Schmidt map* if it can be factored into $E \xrightarrow{\alpha} H_1 \xrightarrow{u} H$, where H_1 is a Hilbert space, α is a continuous linear map and u is a Hilbert-Schmidt map.

The *Hilbert-Schmidt topology* τ_{HS} on E (or the Hilbert-Schmidt topology associated with the topology of E) will be the coarsest topology on E for which all Hilbert-Schmidt maps are continuous.

The Hilbert-Schmidt maps form a directed system. More precisely, if $f: E \to H$ and $g: E \to K$ are Hilbert-Schmidt maps, then the

map $e \longmapsto (f(e), g(e))$ from E into $H \oplus K$ is a Hilbert-Schmidt map. Indeed, we have the commutative diagram



where u and v are Hilbert-Schmidt maps and thus w: $(h, k) \longmapsto (u(h), v(k))$ is also a Hilbert-Schmidt map.

It follows that a fundamental system of neighbourhoods of 0 for τ_{HS} is given by the sets $f^{-1}(B)$, where f: $E \to H$ is a Hilbert-Schmidt map and B is a ball in H. We shall now describe another fundamental system of neighbourhoods of 0 for τ_{HS}, which does not involve the notion of "all" Hilbert-Schmidt maps.

Let f: $E \to H$ be a Hilbert-Schmidt map which, according to the definition, is factored into $f = u \circ \alpha$. Then $N_f = \mathrm{Ker}(f) \supset N_\alpha = \mathrm{Ker}(\alpha)$ and we have a canonical surjection $E/N_\alpha \to E/N_f$. If B is an open ball in H, then $U = f^{-1}(B)$ is a pre-Hilbert neighbourhood of 0 in E. Also the set $u^{-1}(B)$ contains an open ball $B_1 \subset H_1$, hence $V = \alpha^{-1}(B_1)$ is a pre-Hilbert neighbourhood of 0 in E and $V = \alpha^{-1}(B_1) \subset \alpha^{-1}(u^{-1}(B)) = f^{-1}(B) = U$. Now the pre-Hilbert structure of E_U is obtained from that of H, where B is taken to be the new unit ball. If ρ is the radius of B, then the new norm $\|x\|'$ is given in terms of the original norm $\|x\|$ by the formula

$$\|x\|' = \frac{1}{\rho}\|x\|.$$

Similarly, if ρ_1 is the radius of B_1, then the norm $\|y\|'$ defined by B_1 and induced on E_V is given by $\|y\|' = \rho_1^{-1}\|y\|$. It follows that the canonical map $\widehat{E}_V \to \widehat{E}_U$ is Hilbert-Schmidt with a Hilbert-Schmidt norm $= \frac{\rho_1}{\rho}\|u\|_2$.

Conversely let U be a pre-Hilbert neighbourhood of 0 in E and assume that there exists a pre-Hilbert neighbourhood V of 0 in E

such that $V \subset U$ and the canonical map $u: \hat{E}_V \to \hat{E}_U$ is Hilbert-Schmidt. Then the canonical map $f: E \to \hat{E}_U$ can be factored into $f = u \circ \alpha$, where $\alpha: E \to \hat{E}_V$ is the canonical map. Thus $U = f^{-1}(B)$, where B is the open unit ball of $\hat{E}_U$, is a neighbourhood of 0 for τ_{HS}. We have proved the following result.

PROPOSITION 7. *Let E be a locally convex Hausdorff space. A fundamental system of neighbourhoods of 0 in E for the Hilbert-Schmidt topology is given by all balanced, convex, open, pre-Hilbert neighbourhoods U of 0 for which there exists a balanced, convex, open, pre-Hilbert neighbourhood V of 0 such that $V \subset U$ and the canonical map $\hat{E}_V \to \hat{E}_U$ is a Hilbert-Schmidt operator.*

In particular, if E is a nuclear space, then τ_{HS} coincides with the topology of E.

THEOREM 3 (Sazonov-Badrikian)[1]. *Let E be a quasi-complete, locally convex Hausdorff space and let E'_c be its dual equipped with the topology of uniform convergence on compact subsets of E. Assume that every compact subset of E is contained in a balanced, convex, compact Hilbert set. A function of positive type ϕ on E'_c is the Fourier transform of a Radon measure on E if and only if it is continuous for the Hilbert-Schmidt topology associated with the topology of E'_c.*

PROOF. (a) Assume that ϕ is continuous on $(E'_c)_{HS}$. Without restricting the generality we may assume that $\phi(0) = 1$. Given $\eta > 0$, there exist two compact Hilbert sets $A \subset B \subset E$ such that the canonical map $(E'_{B^\circ})^\wedge \to (E'_{A^\circ})^\wedge$ is Hilbert-Schmidt and $|1 - \phi(\xi)| \leqslant \eta$ on A°. By Prop. 2 of Chap. 2, §2, there exists a cylindrical measure μ on E such that $\mathscr{F}(\mu) = \phi$. By Lemma 4 of Chap. 2, §3, the cylindrical measure μ is concentrated scalarly up to $k\eta$ on $\frac{1}{\eta} A$. Now the canonical map $E_A \to E_B$ is the transpose of the map $(E'_{B^\circ})^\wedge \to (E'_{A^\circ})^\wedge$ and therefore a Hilbert-Schmidt operator; let S be its Hilbert-Schmidt norm. It follows from Lemma 3 of Chap. 3, §3 that μ is concentrated cylindrically up to

(1) The Hilbert case is due to Sazonov, [19], the general case to Badrikian [22].

$$\epsilon = C \max. \left(k\eta, \frac{S}{\rho}\right),$$

on $\frac{\rho}{\eta} B$, since the Hilbert-Schmidt norm of the map $E_{(1/\eta)A} \to E_{(\rho/\eta)B}$ is $\frac{1}{\rho} S$. Taking η small and ρ large, ϵ is arbitrarily small. Since $\frac{\rho B}{\eta}$ is compact, μ is cylindrically concentrated on the collection of compact sets of E and thus it is a Radon measure (p. 188).

(b) Let H be a Hilbert space and μ a probability on the unit ball B of H. Let H'_μ be the vector space of all equivalence classes of linear forms on H whose restrictions to B are square-μ-integrable. For any $\dot{f} \in H'_\mu$ let us set

$$\Phi(\dot{f}) = \int e^{-2\pi i f(x)} d\mu(x).$$

Φ is a function of positive type and it is continuous for the topology induced on H'_μ by L^2_μ. Indeed,

$$|\Phi(\dot{f}) - \Phi(0)| \leqslant \int |e^{-2\pi i f(x)} - 1| \, d\mu(x)$$

$$\leqslant 2\pi \|f\|_{L^1_\mu} \leqslant \text{const.} \ \|f\|_{L^2_\mu},$$

since for real u we have $|e^{-iu} - 1| \leqslant |u|$. Now let $\phi = \mathscr{F}(\mu)$. Then for any $\xi \in H'$ we have $\varphi(\xi) = \Phi(\dot{\xi})$, i.e. φ is the inverse image of Φ with respect to the map $\xi \longmapsto \dot{\xi}$ from H' into H'_μ. Since Φ is continuous on H'_μ, it is the Fourier transform of a cylindrical measure λ on the dual H_μ of H'_μ, concentrated scalarly on the balls of H_μ.

The measure μ is the image of λ with respect to the map u: $H_\mu \to H$, transpose of the map $H' \to H'_\mu$. Indeed, by formula (4) of Chap. 2, §2 we have

$$\mathscr{F}(u(\lambda)) = \mathscr{F}(\lambda) \circ {}^t u = \Phi \circ {}^t u = \phi = \mathscr{F}(\mu).$$

On the other hand, the map $H' \to H'_\mu$ is Hilbert-Schmidt. In fact, if $(\xi_i)_{i \in I}$ is an orthonormal basis of H', then

$$\sum_{i \in I} \|\dot{\xi}_i\|^2_{H'_\mu} = \sum_{i \in I} \int |<x, \xi_i>|^2 d\mu(x)$$

$$= \int \sum_{i \in I} |<x, \xi_i>|^2 \, d\mu(x)$$

$$= \int \|x\|^2 \, d\mu(x) < \infty.$$

Thus we have proved that the Fourier transform ϕ of a probability μ on the unit ball B of a Hilbert space H is continuous on its dual H' for the Hilbert-Schmidt topology associated with the strong topology of H'.

Let now μ be a probability on E and $\phi = \mathscr{F}(\mu)$. Given $\epsilon > 0$ there exists a balanced, convex, compact Hilbert set A in E such that $\mu(A) \geqslant 1 - \epsilon$. Let χ_A be the characteristic function of A. By what we have proved above, $\mathscr{F}(\chi_A \mu)$ is a continuous function on $(E_A)'_{HS}$, hence a fortiori on $(E'_c)_{HS}$. Now for every $\xi \in E'$ we have $|\phi(\xi) - \mathscr{F}(\chi_A \mu)(\xi)| \leqslant \int_{\mathscr{C}_A} d\mu(\xi) \leqslant \epsilon$, hence ϕ is the uniform limit of the continuous functions $\mathscr{F}(\chi_A \mu)$, and is therefore itself continuous.

SUMMARY OF CHAPTER V

This long chapter establishes a relationship between cylindrical measures and random functions. No theory of probability is assumed to be known, everything is self-contained. The § 1 gives the basic concepts of probability theory: the sample space (page 244), convergence in probability or in measure (page 247), relationship with almost sure convergence (Proposition 1, page 248), narrow topology (page 249), convergence in probability implies convergence in law (Proposition 2, page 250); a consequence is given for cylindrical probabilities, Theorem 1, page 253, which is rather an exercise.

§2 deals with random functions. The definition is given, page 254. Canonical random functions are defined, page 254, with examples and counterexamples, and the notion of isonomy. Almost sure properties are defined for random functions, page 255. The main result of this § is the bijective correspondence between cylindrical measures on E and isonomy classes of linear random functions on E', page 256. Propositions 4, 5, 6, 7, connect properties of cylindrical measures and properties of the associated random functions. Then comes the most important connection: between scalar concentration of cylindrical measures and continuity of the associated random functions, Theorem 2, page 265. Theorem 4, page 268, inverts the role of the space and its dual. Propositions 4, page 260, and 8, page 267, and Theorem 3, page 267, connect the case of Radon measures with the almost sure properties of the random functions. The § 3 gives applications, mainly to random sequences. The Proposition 9, page 271, is the most important, for cylindrical measures on l^p, $1 < p < +\infty$ while the cases $p = 1$, $p = \infty$, are given later on, page 281; Proposition 10, page 271, gives the corresponding case for Radon measures with almost sure convergence. Theorem 6, page 273, gives then the "only if" part of Sazonov theorem, announced in Chapter III. Theorem 7, page 276, gives an analogous application of Minlos theorem. Then one studies, page 277, the random function associated to a cylindrical probability on the space of continuous functions on a compact space. Random functions and random distributions on a manifold are studied, page 279. On page 281, one comes back to the

case of the spaces l^p, but for $p = 1$ and $p = +\infty$, which are more difficult; Theorems 8, page 282, and 9, page 287, give properties of scalarly l^p sequences in topological vector spaces (not necessarily locally convex), which are useful in all these problems (and also in the theory of p-radonifying operators, not developed in this book).

The §4 is very important. Instead of considering a projective system of measures, as in the first part, page 74, one considers here a projective system of random variables. Theorem 10, page 291, gives the necessary and sufficient condition for such a system to define a random variable, the projective limit; the statement of the theorem is rather long, and probably not all the possibilities developped in it are useful; the main applications are given by the corollary, page 294. Applications are given to decomposed and decomposing continuous linear maps in topological vector spaces, Definition 1, page 295, Theorem 11, page 296; Proposition 11 applies then Minlos's theorem to the case of conuclear spaces. More generally, Theorem 12, page 291, gives the correspondence, through duality, between radonifying and decomposing maps. The end of the section gives complements or exercises.

The §5 studies the random functions from the point of view of category theory; it is of secondary importance.

V

CYLINDRICAL MEASURES AND PROBABILITY THEORY

1. Basic Concepts of Probability Theory. A *sample space* (or probability space) is a pair (Ω, m) consisting of a topological space Ω and a probability, i.e. a finite positive Radon measure m with total mass 1, defined on Ω. If there is no possibility of confusion, we shall speak simply about the sample space Ω without mentioning explicitly the probability m.

An m-measurable subset E of Ω will be called an *event*; a point $\omega \in \Omega$ will be called an *elementary event*. The probability $\mathscr{P}(E)$ of an event E is, by definition, its measure $m(E)$. Two events E and F are said to be *independent* if $\mathscr{P}(E \cap F) = \mathscr{P}(E).\mathscr{P}(F)$.

Let X be a topological space. We say that two m-measurable maps from Ω into X are equivalent if they coincide almost everywhere with respect to m. An equivalence class of m-measurable maps $\Omega \to X$ will be called a *random variable* with values in X, and we shall denote by $\mathrm{Meas}(\Omega, m; X)$† the set of all random variables with values in X. If $X = \mathbf{R}$, we shall speak about real random variables.

Let $x: \Omega \to X$ be a random variable. The image measure $x(m)$ is a probability on X and is called the *probability distribution* of x. If A is an $x(m)$-measurable subset of X, then its measure

$$(x(m))(A) = m(x^{-1}(A)) = \mathscr{P}(\{\omega \mid \omega \in \Omega,\ x(\omega) \in A\})$$

will be denoted simply by $\mathscr{P}\{x \in A\}$ and considered to be the probability that $x(\omega)$ shall lie in A.

Let X and Y be two topological spaces, and $x: \Omega \to X$, $y: \Omega \to Y$ two random variables. They define a random variable (x, y): $\omega \longmapsto (x(\omega), y(\omega))$ with values in $X \times Y$, whose probability distribution is $(x, y)\,(m)$. If A is an $x(m)$-measurable subset of X and

† This space is also denoted $L^0(\Omega, m; X)$

B a $y(m)$-measurable subset of Y, we have $(x, y)\,(m)\,(A \times B) = m(x^{-1}(A) \cap y^{-1}(B))$. The two random variables x and y are said to be *independent* if $(x, y)\,(m) = x(m) \otimes y(m)$. This can be expressed by saying that for arbitrary measurable $A \subset X$ and $B \subset Y$ we have $\mathscr{P}\{(x, y) \in A \times B\} = \mathscr{P}\{x \in A\}.\ \mathscr{P}\{y \in B\}$, i.e. $m(x^{-1}(A) \cap y^{-1}(B)) = m(x^{-1}(A)).\ m(y^{-1}(B))$. The last condition means that the events $x^{-1}(A)$ and $y^{-1}(B)$ are independent.

We can define similarly the independence of a finite or countably infinite family $(x_i)_{i \in I}$ of random variables $x_i: \Omega \to X_i$. If the family is not countable, then this definition is meaningless since the map (x_i): $\omega \longmapsto (x_i(\omega_i))$ from into $\prod_{i \in I} X_i$ is not necessarily m-measurable. In this case we say that $(x_i)_{i \in I}$ is independent if every finite or countable subfamily is independent.

Let now (Ω, m) and (Ω', m') be two sample spaces and X a topological space. Two random variables $x: \Omega \to X$ and $x': \Omega' \to X$ will be *isonomous* if they have the same probability distribution on X, i.e. if $x(m) = x'(m')$.

EXAMPLE 1. We consider a coin being tossed twice. The sample space consists of four points, $\Omega = \{HH, HT, TH, TT\}$ where $H =$ head, $T =$ tail, and for instance the symbol HT means that the outcome of the first toss is head, of the second tail. On Ω we put the discrete topology and each point has measure (probability) $\frac{1}{4}$. Let now X be the space $\{H, T\}$ equipped with the discrete topology. Define two random variables x and $x': \Omega \to X$, where x gives the outcome of the first toss and x' gives the outcome of the second, i.e.

$$x(HH) = H, \quad x(HT) = H, \quad x(TH) = T, \quad x(TT) = T,$$

$$x'(HH) = H, \quad x'(HT) = T, \quad x'(TH) = H, \quad x'(TT) = T.$$

It is easy to check that x and x' are independent and isonomous.

Consider now one single toss counted twice. If the sample space is $\Omega = \{HH, HT, TH, TT\}$, then HH and TT have probability $\frac{1}{2}$, while HT and TH have probability 0. Let the random variable x and x' be defined as before. They are clearly again isonomous,

but they are not independent. Indeed, we have, for instance, $\mathscr{P}\{x = H, x' = H\} = \frac{1}{2}$, while $\mathscr{P}\{x = H\}.\ \mathscr{P}\{x' = H\} = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$.

Assume now that $(X_i)_{i\in I}$ is a family of completely regular spaces and let $(x_i)_{i\in I}$ be a family of random variables $x_i\colon \Omega \to X_i$. We shall say that the family (x_i) is *canonical* (or *regular*) if there exists a random variable x with values in the product $\boldsymbol{X} = \prod_{i\in I} \check{X}_i$ of the Čech compactifications of the spaces X_i, such that $x_i = \pi_i \circ x$ for each $i \in I$, where π_i denotes the projection $\boldsymbol{X} \to \check{X}_i$.

If $x\colon (\Omega, m) \to \boldsymbol{X}$ and $x'\colon (\Omega, m) \to \boldsymbol{X}$ are two random variables such that $\pi_i \circ x = \pi_i \circ x'$ for all $i \in I$, then $x = x'$. Indeed, let K be a compact subset of Ω such that the restrictions to K of both x and x' are continuous and assume also that the support of the restriction m_K of m to K is K, i.e. that K is m-full. Then x and x' coincide almost everywhere with respect to m on K, and therefore also on Ω, i.e. they define the same random variable.

We shall say that two families $(x_i)_{i\in I}$ and $(x'_i)_{i\in I}$ of random variables $x_i\colon \Omega \to X_i$ and $x'_i\colon \Omega' \to X_i$ are isonomous if for every finite subset J of I they define the same probability distribution on $\prod_{i\in J} X_i$. Given a family (x_i) of random variables with values in the completely regular spaces X_i, there exists a canonical family (x'_i) of random variables isonomous to (x_i). Indeed, for every finite subset J of I the map $(x_i)_{i\in J}\colon \Omega \to \boldsymbol{X}_J = \prod_{i\in J} X_i$ defined by $\omega \longmapsto (x_i(\omega))_{i\in J}$ has a probability distribution μ_J on $\boldsymbol{X}_J$. These probabilities μ_J form a projective system and have as their projective limit a probability μ on $X = \prod_{i\in I} \check{X}_i$. If we choose $\Omega' = \boldsymbol{X}$, $m' = \mu$ and for x'_i the projections $\boldsymbol{X} \to \check{X}_i$, then (x'_i) is the required family.

We shall now define two modes of convergence for random variables and their probability distributions.

First let $\boldsymbol{X}$ be a uniform space. We say that a sequence (x_i) of elements of $\mathrm{Meas}(\Omega, m; \boldsymbol{X})$ converges to $x \in \mathrm{Meas}(\Omega, m; \boldsymbol{X})$ *in*

probability or *in measure* if for every $\delta > 0$ and every open uniformity (= "entourage") $\mathfrak{U}$ of X there exists an integer i_0 such that

$$\mathscr{P}\{(x_i, x) \notin \mathfrak{U}\} = m(\{\omega \mid (x_i(\omega), x(\omega)) \notin \mathfrak{U}\}) < \delta$$

for $i \geqslant i_0$. This convergence corresponds to a uniform structure on Meas $(\Omega, m; X)$, called the uniform structure of the convergence in probability, and for which a fundamental system $\mathscr{B}$ of uniformities is defined as follows. For any $\delta > 0$ and for any open uniformity $\mathfrak{U}$ of X let $\mathscr{W} = \mathscr{W}(\delta, \mathfrak{U})$ be the set of all pairs (x, y) of elements of Meas $(\Omega, m; X)$ such that $\mathscr{P}\{(x, y) \notin \mathfrak{U}\} < \delta$. The sets $\mathscr{W}$ form $\mathscr{B}$ [4, Chap. IV, 2nd ed., § 5, n°11]. Except when it is explicity written, Meas $(\Omega, m; X)$ will be always equipped with this uniform structure.

EXAMPLE 2. Let $\Omega = \mathbf{T}$ the one-dimensional torus, m the Lebesgue measure on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, and $X = \mathbf{R}$. Let (I_n) be a sequence of intervals of length $1/n$, and assume that each point of Ω is contained in infinitely many intervals I_n. Denote by χ_A the characteristic function of the set A and consider the random variables $\chi_n = n\chi_{I_n} : \Omega \to X$. The sequence (x_n) converges in probability to 0 but it does not converge at any point, since it is even unbounded at every point.

Suppose that X is a Hausdorff topological vector space. Then two elements of Meas $(\Omega, m; X)$ can be added, and an element of Meas $(\Omega, m; X)$ can be multiplied by a scalar in such a way that Meas $(\Omega, m; X)$ becomes a vector space. Furthermore the uniform structure of the convergence in probability is associated with a topology compatible with the vector space structure of Meas $(\Omega, m; X)$. Indeed, for any balanced, open neighborhood U of the origin in X and for any $\delta > 0$, let $W = W(\delta, U)$ be the set of all those $x \in$ Meas $(\Omega, m; X)$ for which $\mathscr{P}\{x \notin U\} < \delta$. A fundamental system of neighborhoods $\mathscr{N}$ of 0 is then given by all the sets W. It is trivial to check that $\mathscr{N}$ is a filter basis consisting of balanced sets and that given $W \in \mathscr{N}$ there exists $W_1 \in \mathscr{N}$ such that $W_1 + W_1 \subset W$. Thus to show that $\mathscr{N}$ defines a topology compatible with the vector space structure of Meas $(\Omega, m; X)$, it remains to prove that every $W \in \mathscr{N}$ is absorbing [3, Chap. I, 2nd ed,. § 1, n° 5, Prop. 4 and Remark 2]. Thus let x be

an element of Meas $(\Omega,m;X)$ and $W=W(\delta,U)$ an element of $\mathcal{N}$. For every integer $n \geq 1$ let E_n be the set of those $\omega \in \Omega$ for which $\frac{1}{n} x(\omega) \notin U$. Then $E_n \supset E_{n+1}$ and the measure of $\bigcap E_n$ is zero since U is absorbing. There exists an integer n_0 such that $m(E_{n_0}) < \delta$ and consequently $\frac{1}{\lambda} x \in W$ for $|\lambda| \geq n_0$ since $\frac{1}{\lambda} x(\omega) \notin U$ implies $\frac{1}{n_0} x(\omega) \notin U$, i.e. $\omega \in E_{n_0}$.

Let us observe that on the other hand if m has no point mass then Meas $(\Omega, m; X)$ is not locally convex, even if $X = \mathbf{R}$. Indeed, let $\delta > 0$ and let (Ω_j) be a finite partitioning of Ω such that $m(\Omega_j) < \delta$ for each j. Every $x \in$ Meas $(\Omega, m; X)$ can be written in the form $x = 1/n \, \Sigma n . \chi_{\Omega j} x$ and $n\chi_{\Omega j} x \in W(\delta, U)$ for any neighborhood U of 0. This proves that the convex hull of $W(\delta, U)$ is the entire space which is therefore the only convex neighborhood of the origin.

Let now X be again an arbitrary topological space. A sequence (x_i) of elements of Meas $(\Omega, m; X)$ converges *almost surely* (or *almost everywhere*) to $x \in$ Meas $(\Omega, m; X)$ if the set of those points $\omega \in \Omega$ for which $x_i(\omega)$ does not converge to $x(\omega)$ has measure zero.

PROPOSITION 1. *Let X be a uniform space and (x_i) a sequence of elements of* Meas $(\Omega, m; X)$ *which converges almost surely to the random variable x. Then (x_i) converges to x in probability.*

PROOF. Let $\delta > 0$ and $\mathfrak{U}$ be a uniformity of X. For every integer $n \geq 1$ let E_n be the set of those points $\omega \in \Omega$ for which there exists an $i \geq n$ such that $(x_i(\omega), x(\omega)) \notin \mathfrak{U}$, i.e.

$$E_n = \bigcup_{i \geq n} \{(x_i, x) \notin \mathfrak{U}\}.$$

We have $E_n \supset E_{n+1}$ and the hypothesis means that the measure of $\bigcap E_n$ is zero. Hence there exists an integer n_0 such that $m(E_{n_0}) < \delta$ and consequently $\mathcal{P}\{(x_i, x) \notin \mathfrak{U}\} < \delta$ for $i \geq n_0$.

We know from Example 2 that the converse of Prop. 1 is false.

Given a sequence of random variables, their probability distributions form a sequence of Radon measures, and we shall be

interested in their convergence. We shall therefore examine first the convergence of Radon measures in general.

Let X be a completely regular space and (μ_i) a sequence of finite Radon measures on X. We do not assume that the measures μ_i are necessarily positive. We say that (μ_i) converges *narrowly* to the finite Radon measure μ if for every function ϕ belonging to the space $\mathscr{B}(X)$ of bounded continuous functions $\mu_i(\phi)$ converges to $\mu(\phi)$. More generally, the narrow topology on the vector space $\mathscr{M}^1(X)$ of all finite Radon measures is defined as the topology of pointwise convergence on bounded continuous functions: if for any $\phi_1, \ldots, \phi_p \in \mathscr{B}(X)$ and for any $\epsilon > 0$ we write $U(\phi_1, \ldots, \phi_p; \epsilon) = \{\mu \mid \mu \in \mathscr{M}^1(X), |\mu(\phi_i)| < \epsilon, 1 \leqslant i \leqslant p\}$, then the sets $U(\phi_1, \ldots, \phi_p; \epsilon)$ form a fundamental set of neighborhoods of the origin in $\mathscr{M}^1(X)$ for the narrow topology.†

If X is a locally compact space then there exists another kind of convergence, this time for all Radon measures, not necessarily finite. A sequence (μ_i) of Radon measure on X is said to converge *vaguely* to the Radon measure μ if for every function ϕ belonging to the space $\mathscr{K}(X)$ of continuous functions with compact support $\mu_i(\phi)$ converges to $\mu(\phi)$ [4, Chap. III, 2nd ed., § 1, n°9]. The vague topology on the vector space $\mathscr{M}(X)$ of all Radon measures on X is the topology of pointwise convergence on continuous functions with compact support: if for $\phi_1, \ldots, \phi_p \in \mathscr{K}(X)$ and for $\epsilon > 0$ we write $V(\phi_1, \ldots, \phi_p; \epsilon) = \{\mu \mid \mu \in \mathscr{M}(X), |\mu(\phi_i)| \leqslant \epsilon, 1 \leqslant i \leqslant p\}$, then the sets $V(\phi_1, \ldots, \phi_p; \epsilon)$ form a fundamental system of neighborhoods of the origin in $\mathscr{M}(X)$ for the vague topology. Since $\mathscr{K}(X) \subset \mathscr{B}(X)$, the narrow topology is finer than the vague topology. In general it is strictly finer. Indeed, if e.g. (a_n) is a sequence of points of $\mathbf{R}$ which tends to $+\infty$, then the point measures δ_{a_n} converge to 0 vaguely but not narrowly. If, however, (μ_i) is a sequence or filter of finite positive measures which converges vaguely to μ and if $\mu_i(X)$ converges to $\mu(X)$, then (μ_i) converges to μ narrowly (App. §1, Prop. 5).

† The narrow topology will be defined even if X is not completely regular, see Appendix, §1.

PROPOSITION 2. *Let X be a uniform Hausdorff space. The map $x \longmapsto x(m)$ from* $\mathrm{Meas}(\Omega, m; X)$ *into the set $\mathscr{M}^1_+(X)$ of finite positive Radon measures is continuous, if* $\mathrm{Meas}(\Omega, m; X)$ *is equipped with the topology of convergence in probability and $\mathscr{M}^1(X)$ with the narrow topology.*

We have to prove that if $\mathscr{F}$ is a filter of random variables with values in X, which converges in probability to a random variable $x_0: \Omega \to X$, then its image $\mathscr{F}(m)$ under the map $x \longmapsto x(m)$ converges narrowly to the measure $x_0(m)$.

Let $\epsilon > 0$ and assume that $\phi \in \mathscr{B}(X)$ is uniformly continuous. There exists a uniformity $\mathfrak{U}$ of X such that $(x', x'') \in \mathfrak{U}$ implies $|\phi(x') - \phi(x'')| < \epsilon$. Since $\mathscr{F}$ converges in probability to x_0, there exists an $F \in \mathscr{F}$ such that for $x \in F$ the set $A = \{\omega \mid (x(\omega), x_0(\omega)) \notin \mathfrak{U}\}$ has measure $\leqslant \epsilon$. Thus for $x \in F$ we have

$$\left| \int_\Omega \phi(x(\omega)) dm(\omega) - \int_\Omega \phi(x_0(\omega))\, dm(\omega) \right|$$

$$\leqslant \int_A |\phi(x(\omega)) - \phi(x_0(\omega))|\, dm(\omega) + \int_{\Omega - A} |\phi(x(\omega)) - \phi(x_0(\omega))|\, dm(\omega)$$

$$< 2 \max |\phi(x)|\, .\, \epsilon + \epsilon,$$

i.e. $\mathscr{F}(m)(\phi)$ tends to $x_0(m)(\phi)$. In order to pass to the case when ϕ is not necessarily uniformly continuous, we shall use

LEMMA 1. *Let X be a uniform space, K a compact subset of X, ϕ a continuous function on X and $\delta > 0$. There exists as open, symmetric uniformity $\mathfrak{U}$ of X such that $x' \in K$ and $(x', x'') \in \mathfrak{U}$ imply $|\phi(x') - \phi(x'')| < \delta$.*

PROOF. For each point $a \in K$ there exists a symmetric open uniformity $\mathfrak{U}'_a$ of X such that $(x', a) \in \mathfrak{U}'_a$ and $(x'', a) \in \mathfrak{U}'_a$ imply $|\phi(x') - \phi(x'')| < \delta$. Let $\mathfrak{U}''_a$ be a symmetric open uniformity of X such that $\mathfrak{U}''_a \circ \mathfrak{U}''_a \subset \mathfrak{U}'_a$. The neighborhoods $\mathfrak{U}''_a(a)$ cover the compact set K, hence there exist finitely many among these neighborhoods which already cover K; let $\mathfrak{U}''_{a_i}$, $i = 1, \ldots, n$, be the corresponding uniformities. Finally, let $\mathfrak{U}$ be an open symmetric uniformity of X

such that $\mathfrak{U} \subset \bigcap_{i=1}^{n} \mathfrak{U}''_{a_i}$. This $\mathfrak{U}$ has the required property. Indeed, let $x' \in K$ and $(x', x'') \in \mathfrak{U}$. Since the neighborhoods $\mathfrak{U}''_{a_i}(a_i)$ cover K, there exists an index j such that $x' \in \mathfrak{U}''_{a_j}(a_j)$, i.e. $(x', a_j) \subset \mathfrak{U}''_{a_j} \subset \mathfrak{U}'_{a_j}$. Since $(x', x'') \in \mathfrak{U}''_{a_j}$, we have also $(x'', a_j) \subset \mathfrak{U}''_{a_j} \circ \mathfrak{U}''_{a_j} \subset \mathfrak{U}'_{a_j}$ and consequently $|\phi(x') - \phi(x'')| < \delta$.

PROOF OF PROPOSITION 2. Given $\epsilon > 0$, let K be a compact subset of X such that $x_0(m)(K) \geqslant 1 - \frac{\epsilon}{5}$, i.e. $\mathscr{P}\{x_0 \notin K\} < \frac{\epsilon}{5}$. Let $\phi \in \mathscr{B}(X)$, and we may assume without restricting the generality that $|\phi(x)| < 1$. Set $\delta = \frac{\epsilon}{5}$ and let $\mathfrak{U}$ be the uniformity of X which the Lemma associates with K, ϕ and δ. By hypothesis, there exists an $F \in \mathscr{F}$ such that $\mathscr{P}\{(x, x_0) \notin \mathfrak{U}\} < \frac{\epsilon}{5}$ for $x \in F$.

Let x belong to F and let us estimate the integral

$$\int_{\Omega} |\phi(x(\omega)) - \phi(x_0(\omega))| \, dm(\omega).$$

We divide Ω into three subsets.

(1) The $\omega \in \Omega$ such that $x_0(\omega) \notin K$. The measure of this set is $< \frac{\epsilon}{5}$, hence its contribution to the integral is $< \frac{2\epsilon}{5}$.

(2) The $\omega \in \Omega$ such that $(x(\omega), x_0(\omega)) \notin \mathfrak{U}$. The contribution of this set to the integral is also $< \frac{2\epsilon}{5}$.

(3) The $\omega \in \Omega$ such that $x_0(\omega) \in K$ and $(x(\omega), x_0(\omega)) \in \mathfrak{U}$. By virtue of the lemma we have $|\phi(x(\omega)) - \phi(x_0(\omega))| < \frac{\epsilon}{5}$, hence the contribution of this set to the integral is $< \frac{\epsilon}{5}$.

It follows that for $x \in F$ the integral is $< \frac{2\epsilon}{5} + \frac{2\epsilon}{5} + \frac{\epsilon}{5} = \epsilon$, which proves the proposition.

REMARK 1. Only in particular cases is it true that if $\mathscr{F}(m)$ converges narrowly to $x_0(m)$, then $\mathscr{F}$ converges to x_0 in probability. *For instance, if $\mathscr{F}(m)$ converges narrowly to the Dirac measure δ_a on X, then $\mathscr{F}$ converges in probability to the constant random variable $a: \omega \longmapsto a$.* Indeed, given $\phi \in \mathscr{B}(X)$ and $\epsilon > 0$ there exists $F \in \mathscr{F}$ such that

$$\left| \int_\Omega \phi(x(\omega))\, dm(\omega) - \phi(a) \right| \leqslant \epsilon$$

for $x \in F$, Let $\mathfrak{U}$ be a uniformity of X and choose ϕ so that $\phi(a) = 1$, $\phi(x) = 0$ for $x \notin \mathfrak{U}(a)$, and $0 \leqslant \phi(x) \leqslant 1$. Then $\mathscr{P}\{(x, a) \notin \mathfrak{U}\} \leqslant \epsilon$ for $x \in F$.

REMARK 2. Let X and Y be two completely regular spaces, and $h: X \to Y$ a continuous map. If the finite Radon measures μ_i converge narrowly to the finite Radon measure μ on X, then the $h(\mu_i)$ converge narrowly to $h(\mu)$ on Y. Indeed, if ϕ is a bounded continuous function on Y, then $\phi \circ h$ is a bounded continuous function on X and therefore $h(\mu_i)\ (\phi) = \mu_i(\phi \circ h)$ tends to $\mu(\phi \circ h)$.

COROLLARY. *Let X be a Hausdorff space, Y a uniform Hausdorff space, $\mathscr{F} = (h_\alpha)_{\alpha \in A}$ a directed family of continuous maps $h_\alpha: X \to Y$ and μ a probability on X. Assume that the h_α converge with respect to A to a continuous map $h: X \to Y$, uniformly on every compact subset of X. Then $h_\alpha(\mu)$ converges narrowly to $h(\mu)$ with respect to A.*

By putting $\Omega = X$, $m = \mu$, and replacing Y by X, it is a particular case of the previous proposition.

Let us observe that if in the corollary the filter of sections of A has a countable basis, then it is sufficient to assume that the h_α converge to h pointwise. Indeed, since the measure μ is finite and for each $\phi \in \mathscr{B}(Y)$ the functions $\phi \circ h_\alpha$ are uniformly bounded on X, it follows from Lebesgue's dominated convergence theorem that $h_\alpha(\mu)\ (\phi) = \mu(\phi \circ h_\alpha)$ converges to $h(\mu)\ (\phi) = \mu(\phi \circ h)$.

We shall now prove a kind of partial converse of the corollary of Prop. 2 which will give a necessary and sufficient condition for a

cylindrical measure to be a Radon measure in terms of the narrow convergence of its projections.

Let E be a locally convex Hausdorff space, and let $\mathscr{F} = (p_\alpha)_{\alpha \in A}$ be a directed system of continuous projectors $p_\alpha : E \to E$. We set $p_\alpha(E) = E_\alpha$ and assume that all the spaces E_α are finite-dimensional. Furthermore we assume that for $\beta \leqslant \alpha$ we have $E_\beta \subset E_\alpha$ and $p_\beta \circ p_\alpha = p_\beta$.

THEOREM 1. *Using the notation just explained, assume that p_α tends to the identity map $E \to E$ with respect to A, uniformly on every compact subset of E. Let μ be a cylindrical measure on E, and assume that one of the following conditions is satisfied:*

(a) *μ is scalarly concentrated on the compact subsets of E;*

(b) *μ is scalarly concentrated on the convex weakly compact subsets of E and E is quasi-complete.*

Then μ is a Radon measure on E if and only if the Radon measures $p_\alpha(\mu)$ converge to a Radon measure ν with respect to A for the narrow topology on $\mathscr{M}^1(E)$. If this is the case, then $\mu = \nu$.

PROOF. Taking $X = Y = E$, $h_\alpha = p_\alpha$ and $h =$ the identity map in the corollary of Prop. 2, we see that the condition is necessary. Suppose that the $p_\alpha(\mu)$ converge narrowly to a finite Radon measure ν. It follows from Remark 2 that for any fixed $\beta \in A$ the measures $p_\beta p_\alpha(\mu) = p_\beta(\mu)$ converge to $p_\beta(\nu)$, hence $p_\beta(\mu) = p_\beta(\nu)$ for every $\beta \in A$.

(a) We shall apply Theorem 4 of Chap. 2, §3 taking for $\mathfrak{S}$ the collection of all balanced, compact subsets of E. The projections p_α separate the points of E since $p_\alpha(x)$ tends to x for every $x \in E$. On the other hand the union of the sets ${}^t p_\alpha(E'_\alpha)$ is dense in $E'_{\mathfrak{S}}$ since each $\xi \in E'$ is the limit of $(\xi | E_\alpha) \circ p_\alpha = {}^t p_\alpha(\xi | E_\alpha)$, uniformly on every compact subset of E. Now ν as a Radon measure is scalarly concentrated on the sets of $\mathfrak{S}$. Hence from $p_\beta(\mu) = p_\beta(\nu)$ for every $\beta \in A$, and from the theorem referred to above, we obtain $\mu = \nu$.

(b) Assume now that μ is scalarly concentrated on the convex weakly compact subsets of E. But ν is also concentrated on the balanced, convex, weakly compact subsets of E since E is

quasicomplete. Since $p_\beta(\mu) = p_\beta(\nu)$ for every $\beta \in A$, it follows from the corollary of Theorem 4 of Chap. 2, § 3 that $\mu = \nu$.

2. Random Functions. We keep the notation of §1. Thus let (Ω, m) be a sample space, X a topological space and Meas $(\Omega, m; X)$ the set of all random variables with values in X.

Given an arbitrary set T, a *random function* f on T with values in X will be a map $T \to \text{Meas}\,(\Omega, m; X)$. Thus f associates with every $t \in T$ a random variable $f(t)\colon \Omega \to X$. Unless T has some structure, a random function f is therefore the same thing as a family $(f(t))_{t \in T}$ of random variables, whose index set is T. It follows that the concept of isonomous random functions $f\colon T \to \text{Meas}\,(\Omega, m; X)$ and $f'\colon T \to \text{Meas}\,(\Omega', m'; X)$ is well defined (p. 248).

Two random functions $f\colon T \to \text{Meas}\,(\Omega, m; X)$ and $g\colon S \to \text{Meas}\,(\Omega, m; Y)$ corresponding to the same sample space (Ω, m) are said to be *independent* if for arbitrary finite sequences $t_1, \ldots, t_p \in T$ and $s_1, \ldots, s_q \in S$ the random variables $(f(t_1), \ldots, f(t_p))\colon \Omega \to X^p$ and $(g(s_1), \ldots, g(s_q))\colon \Omega \to Y^q$ are independent.

Let X be a completely regular space. In accordance with the general definition (p. 248), we say that the random function $f\colon T \to \text{Meas}\,(\Omega, m; X)$ is *canonical* (or *regular*) if there exists a random variable $\check{f}\colon \Omega \to \check{X}^T = \mathscr{F}(T, \check{X})$ such that $f(t) = \pi_t \circ \check{f}$ for every $t \in T$, i e., $f(t)(\omega) = \check{f}(\omega)(t)$ for every $t \in T$ and almost every $\omega \in \Omega$. Every random function is isonomous to a canonical random function (p. 248), so that we shall often assume that a given random function is canonical. We want to exhibit, however, an example of a random function which is not canonical, and which we shall need later on.

EXAMPLE 3. Let T, Ω and X be the compact interval $[0, 1]$ and m the Lebesgue measure dx on Ω. For $f\colon T \to \text{Meas}\,(\Omega, m; X)$ we take the random function which associates with $t \in [0, 1]$ the random variable $f(t)\colon \Omega \to X$ given by $f(t)(\omega) = Y(\omega - t)$, where Y is the Heaviside function. If f were canonical, there would

exist a random variable $\check{f}: \Omega \to [0, 1]^{[0,1]}$ such that $\check{f}(\omega)(t) = f(t)(\omega)$ for $t \in T$ and almost every $\omega \in \Omega$. Setting $K(\omega, t) = \check{f}(\omega)(t) \in [0,1]$ we see that the existence of $\check{f}$ is equivalent to the existence of a map $(\omega, t) \to K(\omega, t)$ from $[0, 1] \times [0, 1]$ into $[0, 1]$ such that

(a) for every $t \in [0, 1]$ the function $\omega \longmapsto K(\omega, t)$ is equal to $\omega \longmapsto Y(\omega - t)$ almost everywhere with respect to m;

(b) the map which associates with each $\omega \in \Omega$ the map $K(\omega,.)$: $t \longmapsto K(\omega, t)$ is m-measurable from $\Omega = [0, 1]$ into $X^T = [0, 1]^{[0,1]}$.

We can say very roughly that $K(\omega, t)$ must be equal almost everywhere to $Y(\omega - t)$ but more care than this must be taken with the exceptional set.

Let us show that such a map K cannot exist. Let Ω' be an m-full compact subset of Ω such that the restriction to Ω' of the map $\omega \to K(\omega,.)$ from Ω into X^T is continuous. Let us consider a fixed point $s \in \Omega'$. The function $\omega \longmapsto K(\omega, s)$ must be continuous on Ω' as the composition of the continuous map $\omega \longmapsto K(\omega.) = (K(\omega, t))_{t \in T}$ and of the projection $(K(\omega, t))_{t \in T} \longmapsto K(\omega, s)$. On the other hand $K(\omega, s)$ must be equal to $Y(\omega - s)$ for almost every ω. Since Ω' is m-full, the continuity of $\omega \longmapsto K(\omega, s)$ implies that $K(\omega, s) = 0$ for $\omega < s$ and $K(\omega, s) = 1$ for $\omega > s$. But $\omega \longmapsto K(\omega, s)$ is also continuous at the point $\omega = s \in \Omega'$, therefore s must be an isolated point of Ω' either to the left or to the right.

Now s was an arbitrary point of Ω' and therefore every point of Ω' is isolated either to the left or to the right. It follows that the set Ω' is countable and therefore of m-measure zero. But if condition (b) were verified, we should be able to choose sets Ω' whose m-measure is arbitrarily close to 1.

Suppose that T is a topological space. We say that the canonical random function $f: T \to \mathrm{Meas}(\Omega, m; X)$ is *almost surely continuous* if for every ω, which does not belong to some subset of Ω with m-measure zero, the function $\check{f}(\omega)$ maps T into X (and not only into $\check{X}$), and $\check{f}(\omega): T \to X$ is continuous.

If both T and X are vector spaces, then we can speak of *linear random functions*, i.e. of random functions $f: T \to \mathrm{Meas}(\Omega, m; X)$ which for every $\lambda_1, \lambda_2 \in \mathbf{R}$ and $t_1, t_2 \in T$ satisfy the relation

$$f(\lambda_1 t_1 + \lambda_2 t_2) = \lambda_1 f(t_1) + \lambda_2 f(t_2)$$

in the sense of equality of random variables. If f is canonical then this is equivalent to saying that for given $\lambda_1, \lambda_2 \in \mathbf{R}$ and $t_1, t_2 \in T$ one has

$$\check{f}(\omega)(\lambda_1 t_1 + \lambda_2 t_2) = \lambda_1 \check{f}(\omega)(t_1) + \lambda_2 \check{f}(\omega)(t_2) \qquad (1)$$

except for those ω which belong to a subset of m-measure zero of Ω. This exceptional set depends, however, on $\lambda_1, \lambda_2, t_1, t_2$ and therefore we *cannot* say that if f is a linear random function, then $\check{f}(\omega)$ is a linear map $T \to X$ for almost every $\omega \in \Omega$. If the exceptional set outside which (1) holds has m-measure zero and is the same for all $\lambda_1, \lambda_2, t_1, t_2$, then $\check{f}(\omega): T \to X$ is a linear map for almost every $\omega \in \Omega$ and in this case we say that f is *almost surely a linear map* (or, if $X = \mathbf{R}$, that f is almost surely a linear form).

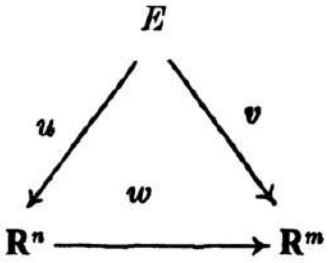
Let us also observe that linearity depends only on the isonomy class of the random function $f: T \to \mathrm{Meas}\,(\Omega, m; X)$. Indeed, if g denotes the random variable $(f(\lambda_1 t_1 + \lambda_2 t_2), \lambda_1 f(t_1), \lambda_2 f(t_2)): \Omega \to \mathbf{R}^3$, and $\sigma: \mathbf{R}^3 \to \mathbf{R}$ is the map $(\alpha, \beta, \gamma) \longmapsto \alpha - \beta - \gamma$, then the linearity of f means that $m(\{\omega \mid \sigma(g(\omega)) = 0\}) = 1$, i.e. that $\sigma(g(m))$ is the Dirac measure δ at the origin, for every choice of $t_1, t_2 \in T$ and $\lambda_1, \lambda_2 \in \mathbf{R}$. But by the very definition of isonomy, the measure $g(m)$ depends only on the isonomy class of f.

Let E be a locally convex Hausdorff space and E' its dual. We want to show that *there exists a bijective correspondence between the cylindrical measures on E and the isonomy classes of linear random functions* $E' \to \mathrm{Meas}\,(\Omega, m; \mathbf{R})$.

Indeed, let $f: E' \to \mathrm{Meas}\,(\Omega, m; \mathbf{R})$ be a linear random function. A continuous linear map $u: E \to \mathbf{R}^n$ is given by n continuous linear forms $\xi_1, \ldots, \xi_n \in E'$. Now f associates with each ξ_j $(1 \leqslant j \leqslant n)$ a random variable $f(\xi_j): \Omega \to \mathbf{R}$ and therefore we can consider the

n-tuple $f(u) = (f(\xi_1), \ldots, f(\xi_n))$ as a random variable with values in $\mathbf{R}^n$. We associate with the continuous linear map u the image measure $f(u)\,(m) = \mu_u$. If f is replaced by an isonomous linear random function g, then we have $f(u)\,(m) = g(u)\,(m)$ for every continuous linear map $u: E \to \mathbf{R}^n$, and therefore the probabilities μ_u depend only on the isonomy class of f. In particular we may assume that f is canonical.

Suppose that we have two continuous linear maps $u: E \to \mathbf{R}^n$ and $v: E \to \mathbf{R}^m$, as well as a linear map $w: \mathbf{R}^n \to \mathbf{R}^m$, such that the diagram



is commutative. The map w is given by its matrix (w_{ij}), and if $v = (\eta_1, \ldots, \eta_m)$, then $\eta_j = \sum_{i=1}^{n} w_{ij}\,\xi_i\ (1 \leqslant j \leqslant m)$. Since f is linear, we have, for almost every $\omega \in \Omega$,

$$f(\eta_j)\,(\omega) = \breve{f}(\omega)\,(\eta_j) = \breve{f}(\omega)\left(\sum_{i=1}^{n} w_{ij}\xi_i\right)$$

$$= \sum_{i=1}^{n} w_{ij}\breve{f}(\omega)(\xi_i) = \sum_{i=1}^{n} w_{ij} f(\xi_i)\,(\omega),$$

and therefore $f(v) = w \circ f(u)$ almost surely. Thus $\mu_v = f(v)\,(m) = w(\mu_u)$ and the family (μ_u) is a cylindrical measure μ on E. For any continuous linear map $u: E \to \mathbf{R}^n$ we have $u(\mu) = f(u)\,(m)$.

Conversely, let μ be a cylindrical measure on E. For Ω we take $\breve{\mathbf{R}}^{E'}$. To every n-tuple $u = (\xi_1, \ldots, \xi_n)$ of elements of E', there corresponds a probability μ_u on $\breve{\mathbf{R}}^n$ and a projection from $\breve{\mathbf{R}}^{E'}$ onto the partial product $\breve{\mathbf{R}}^n = \breve{\mathbf{R}}^u$. It is clear that these probabilities μ_u form a projective system, and therefore they define a probability m on Ω.

Let $\check{f}$ be the identity map $\check{\mathbf{R}}^{E'} \to \check{\mathbf{R}}^{E'}$. It defines a random function $f\colon E' \to \text{Meas}\,(\Omega, m; \mathbf{R})$ which associates with each $\xi \in E'$ the random variable $f(\xi) = \pi_\xi \circ \check{f}$, which is given by $f(\xi)(\omega) = \omega(\xi) \in \check{\mathbf{R}}$ for each $\omega = (\omega(\xi)) \in \check{\mathbf{R}}^{E'}$. For almost every $\omega \in \Omega$ the map $f(\xi)$ takes its value in $\mathbf{R}$ since $m(\pi_\xi^{-1}(\mathbf{R})) = \pi_\xi(m)\,(\mathbf{R}) = \mu_\xi(\mathbf{R}) = 1$. It is clear from the definition of f that the cylindrical measure associated with it is the measure μ originally given.

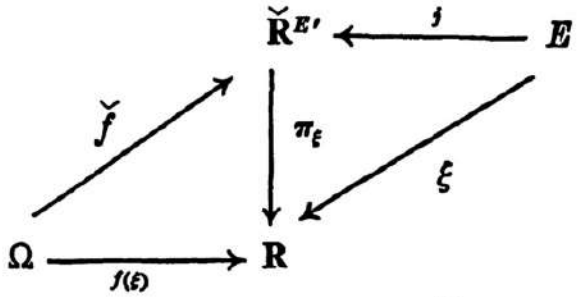
Finally, we show that the random function f is linear. If $\xi, \eta \in E'$, $\lambda_1, \lambda_2 \in \mathbf{R}$, let $u = (\lambda_1 \xi + \lambda_2 \eta, \lambda_1 \xi, \lambda_2 \eta)\colon E \to \mathbf{R}^3$ and $\sigma\colon \mathbf{R}^3 \to \mathbf{R}$ the map $(\alpha, \beta, \gamma) \longmapsto \alpha - \beta - \gamma$. Then $\sigma \circ u = 0$, hence $\sigma(\mu_u) = \mu_{\sigma \circ u} = \delta$. Thus μ_u is concentrated on the subset $\Gamma = \sigma^{-1}(\{0\}) = \{(\alpha, \beta, \gamma) \mid \alpha - \beta - \gamma = 0\}$ of $\mathbf{R}^3$. If g is the random variable $(f(\lambda_1 \xi + \lambda_2 \eta), \lambda_1 f(\xi), \lambda_2 f(\eta))\colon \Omega \to \mathbf{R}^3$, then $g(m) = \mu_u$ and therefore

$$m(\{\omega \mid f(\lambda_1 \xi + \lambda_2 \eta)\,(\omega) - \lambda_1 f(\xi)\,(\omega) - \lambda_2 f(\eta)\,(\omega) = 0\})$$
$$= m(\{\omega \mid g(\omega) \in \Gamma\}) = m(g^{-1}(\Gamma)) = g(m)\,(\Gamma) = \mu_u(\Gamma) = 1.$$

It follows from the preceding considerations that if μ is a cylindrical measure on E and $f\colon E' \to \text{Meas}\,(\Omega, m; \mathbf{R})$ a linear random function, then f is associated with μ if and only if we have

$$f(\xi)(m) = \xi(\mu) \quad \text{for all} \quad \xi \in E'. \tag{2}$$

We can consider E as a subset of $\check{\mathbf{R}}^{E'}$ with the help of the canonical injection $j\colon E \to \check{\mathbf{R}}^{E'}$ defined by $j(x) = (\langle x, \xi \rangle)_{\xi \in E'}$. Also the product topology on $\check{\mathbf{R}}^{E'}$ induces on E the weak topology $\sigma(E, E')$. If $f\colon E' \to \text{Meas}\,(\Omega, m; \mathbf{R})$ is a canonical random function associated with μ, then we have the commutative diagram



for each $\xi \in E'$. It follows from (2) that $\check{\mu} = \check{f}(m)$ is the Radon measure on $\check{\mathbf{R}}^{E'}$ which corresponds as a projective limit to the cylindrical

measure μ on E. This measure $\breve{\mu}$ on $\breve{\mathbf{R}}^{E'}$ should not be confused with the measure on $\prod_F (E/F)^\vee$ we have also denoted by $\breve{\mu}$ in Chap. 2, §1. However, E'^*(and a fortiori E) is imbedded topologically into $\breve{\mathbf{R}}^{E'}$ with the help of the map $x'' \longmapsto (< x', x'' >)_{x' \in E'}$ and also into $\prod_F (E/F)^\vee$ (Chap. 2, § 1, Prop. 1); this will allow us to compare in some way the two Radon measures both called $\breve{\mu}$, one on $\prod_F (E/F)^\vee$, the other on $\breve{\mathbf{R}}^{E'}$. Consider a general projective system (X_i, π_{ij}) with compact spaces X_i and let X be a subspace of the projective limit. Let (μ_i) be a projective system of Radon probabilities on the X_i, μ the projective limit. We saw (Part I, Chap. I, Th. 20) that there exists a largest Radon measure μ on X such that $\pi_i(\bar{\mu}) < \mu_i$; let us show that $\bar{\mu}$ is the induced measure μ_X. For every $i \in I$ we have $\pi_i(\mu) = \mu_i$, therefore $\pi_i(\mu_X) < \mu_i$, hence $\mu_X < \bar{\mu}$ according to the definition of μ. On the other hand, $\pi_i(\bar{\mu}) < \mu_i$ for every $i \in I$, hence, if $\tilde{\bar{\mu}}$ is the image of $\bar{\mu}$ by the injection of X into the projective limit of the X_i, then $\pi_i(\tilde{\bar{\mu}}) < \mu_i$, and therefore $\tilde{\bar{\mu}} < \mu$, which proves that $\bar{\mu} < \mu_X$. Thus finally $\bar{\mu} = \mu_X$. Let us come back now to the cylindrical measure μ on E. We have two projective systems, one with the $E \to E/F$ and the other with the projections of $\breve{\mathbf{R}}^{E'}$ on the products $\breve{\mathbf{R}}^{(\xi_1, \xi_2, \ldots, \xi_n)}$, $\xi_i \in E'$. From that we have two Radon measures, both called $\breve{\mu}$, one on $\prod_F (E/F)^\vee$, the other on $\breve{\mathbf{R}}^{E'}$. They cannot be compared themselves, being on different spaces. But E'^* is a topological subspace of both. These 2 measures $\breve{\mu}$ induce 2 Radon measures $\breve{\mu}_{E'^*}$ on E'^*; according to our previous remark, the first induced measure is the largest Radon measure on E'^* whose image, in every E/F, is $< \mu_{E/F}$, and the second is the largest Radon measure on E'^* whose image on $\breve{\mathbf{R}}^{(\xi_1, \ldots, \xi_n)}$ is $< (\xi_1, \xi_2, \ldots, \xi_n)(\mu)$. But these two properties are equivalent to each other, and equivalent to saying that for every continuous linear map u from E into a finite dimensional vector space G, the image is $< u(\mu)$. Therefore the two Radon measures $\breve{\mu}$ on $\prod_F (E/F)^\vee$ and $\breve{\mathbf{R}}^{E'}$ respectively, although

not comparable, induce the same measure on E'^*; in particular they give the same measure to E'^*, and if one of them is carried by E'^* or E, the other has the same property, so that the common denomination $\breve{\mu}$ for the two Radon measures on different spaces is not very harmful.

We can now explain a little better why the "measure" of a cylindrical set, mentioned briefly in Chap. 2, § 1, is not countably additive. Let L be a finite free subset of E' and let $A \subset E$ be the inverse image of a Borel subset B of $\mathbf{R}^L$ with respect to the composite map u: $E \xrightarrow{j} \breve{\mathbf{R}}^{E'} \xrightarrow{\pi_L} \mathbf{R}^L$. Then the "measure" of the cylindrical set A will be $\mu_u(B) = \breve{\mu}(\pi_L^{-1}(B))$. Now $\pi_L^{-1}(B) \supset A$ but in general $\pi_L^{-1}(B) \neq A$ and therefore $\breve{\mu}(A) \neq \mu_u(B)$ since most often $\breve{\mu}(E) = 0$.

In a general way, if f: $T \to \text{Meas}\ (\Omega, m; X)$ is an arbitrary canonical random function, and A a subset of $\breve{X}^T$, we say that f is *almost surely in* A if $\breve{f}(\omega) \in A$ except for a set of points $\omega \in \Omega$ of m-measure zero. For example, if T and X are vector spaces, then f is almost surely a linear map if and only if f is almost surely in the subset $\text{Hom}(T, X)$, consisting of all linear maps $T \to X$ of the set $\mathscr{F}(T, \breve{X}) = \breve{X}^T$. With this terminology we have

PROPOSITION 4. *Let μ be a cylindrical measure on the locally convex Hausdorff space E and f: $E' \to \text{Meas}\ (\Omega, m; \mathbf{R})$ an associated canonical linear random function. Then the following two conditions are equivalent* :

(a) *μ is a Radon measure on E equipped with the weak topology $\sigma(E, E')$;*

(b) *f is almost surely in E.*

PROOF. To say that μ is a Radon measure on E equipped with $\sigma(E, E')$ is to say that $\breve{\mu}$ is concentrated on E, i.e. $\breve{\mu}(E) = 1$. But this means that $m(\breve{f}^{-1}(E)) = 1$, i.e. f is almost surely in E.

To make Prop. 4 more meaningful, let us observe that condition (b) depends a priori only on the isonomy class of the random variable f, assuming that it is canonical, since it means that $\check{f}(m)$ is carried by E.

The two equivalent conditions of Prop. 4 imply that f is almost surely a continuous linear form on E' if we equip E' with the weak* topology $\sigma(E', E)$, i.e. $\xi \to f(\xi)(\omega) = \check{f}(\omega)(\xi)$ is a continuous linear form on E' for almost every $\omega \in \Omega$. Indeed, if f is almost surely in E, then for almost every $\omega \in \Omega$ we can find an element $x(\omega) \in E$ such that $\check{f}(\omega) = (\langle x(\omega), \xi \rangle)_{\xi \in E'}$. Since $\xi \longmapsto \langle x(\omega), \xi \rangle$ maps E' linearly into $\mathbf{R}$ and is continuous for $\sigma(E', E)$, we see that f is almost surely a continuous linear form on E'.

REMARK. Assume that f is a canonical linear random function on E' which is almost surely continuous for $\sigma(E', E)$. We cannot assert that f is almost surely a continuous linear form on $\sigma(E', E)$. But let us assume that E' has a countable dense subset D for the topology $\sigma(E', E)$. Under this assumption f is almost surely a continuous linear form on E' for $\sigma(E', E)$, i.e. f is almost surely in E and thus the equivalent conditions (a) and (b) of Prop. 4 are satisfied. Indeed, there exists a set of measure zero $N_0 \subset \Omega$ such that $\check{f}(\omega): E' \to \mathbf{R}$ is continuous for $\omega \notin N_0$. Since f is a linear random function, for any two rational numbers λ_1, λ_2 and any two elements $\xi_1, \xi_2 \in D$ there exists a set $N(\lambda_1, \lambda_2, \xi_1, \xi_2)$ of measure zero contained in Ω such that

$$\check{f}(\omega)(\lambda_1 \xi_1 + \lambda_2 \xi_2) = \lambda_1 \check{f}(\omega)(\xi_1) + \lambda_2 \check{f}(\omega)(\xi_2) \tag{3}$$

for $\omega \notin N(\lambda_1, \lambda_2, \xi_1, \xi_2)$. The union N of N_0 and of all the sets $N(\lambda_1, \lambda_2, \xi_1, \xi_2)$, where λ_1, λ_2 vary in $\mathbf{Q}$ and ξ_1, ξ_2 vary in D, is of measure zero. For $\omega \notin N$ the map $\check{f}(\omega)$ is continuous and verifies (3) for all $\lambda_1, \lambda_2 \in \mathbf{Q}$ and $\xi_1, \xi_2 \in D$. The continuity of $\check{f}(\omega)$ implies that (3) is verified for all $\lambda_1, \lambda_2 \in \mathbf{R}$ and $\xi_1, \xi_2 \in E'$, i.e. for $\omega \notin N$ the map $\check{f}(\omega)$ is a linear form on E'.

We shall now investigate how the linear random functions associated with cylindrical measures behave if we perform on the cylindrical measures the operations introduced in Chap. 2, §2.

PROPOSITION 5. *Let E_1 and E_2 be two locally convex Hausdorff spaces, $u: E_1 \to E_2$ a weakly continuous linear map, and μ a cylindrical measure on E_1. If the linear random function*

$$f: E_1' \to \text{Meas}(\Omega, m; \mathbf{R})$$

is associated with μ, then $f \circ {}^t u: E_2' \to \text{Meas}(\Omega, m; \mathbf{R})$ is associated with $u(\mu)$.

PROOF. If $\eta \in E_2'$, then applying (2) with $\xi = {}^t u(\eta) = \eta \circ u$ we have $(f \circ {}^t u)(\eta)(m) = f({}^t u(\eta))(m) = {}^t u(\eta)(\mu) = \eta(u(\mu))$, which proves that $f \circ {}^t u$ is associated with $u(\mu)$.

Under the hypotheses and with the notation of Prop. 5 we want to derive a formula linking the Radon measure $\check{\mu}$ on $\check{\mathbf{R}}^{E_1'}$ with the Radon measure $(u(\mu))^{\check{}}$ on $\check{\mathbf{R}}^{E_2'}$. The map u defines a continuous map $\check{u}: \check{\mathbf{R}}^{E_1'} \to \check{\mathbf{R}}^{E_2'}$ which associates with each element $a = (a_\xi)_{\xi \in E_1'} \in \check{\mathbf{R}}^{E_1'}$ the element $u(a) = (a_{\eta \cdot u})_{\eta \in E_2'} \in \check{\mathbf{R}}^{E_2'}$. If $j_1: E_1 \to \check{\mathbf{R}}^{E_1'}$ and $j_2: E_2 \to \check{\mathbf{R}}^{E_2'}$ denote the canonical injections, then the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{u} & E_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ \check{\mathbf{R}}^{E_1'} & \xrightarrow{\check{u}} & \check{\mathbf{R}}^{E_2'} \end{array}$$

is commutative since

$$\check{u}(j_1(x)) = (\langle x, {}^t u(\eta) \rangle)_{\eta \in E_2'} = (\langle u(x), \eta \rangle)_{\eta \in E_2'} = j_2(u(x))$$

for every $x \in E_1$. We have

$$(u(\mu))^{\check{}} = \check{u}(\mu) \tag{4}$$

for every cylindrical measure on E_1. To prove this it is sufficient to show that for every $\eta \in E_2'$ the image of the two measures with respect to the projection $\pi_\eta: \check{\mathbf{R}}^{E_2'} \to \check{\mathbf{R}}$ is the same. We have $\pi_\eta \circ \check{u} =$

$\pi_{\eta \circ u}$ and therefore $\pi_\eta(\check{\mu}(\check{u})) = \pi_{\eta \circ u}(\check{\mu}) = (\eta \circ u)(\mu)$. Since on the other hand $\pi_\eta\,(\,(u(\mu))^{\check{}}\,) = \eta\,(u(\mu))$, we have proved relation (4).

PROPOSITION 6. *Let E_1 and E_2 be two locally convex Hausdorff spaces, μ a cylindrical measure on E_1 and ν a cylindrical measure on E_2. Let $f: E_1' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ be a linear random function associated with μ and $g: E_2' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ a linear random function associated with ν. If f and g are independent then the linear random function $h: E_1' \times E_2' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$, defined by $h(\xi, \eta) = f(\xi) + g(\eta)$ for $\xi \in E_1'$, $\eta \in E_2'$, is associated with $\mu \otimes \nu$* (Chap. 2, § 2).

PROOF. Let $u = (\xi_1, \ldots, \xi_p)$ be a continuous linear map from E_1 into $\mathbf{R}^p$ and $v = (\eta_1, \ldots, \eta_q)$ a continuous linear map from E_2 into $\mathbf{R}^q$, where $\xi_1, \ldots, \xi_p \in E_1'$ and $\eta_1, \ldots, \eta_q \in E_2'$. Explicitly, these maps are given by

$$u(x) = \sum_{i=1}^{p} \langle x, \xi_i \rangle\, e_i \quad \text{for } x \in E_1,$$

and

$$v(y) = \sum_{j=1}^{q} \langle y, \eta_j \rangle > f_j \quad \text{for } y \in E_2,$$

where $e_1, \ldots, e_p$ is the canonical basis of $\mathbf{R}^p$ and $f_1, \ldots, f_q$ is the canonical basis of $\mathbf{R}^q$. These two maps define a continuous linear map $w = (u, v): E_1 \times E_2 \to \mathbf{R}^{p+q}$ given by

$$\begin{aligned} w(x, y) &= \sum_{i=1}^{p} \langle \xi_i, x \rangle\, e_i + \sum_{j=1}^{q} \langle \eta_j, y \rangle f_j \\ &= \sum_{i=1}^{p} \langle (\xi_i, 0), (x, y) \rangle\, e_i + \sum_{j=1}^{q} \langle (0, \eta_j), (x, y) \rangle f_j. \end{aligned} \tag{5}$$

Let us immediately observe that the collection of these maps w is coinitial with the collection of all continuous linear maps $E_1 \times E_2 \to \mathbf{R}^n$. Indeed, such a map is of the form

$$(x, y) \longmapsto \langle (\xi_1, \eta_1), (x, y) \rangle\, b_1 + \ldots + \langle (\xi_n, \eta_n), (x, y) \rangle\, b_n,$$

where $b_1, \ldots, b_n$ is the canonical basis of $\mathbf{R}^n$, and can therefore be factored into $\rho \circ w$, where w is the map above with $p = q = n$,

and ρ: $\mathbf{R}^{2n} \to \mathbf{R}^n$ is the map $\sum_{i=1}^{n} \alpha_i e_i + \sum_{j=1}^{n} \beta_j f_j \longmapsto \sum_{i=1}^{n} (\alpha_i + \beta_i) b_i$.

The tensor product $\mu \otimes \nu$ is therefore characterized by the relations

$$w(\mu \otimes \nu) = u(\mu) \otimes v(\nu)$$

whenever $w = (u, v)$. Since f is associated with μ and g with ν we have

$$u(\mu) = f(u)\,(m) = (f(\xi_1), \ldots, f(\xi_p))\,(m),$$
$$v(\nu) = g(v)\,(m) = (g(\eta_1), \ldots, g(\eta_q))\,(m),$$

and hence

$$w(\mu \otimes \nu) = (f(\xi_1), \ldots, f(\xi_p))\,(m) \otimes (g(\eta_1), \ldots, g(\eta_q))\,(m).$$

Since the random functions f and g are assumed to be independent this last measure on $\mathbf{R}^{p+q}$ is equal to

$$(f(\xi_1), \ldots, f(\xi_p), g(\eta_1), \ldots, g(\eta_q))\,(m)$$
$$= (h(\xi_1, 0), \ldots, h(\xi_p, 0), h(0, \eta_1), \ldots, h(0, \eta_q))\,(m).$$

By virtue of (5) we obtain finally that

$$w(\mu \otimes \nu) = h(w)\,(m),$$

which proves that h is associated with $\mu \otimes \nu$.

It is easy to construct linear random functions f and g which satisfy the conditions of Proposition 6. Indeed, let f_0: $E_1' \to \mathrm{Meas}(\Omega_1, m_1; \mathbf{R})$ be a linear random function associated with μ, and g_0: $E_2' \to \mathrm{Meas}(\Omega_2, m_2; \mathbf{R})$ a linear random function associated with ν. Define f: $E_1' \to \mathrm{Meas}(\Omega_1 \times \Omega_2, m_1 \otimes m_2; \mathbf{R})$ by $f(\xi)\,(\omega_1, \omega_2) = f_0(\xi)\,(\omega_1)$ for $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$, $\xi \in E_1'$, and similarly define g: $E_2' \to \mathrm{Meas}(\Omega_1 \times \Omega_2, m_1 \otimes m_2; \mathbf{R})$ by $g(\eta)\,(\omega_1, \omega_2) = g_0(\eta)\,(\omega_2)$ for $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$, $\eta \in E_2'$. It is straightforward to check that f is isonomous to f_0, g is isonomous to g_0, and the random functions f and g are independent.

PROPOSITION 7. *Let E be a locally convex Hausdorff space, μ and ν two cylindrical measures on E, f: $E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ a linear random function associated with μ and g: $E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ a*

linear random function associated with ν. If f and g are independent, then the random function $\xi \longmapsto f(\xi) + g(\xi)$ is associated with $\mu * \nu$ (Chap. 2, §2).

PROOF. We have $\mu * \nu = \sigma(\mu \otimes \nu)$, where $\sigma: E \times E \to E$ is the map $(x, y) \longmapsto x + y$. By Prop. 6 the random function $(\xi, \eta) \to h(\xi, \eta) = f(\xi) + g(\eta)$ is associated with $\mu \otimes \nu$, hence by Prop. 5 $h \circ {}^t\sigma$ is associated with $\mu * \nu$. Since ${}^t\sigma: E' \to E' \times E'$ is the diagonal map $\xi \longmapsto (\xi, \xi)$, we have

$$(h \circ {}^t\sigma)(\xi) = h(\xi, \xi) = f(\xi) + g(\xi).$$

Next we want to translate the concentration properties of the cylindrical measure μ on E into properties of the linear random function $f: E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ associated with it.

THEOREM 2. *Let E be a locally convex Hausdorff space, μ a cylindrical measure on E, and $f: E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ a linear random function associated with μ. Let $\mathfrak{S}$ be a saturated collection of subsets of E. The following conditions are equivalent:*

(a) *μ is scalarly $\mathfrak{S}$-concentrated;*

(b) *the map $\xi \longmapsto \xi(\mu)$ from $E'_{\mathfrak{S}}$ into $\mathscr{M}^1(\mathbf{R})$ equipped with the vague or narrow topology is continuous;*

(c) *the map $\xi \longmapsto \xi(\mu)$ from $E'_{\mathfrak{S}}$ into $\mathscr{M}^1(\mathbf{R})$ is continuous at the origin;*

(d) *the random function $f: E'_{\mathfrak{S}} \to \mathrm{Meas}(\Omega; m; \mathbf{R})$ is continuous if $\mathrm{Meas}(\Omega, m; \mathbf{R})$ is equipped with the topology of convergence in probability;*

(e) *the random function f is continuous at the origin.*

PROOF. (a) and (b) are equivalent by the corollary of Theorem 2, Chap. 2, §3. It is clear that (b) implies (c). Next assume that $\xi \to 0$ in $E'_{\mathfrak{S}}$ implies that $\xi(\mu)$ converges to δ vaguely and therefore (p. 249) also narrowly in $\mathscr{M}^1(\mathbf{R})$. By formula (2) we have $f(\xi)(m) = \xi(\mu)$ and $0(m) = \delta$, and therefore, by virtue of Remark 1 (§1), from the fact that ξ converges to 0 in $E'_{\mathfrak{S}}$ it follows that $f(\xi)$ converges in probability to the random variable 0, i.e. f is continuous at the

origin. Since f is linear, it is continuous everywhere, thus (c) implies (d), equivalent to (e).

Finally if (d) is satisfied, then, by virtue of Prop. 2 (§1), $\xi \to \xi_0$ in E' implies that $f(\xi)(m) = \xi(\mu)$ converges to $f(\xi_0)(m) = \xi_0(\mu)$ narrowly, and a fortiori vaguely in $\mathscr{M}^1(\mathbf{R})$, i.e. condition (b) is satisfied.

SECOND PROOF. We shall give another proof, which does not depend on the Fourier transform.

Let us prove first the equivalence of (a) and (c). Let us recall that, for every $A \in \mathfrak{S}$, the set $\xi(A)$ is contained in the closed interval $[-N(\xi), +N(\xi)]$, and contains the corresponding open interval (Lemma 4 of Theorem 1 of Chap. 2, §3), N being the gauge of the polar set A°. On the other hand, the narrow convergence of measures ν_j to δ, on the space $\mathbf{R}$, means that, for every $a > 0$, $\nu_j([-a, +a])$ or $\nu_j(]-a, +a[)$, converges to 1 (see Remark 1 after the proof of Proposition 2). Let us assume μ to be scalarly $\mathfrak{S}$-concentrated; for every $\epsilon > 0$, there exists $A \in \mathfrak{S}$, such that, for every $\xi \in E'$, $\xi(\mu)$ is concentrated up to ϵ on $\xi(A) \subset [-N(\xi), +N(\xi)]$. If ξ converges to 0 in $E'_{\mathfrak{S}}$, then $N(\xi)$ converges to 0, and $\xi(\mu)$ will converge narrowly to δ on $\mathbf{R}$, hence $\xi \longmapsto \xi(\mu)$ is continuous at the origin. Let us assume conversely that $\xi \longmapsto \xi(\mu)$ is continuous at the origin: for a given $\epsilon > 0$, there exists $A \in \mathfrak{S}$ such that, for $\xi \in A^\circ$, $\xi(\mu)$ is concentrated up to ϵ on $]-1, +1[$; therefore, for $\xi \in kA^\circ$, the measure $\xi(\mu)$ is concentrated up to ϵ on $]-k, +k[$; and finally, for every ξ, the measure $\xi(\mu)$ is concentrated up to ϵ on $]-N(\xi), +N(\xi)[$, hence on $\xi(A)$, and μ is scalarly $\mathfrak{S}$-concentrated.

Now (c) and (e) are equivalent, according the remark after the proof of Prop. 2; (d) and (e) are equivalent since $E'_{\mathfrak{S}}$ and $\mathrm{Meas}(\Omega, m; \mathbf{R})$ are topological vector spaces, and finally (d) implies (b) (Prop. 2), which trivially implies (c).

To clarify Theorem 2, we make an observation analogous to the one made after Prop. 4. Let T be a topological space and X a uniform space. Then *the continuity* of a random function $f: T \to$

$\mathrm{Meas}(\Omega, m; X)$ for the topology of convergence in measure on $\mathrm{Meas}(\Omega, m; X)$ *depends only on the isonomy class* of f. Indeed, the continuity of f at a point t_0 means that given $\epsilon > 0$ and a uniformity $\mathfrak{U}$ of X there exists a neighborhood $V(t_0)$ of t_0 such that

$$m(\{\omega \mid (f(t)(\omega), f(t_0)(\omega)) \notin \mathfrak{U}\}) \leqslant \epsilon \tag{6}$$

for $t \in V(t_0)$. But if we introduce the map $(f(t), f(t_0)): \Omega \to X \times X$, then the left-hand side of (6) is equal to

$$(f(t), f(t_0))(m)(\mathscr{C}\mathfrak{U}),$$

and by the very definition of isonomy, the measure $(f(t), f(t_0))(m)$ depends only on the isonomy class of f. Actually we use a condition which is somewhat weaker than isonomy, since the latter involves the measures $(f(t_1), \ldots, f(t_n))(m)$ on X^n for all n.

PROPOSITION 8. *The cylindrical measure μ on the locally convex Hausdorff space E is a Radon measure for the original topology of E if and only if a canonical linear random function $f: E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ associated with μ is almost surely in the union of a countable collection of compact subsets of $E \subset \check{\mathbf{R}}^{E'}$.*

PROOF. We know (Chap. 2, § 1) that μ is a Radon measure on E if and only if there exists a sequence (K_n) of compact subsets of E such that for $L = \bigcup_n K_n$ we have $\check{\mu}(L) = \check{f}(m)(L) = 1$. But this is equivalent to $m(\check{f}^{-1}(L)) = 1$.

THEOREM 3. *Let E be a locally convex Hausdorff space, μ a cylindrical measure on E and $f: E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ a canonical linear random function associated with μ. Let $\mathfrak{S}$ be saturated collection of bounded subsets of E which cover E. If μ is cylindrically $\mathfrak{S}$-concentrated, then f is almost surely a continuous linear form on $E'_{\mathfrak{S}}$.*

PROOF. Since the sets of $\mathfrak{S}$ cover E, the space $E'_{\mathfrak{S}}$ is Hausdorff and we have $E \subset (E'_{\mathfrak{S}})' \subset E''$. The sets belonging to $\mathfrak{S}$ are relatively weak*-compact in $(E'_{\mathfrak{S}})'$, and $(E'_{\mathfrak{S}})'$ is the union of the balanced, convex, weak*-closed hulls in E'' of the sets $A \in \mathfrak{S}$. If μ is cylindrically $\mathfrak{S}$-concentrated on E, then on $(E'_{\mathfrak{S}})'$ it is cylindrically

concentrated on compact sets and therefore a Radon measure on $(E'_{\mathfrak{S}})'$ equipped with the weak*-topology $\sigma((E'_{\mathfrak{S}})', E'_{\mathfrak{S}})$. It follows from a remark made after Prop. 4 that f is almost surely a continuous linear form on E' equipped with the topology $\sigma(E'_{\mathfrak{S}}, (E'_{\mathfrak{S}})')$. A fortiori f is almost surely a continuous linear form on $E'_{\mathfrak{S}}$.

We have a partial converse to Theorem 3. Assume that E is quasi-complete and let $\mathfrak{S}$ be the collection of *all* balanced, convex, $\sigma(E, E')$-compact subsets of E, so that the $\mathfrak{S}$-topology on E' is the Mackey topology $\tau(E', E)$ [3, Chap. IV, §2, $n°$ 3, Corol. of Th. 2]. If f is almost surely a continuous linear form on $E'_{\mathfrak{S}}$, then μ is cylindrically $\mathfrak{S}$-concentrated and a Radon measure on E equipped with the weak topology $\sigma(E, E')$. Indeed, by Mackey's theorem [3, Chap. IV, §2, $n°$ 3, Th. 2] f is almost surely in E. It follows from Prop. 4 that μ is a Radon measure on E equipped with the topology $\sigma(E, E')$, i.e. μ is cylindrically concentrated on the weakly compact subsets of E. Since E is quasi-complete, by the Krein-Šmulian theorem (see the proof of Lemma 7 of Chap. 1, §4) the balanced, convex, closed hull of a weakly compact set is weakly compact. Hence μ is cylindrically concentrated on the sets of $\mathfrak{S}$.

To conclude this section we want to invert the roles of E and E'.

THEOREM 4. *Let F be a locally convex Hausdorff space and F' its dual equipped with the weak*-topology $\sigma(F', F)$. There exists a bijective correspondence between the cylindrical measures μ on F' and the isonomy classes of linear random functions $f: F \to \mathrm{Meas}(\Omega, m; \mathbf{R})$. Then*

(i) *μ is scalarly ϵ-concentrated if and only if f is continuous.*

(ii) *Let us assume that f is canonical, then μ is a Radon measure if and only if f is almost surely in F'.*

(iii) *If μ is cylindrically ϵ-concentrated, then f is almost surely in F'.*

PROOF. For the weak*-topology $\sigma(F', F)$ the dual of $E = F'$ is the space F itself. Hence the bijective correspondence between the cylindrical measures μ and the isonomy classes of random functions f follows. If $\mathfrak{S}$ is the collection ϵ of all balanced, convex,

closed, equicontinuous subsets of $E = F'$, then the $\mathfrak{S}$-topology on $F = E'$ is precisely its original topology [3, Chap. IV, §2, n^0 2, Prop. 1]. Thus (i) follows from Theorem 2, (ii) from Prop. 4 and (iii) from Theorem 3.

3. Examples and Applications. 1) Let μ be a cylindrical measure on the Banach space l^p $(1 < p < \infty)$ and assume that μ is scalarly concentrated on the balls of l^p. We write, as usual, $1/p + 1/p' = 1$, and we know that $l^{p'}$ is the dual of l^p. To μ there corresponds a linear random function $f\colon l^{p'} \to \mathrm{Meas}\,(\Omega, m; \mathbf{R})$, which by Theorem 2 is continuous if $l^{p'}$ is equipped with its norm topology and Meas $(\Omega, m; \mathbf{R})$ with the topology of convergence in probability. Thus for each $\xi \in l^{p'}$ we have a random variable $f(\xi)\colon \Omega \to \mathbf{R}$, if $\|\xi - \xi_0\| \to 0$, then $f(\xi)$ tends to $f(\xi_0)$ in probability, and for $\kappa, \lambda \in \mathbf{R}$, $\xi, \eta \in l^{p'}$ we have $f(\kappa\xi + \lambda\eta) = \kappa f(\xi) + \lambda f(\eta)$ almost surely.

For each $n \in \mathbf{N}$ let $\epsilon_n = (0,0, \ldots, 0, \underbrace{1}_{n}, 0, \ldots)$ be the n^{th} unit vector in $l^{p'}$. To ϵ_n there corresponds a random variable $f(\epsilon_n) = x_n\colon \Omega \to \mathbf{R}$. If $\xi = (\alpha_k)$ is an arbitrary element of $l^{p'}$, i.e., if $\sum_{k=0}^{\infty} |\alpha_k|^{p'} < \infty$, then the sequence $\sum_{k=0}^{n} \alpha_k \epsilon_k$ converges to ξ in $l^{p'}$ since $\|\xi - \sum_{k=0}^{n} \alpha_k \epsilon_k\|^{p'} \leqslant \sum_{k>m} |\alpha_k|^{p'}$, and therefore $f(\sum_{k=0}^{n} \alpha_k \epsilon_k) = \sum_{k=0}^{n} \alpha_k f(\epsilon_k) = \sum_{k=0}^{n} \alpha_k x_k$ converges in measure to $f(\xi)$. In other words, each $f(\xi)$ can be represented as a series $\sum_{k=0}^{\infty} \alpha_k x_k$, convergent in probability.

Conversely, let (x_n) be a sequence of real random variables and assume that for every $\xi = (\alpha_k) \in l^{p'}$ the series $\sum_{k=0}^{\infty} \alpha_k x_k$ converges in probability to some random variable $f(\xi) \in \mathrm{Meas}\,(\Omega, m; \mathbf{R})$. Clearly $f\colon \xi \longmapsto f(\xi)$ is a linear random function $l^{p'} \to \mathrm{Meas}\,(\Omega, m; \mathbf{R})$ and we want to prove that it is continuous in probability. For any integer $N \geqslant 0$ let f_N be the linear map $\xi \longmapsto f_N(\xi) = \sum_{k=0}^{N} \alpha_k x_k$ from

$l^{p'}$ into Meas $(\Omega, m; \mathbf{R})$, and each of these maps is continuous. Indeed, f_N is the composition of the projection $\xi \longmapsto (\alpha_0, \alpha_1, \ldots, \alpha_N)$ from $l^{p'}$ onto $\mathbf{R}^{N+1}$ and of the linear map $(\alpha_0, \alpha_1, \ldots, \alpha_N) \longmapsto \sum_{k=0}^{N} \alpha_k x_k$ from $\mathbf{R}^{N+1}$ into Meas $(\Omega, m; \mathbf{R})$. The first of these maps is obviously continuous, and the second one is continuous because $\mathbf{R}^{N+1}$ is finite-dimensional [3, Chap. I, 2nd ed., §2, n^0 3, Corol. 2 of Th. 2].

On the other hand f_N tends pointwise to f and we can use the following form of the Banach-Steinhaus theorem for spaces which are not necessarily locally convex spaces.

Let E and F be two topological vector spaces and assume that E is a Baire space. If H is a subset of $\mathscr{L}(E, F)$ bounded for the topology of simple convergence, then H is equicontinuous [3, Chap. III, §3, Exerc. 15(*a*)]. *Proof.* Let V be a neighborhood of 0 in F and let W be a balanced closed neighborhood of 0 in F such that $W + W \subset V$. For every $n \in \mathbf{N}$ the set $M_n = \bigcap_{u \in H} u^{-1}(nW)$ is closed and the assumption that H is simply bounded implies $\bigcup_{n=1}^{\infty} M_n = E$. Since E is a Baire space [2, Chap. IX, 2nd ed., §5, $n^0$3, Def. 3], one of the sets M_n has an interior point and therefore $M = \bigcap_{u \in H} u^{-1}(W)$ has an interior point y. Hence $0 = y - y$ is an interior point of $M - M$, and since $M - M \subset \bigcap_{u \in H} u^{-1}(W + W) \subset \bigcup_{u \in H} u^{-1}(V)$, the set $\bigcap_{u \in H} u^{-1}(V)$ is a neighborhood of 0 in E, which means that H is equicontinuous.

From this theorem one obtains the usual corollary:

Let $u_n: E \to F$ be a sequence of continuous linear maps which converges pointwise to a linear map $u: E \to F$. Then u is continuous and the convergence is uniform on every compact subset of E [2, Chap. X, 2nd ed., §2, n^0 4, Th. 1].

Returning to our problem we choose $E = l^{p'}$, $F = \text{Meas } (\Omega, m; \mathbf{R})$, and we obtain that $f: \xi \longmapsto f(\xi)$ is a continuous linear map from $l^{p'}$

into Meas $(\Omega, m; \mathbf{R})$, i.e. a continuous linear random function. We have proved therefore

PROPOSITION 9. *There is a bijective correspondence between the cylindrical measures μ on $l^p (1 < p < \infty)$ scalarly concentrated on the balls, and the isonomy classes of sequences (x_n) of real random variables with the following property: for each numerical sequence $(\alpha_n) \in l^{p'}$, $1/p + 1/p' = 1$, the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges in probability.*

REMARK 3. If $(x_n)_{n \in \mathbf{N}}$ is a sequence of real random variables such that for every $\xi = (\alpha_n) \in l^{p'}$ the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges in Meas $(\Omega, m; \mathbf{R})$, then for every $\xi = (\alpha_n) \in l^{p'}$ the family $(\alpha_n x_n)_{n \in \mathbf{N}}$ is summable [2, Chap. III, 3rd ed., § 5, n° 1, Def. 1] in Meas $(\Omega, m; \mathbf{R})$. Indeed, we have seen in the proof of Prop. 9 that $\xi = (\alpha_n) \longmapsto f(\xi) = \sum_{n=0}^{\infty} \alpha_n x_n$ is a continuous linear map from $l^{p'}$ into Meas $(\Omega, m; \mathbf{R})$. But the family $(\alpha_n \epsilon_n)_{n \in \mathbf{N}}$ is summable in $l^{p'}$, therefore its image by f, the family $(\alpha_n x_n)_{n \in \mathbf{N}}$, is summable in Meas $(\Omega, m; \mathbf{R})$.

A similar remark can be made in connection with several other results of this section.

We shall now characterize those sequences (x_n) of random variables for which the associated cylindrical measure μ is a Radon measure on l^p.

PROPOSITION 10. *The cylindrical measure μ on l^p, scalarly concentrated on balls, is a Radon measure if and only if the associated sequence (x_n) of real random variables satisfies $\sum_{n=0}^{\infty} |x_n(\omega)|^p < \infty$ for almost every $\omega \in \Omega$.*

PROOF. Let us first observe that the condition that $\sum_{n=0}^{\infty} |x_n(\omega)|^p$ converges almost surely, depends only on the isonomy class of the random function $f: l^{p'} \to \text{Meas}(\Omega, m; \mathbf{R})$ associated with the cylindrical measure μ. Indeed, if f is replaced by an isonomous random function, then (x_n) is replaced by an isonomous sequence. Now

the sequence (x_n) defines a map $\Omega \to \mathbf{R}^{\mathbf{N}}$, and the condition that $\Sigma |x_n(\omega)|^p < \infty$ for almost every $\omega \in \Omega$ means that the image of the measure m with respect to the map $(x_n): \Omega \to \mathbf{R}^{\mathbf{N}}$ is concentrated on the subspace l^p of $\mathbf{R}^{\mathbf{N}}$. By the very definition of isonomy, the projections of (x_n) (m) onto the products $\mathbf{R}^J$, where J is a finite subset of $\mathbf{N}$, depends only on the isonomy class of (x_n), and therefore (x_n) (m) depends only on the isonomy class of (x_n).

We may therefore consider the "fundamental" random function f for which $\Omega = \check{\mathbf{R}}^{l^{p'}}$, $m = \check{\mu}$, and which assigns to $\xi \in l^{p'}$ the random variable $f(\xi)$ which is simply the ξ-th projection π_ξ (cf. p. 258). There is a canonical injection $j: l^p \to \check{\mathbf{R}}^{l^{p'}}$ given by $j(c) = (\langle c, \xi \rangle)_{\xi \in l^{p'}}$ for $c = (c_k) \in l^p$, and the value of $x_n = f(\epsilon_n)$ at a point $\omega = j(c) \in j(l^p)$ is given by $x_n(\omega) = \pi_{\epsilon_n}(\langle c, \xi \rangle) = \langle c, \epsilon_n \rangle = c_n$. Thus $\check{\mu}$ is concentrated on $j(l^p)$ if and only if $(x_n(\omega))_{n \in \mathbf{N}}$ belongs to l^p for almost every $\omega \in \Omega$.

Consider now the case when $p = p' = 2$ and l^2 is a Hilbert space. Let $(\lambda_n) \in l^2$ and let the map $u: l^2 \to l^2$ be defined by $u: (c_n) \longmapsto (\lambda_n c_n)$. Then u is a Hilbert-Schmidt map since if (e_n) is the orthonormal system of unit vectors $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$, then $\Sigma \| u(e_n) \|^2 = \Sigma |\lambda_n|^2 < \infty$. If μ is a cylindrical measure on l^2, scalarly concentrated on the balls of l^2, then it follows from Sazonov's theorem (Chap. 3, § 3, Theorem 2) that $u(\mu)$ is a Radon measure on l^2. The transpose ${}^t u$ of u is the map $(\alpha_n) \longmapsto (\lambda_n \alpha_n)$ since it satisfies the relation $\langle u(c), \xi \rangle = \Sigma \lambda_n c_n \alpha_n = \langle c, {}^t u(\xi) \rangle$ for $c = (c_n) \in l^2$ and $\xi = (\alpha_n) \in l^2$. Therefore if f is the real random function associated with μ, by Prop. 5 (§2) the real random function associated with $u(\mu)$ is $f \circ {}^t u$, and the sequence of real random variables associated with $u(\mu)$ is given by $(f \circ {}^t u)(\epsilon_n) = f(\lambda_n \epsilon_n) = \lambda_n f(\epsilon_n) = \lambda_n x_n$. We have therefore the following consequence of Sazonov's theorem in which cylindrical measures do not figure explicitly.

THEOREM 5. *Let (x_n) be a sequence of real random variables and assume that for every numerical sequence (α_n) such that $\Sigma |\alpha_n|^2 < \infty$*

the series $\sum_{n=0}^{\infty} \alpha_n x_n$ *converges in probability. Then for every sequence* (α_n) *such that* $\Sigma |\alpha_n|^2 < \infty$ *the series* $\Sigma |\alpha_n x_n|^2$ *converges almost surely.*

Let us observe that if (x_n) is a sequence of independent random variables, then the assertion is well known.

The methods developed so far in this section permit us to prove the converse of Sazonov's theorem which we state in the following form.

THEOREM 6. *Let E and F be two Hilbert spaces, and $u: E \to F$ a continuous linear map. If the image $u(\mu)$ of every cylindrical measure μ scalarly concentrated on the balls of E is a Radon measure on F, then u is a Hilbert-Schmidt operator.*

PROOF. For the sake of brevity let us say that the continuous linear map $u: E \to F$ is radonifying† if it transforms every cylindrical measure μ scalarly concentrated on the balls of E into a Radon measure $u(\mu)$ on F.

(a) It follows from the considerations which lead to the proof of Theorem 5 that the map $(c_n) \to (\lambda_n c_n)$ from l^2 into l^2, defined by some sequence (λ_n) is radonifying if and only if for any sequence (x_n) of real random variables such that $\Sigma \alpha_n x_n$ converges in probability for every $(\alpha_n) \in l^2$ we have $\Sigma |\lambda_n x_n(\omega)|^2 < \infty$ almost surely.

It follows that the identity map $l^2 \to l^2$, corresponding to $\lambda_n = 1$ for $n \geqslant 0$ is not radonifying. Indeed, let us consider a sequence (x_n) of independent real random variables taking the values ± 1 with probabilities $\frac{1}{2}$. A classical theorem of probability theory tells us that $\sum_{n=0}^{\infty} \alpha_n x_n$ converges in probability if and only if $(\alpha_n) \in l^2$. However, the series $\sum_{n=0}^{\infty} |x_n(\omega)|^2 = \sum_{n=0}^{\infty} 1$ does not converge almost surely.

(b) If $u: E \to F$ is a continuous linear map from the Hilbert space E into the Hilbert space F, then we have the decomposi-

† The notion of radonifying maps will be introduced more systematically later on (Chap. 5, § 4, Definition 2).

tions $u = UT$ and $T = Vu$, where $T = \sqrt{u^*u}$ is a positive hermitian operator $E \to E$, and $U: E \to F$, $V: F \to E$ are partial isometries (p. 224). If u is radonifying, then so is T, and if T is a Hilbert-Schmidt operator, then so is u.

Thus to prove Theorem 6 it is sufficient to prove that if a positive hermitian operator u on the Hilbert space E is radonifying, then it is Hilbert-Schmidt.

(c) Assume that u is a radonifying positive hermitian operator on E. For any $\epsilon > 0$ let $K_\epsilon \subset E$ be the spectral subspace of u corresponding to the part of the spectrum of u for which $\lambda \geqslant \epsilon$. The restriction u_ϵ of u to K_ϵ can be factored into

$$K_\epsilon \xrightarrow{j} E \xrightarrow{u} E \xrightarrow{p} K_\epsilon$$

where j is the canonical injection and p the orthogonal projector onto K_ϵ. Hence $u_\epsilon: K_\epsilon \to K_\epsilon$ is radonifying. On the other hand, u_ϵ is inversible and so the identity map $u_\epsilon^{-1} \circ u_\epsilon: K_\epsilon \to K_\epsilon$ is also radonifying. If K_ϵ were infinite-dimensional, then the restriction of the identity map to any separable infinite-dimensional subspace of K_ϵ would also be radonifying, which is impossible by part (a). Thus for every $\epsilon > 0$ the space K_ϵ is finite-dimensional, i.e. u is a compact operator.

(d) Since only countably many eigenvalues of u are different from zero, we may assume that $E = l^2$, and that u is the map $(c_n) \longmapsto (\lambda_n c_n)$, where the sequence λ_n converges to zero. Let again (x_n) be the sequence of independent random variables with values $\pm$ considered in part (a) of the proof. Since u is radonifying, we have $\sum_{n=0}^{\infty} |\lambda_n x_n(\omega)|^2 = \sum_{n=0}^{\infty} |\lambda_n|^2 < \infty$, which proves that u is a Hilbert-Schmidt operator.

2) Let s be the vector space of all numerical sequences $c = (c_n)$ which are rapidly decreasing, i.e. for every $k \geqslant 0$ the sequence $(n^k c_n)$ converges to 0 as $n \to \infty$. If we equip s with the family of semi-norms

$$q_k(c) = \max_n |n^k c_n|$$

we obtain a nuclear Fréchet space [9, Chap. II, §2, n^0 3, Th. 10, p. 55]. Let s' be the space of all numerical sequences $\xi = (\alpha_n)$ which are slowly increasing, i.e. such that there exists an integer $k \geqslant 0$ for which the sequence $(n^{-k}\,\alpha_n)_{n\geqslant 1}$ is bounded. The bilinear form

$$(\xi, c) \longmapsto \langle \xi, c\rangle = \Sigma\, \alpha_n\, c_n$$

establishes a canonical isomorphism between s' and the dual of s, and in particular s' equipped with the strong topology is a nuclear space [9, Chap. II, §2, n^0 1, Th. 7, p. 40].

Now let μ be a cylindrical measure on s', scalarly concentrated on the bounded (i.e. equicontinuous) subsets of s'. To μ there corresponds a random function f: $s \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ which is continuous in probability. To each unit vector e_n in s there corresponds a random variable $f(e_n) = x_n$: $\Omega \to \mathbf{R}$. If $c = (c_n)$ is an arbitrary element in s, then $\sum_{n=0}^{N} c_n\, e_n$ converges to c in s as $N \to \infty$ since $q_k\left(c - \sum_{n=0}^{N} c_n\, e_n\right) = \max_{n>N} |n^k c_n|$, and therefore by continuity $f\left(\sum_{n=0}^{N} c_n\, e_n\right) = \sum_{n=0}^{N} c_n\, x_n$ converges in probability to $f(c)$. In other words, each $f(c)$: $\Omega \to \mathbf{R}$ is the sum of a series $\sum_{n=0}^{\infty} c_n\, x_n$ which converges in probability.

Conversely, if (x_n) is a sequence of real random variables such that for each $c = (c_n) \in s$ the series $\Sigma\, c_n\, x_n$ converges in probability to some $f(c)$: $\Omega \to \mathbf{R}$, then f: $c \longmapsto f(c)$ is clearly a linear random function, and we prove similarly as in the preceding example that f is continuous in probability. Thus there exists a bijective correspondence between the ϵ-concentrated cylindrical measures on s' and the sequences (x_n) of real random variables such that $\Sigma\, c_n\, x_n$ converges in probability for all $(c_n) \in s$.

On the other hand μ is a Radon measure on s' if and only if the associated sequence (x_n) of real random variables belongs almost surely to s'. In the first place one sees exactly as in the proof of Prop. 10 that the condition "$(x_n(\omega)) \in s'$ almost surely" depends only

on the isonomy class of f. We may therefore reason on the "fundamental" random function f for which $\Omega = \check{\mathbf{R}}^s$, $m = \check{\mu}$ and $f(c) = \pi_c$ for every $c \in s$. The canonical injection $j: s' \to \check{\mathbf{R}}^s$ is given by $j(\xi) = (\langle c, \xi \rangle)_{c \in s}$ for $\xi = (a_n) \in s'$, and at $\omega = j(\xi)$ we have $x_n(\omega) = f(e_n)(\omega) = \pi_{e_n}(\langle c, \xi \rangle) = \langle e_n, \xi \rangle = a_n$. Thus $\check{\mu}$ is concentrated on $j(s')$ if and only if $(x_n(\omega))_{n \in \mathbf{N}}$ belongs to s' for almost every $\omega \in \Omega$.

Now by the theorem of Minlos (Chap. 4, § 3, Th. 2), every scalarly ϵ-concentrated cylindrical measure on s' is a Radon measure, and we obtain

THEOREM 7. *Let (x_n) be a sequence of real random variables and assume that for every rapidly decreasing numerical sequence (c_n) the series $\sum_{n=0}^{\infty} c_n x_n$ converges in probability. Then the sequence (x_n) is almost surely slowly increasing.*

COROLLARY. *Let (x_n) be a sequence of real random variables such that for every rapidly decreasing numerical sequence (c_n) the sequence $(c_n x_n)$ is almost surely bounded. Then (x_n) is almost surely slowly increasing.*

PROOF. For any sequence (c_n) belonging to s the sequence $(n^2 c_n)$ also belongs to s and therefore

$$\sum_{n=0}^{\infty} |c_n x_n(\omega)| = \sum_{n=0}^{\infty} \frac{1}{n^2} |n^2 c_n x_n(\omega)| < \infty$$

almost surely. Hence $\sum_{n=0}^{\infty} c_n x_n(\omega)$ converges almost surely and thus by Prop. 1 (§1) also in probability. It follows from the theorem that (x_n) is almost surely slowly increasing.

The corollary is much less deep than the theorem and can be proved also directly in an elementary way.

3) Let μ be a cylindrical measure on the Banach space $\mathscr{C}(K)$ of all continuous functions on the compact space K. Assume that μ is scalarly concentrated on the compact subsets of $\mathscr{C}(K)$. To μ there corresponds a continuous linear random function $f: \mathscr{C}'_c(K) \to \mathrm{Meas}(\Omega, m; \mathbf{R})$, where $\mathscr{C}'_c(K)$ is the space of Radon measures on

K equipped with the topology of uniform convergence on compact sets, and $\mathrm{Meas}(\Omega, m; \mathbf{R})$ is equipped with the topology of convergence in measure. For each Radon measure $\theta \in \mathscr{C}'_c(K)$ we have a random variable $f(\theta)\colon \Omega \to \mathbf{R}$ and if $\theta \to \theta_0$ in $\mathscr{C}'_c(K)$, then $f(\theta)$ tends to $f(\theta_0)$ in probability. Furthermore for any $\lambda_1, \lambda_2 \in \mathbf{R}$ and $\theta_1, \theta_2 \in \mathscr{C}'_c(K)$ we have the relation $f(\lambda_1\theta_1 + \lambda_2\theta_2) = \lambda_1 f(\theta_1) + \lambda_2 f(\theta_2)$ between random variables.

Next we want to show that the random variables $f(\theta)$ can be represented as Riemann integrals in a sense we shall explain now. Let h be a continuous numerical function and θ a positive Radon measure on the compact space K. If (K_i) is a finite partitioning of $K = \bigcup_i K_i$ by Borel sets K_i, and if $x_i \in K_i$ for each i, then the corresponding Riemann sum is defined by

$$\sum_i h(x_i)\,\theta(K_i) = \langle h, \sum_i \theta(K_i)\,\delta_{x_i} \rangle.$$

Clearly these Riemann sums converge to $\theta(h) = \int h(x)\,d\,\theta(x)$ with respect to the directed system of all finite Borel partitions of K. Since the measures $\sum \theta(K_i)\,\delta_{x_i}$ are all bounded in norm by $\theta(K)$ and thus form an equicontinuous set in $\mathscr{C}'(K)$, by Ascoli's theorem [2, Chap. IX, 2nd ed., §2, $n°$ 4, Th. 1] the convergence is uniform on every compact subset of $\mathscr{C}(K)$, i.e. the measures $\sum_i \theta(K_i)\delta_{x_i}$ converge to θ in $\mathscr{C}'_c(K)$. Hence if $f\colon \mathscr{C}'_c(K) \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ is a linear random function associated with a cylindrical measure scalarly concentrated on the compact subsets of $\mathscr{C}(K)$, then for each $\theta \in \mathscr{C}'_c(K)$ the random variable $\sum_i f(\delta_{x_i})\,\theta(K_i)$ converges in measure to $f(\theta)$. Define the random function $\bar{f}\colon K \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ by $\bar{f}(x) = f(\delta_x)$ for $x \in K$. The function $\bar{f}$ is continuous for the topology of convergence in measure on $\mathrm{Meas}(\Omega, x; \mathbf{R})$, and for each $\theta \in \mathscr{C}'_c(K)$ the sums $\sum_i \bar{f}(x_i)\,\theta(K_i)$ converge in measure to $f(\theta)$. We shall indicate this last condition by saying that $\bar{f}$ is Riemann integrable for every $\theta \in \mathscr{C}'_c(K)$ and $f(\theta)$ is equal to the Riemann integral $\int \bar{f}(x) d\,\theta(x)$ of $\bar{f}$.

Conversely, let $\bar{f}: K \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ be a random function, continuous for the topology of convergence in measure on $\mathrm{Meas}(\Omega, m_1 \mathbf{R})$. Assume that

(a) for each $\theta \in \mathscr{C}'_c(K)$ the sums $\sum \bar{f}(x_i)\,\theta(K_i)$ converge in measure to some random variable $f(\theta) = \int \bar{f}(x)d\theta(x): \Omega \to \mathbf{R}$;

(b) the linear random function $f: \theta \longmapsto f(\theta)$ so obtained is continuous from $\mathscr{C}'_c(K)$ into $\mathrm{Meas}(\Omega, m; \mathbf{R})$ equipped with the topology of convergence in measure.

Then to $\bar{f}$ there corresponds a cylindrical measure on $\mathscr{C}(K)$, scalarly concentrated on the compact sets. There is no reason to believe that if $\bar{f}$ is a continuous random function on K, Riemann-integrable for every Radon measure, then the associated random function f will automatically satisfy condition (*b*).

One can also see the previous fact in another way. The random function $\bar{f}$ on K is nothing else than a continuous function on K with values in $\mathrm{Meas}(\Omega, m; \mathbf{R})$. This last space *is not locally convex*, therefore such a function need not be Riemann integrable. The hypothesis (a) means exactly that $\bar{f}$ is Riemann integrable, for every Radon measure θ on K.

Of course one can replace K by a locally compact space T, countable at infinity; then $\mathscr{C}(T)$ will be the space of real continuous function on T, and $\mathscr{C}'(T)$ the space of Radon measures on T with compact support.

Suppose now that K is for instance the n-dimensional torus $\mathbf{T}^n$. With every function $p \in \mathscr{C}(K)$ we associate the distribution T_p given by $\langle T_p, \phi \rangle = \int p(x)\phi(x)dx$ for every $\phi \in \mathscr{D}(K)$. Thus we obtain a linear map $u: p \longmapsto T_p$ from $\mathscr{C}(K)$ into $\mathscr{D}'(K)$, and if μ is a cylindrical measure, scalarly concentrated on compact subsets of $\mathscr{C}(K)$, then $u(\mu)$ is a Radon measure on $\mathscr{D}'(K)$ by virtue of Corol. 1 of Theorem 2 of Chap. 4 §3. The random function $g = f \circ {}^t u: \mathscr{D}(K) \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ associated with $u(\mu)$, if it is canonical†, is almost

† We shall see, in fact, that it is always canonical (Prop. 11, Chap. 5, §4).

surely in $\mathscr{D}'(K)$ (Th. 4, §2), i.e. it is a "random distribution": for almost every $\omega \in \Omega$ there exists $T(\omega) \in \mathscr{D}'(K)$ such that $\check{g}(\omega)(\phi) = \langle T(\omega), \phi \rangle$ for every $\phi \in \mathscr{D}(K)$. (The same result holds for $\mathscr{E}(\mathbf{R}^n)$ and $\mathscr{D}'(\mathbf{R}^n)$).

As a matter of fact we shall now see that g is almost surely in a much smaller subspace of $\mathscr{D}'(\mathbf{T}^n)$. For any multi-index $p = (p_1, \ldots, p_n) \in \mathbf{Z}^n$ the Fourier coefficient $\hat{T}(p) = c_p$ of a distribution $T \in \mathscr{D}'(\mathbf{T}^n) = \mathscr{D}'(K)$ is defined by

$$c_p = \langle T, e^{-2\pi i \langle p,t \rangle} \rangle,$$

where $t = (t_1, \ldots, t_n) \in \mathbf{T}^n$ and $\langle p, t \rangle = p_1 t_1 + \cdots + p_n t_n$. Given a real number s we denote by $\mathscr{H}^s = \mathscr{H}^s(\mathbf{T}^n)$ the vector space of all distributions $T \in \mathscr{D}'(K)$ such that the series $\sum_{p \in \mathbf{Z}^n} |p|^{2s} |c_p|^2$ converges, where $|p| = |p_1| + \cdots + |p_n|$. If we equip $\mathscr{H}^s$ with the inner product

$$\langle S, T \rangle = \hat{S}(0)\hat{T}(0) + \sum_{p \in \mathbf{Z}^n}{}' |p|^{2s} \hat{S}(p)\hat{T}(p),$$

it becomes a Hilbert space (the dash of the summation symbol indicates that the term corresponding to $p = 0 = (0, \ldots, 0)$ is omitted). The corresponding norm will be denoted by $\|T\|_s$.

The exponential functions $t \longmapsto e^{2\pi i \langle p,t \rangle}$ $(p \in \mathbf{Z}^n)$ form an orthogonal basis of $\mathscr{H}^s$ and we have $\|e^{2\pi i \langle p,t \rangle}\|_s = |p|^s$ if $p \neq 0$ and $= 1$ if $p = 0$. If we express the elements of $\mathscr{H}^s$ in terms of the corresponding normalized basis, the identical injection j from $\mathscr{H}^s$ into $\mathscr{H}^{s-k}$ is given by

$$\sum_p \gamma_p \frac{e^{2\pi i \langle p,t \rangle}}{|p|^s} \longmapsto \sum_p |\gamma_p |p|^{-k} \frac{e^{2\pi i \langle p,t \rangle}}{|p|^{s-k}}$$

and in particular

$$j\left(\frac{e^{2\pi i \langle p,t \rangle}}{|p|^s}\right) = |p|^{-k} \frac{e^{2\pi i \langle p,t \rangle}}{|p|^{s-k}}.$$

Hence j is a Hilbert-Schmidt map if and only if

$$\sum \left\| j\left(\frac{e^{2\pi i\langle p,t\rangle}}{|p|^s}\right)\right\|^2_{s-k} = \sum \frac{1}{|p|^{2k}} < \infty,$$

i.e. if and only if $k > n/2$.

Let now μ be a cylindrical measure on $\mathscr{C}(\mathbf{T}^n)$ scalarly concentrated on the balls of $\mathscr{C}(\mathbf{T}^n)$. Since $\mathscr{C}(\mathbf{T}^n) \subset L^2(\mathbf{T}^n) = \mathscr{H}^0$ and the unit ball of $\mathscr{C}(\mathbf{T}^n)$ is contained in the unit ball of $\mathscr{H}^0$, we can consider μ as a cylindrical measure on $\mathscr{H}^0$, scalarly concentrated on the balls of $\mathscr{H}^0$. By Sazonov's theorem (Chap. 3, §3, Th. 1) and the preceding considerations μ is a Radon measure on $\mathscr{H}^{-k}$ for every $k > n/2$. In particular the canonical random functions associated with μ are almost surely in any $\mathscr{H}^s$ with $s < -n/2$.

The result still holds if $\mathbf{T}^n$ is replaced by any compact, n-dimensional, infinitely differentiable manifold V. In this case the norm on $\mathscr{H}^s(V)$ is defined only up to equivalence. Let $(\alpha_i)_{i\in I}$ be a finite, infinitely differentiable partition of unity on V, where the supports of the α_i are contained in coordinate patches. We may assume that each local chart is a diffeomorphism from an open subset of V onto an open subset of $\mathbf{T}^n$. In $\mathscr{H}^s(V)$ we take for norm $T \longmapsto \|T\|_s = (\sum_{i\in I} \|\alpha_i T\|_s^2)^{1/2}$, where $\|\alpha_i T\|_s$ is the norm in $\mathscr{H}^s(\mathbf{T}^n)$. Then $\mathscr{H}^s(V) \to \mathscr{H}^{s-k}(V)$ is a Hilbert-Schmidt map if and only if $k > n/2$. Indeed, let first $k > n/2$. Then the injection $\mathscr{H}^s(\mathbf{T}^n) \to \mathscr{H}^{s-k}(\mathbf{T}^n)$ is a Hilbert-Schmidt map. Now $\mathscr{H}^s(V) \to \mathscr{H}^{s-k}(V)$ is the restriction to $\mathscr{H}^s(V)$ of the map $\bigoplus_{i\in I} \mathscr{H}^s(\mathbf{T}^n) = \bigoplus_{i\in I} \mathscr{H}^{s-k}(\mathbf{T}^n)$, since $T \longmapsto (\alpha_i T)_{i\in I}$ is an isometry of $\mathscr{H}^s(V)$ onto a closed subspace of $\bigoplus_{i\in I} \mathscr{H}^s(\mathbf{T}^n)$. Thus $\mathscr{H}^s(V) \to \mathscr{H}^{s-k}(V)$ is a Hilbert-Schmidt map. Conversely, if $k \leqslant n/2$, then the map $\bigoplus_{i\in I} \mathscr{H}^s(\mathbf{T}^n) \to \bigoplus_{i\in I} \mathscr{H}^{s-k}(\mathbf{T}^n)$ is not Hilbert-Schmidt, but it could a priori still induce a Hilbert-Schmidt map on $\mathscr{H}^s(V)$. Let us, however, invert the roles of V and $\mathbf{T}^n$ and let us choose a finite, infinitely differentiable partition of unity $(\beta_j)_{j\in J}$ on $\mathbf{T}^n$ such that the support of each β_j has an open neighborhood diffeomorphic to an open subset of V. Then $\mathscr{H}^s(\mathbf{T}^n)$ is a closed subspace of $\bigoplus_{j\in J} \mathscr{H}^s(V)$, and if $\mathscr{H}^s(V) \to \mathscr{H}^{s-k}(V)$ were a Hilbert-Schmidt map, then the

same would be true for $\mathscr{H}^s(\mathbf{T}^n) \to \mathscr{H}^{s-k}(\mathbf{T}^n)$, which is impossible if $k \leqslant n/2$.

4) We shall now consider cylindrical measures on the spaces l^p in the cases $p = 1$ or ∞, which were omitted in example 1). It does not seem to be easy to characterize the cylindrical measures on l^1 equipped with the normed topology, scalarly concentrated on the balls of l^1. It is still true that if μ is such a cylindrical measure on l^1 we can associate with it an isonomy class of continuous linear random functions f: $l^\infty \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ and therefore also a sequence of random variables $x_n = f(\epsilon_n)$, where $\epsilon_n = (0,\ldots,0,\overset{n}{\smile}{1}, 0\ldots.)$ is the n-th unit vector in l^∞. But if $\xi = (\alpha_n) \in l^\infty$, then the sections $\sum_{k=0}^{n} \alpha_k \epsilon_k$ do not necessarily converge to ξ in l^∞ and therefore we do not know whether $\sum_{k=0}^{n} \alpha_k x_k$ converges to $f(\xi)$ in probability. It is, however, obviously true that if conversely (x_n) is a sequence of real random variables such that for every $\xi = (\alpha_n) \in l^\infty$ the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges in probability to some random variable $f(\xi)$, then $\xi \longmapsto f(\xi)$ is a continuous linear random function (see the proof of Prop. 9).

The situation becomes much simpler if we equip l^1 with the weak*-topology $\sigma(l^1, c_0)$. Then the dual of l^1 is c_0 and exactly as in Prop. 9 we obtain a bijective correspondence between the cylindrical measures μ on l^1 scalarly concentrated on balls and the isonomy classes of sequences (x_n) of real random variables such that for every $\xi = (\alpha_n) \in c_0$ the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges in measure.

Denote by F the space l^1 equipped with the normed topology, and by E the space l^1 equipped with the topology $\sigma(l^1, c_0)$. The canonical injection j: $F \to E$ is continuous and associates with each cylindrical measure μ on F, scalarly concentrated on balls, a cylindrical measure $j(\mu)$ on E, scalarly concentrated on balls. We know (Chap. 2, §3, Example 3) that the map $\mu \longmapsto j(\mu)$ is not injective. It is, however, surjective as an immediate consequence

of the following result which we shall prove later: If a sequence of real random variables (x_n) is such that $\sum_{n=0}^{\infty} \alpha_n x_n$ converges in probability for every $\xi = (\alpha_n) \in c_0$, then $\sum_{n=0}^{\infty} \alpha_n x_n$ converges in probability also for every $\xi = (\alpha_n) \in l^\infty$.

We see as in the proof of Prop. 10 that μ is a Radon measure on E if and only if the associated sequence (x_n) is almost surely in l^1. If μ is a Radon measure on E, then it is also a Radon measure on F since F is a Polish space.

Cylindrical measures on l^∞ equipped with the normed topology seem also to be difficult to handle, but if we equip l^∞ with the weak*-topology $\sigma(l^\infty, l^1)$ we obtain a bijective correspondence between the cylindrical measures μ on l^∞, scalarly (i.e., with respect to the dual l^1) concentrated on balls, and the isonomy classes of sequences (x_n) of real random variables such that for every $\xi = (\alpha_n) \in l^1$ the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges in probability. The cylindrical measure μ is a Radon measure on l^∞ equipped with $\sigma(l^\infty, l^1)$ if and only if (x_n) is almost surely in l^∞, but by Example 2 of Chap. 1, §5 if μ is a Radon measure on l^∞ equipped with $\sigma(l^\infty, l^1)$, it is not necessarily a Radon measure on l^∞ equipped with the normed topology.

5) The considerations of this section motivate the following theorem which is known for locally convex spaces.

THEOREM 8. *Let E be a Hausdorff, quasi-complete, topological vector space, $(x_n)_{n \in \mathbf{N}}$ a sequence of elements of E and $1 < p < \infty$. The following conditions are equivalent.*

(i) *For every sequence $\xi = (\alpha_n)_{n \in \mathbf{N}} \in l^{p'}$ the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges.*

(i') *For every sequence $\xi = (\alpha_n)_{n \in \mathbf{N}} \in l^{p'}$ the family $(\alpha_n x_n)$ is summable.*

(ii) *There exists a bounded, balanced, convex, closed subset B of E such that the sequence (x_n) belongs to B and that in the Banach space E_B* (Chap. 4, §2) *the sequence is scalarly in l^p.*

(ii′) *There exists a bounded, balanced, convex, closed subset B of E such that the sequence (x_n) belongs to B and that in the Banach space E_B the series $\sum_{n=0}^{\infty} \alpha_n x_n$ is scalarly convergent for every $\xi = (\alpha_n) \in l^{p'}$.*

(ii″) *There exists a bounded, balanced, convex, closed subset B of E such that the sequence (x_n) belongs to B and that in the Banach space E_B the family $(\alpha_n x_n)$ is scalarly summable for every $\xi = (\alpha_n) \in l^{p'}$.*

(iii) *The map $\xi = (\alpha_n) \longmapsto \sum_{n=0}^{\infty} \alpha_n x_n$ defined for the "finite" sequences (i.e. those for which $\alpha_n = 0$ if n is sufficiently large) can be extended into a continuous linear map from $l^{p'}$ into E.*

If furthermore E is locally convex, then the preceding conditions are also equivalent to the following.

(iv) *The sequence (x_n) is scalarly in l^p.*

(iv′) *For every sequence $\xi = (\alpha_n) \in l^{p'}$ the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges scalarly.*

(iv″) *For every sequence $\xi = (\alpha_n) \in l^{p'}$ the family $(\alpha_n x_n)$ is scalarly summable.*

PROOF. We shall give the proof in the following order:

(1) if E is locally convex, (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii);

(2) for E arbitrary, (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

As a conclusion, (ii) $\Leftrightarrow$ (iii) if E is locally convex, (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv);

(3) (iii) $\Rightarrow$ (i′) $\Rightarrow$ (i) $\Rightarrow$ (iii); consequently, (i) $\Leftrightarrow$ (i′) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), $\Leftrightarrow$ (iv) if E is locally convex;

(4) if E is locally convex, (iv) $\Leftrightarrow$ (iv′) $\Leftrightarrow$ (iv″); therefore, for E arbitrary, (ii) $\Leftrightarrow$ (ii′) $\Leftrightarrow$ (ii″); the proof will be complete.

(ii) $\Rightarrow$ (iv). In general, if $u: G \to H$ is a continuous linear map from the locally convex space G into the locally convex space H, and if (x_n) is a sequence of elements of G such that for every $x' \in G'$ the numerical sequence $(\langle x_n, x' \rangle)$ is in l^p, then it follows from

$\langle u(x_n), y' \rangle = \langle x_n, {}^t u(y') \rangle$ that for every $y' \in H'$ the numerical sequence $(\langle u(x_n), y' \rangle)$ is in l^p. Applying this to the continuous injection $E_B \to E$ we see that (ii) implies (iv).

Next let us show that if E is locally convex, then (iv) implies (iii). Let us assume that for every $x' \in E'$ we have $(\sum_{n=0}^{\infty} |\langle x_n, x' \rangle|^p)^{1/p} < +\infty$. For every $n \in \mathbf{N}$ denote by u_n the continuous linear map from $l^{p'}$ into E which associates with each $\xi = (\alpha_n) \in l^{p'}$ the element $\sum_{k=0}^{n} \alpha_k x_k \in E$. For every $\xi = (\alpha_n) \in l^{p'}$ the sequence $(u_n(\xi))$ is weakly bounded in E since for every $x' \in E'$ we have by assumption

$$|\langle u_n(\xi), x' \rangle| \leqslant \sum_{k=0}^{n} |\alpha_k \langle x_k, x' \rangle|$$

$$\leqslant \left(\sum_{k=0}^{\infty} |\alpha_k|^{p'} \right)^{1/p'} \left(\sum_{k=0}^{\infty} |\langle x_k, x' \rangle|^p \right)^{1/p},$$

hence the sequence $(u_n(\xi))$ is also bounded in E for the original topology [3, Chap. IV, § 2, $n°4$, Corol. of Th. 3]. In other words, the sequence (u_n) is bounded in $\mathscr{L}(l^{p'}, E)$ for the topology of simple convergence, and therefore by the usual Banach-Steinhaus theorem [3, Chap. III, § 3, $n°6$, Th. 2] it is equicontinuous. For evey "finite" sequence $\xi = (a_n)$, the sequence $(u_n(\xi))_{n \in \mathbf{N}}$ has a limit $u(\xi)$. Now since $1 \leqslant p' < \infty$, the "finite" sequences form a dense subset of $l^{p'}$, and on this dense subset the sequence (u_n) has a limit u for the topology of pointwise convergence. Since E is quasi-complete, it follows [3, Chap. III, § 3, $n°7$, Corol. 1 of Th. 4] that the $u_n(\xi)$ converge to $u(\xi) = \sum_{k=0}^{\infty} \alpha_k x_k$ as $n \to \infty$ for every $\xi \in l^{p'}$, and that u is a continuous linear map from $l^{p'}$ into E.

We can now prove that (ii) implies (iii). Indeed, applying the preceding result to the Banach space E_B, we see that if (ii) is true, then $\xi \longmapsto u(\xi) = \sum_{n=0}^{\infty} \alpha_n \xi_n$ is a continuous linear map from l^p into E_B, and therefore a fortiori from l^p into E.

(iii)$\Rightarrow$ (ii). Let $u: l^{p'} \to E$ be the continuous linear map of (iii). It maps the unit ball of $l^{p'}$ into a bounded, balanced, convex subset of E, and we denote by B the closure of this set. Then $u: l^{p'} \to E_B$ is a continuous linear map with norm 1 and the elements $x_n = u(\epsilon_n)$ belong to B. Also the sequence (ϵ_n) in $l^{p'}$ belongs scalarly to l^p, hence, by the observation made at the very beginning of the proof, the sequence (x_n) in E_B belongs scalarly to l^p.

(iii)$\Rightarrow$ (i'). Assume that $u: \xi \longmapsto \Sigma \alpha_n x_n$ is a continuous linear map from $l^{p'}$ into E. Given a neighborhood V of 0 in E there exists $\epsilon > 0$ such that $\|\xi - \xi'\| \leqslant \epsilon$ implies $u(\xi) - u(\xi') \in V$. For any $\xi = (\alpha_n) \in l^{p'}$ and any finite subset F of $\mathbf{N}$ denote by $\xi_F = (\alpha_n^F)$ the sequence defined by $\alpha_n^F = \alpha_n$ if $n \in F$ and $\alpha_n^F = 0$ if $n \notin F$. There exists a finite set $F_0 \subset \mathbf{N}$ such that $\|\xi - \xi_F\| \leqslant \epsilon$ for every $F \supset F_0$. Hence $u(\xi) - \sum_{n \in F} \alpha_n x_n = u(\xi) - u(\xi_F) \in V$ for every $F \supset F_0$.

It is trivial that (i') implies (i). On the other hand (i) implies (iii) by virtue of the form of the Banach-Steinhaus theorem used in the proof of Prop. 9.

To conclude the proof it is sufficient to observe that if F is a locally convex Hausdorff space, (y_n) a sequence of elements of F, and $1 \leqslant p \leqslant \infty$(1), then the following conditions are equivalent.

(α) The sequence (y_n) is scalarly in l^p, i.e. for every $y' \in F'$ the numerical sequence $(\langle y_n, y' \rangle)$ belongs to l^p.

(β) For every sequence $(\alpha_n) \in l^{p'}$ the series $\sum_{n=0}^{\infty} \alpha_n y_n$ is scalarly convergent, i.e. for every $y' \in F'$ the series $\sum_{n=0}^{\infty} \alpha_n \langle y_n, y' \rangle$ is convergent.

(γ) For every sequence $(\alpha_n) \in l^{p'}$ the family $(\alpha_n y_n)$ is scalarly summable, i.e. for every $y' \in F'$ the family $(\alpha_n \langle y_n, y' \rangle)$ is summable.

(1) The previous results did not hold for $p = 1$. For instance, in the Banach space c_0 of the real sequences converging to 0, the sequence of elements ϵ_n, $n \in \mathbf{N}$ (where ϵ_n is the sequence (0,...0,1,0,...), the nth term being 1) is scalarly in l^1, but not summable.

Indeed, the sequence $(\langle y_n, y' \rangle)$ belongs to l^p if and only if for every $(\alpha_n) \in l^{p'}$ the series $\sum_{n=0}^{\infty} |\alpha_n| \cdot |\langle y_n, y' \rangle|$ is convergent.

Thus the conditions (ii), (ii′) and (ii″) and also the conditions (iv), (iv′), (iv″) are equivalent, and the proof of the theorem is complete.

The theorem holds in particular for $p = \infty$. Now that a sequence in a locally convex space F belongs scalarly to l^∞ means that it is weakly bounded, i.e. bounded in F [3, Chap. IV, § 2, n^0 4, Corol. of Th. 3], and therefore we obtain

COROLLARY 1. *Let E be a Hausdorff quasi-complete topological vector space and (x_n) a sequence of elements of E. The following conditions are equivalent.*

(i) *For every sequence $\xi = (\alpha_n) \in l^1$ the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges.*

(i′) *For every sequence $\xi = (\alpha_n) \in l^1$ the family $(\alpha_n x_n)$ is summable.*

(ii) *The convex (or balanced, closed, convex) hull B of the sequence (x_n) is bounded in E.*

(iii) *The map $\xi = (\alpha_n) \longmapsto \sum_{n=0}^{\infty} \alpha_n x_n$ defined for "finite" sequences can be extended to a continuous linear map $l^1 \to E$.*

If E is locally convex, then the preceding conditions are also equivalent to

(iv) *The sequence (x_n) is bounded in E.*

The equivalence of (ii) and (iv) is, of course, a special case of the result that in a locally convex space a set is bounded if and only if its balanced, closed, convex hull is bounded.

Applying the equivalence of (i) and (ii) to $E = \mathbf{Meas}(\Omega, m; \mathbf{R})$, and taking into account the final remarks of 4), we obtain

COROLLARY 2. *There exists a bijective correspondence between the cylindrical measures on l^∞ equipped with the topology $\sigma(l^\infty, l^1)$, scalarly concentrated on the balls of l^∞, and the isonomy classes of*

sequences of real random variables (x_n) *whose convex (or balanced, closed, convex) hull in* $\mathrm{Meas}(\Omega, m; \mathbf{R})$ *is bounded.*

For $p = 1$ Theorem 8 has to be modified since in l^∞ the "finite" sequences do not form a dense subset. As in 4), we consider the topology $\sigma(l^1, c_0)$ on l^1 and we obtain

THEOREM 9. *Let* E *be a Hausdorff, quasi-complete topological vector space and* (x_n) *a sequence of elements of* E. *The following conditions are equivalent.*

(i) *For every sequence* $\xi = (\alpha_n) \in c_0$ *the series* $\sum_{n=0}^{\infty} \alpha_n x_n$ *converges.*

(i′) *For every sequence* $\xi = (\alpha_n) \in c_0$ *the family* $(\alpha_n x_n)$ *is summable.*

(ii) *There exists a bounded, balanced, convex, closed subset* B *of* E *such that the sequence* (x_n) *belongs to* B *and that in the Banach space* E_B *the sequence is scalarly in* l^1.

(ii′) *There exists a bounded, balanced, convex, closed subset* B *of* E *such that the sequence* (x_n) *belongs to* B *and that in the Banach space* E_B *the series* $\sum_{n=0}^{\infty} \alpha_n x_n$ *is scalarly convergent for every* $\xi = (\alpha_n) \in c_0$, *or, equivalently, for every* $\xi = (\alpha_n) \in l^\infty$.

(ii″) *There exists a bounded, balanced convex, closed subset* B *of* E *such that the sequence* (x_n) *belongs to* B *and that in the Banach space* E_B *the family* $(\alpha_n x_n)$ *is scalarly summable for every* $\xi = (\alpha_n) \in c_0$, *or equivalently for every* $\xi = (\alpha_n) \in l^\infty$.

(iii) *The map* $\xi = (\alpha_n) \longmapsto \sum_{n=0}^{\infty} \alpha_n x_n$ *defined for finite sequences* ξ *can be extended into a continuous linear map from* c_0 *into* E.

If furthermore E *is locally convex, then the preceding conditions are equivalent to the following.*

(iv) *The sequence* (x_n) *is scalarly in* l^1.

(iv′) *The series* $\sum_{n=0}^{\infty} \alpha_n x_n$ *converges scalarly for every sequence* $\xi = (\alpha_n) \in c_0$, *or equivalently for every sequence* $\xi = (\alpha_n) \in l^\infty$.

(iv″) *The family* $(\alpha_n x_n)$ *is scalarly summable for every sequence* $\xi = (\alpha_n) \in c_0$, *or equivalently for every sequence* $\xi = (\alpha_n) \in l^\infty$.

PROOF. It is sufficient to repeat the proof of Theorem 8 replacing $l^{p'}$ everywhere by c_0, and to observe that a numerical sequence $c = (c_n)$ belongs to l^1 if and only if $\sum_{n=0}^{\infty} |\alpha_n c_n| < \infty$ for every $(\alpha_n) \in c_0$ or for every $(\alpha_n) \in l^\infty$.

Let us observe that if (x_n) is such that $\sum_{n=0}^{\infty} \alpha_n x_n$ converges for every $(\alpha_n) \in c_0$, then it does not follow necessarily that $\sum_{n=0}^{\infty} \alpha_n x_n$ converges for every $(\alpha_n) \in l^\infty$. For example let $E = c_0$ and $x_n = \epsilon_n$. Then for $\xi = (\alpha_n) \in c_0$ the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges in c_0 to the sum ξ, but for $\xi = (1, 1, \ldots, 1, \ldots) \in l^\infty$ the series $\sum_{n=0}^{\infty} \alpha_n x_n = \sum_{n=0}^{\infty} \epsilon_n$ is not convergent. However, if $E = \mathrm{Meas}(\Omega, m; \mathbf{R})$, then — as we have observed in 4) — the property does hold.

REMARK 4. In Theorems 8 and 9 and the corollaries we could replace sequences $(x_n)_{n \in \mathbf{N}}$ by arbitrary families $(x_\iota)_{\iota \in I}$. In this case the assertions concerning convergent series become meaningless, only the assertions concerning summable families remain, and the spaces l^p, c_0 have to be replaced by $l^p(I)$, $c_0(I)$.

4. An analogue of Prokhorov's Theorem for random variables with values in a Suslin Space. The main result of this section (Theorem 10) is unrelated to topological vector spaces, but we have preferred to present it here because it uses the notion of a random variable which we introduced only in this chapter.

Let I be an increasingly directed ordered set and $(X_\iota)_{\iota \in I}$ a family of topological spaces. Assume that for $\iota \leqslant \kappa$ we are given a continuous map $\pi_{\iota\kappa} : X_\kappa \to X_\iota$ such that $\pi_{\iota\iota}$ is the identity map and $\pi_{\iota\kappa} \circ \pi_{\kappa\lambda} = \pi_{\iota\lambda}$ for $\iota \leqslant \kappa \leqslant \lambda$. Denote by $Z = Z_I = \varprojlim X_\iota$ the

projective limit of the spaces X_ι, and by π_ι the canonical map $Z \to X_\iota$. We have $\pi_\iota = \pi_{\iota\kappa} \circ \pi_\kappa$ for $\iota \leqslant \kappa$ and π_ι is the restriction to Z of the canonical projection $X_I = \prod_{\kappa \in I} X_\kappa \to X_\iota$.

Let X be a topological space and suppose that for each $\iota \in I$ we have a continuous map $\rho_\iota : X \to X_\iota$ such that $\rho_\iota = \pi_{\iota\kappa} \circ \rho_\kappa$ for $\iota \leqslant \kappa$. The maps ρ_ι define a unique continuous map $\rho : X \to Z$ such that $\rho_\iota = \pi_\iota \circ \rho$ for every $\iota \in I$. For any directed subset J of I we shall denote by Z_J the projective limit of the spaces $(X_\iota)_{\iota \in J}$, a subspace of the product space $X_J = \prod_{\iota \in J} X_\iota$, and by ρ_J the unique continuous map $X \to Z_J$ which for every $\iota \in J$ verifies $\rho_\iota = \pi_{\iota J}\, \rho_J$, where $\pi_{\iota J}$ is the canonical map $Z_J \to X_\iota$.

Let now (Ω, m) be a sample space, and assume that for each $\iota \in I$ we are given a random variable $f_\iota \in \mathrm{Meas}(\Omega, m; X_\iota)$. We assume furthermore that for $\iota \leqslant \kappa$ we have $f_\iota = \pi_{\iota\kappa} \circ f_\kappa$ in the sense of random variables, i.e. as an equivalence class of functions defined on Ω, with values in X_ι, which are equal almost everywhere with respect to the measure m. We ask whether there exists a random variable $f \in \mathrm{Meas}(\Omega, m; X)$ such that $f_\iota = \rho_\iota \circ f$ for every $\iota \in I$.

Prokhorov's theorem gave us an answer to the question whether, given probabilities μ_ι on the X_ι, there exists a probability μ on X such that $\mu_\iota = \rho_\iota(\mu)$ for each $\iota \in I$. We now ask the analogous question for random variables. In particular if the spaces X_ι are the quotient spaces E/F of a locally convex Hausdorff space E modulo the closed subspaces of finite codimension F, then a family $(f_{E/F})$ of random variables $f_{E/F} \in \mathrm{Meas}(\Omega, m; E/F)$, with the appropriate coherence conditions, can be called a "cylindrical random variable with values in E". Our question formulated above asks in this case whether the family $(f_{E/F})$ defines a random variable f with values in E.

An obvious necessary condition for the existence of f is that the projective system $\mu_\iota = f_\iota(m)$ of probabilities satisfy the condition of Prokhorov: Given $\epsilon > 0$ there exists a compact subset K of X such that $\mu_\iota(\rho_\iota(K)) \geqslant 1 - \epsilon$ for every $\iota \in I$. Indeed, if f exists

then $\mu = f(m)$ is a Radon measure on X such that $\rho_\iota(\mu) = \mu_\iota$ for every $\iota \in I$.

Also if the map ρ is injective, i.e. if the maps ρ_ι separate the points of X, then a random variable $f: \Omega \to X$ satisfying our conditions, if it exists, must be unique. Indeed, suppose that f and g are two elements of $\mathrm{Meas}(\Omega, m; X)$ such that $\rho_\iota \circ f = \rho_\iota \circ g$ for every $\iota \in I$. Let $\delta > 0$ and let Ω_δ be a compact subset of Ω such that $m(\Omega - \Omega_\delta) \leqslant \delta$ and the restrictions of both f and g to Ω_δ are continuous. Replacing, if necessary, Ω_δ by the support of the restriction of m to Ω_δ, we may assume that Ω_δ is m-full, i.e. the support of the restriction of m to Ω_δ is Ω_δ. Since $\rho_\iota \circ f$ and $\rho_\iota \circ g$ are continuous and equal almost everywhere in Ω_δ, and since Ω_δ is m-full, we have $\rho_\iota \circ f = \rho_\iota \circ g$ everywhere in Ω_δ. But the maps ρ separate the points of X, hence $f = g$ everywhere in Ω_δ. Now $\delta > 0$ is arbitrary, hence f and g are equal almost everywhere with respect to m.

We can furthermore assert that if X is the projective limit Z of the spaces X_ι and if the set I is countable (in which case, as we know, the condition of Prokhorov is automatically satisfied) then f exists. Indeed, in this case we can simply take $f = (f_\iota)_{\iota \in I} : \Omega \to X_I$. Since the f_ι form a countable family of measurable maps, the map f is measurable, i.e. a random variable. Because I is countable, for every $\delta > 0$ we can find an m-full compact subset Ω_δ of Ω such that $m(\Omega_\delta) \geqslant 1 - \delta$ and that the restrictions of f and of each f_ι to Ω_δ are continuous. Since $\pi_\iota \circ f$ and f_ι are continuous and equal almost everywhere in Ω_δ, and since Ω_δ is m-full, we have $\pi_\iota \circ f = f_\iota$ everywhere in Ω_δ. It follows that for $\iota \leqslant \kappa$ we have $\pi_{\iota\kappa} \circ f_\kappa = \pi_{\iota\kappa} \circ \pi_\kappa \circ f = \pi_\iota \circ f = f_\iota$ everywhere in Ω_δ, i e. f maps Ω_δ into Z. Since $\delta > 0$ is arbitrary, for almost every $\omega \in \Omega$ the map f takes its values in Z, hence f satisfies our requirements.

We shall now show by a counterexample that if I is not countable, then the above-mentioned necessary condition is not sufficient for the existence of f.

COUNTEREXAMPLE. We shall consider an index set A, a family $(X_\alpha)_{\alpha\in A}$ of compact topological spaces and their product $X = \prod_{\alpha\in A} X_\alpha$. For the index set I we shall take the set of all finite subsets of A. Suppose that on each space X_α we are given a probability μ_α. Then for each $\iota \in I$ we have the probability $\mu_\iota = \bigotimes_{\alpha\in\iota} \mu_\alpha$ on the compact space $X_\iota = \prod_{\alpha\in\iota} X_\alpha$. If for $\kappa \supset \iota$ we denote by $\pi_{\iota\kappa}$ the canonical projection $X_\kappa \to X_\iota$, then clearly $\mu_\iota = \pi_{\iota\kappa}(\mu_\kappa)$. The projective limit of the probabilities μ_ι is simply the tensor product $\bigotimes_{\alpha\in A} \mu_\alpha$ on X and we know that in this case the condition of Prokhorov is automatically satisfied for the probabilities μ_ι, for any choice of the probabilities μ_α.

If for each $\alpha \in A$ we are given a random variable $f_\alpha: (\Omega, m) \to X_\alpha$, then for each $\iota \in I$ we have a random variable $f_\iota = (f_\alpha)_{\alpha\in\iota}: \Omega \to X_\iota$ and our problem consists in finding a random variable $f: \Omega \to X$ such that $f_\alpha = \pi_\alpha \circ f$ for every $\alpha \in A$, where π_α is the canonical projection $X \to X_\alpha$.

For (Ω, m) we choose the interval $[0, 1]$ and the Lebesgue measure dx. For A we choose $[0, 1]$ and also the space X_α will be the interval $[0, 1]$ for every index $\alpha \in [0, 1]$. Finally for any $\alpha \in A = [0, 1]$ let f_α be the random variable defined by $\omega \longmapsto Y(\omega - \alpha)$, where Y is the Heaviside function. Now we proved in Example 3 (§2) that there exists no random variable $f: \Omega \to X = [0, 1]^{[0,1]}$ such that $\pi_\alpha \circ f = f_\alpha$ for every $\alpha \in A = [0, 1]$.

In view of this counterexample it will be interesting to give some sufficient conditions for the existence of f. This is the purpose of

THEOREM 10.† *Let $(X_\iota, \pi_{\iota\kappa})_{\iota,\kappa\in I}$ be a projective system of topological spaces and suppose that for each $\iota \in I$ we are given a continuous map ρ_ι from a topological space X into X_ι such that $\rho_\iota = \pi_{\iota\kappa} \circ \rho_\kappa$ for $\iota \leqslant \kappa$. Suppose that for every $\iota \in I$ we are given a random variable*

† The statement of this theorem is rather long; in (i) and (ii) there is no condition of separation by the ρ_ι, but conditions on X and μ, while in (iii) and (iv) there are conditions on the separation of the points of X. The main applications are given in the corollary.

$f_\iota: (\Omega, m) \to X_\iota$ such that $f_\iota = \pi_{\iota\kappa} \circ f_\kappa$ for $\iota < \kappa$ and assume that one of the following conditions is satisfied:

(i) *X is a Suslin space, and for every countable directed subset J of I there exists a probability μ_J on the image $\rho_J(X)$ of X in the projective limit $Z_J = \varprojlim_{\iota \in J} X_\iota$ such that $\pi_{\iota J}(\mu_J) = f_\iota(m)$ for every $\iota \in J$.*

(ii) *There exists on X a probability μ such that $\rho_\iota(\mu) = f_\iota(m)$ for every $\iota \in I$, and μ is concentrated on a countable union of compact metrizable subsets of X.*

(iii) *There exists a countable subset J_0 of I such that the maps ρ_κ with $\kappa \in J_0$ separate the points of X, and for every countable directed subset J of I there exists a probability μ_J on X such that $\rho_\kappa(\mu_J) = f_\kappa(m)$ for every $\kappa \in J$.*

(iv) *There exists on X a probability μ such that $\rho_\iota(\mu) = f_\iota(m)$ for every $\iota \in I$, the probability μ is concentrated on a subspace X' of X, and there exists a countable family $(\rho_\kappa)_{\kappa \in J_0}$ of maps which separate the points of X'.*

Under any one of these conditions there exists a random variable $f: (\Omega, m) \to X$ such that $f_\iota = \rho_\iota \circ f$ for every $\iota \in I$.

PROOF. (a) Let us first assume that I is countable, that X is a subspace of $X_I = \prod_{\kappa \in I} X_\kappa$, that $\rho_\iota: X \to X_\iota$ is the restriction to X of the ι-th projection $X_I \to X_\iota$, and that on X there exists a probability μ such that $\rho_\iota(\mu) = f_\iota(m) = \mu_\iota$ for every $\iota \in I$. Then X is a subspace of the projective limit Z of the spaces X_ι. Let f be the m-measurable map $(f_\iota)_{\iota \in I}$ from Ω into X_I. The measures $f(m)$ and μ on X_I coincide since they have the same projections on every factor space X_ι and therefore also on every space X_J where J is a finite subset of I. It follows that $m(f^{-1}(X)) = f(m)(X) = \mu(X) = 1$, i.e. f takes its values in X for almost every $\omega \in \Omega$.

(b) Next let us not assume any more that X is a subspace of X_I but only that the map $\rho: X \to Z$ is injective, and that on X there exists a probability μ such that $\rho_\iota(\mu) = f_\iota(m) = \mu_\iota$ for every $\iota \in I$ (I still countable). In this case $\rho(X)$ is a subspace of $Z \subset X_I$ and $\rho(\mu)$ is a

probability on $\rho(X)$ such that $\pi_\iota(\rho(\mu)) = f_\iota(m)$ for every $\iota \in I$. By part (a) of the proof there exists a random variable $g: \Omega \to X_I$ which takes almost all its values in $\rho(X)$ and such that $\pi_\iota \circ g = f_\iota$ for every $\iota \in I$. Now the map $\rho^{-1}: X_I \to X$ is defined almost everywhere and is measurable with respect to $\rho(\mu) = g(m)$ because $\rho^{-1} \circ \rho = Id_X$ is μ-measurable hence $f = \rho^{-1} \circ g: \Omega \to X$ is m-measurable, and $\rho_\iota \circ f = \rho_\iota \circ \rho^{-1} \circ g = \pi_\iota \circ g = f_\iota$ for every $\iota \in I$.

(c) We can now prove the theorem under the hypothesis (iii). Let J be any countable directed subset of I containing J_0. Then the map $\rho_J: X \to Z_J$ is injective and therefore by part (b) of the proof there exists a random variable $f_J: \Omega \to X$ such that $\rho_\kappa \circ f_J = f_\kappa$ for every $\kappa \in J$. Since the last relation holds in particular for every $\kappa \in J_0$, and since the ρ_κ with $\kappa \in J_0$ separate the points of X, we know from an earlier remark that the random variable f_J is independent of J and can therefore be denoted by f. For any $\iota \in I$ there exists a countable directed subset of I containing both ι and J_0, hence we have $\rho_\iota \circ f = f_\iota$.

(d) The theorem with hypothesis (iv) now follows easily, we have only to replace X by X' and μ by its restriction to X' in the preceding consideration.

(e) Next we prove the theorem with hypothesis (i). The maps π_ι separate the points of $\rho(X)$, therefore, since $\rho(X)$ is a Suslin space, there exists a countable directed subfamily J_0 of I such that the maps π_κ with $\kappa \in J_0$ already separate the points of $\rho(X)$ (p. 105). Now, using again the fact that $\rho(X)$ is a Suslin space, we obtain that for any countable directed subset J of I the projection $\pi_J \rho(X) \to \rho_J(X)$ has a universally measurable section (p. 127), i.e. there exists a universally measurable map $s_J: \rho_J(X) \to \rho(X)$ such that $\pi_J \circ s_J$ is the identity map on $\rho_J(X)$. For the measure $s_J(\mu_J)$ on $\rho(X)$ we have $\pi_\kappa(s_J(\mu_J)) = \pi_{\kappa J}(\pi_J(s_J(\mu_J))) = \pi_{\kappa J}(\mu_J) = f_\kappa(m)$ for every $\kappa \in I$. Thus the space $\rho(X)$ and the probabilities $s_J(\mu_J)$ satisfy all the conditions of (iii) and therefore there exists a random variable $g: \Omega \to \rho(X)$ such that $\pi_\iota \circ g = f_\iota$ for every $\iota \in I$. Using once again the property just mentioned of Suslin spaces, we obtain a

universally measurable map $s: \rho(X) \to X$ such that $\rho \circ s$ is the identity on $\rho(X)$. Then $f = s \circ g: \Omega \to X$ is a random variable such that $\rho_\iota \circ f = \rho_\iota \circ s \circ g = \pi_\iota \circ \rho \circ s \circ g = \pi_\iota \circ g = f_\iota$ for every $\iota \in I$.

(f) Finally if μ is concentrated on a countable union X' of compact metrizable subspaces of X, then X' is a Lusin space, hence *a fortiori* a Suslin space, and applying the preceding case to X' and to the probabilities $\rho_J(\mu)$ we see that there exists a random variable $f: \Omega \to X'$ such that $\rho_\iota \circ f = f_\iota$ for every $\iota \in I$. But f is also a random variable with values in X and therefore we proved the theorem also under hypothesis (ii).

REMARK 5. By Prokhorov's theorem the measures μ_J which figure in hypothesis (i) exist if and only if for every countable directed subset J of I and every $\epsilon > 0$ there exists a compact subset K of $\rho_J(X)$ such that $(f_\kappa(m))\,(\pi_{\kappa J}(K)) \geqslant 1 - \epsilon$ for every $\kappa \in J$. It follows from the theorem that if X is a Suslin space then this weaker condition implies the analogous condition for the whole set I, which we know is necessary for the existence of the random variable $f: \Omega \to X$.

COROLLARY. *Let $(X_\iota, \pi_{\iota k})$ be a projective system of topological spaces, and suppose that for each $\iota \in I$ we are given a continuous map ρ_ι from a topological space X into X_ι such that $\rho_\iota = \pi_{\iota\kappa} \circ \rho_\kappa$ for $\iota \leqslant \kappa$. Suppose that for each ι we are given a random variable $f_\iota: (\Omega, m) \to X_\iota$ such that $f_\iota = \pi_{\iota\kappa} \circ f_\kappa$ for $\iota \leqslant \kappa$. Assume that there exists a Radon measure μ on X such that $\rho_\iota(\mu) = f_\iota(m)$ for every $\iota \in I$. If μ is concentrated on a countable union of compact metrizable subspaces of X (in particular, if X is metrizable, or Suslin, or if it becomes metrizable for some coarser topology, or if all the compact subspaces of X are metrizable, or if there exists a countable collection of numerical continuous functions on X which separate the points, or if it is a locally convex Hausdorff vector space whose dual possesses a countable weak-dense subset), or if there exists a countable subset J of I such that the π_ι, $\iota \in J$, separate the points of X, then there exists a random function $f: (\Omega, m) \to X$ such that $f_\iota = \rho_\iota \circ f$ for every $\iota \in I$. If the π_ι, $\iota \in I$, separate the points of X, then f is unique.*

Under any one among the hypotheses of Theorem 10 the family $(f_\iota)_{\iota \in I}$ of random variables is canonical. In fact $\check{f} = \rho_I \circ f$ is an m-measurable map from Ω into $X_I = \prod_{\iota \in I} X_\iota$, and not only into $\prod_{\iota \in I} \check{X}_\iota$, such that $f_\iota = \pi_\iota \circ \check{f}$ for every $\iota \in I$, and we do not even have to assume that the spaces X_ι are completely regular.

DEFINITION 1. *Let E and E' be two vector spaces in separating duality with respect to the bilinear form $(e, e') \longmapsto \langle e, e' \rangle$* [3, Chap. II, 2nd ed., §6, n° 1]. *Let $f: E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ be a linear random function. We shall say that f is decomposed if there exists a function $\phi: \Omega \to E$ such that for each $e' \in E'$ the function $\langle \phi, e' \rangle : \omega \longmapsto \langle \phi(\omega), e' \rangle$ is equal to $f(e'): \Omega \to \mathbf{R}$ as a random variable. We shall say that f is measurably decomposed if there exists such a ϕ which is Lusin m-measurable.*

If such a function ϕ exists, it is scalarly measurable; therefore, if E is Suslin, ϕ is measurable, and every decomposed linear randon function is measurably decomposed if E is Suslin (Chap. I, §4, Th. 4). Not every linear random function f is decomposed since, as we shall see in the Corollaries of Theorem 11, for the usual spaces of analysis f will be decomposed if and only if the associated cylindrical measure is a Radon measure. Conversely, of course, every scalarly m-measurable map $\phi : \Omega \to E$ defines a linear random function $f: E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ by way of $f(e') = \langle \phi, e' \rangle$. Observe also that a decomposed linear random function f is not necessarily canonical since ϕ does not have to be m-measurable; if f is measurably decomposed, it is canonical.

If E' possesses a countable subset D which is dense for the weak*-topology $\sigma(E', E)$, then the function $\phi : \Omega \to E$ associated with a decomposed linear random function $f : E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ is uniquely determined almost everywhere. Indeed, if ϕ_1 and ϕ_2 are both associated with f, then for each $e' \in D$ there exists a set $N(e')$ of m-measure zero such that $\langle \phi_1(\omega), e' \rangle = \langle \phi_2(\omega), e' \rangle$ for $\omega \notin N(e')$. The union N of the sets $N(e')$ as e' varies in D has m-measure zero

and $\langle \phi_1(\omega), e' \rangle = \langle \phi_2(\omega), e' \rangle$ for every $e' \in E'$ if $\omega \notin N$. Thus $\phi_1(\omega) = \phi_2(\omega)$ for every $\omega \in \Omega - N$. On the other hand, we already know that, if f is measurably decomposed, the measurable function ϕ is uniquely determined almost everywhere, without assumption on E'.

According to §2, to a linear random function $f: E' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ there corresponds a cylindrical measure μ on E equipped with the topology $\sigma(E, E')$, defined as follows: if $u: E \to \mathbf{R}^n$ is a continuous linear map defined with the help of the linear forms $\xi_1, \ldots, \xi_n \in E'$, then μ_u is the image of m with respect to the map $f(u) = (f(\xi_1), \ldots, f(\xi_n)): \Omega \to \mathbf{R}^n$. If f is decomposed, then $f(u)$ is equivalent to $u \circ \phi$ since

$$\begin{aligned} u(\phi(\omega)) &= (\langle \phi(\omega), \xi_1 \rangle, \ldots, \langle \phi(\omega), \xi_n \rangle) \\ &= (f(\xi_1)\ (\omega), \ldots, f(\xi_n)\ (\omega)) = f(u)\ (\omega) \end{aligned}$$

for almost every $\omega \in \Omega$. In other words, for a decomposed linear random function $f: E' \to \mathrm{Meas}\ (\Omega, m; \mathbf{R})$ the associated cylindrical measure $\mu = (\mu_g)$ on E equipped with $\sigma(E, E')$ is given by $\mu_g = (g \circ \phi)\ (m)$ for any continuous linear map $g: E \to G$ into a finite-dimensional Hausdorff vector space G. The map $g \circ \phi: \Omega \to G$ is m-measurable since ϕ is scalarly m-measurable.

THEOREM 11. *Let E be a locally convex Hausdorff space equipped with the weak topology $\sigma(E, E')$. Let $f: E' \to \mathrm{Meas}\ (\Omega, m; \mathbf{R})$ be a linear random function and μ the associated cylindrical measure on E.*

(a) *If f is measurably decomposed (in particular, if f is decomposed and E Suslin), then μ is a Radon measure on E, the equivalence class of ϕ is a random variable with values in E, and f is canonical.*

(b) *If μ is a Radon measure on E, concentrated on a subset of E whose points are separated by a countable set of elements of E' (or equivalently on the countable union of compact metrizable subsets of E), then f is measurably decomposed.*

PROOF. (a) is trivial ; the map $\phi: \Omega \to E$, which satisfies $f(e') = \langle \phi, e' \rangle$ for every $e' \in E'$ is m-measurable and defines a random

variable with values in E and the Radon measure $\phi(m)$ coincides with μ since for every continuous linear map $g: E \to G$ we have $(g \circ \phi)\ (m) = \mu_g$.

(b) If in Theorem 10 we take $X = E$, for the X_ι the finite-dimensional vector spaces $G \simeq \mathbf{R}^n$, for the ρ_ι the continuous linear maps $g: E \to G$, and for the f_ι the random variables $f(g) = f(e'_1, \ldots, e'_n)$: $\Omega \to G$, then we see that there exists a random variable $\phi: \Omega \to E$ such that $e \circ \phi, = \langle \phi, e' \rangle = f(e')$ for every $e' \in E'$.

COROLLARY 1. *Assume there exists a countable dense subset in E' for $\sigma(E', E)$. Then f is measurably decomposed if and only if μ is a Radon measure on $\sigma(E, E')$.*

COROLLARY 2. *Assume E is Suslin. Then f is decomposed if and only if it is measurably decomposed, if and only if μ is a Radon measure on E for $\sigma(E, E')$ or the initial topology.*

Assume we now equip E with the weak topology $\sigma(E, E')$. We know from Prop. 4 (§2) that, if f is canonical, the following properties are equivalent: a) μ is a Radon measure; b) f is almost surely in E; c) f is measurably decomposed, which implies c') f is decomposed. If f is not canonical b) is meaningless; it remains true however that c) implies a) and c'). Now if E is Suslin, we have always the equivalence between a), b), c), c'), f is always canonical, and the topology $\sigma(E, E')$ can be replaced by the initial topology.

REMARK 7. We can also consider on E a locally convex topology τ finer than $\sigma(E, E')$, and ask whether, given a decomposed linear random function $f: E' \to \text{Meas}\ (\Omega, m; \mathbf{R})$, the corresponding function $\phi: \Omega \to E_\tau$ is m-measurable. If E_σ is a Suslin space, then $\phi: \Omega \to E_\sigma$ is m-measurable and $\phi(m) = \mu$ is a Radon measure on E_σ (Th. 11), hence $\phi: \Omega \to E_\tau$ will be m-measurable if and only if $E_\sigma \to E_\tau$, is μ-measurable. This is also necessary and sufficient for μ to be a Radon measure on E_τ. The canonical map $E_\sigma \to E_\tau$, will certainly be μ-measurable if the topologies $\sigma(E, E')$ and τ are Radon-equivalent (Chap. 1, §2), in particular if τ is the original topology of a locally convex metrizable space (Chap. 1, § 4, Th. §3),

or if E_τ is a Suslin space. As we already observed (Chap. 1, § 3), the usual spaces of analysis are Suslin spaces as soon as they are separable. Also in most practical cases if F is a Suslin space then F'_{cc} (p. 233) is one too.

PROPOSITION 11. *Let $\mathfrak{S}$ be a saturated collection of subsets of E. Suppose that E is a Suslin space for $\sigma(E, E')$ and that it is $\mathfrak{S}$-conuclear. Then every continuous linear random function $f: E'_{\mathfrak{S}} \to \mathrm{Meas}\,(\Omega, m; \mathbf{R})$, is measurably decomposed and canonical.*

PROOF. If f is continuous then by Theorem 2 (§2) the cylindrical measure μ on E associated with f is scalarly concentrated on the sets of $\mathfrak{S}$. By Minlos' theorem (Chap. 4, §3, Theorem 2) μ is a Radon measure on E and thus by Theorem 11 the random function f is decomposed and canonical.

EXAMPLE 4. If $E = \mathscr{D}'$, $E' = \mathscr{D}$ and for $\mathfrak{S}$ we take the collection of all bounded subsets of $\mathscr{D}'$ then the conditions of Prop. 11 are verified. Thus if $f: \mathscr{D} \to \mathrm{Meas}\,(\Omega, m; \mathbf{R})$ is a continuous random function, then there exists a "random distribution" (§3, 3) $\omega \longmapsto T(\omega) \in \mathscr{D}'$ such that for every $\alpha \in \mathscr{D}$ we have $f(\alpha)\,(\omega) = \langle T(\omega), \alpha \rangle$ for almost every $\omega \in \Omega$. In other words, f is canonical and $\check{f}$ is almost surely a continuous linear form on $\mathscr{D}$.

EXAMPLE 5. Let G be a locally convex Hausdorff space and $f: G \to L^\infty(\Omega, m)$ a continuous linear map. It follows from a well-known result due to Dorothy Maharam that there exists a lifting $\bar{f}: G \to \mathscr{L}^\infty(\Omega, m)$, i.e. a continuous linear map $\bar{f}$ such that $\pi \circ \bar{f} = f$, where π is the canonical surjection from $\mathscr{L}^\infty(\Omega, m)$ onto $L^\infty(\Omega, m)$. Hence we can define a function $\phi: \Omega \to G'_\sigma$ by setting $\langle \phi(\omega), g \rangle = \bar{f}(g)\,(\omega)$ for every $g \in G$ and $\omega \in \Omega$. Then for every $g \in G$ the elements $\langle \phi, g \rangle$ and $f(g)$ of $L^\infty(\Omega, m)$ are equal [4, Chap. IV, §2, n^0 5, Th. 1] and therefore the random function obtained by composing f with the canonical injection $L^\infty(\Omega, m) \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ is decomposed. Let V be the inverse image in G of the unit ball of $L^\infty(\Omega, m)$ with respect to the continuous map f. Then $|\langle \phi(\omega), g \rangle| \leqslant 1$ for $g \in V$ and $\omega \in \Omega$, i.e. ϕ maps Ω into the polar V° of V. If now

G'_σ is a Suslin space, then we know that G is separable, furthermore ϕ is m-measurable and the cylindrical measure associated with f is the Radon measure $\phi(m)$ carried by the compact subset V° of G'_σ. *In other words, every continuous linear map $G \to L^\infty$ is decomposed, and measurably decomposed if G'_σ is Suslin.*

DEFINITION 2. (1) *Let (E, E') be a pair of vector spaces in separating duality, F a locally convex Hausdorff space, and $\mathfrak{S}$ a collection of bounded subsets of E. A weakly continuous linear map $u: E \to F$ is said to be $\mathfrak{S}$-radonifying, if for every scalarly $\mathfrak{S}$-concentrated cylindrical measure μ on E, the image $u(\mu)$ is a Radon measure on F.*

(2) *Let (E, E') be a pair of vector spaces in separating duality, and F be a locally convex (not necessarily Hausdorff) space. A linear map $u: E \to F$ is said to be decomposing, if for every continuous linear random function $f: F \to \mathrm{Meas}\,(\Omega, m; \mathbf{R})$, the map $f \circ u$ is decomposed by some $\phi: \Omega \to E'$.*

(3) *With the same conditions as in (2) if, furthermore, τ is a given topology on E', μ is said to be τ-decomposing, if for every linear continuous $f: F \to \mathrm{Meas}\,(\Omega, m; \mathbf{R})$ the map $f \circ u$ is measurably decomposed, i.e. decomposed by some $\phi: \Omega \to E'$ which is m-measurable for the topology τ.*

In (1), if $\mathfrak{S}$ is not mentioned, it means that we take the collection of all the bounded subsets of E (or, equivalently, all the balanced bounded subsets). In (3), if τ is not mentioned, it means that we take the strong topology on E'.

THEOREM 12. *Let (U, U') and (V, V') be two pairs of vector spaces in separating duality. Let $\mathfrak{S}$ be a saturated collection of $\sigma(U, U')$-bounded sets of U. Assume that V'_σ has a countable dense subset and u is a weakly continuous linear map from U into V. Then u is $\mathfrak{S}$-radonifying if and only if ${}^t u$ is measurably decomposing from V'_σ into $U'_{\mathfrak{S}}$, relatively to the topology $\tau = \sigma(V, V')$ on V. If moreover V_σ is Suslin, this is true if and only if ${}^t u$ is decomposing from V'_σ into $U'_{\mathfrak{S}}$.*

PROOF. First of all, we know that, if V is Suslin, "decomposing" is equivalent to "measurably decomposing". Therefore it is sufficient

to prove the equivalence of "u radonifying" and "${}^t u$ measurably decomposing".

(a) Assume that ${}^t u$ is measurably decomposing from V'_σ into $U'_{\mathfrak{S}}$, and let μ be a cylindrical measure on U, scalarly concentrated on the sets of $\mathfrak{S}$. To μ there corresponds a linear random function $f: U' \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ (§2), which according to Theorem 2 (§2) is continuous for the $\mathfrak{S}$-topology on U'. By assumption, $f \circ {}^t u$ is measurably decomposed. Now by Prop. 5 (§2) the cylindrical measure associated with the random function $f \circ {}^t u$ is $u(\mu)$, hence it follows that $u(\mu)$ is a Radon measure on V_σ. Thus u is $\mathfrak{S}$-radonifying.

(b) Assume that u is $\mathfrak{S}$-radonifying and let $f: U'_{\mathfrak{S}} \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ be a continuous linear random function. By Theorem 2 (§2) to f there corresponds a cylindrical measure μ on U scalarly concentrated on the sets of $\mathfrak{S}$. By assumption $u(\mu)$ is a Radon measure on V_σ and since $f \circ {}^t u$ is the random function associated with $u(\mu)$, it follows from Corollary 1 of Theorem 11 that if V'_σ has a countable dense subset, then $f \circ {}^t u$ is measurably decomposed. Thus ${}^t u$ is measurably decomposing, from V'_σ into $U'_{\mathfrak{S}}$.

REMARK 8. Similarly as in Remark 7, we can consider on V a locally convex topology τ finer than $\sigma(V, V')$. If V_σ and V_τ are Radon-equivalent (in particular if V_τ is a Suslin space and if $u: U \to V_\tau$ is weakly continuous), then every $\mathfrak{S}$-radonifying map $u: U \to V$ will transform every cylindrical measure on U, scalarly concentrated on the sets of $\mathfrak{S}$, into a Radon measure on V, and u will be $\mathfrak{S}$-radonifying from U into V_τ.

EXAMPLE. Let E be a locally convex vector space, and $\mathfrak{S}$ a saturated collection of completing subsets of E such that if A is a bounded balanced convex completing subset of E contained in $B \in \mathfrak{S}$, then $A \in \mathfrak{S}$. If E is $\mathfrak{S}$-conuclear, the identity map of E is $\mathfrak{S}$-radonifying, and any weakly continuous linear map of E into a locally convex vector space is $\mathfrak{S}$-radonifying.

This is exactly Theorem 2 of Chap. 4, §3 (Minlos). In particular, if we take for $\mathfrak{S}$ the collection of all the bounded balanced convex compact subsets, then if E is cc-conuclear, every weakly continuous

linear map from E into F, and every continuous linear map from F into E is cc-radonifying.

COROLLARY. *Let u be a continuous linear map from a Hilbert space into a Hilbert space. Then u is radonifying, decomposing or measurably decomposing, if and only if u is a Hilbert-Schmidt map.*

PROOF. For the radonifying case this is Sazonov's theorem (Chap. III, §3, Th. 2, and Chap V §3, Th. 6). Let us consider the decomposing case. A map is Hilbert-Schmidt, if and only if its transposed map is Hilbert-Schmidt. Then, by a *transposition* of the statement, it will be sufficient to prove that, ${}^tu$ is decomposing or measurably decomposing if and only if u is Hilbert-Schmidt.

Let $u\colon E \to F$ be a Hilbert-Schmidt map, and let

$$f\colon E' \to \text{Meas}\,(\Omega, m; \mathbf{R})$$

be a linear random function which is continuous for the strong topology on E'. To f there corresponds a cylindrical measure μ on E, which by Theorem 2 (§2) is scalarly concentrated on the balls of E. By Sazonov's theorem $u(\mu)$ is a Radon measure on F, hence taking into account that F is metrizable, it follows from Theorem 11 that the random function $f \circ {}^tu$ associated with $u(\mu)$ (§2, Prop. 5) is decomposed. Thus ${}^tu$ is measurably decomposing.

Conversely, assume that ${}^tu\colon F' \to E'$ is decomposing. Then for any separable closed subspace V' of F' the restriction of ${}^tu$ to V' is also decomposing, Indeed, let $f\colon E' \to \text{Meas}\,(\Omega, m; \mathbf{R})$ be a continuous linear random function and $\phi\colon \Omega \to F$ such that for every $e' \in F'$ we have $(f \circ {}^tu)\,(e')\,(\omega) = \langle \phi(\omega), e' \rangle$ for almost every $\omega \in \Omega$. Let V be the orthogonal complement of $V'^{\perp}$ in F, and $p\colon F \to V$ the orthogonal projector. Then ${}^tp = j$ is the canonical injection from V' into F', and for every $v' \in V'$ we have

$$(f \circ {}^tu \circ j)\,(v')\,(\omega) = \langle \phi(\omega), j\,(v') \rangle = \langle (p \circ \phi)\,(\omega), v' \rangle$$

for almost every $\omega \in \Omega$. Since $p \circ \phi$ is a map from Ω into V, we see that $f \circ {}^tu \circ j = f \circ {}^t(p \circ u)$ is decomposed.

† We shall study in another paper various classes of radonifying maps, for instance p-radonifying maps; the cases considered here are the 0-radonifying maps.

It follows from Theorem 12 that $p \circ u$ is radonifying with respect to the balls of E, hence by the converse of Sazonov's theorem it is a Hilbert-Schmidt map. By Prop. 1 of Chap. 3, §1 the map ${}^t(p \circ u) = {}^t u \circ j$ is of Hilbert-Schmidt class, and this is true for every injection j of a separable closed subspace V' of F' into F'. But then ${}^t u$ must be a Hilbert-Schmidt map, since if $(e'_\iota)_{\iota \in I}$ is an orthonormal family in F' such that $\sum_{\iota \in I} \| {}^t u(e'_\iota) \|^2 = \infty$, then there exists a countable subset J of I such that $\sum_{\iota \in J} \| {}^t u(e'_\iota) \|^2 = \infty$. Using again Prop. 1 of Chap. 3, §1, we obtain that $u: E \to F$ is a Hilbert-Schmidt map.

REMARK 9. In the corollary of Theorem 12 we proved, with the help of the converse of Sazonov's theorem (Theorem 6, §3), that if ${}^t u$ is decomposing then u is a Hilbert-Schmidt map. One can, however, also deduce the converse of Sazonov's theorem from the fact just stated and we shall therefore sketch briefly a direct proof for it which has some points in common with the proof given for Theorem 6 of §3, but which does, of course, not depend upon that theorem.

Let u be a continuous linear map from the Hilbert space E into the Hilbert space F such that ${}^t u$ is decomposing. We may assume that E is infinite-dimensional and separable. Using the polar decomposition $u = UT$ we see that we may also assume $E = F$ and that u is a positive definite hermitian operator on E. For any $\epsilon > 0$ let K_ϵ be the spectral subspace of u corresponding to that part of the spectrum of u for which $\lambda \geqslant \epsilon$. The restriction u_ϵ of u to K_ϵ can be factored into $K_\epsilon \xrightarrow{j} E \xrightarrow{u} E \xrightarrow{p} K_\epsilon$, where j is the canonical injection and p the orthogonal projector onto K_ϵ. Hence ${}^t u_\epsilon = u_\epsilon$ is also decomposing. But u_ϵ is inversible, and therefore the identity map $u_\epsilon^{-1} \circ u_\epsilon: K_\epsilon \to K_\epsilon$ is decomposing.

Now the identity map $\mathscr{H} \to \mathscr{H}$ for any Hilbert space of countable type $\mathscr{H}$ is not decomposing unless $\mathscr{H}$ is finite-dimensional. Indeed $\mathscr{H}$ is isomorphic to some space $L^2(\Omega, m)$ with a compact Ω, and if the identity map is decomposing, then the canonical injection $L^2(\Omega, m) \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ is decomposed. But the canonical map

$\mathscr{C}(\Omega) \to L^2(\Omega, m) \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ is obviously decomposed since we can take for $\phi: \Omega \to \mathscr{C}'(\Omega)$ the map which associates with each $\omega \in \Omega$ the Dirac measure δ_ω at ω. By the uniqueness of the decomposing map, if $L^2(\Omega, m) \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ is decomposed, then there exists a map $\phi: \Omega \to (L^2(\Omega, m))'$ such that for almost every $\omega \in \Omega$ the linear form $\phi(\omega)$ induces δ_ω on $\mathscr{C}(\Omega)$. In other words, for almost every $\omega \in \Omega$ the linear form δ_ω on $\mathscr{C}(\Omega)$ has to be continuous for the topology induced by $L^2(\Omega, m)$ on $\mathscr{C}(\Omega)$. Now this is only possible if m has a point measure at ω; therefore m has a point measure almost everywhere and is therefore atomic. But if $\mathscr{H}$ is an infinite-dimensional Hilbert space of countable type, we can choose $\Omega = [0, 1]$ and for m the Lebesgue measure dx.

It follows therefore that for every $\epsilon > 0$ the space K_ϵ is finite-dimensional, i.e. u is a compact map. We may therefore assume that $E = L^2(\mathbf{T}, dx)$, where $\mathbf{T}$ is the one-dimensional torus and dx is the Lebesgue measure, and that u is the multiplication operator $\sum_{n \in \mathbf{Z}} c_n e^{in\theta} \longmapsto \sum_{n \in \mathbf{Z}} \alpha_n c_n e^{in\theta}$, where $(\alpha_n)_{n \in \mathbf{Z}}$ is a sequence of positive numbers which tends to zero as $|n| \to \infty$. Now u considered as a map from $\mathscr{D}(\mathbf{T})$ into $\mathrm{Meas}(\mathbf{T}, dx; \mathbf{R})$ is decomposed since, if α is the distribution on $\mathbf{T}$ whose Fourier coefficients are the α_n, then for $\phi: T \to \mathscr{D}'(\mathbf{T})$ we can take the function which associates with each $\omega \in \mathbf{T}$ the distribution $\tau_\omega \check{\alpha}$. Indeed, for any $\theta \in \mathscr{D}(\mathbf{T})$ we have $u(\theta)(\omega) = (\alpha * \theta)(\omega) = \langle \tau_\omega \check{\alpha}, \theta \rangle$.

By our hypothesis the map u: $L^2(\mathbf{T}, dx) \to \mathrm{Meas}(\mathbf{T}, dx; \mathbf{R})$ is decomposed. By the uniqueness of the map ϕ, for almost every $\omega \in \mathbf{T}$ the value $\phi(\omega)$ must be the distribution $\tau_\omega \check{\alpha}$. Therefore $\tau_\omega \check{\alpha}$ is a continuous linear form on $\mathscr{D}(T)$ for the topology induced by $L^2(\mathbf{T}, dx)$, i.e. α must belong to $L^2(\mathbf{T}, dx)$. Thus $\sum_{n \in \mathbf{Z}} |\alpha_n|^2 < \infty$ and therefore u is a Hilbert-Schmidt map.

Let us give some elementary properties of radonifying maps.

PROPOSITION 12. *Let $(E_n)_{n \in \mathbf{N}}$ be a countable family of locally convex vector spaces, and let $\mathfrak{S}_n$ be a collection of subsets of E_n. Let $(F_n)_{n \in \mathbf{N}}$ be another countable family of locally convex spaces, and let u_n be*

a radonifying map from E_n into F_n with respect to $\mathfrak{S}_n$. Then the map $u = \prod_n u_n$ from $E = \prod_n E_n$ into $F = \prod_n F_n$ is radonifying relatively to the collection $\mathfrak{S} = \prod_n \mathfrak{S}_n$ of subsets of the form $\prod_n A_n$, $A_n \in \mathfrak{S}_n$.

PROOF. Let μ be a cylindrical measure on E, scalarly concentrated on the sets of $\mathfrak{S}$. If p_n is the projection on the n-th factor of E and q_n is the projection on the n-th factor in F, then we have $q_n \circ u = u_n \circ p_n$; but $p_n \mu$ is scalarly concentrated on the collection $\mathfrak{S}_n$, and therefore $u_n\, p_n\, \mu_n$ is a Radon measure since u_n is $\mathfrak{S}_n$-radonifying. Therefore $q_n u \mu$ is a Radon measure, and Prop. 1 bis of Chap. 2, §2, proves that $u\mu$ is a Radon measure, and the proof is complete.

PROPOSITION 13. *If u and v are 2 linear weakly continuous $\mathfrak{S}$-radonifying maps from E into F, then $u + v$ also is $\mathfrak{S}$-radonifying.*

PROOF. Let μ be a cylindrical measure on E, scalarly concentrated on the subsets of the collection $\mathfrak{S}$. Then $u(\mu)$ and $v(\mu)$ are Radon measures, therefore $(u + v)(\mu)$ is a Radon measure too, according to the corollary of Prop. 1 bis of Chap. 2, §2.

COROLLARY. *Let E be a locally convex vector space. The cc-radonifying continuous linear maps of E into E form an ideal in $\mathscr{L}(E; E)$.*

Namely, if u and v are *cc*-radonifying, $u + v$ is *cc*-radonifying, and for every $w \in \mathscr{L}(E, E)$, uw and wv are trivially *cc*-radonifying.

EXAMPLES. We saw (p. 280) that, if V is a compact C^∞ manifold, the natural injection $\mathscr{H}^s(V) \to \mathscr{H}^{s-k}(V)$ is radonifying if and only if $k > n/2$. Assume now V is a C^∞ manifold, not compact but countable at infinity. The spaces $\mathscr{H}^s(V)$ have no meaning anymore; but one can define the spaces $\mathscr{H}^s_{\mathrm{loc}}(V)$. They are no longer Hilbert spaces; let us prove however that the natural injection $\mathscr{H}^s_{\mathrm{loc}}(V) \to \mathscr{H}^{s-k}_{\mathrm{loc}}(V)$ is radonifying if and only if $k > n/2$.

Take the same notations as on p. 280, but with a countable C^∞ partition of unity $(\alpha_i)_{i \in I}$. This partition gives an isomorphism of $\mathscr{H}^s_{\mathrm{loc}}(V)$ with a closed vector subspace of $\prod_{i \in I} \mathscr{H}^s(\mathbf{T}^n)$. Then the injection $u\colon \mathscr{H}^s_{\mathrm{loc}}(V) \to \mathscr{H}^{s-k}_{\mathrm{loc}}(V)$ is identified with the restriction to the

subspace $\mathscr{H}^s_{\mathrm{loc}}(V)$ of $\prod_{i\in I} \mathscr{H}^s(\mathbf{T}^n)$ into the closed subspace $\mathscr{H}^{s-k}_{\mathrm{loc}}(V)$ of $\prod_{i\in I} \mathscr{H}^{s-k}(\mathbf{T}^n)$, of the map $\prod_{i\in I} u_i$, where u_i is the injection $\mathscr{H}_s(\mathbf{T}^n) \to \mathscr{H}^{s-k}(\mathbf{T}^n)$. We know that u_i is Hilbert-Schmidt and therefore radonifying if $k > n/2$. Therefore Prop. 12 shows that u is radonifying $\prod_{i\in I} \mathscr{H}^s(\mathbf{T}^n) \to \prod_{i\in I} \mathscr{H}^{s-k}(\mathbf{T}^n)$ and its restriction to $\mathscr{H}^s_{\mathrm{loc}}(V)$ is radonifying $\mathscr{H}^s_{\mathrm{loc}}(V) \to \prod_{i\in I} \mathscr{H}^{s-k}(\mathbf{T}^n)$. But the image of $\mathscr{H}^s_{\mathrm{loc}}(V)$ is in $\mathscr{H}^{s-k}_{\mathrm{loc}}(V)$, and therefore u is radonifying $\mathscr{H}^s_{\mathrm{loc}}(V) \to \mathscr{H}^{s-k}_{\mathrm{loc}}(V)$ for $k > n/2$. [If u is $\mathfrak{S}$-radonifying $E \to F$, and $u(E) \subset F_1$, a closed vector subspace of F, then u is $\mathfrak{S}$-radonifying $E \to F_1$. Namely, if μ is a cylindrical measure in E, scalarly $\mathfrak{S}$-concentrated, $u(\mu)$ is a cylindrical measure in F_1, whose image in F is a Radon measure cylindrically concentrated in F_1, therefore concentrated in F_1 (Prop. 3 of Chap. 2, §3), and it comes from a Radon measure ν of F_1. Then $u(\mu)$ and ν, cylindrical measures of F_1, have the same image in F, and $F_1 \to F$ is a strict morphism, therefore they coincide (Remark, p. 202) and $u(\mu)$ in F_1 is a Radon measure].

Conversely assume that the natural injection $\mathscr{H}^s_{\mathrm{loc}}(V) \to \mathscr{H}^{s-k}_{\mathrm{loc}}(V)$ is radonifying. Exactly in the same way as p. 280, it would imply that $\mathscr{H}^s(\mathbf{T}^n) \to \mathscr{H}^{s-k}_{\mathrm{loc}}(\mathbf{T}^n)$ is radonifying, hence Hilbert-Schimdt, and necessarily $k > n/2$.

5. Categories of Random Functions. Let T be a set and X a completely regular topological space. We want to define a category Rand $(T; X)$ whose objects will be the triples $(\Omega, m; f)$, where (Ω, m) is a sample space and $f: T \to \mathrm{Meas}\,(\Omega, m; X)$ a random function. If there is no possibility of confusion, we shall write simply f for $(\Omega, m; f)$. A *morphism* ϕ: $(\Omega, m; f) \to (\Omega', m'; f')$ is a random variable $\Omega \to \Omega'$, i.e. an equivalence class of m-measurable functions equal almost everywhere, which verifies $\phi(m) = m'$ and is such that for every $t \in T$ the random variables $f'(t) \circ \phi$ and $f(t)$ are the same. If this is the case, we shall write symbolically $f = f' \circ \phi$ or $f = \phi^*(f')$. For any $(\Omega, m; f)$ the identity morphism 1_f is the equivalence class of maps $\Omega \to \Omega$ which

contains the identity map. It is clear that with these definitions Rand $(T; X)$ becomes a category.

If there exists a morphism ϕ such that $f = \phi^*(f')$, i.e. if the set $\mathrm{Mor}(f, f')$ of morphisms $f \to f'$ is not empty, then the random functions f and f' are isonomous. Indeed, if for $t_1, \ldots, t_p \in T$ we denote by $(f(t_1), \ldots, f(t_p))$ the random variable $\omega \longmapsto (f(t_1)(\omega), \ldots, f(t_p)(\omega))$ with values in X^p, then we have

$$(f(t_1), \ldots, f(t_p))(m) = (f'(t_1) \circ \phi, \ldots, f'(t_p) \circ \phi)(m)$$
$$= (f'(t_1), \ldots, f'(t_p))\,\phi(m) = (f'(t_1), \ldots, f'(t_p))(m').$$

Conversely, if f and f' are isonomous there does not necessarily exist a morphism $\phi: f \to f'$. Indeed, let T consist of one element. Then every $(\Omega, m; f)$ is simply a random variable $x: \Omega \to X$. Consider two random variables $x: \Omega \to X$ and $x': \Omega' \to X$ which have both the same constant value $a \in X$. Then x and x' are isonomous since the measures $x(m)$ and $x'(m')$ are both equal to the Dirac measure δ_a at the point a. If, however, Ω has one point and Ω' has two points, each with non-zero probability, then there exists no map $\phi: \Omega \to \Omega'$ for which $\phi(m) = m'$.

In particular, each isonomy class of $\mathrm{Rand}(T; X)$ is a full subcategory of $\mathrm{Rand}(T; X)$, and there are no morphisms going from an object in one class to an object in a different class. Therefore, in what follows, we can restrict ourselves to considering only one isonomy class.

If there exists a morphism $\phi: f \to f'$, i.e. if $f = \phi^*(f')$, then we say that f is *representable* by f' and that f' *represents* f. Here we encounter the following problem: if f is representable by f' and f' is representable by f, does it follow that f and f' are isomorphic? Assuming that $f = f' \circ \phi$ and $f' = f \circ \phi'$ we obtain $f = f \circ \theta$, where $\theta = \phi' \circ \phi$ is a morphism $f \to f$ but such a morphism is not necessarily the identity 1_f. For instance let T consist of only one element. Then f is simply a random variable x, and if this random variable x is a constant $\omega \longmapsto a$, then $\theta: \Omega \to \Omega$ such that $\theta(m) = m$ will be a morphism $f \to f$. But it is easy to construct

such maps θ which are not almost surely equal to the identity map. Take, e.g. $\Omega = [0,1]$ and for m the Lebesgue measure $d\omega$. Then the map θ defined by

$$\theta(\omega) = 2\omega, \qquad \text{if } 0 \leqslant \omega \leqslant \tfrac{1}{2},$$

$$\theta(\omega) = 2\omega - 1, \qquad \text{if } \tfrac{1}{2} < \omega \leqslant 1,$$

satisfies $\theta(m) = m$ since

$$\int_0^1 x(\omega)\, d\theta(m) = \int_0^{1/2} x(2\omega)\, d\omega + \int_{1/2}^1 x\,(2\omega - 1)\, d\omega = \int_0^1 x(\omega)\, d\omega,$$

but clearly θ is not equal to $\omega \longmapsto \omega$ almost everywhere.

We shall consider certain special objects $(\Omega, m; f)$ in Rand $(T; X)$, which we shall call *fundamental random functions*, for which $\Omega = \breve{X}^T$ and for which $f(t)$ is the equivalence class of the projection map $\pi_t\colon \breve{X}^T \to \breve{X}$. The probability on $\breve{X}^T$ will of course have to be such that each π_t takes almost all its values in X. In every isonomy class of Rand $(T; X)$ there exists exactly one fundamental random function. Indeed, let $f\colon T \to \text{Meas}\,(\Omega, m; X)$ be an arbitrary random function and for any finite subset $J = (t_1, \ldots, t_p)$ of T let μ_J be the probability $(f(t_1), \ldots, f(t_p))(m)$ on X^J. The measures μ_J depend only on the isonomy class of f and their projective limit $\breve{\mu}$ is a probability on $\breve{X}^T$. Each projection $\pi_t\colon \breve{X}^T \to \breve{X}$ takes its values $\breve{\mu}$-almost everywhere in X since $\breve{\mu}(\pi_t^{-1}(X)) = \pi_t(\breve{\mu})(X) = f(t)(m)(X) = 1$. Clearly the fundamental random function $(\breve{X}^T, \breve{\mu}; (\pi_t))$ is isonomous to $(\Omega, m; f)$, and it is the only such fundamental random function. We can furthermore say that there exists a bijective correspondence between the isonomy classes of Rand $(T; X)$ and the positive Radon measures $\breve{\mu}$ on $\breve{X}^T$ such that $\breve{\mu}(\breve{X}^T) = 1$, and such that for every $t \in T$ the measure $\pi_t(\breve{\mu})$ is concentrated on X.

With the present terminology we can say that a random function $(\Omega, m; f)$ is canonical (§2) if and only if it is representable by the unique fundamental random function $(\breve{X}^T, \breve{\mu}; (\pi_t))$ isonomous to it. For this reason we shall also say that a canonical random

function is *initial*. We have seen in Example 3 (§2) that not every $(\Omega, m; f)$ is initial†.

If the random function f is initial, i.e. if there exists a morphism $\phi: \Omega \to \breve{X}^T$ such that $f(t) = \pi_t \circ \phi$ for every $t \in T$, then we know (see §1) that the random variable ϕ is uniquely determined by f. In accordance with our previous notation, we shall denote it by $\breve{f}$.

Dually we shall say that an object $(\Omega, m; f)$ of Rand $(T; X)$ is *final*† if it represents the unique fundamental random function $(\breve{X}^T, \breve{\mu}; (\pi_t))$ isonomous to it, i.e. if there exists a random variable $\phi: \breve{X}^T \to \Omega$ such that $\phi(\breve{\mu}) = m$ and $\pi_t = f(t) \circ \phi$ for every $t \in T$. The morphism ϕ is here probably not unique.

A random function is initial if and only if it is representable by every isonomous final random function. A random function is final if and only if it represents every isonomous initial random function. These statements follow trivially from the obvious fact that a fundamental random function is at the same time initial and final. Conversely, we have

THEOREM 13. *If a random function is both initial and final, then it is isomorphic to the fundamental random function isonomous to it.*

PROOF. Suppose that $f: T \to \text{Meas}(\Omega, m; X)$ is initial and final. Let $\breve{f}: \Omega \to \breve{X}^T$ be the random variable such that $f(t) = \pi_t \circ \breve{f}$ for every $t \in T$, and let $\phi: \breve{X}^T \to \Omega$ be a morphism such that $\pi_t = f(t) \circ \phi$ for every $t \in T$. We have to prove that $\breve{f} \circ \phi$ and $\phi \circ \breve{f}$ are both equivalent to the corresponding identity maps.

For $\breve{f} \circ \phi$ this is easy since $\pi_t = f(t) \circ \phi = \pi_t \circ \breve{f} \circ \phi$ shows that $\breve{f} \circ \phi$ is a morphism of $(\breve{X}^T, \breve{\mu}; (\pi_t))$ into itself, and we know that this morphism is unique, hence necessarily the identity.

† Initial and final do not have here the same meaning as in the theory of categories.

Set $\theta = \phi \circ \check{f}: \Omega \to \Omega$. By what we have just seen, $\theta^2 = \theta \circ \theta = (\phi \circ \check{f}) \circ (\phi \circ \check{f}) = \phi \circ (\check{f} \circ \phi) \circ \check{f} = \phi \circ \check{f} = \theta$, i.e. the map θ^2 is almost surely equal to θ. Let Ω_1 be a subset of Ω such that $m(\Omega_1) = 1$ and $\theta(\omega) = \theta^2(\omega)$ for $\omega \in \Omega_1$. Since $\theta(m) = m$ the set $\Omega_2 = \theta(\Omega_1)$ also satisfies $m(\Omega_2) = 1$. Furthermore every $\omega_2 \in \Omega_2$ is of the form $\omega_2 = \theta(\omega_1)$ with $\omega_1 \in \Omega_1$, and therefore $\theta(\omega_2) = \theta^2(\omega_1) = \theta(\omega_1) = \omega_2$, i.e. θ is the identity map on Ω_2.

Now assume that T is a topological space and X is a uniform space. We shall consider the full subcategory Cont Rand $(T; X)$ of Rand $(T; X)$ whose objects are the continuous random functions $f: T \to \text{Meas}(\Omega, m; X)$, where $\text{Meas}(\Omega, m; X)$ is equipped with the topology of convergence in probability. We know from the observation made after Theorem 2 (§2) that the continuity of f depends only on its isonomy class, hence all the considerations made so far in this section apply to the category Cont Rand $(T; X)$.

Let now D be a dense subset of T and for every continuous random function $f: T \to \text{Meas}(\Omega, m; X)$ denote by f_D its restriction to D. Then the map $f \longmapsto f_D$ defines a functor from the category Cont Rand $(T; X)$ into the category Cont Rand $(D; X)$. If $\phi: f \to f'$ is a morphism in Cont Rand $(T; X)$ then of course the corresponding morphism $\phi_D: f_D \to f'_D$ is the same random variable $\Omega \to \Omega'$. Furthermore the map $f \longmapsto f_D$ is injective since D is dense in T. Conversely given an object g of Cont Rand $(D; X)$, there exists an object f of Cont Rand $(T; X)$ such that $g = f_D$ if and only if whenever $d \in D$ converges to a point $t \in T$ the random variable $g(d)$ converges in $\text{Meas}(\Omega, m; X)$ to a random variable $g^T(t)$. Namely if this condition is satisfied, then the random function $g^T: t \longmapsto g^T(t)$ will be continuous [2, Chap. I, 3rd ed., §8, n°5, Th. 1], and clearly g^T will be the required function f.

It is reasonable to ask whether the possibility of extending g to g^T depends only on the isonomy class of g. We shall show that the answer to this question is affirmative if we impose on X a mild condition, but is negative if no restriction is made concerning X.

THEOREM 14. *Assume that on the uniform space X there exists a coarser uniform structure which is metrizable. Then the possibility to extend a continuous random function $g: D \to \mathrm{Meas}(\Omega, m; X)$ into a continuous random function $g^T: T \to \mathrm{Meas}(\Omega, m; X)$ depends only on the isonomy class of g.*

PROOF. (a) Assume first that X is metrizable and complete. Then the space $\mathrm{Meas}(\Omega, m; X)$ is complete [4, Chap. IV, 2nd ed., §5, n° 11, Prop. 19], and therefore the above necessary and sufficient condition for the existence of g^T can be reformulated as follows: if $d \in D$ converges to $t \in T$, then the corresponding $g(d)$ form a Cauchy filter in $\mathrm{Meas}(\Omega, m; X)$. Now the $g(d)$ form a Cauchy filter if and only if given a uniformity $\mathfrak{U}$ in X and a number $\epsilon > 0$ we have $m(\{\omega \mid (g(d')(\omega), g(d'')(\omega)) \notin \mathfrak{U}\}) < \epsilon$, i.e. $(g(d'), g(d''))(m)(\complement \mathfrak{U}) < \epsilon$ provided d' and d'' are sufficiently close to t, where $(g(d'), g(d''))$ is the map $\omega \longmapsto (g(d')(\omega), g(d'')(\omega))$ from Ω into $X \times X$. Now, by definition, the measure $(g(d'), g(d''))(m)$ depends only on the isonomy class of g.

(b) Let us assume next that X is only metrizable and let $\hat{X}$ be its completion. We have just seen that the fact that g can be extended to a random function $g^T: T \to \mathrm{Meas}(\Omega, m; \hat{X})$ depends only on the isonomy class of g. We have to show that the fact that for every $t \in T$ the random variable $g^T(t)$ takes almost all its values in X, i.e. that $g^T(t)(m)$ is concentrated on X, depends only on the isonomy class of g. But by Prop. 2 (§1) the measure $g^T(t)(m)$ is the narrow limit of the measures $g(d)(m)$ as d tends to t, and $g(d)(m)$ depends only on the isonomy class of g. Let us observe that the isonomy class of g^T depends only on the isonomy class of g. Indeed, for $t_j \in T (1 \leqslant j \leqslant p)$ the measure $(g^T(t_1), \ldots, g^T(t_p))(m)$ is the narrow limit of the measures $(g(d_1), \ldots, g(d_p))(m)$, where $d_j \in D (1 \leqslant j \leqslant p)$ and d_j tends to t_j.

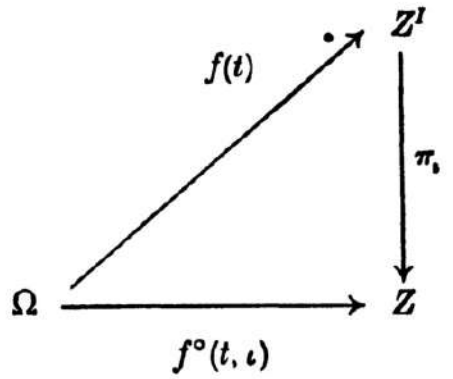
(c) Let us finally assume that there exists a metrizable space $\dot{X}$ whose underlying set is the same as that of X, and such that the uniform structure of $\dot{X}$ is coarser than that of X. We know already

that the fact that g: $D \to$ Meas $(\Omega, m; \dot{X})$ extends to a continuous random function $g^T : T \to$ Meas $(\Omega, m; \dot{X})$ depends only on the isonomy class of g, and by virtue of the last remark the isonomy class of g^T depends only on the isonomy class of g. Now for every $t \in T$ the map $g^T(t)$ will be a random variable with values in X if and only if the identity map $\dot{X} \to X$ is measurable with respect to the measure $g^T(t)\,(m)$ on $\dot{X}$, but this depends only on the isonomy class of g^T, and therefore only on the isonomy class of g. Finally, we know from §2 that whether g^T: $T \to$ Meas $(\Omega, m; X)$ is continuous depends only on the isonomy class of g^T, i.e. on the isonomy class of g.

REMARK 10. - If the uniform space X is a Suslin space, then there exists on X a coarser uniform structure which is metrizable. Indeed, the uniformly continuous functions separate the points of X, therefore by a property of Suslin spaces we used repeatedly, there exists a countable family of uniformly continuous functions which separate the points of X.

EXAMPLE 6. To construct an example where the possibility to extend g does not depend only on the isonomy class of g we shall take $D = \mathbf{N}$, for T the Alexandrov compactification of $\mathbf{N}$, and $X = Z^I$, where $Z = I = [0, 1]$ and X is equipped with the product uniform structure.

Observe first that any random function f: $T \to$ Meas $(\Omega, m; Z^I)$ is determined by a random function $f^\circ : T \times I \to$ Meas $(\Omega, m; Z)$ which is "partially canonical with respect to I". Indeed, for any $t \in T$ and $\iota \in I$ the random variable $f^\circ(t, \iota) : \Omega \to Z$ is the composition of the random variable $f(t) : \Omega \to Z^I$ and of the ι^{th} projection $\pi_\iota : Z^I \to Z$, hence for any fixed t the random function $\iota \longmapsto f^\circ(t, \iota)$ is indeed canonical as $Z = \breve{Z}$ since Z is compact. Now clearly the isonomy class of f° depends only on the isonomy class of f but the converse is also true. Indeed, the image measure $(f(t_1), \ldots, f(t_p))(m)$ on $(Z^I)^p$ depends only on its projections onto the finite products $Z^{J_1} \times Z^{J_2} \times \ldots \times Z^{J_p}$ and this projection is the image of m by the system of maps $f^\circ(t_k, \iota_{kl})$, where $1 \leq k \leq p$ and $J_k = (\iota_{kl})$.



The continuity of $f: T \to \text{Meas}(\Omega, m;\ Z^I)$ depends only on the continuity of $f^\circ : T \times I \to \text{Meas}\ (\Omega, m; Z)$, where $T \times I$ is the product of T and of I equipped with the *discrete* topology. In fact, $f(t)$ converges in measure to $f(t_0)$ as $t \to t_0$ if and only if for every finite subset J of I the random variable $\pi_J \circ f(t)$ converges in measure to $\pi_J \circ f(t_0)$ as $t \to t_0$, i.e. for every $\iota \in I$ the random variable $f^\circ(t, \iota)$ converges in measure to $f^\circ(t_0, \iota)$ as $t \to t_0$.

Let us also note the obvious relation $(f_D)^\circ = (f^\circ)_{D \times I}$.

Suppose then that we have a continuous random function h: $T \times I \to \text{Meas}\ (\Omega,\ m\ ;\ Z)$ which is *not* partially canonical with respect to I but whose restriction $h_{D \times I}$ to $D \times I$ *is* partially canonical with respect to I. Then h cannot be equal to f° for any continuous random function f: $T \to \text{Meas}\ (\Omega,\ m\ ;\ Z^I)$, but $h_{D \times I} = g^\circ$, where g: $D \to \text{Meas}\ (\Omega, m; Z^I)$ is a continuous random function which cannot be extended to a random function g^T. Let now h': $T \times I \to \text{Meas}\ (\Omega'$, m'; $Z)$ be a *canonical* random function with $\Omega' = Z^{T \times I} = X^T$, which is isonomous to h. A fortiori h' is then partially canonical with respect to I and therefore $h' = f'^\circ$ for some continuous random function f': $T \to \text{Meas}\ (\Omega', m'; Z^I)$. Also $h'_{D \times I}$ is isonomous to $h_{D \times I}$ and $h'_{D \times I} = g'^\circ$ where g' is a continuous random function $D \to \text{Meas}\ (\Omega', m'; Z^I)$. By the remark made above, the random functions g and g' are isonomous but g has no extension to T while g' has the extension $g'^T = f'$.

Recalling that $D = \mathbf{N}$ and $T = \mathbf{N} \cup \{\infty\}$ is the Alexandrov compactification of $\mathbf{N}$, to obtain such a random function h it is

sufficient to exhibit a sequence of random functions $u_n: I \to$ Meas $(\Omega, m; Z)$ which are all canonical and such that for every $\iota \in I$ the sequence $u_n(\iota): \Omega \to Z$ of random variables converges in measure to a random variable $u(\iota): \Omega \to Z$ such that the random function $\iota \longmapsto u(\iota)$ is not canonical. The random function h defined by $h(n, \iota) = u_n(\iota)$ for $n \in \mathbf{N}, \iota \in I$, and by $h(\infty, \iota) = u(\iota)$ for $\iota \in I$ has then the required properties. In particular, it is continuous since I is considered to be equipped with the discrete topology.

We have seen in Example 3 (§2) that the random function $u: I \to \text{Meas}(\Omega, m; Z)$ given by $u(\iota)(\omega) = Y(\omega - \iota)$ for $\iota \in I$ and almost every $\omega \in \Omega = [0, 1]$ is not canonical. On the other hand, define the continuous function Y_n on $\mathbf{R}$ by $Y_n(\omega) = 0$ if $\omega \leqslant 0$, $Y_n(\omega) = 1$ if $\omega \geqslant 1/n$ and Y_n linear in $0 \leqslant \omega \leqslant 1/n$, and let the random function $u_n: I \to \text{Meas}(\Omega, m; Z)$ be given by $u_n(\iota)(\omega) = Y_n(\omega - \iota)$ for $\iota \in I$ and almost every $\omega \in \Omega = [0, 1]$ with respect to the Lebesgue measure m. The random function u_n is canonical since for $u_n: \check{\Omega} \to Z^I$ we can take the map $\omega \longmapsto (Y_n(\omega - \iota))_{\iota \in I}$ which is continuous since the function Y_n is continuous. Finally, for every $\iota \in I$ the function $u_n(\iota)$ converges to $u(\iota)$ almost everywhere, hence a fortiori in measure (Prop. 1, §1).

Let T be a topological space, D a dense subspace of T and X a uniform space. Let $\psi: (\Omega, m; g) \to (\Omega', m'; g')$ be a morphism in the category Cont Rand $(D; X)$, i.e. a random variable $\psi: \Omega \to \Omega'$ such that $\psi(m) = m'$ and $g(d) = g'(d) \circ \psi$ for every $d \in D$. Assume now that g and g' both have extensions g^T and g'^T; since g' is isonomous to g, this will certainly be the case if g has an extension and if X satisfies the condition of Theorem 14, i.e. there exists on X a coarser uniform structure which is metrizable. The two functions $t \longmapsto g^T(t)$ and $t \to g'^T(t) \circ \psi$ are both continuous from T into Meas$(\Omega, m; X)$ and they coincide on D. Therefore we have $g^T(t) = g'^T(t) \circ \psi$ for every $t \in T$, i.e. the random variable ψ yields a morphism $\psi^T: g^T \to g'^T$.

We now make the following two observations.

(a) If g is a fundamental random function in the category Cont Rand $(D; X)$ and if g^T exists, then g^T is a final random function in Cont Rand $(T; X)$. Indeed, the sample space Ω corresponding to g is $\breve{X}^D$, and the canonical projection $\pi_D: \breve{X}^T \to \breve{X}^D$ is continuous, hence measurable. If $\breve{\mu}$ is the probability on $\breve{X}^T$ associated with the fundamental random function of the isonomy class of g^T, then $\pi_D(\breve{\mu})$ is clearly the probability associated with g on $\breve{X}^D$. Furthermore the two random functions $t \longmapsto \pi t$ and $t \longmapsto g^T(t) \circ \pi_D$ are both continuous on T with values in $\mathrm{Meas}(\breve{X}^T, \breve{\mu}; X)$. Since for any $d \in D$ they are both equal to π_d, and since D is dense in T, we have $\pi_t = g^T(t) \circ \pi_D$ for every $t \in T$, i.e. g^T is final with $\phi = \pi_D$.

(b) If f is a fundamental random function in the category Cont Rand $(T; X)$, then f_D is an initial (i.e. canonical) random function in Cont Rand$(D; X)$. Indeed, $f(t) = \pi_t: \breve{X}^T \to X$ for every $t \in T$, hence if π_D is the canonical projection $\breve{X}^T \to \breve{X}^D$, we have $f_D(d) = \pi_d \circ \pi_D$ for every $d \in D$, which means precisely that π_D is a morphism from f_D into the fundamental canonical random function of its isonomy class. More generally, we can state

PROPOSITION 14. *Let T be a topological space, D a dense subspace of T and X a uniform space. If the random function $f: T \to \mathrm{Meas}(\Omega, m; X)$ is initial, then $f_D: D \to \mathrm{Meas}(\Omega, m; X)$ is initial; if f_D is final, then f is final.*

PROOF. (a) Assume that $g = f_D$ is final. Then there exists a morphism ψ such that $h = g \circ \psi$, where h is the unique fundamental random function isonomous to g. It follows that $h^T = g^T \circ \psi^T$, and we know from observation (a) that h^T is final. There exists therefore a morphism ϕ such that $h' = h^T \circ \phi$, where h' is the fundamental random function isonomous to h^T. Thus we have $h' = g^T \circ \psi^T \circ \phi$, and since $g^T = f$, we see that f is final.

(b) If f is initial, then there exists a morphism ϕ such that $f = h \circ \phi$, where h is the fundamental random function of the

isonomy class of f. We have therefore $f_D = h_D \circ \phi_D$ and since h_D is initial by observation (b), there exists a morphism ψ such that $h_D = h' \circ \phi$, where h' is the fundamental random function of the isonomy class of h_D, i.e. of f_D. Thus $f_D = h' \circ \psi \circ \phi_D$, i.e. f_D is initial.

Assume that on the uniform space X there exists a coarser uniform structure which is metrizable. Then by virtue of Theorem 14 the injective map $f \longmapsto f_D$ from Cont Rand $(T; X)$ into Cont Rand $(D; X)$ transforms an isonomy class of Cont Rand $(T; X)$ into exactly one isonomy class of Cont Rand $(D; X)$. If $\mathscr{F}$ is an isonomy class in Cont Rand $(T; X)$ and if $\mathscr{F}_D$ is its image in Cont Rand $(D; X)$, then the maps $f \longmapsto f_D$, $\phi \longmapsto \phi_D$ define a functorial isomorphism from $\mathscr{F}$ onto $\mathscr{F}_D$, whose inverse is given by the maps $g \longmapsto g^T$, $\psi \to \psi^T$.

Changing the situation assume that T is a vector space over $\mathbf{R}$ and that X is a topological vector space over $\mathbf{R}$. We consider the full subcategory Lin Rand $(T; X)$ of Rand $(T; X)$ whose objects are the linear random functions $(\Omega, m; f)$ from T into the vector space Meas $(\Omega, m; X)$. As we have observed in §2, the linearity of a random function depends only on its isonomy class, and therefore the definitions and assertions made at the beginning of this section concerning the category Rand $(T; X)$ are meaningful and true for Lin Rand $(T; X)$.

Let D be a system of generators of the vector space T, and for any linear random function $f: T \to$ Meas $(\Omega, m; X)$ denote by f_D the restriction of f to D. Also if $\phi: f \to f'$ is a morphism in Lin Rand $(T; X)$, the same map $\phi: \Omega \to \Omega'$ yields a morphism $\phi_D: f_D \to f'_D$ in Rand $(D; X)$, and it is clear that the maps $f \longmapsto f_D$ and $\phi \longmapsto \phi_D$ define a functor from Lin Rand $(T; X)$ into Rand $(D; X)$. Furthermore, the map $f \longmapsto f_D$ is injective since D generates T. Conversely, given an object g in Rand $(D; X)$, there exists an object f in Lin Rand $(T; X)$ such that $g = f_D$ if and only if for any finite family $(d_j)_{1 \leqslant j \leqslant n}$ of vectors in D such that $\sum_{j=1}^{n} \alpha_j d_j = 0$ for $\alpha_j \in \mathbf{R}$ $(1 < j < n)$, we have $\sum_{j=1}^{n} \alpha_j g(d_j) = 0$. If this condition is satisfied, then the

extension f will be denoted by g^T, and for any $t = \sum_{j=1}^{n} \tau_j d_j \in T$ ($\tau_j \in \mathbf{R}$, $d_j \in D$, $1 \leqslant j \leqslant n$) it is defined by $f(t) = \sum_{j=1}^{n} \tau_j g(d_j)$.

If g is a random function in Rand $(D; X)$ which has an extension g^T, and if g' is isonomous to g, then g' also has an extension g'^T. Indeed, let $(d_j)_{1 \leqslant j \leqslant n}$ be a finite family of vectors, and $(\alpha_j)_{1 \leqslant j \leqslant n}$ a finite family of scalars such that $\sum_{j=0}^{n} \alpha_j d_j = 0$. Denote by H_α the subset of X^n consisting of those n-tuples $(x_1, \ldots, x_n)$ for which $\sum_{j=1}^{n} \alpha_j x_j = 0$. Then the condition $m(\{\omega \mid \sum_{j=1}^{n} \alpha_j g(d_j)(\omega) = 0\}) = 1$ is equivalent to $(g(d_1), \ldots, g(d_n))(m)(H_\alpha) = 1$. But $(g(d_1), \ldots, g(d_n))(m) = (g'(d_1), \ldots, g'(d_n))(m)$, hence if the condition of extendability is satisfied by g, then it is also satisfied by g'.

Let $\psi: g \to g'$ be a morphism in the category Rand $(D; X)$, i.e. a random variable $\psi: \Omega \to \Omega'$ such that $g(d) = g'(d) \circ \psi$ for every $d \in D$. Assume that g has an extension g^T. Then g' also has an extension g'^T since it is isonomous to g. The two functions $t \longmapsto g^T(t)$ and $t \longmapsto g'^T(t) \circ \psi$ are both linear and they coincide on the system of generators D. Therefore we have $g^T(t) = g'^T(t) \circ \psi$ for every $t \in T$ and the map ψ defines a morphism $\psi^T: g^T \to g'^T$. We see exactly as in Prop. 14 that if f is initial, then f_D is initial, and if f_D is final, then f is final.

We can combine the last two situations and suppose that T and X are topological vector spaces over $\mathbf{R}$ and D a total subset of T. In this situation we consider continuous linear random functions $f: T \to \text{Meas}(\Omega, m; X)$ and their restrictions f_D to D. This is precisely what we have done in §3, where we considered for example $T = l^p$ $(1 < p < \infty)$, $X = \mathbf{R}$ and D was the set of unit vectors $(\epsilon_n)_{n \in \mathbf{N}}$. If we identify D with $\mathbf{N}$ then f_D becomes a random function $n \longmapsto x_n = f(\epsilon_n)$, i.e. a sequence (x_n) of random variables.

If, as in the examples of §3, the set D is countable, then to every fundamental random function in Rand $(D; X)$ we can find an isomorphic random function whose sample space is X^D rather than $\check{X}^D$.

Indeed, if D is countable, then the measure $\breve{\mu}$ on $\breve{X}^D$, the projective limit of the measures $(f(d_1),\dots,f(d_p))(m)$, is concentrated on X^D and thus the isomorphism in question is the equivalence class of the identity map $\breve{X}^D \to X^D$, which is defined almost everywhere. Thus in the example above each x_n can be considered to be a real random variable defined on $\mathbf{R}^{\mathbf{N}}$.

A similar situation arises if E is a locally convex Hausdorff space, E' its dual, μ a cylindrical measure on E, $f_0: E' \to \mathrm{Meas}\,(\breve{\mathbf{R}}^{E'}, \breve{\mu}; \mathbf{R})$ the fundamental linear random function associated with μ, and μ is a Radon measure on E for the weak topology $\sigma(E, E')$. In this case the linear random function $I: E' \to \mathrm{Meas}\,(E, \mu; \mathbf{R})$ given by $I(\xi)$: $x \longmapsto \langle x, \xi \rangle$ for $\xi \in E'$ and $x \in E$, is isomorphic to f_0, and in particular it is also associated with the cylindrical measure μ. Indeed, since μ is concentrated on E, the injective, continuous, open map $j: x \longmapsto (\langle x, \xi \rangle)_{\xi \in E'}$ from E into $\hat{\mathbf{R}}^{E'}$ has an inverse which is defined almost everywhere on $\breve{\mathbf{R}}^{E'}$. Since $I(\xi) = \pi_\xi \circ j$ for every $\xi \in E'$, the random variable j defines an isomorphism $I \to f_0$. Clearly the random function $(E, \mu; I)$ is much nicer than $(\breve{\mathbf{R}}^{E'}, \breve{\mu}; f_0)$.

Let $(\breve{X}^T, \breve{\mu}; (\pi_t)_{t \in T})$ be a fundamental random function in $\mathrm{Rand}(T; X)$. Assume that $\breve{\mu}$ is concentrated on a countable union of compact metrizable subsets of $\breve{X}^T$. If $(\Omega, m; f)$ is an arbitrary random function of the isonomy class $\mathscr{F}$ of $(\breve{X}^T, \breve{\mu}; (\pi_t)_{t \in T})$, then $\pi_t(\breve{\mu}) = f(t)$ (m) for every $t \in T$, i.e. condition (ii) of Theorem 10 (§4) is satisfied. Therefore there exists a random variable $\phi: \Omega \to \breve{X}^T$ such that $f(t) = \pi_t \circ \phi$ for every $t \in T$. In other words, every random function in $\mathscr{F}$ is initial.

Next let us recall the following fact from general topology: the image of a compact metrizable space with respect to a continuous map is a compact metrizable space. Indeed, a compact metrizable space is a Suslin space, hence its image with respect to a continuous map is also a Suslin space. But this image is also compac tand therefore it is metrizable.

Now Let $f\colon T \to \mathrm{Meas}\,(\Omega, m; X)$ be an initial random function in $\mathrm{Rand}\,(T; X)$ and assume that m is concentrated on a countable union of compact metrizable subsets of Ω (cf. Remark 6, §4). By the Lusin-measurability of $f(t)$ for each $t \in T$ the measure m is concentrated on a countable union of compact metrizable subsets of Ω on which $f(t)$ is continuous, hence, by the result from general topology just recalled, $f(t)\,(m)$ is concentrated on a countable union of compact metrizable subsets of X. It follows that the fundamental random function $(\breve{X}^T, \breve{\mu}; (\pi_t))$ isonomous to $(\Omega, m; f)$ is such that $\breve{\mu}$ is concentrated on a countable union of compact metrizable subspaces of $\breve{X}^T$, and therefore, as we saw above, every random function in the isonomy class $\mathscr{F}$ of f is initial. By the same argument for every final random function $(\Omega', m'; f')$ of $\mathscr{F}$ the measure m' is concentrated on a countable union of compact metrizable subsets of Ω', and since f' is also initial, by Theorem 13 it is isomorphic to the fundamental random function.

Let us say that an isonomy class $\mathscr{F}$ in $\mathrm{Rand}\,(T; X)$ is *universally initial* if every random function in $\mathscr{F}$ is initial. If $\mathscr{F}$ is universally initial, then, as we have just observed, every final random function f' is isomorphic to the fundamental random function of $\mathscr{F}$, and for every f in $\mathscr{F}$ there exists a unique morphism $f \to f'$.

We shall say that a random function f' in $\mathrm{Rand}\,(X; T)$ is *absolutely final* if for any random function f isonomous to f' there exists a unique morphism $f \to f'$. Thus in a universally initial isonomy class every final random function is absolutely final.

Let us return now to a universally initial isonomy class $\mathscr{F}$ in which there exists an initial random function $(\Omega, m; f)$ such that m is concentrated on a countable union of compact metrizable subsets of Ω. We shall show that in such an isonomy class $\mathscr{F}$, there exists a very simple final (hence also absolutely final and also initial) random function. Indeed, if $(\Omega', m'; f')$ is a final random function in $\mathscr{F}$, then we know that m' is concentrated on a countable union of compact metrizable subsets of Ω', hence we are allowed to take $\Omega' = [0, 1]^{\mathbf{N}}$ since every compact metrizable space can be imbedded

into $[0, 1]^{\mathbf{N}}$. According to Peano, there exists a bijection $[0, 1]^{\mathbf{N}} \to [0, 1]$ which is a Borel map together with its inverse, and therefore we may even take $\Omega' = [0, 1]$. In particular, there exists in $\mathscr{F}$ an initial random function with sample space $[0, 1]$. Conversely, if in the isonomy class $\mathscr{F}$ there exists an initial random function with sample space $[0, 1]$, then every random function in $\mathscr{F}$ is initial and every final random function in $\mathscr{F}$ is absolutely final.

We shall now see a more general (cf. Example 7) case of isonomy classes in which there exist absolutely final random functions. Given a set T, a completely regular space X and an isonomy class $\mathscr{F}$ in Rand $(T; X)$, we shall say that a subset D of T is *total* for $\mathscr{F}$ if the following two conditions are satisfied:

a) the restriction map $f \longmapsto f_D$ from Rand $(T; X)$ into Rand $(D; X)$ maps $\mathscr{F}$ bijectively onto exactly one isonomy class $\mathscr{F}_D$ in Rand $(D; X)$;

b) a random variable $\phi: \Omega \to \Omega'$ is a morphism $(\Omega, m; f) \to (\Omega', m'; f')$ in $\mathscr{F}$ if and only if it is a morphism $(\Omega, m; f_D) \to (\Omega, m'; f'_D)$ in $\mathscr{F}_D$.

We have seen two examples of such total subsets.

1) T a topological space, X a uniform space with a coarser metrizable uniform structure, $\mathscr{F}$ an isonomy class of continuous random functions and D a dense subset of T;

2) T is a vector space, X a topological vector space, $\mathscr{F}$ an isonomy class of linear random variables and D a system of generators of T.

Let us now assume that D is total for $\mathscr{F}$ and that $\mathscr{F}_D$ is universally initial. This will happen, for instance, if D is countable and every compact subset of X is metrizable, since then there exists a random function, isomorphic to the fundamental random function of $\mathscr{F}_D$, whose sample space is $\Omega = X^D$ (p. 316), and therefore the corresponding probability m is concentrated on a countable union of compact metrizable subsets of Ω.

Since $\mathscr{F}_D$ contains final random functions, e.g. the fundamental random function, and since in a universally initial isonomy class every final random function is absolutely final, there exists in $\mathscr{F}_D$ an absolutely final random function g'. Let $f' = g'^T$, i.e. $f'_D = g'$. Then, given any random function f in $\mathscr{F}$, there exists a unique morphism $f_D \to g'$ and therefore, by condition b), a unique morphism $f \to f'$. Thus f' is an absolutely final random function in $\mathscr{F}$. Furthermore, in $\mathscr{F}$ all absolutely final random functions are isomorphic. Indeed let f and f' be two absolutely final random functions in $\mathscr{F}$. Then f_D and f'_D are final random functions in $\mathscr{F}_D$ and since in a universally initial isonomy class all final random functions are isomorphic, we have an isomorphism $\phi : f_D \to f'_D$. But then, by condition b) above, ϕ is also an isomorphism $f \to f'$. It remains an open problem whether in an arbitrary isonomy class all absolute by final random functions are isomorphic.

EXAMPLE 7. We want to show that there exist isonomy classes which possess absolutely final random functions but which are not universally initial. Indeed, let us consider the non-canonical, i.e. non-initial random function $f: T \to \mathrm{Meas}\,(\Omega, m; X)$, where $T = \Omega = X = [0, 1]$, $m = dx$ and $f(t)(\omega) = Y(\omega - t)$ for all $t \in T$ and almost every $\omega \in \Omega$(§2, example 3). If t converges to t_0 in T, then the function $\omega \longmapsto Y(\omega - t)$ converges almost surely, thus also in probability, to $\omega \longmapsto Y(\omega - t_0)$, i.e. f is continuous. The subset $D = T \cap \mathbf{Q}$ is countable and dense in T, and X is metrizable. Therefore, if we denote by $\mathscr{F}$ the isonomy class of f, by our remarks above, $\mathscr{F}_D$ will be universally initial, and since D is total in T for $\mathscr{F}$, there exists an absolutely final random function in $\mathscr{F}$.

On the other hand, not every random function in $\mathscr{F}$ is initial since f is not. Furthermore the probability $\breve{\mu}$ associated with the fundamental random function $(\breve{X}^T, \breve{\mu}; (\pi_t))$ of $\mathscr{F}$ is not concentrated on a union of compact metrizable subsets of $\breve{X}^T = [0, 1]^{[0,1]}$. Finally, not all the final random functions of $\mathscr{F}$ are isomorphic to the fundamental random function f_0 of $\mathscr{F}$. Indeed, if the absolutely final random function f' of $\mathscr{F}$, whose existence was proved above,

were isomorphic to f_0, then the morphisms $f \to f'$ and $f' \to f_0$ would yield by composition a morphism $f \to f_0$, which cannot exist since f is not initial.

EXAMPLE 8. If $T = l^{p'}$ $(1 < p' < \infty)$, $X = \mathbf{R}$ and $\mathscr{F}$ is an isonomy class of continuous random functions f: $l^{p'} \to$ Meas $(\Omega, m; \mathbf{R})$, then an absolutely final random function of $\mathscr{F}$ is very much in evidence (and was used implicitly in the proof of Prop. 10, § 3). The set $D = \{\epsilon_n\}$ of unit vectors of $l^{p'}$ is countable, and total in $l^{p'}$ for $\mathscr{F}$. The fundamental random function g of the isonomy class $\mathscr{F}_D$ in Rand $(D; \mathbf{R})$ associates with every $\epsilon_n \in D$, i.e. with every $n \in \mathbf{N}$, the n^{th} projection π_n: $\mathbf{R}^D \simeq \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$, and the measure m_0 on $\mathbf{R}^{\mathbf{N}}$ is given by $\pi_n(m_0) = \epsilon_n(\mu)$, where μ is the cylindrical measure on l_p' associated with f and $\epsilon_n \in l^{p'}$ is considered as a continuous linear form on l^p. We know that the random function $(\mathbf{R}^{\mathbf{N}}, m_0; g^T)$ is absolutely final in $\mathscr{F}$. Given f in $\mathscr{F}$, the morphism ϕ: $f \to g^T$ is easily described. The corresponding random function f_D is given by the sequence $x_n = f(\epsilon_n)$ of random variables $\Omega \to \mathbf{R}$. The sequence (x_n) defines a map ψ: $\omega \longmapsto (x_n(\omega))_n$ from Ω into $\mathbf{R}^{\mathbf{N}}$ such that $x_n = \pi_n \circ \psi$ and $x_n(m) = f(\epsilon_n)(m) = \epsilon_n(\mu) = \pi_n(m_0)$ for every $n \in \mathbf{N}$. Thus $\psi(m) = m_0$, the map ψ is a morphism $f_D \to g$ and $\phi = \psi^T$ is the required morphism $f \to g^T$. In the proof of Prop. 10 we wanted to show that the condition "$\sum_{n=0}^{\infty} |x_n(\omega)|^p < \infty$ almost surely" depended only on the isonomy class of f and we did this by showing that the condition was equivalent to the condition that m_0 is concentrated on the subset l^p of $\mathbf{R}^{\mathbf{N}}$.

To conclude, we want to give an example which will show that initial and final random functions behave in a completely different way.

EXAMPLE 9. Let $(\Omega, m; f)$ be an arbitrary random function in Rand $(T; X)$. We shall construct a random function $(\Omega', m'; f')$ in Rand $(T; X)$ such that there exists a morphism $f' \to f$, and in particular f and f' are isonomous, but there exists no morphism $f \to f'$.

Let A be an index set and set $\Omega' = \Omega \times [0, 1]^A$, $m' = m \otimes (dx)^A$. If ϕ is the projection $\Omega' \to \Omega$ then we have $\phi(m') = m$. Define f' by $f'(t) = f(t) \circ \phi$ for every $t \in T$. Then ϕ is a morphism $f' \to f$. Let us show that if the cardinality of the index set A is sufficiently large, there exists no random variable $\phi' : \Omega \to \Omega'$ such that $\phi'(m) = m'$. Indeed, assume that such a map ϕ' exists. Then $\Xi = \phi'(\Omega)$ has m'-measure one in Ω'. By Fubini's theorem, for almost every $\omega \in \Omega$ with respect to m the set Ξ meets the section $\{\omega\} \times [0, 1]^A$ in a set whose $(dx)^A$-measure is one. In particular, there exists at least one $\omega \in \Omega$ such that the intersection of $\{\omega\} \times [0, 1]^A$ with Ξ is dense in $\{\omega\} \times [0, 1]^A$.

Now card $\Xi \leqslant$ card Ω and card $A <$ card $[0,1]^A$. On the other hand, if a set H has cardinality α, then its closure $\bar{H}$ has at most cardinality 2^{2^α}. Indeed, the cardinality of the set of all subsets of H is 2^α, the set $\mathscr{F}$ of all filters on H has cardinality $\leqslant 2^{2^\alpha}$, and there exists a map from $\mathscr{F}$ onto $\bar{H}$. Therefore, if we choose card $A > 2^{2^{\text{card } \Omega}}$ we obtain a contradiction since we have card $A <$ card $[0, 1]^A \leqslant$ card $\bar{\Xi} \leqslant 2^{2^{\text{card } \Xi}} \leqslant 2^{2^{\text{card } \Omega}}$.

We can now draw our conclusions.

(i) We have found isonomy classes in which every random function is initial. There exist no isonomy classes in which every random function is final. Indeed, choosing for f the fundamental random function of any class $\mathscr{F}$, the example shows the existence of an $f' \in \mathscr{F}$ such that there exists no morphism $f \to f'$.

(ii) We have found isonomy classes in which all final random functions are isomorphic to each other. There exist no isonomy classes in which all initial random functions are isomorphic to each other. Indeed, let f be an initial random function in the isonomy class $\mathscr{F}$ and let f' in $\mathscr{F}$ be such that there exists a morphism $f' \to f$ but there exists no morphism $f \to f'$. Then f' is also initial but f and f' cannot be isomorphic.

(iii) We have found isonomy classes in which there exist absolutely final random functions. In no isonomy class can there

exist "absolutely initial" random functions. Indeed, for any f there exists an f' isonomous to f such that there is no morphism $f \to f'$, i.e. f is not "absolutely initial".

These are "negative conclusions". We can draw the positive ones:

(a) if $\mathscr{F}$ contains some initial f for which m is carried by a countable union of metrizable compact subspaces, then $\mathscr{F}$ is universally initial (all the elements of $\mathscr{F}$ are initial);

(b) if $\mathscr{F}$ is universally initial, every final element of $\mathscr{F}$ is absolutely final, and all the final elements are isomorphic to each other;

(c) if D is total and if $\mathscr{F}_D$ is universally initial, $\mathscr{F}$ has absolutely final elements, and all of them are isomorphic.

SUMMARY OF CHAPTER VI

The gaussian measures play a fundamental role in probability theory; here one studies them in terms of cylindrical measures. §1 defines Gauss cylindrical measures on Hilbert spaces; from it one defines gaussian cylindrical measures in topological vector spaces Definition 1, page 329, using the notion of Hilbert subspaces. The §2, gives the main results of the theory of reproducing kernels of Hilbert subspaces: the cone Hilb (E), page 331, the relationship between Hilbert subspaces and their gaussian measures, page 333; the kernels page 336, the reproducing kernel of a Hilbert subspace, page 337; the bijective correspondence between Hilbert subspaces and positive kernels, page 338. This gives the Fourier transform of a gaussian law in terms of the reproducing kernel of its Hilbert subspace, page 340.

§3 gives a second proof on the "only if" part of the theorem of Sazonov, and a converse of the theorem of Minlos, using Gauss laws. The fundamental theorem is Theorem 1, page 341; the image of the canonical Gauss law of a Hilbert space, through a continuous linear map u into another Hilbert space, is Radon, if and only if u is Hilbert-Schmidt. This is, by the way, a particular theorem of a more general one, relative to p-radonifying maps, which is not given in this book. Then Theorem 2, page 344, gives the converse of Minlos' theorem.

§4 studies the extension of Sazonov's theorems for non-hilbertian cases, namely the spaces of sequences l^p. We do not give here the proof of the Theorem 3, page 347, which has been considerably developed and generalized in further publications. We are interested here only in applications of this theorem to radonifying and decomposing maps, Theorem 4, page 349.

§5 gives an application of the general methods used here to the regularity of brownian motion. The main theorems are, Theorem 5, page 351, and 6, page 352, from which one deduces the almost sure continuity of the brownian motion, Theorem 7, page 364. This is essentially an example; it enters into the recent methods using ϵ-entropy.

VI

GAUSS MEASURES

1. The definition of Gauss measures. On the real line $\mathbf{R}$ the Gauss measure γ is defined by

$$\gamma = e(x)dx = e^{-\pi x^2}dx.$$

It is obviously positive and it is well known that $\gamma(\mathbf{R}) = \int_{-\infty}^{\infty} e(x)dx = 1$, i.e. γ is a probability on $\mathbf{R}$. We have furthermore $\int_{\mathbf{R}} xe(x)\,dx = 0$, $\int_{\mathbf{R}} x^2 e(x)\,dx = \frac{1}{2\pi}$ and the Fourier transform of γ is given by

$$\int_{\mathbf{R}} e^{-\pi x^2} e^{-2\pi i x\cdot\xi} dx = e^{-\pi\xi^2}.$$

Let E be a real Euclidean space, i.e. a finite-dimensional real Hilbert space, and let us denote the norm of E by $x \longmapsto |x|$. If $(e_i)_{1\leqslant i\leqslant n}$ is an orthonormal basis of E, then the mapping $x = \sum_{i=1}^{n} x_i e_i \longmapsto (x_i)$ defines an isomorphism from E onto $\mathbf{R}^n$. We denote by dx the measure on E which corresponds to the Lebesgue measure $dx_1 dx_2 \ldots dx_n$ on $\mathbf{R}^n$ under this isomorphism; clearly the measure dx is independent of the particular choice of the orthonormal basis (e_i). The Gauss measure γ on E is defined by

$$\gamma = e^{-\pi|x|^2}dx.$$

If E and F are two Euclidean spaces with norms $x \longmapsto |x|$ and $y \longmapsto |y|$ respectively, then their Hilbert sum [3, Chap. V, §2, $n^0$1] $E \oplus F$ is a Euclidean space equipped with the norm $(x, y) \longmapsto (|x|^2 + |y|^2)^{1/2}$. It follows that the Gauss measure on $E \oplus F$ is

$$e^{-\pi(|x|^2+|y|^2)}\,dxdy = e^{-\pi|x|^2}\,dx \otimes e^{-\pi|y|^2}\,dy,$$

i.e. the tensor product of the Gauss measures on E and F. In particular the Euclidean space $\mathbf{R}^n$ is the Hilbert sum of n real lines $\mathbf{R}$ and so the Gauss measure γ on $\mathbf{R}^n$ is given by

$$\gamma = e^{-\pi|x|^2}\,dx = e^{-\pi x_1^2}\,dx_1 \otimes \ldots \otimes e^{-\pi x_n^2}\,dx_n,$$

where $|x|^2 = x_1^2 + \ldots + x_n^2$.

Let E be a real Euclidean space, E' its dual and $(x, \xi) \longmapsto \langle x, \xi \rangle$ the canonical bilinear form on $E \times E'$. We know [3, Chap. V, §1, n^0 6, Th. 3] that E' is equipped canonically with a Euclidean structure; denote by $\xi \longmapsto |\xi|$ the norm on E'. The Fourier transform $\phi = \mathscr{F}(\gamma)$ of the Gauss measure γ on E is given by

$$\phi(\xi) = e^{-\pi|\xi|^2}.$$

Indeed, let $(e_i)_{1 \leqslant i \leqslant n}$ be an orthonormal basis of E and let $(e_i')_{1 \leqslant i \leqslant n}$ be the dual basis of E'. Then for $x = \Sigma\, x_i e_i$ and $\xi = \Sigma\, \xi_i e_i'$ we have $|x|^2 = \Sigma\, x_i^2$, $|\xi|^2 = \Sigma\, \xi_i^2$ and $\langle x, \xi \rangle = \Sigma\, x_i \xi_i$, hence

$$\phi(\xi) = \int_E e^{-\pi|x|^2}\, e^{-2\pi i \langle x, \xi \rangle}\,dx = \prod_{i=1}^{n} \int_{\mathbf{R}} e^{-\pi x_i^2}\, e^{-2\pi i x_i \xi_i}\,dx_i$$

$$= \prod_{i=1}^{n} e^{-\pi \xi_i^2} = e^{-\pi|\xi|^2}.$$

Let E be a Euclidean space, M a subspace of E and p the orthogonal projector from E onto M. If γ is the Gauss measure on E, then $p(\gamma)$ is the Gauss measure on M. Indeed, let $(e_i)_{1 \leqslant i \leqslant n}$ be an orthonormal basis of E such that the first m vectors e_i span M. Then for any bounded continuous function f on M we have

$$p(\gamma)(f) = \int f(x_1, \ldots, x_m) \exp\left(-\pi \sum_{i=1}^{n} x_i^2 \right) dx_1 \ldots dx_n$$

$$= \left(\int f(x_1, \ldots, x_m) \exp\left(-\pi \sum_{i=1}^{m} x_i^2 \right) dx_1 \ldots dx_m \right) \times$$

$$\times \left(\int \exp\left(-\pi \sum_{i=m+1}^{n} x_i^2 \right) dx_{m+1} \ldots dx_n \right).$$

Now the value of the second integral is 1 and since the Gauss measure on M is

$$\exp\left(-\pi \sum_{i=1}^{m} x_i^2 \right) dx_1 \ldots dx_m,$$

the assertion follows.

Let E be a Euclidean space, F a finite-dimensional vector space, $u: E \to F$ a surjective linear map and γ the Gauss measure on E. If we equip F with the image Euclidean structure with respect to u, then $u(\gamma)$ is the Gauss measure on F. Indeed, if N is the kernel of u and $N^\perp$ its orthogonal complement, then $u = \bar{u} \circ p$, where p is the orthogonal projector onto $N^\perp$ and $\bar{u}: N^\perp \to F$ is a bijection. The image Euclidean structure on F is obtained by carrying over that of $N^\perp$ with the help of $\bar{u}$. On the other hand, we have just seen that $p(\gamma)$ is the Gauss measure on $N^\perp$.

Now let E be an arbitrary Hilbert space. The Gauss measure γ on E will be a cylindrical measure defined as follows. Let G be a finite-dimensional vector space and $g: E \to G$ a continuous surjective linear map. If N is the kernel of g, then the bijection $N^\perp \to G$ defines as before a Euclidean structure on G and therefore also a Gauss measure γ_g. Furthermore, let H be a second finite-dimensional space, $h: E \to H$ a continuous surjective linear map, and $w: G \to H$ a linear map such that $w \circ g = h$. Then the Euclidean structure on H defined by h is the image with respect to w of the Euclidean structure on G defined by g. It follows from the preceding remark that $\gamma_h = w(\gamma_g)$ and therefore the measures γ_g form a projective system which defines the cylindrical measure γ on E.

According to this definition, γ can be characterized as the cylindrical measure on E such that for any finite-dimensional subspace F of E one has $p(\gamma) = \gamma_F$, where p is the orthogonal projector onto F, and γ_F is the Gauss measure on F for the Euclidean structure induced on F by E. Indeed, it is sufficient to observe that $E/F^\perp$ is isomorphic to F ($F^\perp$ is the orthogonal complement of F), the canonical projection $E \to E/F^\perp$ being identified with the orthogonal projection $E \to F$; if $G^\perp \subset F^\perp$ or $F \subset G$, then $E/G^\perp \to E/F^\perp$ is identified with the orthogonal projection $G \to F$.

The Gauss measure γ on E is scalarly concentrated on the balls of E since for any $\epsilon > 0$ there exists $R > 0$ such that $\int_{|x| \geq R} e(x)\, dx \leq \epsilon$. On the other hand, γ is not concentrated cylindrically on the balls of E. Indeed, if $g: E \to G$ is a continuous surjective linear map onto

the n-dimensional vector space G, then g maps the ball with radius R of E onto the ball with radius R of G. Now the ball with radius R in G is contained in the cube with edge $2R$, and the measure of this cube with respect to the image measure γ_g of γ is

$$\left(\int_{-R}^{R} e^{-\pi t^2}\, dt\right)^n.$$

Since the integral has value < 1, this expression tends to 0 for fixed R as $n \to \infty$.

The Gauss cylindrical measure gives also a simple counterexample to prove that the "measure" defined by γ on the cylindrical sets (see Chap. 2, §1, p. 173) is not countably additive. Let E a separable Hilbert space, $(e_n)_{n\in\mathbf{N}}$ a Hilbert basis. Let E_n be the cylindrical set $E_n = \{x \in E;\ |x_i| \leqslant n,\ 1 \leqslant i \leqslant a_n\}$. If a_n is increasing and goes to infinity with n, the pointwise limit of the characteristic functions on the E_n is 1 (i.e. every $x \in E$ belongs to E_n for n large enough). However, the measure of E_n is $(\int_{-n}^{+n} e^{-\pi t} dt)^{a_n}$ which tends to 0 if a_n tends to infinity sufficiently fast as $n \to \infty$.

If the Hilbert space E is the Hilbert sum of the Hilbert spaces $E_1, \ldots, E_k$, then the Gauss measure γ on E is the tensor product (Chap. 2, §2 (a)) of the Gauss measures γ_j on the spaces $E_j (1 \leqslant j \leqslant k)$. Indeed, let G be a finite-dimensional vector space, the direct sum of the vector spaces $G_1, \ldots, G_k$, and for each index j let $g_j : E_j \to G_j$ be a continuous surjective linear map. The linear map $g = g_1 \oplus \ldots \oplus g_k$: $E \to G$ [1, Chap. II, 3rd ed., §1, n^0 6, Prop. 7] is continuous and surjective [loc. cit., Corol. 1] and the Euclidean structure on G defined by g is the Hilbert sum of the Euclidean structures defined on the G_j by the g_j. Therefore, by a remark made above, $\gamma_g = \gamma_{g_1} \otimes \ldots \otimes \gamma_{g_k}$, from where the assertion follows.

Let E_1 and E_2 be two Hilbert spaces, $u : E_1 \to E_2$ a continuous surjective linear map and assume that the Hilbert structure on E_2 is the image with respect to u of the Hilbert structure on E_1. If γ_i denotes the Gauss measure on $E_i (i = 1, 2)$, then $\gamma_2 = u(\gamma_1)$. This

follows easily from the definition of the image of a cylindrical measure (Chap. 2, §2(b)).

If E is a Hilbert space, E' its dual and γ the Gauss measure on E, then its Fourier transform $\varphi = \mathscr{F}(\gamma)$ (Chap. 2, §2(d)) is given by

$$\varphi(\xi) = e^{-\pi\|\xi\|^2}$$

for $\xi \in E'$. Indeed, if F is a closed subspace with finite codimension of E, then the Fourier transform of the Gauss measure on E/F, equipped with the Euclidean structure defined by $\pi_{E/F}$, is the function $\xi \longmapsto e^{-\pi|\xi|^2}$ on $F^\perp \subset E'$.

Now we come to the definition of the general concept of Gauss measure.

DEFINITION 1. *Let E be a locally convex Hausdorff space and μ a cylindrical measure on E. We shall say that μ is a Gauss measure if there exists a Hilbert subspace* (Chap. 4, §2, Def. 4) $\mathscr{H}$ *of E such that μ is the image of the Gauss measure $\gamma_{\mathscr{H}}$ on $\mathscr{H}$ with respect to the canonical injection $j\colon \mathscr{H} \to E$.*

If there is no possibility of confusion, we shall write simply $\gamma_{\mathscr{H}}$ instead of $j(\gamma_{\mathscr{H}})$. According to this definition there exist many Gauss measures on a locally convex Hausdorff space. For example the Dirac measure δ is a Gauss measure, corresponding to the Hilbert subspace $\mathscr{H} = \{0\}$ of E.

If E is a Hilbert space there is a distinguished Gauss measure on E associated with E as a Hilbert subspace of itself. We shall refer to this Gauss measure as the canonical one.

EXAMPLE 1. Let E be the Banach space $\mathscr{C}[0, 1]$ of all real-valued continuous functions on the interval [0,1]. Let A be the collection of all finite subsets of [0,1] containing the points 0 and 1, and for each $\alpha = \{0 = t_0 < t_1 < \ldots < t_n = 1\} \in A$ denote by p_α the projector which associates with each function $f \in \mathscr{C}[0, 1]$ the function which takes the value $f(t_i)$ at the points $t_i \in \alpha$ and is linear between two such points. The projectors p_α tend to the identity map of $\mathscr{C}[0,1]$ with respect to the directed set A, uniformly on every compact subset of $\mathscr{C}[0, 1]$. Let μ be a cylindrical measure on $\mathscr{C}[0, 1]$ scalarly concentrated on

the weakly compact subsets of E. It follows from Theorem 1 of Chap. 5, §1 that μ is a Radon measure on $\mathscr{C}[0, 1]$ if and only if $p_\alpha \mu$ has a limit with respect to A for the narrow topology on $\mathscr{M}^1(E)$.

Let $\mathscr{H} = \mathscr{H}^1([0, 1]; 0)$ be the vector space of all real-valued continuous functions on $[0, 1]$ which vanish at the origin and whose derivative in the sense of distributions in the open interval $[0, 1]$ belongs to $L^2[0, 1]$. Equipped with the inner product

$$(f \mid g) = \int_0^1 f'(t)g'(t)\,dt$$

the space $\mathscr{H}$ becomes a Hilbert subspace of $\mathscr{C}[0, 1]$. The corresponding Gauss measure $\gamma_{\mathscr{H}}$ on $\mathscr{C}[0, 1]$ is the Wiener-Lévy measure of Brownian motion. One can prove that $\gamma_{\mathscr{H}}$ is a Radon measure by showing that the $p_\alpha \gamma_{\mathscr{H}}$ converge narrowly with respect to A.

Because of the relation between the Gauss measures on E and the Hilbert subspaces of E it will be useful to take a closer look at the set of the latter.

2. **The Hilbert subspaces of a locally convex space.** Let Γ be a set and assume that we have on Γ two algebraic operations:

(a) a map $\Gamma \times \Gamma \to \Gamma$ called addition and denoted by $(x, y) \longmapsto x + y$,

(b) a map $\mathbf{R}_+ \times \Gamma \to \Gamma$ called multiplication by a positive number and denoted by $(\lambda, x) \longmapsto \lambda x$.

We shall say that Γ is a *convex cone* [15, p. 187] if these two operations satisfy the following axioms:

(1) addition is associative: $x+(y+z)=(x+y)+z$ for all $x, y, z \in \Gamma$;

(2) addition is commutative: $x + y = y + x$ for all $x, y \in \Gamma$;

(3) there exists an element $0 \in \Gamma$ such that $0 + x = x$ for all $x \in \Gamma$;

(4) $0x = \lambda 0 = 0$ for all $\lambda \in \mathbf{R}_+$ and $x \in \Gamma$;

(5) $1.x = x$ for all $x \in \Gamma$;

(6) $(\lambda\, \mu)\, x = \lambda\, (\mu\, x)$ for $\lambda,\ \mu \in \mathbf{R}_+$ and $x \in \Gamma$;

(7) $(\lambda + \mu)\, x = \lambda x + \mu x$ for $\lambda, \mu \in \mathbf{R}_+$ and $x \in \Gamma$;

(8) $\lambda(x + y) = \lambda x + \lambda y$ for $\lambda \in \mathbf{R}_+$ and $x, y \in \Gamma$.

Let now E be a locally convex Hausdorff space and denote by Hilb(E) the set of all Hilbert subspaces of E. We shall define two algebraic operations which will make a convex cone out of Hilb(E) [12, §2, pp. 132-138].

Let $\mathscr{H} \in$ Hilb(E). If $\lambda = 0$ we define $\lambda\mathscr{H} = \{0\}$. If $\lambda > 0$ then $\lambda\mathscr{H}$ will have the same underlying vector space as $\mathscr{H}$ but the scalar product on $\lambda\mathscr{H}$ will be defined by

$$(x \mid y)_{\lambda\mathscr{H}} = \frac{1}{\lambda}\,(x \mid y)_{\mathscr{H}}.$$

We have in particular $\|x\|_{\lambda\mathscr{H}} = \dfrac{1}{\sqrt{\lambda}}\,\|x\|_{\mathscr{H}}$. We can also say that the Hilbert structure on $\lambda\mathscr{H}$ is the image of the Hilbert structure on $\mathscr{H}$ with respect to the homothecy $x \longmapsto (\sqrt{\lambda})x$ (see below, p. 333).

The sum of two spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ belonging to Hilb(E) is more difficult to define. We take the subspace $\mathscr{H}_1 + \mathscr{H}_2$ of E and define on it the norm

$$\|h\|_{\mathscr{H}_1+\mathscr{H}_2} = \inf_{h=h_1+h_2} (\|h_1\|^2 + \|h_2\|^2)^{1/2}. \tag{1}$$

We have to show that this norm arises from a Hilbert structure on $\mathscr{H}_1 + \mathscr{H}_2$. To this end consider the Hilbert sum $\mathscr{H}_1 \oplus \mathscr{H}_2$ of the two spaces, on which, as we know, the norm is defined by

$$\|(h_1, h_2)\| = (\|h_1\|^2_{\mathscr{H}_1} + \|h_2\|^2_{\mathscr{H}_2})^{1/2}.$$

Of course $\mathscr{H}_1 \oplus \mathscr{H}_2$ is not a subspace of E. The linear map $(h_1, h_2) \longmapsto h_1 + h_2$ from $\mathscr{H}_1 \oplus \mathscr{H}_2$ into E sends $\mathscr{H}_1 \oplus \mathscr{H}_2$ onto $\mathscr{H}_1 + \mathscr{H}_2$. If $\mathscr{N}$ is the kernel of this map, then $\mathscr{H}_1 \oplus \mathscr{H}_2/\mathscr{N}$ is a Hilbert space whose norm for an element $\dot{h} \in \mathscr{H}_1 \oplus \mathscr{H}_2/\mathscr{N}$ is given by

$$\|\dot{h}\| = \inf_{(h_1,h_2)\in\dot{h}} (\|h_1\|^2_{\mathscr{H}_1} + \|h_2\|^2_{\mathscr{H}_2})^{1/2}.$$

The map $(h_1, h_2)^{\cdot} \longmapsto h_1 + h_2$ is an isometry from the normed space $\mathscr{H}_1 \oplus \mathscr{H}_2/\mathscr{N}$ onto the normed space $\mathscr{H}_1 + \mathscr{H}_2$, which proves that (1)

is a Hilbert norm. There is a unique element (h_1, h_2) belonging to the orthogonal complement $\mathcal{N}^\perp \subset \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $h_1 + h_2 = h$ and for this element we have

$$\|h\|_{\mathcal{H}_1+\mathcal{H}_2} = (\|h_1\|^2_{\mathcal{H}_1} + \|h_2\|^2_{\mathcal{H}_2})^{1/2}.$$

All the axioms of convex cones are easily checked except (7), i.e. $(\lambda + \mu)\mathcal{H} = \lambda\mathcal{H} + \mu\mathcal{H}$ for $\lambda \geqslant 0$, $\mu \geqslant 0$. To prove this formula, we may assume that $\lambda > 0$ and $\mu > 0$ since otherwise the result is trivial. Let $h \in \mathcal{H}$. An element $(h_1, h_2) \in \lambda\mathcal{H} \oplus \mu\mathcal{H}$ will belong to the orthogonal complement $\mathcal{N}^\perp$ of the kernel $\mathcal{N}$ of $(h_1, h_2) \longmapsto h_1 + h_2$ if and only if

$$(h_1 \mid k_1)_{\lambda\mathcal{H}} + (h_2 \mid k_2)_{\mu\mathcal{H}} = 0,$$

i.e.

$$\frac{1}{\lambda}(h_1 \mid k_1)_{\mathcal{H}} + \frac{1}{\mu}(h_2 \mid k_2)_{\mathcal{H}} = 0$$

for all $(k_1, k_2) \in \mathcal{H} \times \mathcal{H}$ such that $k_1 + k_2 = 0$. This condition is equivalent to

$$\left(\frac{1}{\lambda}h_1 - \frac{1}{\mu}h_2 \mid k\right) = 0 \quad \text{for all } k \in \mathcal{H},$$

i.e., to $\frac{1}{\lambda}h_1 - \frac{1}{\mu}h_2 = 0$. If now $h_1 + h_2 = h$, we obtain

$$h_1 = \frac{\lambda}{\lambda + \mu}h, \quad h_2 = \frac{\mu}{\lambda + \mu}h$$

and thus

$$\begin{aligned}
\|h\|^2_{\lambda\mathcal{H}+\mu\mathcal{H}} &= \|h_1\|^2_{\lambda\mathcal{H}} + \|h_2\|^2_{\mu\mathcal{H}} \\
&= \frac{\lambda^2}{(\lambda+\mu)^2}\|h\|^2_{\lambda\mathcal{H}} + \frac{\mu^2}{(\lambda+\mu)^2}\|h\|^2_{\mu\mathcal{H}} \\
&= \frac{\lambda}{(\lambda+\mu)^2}\|h\|^2_{\mathcal{H}} + \frac{\mu}{(\lambda+\mu)^2}\|h\|^2_{\mathcal{H}} \\
&= \frac{1}{\lambda+\mu}\|h\|^2_{\mathcal{H}} = \|h\|^2_{(\lambda+\mu)\mathcal{H}}.
\end{aligned}$$

We can also define an order relation on the set Hilb (E). If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert subspaces of E, we shall say that $\mathcal{H}_1 <$

$\mathscr{H}_2$ if $\mathscr{H}_1$ is contained in $\mathscr{H}_2$ and the canonical injection $\mathscr{H}_1 \to \mathscr{H}_2$ has norm $\leqslant 1$, i.e. $\|h\|_{\mathscr{H}_2} \leqslant \|h\|_{\mathscr{H}_1}$ for all $h \in \mathscr{H}_1$. Clearly $\lambda\mathscr{H} \leqslant \mathscr{H}$ is equivalent to $\lambda \leqslant 1$ if $\mathscr{H} \neq \{0\}$. Furthermore we have $\mathscr{H}_1 \leqslant \mathscr{H}_1 + \mathscr{H}_2$ since $h_1 + 0$ is one possible representation of $h_1 \in \mathscr{H}_1$ and therefore $\|h_1\|_{\mathscr{H}_1+\mathscr{H}_2} = \inf_{k_1+k_2=h_1} (\|k_1\|^2 + \|k_2\|^2)^{\frac{1}{2}} \leqslant \|h_1\|_{\mathscr{H}_1}$. We shall see soon that there is a close connection between the convex cone structure of Hilb (E) and the order on Hilb (E).

Let Γ_1 and Γ_2 be two convex cones. A map $f\colon \Gamma_1 \to \Gamma_2$ will be a morphism of convex cones if $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$ for $\lambda, \mu \in \mathbf{R}_+$ and $x, y \in \Gamma_1$. With these morphisms we can speak of the category $\mathscr{C}$ of convex cones.

Now let E and F be two locally convex Hausdorff spaces and $u\colon E \to F$ a continuous linear map. Let $\mathscr{H}$ be a Hilbert subspace of E and $\mathscr{N} = \operatorname{Ker}(u|\mathscr{H}) = \mathscr{H} \cap \operatorname{Ker}(u)$. The map $u|\mathscr{H}\colon \mathscr{H} \to F$ can be factored into $\mathscr{H} \to \mathscr{H}/\mathscr{N} \xrightarrow{\bar{u}} F$, where $\bar{u}$ is a continuous injective map, i.e. a bijection from $\mathscr{H}/\mathscr{N}$ onto $u(\mathscr{H})$. We shall equip $u(\mathscr{H})$ with the Hilbert structure obtained from that of $\mathscr{H}/\mathscr{N}$ with the help of the bijection $\bar{u}$. Then $u(\mathscr{H}) \in \text{Hilb}\ (F)$ and $\|k\|_{u(\mathscr{H})} = \inf_{u(h)=k} \|h\|_{\mathscr{H}}$ [12, § 8, p. 174]. It is easy to check that if we associate with each locally convex Hausdorff space E the convex cone Hilb (E), and with every continuous linear map $u\colon E \to F$ the morphism Hilb $(u)\colon \mathscr{H} \to u(\mathscr{H})$ from Hilb (E) into Hilb (F), then Hilb is a functor from the category of locally convex Hausdorff spaces into the category $\mathscr{C}$.

According to Def. 1 (§1) to each $\mathscr{H} \in \text{Hilb}(E)$ there corresponds a Gauss measure $\gamma_{\mathscr{H}}$ on E. If $\lambda \geqslant 0$, then $\gamma_{\lambda\mathscr{H}} = [\sqrt{\lambda}]\ \gamma_{\mathscr{H}}$, where $[\sqrt{\lambda}]\ \gamma_{\mathscr{H}}$ denotes the image of the measure $\gamma_{\mathscr{H}}$ with respect to the homothecy with ratio $\sqrt{\lambda}$. More generally:

PROPOSITION 1. *If $u\colon E \to F$ is a continuous linear map and $\gamma_{\mathscr{H}}$ a Gauss measure on E, then $u(\gamma_{\mathscr{H}}) = \gamma_{u(\mathscr{H})}$.*

This is an immediate consequence of the definition of the image of a cylindrical measure (Chap. 2, § 2(b)), of the definition of a

Gauss measure and of the definition of $u(\mathscr{H})$. The statement made before Prop. 1 follows from the proposition if we observe (cf. p. 331) that $\lambda\mathscr{H}$ is the image of $\mathscr{H}$ with respect to the homothecy with ratio $\sqrt{\lambda}$.

Now let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert subspaces of E. Then we have

$$\gamma_{\mathscr{H}_1+\mathscr{H}_2} = \gamma_{\mathscr{H}_1} * \gamma_{\mathscr{H}_2}.$$

Indeed, the Gauss measure on the Hilbert sum $\mathscr{H}_1 \oplus \mathscr{H}_2$ is $\gamma_{\mathscr{H}_1} \otimes \gamma_{\mathscr{H}_2}$ (p. 328). Now $\mathscr{H}_1 + \mathscr{H}_2$ is the image of $\mathscr{H}_1 \oplus \mathscr{H}_2$ with respect to the map $(h_1, h_2) \longmapsto h_1 + h_2$ (p. 331) and $\gamma_{\mathscr{H}_1} * \gamma_{\mathscr{H}_2}$ is the image of $\gamma_{\mathscr{H}_1} \otimes \gamma_{\mathscr{H}_2}$ with respect to the same map (Chap. 2, § 2 (c)). Thus formula (2) follows from Prop. 1.

Denote by $\mathrm{Cyl}(E)$ the set of all cylindrical measures on the locally convex Hausdorff space E. On the set $\mathrm{Cyl}(E)$ we define two algebraic operations:

(a) if $\mu \in \mathrm{Cyl}(E)$ and $\lambda \geqslant 0$, denote by $\lambda . \mu$ the image $[\sqrt{\lambda}]\ \mu$ of μ with respect to the homothecy $x \longmapsto (\sqrt{\lambda})x$;

(b) for $\mu, \nu \in \mathrm{Cyl}(E)$ the element $\mu * \nu \in \mathrm{Cyl}(E)$ is well defined.

With these two operations the set $\mathrm{Cyl}(E)$ is not a convex cone since axiom (7) is not satisfied. However, $\mathrm{Cyl}(E)$ is a commutative semigroup with the positive real numbers as operators.

The meaning of our preceding considerations is then that the map $\mathscr{H} \longmapsto \gamma_{\mathscr{H}}$ from $\mathrm{Hilb}(E)$ into $\mathrm{Cyl}(E)$ is compatible with the algebraic operations defined on these sets. We shall prove (p. 340) that this map is *injective* but, before we can do that, we shall have to speak about the kernels associated with the Hilbert subspaces of E.

We say that a convex cone Γ is *salient* [12, §8, p. 187] if $x + y = 0$ implies $x = y = 0$. We say that Γ is *regular* if $x + z = y + z$ implies $x = y$.

If Γ is a convex cone with vertex at the origin in a real vector space G then it is clearly a convex cone in the sense defined above. When speaking of convex cones in vector spaces, we shall always assume tacitly that their vertices are at 0. A convex cone $\Gamma \subset G$

is salient if and only if it contains no straight line going through 0, or if $\Gamma \cap (-\Gamma) = \{0\}$. A salient convex cone $\Gamma \subset G$ defines an order compatible with the vector space structure of G on G if we set $x \leqslant p$ whenever $y - x \in \Gamma$. Clearly every cone in a vector space is regular.

Conversely, let Γ be a convex cone. On the set $\Gamma \times \Gamma$ introduce an equivalence relation R by considering the pairs (x, y) and (x', y') equivalent if there exists $z \in \Gamma$ such that $x + y' + z = x' + y + z$. It is trivial to check that R is indeed an equivalence relation. Denote by G_0 the quotient set $\Gamma \times \Gamma / R$, and let $[(x, y)]$ be the image of $(x, y) \in \Gamma \times \Gamma$ in G_0. On G_0 we define an addition by setting $[(x, y)] + [(x', y')] = [(x + x', y + y')]$, and a multiplication by a scalar $\lambda \in \mathbf{R}$ by setting $\lambda[(x, y)] = [(\lambda x, \lambda y)]$ if $\lambda \geqslant 0$ and $\lambda[(x, y)] = [(\lambda y, \lambda x)]$ if $\lambda < 0$. These two operations are well defined, e.g. if $(x', y') \sim (x'', y'')$ then $x' + y'' + z = x'' + y' + z$ for some $z \in \Gamma$ and therefore also $x + x' + y + y'' + z = x + x'' + y + y' + z$, i.e., $(x + x', y + y') \sim (x + x'', y + y'')$. One can easily check that these operations satisfy the axioms of a vector space. In particular the zero element of G_0 is the equivalence class consisting of all pairs $(x, x) \in \Gamma \times \Gamma$, and the opposite of $[(x, y)]$ is $[(y, x)]$. Denote by Γ_0 the set of all elements $[(x, 0)] \in G_0$ as x varies in Γ. Clearly Γ_0 is a convex cone in G_0. Also $G_0 = \Gamma_0 - \Gamma_0$ since $[(x, y)] = [(x, 0)] - [(y, 0)]$.

The map $\phi : x \longmapsto [(x, 0)]$ is clearly a surjective morphism from Γ onto Γ_0. It is bijective if and only if Γ is regular. Indeed, Γ_0 is regular as a convex cone in the vector space G_0. Therefore if ϕ is an isomorphism, Γ must be regular. Conversely, let Γ be regular and $[(x, 0)] = [(y, 0)]$. Then there exists $z \in \Gamma$ such that $x + z = y + z$ and thus $x = y$.

Let f be a homomorphism of Γ into some vector space G, i.e. a map $f : \Gamma \to G$ such that $f(\lambda x) = \lambda f(x)$ and $f(x + y) = f(x) + f(y)$ for $\lambda \geqslant 0$, $x, y \in \Gamma$. Setting $\bar{f}[(x, 0)] = f(x)$ we obtain a homomorphism $\bar{f} : \Gamma_0 \to G$ such that $f = \bar{f} \circ \phi$. The map $\bar{f}$ is well defined since $(x, 0) \sim (y, 0)$, i.e. $x + z = y + z$ implies $f(x) + f(z) = f(y) + f(z)$, i.e. $f(x) = f(y)$. It is uniquely determined by f, and f can be extended into a unique homomorphism of vector spaces $G_0 \to G$. Thus the pair

(ϕ, G_0) is a solution of the universal problem of homomorphisms of Γ into vector spaces. In other words, the functor which associates with each vector space G the set of homomorphisms $\Gamma \to G$ is representable, and represented by the vector space G_0. The map ϕ is injective, i.e. Γ is regular, if and only if there exists an injective homomorphism from Γ into some vector space.

We shall see that Hilb(E) is a salient, regular, convex cone.

Let E be a locally convex Hausdorff space over the field $\mathbf{R}$, and let E' be the dual of E. A *kernel* with respect to E[12, §3, p. 139] is a linear map $H: E' \to E$ which is continuous with respect to the weak* topology $\sigma(E', E)$ on E' and the weak topology $\sigma(E, E')$ on E. A fortiori H is then a continuous linear map from E' equipped with the strong topology into E equipped with the original topology. If E is a vector space over $\mathbf{C}$, then a kernel is a linear map H from the anti-dual $\bar{E}'$ of E into E which is continuous for the topologies $\sigma(\bar{E}', \bar{E})$ and $\sigma(E, E')$. Since we are mainly interested in the real case, we refer to [15] for the complex case.

We say that a linear map $H: E' \to E$ is positive if

$$\langle He', e' \rangle \geqslant 0 \quad \text{for all } e' \in E'.$$

If a positive linear map $H: E' \to E$ is continuous for the strong topology on E' and the original topology on E, then it is also continuous for the topologies $\sigma(E', E)$ and $\sigma(E, E')$, i.e. it is a kernel [12, §3, Prop. 4, p. 139].

If $\mathscr{H}$ is a Hilbert space, then we have a kernel $\theta: \mathscr{H}' \to \mathscr{H}$ defined by

$$(h \mid \theta k')_{\mathscr{H}} = \langle h, k' \rangle \quad \text{for } h \in \mathscr{H},\ k' \in \mathscr{H}',$$

and by the Fréchet-Riesz theorem [3, Chap. V. §1, n° 6, Th. 3] θ is an isomorphism.

Let now $\mathscr{H}$ be a Hilbert subspace of the locally convex Hausdorff space E and $j: \mathscr{H} \to E$ the canonical injection. Its transpose ${}^t j: E' \to \mathscr{H}'$ is a continuous linear map and therefore

$$H = j \circ \theta \circ {}^t j: E' \to \mathscr{H}' \to \mathscr{H} \to E$$

is a kernel. We shall call it the kernel associated with $\mathscr{H}$. It is positive since for $e' \in E'$ we have

$$\langle He', e' \rangle = \langle j\,\theta^t_j(e'), e' \rangle = \langle \theta^t_j(e'), {}^t j(e') \rangle$$
$$= (\theta^t_j(e') \mid \theta^t j(e'))_{\mathscr{H}} = \| \theta^t_j(e') \|^2_{\mathscr{H}} \geqslant 0.$$

The kernel H is the only map from E' into $\mathscr{H}$ such that

$$(h \mid He')_{\mathscr{H}} = \langle h, e' \rangle \text{ for all } h \in \mathscr{H} \text{ and } e' \in E'. \tag{3}$$

Indeed,

$$\langle h, e' \rangle = \langle j(h), e' \rangle_{E,E'} = \langle h, {}^t j(e') \rangle_{\mathscr{H},\mathscr{H}'}$$
$$= (h \mid \theta^t j(e'))_{\mathscr{H}} = (h \mid He')_{\mathscr{H}}.$$

Furthermore, given e' the element $He' \in \mathscr{H}$ is completely determined if we know its scalar product with each $h \in \mathscr{H}$.

We have thus obtained a map $\mathscr{H} \longmapsto H$ from Hilb (E) into the set $\mathscr{L}^+(E) = \mathscr{L}^+(E'; E)$ of positive kernels with respect to E. This map is injective as follows immediately from the following proposition [9, §4, Prop. 8, p. 146]:

PROPOSITION 2. *Let $\mathscr{H}$ be a Hilbert subspace of E and H the kernel associated with $\mathscr{H}$. An element $h \in E'^*$ belongs to $\mathscr{H}$ if and only if*

$$l(h) = \sup_{e' \in E'} \frac{|\langle h, e' \rangle|}{\langle He', e' \rangle^{1/2}} < \infty.$$

PROOF. (a) If $h \in \mathscr{H}$, then

$$|\langle h, e' \rangle| = |(h \mid He')_{\mathscr{H}}| \leqslant \| h \|_{\mathscr{H}} \cdot \| He' \|_{\mathscr{H}} = \| h \|_{\mathscr{H}} \langle He', e' \rangle^{1/2}$$

by (3). Thus $l(h) \leqslant \| h \|_{\mathscr{H}}$.

(b) Let $l(h) < \infty$. The linear form $e' \longmapsto \langle h, e' \rangle$ is zero on every e' such that $He' = 0$. Thus it defines a linear form $He' \longmapsto \langle h, e' \rangle$ on $\mathscr{H}_0 = H(\bar{E}')$. Now $j: \mathscr{H} \to E$ is injective, hence $\mathscr{H}_0$ is dense in $\mathscr{H}$ [3, Chap. IV, §4, n^0 1, Prop. 3]. Furthermore $|\langle h, e' \rangle| \leqslant l(h) \langle He', e' \rangle^{1/2} = l(h) \| He' \|_{\mathscr{H}}$. There exists therefore an element $k \in \mathscr{H}$ such that

$$\langle h, e' \rangle = (k \mid He')_{\mathscr{H}} \text{ for } e' \in E'.$$

The fundamental relation (3) shows that $\langle k, e' \rangle = \langle h, e' \rangle$ for all $e' \in E'$. Thus $h = k \in \mathscr{H}$.

It can be shown furthermore that if E is quasi-complete the map $\mathscr{H} \longmapsto H$: $\mathrm{Hilb}(E) \to \mathscr{L}^+(E)$ is surjective [12, §5, Prop. 10, pp. 154-157] and that it has the following properties [12, §6, pp. 159-161].

1) Let $\mathscr{H}$ be a Hilbert subspace of E and H the associated kernel. Then for $\lambda \geqslant 0$ the kernel associated with $\lambda\mathscr{H}$ is λH.

2) Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert subspaces of E and H_1, H_2 the associated kernels. Then H_1+H_2 is the kernel associated with $\mathscr{H}_1 + \mathscr{H}_2$.

3) Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert subspaces of E and H_1, H_2 the associated kernels. Then the relation $\mathscr{H}_1 < \mathscr{H}_2$ is equivalent to $H_1 \leqslant H_2$.

The results mean that the map $\mathscr{H} \longmapsto H$ from $\mathrm{Hilb}(E)$ onto $\mathscr{L}^+(E)$ is an isomorphism for the convex cone structure on these sets and also an isomorphism of ordered sets.

It follows now easily that $\mathrm{Hilb}(E)$ is a salient and regular convex cone. Indeed, $\mathscr{L}^+(E)$ is the convex cone of the positive elements of the vector space $\mathscr{L}(E'; E)$. Therefore to show that $\mathscr{L}^+(E)$, and thus also $\mathrm{Hilb}(E)$, is salient, we have to show that $H \geqslant 0$ and $-H \geqslant 0$ imply $H = 0$. In the first place $\langle He', e\rangle \geqslant 0$ and $\langle He', e'\rangle \leqslant 0$ imply $\langle He', e'\rangle = 0$. But if $\langle He', e'\rangle = 0$ for every $e' \in E'$, then by virtue of

$$\langle He', f'\rangle = \tfrac{1}{2}\langle H(e'+f'), e'+f'\rangle + \tfrac{1}{2}\langle H(e'-f'), e'-f'\rangle$$

we have $\langle He', f'\rangle = 0$ for every $f' \in E'$, hence $He' = 0$ for every $e' \in E'$, i.e., $H = 0$.

Next we can show that $\mathscr{H}_1 < \mathscr{H}$ if and only if there exists $\mathscr{H}_2 \in \mathrm{Hilb}(E)$ such that $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$ (and $\mathscr{H}_2$ is then unique since $\mathrm{Hilb}(E)$ is a regular convex cone). Indeed $\mathscr{H}_1 < \mathscr{H}$ is equivalent to the relation $H_1 \leqslant H$ between the corresponding kernels, i.e. to the existence of $H_2 \in \mathscr{L}^+(E)$ such that $H = H_1 + H_2$.

A kernel $H: E' \to E$ defines a bilinear form B on $E' \times E'$ according to the formula

$$B(e', f') = \langle He', f'\rangle. \tag{4}$$

It is easy to show [12, §3, Prop. 5, p. 140] that a bilinear form B on $E' \times E'$ is defined with the help of a kernel H according to (4) if and only if it is separately weak *—continuous. Furthermore B is positive if and only if H is positive.

Thus to every $\mathscr{H} \in \mathrm{Hilb}(E)$ there corresponds a kernel $H \in \mathscr{L}^+(E)$ and a positive quadratic form $Q_{\mathscr{H}}$ on E' given by

$$Q_{\mathscr{H}}(e') = B(e', e') = \langle He', e' \rangle = \| He' \|^2_{\mathscr{H}},$$

where, in the last term, we have written, by abuse of notation, $\| He' \|^2_{\mathscr{H}}$ instead of $\| \theta^t j(e') \|^2_{\mathscr{H}}$. Conversely let Q be a positive quadratic form on E' such that the bilinear form

$$(e', f') \longmapsto B(e', f') = \tfrac{1}{2}\{Q(e' + f') - Q(e') - Q(f')\}$$

is separately continuous. Then there exists a Hilbert subspace $\mathscr{H} \in \mathrm{Hilb}(E)$ such that $Q = Q_{\mathscr{H}}$.

If B is the positive bilinear form corresponding to the Hilbert subspace $\mathscr{H}$ of E, then B defines a pre-Hilbert structure on E'. The completion of the associated Hausdorff space is a Hilbert space whose dual considered as a subspace of E'^* is precisely the original Hilbert subspace $\mathscr{H}$ [12, §5, second proof of Prop. 10, pp. 156-157].

Let u be a weakly continuous linear map from the locally convex Hausdorff space E into the locally convex Hausdorff space F. If $\mathscr{H} \in \mathrm{Hilb}(E)$, then we have the sequence of weakly continuous linear maps

$$F' \xrightarrow{{}^t u} E' \xrightarrow{{}^t j} \mathscr{H}' \xrightarrow{\theta} \mathscr{H} \xrightarrow{j} E \xrightarrow{u} F,$$

from where it follows easily [12, §8, Prop. 21, p. 176] that the kernel associated with $u(\mathscr{H}) \in \mathrm{Hilb}(F)$ is $u(H) = u \circ H \circ {}^t u$, where H is the kernel associated with $\mathscr{H}$. Consequently we have $Q_{u(\mathscr{H})}(f') = \langle u(H) f', f' \rangle = \langle H \circ {}^t u(f'), {}^t u(f') \rangle = Q_{\mathscr{H}}({}^t u(f'))$, i.e.

$$Q_u(\mathscr{H}) = Q_{\mathscr{H}} \circ {}^t u. \tag{5}$$

Let E be a locally convex Hausdorff space, $\mathscr{H} \in \mathrm{Hilb}(E)$ and $\gamma_{\mathscr{H}} = j(\gamma_{\mathscr{H}})$ the Gauss measure on E associated with $\mathscr{H}$. The Fourier transform of $\gamma_{\mathscr{H}}$ is then given by

$$\mathscr{F}(\gamma_{\mathscr{H}}) = e^{-\pi Q}{}_{\mathscr{H}}. \tag{6}$$

Indeed, we know (p. 329), that the Fourier transform of $\gamma_{\mathscr{H}}$ as a cylindrical measure on $\mathscr{H}$ is given by $\xi \longmapsto e^{-\pi\|\xi\|^2}$. It follows therefore from formula (2) of Chap. 2, § 2, that for $e' \in E'$ we have

$$\mathscr{F}(j(\gamma_{\mathscr{H}})(e') = \mathscr{F}(\gamma_{\mathscr{H}}) \circ {}^t j(e') = e^{-\pi\|{}^t j(e')\|^2_{\mathscr{H}}} = e^{-\pi Q}{}_{\mathscr{H}}(e').$$

We can now prove that the map $\mathscr{H} \longmapsto \gamma_{\mathscr{H}}$ from Hilb(E) into Cyl(E) is injective (p. 334). Indeed, a Gauss measure γ determines uniquely its Fourier transform $\mathscr{F}(\gamma)$ which by (6) determines uniquely the quadratic form Q on E'. But this quadratic form Q determines uniquely the kernel $H \in \mathscr{L}^+(E)$ and therefore also $\mathscr{H} \in \mathrm{Hilb}(E)$.

EXAMPLE 2. Let I be an index set and for each finite subset J of I let γ_J be the canonical Gauss measure on the Euclidean space $\mathbf{R}^J$. If $K \subset J$ and π is the orthogonal projector $\mathbf{R}^J \to \mathbf{R}^K$, then $\pi(\gamma_J) = \gamma_K$, hence by Remark 1 of Chap. 2, §1, we obtain a cylindrical measure μ on the space $\mathbf{R}^I$.

The space $\mathscr{H} = l^2(I)$ is a Hilbert subspace of $\mathbf{R}^I$ and μ is the Gauss measure γ_H on $\mathbf{R}^I$. Indeed, if j is the canonical injection $l^2(I) \to \mathbf{R}_I$ and π_J the canonical projection from $\mathbf{R}^I$ onto the finite-dimensional subspace $\mathbf{R}^J$, then the natural Euclidean structure of $\mathbf{R}^J$ coincides with the Euclidean structure obtained with the help of the map $\pi_J \circ j$. It follows from the definition of an image measure (Chap. 2, §2(b)) that $\mu = j(\gamma_H)$. The transpose of j is the canonical injection $\mathbf{R}^{(I)} \to l^2(I)$, and therefore the kernel H associated with $\mathscr{H}$ is the canonical injection $\mathbf{R}^{(I)} \to \mathbf{R}^I$.

We know from Chap. 2, §1, Example 1 that if I is uncountable, then $\check{\mu}(E'^*) = 0$ for $E'^* = E = \mathbf{R}^I$. From here we shall deduce that if $\mathscr{H}$ is any nonseparable Hilbert space and γ the canonical Gauss measure on $\mathscr{H}$, then $\check{\gamma}(\mathscr{H}'^*) = 0$. Indeed, let u: $(l^2(I))'^* \to \mathbf{R}^I$ be the non-injective map obtained by extending the canonical injection $l^2(I) \to \mathbf{R}^I$ by continuity. If I is uncountable we have $(j(\gamma))^\vee(\mathbf{R}^I) = 0$. Formula (4) of Chap. 5, §2 yields

$$\check{u}(\check{\gamma}) = (u(\gamma))^\vee = (j(\gamma))^\vee,$$

hence $\breve{\gamma}(u^{-1}(\mathbf{R}^I)) = 0$ and a fortiori $\breve{\gamma}((l^2(I))'^*) = 0$. If we take into account that every non-separable Hilbert space is of the form $l^2(I)$ with an uncountable index set I, the assertion follows.

It seems to be an open question whether, if $\mathscr{H}$ is a separable Hilbert space and γ the canonical Gauss measure on $\mathscr{H}$, we have $\breve{\gamma}(\mathscr{H}'^*) = 0$.

3. The converse of the theorems of Sazonov and Minlos.

THEOREM 1. *Let E and F be two Hilbert spaces, $u: E \to F$ a continuous linear map and γ the canonical Gauss measure on E. If u is not a Hilbert-Schmidt map (Chap. 3, §1, Def. 1) then $u(\gamma)$ is not concentrated cylindrically on the balls of F (and, in particular, $u(\gamma)$ is not a Radon measure on F).*

The proof of this theorem will follow after a few lemmas of which the first one is crucial.

LEMMA 1. *There exist two universal constants $\Lambda > 0$ and $0 < \delta < 1$ with the following property. Let $B_1 = \{x \mid \Sigma x_i^2 < 1\}$ be the uniball of $\mathbf{R}^n$ and $B_2 = \{x \mid \Sigma \alpha_i^2 x_i^2 < 1\}$, where $\Sigma \alpha_i^2 = S^2$. If γ is the Gauss measure on $\mathbf{R}^n$ with B_1 as unit ball and if $S > \Lambda$, then γ is not concentrated up to δ on B_2. i.e. $\gamma(\mathbf{C}\, B_1) > \delta$.*

PROOF. For every $t \in \mathbf{R}$ we have

$$\int_{\mathbf{R}^n} (1 - e^{-\pi t^2 \Sigma \alpha_i^2 x_i^2})\, e^{-\pi \Sigma x_i^2}\, dx$$

$$= 1 - \int_{\mathbf{R}^n} e^{-\pi \Sigma (1 + t^2 \alpha_i^2 x_i^2)}\, dx$$

$$= 1 - \prod_{i=1}^{n} \frac{1}{\sqrt{1 + t^2 \alpha_i^2}}.$$

On the other hand

$$\int_{\mathbf{R}^n} \left(1 - e^{-\pi t^2 \Sigma \alpha_i^2 x_i^2}\right) e^{-\pi \Sigma x_i^2}\, dx$$

$$= \left(\int_{B_2} + \int_{CB_2} \right) \left(1 - e^{-\pi t^2 \Sigma \alpha_i^2 x_i^2}\right) e^{-\pi \Sigma x_i^2} \, dx$$

$$< 1 - e^{-\pi t^2} + \gamma(CB_2).$$

If follows that

$$\gamma(CB_2) > e^{-\pi t^2} - \prod_{i=1}^{n} \frac{1}{\sqrt{(1+t^2 \alpha_i^2)}}$$

for every $t \in \mathbf{R}$. Let us introduce the new parameter τ by setting $\tau = St$. Since $\alpha_i < S$ we have $\alpha_i t < \tau$ for $t, \tau > 0$ and $1 < i < n$. From the inequality

$$\log (1 + u) > u - \frac{u^2}{2} = u(1 - \tfrac{1}{2} u)$$

we deduce

$$\tfrac{1}{2} \log (1 + t^2 \alpha_i^2) > \tfrac{1}{2} t^2 \alpha_i^2 (1 - \tfrac{1}{2} t^2 \alpha_i^2) > \tfrac{1}{2} t^2 \alpha_i^2 (1 - \tfrac{1}{2} \tau^2)$$

and therefore

$$\log \prod_{i=1}^{n} \sqrt{(1 + t^2 \alpha_i^2)} > \tfrac{1}{2} t^2 S^2 (1 - \tfrac{1}{2} \tau^2) = \tfrac{1}{2} \tau^2 (1 - \tfrac{1}{2} \tau^2).$$

Thus we obtain

$$\gamma(C\, B_2) > e^{-\pi \tau^2 / S^2} - e^{-\frac{1}{2} \tau^2} (1 - \tfrac{1}{2} \tau^2).$$

Now the function $\tau \longmapsto e^{-\frac{1}{2}\tau^2(1-\frac{1}{2}\tau^2)}$ attains its absolute minimum at $\tau = 1$ and therefore the strongest consequence of the preceding inequality is

$$\gamma(C\, B_2) > e^{-\pi/S^2} - e^{-1/4}.$$

The right hand side tends increasingly to $1 - e^{-1/4}$ as $S \to \infty$. Thus for every $\delta < 1 - e^{-1/4}$ we can determine a $\Lambda > 0$ such that $S > \Lambda$ implies $e^{-\pi/S^2} - e^{-1/4} > \delta$.

LEMMA 2. *Let Λ and δ be the two universal constants which figure in Lemma* 1. *Let E_1 and E_2 be two finite-dimensional Hilbert spaces and $u: E_1 \to E_2$ a linear bijection. Assume that the Hilbert-Schmidt norm of u satisfies the inequality $\|u\|_2 > \Lambda$. If γ_1 is the Gauss measure on E_1, then $u(\gamma_1)$ is not concentrated up to δ on the unit ball of E_2.*

PROOF. (cf. the proof of Lemma 2, Chap. 3, §3). Let $(e_i)_{1\leqslant i\leqslant n}$ be an orthonormal basis of E_1 such that the vectors $u(e_i)$ are orthogonal in E_2, and set $\alpha_i = \|u(e_i)\|$. For any $x = \Sigma x_i e_i \in E_1$ we have

$$\|u(x)\|^2 = \Sigma x_i^2 \|u(e_i)\|^2 = \Sigma \alpha_i^2 x_i^2.$$

On the other hand

$$\Sigma \alpha_i^2 = \Sigma \|u(e_i)\|^2 = \|u\|_2^2 > \Lambda^2.$$

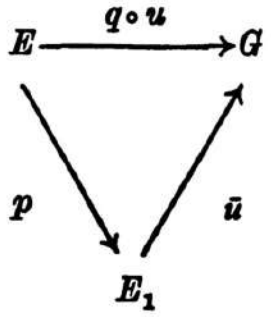
The map $(x_1, \ldots, x_n) \longmapsto \Sigma x_i e_i = x$ is an isomorphism of the Hilbert spaces $\mathbf{R}^n$ and E_1 and the linear bijection $(x_1, \ldots, x_n) \longmapsto u(x)$ sends the set $B_2 = \{(x_i) \mid \Sigma \alpha_i^2 x_i^2 < 1\}$ onto the unit ball of E_2. The assertion now follows from Lemma 1.

LEMMA 3. *Again let Λ and δ be the two universal constants introduced in Lemma* 1. *Let E and F be two Hilbert spaces and $u: E \to F$ a continuous linear map such that $\Lambda < \|u\|_2 < \infty$, where $\|u\|_2 = \infty$ means that u is not of Hilbert-Schmidt type. If γ is the Gauss measure on E, then $u(\gamma)$ is not concentrated cylindrically up to δ on the unit ball of F.*

PROOF. Let $(e_i)_{i\in I}$ be an orthonormal basis of E. We have by assumption $\|u\|_2^2 = \sum_{i\in I} \|u(e_i)\|^2 > \Lambda^2$. There exists a finite subset J of I such that $\sum_{i\in J} \|u(e_i)\|^2 > \Lambda^2$. Let G be the finite-dimensional subspace of F generated by the vectors $u(e_i)$ with $i \in J$, and let $q: F \to G$ be the orthogonal projector. For the composed map $q \circ u$: $E \to F \to G$ we have

$$\|q \circ u\|_2^2 = \sum_{i\in I} \|q(u(e_i))\|^2 > \sum_{i\in J} \|q(u(e_i))\|^2 > \Lambda^2.$$

Let E_1 be the orthogonal complement of $\mathrm{Ker}(q \circ u)$ and let $p: E \to E_1$ be the orthogonal projection. We have the commutative diagram



where $\bar{u}$ is a bijection. Furthermore $\|\bar{u}\|_2 = \|q \circ u\|_2 > \Lambda$ and $p(\gamma)$ is the Gauss measure on E_1. By Lemma 2 the measure $\bar{u}(p(\gamma))$ is not concentrated up to δ on the unit ball of G. Since $\bar{u}(p(\gamma)) = q(u(\gamma))$, this means that $u(\gamma)$ is not concentrated cylindrically up to δ on the unit ball of F.

PROOF OF THEOREM 1. Let B be a ball in F. If ρ is the radius of B, then the Hilbert space $\rho^2 F = F_B$ has B as its unit ball. Since $\|u\|_2 = \infty$ the map u considered as a map from E into F_B has an infinite Hilbert-Schmidt norm. It follows from Lemma 3 that $u(\gamma)$ is not concentrated up to δ on B. Since B is an arbitrary ball in F, the theorem is proved.

Next we prove the converse of Theorem 2 of Chap. 4, §3.

THEOREM 2. *Let E be a locally convex Hausdorff space and let $\mathfrak{S}$ be a saturated collection of Hilbert subsets of E. If every cylindrical measure on E which is concentrated scalarly on the sets of $\mathfrak{S}$ (Chap. 2, §3, Def. 1) is concentrated cylindrically on the sets of $\mathfrak{S}$, then E is $\mathfrak{S}$-conuclear.*

PROOF. Suppose that E is not $\mathfrak{S}$-conuclear. Then according to Prop. 6 of Chap. 4, §2 there exists a Hilbert set $A \in \mathfrak{S}$ such that for no Hilbert set $B \in \mathfrak{S}$ containing A is the canonical map $E_A \to E_B$ of Hilbert-Schmidt type. Let γ_A be the canonical Gauss measure on E_A. It defines a Gauss measure $j_A(\gamma_A)$ on E, where $j_A: E_A \to E$ is the canonical injection. The cylindrical measure $j_A(\gamma_A)$ is scalarly concentrated on the sets of $\mathfrak{S}$ since γ_A is concentrated scalarly on the balls of E_A (p. 327). We shall show that $j_A(\gamma_A)$ is not concentrated cylindrically on the sets of $\mathfrak{S}$, which is a contradiction to the hypothesis of the theorem.

Let δ be the constant which figures in Lemmas 1–3 and assume that there exists some set $B \in \mathfrak{S}$ such that $j_A(\gamma_A)$ is concentrated cylindrically up to δ on B. There exists a Hilbert set $B' \in \mathfrak{S}$ such that $A \subset B'$ and $B \subset B'$. Since the canonical map $j_{B',A}: E_A \to E_{B'}$ is not of Hilbert-Schmidt type, it follows from Lemma 3 that $j_{B',A}(\gamma_A)$ is not concentrated cylindrically up to δ on the unit ball B' of $E_{B'}$.

We can now apply Corollary 2 of Theorem 3 of Chap. 2, §3 to the case $F = E_{B'}$, $D = B'$, $j = j_{B'}: E_{B'} \to E$, $\mu = j_{B',A}(\gamma_A)$ and $\epsilon = \delta$ since B' is a Hilbert set and therefore weakly compact. By assumption $j(\mu) = j_{B'}(j_{B',A}(\gamma_A)) = j_A(\gamma_A)$ is concentrated cylindrically up to δ on B and thus a fortiori on B'. Hence by the quoted result $j_{B',A}(\gamma_A)$ is concentrated cylindrically up to δ on B', which contradicts what we have said above.

COROLLARY 1. *Let E be a quasi-complete locally convex Hausdorff space and assume that every balanced, convex, compact subset of E is contained in a balanced, convex, compact Hilbert set. If every cylindrical measure on E which is concentrated scalarly on the balanced, convex, compact subsets of E is a Radon measure, then E is cc-conuclear* (p. 233) (*or c-conuclear*).

PROOF. Every cylindrical measure concentrated scalarly on the balanced, convex, compact sets is then in particular cylindrically concentrated on the compact sets, hence also on the balanced, convex, compact Hilbert sets, since E is quasi-complete.

Taking for $\mathfrak{S}$ the collection of all balanced, convex, compact Hilbert subsets of E, the conditions of Theorem 2 are satisfied.

COROLLARY 2. *Let E be a quasi-complete locally convex Hausdorff space and assume that every balanced, convex, compact subset of E is contained in a balanced, convex, compact Hilbert set. If every function of positive type* (p. 187), *continuous on E'_{cc}* (p. 233), *is the Fourier transform of a Radon measure on E, then E is cc-conuclear* (*or c-conuclear*).

This follows from Theorem 1 of Chap. 2, §3 and from Corollary 1.

The methods we used to prove Theorem 2 of Chap. 4, §3 and Theorem 2 of the present section yield also a result which is more general in appearance. Let E and F be two locally convex Hausdorff spaces, $\mathfrak{S}$ a saturated collection of Hilbert subsets of E and let $\mathfrak{T}$ be a similar collection of subsets of F. We say that a linear map $u: E \to F$ is an $\mathfrak{S}$-$\mathfrak{T}$-Hilbert-Schmidt map if for every Hilbert set $A \in \mathfrak{S}$ there exists a Hilbert set $B \in \mathfrak{T}$ such that u maps

E_A into F_B and such that the corresponding map $E_A \to F_B$ is of Hilbert-Schmidt type.

PROPOSITION 3. (a) *If the continuous linear map $u: E \to F$ is an $\mathfrak{S}$-$\mathfrak{T}$-Hilbert-Schmidt map, and μ a cylindrical measure on E scalarly concentrated on the sets of $\mathfrak{S}$, then $u(\mu)$ is a Radon measure on F, cylindrically concentrated on the sets of $\mathfrak{T}$.*

(b) *Conversely, if $u: E \to F$ is a continuous linear map such that $u(A) \in \mathfrak{T}$ for every $A \in \mathfrak{S}$, and for every cylindrical measure μ on E, scalarly concentrated on the sets of $\mathfrak{S}$, the image $u(\mu)$ is concentrated cylindrically on the sets of $\mathfrak{T}$, then u is a $\mathfrak{S}$-$\mathfrak{T}$-Hilbert-Schmidt map.*

PROOF. (a) Assume that $u: E \to F$ is an $\mathfrak{S}$-$\mathfrak{T}$-Hilbert-Schmidt map, and let μ be a cylindrical measure on E, scalarly concentrated on the sets of $\mathfrak{S}$. Given $\epsilon > 0$, set $\eta = S = \epsilon/C$, where C is the universal constant of Lemma 1 of Chap. 4, §3. By assumption there exists a Hilbert set $A \in \mathfrak{S}$ such that μ is concentrated scalarly up to η on A, and a Hilbert set $B \in \mathfrak{T}$ such that the map $u: E_A \to F_B$ is of Hilbert-Schmidt type. Since $\mathfrak{T}$ is invariant with respect to homothecies, we can choose B so large that the Hilbert-Schmidt norm of u is $\leqslant S$. By Lemma 2 of Chap. 4, §3 the cylindrical measure $u(\mu)$ is concentrated cylindrically up to $\epsilon = C \operatorname{Max}(\eta, S)$ on B. Still by the same lemma, for any $\epsilon' > \epsilon$ the cylindrical measure $u(\mu)$ is concentrated cylindrically up to ϵ' on a compact subset of F_B, hence a fortiori on a compact subset of F. Thus $u(\mu)$ is a Radon measure.

(b) Assume now that the continuous linear map $u: E \to F$ satisfies $u(\mathfrak{S}) \subset \mathfrak{T}$ and that it transforms every cylindrical measure μ scalarly concentrated on the sets of $\mathfrak{S}$ into a cylindrical measure $u(\mu)$ cylindrically concentrated on the sets of $\mathfrak{T}$. Let A be a Hilbert set in $\mathfrak{S}$, and let γ_A be the canonical Gauss measure on the Hilbert space E_A. It defines a Gauss measure $j_A(\gamma_A)$ on E, where $j_A: E_A \to E$ is the canonical injection, and $j_A(\gamma_A)$ is scalarly concentrated on the sets of $\mathfrak{S}$, since γ_A is scalarly concentrated on the balls of E_A. Let $\delta > 0$ be the constant of Lemmas 1–3. By

our assumption, there exists a Hilbert set $B \in \mathfrak{T}$ such that $u(\gamma_A)$ is cylindrically concentrated up to δ on B.

Now $u(A)$ belongs to $\mathfrak{T}$ and since $\mathfrak{T}$ is increasingly directed, there exists a Hilbert set $B' \in \mathfrak{T}$ which contains both $u(A)$ and B. If the map $u: E_A \to F_{B'}$ were not of Hilbert-Schmidt type, then it would follow from Lemma 3 that $u(\gamma_A)$ is not concentrated cylindrically up to δ on B'. On the other hand, we can apply Corollary 2 of Theorem 3 of Chap. 2, § 3 taking $F_{B'}$ for F, F for E, the canonical injection $F_{B'} \to F$ for j, B' for D, $u(\gamma_A)$ for μ and δ for ϵ. Since the cylindrical measure $u(\gamma_A)$ on F is concentrated cylindrically up to δ on B, hence a fortiori on B', it follows from the quoted corollary that the cylindrical measure $u(\gamma_A)$ on $F_{B'}$ is concentrated cylindrically up to δ on the unit ball B' of $F_{B'}$. Therefore the map $u: E_A \to F_{B'}$ must be of Hilbert-Schmidt type, i.e. u is an $\mathfrak{S}$-$\mathfrak{T}$-Hilbert-Schmidt map.

4. The analogue of Sazonov's theorem for l^p spaces $(0<p<+\infty)$, and applications. Let $\alpha = (\alpha_n)_{n\in\mathbf{N}}$ be a sequence of real numbers. Then the multiplication $(x_n)_{n\in\mathbf{N}} \longmapsto (\alpha_n x_n)_{n\in\mathbf{N}}$ operates from l^p into l^q under suitable conditions on the sequence α. When is this operation radonifying or decomposing? The radonifying cases have been completely solved, and will be found in [26]. Let us simply give here one case, without proof, in order to give some applications.

THEOREM 3. *In order that the map $x \longmapsto \alpha x$ be radonifying (relatively to the bounded subsets of l^∞) from $\sigma(l^\infty, l^1)$ into l^1, it is necessary and sufficient that*

$$\sum_{n\geq 0} |\alpha_n| \left(1 + \left|\log \frac{1}{|\alpha_n|}\right|\right) < +\infty. \tag{1}$$

For the proof, see [26].

Now let us apply Theorem 12 of Chap. 5, § 4 and Remark 8 following it with $(U, U') = (l^\infty, l^1)$ and $(V, V') = (l^1, l^\infty)$. The spaces l^1 and $\sigma(l^\infty, l^1)$ are Suslin, therefore we have

COROLLARY. *The map $x \longmapsto \alpha x$ from $\sigma(l^\infty, l^1)$ into l^1 is decomposing or measurably decomposing with respect to the topology of l^1, if*

and only if $\sum_{n\geq 0} |\alpha_n| \left(1 + \left|\log \frac{1}{|\alpha_n|}\right|\right) < +\infty$.

Before giving applications to general radonifying and decomposing maps, let us give some complements concerning completing subsets of a locally convex Hausdorff topological vector space (Def. p. 230). Of course the completing subsets of E and $\sigma(E, E')$ are the same, because the bounded subsets are the same. Now let us prove that the completing subsets are the same also for the weak topology $\sigma(E', E)$ of E' and the strong topology of the dual E'. Indeed, let $\mathscr{B} \to E'$ be a linear map from a Banach space $\mathscr{B}$ into E', continuous from $\mathscr{B}$ into E'_σ. Its transpose is continuous from $\sigma(E, E')$ into $\sigma(\mathscr{B}', \mathscr{B})$, hence the map is also continuous from $\mathscr{B}$ into E' with the strong topology, which proves our affirmation. As an example, the equicontinuous subsets of E' are always completing; and therefore also the bounded subsets of E' if E is barrelled.

PROPOSITION 4. *Let $(e_n)_{n\in\mathbf{N}}$ be a sequence of elements of E. It is completing if and only if, for every sequence $c = (c_n)_{n\in\mathbf{N}} \in l^2$, the series $\sum_n c_n e_n$ is convergent (or summable) in E; or if and only if it is contained in the image, by some continuous linear map, of the unit ball of the space l^1.*

PROOF. (Observe that this result is a complement, relative to the case $p = \infty$ of Theorem 8 of Chap. 5, § 3, which concerned only quasi-complete spaces).

(a) If $(e_n)_{n\in\mathbf{N}}$ is contained in the image of the unit ball of l^1 by a continuous linear map, it is completing by definition, l^1 being a Banach space.

(b) Assume now that the sequence is completing, therefore contained in the image $u(B)$ of the unit ball B of a Banach space $\mathscr{B}$ by a continuous linear map u: $e_n = u(b_n)$, $b_n \in B$. Then, if $(c_n)_{n\in\mathbf{N}} \in l^1$, the series $\sum_n c_n b_n$ is summable in $\mathscr{B}$, therefore its image $\sum_n c_n e_n$ is summable in E.

(c) Finally, assume that, for every sequence $(c_n)_{n\in\mathbf{N}} \in l^1$, the series $\sum_n c_n e_n$ converges. It defines a linear map u: $(c_n)_{n\in\mathbf{N}} \longmapsto \sum_n c_n e_n$ from l^1 into E; this map u is a pointwise limit of the truncated maps, which are continuous, and u is therefore continuous since, l^1 is a Baire space (exactly as in the proof of Th. 8 of Chap. 5 of § 4); $(e_n)_{n\in\mathbf{N}}$ is contained in the image of the unit ball of l^1 by the continuous linear map u: e_n is the image $u(\epsilon_n)$ of the element ϵ_n of l^1 where $\epsilon_n = (0, 0\ldots, 0, 1, 0, \ldots)$ (the nth term being 1).

THEOREM 4. *Let E, F be two locally convex Hausdorff vector spaces, u a weakly continuous linear map $E \to F$, which can be written in the form*

$$u = \sum_{n\in\mathbf{N}} \alpha_n e'_n \otimes f_n,$$

where the sequences $(e'_n)_{n\in\mathbf{N}}$ and $(f_n)_{n\in\mathbf{N}}$ are completing in E' and F respectively, and $\sum_{n\in\mathbf{N}} |\alpha_n|(1 + \log|\frac{1}{|\alpha_n|}|) < +\infty$. Then u is radonifying, with respect to the bounded subsets of E, and measurably decomposing, with respect to the strong topology of E'. The transpose ${}^t u$ is radonifying from $\sigma(F', F)$ into E', with respect to the bounded subsets of $\sigma(F', F)$, and measurably decomposing with respect to the topology of F. The condition on the α_n's is the best possible.

PROOF. First of all, u is well defined. Namely, if $e \in E$, the sequence $(\langle e'_n, e\rangle)_{n\in\mathbf{N}}$ is bounded; thus the sequence $(\alpha_n \langle e'_n, e\rangle)_{n\in\mathbf{N}}$ is l^1, and since the sequence $(f_n)_{n\in\mathbf{N}}$ is completing, $\sum_{n\in\mathbf{N}} \alpha_n \langle e'_n, e\rangle f_n$ converges to an element $u(e)$ in F (Prop. 4). Now u is weakly continuous. Namely, if $f' \in F'$, $\langle u(e), f'\rangle = \sum_{n\in\mathbf{N}} \alpha_n \langle f', f_n\rangle \langle e'_n, e\rangle$; the sequence $(\langle f', f_n\rangle)_{n\in\mathbf{N}}$ is bounded, the sequence $(\alpha_n \langle f', f_n\rangle)_{n\in\mathbf{N}}$ is l^1, therefore $\sum_{n\in\mathbf{N}} \alpha_n \langle f', f_n\rangle e'_n$ converges in E' to an element η' because $(e'_n)_{n\in\mathbf{N}}$ is completing, and therefore, if e converges weakly to 0 in E, then $\langle u(e), f'\rangle = \langle \eta', e\rangle$ converges to 0, which means that $u(e)$ converges to 0 weakly in F, and u is weakly continuous. Now u can be factorized as

$$E \xrightarrow{v} \sigma(l^\infty, l^1) \xrightarrow{\alpha} l^1 \xrightarrow{w} F.$$

The map v: $E \to \sigma(l^\infty, l^1)$ is given by $e \longmapsto (\langle e'_n, e\rangle)_{n\in\mathbf{N}}$; it is weakly continuous, just by the same reasoning as before, using the fact that $(e'_n)_{n\in\mathbf{N}}$ is completing. The second map is $(c_n)_{n\in\mathbf{N}} \longmapsto (\alpha_n c_n)_{n\in\mathbf{N}}$; according to the hypothesis concerning the α_n's, it is radonifying and decomposing (Theorem 3 and corollary). The third map w is $(t_n)_{n\in\mathbf{N}} \longmapsto \sum_{n\in\mathbf{N}} t_n f_n$, continuous from l^1 into F since the sequence $(f_n)_{n\in\mathbf{N}}$ is completing.

The map $\sigma(l^\infty, l^1) \to l^1$ being radonifying with respect to the bounded subsets of $\sigma(l^\infty, l^1)$, u is radonifying with respect to the bounded subsets of E. The map $\sigma(l^\infty, l^1) \to l^1$ being measurably decomposing with respect to the topology l^1 of the dual of $\sigma(l^\infty, l^1)$, u is measurably decomposing with respect to the strong topology E' of the dual of E (since the transpose $v: l^1 \to E'$ of the map v is strongly continuous).

Now if we consider the transpose ${}^t u$ of u, it has exactly the same decomposition, as a map $\sigma(F', F) \to E'$; therefore ${}^t u$ is radonifying from $\sigma(F', F)$ into E' with respect to the bounded subsets of $\sigma(F', F)$, and measurably decomposing, with respect to the strong topology of the dual of $\sigma(F', F)$, a fortiori with respect to the given topology of F.

The condition on the α_n's is the best possible, according to Theorem 3 and its corollary.

5. An application to Brownian motion†. Let X be a complete metric space and d the distance function on X. We denote by T the interval $[0,1]$ of the real line. Let μ be a projective system of probabilities on X^T. This means that for every finite subset A of T we are given a probability μ_A on X^A, so that for $A \subset B$ we have $\pi_{A,B}(\mu_B) = \mu_A$, where $\pi_{A,B}$ is the canonical projection $X^B \to X^A$. We shall give a sufficient condition for μ to be a Radon measure

†See analogous methods in Simone Chevet [24].

on X^T concentrated on the set of continuous maps from T into X.

It will be convenient to state separately some notations and conditions which will be common to Theorems 5 and 6 and their corollary.

HYPOTHESES. *We shall denote by ϕ and ψ two positive, continuous, increasing functions defined in $[0, \tau]$, where τ is some number $0 < \tau < 1$. We assume that $\phi(0) = \psi(0) = 0$ and that ψ is concave from below.*

(ϵ_n) and (η_n) will be two decreasing sequences of strictly positive numbers defined for $n > n_0$, and tending to zero as $n \to \infty$. We assume that η_n is an integral multiple of $1/2^n$ and that $\eta_n < \tau$. Since this implies in particular that $1/2^n < \eta_n < \tau$, we see why η_n can be defined only for indices n larger than an appropriate n_0.

We assume furthermore that the function ϕ and ψ, the sequences (ϵ_n) and (η_n) satisfy the following relations:

$$\epsilon_n = \phi(\eta_n), \tag{1}$$

$$\frac{1}{2^n} = o(\eta_n) \quad as\ n \to \infty, \tag{2}$$

$$\sum_{r \in \mathbf{N}} \psi\left(\frac{1}{2^{n+r}}\right) = o(\psi(\eta_{n+1})) \quad as\ n \to \infty, \tag{3}$$

$$\sum_{n > n_0} 2^{2n} \eta_n \epsilon_n < \infty. \tag{4}$$

THEOREM 5. *Let μ be a projective system of probabilities on X^T. Let us suppose that for every pair of points $t, t' \in T$ such that $|t' - t| < \tau$ we have*

$$\mu_{(t,t')}(\{(u, v) \in X \times X \mid d(u, v) > \psi(|t' - t|)\}) < \phi(|t' - t|). \tag{5}$$

Then μ is a Radon measure on X^T concentrated on the set of continuous maps from T into X. Furthermore, given an arbitrary $a > 1$, for almost every function $f \in X^T$ with respect to the measure μ, the Hölder condition

$$d(f(t), f(t')) < a\psi(|t'-t|) \tag{6}$$

is satisfied for sufficiently small $|t'-t|$.

We observe that (6) is not necessarily true for $a = 1$, and that the "sufficiently small" depends on f and on a.

In terms of random functions the theorem can be restated as follows

THEOREM 6. *Let* $f: T \to \text{Meas}(\Omega, m; X)$ *be a canonical random function and* $\check{f}: \Omega \to \check{X}^T$ *the associated random variable such that* $f(t) = \pi_t \circ \check{f}$ *for every* $t \in T$. *Let us suppose that for every pair of points* $t \in T$ *and* $t' \in T$ *such that* $|t'-t| < \tau$ *we have*

$$m(\{\omega \in \Omega \mid d(f(t)(\omega), f(t')(\omega)) > \psi(|t'-t|)\}) < \phi(|t'-t|). \tag{7}$$

Then f *is almost surely continuous. Furthermore, given an arbitrary* $a > 1$, *the random function* f *satisfies almost surely the Hölder condition*

$$d(f(t), f(t')) < a\psi(|t'-t|) \tag{8}$$

for sufficiently small $|t'-t|$.

The last condition means that for almost every $\omega \in \Omega$ with respect to m, there exists a number $\eta = \eta(\omega, a) > 0$ such that $|t'-t| < \eta$ implies

$$d(f(t)(\omega), f(t')(\omega)) < a\psi(|t'-t|).$$

To show that the two statements are equivalent, let us first assume that Theorem 5 is true. With the notation of Theorem 6 set $\check{\mu} = \check{f}(m)$, and for every finite subset $A = \{t_1, \ldots, t_p\}$ of T set $\mu_A = (f(t_1), \ldots, f(t_p))(m)$. The measures μ_A form a projective system of probabilities on X^T whose projective limit is the measure $\check{\mu}$ on $\check{X}^T$, and conditions (5) and (7) mean the same thing. Hence for almost every $\omega \in \Omega$ the function $t \longmapsto \check{f}(\omega)(t) = f(t)(\omega)$ is continuous and satisfies the equivalent conditions (6) and (8).

Conversely, assume that Theorem 6 is true. Choose $\Omega = \check{X}^T$, take for m the projective limit $\check{\mu}$ of the probabilities μ_A, and let $\check{f}$ be

the identity map $\check{X}^T \to \check{X}^T$. The elements ω of Ω are the maps $t \longmapsto \omega(t)$ from T into $\check{X}$, and for each $t \in T$ and $\omega \in \Omega$ we have $f(t)(\omega) = \pi_t \circ \check{f}(\omega) = \omega(t)$. Conditions (5) and (7) are again equivalent, and if they are satisfied, then for almost every ω with respect to $\check{\mu}$, the function $t \longmapsto \omega(t)$ maps T continuously into X and satisfies the Hölder condition (6).

Theorem 5 can be proved with the help of Prokhorov's theorem, without using random functions (cf. Remark 1 below). It is, however, simpler to prove Theorem 6, which we shall do after some preliminary considerations.

For every $n \in \mathbf{N}$ we denote by Δ_n the set $\left\{ \frac{k}{2^n} \,\middle|\, 0 < k < 2^n \right\}$ of dyadic numbers with denominator 2^n in T, and by Δ the set $\bigcup_{n \in \mathbf{N}} \Delta_n$ of all dyadic numbers in T.

Let $f: T \to \text{Meas}\,(\Omega, m; X)$ be a random function satisfying the conditions of Theorem 6. The restriction f_Δ of f to Δ is a canonical random function satisfying (7) for $t, t' \in \Delta$ and $|t' - t| < \tau$. If $\check{f}_\Delta : \Omega \to \check{X}^\Delta$ is the corresponding random variable, then for m-almost every $\omega \in \Omega$ the value $\check{f}_\Delta(\omega)$ belongs to X^Δ since Δ is countable.

For every $t \in T$ let us denote by t_n the smallest dyadic number in Δ_n such that $t_n \geqslant t$. The sequence (t_n) is decreasing and converges to t as $n \to \infty$.

LEMMA 4. *For m-almost every $\omega \in \Omega$ and for every $t \in T$ the sequence $(\check{f}_\Delta(\omega)(t_n))$ converges in X as $n \to \infty$.*

PROOF. For each $n \in \mathbf{N}$ we have $t_n - t_{n+1} = 0$ or $1/2^{n+1}$, and therefore $t_n - t_{n+1} \leqslant \eta_{n+1}$ for $n \geqslant n_0$, and more particularly $t_n - t_{n+1} < \tau$. It follows from (7) that for $n \geqslant n_0$ we have

$$m\left(\left\{ \omega \in \Omega \mid d(\check{f}_\Delta(\omega)(t_{n+1}), \check{f}_\Delta(\omega)(t_n)) > \psi\left(\frac{1}{2^{n+1}}\right) \right\}\right)$$

$$< \phi\left(\frac{1}{2^{n+1}}\right) < \phi(\eta_n) = \epsilon_n.$$

As t varies in T, there exist only finitely many pairs (t_n, t_{n+1}) such that $t_n \in \Delta_n$, $t_{n+1} \in \Delta_{n+1}$, $0 < t_n - t_{n+1} < 1/2^{n+1}$, and the number of those pairs of this kind for which $t_{n+1} < t_n$ is the number of non-zero elements of Δ_n, i.e. 2^n. Consequently the set of those points $\omega \in \Omega$ for which there exists a point $t \in T$ such that

$$d(\check{f}_\Delta(\omega)\,(t_{n+1}), \check{f}_\Delta(\omega)\,(t_n)) > \psi\left(\frac{1}{2^{n+1}}\right)$$

has m-measure $< 2^n \epsilon_n$. From the inequality $2^n \epsilon_n < 2^{2n} \eta_n \epsilon_n (n > n_0)$ and from hypothesis (4) it follows that the series $\sum_{n \geqslant n_0} 2^n \epsilon_n$ converges. Since

$$m\left(\left\{\omega \in \Omega \mid \exists\, n \geqslant N, \exists\, t \in T,\ d(\check{f}_\Delta(\omega)\,(t_{n+1}), \check{f}_\Delta(\omega)\,(t_n)) > \psi\left(\frac{1}{2^{n+1}}\right)\right\}\right) < \sum_{n \geqslant N} 2^n \epsilon_n, \tag{9}$$

we see that the left-hand side of (9) is as small as we wish, provided we take N sufficiently large. Therefore the set

$$\bigcap_{N \in \mathbf{N}} \left\{\omega \in \Omega \mid \exists n > N,\ \exists t \in T,\ d(\check{f}_\Delta(\omega)\,(t_{n+1}), \check{f}_\Delta(\omega)\,(t_n)) > \psi\left(\frac{1}{2^{n+1}}\right)\right\}$$

has m-measure zero and consequently for m-almost every $\omega \in \Omega$ there exists an integer $N = N(\omega)$ such that for every $n > N$ and every $t \in T$ we have

$$d(\check{f}_\Delta(\omega)\,(t_{n+1}), \check{f}_\Delta(\omega)\,(t_n)) < \psi\left(\frac{1}{2^{n+1}}\right).$$

For any such $\omega \in \Omega$ and for any $t \in T$ the sequence $(\check{f}_\Delta(\omega)\ (t_n))_{n \in \mathbf{N}}$ is a Cauchy sequence, since for $n > N$ and $p \in \mathbf{N}$ we have by (3)

$$d(\check{f}_\Delta(\omega)\,(t_{n+p}), \check{f}_\Delta(\omega)\,(t_n)) < \sum_{\nu \geqslant 1} \psi\left(\frac{1}{2^{n+\nu}}\right) < C\,\psi(\eta_{n+2}),$$

and $(\psi(\eta_n))$ tends to zero as $n \to \infty$. Since X is a complete metric space, the sequence $(\check{f}_\Delta(\omega)\ (t_n))$ has a limit in X.

For any $\omega \in \Omega$ which does not belong to the exceptional set of m-measure zero, and for any $t \in T$, let us denote by $\check{g}(\omega)\,(t)$ the limit

of $(\check{f}_\Delta(\omega)\,(t_n))$ as $n \to \infty$. It yields a map $\check{g}\colon \omega \longmapsto \check{g}(\omega)$ from Ω into X^T, defined m-almost everywhere on Ω. The relation between $\check{g}$ and $\check{f}$ will be studied in the proof of Theorem 6 below.

LEMMA 5. *For m-almost every $\omega \in \Omega$ there exists an integer $M = M(\omega)$ such that for all integers $n \geqslant M$ and for all $t, t' \in \Delta_n$ such that $|t' - t| < \eta_n$ we have*

$$d(\check{f}_\Delta(\omega)\,(t), \check{f}_\Delta(\omega)\,(t')) < \psi\,(|t' - t|). \tag{10}$$

PROOF. If $n > n_0$ then by (7) for any pair of points $t, t' \in \Delta_n$ such that $|t' - t| < \eta_n$ we have (10) except on a set of points $\omega \in \Omega$ whose m-measure is $< \phi(|t' - t|) < \phi(\eta_n) = \epsilon_n$. For a fixed $t \in \Delta_n$ the number of points $t' \in \Delta_n$, $t' \neq t$, with $|t' - t| \leqslant \eta_n$ is $< 2^{n+1}\,\eta_n$. Hence the number of all pairs $(t, t') \in \Delta_n \times \Delta_n$, $t \neq t'$, such that $|t' - t| \leqslant \eta_n$ is less than $(2^n + 1)2^{n+1}\,\eta_n < 2^{2n+2}\eta_n$, and therefore for every couple $(t, t') \in \Delta_n \times \Delta_n$ such that $|t' - t| < \eta_n$ we have the inequality (10) except on a set of points $\omega \in \Omega$ whose m-measure is $< 2^{2n+2}\eta_n \epsilon_n$.

If for any integer $M > n_0$ we denote by E_M the set of all those points $\omega \in \Omega$ for which there exists an integer $n \geqslant M$ and two points $t, t' \in \Delta_n$ such that $|t' - t| < \eta_n$ but (10) is not satisfied, then $m(E_M) < \sum_{n \geqslant M} 2^{2n+2}\eta_n \epsilon_n$. By hypothesis (4) the last sum can be made arbitrarily small, provided that M is taken sufficiently large, and therefore the set $\bigcap_{M > n_0} E_M$ has m-measure zero. Thus for m-almost every $\omega \in \Omega$ there exists an integer $M = M(\omega)$ such that $\omega \notin E_M$, i.e. such that for every $n \geqslant M$ and any two points $t, t' \in \Delta_n$ with $|t' - t| < \eta_n$, the relation (10) is satisfied.

LEMMA 6. *Let h be a function defined on T with values in X, such that for every $t \in T$ the value $h(t)$ is the limit of the values $h(t_n)$ as $n \to \infty$. Let N be an integer $> n_0$ and assume that for $n \geqslant N$ and for any pair of points $t, t' \in \Delta_n$ such that $|t' - t| < \eta_n$ we have*

$$d(h(t), h(t')) < \psi(|t' - t|). \tag{11}$$

Then there exists a number $a_N > 1$ such that for any pair of points $t, t' \in T$ with $|t' - t| < \eta_N$ we have

$$d(h(t),\ h(t')) \leqslant a_N\,\psi(|\,t'-t\,|). \tag{12}$$

The numbers a_N depend on ψ, (η_n) and N but not on h, and they converge to 1 as $N \to \infty$.

PROOF. Let $t,\ t' \in T$ be such that $t < t'$ and $t' - t < \eta_N$. We denote, of course, by t'_n the smallest number in Δ_n which is $\geqslant t'$. For any integer $p \geqslant 0$ we have $t' - t_{N+p} \leqslant t' - t \leqslant \eta_N$, and since η_N is an integral multiple of $1/2^N$, and therefore also of $1/2^{N+p}$, it follows from $t' \leqslant t_{N+p} + \eta_N$ that $t'_{N+p} \leqslant t_{N+p} + \eta_N$, i.e. $t'_{N+p} - t_{N+p} \leqslant \eta_N$. Since $h(t)$ is the limit of $h(t_n)$ and $h(t')$ is the limit of $h(t'_n)$ as $n \to \infty$, we have

$$d(h(t),\ h(t')) \leqslant d(h(t_N),\ h(t'_N)) +$$

$$+ \sum_{p=0}^{\infty} d(h(t_{N+p+1}),\ h(t_{N+p})) + \sum_{p=0}^{\infty} d(h(t'_{N+p+1}),\ h(t'_{N+p})).$$

Taking into account the relations $t'_N - t_N \leqslant \eta_N$, $t_{N+p} - t_{N+p+1} \leqslant \frac{1}{2^{N+p+1}} \leqslant \eta_{N+p+1}$ and $t'_{N+p} - t'_{N+p+1} \leqslant \frac{1}{2^{N+p+1}} \leqslant \eta_{N+p+1}$, it follows therefore from (11) that

$$d(h(t),\ h(t')) \leqslant \psi\,(|\,t'_N - t_N|\,) + 2\sum_{p=1}^{\infty} \psi\left(\frac{1}{2^{N+p}}\right). \tag{13}$$

Since ψ is concave from below and $\psi(0) = 0$ we have $\psi(\alpha+\beta) \leqslant \psi(\alpha) + \psi(\beta)$ and thus from $t'_N - t_N \leqslant t' - t + 1/2^N$ we obtain

$$\psi(\,|\,t'_N - t_N\,|\,) \leqslant \psi(\,|\,t' - t\,|\,) + 2\psi\left(\frac{1}{2^N}\right). \tag{14}$$

Combining (13) and (14) we get

$$d(h(t),\ h(t')) \leqslant \psi(\,|\,t' - t\,|\,) + 2\sum_{p=0}^{\infty} \psi\left(\frac{1}{2^{N+p}}\right).$$

By virtue of hypothesis (3) there exists a decreasing sequence of strictly positive numbers (c_N) converging to zero as $N \to \infty$, and such that $2\sum_{p=0}^{\infty} \psi\left(\frac{1}{2^{N+p}}\right) = c_N\,\psi(\eta_{N+1})$ for every $N > \eta_0$. Thus for $t,\ t' \in T$ such that $|\,t' - t\,| \leqslant \eta_N$ we have

$$d(h(t),\ h(t')) \leqslant \psi(\,|\,t' - t\,|\,) + c_N\,\psi(\eta_{N+1}).$$

In all the preceding argument we may replace N by $N + p$, where p is any integer > 0, and obtain that for $t, t' \in T$ such that $|t' - t| < \eta_{N+p}$ we have

$$d(h(t), h(t')) \leqslant \psi(|t' - t|) + c_{N+p}\,\psi(\eta_{N+p+1}). \tag{15}$$

Let $t, t' \in T$, $|t' - t| < \eta_N$, and assume that $t \neq t'$. Let p be the smallest integer $\geqslant 0$ such that $\eta_{N+p+1} < |t' - t|$. Then $|t' - t| \leqslant \eta_{N+p}$ and therefore (15) holds. On the other hand $\psi(\eta_{N+p+1}) \leqslant \psi(|t' - t|)$ and so it follows from (15) that

$$d(h(t), h(t')) \leqslant (1 + c_{N+p})\,\psi(|t' - t|).$$

Thus we see that if we choose $a_N = 1 + c_N$, then all the assertions of Lemma 6 are verified.

Proof of Theorem 6. It follows from the preceding three lemmas that for m-almost every $\omega \in \Omega$ and for every number $a > 1$ there exists an integer $N = N(\omega, a) \geqslant n_0$ such that for any pair $t, t' \in T$ with $|t' - t| < \eta_N$ we have

$$d(\breve{g}(\omega)\,(t), \breve{g}(\omega)\,(t')) \leqslant a\,\psi(|t' - t|). \tag{16}$$

We have to compare now the functions $\breve{g}(\omega)$ and $\breve{f}(\omega)$.

(a) Let us first suppose that X is separable, hence a Polish space. The map $\breve{g}: \omega \longmapsto \breve{g}(\omega)$ from Ω into the space $\mathscr{C}_{s\Delta}(T; X)$ of continuous functions on T with values in X, equipped with the topology of pointwise convergence on Δ, is measurable. The space $\mathscr{C}_u(T; X)$ of all continuous maps $T \to X$, equipped with the topology of uniform convergence, is Polish, hence the map $\omega \longmapsto \breve{g}(\omega)$ from Ω into $\mathscr{C}_u(T; X)$ is also measurable, and a fortiori it is measurable as a map from Ω into $\breve{X}^T$. Consequently the map g, which associates with every $t \in T$ the random variable $\omega \longmapsto \breve{g}\,(\omega)(t)$, is a canonical random function.

Now f and g are both maps from T into Meas $(\Omega, m; X)$. The function f is continuous by condition (7), and g is continuous by what has been said above. Since they coincide on the dense subset Δ of T, they must coincide on all of T. By the uniqueness of the

associated random variables (Chap. 5, § 1) we have $\check{f} = \check{g}$, and therefore (16) is the same as (8).

(b) In the general case, for every $t \in \Delta$ the function $\omega \longmapsto f_\Delta(t)(\omega) = \check{f}_\Delta(\omega)(t)$ takes almost all its values in a countable union of compact subsets of X, i.e. in a separable subspace of X, since it is assumed to be Lusin-measurable. Thus there exists a subset Ω' of Ω, whose complement has m-measure zero, such that the set $\bigcup_{\omega \in \Omega'} \check{f}_\Delta(\omega)(\Delta)$ is separable in X. Hence also its closure $\bigcup_{\omega \in \Omega'} \check{g}(\omega)(T)$ is a separable subset of X. In other words, almost all the functions $\check{g}(\omega): T \to X$ take their values in a Polish subspace Z of X. By virtue of part (a) of the proof, the map $\omega \longmapsto \check{g}(\omega)$ from Ω into $\check{Z}^T$ is measurable, hence it is a fortiori measurable as a map from Ω into $\check{X}^T$. The proof now concludes as in case (a): $\check{g}$ defines a canonical random function g given by $g(t)(\omega) = \check{g}(\omega)(t)$. Since f and g are continuous and coincide on Δ, they are equal. Thus the random variables $\check{f}$ and $\check{g}$ are equal and (8) follows from (16).

REMARK 1. Theorem 6 implies in particular that f is almost surely in a countable union of subsets of $\mathscr{C}(T; X)$ which are compact for the topology induced by X^T, i.e. for the topology of pointwise convergence on T. Let us determine explicitly such a countable family of subsets.

For every $t \in \Delta$ choose $\alpha_t > 0$ so that $\sum_{t \in \Delta} \alpha_t < 1$. For every $\delta > 0$ and $t \in \Delta$ there exists a compact subset X_t^δ of X such that $f(t)(\omega) = \check{f}(\omega)(t) \in X_t^\delta$ except for a set of points $\omega \in \Omega$ whose m-measure is $< \delta \alpha_t$. For $N \geqslant n_0$ and $\delta > 0$ let $K_{\delta,N}$ be the subset of $\mathscr{C}(T; X)$ formed by all the functions h for which $h(t) \in X_t^\delta$ for every $t \in \Delta$, and which are such that for any pair of points $t, t' \in T$ verifying $|t' - t| < \eta_N$ we have $d(h(t), h(t')) < a_N \psi(|t' - t|)$, where a_N is the coefficient introduced in Lemma 6. It follows from the proof of Lemma 5 and from Lemma 6 that the measure $\check{\mu} = \check{f}(m)$ is concentrated up to $\delta + \sum_{n \geqslant N} 2^{2n+2} \eta_n \epsilon_n$ on the set $K_{\delta,N}$. By

Ascoli's theorem [2, Chap. X, 2nd ed., §2, n°5, Corol. 2 of Th. 2] each set $K_{\delta,N}$ is compact in the space $\mathscr{C}_u(T; X)$ of continuous functions $T \to X$ equipped with the topology of uniform convergence. Letting δ run through a sequence (δ_N) which tends to zero as $N \to \infty$, we obtain the required family.

This construction permits us to avoid the separability argument and the properties of Polish spaces in the proof of Theorem 6. In fact, by what we have just said, $\check{g}$ takes almost all its values in a set Σ which is a countable union of compact subsets of $\mathscr{C}_u(T; X)$, i.e. the measure $\check{g}(m)$ is concentrated on Σ. Therefore the Radon measure $\check{g}(m)$ on $\mathscr{C}_{s\Delta}(T; X)$ (cf. the proof of Theorem 6) is the image of a Radon measure on $\mathscr{C}_u(T; X)$ under the identity map $\mathscr{C}_u(T; X) \to \mathscr{C}_{s\Delta}(T; X)$, and therefore the identity map $\mathscr{C}_{s\Delta}(T; X) \to \mathscr{C}_u(T; X)$ is $\check{g}(m)$-measurable. Hence $\check{g}: \Omega \to \mathscr{C}_u(T; X)$ is m-measurable, and we conclude as in part (a) of the preceding proof.

Next we apply Theorem 5 in the case $X = \mathbf{R}$ to cylindrical measures on $\mathscr{C}(T) = \mathscr{C}[0, 1]$.

COROLLARY. *Let μ be a cylindrical measure on $\mathscr{C}[0, 1]$, scalarly concentrated on the balanced, convex, weakly compact sets. For $t, t' \in [0, 1]$ denote by $\mu_{\{t,t'\}}$ the image of μ under the map $(\delta_t, \delta_{t'}): f \longmapsto (f(t), f(t'))$ from $\mathscr{C}[0, 1]$ into $\mathbf{R}^2$. Assume that for $|t' - t| < \tau$ we have*

$$\mu_{\{t,t'\}}(\{(u, v) \in \mathbf{R}^2 \mid |u - v| > \psi(|t' - t|)\}) \leqslant \phi(|t' - t|). \quad (17)$$

Then μ is a Radon measure on $\mathscr{C}[0, 1]$.

PROOF. Let $\mathscr{C}_s[0, 1]$ be the vector space $\mathscr{C}[0, 1]$ equipped with the topology of pointwise convergence, $\mathscr{C}_u[0, 1]$ the same vector space equipped with the topology of uniform convergence, and $j: \mathscr{C}_u[0, 1] \to \mathscr{C}_s[0, 1]$ the identity map. The map $f \longmapsto (f(t))_{t\in[0,1]}$ is a strict morphism from $\mathscr{C}_s[0, 1]$ into $\mathbf{R}^{[0,1]}$, hence the cylindrical measure $j(\mu)$ can be considered as a projective system of probabilities on $\mathbf{R}^{[0,1]}$ which satisfies condition (5), and is therefore by Theorem 5 a Radon measure on $\mathscr{C}_s[0, 1]$. It follows from Remark 1 or from the fact that $\mathscr{C}_u[0, 1]$ is a Polish space that $j(\mu)$ is the

image of a Radon measure μ' on $\mathscr{C}_u[0, 1]$. Since μ and μ' are both scalarly concentrated on the balanced, convex, weakly compact subsets of $\mathscr{C}_u[0, 1]$, by the corol. of Prop. 6 of Chap. 2, §3 it follows from $j(\mu) = j(\mu')$ that $\mu = \mu'$.

We come now to the application of the preceding results to the Wiener-Lévy measure of Brownian motion. Let γ be the canonical Gauss measure on the Hilbert space $L^2[0, 1]$, and denote by $\mathscr{J}$ the continuous linear map from $L^2[0, 1]$ into $\mathscr{C}[0, 1]$ which associates with each $\phi \in L^2[0, 1]$ its primitive $\mathscr{J}\phi = \Phi$ given by $\Phi(x) = \int_0^x \phi(t)\, dt$. The image of $L^2[0, 1]$ with respect to $\mathscr{J}$ is the Hilbert subspace $\mathscr{H}^1 = \mathscr{H}^1([0, 1]; 0)$ of $\mathscr{C}[0, 1]$ introduced in Example 1 (§1). Recall that the inner product of two elements $\Phi, \Psi \in \mathscr{H}^1$ is given by

$$(\Phi \mid \Psi)_{\mathscr{H}^1} = \int_0^1 \Phi'(t)\, \Psi'(t)\, dt.$$

The image $\mu = \mathscr{J}\gamma$ is the Gauss measure on $\mathscr{C}[0, 1]$ associated with the Hilbert subspace $\mathscr{H}^1$ of $\mathscr{C}[0, 1]$ (Def. 1, § 1): it is called the Wiener-Lévy measure. Since γ is scalarly concentrated on the balls of $L^2[0, 1]$, and since these balls are balanced, convex, weakly compact sets, it follows that μ is scalarly concentrated on the balanced, convex, weakly compact subsets of $\mathscr{C}[0, 1]$.

Let us find the kernel H associated with the Hilbert subspace $\mathscr{H}^1$ of $\mathscr{C}[0, 1]$ (§2). We know that H is the unique map from the dual $E' = \mathscr{M}[0, 1]$ of $E = \mathscr{C}[0, 1]$ into $\mathscr{H}^1 \subset \mathscr{C}[0, 1]$ which for every $\nu \in E'$ and $\Phi \in E$ satisfies the relation

$$(H\nu \mid \Phi)_{\mathscr{H}^1} = \langle \Phi, \nu \rangle_{E, E'},$$

i.e.

$$\int_0^1 [H\nu]'(t)\, \Phi'(t)\, dt = \int_0^1 \Phi(t)\, d\nu(t)$$

(cf. Formula (3) of §2). In particular, if we take for ν the Dirac measure δ_x at the point x, we obtain

$$\int_0^1 [H\delta_x]'(t)\,\Phi'(t)\,dt = \Phi(x) = \int_0^1 \chi_{[0,x]}(t)\;\Phi'(t)\,dt,$$

where $\chi_{[0,x]}$ is the characteristic function of the interval $[0, x]$. Thus we have

$$H(\delta_x)(y) = \int_0^y \chi_{[0,x]}(t)\;dt = \min(x, y).$$

The reproducing kernel A associated with H [12, §9, Prop. 23, p. 189] is the numerical function defined on $[0, 1]\times[0, 1]$ by

$$A(x, y) = \langle H\delta_x, \delta_y\rangle = \min(x, y).$$

It follows [12, §9, Prop. 24, p. 191] finally that the kernel H is given by

$$(H\nu)(x) = \int_0^1 A(x, y)\,d\nu(y) = \int_0^1 \min(x, y)\;d\nu(y)$$

for every $\nu \in \mathscr{M}[0, 1]$. In particular, the positive quadratic form $Q_{\mathscr{H}^1}$ on $\mathscr{M}[0, 1]$ is given by

$$Q_{\mathscr{H}^1}(\nu) = \langle H\nu, \nu\rangle = \int_0^1\int_0^1 \min(x, y)\,d\nu(x)\,d\nu(y),$$

and, according to formula (6) of §2, the Fourier transform of μ is the function

$$\nu \longmapsto \exp\left(-\pi\int_0^1\int_0^1 \min(x, y)\,d\nu(x)\,d\nu(y)\right)$$

on $\mathscr{M}[0, 1]$.

The map $h \longmapsto (h(t))_{t\in[0,1]}$ identifies $\mathscr{C}[0, 1]$ with a subset of $\mathbf{R}^{[0,1]}$ and in particular μ can be considered as a projective system of probabilities on $\mathbf{R}^{[0,1]}$. To be able to apply the corollary of Theorem 5, we have to calculate the image $\mu_{\{t,t'\}} = \pi_{\{t,t'\}}(\mu)$ of μ under the projection $\pi_{\{t,t'\}}$ from $\mathbf{R}^{[0,1]}$ onto $\mathbf{R}^2$, where $t, t' \in [0, 1]$, and where, to fix ideas, we may assume that $t < t'$.

In the first place, let us find the Hilbert structure on $\mathbf{R}^2$ obtained from that of $\mathscr{H}^1$ with the help of $\pi_{\{t,t'\}}$, $t \neq 0$. For any $\Phi \in \mathscr{H}^1$ we have $\pi_{\{t,t'\}}(\Phi) = (\Phi(t), \Phi(t'))$, i.e. given $(u, v) \in \mathbf{R}^2$, the norm of (u, v) is the norm in $\mathscr{H}^1$ of a function $\Phi_0 \in \mathscr{H}^1$ such that $\Phi_0(t) = u$, $\Phi_0(t) = v$, and Φ_0 is orthogonal to $\operatorname{Ker} \pi_{\{t,t'\}}$. Such a function $\Phi_0 = \Phi_{u,v}$ is given by the following conditions:

$$\Phi_0(0) = 0,\ \Phi_0(t) = u,\ \Phi_0(t') = v.$$

Φ_0 is linear in the intervals $[0, t]$ and $[t, t']$,

Φ_0 is constant in the interval $[t', 1]$.

Indeed, let $\Psi \in \operatorname{Ker} \pi_{\{t,t'\}}$, i.e. $\Psi(t) = \Psi(t') = 0$. Then

$$(\Phi_0 | \Psi)_{\mathscr{H}^1} = \int_0^1 \Phi_0'(s)\Psi'(s)\,ds,$$

and

$$\int_0^t \Phi_0'(s)\Psi'(s)\,ds = \frac{u}{t}\int_0^t \Psi'(s)\,ds = \frac{u}{t}[\Psi(t) - \Psi(0)] = 0,$$

$$\int_t^{t'} \Phi_0'(s)\Psi'(s)\,ds = \frac{v-u}{t'-t}\int_t^{t'} \Psi'(s)\,ds = \frac{v-u}{t'-t}[\Psi(t') - \Psi(t)] = 0,$$

$$\int_{t'}^1 \Phi_0'(s)\Psi'(s)\,ds = 0,$$

since $\Phi_0'(s) = 0$ for $t' < s < 1$. Thus $(\Phi_0 | \Psi)_{\mathscr{H}^1} = 0$. It follows that the norm of (u, v) we were looking for is given by

$$\|(u, v)\|^2 = \int_0^1 (\Phi_0'(s))^2\,ds$$

$$= \left(\frac{u}{t}\right)^2 t + \left(\frac{v-u}{t'-t}\right)^2 (t'-t) = \frac{u^2}{t} + \frac{(v-u)^2}{t'-t}.$$

An orthonormal base of $\mathbf{R}^2$ with respect to this Hilbert structure is given by the vectors $e_1 = (\sqrt{t}, \sqrt{t})$ and $e_2 = (0, \sqrt{(t'-t)})$, and an arbitrary vector (u, v) of $\mathbf{R}^2$ can be expressed in the form

$$(u, v) = \frac{u}{\sqrt{t}}\, e_1 + \frac{v-u}{\sqrt{(t'-t)}}\, e_2.$$

Now $\mu_{(t,t')}$ is the canonical Gauss measure on $\mathbf{R}^2$, equipped with the Hilbert structure just obtained. Therefore the change of variables

$$x_1 = \frac{u}{\sqrt{t}},\ x_2 = \frac{v-u}{\sqrt{(t'-t)}}$$

yields

$$\mu_{(t,t')} = e^{-\pi\|x\|^2}\, dx_1\, dx_2$$

$$= e^{-\pi(u^2/t + (v-u)^2/(t'-t))}\, \frac{du\, dv}{\sqrt{t(t'-t)}},\ t \neq 0. \qquad (18)$$

We shall need later the expression of the measure $\mu_{(t)} = \pi_{(t)}(\mu)$ for any $t \in [0, 1]$, which can be calculated similarly. Take $t \neq 0$. For any $\phi \in \mathscr{H}^1$ we have $\pi_{(t)}(\Phi) = \Phi(t)$, i.e. the norm of $u \in \mathbf{R}$ with respect to the Hilbert structure on $\mathbf{R}$ deduced from that of $\mathscr{H}^1$ with the help of $\pi_{(t)}$ is the norm in $\mathscr{H}^1$ of a function $\Phi_u \in \mathscr{H}^1$ such that $\Phi_u(t) = u$, and which is orthogonal to $\operatorname{Ker} \pi_{(t)}$. Such a function is given by the following conditions:

$\Phi_u(0) = 0$, $\Phi_u(t) = u$,

Φ_u is linear in the interval $[0, t]$,

Φ_u is constant in the interval $[t, 1]$.

Indeed, if $\Psi \in \operatorname{Ker} \pi_{(t)}$, i.e. $\Psi(t) = 0$, then

$$(\Phi_u \mid \Psi)_{\mathscr{H}^1} = \int_0^1 \Phi_u'(s)\, \Psi'(s)\, ds = \frac{u}{t} \int_0^t \Psi'(s) ds = \frac{u}{s} [\Psi(t) - \Psi(0)] = 0.$$

Hence

$$\| u \|^2 = \int_0^1 (\Phi_u'(s))^2\, ds = \left(\frac{u}{t}\right)^2 t = \frac{u^2}{t}.$$

A vector of unit length is $e = \sqrt{t}$, and writing $u = \frac{u}{\sqrt{t}}\sqrt{t} = x \cdot e$, we see that the canonical Gauss measure for this Hilbert structure is

$$\mu_{\{t\}} = e^{-\pi u^2/t} \frac{du}{\sqrt{t}}, \quad t \neq 0. \tag{19}$$

The formulae (18) and (19) have to be modified if $t = 0$. In this case, one sees easily that

$$\mu_{\{t,t'\}} = \delta_u \otimes e^{-\pi v^2/t'} \frac{dv}{\sqrt{t'}},$$

and that

$$\mu_t = \delta_u.$$

Let us observe that the map $\pi_{\{t\}} \circ \mathscr{I}$ from $L^2[0, 1]$ into $\mathbf{R}$ associates with each $\phi \in L^2[0, 1]$ the number $\int_0^t \phi(s)ds$. An element $\phi_u \in L^2[0, 1]$ for which $(\pi_{\{t\}} \circ \mathscr{I})(\phi_u) = \int_0^t \phi_u(s)\, ds$ has the given value u, and which is orthogonal to $\mathrm{Ker}(\pi_{\{t\}} \circ \mathscr{I})$, is the step function

$$\phi_u(s) = \begin{cases} \dfrac{u}{t} & \text{if } 0 < s < t, \\ 0 & \text{if } t < s < 1. \end{cases}$$

Thus we find again that $\|u\|^2 = \dfrac{u^2}{t}$, and $\Phi_u = \mathscr{I}\phi_u$ is, of course, the function found above. A similar remark applies to the map $\pi_{\{t,t'\}} \circ \mathscr{I}$.

THEOREM 7. *The Wiener-Lévy measure μ is a Radon measure on $\mathscr{C}[0,1]$.*

PROOF. We have to prove that the measure (18) satisfies condition (17) for appropriately chosen functions ϕ and ψ which verify the hypotheses introduced at the beginning of this section. Let θ be an arbitrary number > 1 and set

$$\psi(\xi) = \left(\frac{\theta}{\pi} \xi \log \frac{1}{\xi} \right)^{1/2}. \tag{20}$$

The function $\xi \longmapsto -\xi \log \xi$ is increasing and concave from below for $0 < \xi \leqslant 1/e$. Since $\eta \longmapsto \sqrt{\eta}$ is an increasing function, concave from below, we see that ψ positive, continuous, increasing, concave from below in $[0, \tau]$, with $\tau = 1/e$, and $\psi(0) = 0$.

Next define the function ϕ by

$$\phi(\xi) = \int_{|w|>\psi(\xi)} e^{-\pi w^2/\xi} \frac{dw}{\sqrt{\xi}} = \frac{1}{\sqrt{\pi}} \int_{|x|>\sqrt{\theta \log 1/\xi}} e^{-x^2} dx$$

for $0 < \xi < 1$. We have

$$\mu_{\{t,t'\}}(\{(u, v) \in \mathbf{R}^2 \mid |u - v| > \psi(|t' - t|)\})$$

$$= \int\int_{|u-v|>\psi(|t'-t|)} e^{-\pi(u^2/t + (v-u)^2/(t'-t))} \frac{du\, dv}{\sqrt{t(t'-t)}}.$$

Taking u and $w = v - u$ for independent variables, and integrating with respect to u, the last expression becomes

$$\int_{|w|>\psi(|t'-t|)} e^{-\pi w^2/(t'-t)} \frac{dw}{\sqrt{(t'-t)}} = \phi(|t' - t|).$$

If we set $\phi(0) = 0$ then ϕ is a positive, continuous, increasing function in [0,1]. For $\zeta > 0$ the inequality

$$\int_{|x|>\zeta} e^{-x^2}\, dx < C_1 \frac{e^{-\zeta^2}}{\zeta}$$

is valid ($C_1, \ldots, C_4$ denote constants), hence

$$\phi(\xi) < C_1 \frac{e^{-(\theta \log 1/\xi)}}{\sqrt{\left(\theta \log \frac{1}{\xi}\right)}} < C_1 \frac{\xi^\theta}{\sqrt{\left(\log \frac{1}{\xi}\right)}}. \tag{21}$$

For $n > 4$ we define $\eta_n = \frac{n}{2^n}$. Then η_n is an integral multiple of $\frac{1}{2^n}$, the sequence (η_n) converges to zero and $\frac{1}{2^n} = o(\eta_n)$ as $n \to \infty$, i.e. condition (2) is satisfied. According to (1) we define the sequence (ϵ_n) by $\epsilon_n = \phi(\eta_n)$. By (21) we have

$$\epsilon_n = \phi(\eta_n) < C_1 \frac{\eta_n^\theta}{\sqrt{(\log 1/\eta_n)}} < C_2 \frac{n^{\theta - 1/2}}{2^{\theta n}},$$

and therefore

$$2^{2n}\eta_n\epsilon_n < 2^{2n}\frac{n}{2^n}C_2\frac{n^{\theta-1/2}}{2^{\theta n}} = C_2\frac{n^{\theta+1/2}}{2^{(\theta-1)n}}.$$

Since $\theta > 1$ the series $\sum_{n\geqslant 4} 2^{2n}\eta_n\epsilon_n$ converges, i.e. condition (4) is satisfied. Finally, we have

$$\psi\left(\frac{1}{2^p}\right) = \sqrt{\left(\frac{\theta}{\pi}\frac{1}{2^p}\log 2^p\right)} = C_3\left(\frac{p^{1/2}}{2^{p/2}}\right)$$

and so

$$\sum_{\nu\in\mathbf{N}}\psi\left(\frac{1}{2^{n+\nu}}\right) = C_3\sum_{\nu\in\mathbf{N}}\frac{\sqrt{(n+\nu)}}{2^{(n+\nu)/2}} = O\left(\frac{n^{1/2}}{2^{n/2}}\right)$$

as $n\to\infty$. Since on the other hand

$$\psi(\eta_{n+1}) = \sqrt{\left(\frac{\theta}{\pi}\frac{n+1}{2^{n+1}}\log\frac{2^{n+1}}{n+1}\right)} \sim C_4\frac{n}{2^{n/2}}$$

as $n\to\infty$, we see that condition (3) is also satisfied.

As we have said above, the injection $h \longmapsto (h(t))_{t\in[0,1]} = (\langle h, \delta_t\rangle)_{t\in[0,1]}$ from $\mathscr{C}[0, 1]$ into $\mathbf{R}^{[0,1]}$ associates with the Wiener-Lévy measure μ a projective system of probabilities on $\mathbf{R}^{[0,1]}$, which we shall still denote by μ. On the other hand, if g: $\mathscr{M}[0, 1] \to \mathrm{Meas}(\Omega, m; \mathbf{R})$ is a continuous, linear random function associated with the Wiener-Lévy measure, and if j is the injection $t \longmapsto \delta_t$ from $[0, 1]$ into $\mathscr{M}[0, 1]$, then $f = g\circ j$ is a random function $[0, 1] \to \mathrm{Meas}(\Omega, m; \mathbf{R})$. This random function f is associated with the projective system μ on $\mathbf{R}^{[0,1]}$ in the sense that for every finite subset $A = \{t_1, \ldots, t_p\}$ of $[0, 1]$ we have

$$(f(t_1), \ldots, f(t_p))(m) = \mu_A = \pi_A(\mu)$$

(cf. the proof of the equivalence of Theorems 5 and 6). Indeed, assume that g is canonical and let $\breve{g}$ be the random variable $\Omega \to \mathbf{R}^{\mathscr{M}[0,1]}$ such that $g(\nu) = \pi_\nu\circ\breve{g}$ for every $\nu\in\mathscr{M}[0, 1]$ (Chap. 5, § 2). Then for any $A = \{t_1, \ldots, t_p\} \subset [0, 1]$ we have

$$\begin{aligned}(f(t_1), \ldots, f(t_p))(m) &= (g(\delta_{t_1}), \ldots, g(\delta_{t_p}))(m)\\ &= (\pi_{j(A)}\circ\breve{g})(m) = \pi_A(\mu) = \mu_A.\end{aligned}$$

In particular, for any $t \in [0, 1]$ the probability $f(t)\,(m) = \mu_{\{t\}}$ is given by (19). We call f *the random function on Brownian motion.*

Taking into consideration the form (20) of ψ in the preceding proof, and the fact that both θ in (20) and a in Theorem 5 are arbitrary numbers > 1, we obtain the

COROLLARY. *Given any number $a > 1$, for almost all $\omega \in \Omega$ the random function f of Brownian motion satisfies the Hölder condition*

$$|f(t)\,(\omega) - f(t')\,(\omega)| < \frac{a}{\sqrt{\pi}} \left(|t' - t| \log \frac{1}{|t' - t|} \right)^{1/2}$$

for sufficiently small $|t' - t|$ (depending on a and ω).

SUMMARY OF APPENDIX

Narrow topology has been defined in Chapter V, page 249, only for completely regular spaces; it is defined here in § 1 in the general case, Definition 1, page 370. Proposition 2, page 371, proves that it is Hausdorff. § 2 introduces the notion of Riemann-integrable functions, Definition 2, page 374; then the main theorem of the § is Theorem 1, page 375, connecting Riemann-integrable functions and passage to the limit in narrow convergence.

§ 3 studies compact subspaces of the space of finite measures, equipped with narrow topology. Theorem 3, page 379, and its converse by Prokhorov, Theorem 4, page 381, are the main results; they are fundamental in the applications.

In § 4, the Theorem 5 of Paul Lévy, page 382, is the fundamental tool in the theory of addition of random variables.

§ 5 studies the particular case of the measures on a Suslin space. The main result is the following: if X is Polish or Lusin or Suslin, then the space of finite measures on X, equipped with the narrow topology, is also Polish or Lusin or Suslin, Theorem 7, page 385. Theorem 8 gives an important consequence relative to the Borel functions.

APPENDIX

COMPLEMENTS ON NARROW CONVERGENCE

1. Narrow convergence on an arbitrary topological space. In this appendix we shall be mainly concerned with finite Radon measures on a Hausdorff topological space X. We shall denote by $\mathcal{M}^1(X)$ the vector space of these measures and by $\mathcal{M}^1_+(X)$ the cone of positive finite measures.

If X is completely regular, we defined in Chap. 5, §1 the narrow topology on $\mathcal{M}^1(X)$. Let us recall that a sequence or a filter (μ_j) converges narrowly to μ if $(\mu_j(\phi))$ converges to $\mu(\phi)$ for every ϕ belonging to the space $\mathcal{B}(X)$ of bounded continuous functions. This definition is not appropriate for an arbitrary Hausdorff space X since there might not exist sufficiently many continuous functions on it. Using a paper of F. Topsoe ("A criterion for weak convergence of measures with an application to measures on $D[0, 1]$", *Math. Scandinavica*, 25 (1969), pp. 97-104), we shall define the narrow topology on $\mathcal{M}^1_+(X)$ in a way which is suitable for any Hausdorff space X. The definition is based on the following observation

PROPOSITION 1. *Let (μ_j) be a sequence or a filter of positive finite Radon measures on the Hausdorff space X, and let μ be a positive finite Radon measure on X. If X is completely regular and (μ_j) converges narrowly to μ, then*

$$\liminf_j \mu_j(f) \geqslant \mu(f) \tag{1}$$

for every real-valued, bounded, lower semi-continuous function f on X. Conversely, if X is an arbitrary Hausdorff space and if (1) holds for every real-valued, bounded, lower semi-continuous function f, then $\lim_j \mu_j(\phi) = \mu(\phi)$ for any bounded continuous function ϕ on X.

PROOF. (a) For any $\phi \in \mathcal{B}(X)$ the map $\lambda \longmapsto \lambda(\phi)$ from $\mathcal{M}^1_+(X)$ into $\mathbf{R}$ is continuous if $\mathcal{M}^1_+(X)$ is equipped with the narrow topology. Since X is completely regular, for any bounded, lower semi-continuous function f the map $\lambda \longmapsto \lambda(f)$ is the least upper bound

of all the continuous maps $\lambda \longmapsto \lambda(\phi)$, where ϕ runs through all functions belonging to $\mathscr{B}(X)$ which verify $\phi < f$. Hence the map $\lambda \longmapsto \lambda(f)$ from $\mathscr{M}^1_+(X)$ into $\mathbf{R}$ is lower semi-continuous, and therefore if (μ_j) converges to μ we have (1).

(b) Conversely, let X be an arbitrary Hausdorff space, and assume that (1) holds for any bounded, lower semi-continuous function f. By changing signs we obtain that

$$\limsup_j \mu_j(f) < \mu(f)$$

holds for any bounded, upper semi-continuous function f. Hence for any bounded continuous function ϕ we have

$$\mu(\phi) < \liminf_j \mu_j(\phi) < \limsup_j \mu_j(\phi) < \mu(\phi), \text{ i.e. } \lim_j \mu_j(\phi) = \mu(\phi).$$

It cannot be expected that on an arbitrary Hausdorff space X the condition $\lim_j \mu_j(\phi) = \mu(\phi)$ for every $\phi \in \mathscr{B}(X)$ implies that (1) holds for every bounded, lower semi-continuous function f, since there might be very few continuous functions, but there are always many bounded lower semi-continuous functions, namely the characteristic functions of open sets.

DEFINITION 1 (Topsoe). *Let X be a Hausdorff topological space and $\mathscr{M}^1_+(X)$ the set of all positive finite Radon measures on X. The narrow topology on $\mathscr{M}^1_+(X)$ is defined by assigning to each $\mu_0 \in \mathscr{M}^1_+(X)$ as a system of generators (sub-basis) of the filter of neighborhoods of μ_0 the sets $V(\mu_0; \epsilon, f)$, where ϵ is a number > 0, f a bounded lower semi-continuous function on X, and $V(\mu_0; \epsilon, f)$ is the set of all measures $\mu \in \mathscr{M}^1_+(X)$ which verify $\mu(f) > \mu_0(f) - \epsilon$.*

For every $a \in \mathbf{R}$ and every bounded lower semi-continuous function f on X, the set $\{\mu \in \mathscr{M}^1_+(X);\ \mu(f) < a\}$ is narrowly closed.

A fundamental system of neighborhoods of $\mu_0 \in \mathscr{M}^1_+(X)$ is given by all finite intersections of sets $V(\mu; \epsilon, f)$, or, equivalently, by all sets $V(\mu_0; \epsilon; f_1, \ldots, f_n) = \{\mu \in \mathscr{M}^1_+(X) \mid \mu(f_j) > \mu_0(f_j) - \epsilon,\ 1 < j < n\}$, where $\epsilon > 0$ and the f_j $(1 < j < n)$ are bounded, lower semi-continuous functions on X. The definition could also be formulated in terms of bounded, upper semi-continuous functions f, provided we replace $\mu(f) > \mu_0(f) - \epsilon$ by $\mu(f) < \mu_0(f) + \epsilon$.

It is clear that in this way we indeed obtain a topology on $\mathscr{M}^1_+(X)$. A sequence or a filter (μ_j) of elements of $\mathscr{M}^1_+(X)$ converges narrowly, i.e. for the narrow topology, to $\mu \in \mathscr{M}^1_+(X)$ if and only if $\liminf_j \mu_j(f) \geqslant \mu(f)$ for every bounded, lower semi-continuous function f on X. It follows from Prop. 1 that if X is completely regular then the narrow topology as defined here is the same as the one defined in Chap. 5, §1. The narrow topology on $\mathscr{M}^1_+(X)$ can also be described as the coarsest topology on $\mathscr{M}^1_+(X)$ for which all the maps $\mu \longmapsto \mu(f)$ from $\mathscr{M}^1_+(X)$ into $\mathbf{R}$ are lower semi-continuous, as f varies in the set of all bounded, lower semi-continuous functions on X.

It is fairly simple to see that the intersection of all narrow neighborhoods of a point $\mu_o \in \mathscr{M}^1_+(X)$ is $\{\mu_o\}$. Indeed, if μ belongs to every neighborhood of μ_o, then we have $\mu(f) \geqslant \mu_o(f)$ for every bounded, lower semi-continuous function f, and therefore also for any bounded Borel function f, since if λ is a positive finite Radon measure, and f a bounded Borel function, then $\lambda(f) = \lambda^*(f)$ is the greatest lower bound of all values $\lambda(g)$, where g runs through all bounded, lower semi-continuous functions verifying $g \geqslant f$. Taking $f \geqslant 0$ we see that $\mu \geqslant \mu_o$, and taking $f \leqslant 0$ we obtain $\mu \leqslant \mu_o$. Thus we have $\mu = \mu_o$.

But actually much more is true: we have

PROPOSITION 2. *The narrow topology on $\mathscr{M}^1_+(X)$ is Hausdorff.*

PROOF. Let μ_1 and μ_2 be two distinct elements of $\mathscr{M}^1_+(X)$, and assume that every neighborhood of μ_1 meets every neighborhood of μ_2. Since the constant functions $+1$ and -1 on X are bounded and lower semi-continuous, the neighborhood $V(\mu_1; \epsilon; +1, -1)$ meets the neighborhood $V(\mu_2; \epsilon; +1 -1)$, i.e. the intervals $[\mu_1(1) - \epsilon, \mu_1(1) + \epsilon]$ and $[\mu_2(1) - \epsilon, \mu_2(1) + \epsilon]$ have a point in common. Now this is true for every $\epsilon > 0$, hence $\mu_1(1) = \mu_2(1)$, and we may assume that this common value is equal to 1.

Since $\mu_1 \neq \mu_2$, there exists a Borel subset B of X such that $\mu_1(B) \neq \mu_2(B)$. Suppose for instance that $\mu_1(B) > \mu_2(B)$ and set

$\alpha = \mu_1(B) - \mu_2(B) > 0$. For any $\delta > 0$ there exist two compact sets K and K' such that $K \subset B$, $K' \subset \mathbf{C}B$, $\mu_1(K) \geqslant \mu_1(B) - \delta$ and $\mu_2(K') \geqslant \mu_2(\mathbf{C}B) - \delta$. Since the sets K and K' are compact, there exist two disjoint open sets U and U' such that $K \subset U$ and $K' \subset U'$. By our assumption the neighborhoods $V(\mu_1; \delta; \chi_U, -1)$ and $V(\mu_2; \delta; \chi_{U'})$ have a non-void intersection. For a common element μ of these two neighborhoods we have the inequalities $\mu(1) < 1 + \delta$,

$$\mu(U) \geqslant \mu_1(U) - \delta \geqslant \mu_1(K) - \delta \geqslant \mu_1(B) - 2\delta,$$

$$\mu(U') \geqslant \mu_2(U') - \delta \geqslant \mu_2(K') - \delta \geqslant \mu_2(\mathbf{C}B) - 2\delta,$$

and thus also

$$\mu(1) \geqslant \mu(U) + \mu(U') \geqslant \mu_1(B) + \mu_2(\mathbf{C}B) - 4\delta$$

$$= \mu_2(B) + \alpha + \mu_2(\mathbf{C}B) - 4\delta = 1 + \alpha - 4\delta.$$

It follows that $1 + \alpha - 4\delta \leqslant 1 + \delta$, i.e. $\alpha \leqslant 5\delta$, which is absurd since δ can be chosen arbitrarily close to 0.

PROPOSITION 3. *Let X and Y be two Hausdorff spaces, and F a continuous map from X into Y. If we equip the sets $\mathscr{M}^1_+(X)$ and $\mathscr{M}^1_+(Y)$ with their narrow topologies, then the map $\mu \longmapsto F(\mu)$ from $\mathscr{M}^1_+(X)$ into $\mathscr{M}^1_+(Y)$ is continuous.*

PROOF. If f is a bounded, lower semi-continuous function on Y, then $f \circ F$ is a bounded, lower semi-continuous function on X. Assume that $\mu \in \mathscr{M}^1_+(X)$ belongs to the neighborhood $V(\mu_0; \epsilon; f \circ F)$ of $\mu_0 \in \mathscr{M}^1_+(X)$. Then $\mu(f \circ F) \geqslant \mu_0(f \circ F) - \epsilon$, i.e. $F(\mu)(f) \geqslant F(\mu_0)(f) - \epsilon$, and therefore $F(\mu)$ belongs to the neighborhood $V(F(\mu_0); \epsilon; f)$ of $F(\mu_0) \in \mathscr{M}^1_+(Y)$.

If A is a Borel subset of the Hausdorff space X, we denote by μ_A the measure induced on A by the Radon measure μ on X.

PROPOSITION 4. *Let (μ_j) be a sequence or a filter basis of positive, finite measures which converges narrowly to the positive, finite measure μ on the Hausdorff space X. Assume that the measures μ_j and μ are all concentrated on the Borel subset A of X. Then the induced measures $((\mu_j)_A)$ converge narrowly to μ_A on A.*

PROOF. Let f be a bounded, lower semi-continuous function on A. It has a bounded, lower semi-continuous extension $\bar{f}$ to all of X. Indeed, if $f(x) < M$ for $x \in A$ we can set

$$\bar{f}(x) = \liminf_{y \in A,\, y \to x} f(y) \qquad \text{if } x \in \bar{A},$$

$$\bar{f}(x) = M \qquad \text{if } x \notin \bar{A}.$$

Since $(\mu_j)_A(f) = \mu_j(\bar{f})$ and $\mu_A(f) = \mu(\bar{f})$, we have $\liminf_j (\mu_j)_A(f) \geqslant \mu_A(f)$, i.e. $((\mu_j)_A)$ converges narrowly to μ_A.

REMARK 1. Assume that X is completely regular, denote by $\mathscr{M}^1_A(X)$ the vector space of all finite Radon measures on X concentrated on the Borel subset A of X, and by $(\mathscr{M}^1_+)_A(X)$ the positive cone of $\mathscr{M}^1_A(X)$. Prop. 4 states that $\mu \longmapsto \mu_A$ is a continuous map from $(\mathscr{M}^1_+)_A(X)$ onto $\mathscr{M}^1_+(A)$. Since it is bijective and the inverse map $\mu_A \longmapsto \mu$ is clearly continuous, we actually have a bicontinuous isomorphism from $(\mathscr{M}^1_+)_A(X)$ onto $\mathscr{M}^1_+(A)$.

The corresponding statement is false for $\mathscr{M}^1_A(X)$ and $\mathscr{M}^1(A)$. Take e.g. for X the extended real line $\overline{\mathbf{R}}$ and $A = \mathbf{R}$. Then $\delta_{(n)} - \delta_{(n+1/n)}$ converges to zero narrowly in $\mathscr{M}^1_A(X)$ but not in $\mathscr{M}^1(A)$.

Proposition 4 implies a well-known result:

PROPOSITION 5. *Let X be a locally compact space. The filter $\mathscr{F}$ of positive finite Radon measures on X converges narrowly to the positive finite Radon measure ν if and only if $\mathscr{F}$ converges vaguely to ν and its image $\mathscr{F}(1)$ under the map $\mu \longmapsto \mu(1)$ converges to $\nu(1)$.*

PROOF. It is clear that if $\mathscr{F}$ converges narrowly to ν, then it also converges vaguely, and $\mathscr{F}(1)$ converges to $\nu(1)$. Conversely, assume that for every $\phi \in \mathscr{K}(X)$ the image $\mathscr{F}(\phi)$ of $\mathscr{F}$ under the map $\mu \longmapsto \mu(\phi)$ converges to $\nu(\phi)$ and that $\mathscr{F}(1)$ converges to $\nu(1)$. Denote by $\mathscr{B}^\circ(X)$ the space of all continuous functions on X which tend to zero at infinity. There exists $M \in \mathscr{F}$ such that $\mu(1) < \nu(1) + 1$ for all $\mu \in M$. Since $|\mu(\psi)| < \mu(1) \cdot \max_{x \in X} |\psi(x)|$ for $\psi \in \mathscr{B}^\circ(X)$, it follows that M is an equicontinuous set of linear forms on $\mathscr{B}^\circ(X)$. By hypothesis, $\mathscr{F}$ converges to ν pointwise on the dense subspace

$\mathscr{K}(X)$ of $\mathscr{B}^\circ(X)$, and since on M the topology of pointwise convergence in $\mathscr{K}(X)$ and the topology of pointwise convergence in $\mathscr{B}^\circ(X)$ coincide [2, Chap. X, 2nd ed., §2, $n^\circ 4$, Th. 1], we see that $\mathscr{F}$ also converges to ν pointwise in $\mathscr{B}^\circ(X)$. Using again that $\mathscr{F}(1)$ converges to $\nu(1)$, we obtain that $\mathscr{F}$ converge to ν pointwise on the space of continuous functions which have a limit at infinity. In other words, the filter basis $\mathscr{F}$ converges narrowly to ν on the Alexandrov compactification $\overset{\circ}{X}$ of X. Since the measures belonging to the sets which form the filter basis $\mathscr{F}$ on $\mathscr{C}(\overset{\circ}{X})$, and also the measure ν, are all concentrated on $X \subset \overset{\circ}{X}$, it follows from Prop. 4 that $\mathscr{F}$ converges narrowly to ν on X.

2. **Riemann-integrable functions.** Let (μ_j) be a sequence or a filter of finite Radon measures which converges narrowly to a finite Radon measure μ. We shall be interested in finding conditions on the bounded, not necessarily continuous function f, defined on the Hausdorff space X and taking its values in a Banach space F, which ensure that $\mu_j(f)$ converges to $\mu(f)$.

Let us first point out the following difficulties.

(a) If X is a completely regular space, the maps $\mu \longmapsto \mu^+$, $\mu \longmapsto \mu^-$ and $\mu \longmapsto |\mu|$ from $\mathscr{M}^1(X)$ into $\mathscr{M}^1_+(X)$ are in general not continuous for the narrow topology.

(b) If A is a Borel subset of the completely regular space X, then the map $\mu \longmapsto \mu_A$ from $\mathscr{M}^1(X)$ into $\mathscr{M}^1(A)$, and the map $\mu \longmapsto \mu(A)$ from $\mathscr{M}^1(X)$ into $\mathbf{R}$ are not necessarily continuous (see §1, Remark 1).

Therefore the results of this section will concern only positive measures, which has the added advantage that we are not restricted to completely regular spaces X.

Motivated by a classical property of Riemann-integrable functions, we introduce the following

DEFINITION 2. *Let μ be a positive, finite Radon measure on the Hausdorff space X. We say that the bounded function f, defined on*

X and taking its values in the Banach space F, is Riemann-integrable with respect to μ, if the set of points of discontinuity of f is negligible with respect to μ.

Let $f: X \to F$ be Riemann-integrable with respect to μ. The measure μ is concentrated on the set Y of those points at which f is continuous. The restriction f_Y of f to Y is continous on Y, hence it is μ_Y-measurable and so f is μ-measurable. Since f is bounded and μ finite, f is μ-integrable.

Consider now a real-valued, bounded function f defined on the Hausdorff space X. Let f^+ be the smallest upper semi-continuous function which is $\geqslant f$, given by

$$f^+(x) = \lim_{y \to x} \sup f(y),$$

and let f^- be the largest lower semi-continuous function which is $\leqslant f$, given by

$$f^-(x) = \lim_{y \to x} \inf f(y)$$

[2, Chap. IV, 3rd ed., §6, n° 2, Prop. 4]. The function f is μ-Riemann-integrable if and only if $f(x) = f^+(x) = f^-(x)$ for μ-almost every x. In particular we see again that if a bounded, numerical function is μ-Riemann-integrable, then it is μ-measurable and μ-integrable.

We now give the first answer to the question raised at the beginning of this section.

THEOREM 1. *Let (μ_j) be a sequence or a filter basis of positive, finite measures which converges narrowly to the positive, finite measure μ on the Hausdorff space X. Assume that the measures μ_j and μ are all concentrated on the Borel subset A of X. Let f be a bounded, function taking its values in a Banach space F, whose restriction f_A to A is $(\mu_j)_A$-measurable for every j, and Riemann-integrable with respect to μ_A. Then*

$$\lim_j \mu_j(f) = \mu(f).$$

PROOF. First case: $F = \mathbf{R}$. By virtue of Prop. 4 (§1) we may assume that $A = X$. Let f^+ and f^- be the semicontinuous functions introduced above. Then

$$\liminf_j \mu_j(f) \geqslant \liminf_j \mu_j(f^-) \geqslant \mu(f^-) = \mu(f)$$

and

$$\limsup_j \mu_j(f) \leqslant \limsup_j \mu_j(f^+) \leqslant \mu(f^+) = \mu(f),$$

hence indeed $\lim_j \mu_j(f) = \mu(f)$.

Observe that if X is completely regular, then Prop. 4 follows from Theorem 1 since then any bounded continuous function on A can be extended to a function on X which satisfies the conditions of the theorem.

A subset Y of a Hausdorff space is said to be Riemann-integrable with respect to the positive, finite Radon measure μ if its characteristic function χ_Y is μ-Riemann-integrable, i.e. if the boundary of Y is negligible with respect to μ. We have the following consequence of this first case of Theorem 1.

COROLLARY. *If the positive, finite measures μ_j converge narrowly to the positive, finite measure μ on the Hausdorff space X, then $\lim_j \mu_j(Y) = \mu(Y)$ for every μ-Riemann-integrable subset Y of X which is measurable with respect to every μ_j.*

Second Case (general case): F is arbitrary. We may still assume that $A = X$. Choose $\epsilon > 0$, and $\epsilon_1 = \epsilon/6$. There exists a separable Banach subspace F_0 of F such that $f(x) \in F_0$ for μ-almost all values of x in X. Let $(\Omega_i)_{i \in I}$ be an open covering of F by balls of radius $\leqslant \epsilon_1$, such that only a countable infinity of them intersect F_0, say the Ω_i, $i \in J$, J countable. Let $(\alpha_i)_{i \in I}$ be an associated partition of unity, by continuous functions $\alpha_i \geqslant 0$; this is possible because a metric space is paracompact. Choose a point f_i in every Ω_i, and put $g(x) = \sum_{i \in I} \alpha_i(f(x))\, f_i$.

Since f is bounded, g is also bounded; namely, if $\|f\| < M$, the only terms which intervene in the previous sum are those for which $\alpha_i(f(x)) \neq 0$, and for them $\|f_i\| \leqslant M + \epsilon_1$, and we have $\|g\| \leqslant M + \epsilon_1$. Clearly one has $f(x) = \sum_{i \in I} \alpha_i(f(x))\, f(x)$, hence $\|f(x) - g(x)\| < \epsilon_1$ for every x in X.

Now g is measurable for μ; namely, it coincides μ-almost everywhere with the partial sum $\sum_{i\in J}$, a countable sum of measurable functions (f is measurable, α_i is continuous). Therefore g is μ-integrable. Applying Lebesgue's theorem of dominated convergence, we may write

$$1 = \sum_i \mu(\alpha_i \circ f) \text{ or } \sum_J,$$

$$\mu(g) = \sum_i \mu(\alpha_i \circ f) f_i \text{ or } \sum_J.$$

Therefore there exists a finite subset K of J such that,

$$\sum_{i\in K} \mu(\alpha_i \circ f) < 1 - \eta, \eta = \frac{\epsilon}{6(M+\epsilon_1)}.$$

Put $g' = \sum_{i\in K} (\alpha_i \circ f) f_i$. Now we can write

$$\|\mu(f) - \mu_j(f)\| < \mu(\|f-g\|) + \mu(\|g-g'\|) + \|\mu(g') - \mu_j(g')\| + \\ + \mu_j^*(\|g'-g\|) + \mu_j^*(\|g-f\|).$$

And we have the following inequalities :

(a) $\mu(\|f-g\|) < \epsilon_1$.

(b) $$\mu(\|g-g'\|) = \mu\left(\left\|\sum_{i\in I-K} (\alpha_i \circ f) f_i\right\|\right) \\ < (M+\epsilon_1)\,\mu\left(\sum_{i\in I-K} (\alpha_i \circ f)\right) < (M+\epsilon_1)\,\eta.$$

(c) $\|\mu(g') - \mu_j(g')\| < \epsilon_1$ for $j > j_1$ suitably chosen, since μ_i converges narrowly to μ and g' is a finite linear combination (with vector-valued coefficients f_i) of real μ-Riemann-integrable and μ_j-integrable functions $\alpha_i \circ f$.

(d) $$\mu_j^*(\|g'-g\|) = \mu_j^*\left(\left\|\sum_{i\in I-K} (\alpha_i \circ f) f_i\right\|\right) \\ < (M+\epsilon)\,\mu_j^*\left(\sum_{i\in I-K} (\alpha_i \circ f)\right);$$

but $\sum_{i\in K} (\alpha_i \circ f)$ is μ_j-integrable and μ-Riemann-integrable; therefore $\sum_{i\in I-K} (\alpha_i \circ f)$, its complement to 1, has the same property, and $\lim_j \mu_j\left(\sum_{i\in I-K} (\alpha_i \circ f)\right) = \mu\left(\sum_{i\in I-K} (\alpha_i \circ f)\right) \leqslant \eta$; therefore, for $j \geqslant j_2$ suitably chosen, $\mu_j\left(\sum_{i\in I-K} (\alpha_i \circ f)\right) \leqslant 2\eta$, and $\mu_j^*(\|g' - g\|) \leqslant (M + \epsilon_1)\, 2\eta$.

(e) $\mu_j^*(\|g - f\|) \leqslant \epsilon_1$.

By addition, we get

$$\|\mu(f) - \mu_j(f)\| \leqslant \epsilon_1 + (M + \epsilon_1)\,\eta + \epsilon_1 + (M + \epsilon_1)\, 2\eta + \epsilon_1$$
$$= 3\epsilon_1 + 3\eta(M + \epsilon_1) = \epsilon$$

for $j \geqslant j_1$ and j_2, which completes the proof.

The first proof for F Banach has been given by Maurey [25]; the present proof is due to Royer (unpublished).

EXAMPLE 1. Let (a_j) be a sequence of points of X which converges to the point $a \in X$, and is such that $a_j \neq a$ for every j. Then the Dirac measures $\delta_{(a_j)}$ converge narrowly to the Dirac measure $\delta_{(a)}$ but $\delta_{(a_j)}(\{a\}) = 0$ does not converge to $\delta_{(a)}(\{a\}) = 1$. If a is not an isolated point of X then the boundary of $\{a\}$ is $\{a\}$, which is not negligible with respect to $\delta_{(a)}$.

DEFINITION 3. *Let X be a Hausdorff space, F a Banach space, μ a positive finite Radon measure on X, and f a bounded function defined on X and taking its values in F. We say that f satisfies condition (R, μ) if for every $\epsilon > 0$ there exists a bounded, continuous $g: X \to F$ belonging to $\mathscr{B}(X) \otimes F$, and a bounded, positive, continuous function $h: X \to \mathbf{R}$ such that $\|f - g\| \leqslant h$ and $\mu(h) \leqslant \epsilon$.*

The condition $g \in \mathscr{B}(X) \otimes F$ means that g is of the form $x \longmapsto g(x) = \sum_{i=1}^{n} g_i(x)\, v_i$, where $(g_i)_{1\leqslant i\leqslant n}$ is a finite family of functions belonging to $\mathscr{B}(X)$, and $(v_i)_{1\leqslant i\leqslant n}$ is a finite family of elements of F.

THEOREM 2. *Let μ be a positive, finite Radon measure on the Hausdorff space X, let F be a Banach space, and f a bounded function defined on X and taking its values in F.*

(a) *If f satisfies condition (R, μ), then it is Riemann-integrable with respect to μ.*

(b) *If X is completely regular and f Riemann-integrable with respect to μ, then f satisfies condition (R, μ).*

PROOF. (a) We introduce the oscillation ω_f of f defined by

$$\omega_f(x) = \lim_{y,z \to x} \sup \| f(y) - f(z) \|.$$

The function f has a discontinuity at $x \in X$ if and only if $\omega_f(x) > 0$. Let $\epsilon > 0$, and choose $g \in \mathscr{B}(X) \otimes F$ and $h \in \mathscr{B}(X)$ so that $\| f - g \| \leqslant h$, $\mu(h) \leqslant \epsilon/2$. Since g and h are continuous, we have $\omega_f \leqslant \omega_{f-g} \leqslant 2h$, and therefore $\mu(\omega_f) \leqslant 2\mu(h) \leqslant \epsilon$. Since ϵ is arbitrary, we have $\mu(\omega_f) = 0$, i.e. the set $\{x \mid \omega_f(x) > 0\}$ is μ-negligible.

(b) Since f is μ-integrable, for any $\epsilon > 0$ there exists a function $g \in \mathscr{B}(X) \otimes F$ such that $\mu^*(\| f - g \|) \leqslant \epsilon/2$. Let k be the function $x \longmapsto \| f(x) - g(x) \|$, and let k^+ be smallest upper semi-continuous function which is $\geqslant k$. Since f is μ-Riemann-integrable, we have $k(x) = k^+(x)$ for μ-almost every x, and thus $\mu(k^+) \leqslant \epsilon/2$. The complete regularity of X implies that $\mu(k^+)$ is the greatest lower bound of all values $\mu(h)$, where $h \in \mathscr{B}(X)$ and $h \geqslant k^+$. Hence there exists $h \in \mathscr{B}(X)$ such that $h \geqslant k^+ \geqslant k = \| f - g \|$ and $\mu(h) < \epsilon$.

3. A compactness criterion for the narrow topology. We shall generalize in part a theorem of Prokhorov ('Convergence of random processes and limit theorems in probability theory', Theory of Probability and its Applications, 1 (1956), pp. 157-214; Chap. 1, §5, Theorem 1.12, p. 170) to arbitrary Hausdorff spaces, according to Topsoe's ideas.

DEFINITION 4. *A subset A of $\mathscr{M}^1_+(X)$ is said to be equally interiorly regular (or equally tight) if for every $\epsilon > 0$ there exists a compact subset K of X such that $\mu(X - K) < \epsilon$ for every $\mu \in A$.*

THEOREM 3. *Let X be a Hausdorff space. If a subset A of $\mathscr{M}^1_+(X)$ is equally interiorly regular, and if the set $\{\mu(X) \mid \mu \in A\}$ is bounded, then A is relatively compact for the narrow topology,* (§1, *Def.* 1).

PROOF. Let $\mathfrak{U}$ be an ultrafilter on A, and denote by K an arbitrary compact subset of X. If for every $F \in \mathfrak{U}$ we denote by F_K the set of restrictions μ_K to K of the measures $\mu \in F$, then the sets F_K generate an ultrafilter $\mathfrak{U}_K$ of positive Radon measures on K [2, Chap. I, 3rd ed., §6, $n°6$, Prop. 10]. Since $\mu_K(K)$ is bounded as μ varies in A, by the weak*-compactness of the unit ball of $\mathscr{C}'(K)$, the ultrafilter $\mathfrak{U}_K$ converges vaguely (i.e. narrowly, or for the weak*-topology) to a positive Radon measure $\mu^{(K)} \in \mathscr{C}'(K)$. Let μ^K be the image of $\mu^{(K)}$ under the canonical injection $K \to X$.

The measures μ^K form an ordered, increasingly directed set of positive, finite measures on X, indexed by the collection of compact subsets of X. Indeed, let K and H be two compact subsets of X such that $K \subset H$, and let φ be a positive continuous function on K. The extension $\bar{\varphi}$ of φ to X, which takes the value zero outside K, is a positive, upper semi-continuous function on X. We have

$$(\mu^{(H)})_K(\varphi) = \mu^{(H)}(\bar{\varphi}) \geqslant \limsup_{\mathfrak{U}} \mu_H(\bar{\varphi}) = \lim_{\mathfrak{U}} \mu_K(\varphi) = \mu^{(K)}(\varphi),$$

hence $(\mu^{(H)})_K \geqslant \mu^{(K)}$ and therefore $\mu^H \geqslant \mu^K$.

Since the total masses of the measures in A are bounded, the μ^K have a least upper bound μ_0, which is also the limit of the directed set $\{\mu^K\}$. This limit μ_0 is a positive, finite measure on X which verifies $\mu_0(f) = \sup_K \mu^K(f)$ for every bounded, positive Borel function f on X.

For any positive, bounded, lower semi-continuous function f on X, and for every compact subset K of X inequalities

$$\liminf_{\mathfrak{U}} \mu(f) \geqslant \liminf_{\mathfrak{U}} \mu_K(f_K) \geqslant \mu^{(K)}(f_K) = \mu^K(f) \tag{3}$$

hold. Since this is true for any K, it follows that

$$\liminf_{\mathfrak{U}} \mu(f) \geqslant \mu_0(f).$$

We have to prove that the same inequality holds for any bounded, lower semi-continuous function f, and not just for

positive ones (the trouble is with the first inequality in the chain of inequalities (3) above).

To do this, let us first show that $\lim_{\mathfrak{U}} \mu(1) = \mu_0(1)$. We already know that $\liminf_{\mathfrak{U}} \mu(1) \geqslant \mu_0(1)$. Given $\epsilon > 0$, there exists, by the hypothesis of equal interior regularity, a compact subset K of X such that $\mu(X - K) \leqslant \epsilon$ for every $\mu \in A$. Hence

$$\limsup_{\mathfrak{U}} \mu(1) \leqslant \limsup_{\mathfrak{U}} \mu(K) + \epsilon = \lim_{\mathfrak{U}} \mu_K(K) + \epsilon$$

$$= \mu^{(K)}(K) + \epsilon = \mu^K(K) + \epsilon \leqslant \mu_0(K) + \epsilon \leqslant \mu_0(1) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\limsup_{\mathfrak{U}} \mu(1) \leqslant \mu_0(1)$, from where the assertion follows.

Let now f be any bounded, lower semi-continuous function on X. If $f > -M$, then $f + M > 0$ and therefore we have

$$\liminf_{\mathfrak{U}} \mu(f) = \liminf_{\mathfrak{U}} \mu((f + M) - M)$$

$$= -\mu_0(M) + \liminf_{\mathfrak{U}} \mu(f + M) \geqslant -\mu_0(M) + \mu_0(f + M) = \mu_0(f),$$

and this expresses the fact that $\mathfrak{U}$ converges narrowly to μ_0.

Prokhorov also proved a converse of the previous theorem, under suitable countability conditions:

THEOREM 4. *Let $\mathscr{A}$ be a subset of $\mathscr{M}^1_+(X)$, relatively compact for the narrow topology. Then $\mu(X)$ is bounded for $\mu \in \mathscr{A}$, and if X is locally compact or Polish, for every $\epsilon > 0$ there exists a compact subset K of X such that $\mu(\mathbf{C}\, K) < \epsilon$ for $\mu \in \mathscr{A}$.*

PROOF. The function $\mu \longmapsto \mu(X)$ is continuous on the compact subset $\bar{\mathscr{A}}$ of $\mathscr{M}^1_+(X)$, hence it is bounded.

Assume first that X is locally compact. For every $\mu \in \mathscr{A}$ there exists a compact subset K_μ of X such that $\mu(\mathbf{C}\, K_\mu) < \epsilon$. Let U_μ be any relatively compact open neighborhood of K_μ; we have $\mu(\mathbf{C}\, U_\mu) < \epsilon$. But the function $\mu \longmapsto \mu(\mathbf{C}\, U_\mu)$ is upper semi-continuous on $\mathscr{M}^1_+(X)$, therefore there exists an open neighborhood U_μ of μ in $\mathscr{M}^1_+(X)$ such that for $\mu \in U_\mu$ we have $\mu(\mathbf{C}\, U_\mu) < \epsilon$. When μ

varies, the open sets U_μ cover the compact space $\mathscr{A}$, therefore $\mathscr{A}$ is covered by a finite number of them, corresponding to $\mu_1, \mu_2, \dots \mu_n$. If K is the compact subspace of X defined by $K = \bigcup_{j=1}^{n} \bar{U}_{\mu_j}$, then $\mu(\mathbf{C}K) \leqslant \epsilon$ for $\mu \in \mathscr{A}$.

Assume now that X is a subspace of a topological space Y, and that it is the intersection of a countable family of locally compact subspaces Y_n of Y. If $\bar{\mathscr{A}}$ is the closure of $\mathscr{A}$ in $\mathscr{M}^1_+(X)$, it can be identified (through the injections $X \subset\!\!\rightarrow Y_n \subset\!\!\rightarrow Y$) with a subspace of $\mathscr{M}^1_+(Y)$ or $\mathscr{M}^1_+(Y_n)$, with the induced topology, and remains compact in each of these spaces. For every $\epsilon > 0$, there exists a compact subspace K_n of Y_n (assumed to be locally compact) such that $\mu(\mathbf{C}K_n) \leqslant \epsilon/2^{n+1}$ for $\mu \in \mathscr{A}$. If we put $K = \bigcap_{n \geqslant 0} K_n$, we have $\mu(\mathbf{C}K) \leqslant \epsilon$ for $\mu \in \mathscr{A}$, which proves the result in this case. But if X is Polish, it is a G_δ in a compact space Y (Part I, Ch. II, Th. 1), and an open subspace of a compact space is locally compact, which completes the proof.

4. A theorem of Paul Levy.

THEOREM 5. *Assume that the probabilities μ_j on $\mathbf{R}^n$ converge narrowly to the probability μ. Then the Fourier transforms $\varphi_j = \mathscr{F}(\mu_j)$ converge to $\varphi = \mathscr{F}(\mu)$, uniformly on every compact subset of $\mathbf{R}^n$.*

We shall use the following lemma, whose proof will be given after the proof of the theorem.

LEMMA 1. *Suppose that the positive, finite measures μ_j converge narrowly to the positive, finite measure μ on the completely regular space X, and let Y be a compact topological space. Let f be a bounded, continuous function defined on $X \times Y$ with values in the Banach space F. Then the functions*

$$y \longmapsto \int_X f(x, y)\, d\mu_j(x)$$

converge to the function

$$y \longmapsto \int_X f(x, y)\, d\mu(x),$$

uniformly on Y.

PROOF OF THEOREM 5. Let $X = \mathbf{R}^n$, take for Y a compact subset of the dual of $\mathbf{R}^n$, and $f(x, y) = e^{-2\pi i\langle x, y\rangle}$. Then by Lemma 1, the function

$$y \longmapsto \mathscr{F}(\mu_j)(y) = \int e^{-2\pi i\langle x, y\rangle}\, d\mu_j(x)$$

converges to

$$y \longmapsto \mathscr{F}(\mu)(y) = \int e^{-2\pi i\langle x\ y\rangle}\, d\mu(x),$$

uniformly on Y.

PROOF OF LEMMA 1. For any $x \in X$, the function $f_x : y \longmapsto f(x, y)$ belongs to the Banach space $\mathscr{C}(Y; F)$ of continuous functions on Y with values in F. The map $g : x \longmapsto f_x$ from X into $\mathscr{C}(Y; F)$ is bounded, and since Y is compact, g is also continuous. It follows from Theorem 1 (§2) that $\mu_j(g)$ converges to $\mu(g)$ in $\mathscr{C}(Y; F)$.

For any $y \in Y$, the map, which associates with each element of $\mathscr{C}(Y; F)$ its value at y, is a continuous linear map from $\mathscr{C}(Y; F)$ onto F. Therefore, for any $y \in Y$ and $\nu \in \mathscr{M}^1(X)$ the diagram

$$\begin{array}{ccc} \mathscr{B}(X; \mathscr{C}(Y; F)) & \xrightarrow{\ u\ } & \mathscr{C}(Y; F) \\ \alpha \downarrow & & \downarrow \beta \\ \mathscr{B}(X; F) & \xrightarrow{\ v\ } & F \end{array}$$

is commutative, where $u : h \longmapsto \int h(x)\, d\nu(x)$, $v : \varphi \longmapsto \int \varphi(x)\, d\nu(x)$, $\alpha : h \longmapsto (x \longmapsto h(x)(y))$, and $\beta : \psi \longmapsto \psi(y)$. Consequently, we have $\mu_j(g)(y) = \int_X f(x, y)\, d\mu_j(x)$ and $\mu(g)(y) = \int_X f(x, y)\, d\mu(x)$.

Next we prove the converse of Theorem 5.

THEOREM 6. *Consider the sequence or the filter basis* (μ_j) *of probabilities on* $\mathbf{R}^n$, *and let* $\varphi_j = \mathscr{F}(\mu_j)$ *be the Fourier transform of* μ_j. *Assume that the* φ_j *converge in the space of* $\mathscr{S}'(\mathbf{R}^n)$ *of tempered distributions to a limit* $\varphi \in \mathscr{S}'(\mathbf{R}^n)$ *which is a function, and suppose furthermore that one of the following hypotheses is satisfied.*

(i) φ *is the Fourier transform (in the sense of tempered distributions) of a probability on* $\mathbf{R}^n$.

(ii) *The functions* φ_j *converge pointwise to a limit* φ_0 *on a set* $A \subset \mathbf{R}^n$, *which contains, up to a set of Lebesgue measure zero, a neighborhood of* 0. *Furthermore in this neighborhood* φ_0 *is equal to* φ *almost everywhere with respect to Lebesgue measure, and* $\lim_{y \in A, y \to 0} \varphi_0(y) = 1$.

(iii) *The functions* φ_j *converge to a limit* φ_0, *uniformly in a neighborhood of* 0. (*This is the case in particular if the* φ_j *converge to a limit* φ_0, *uniformly on every compact subset of* $\mathbf{R}^n$. *This formulation is the exact converse of Theorem* 5.)

Then (μ_j) *converges narrowly to a probability* μ, *and* φ *is the Fourier transform of* μ *(in the sense of distributions).*

PROOF. Since (φ_j) converges to φ in $\mathscr{S}'(\mathbf{R}^n)$, there exists a $\mu \in \mathscr{S}'(\mathbf{R}^n)$ such that $\varphi = \mathscr{F}(\mu)$ and (μ_j) converges to μ in $\mathscr{S}'(\mathbf{R}^n)$. For every positive function $\alpha \in \mathscr{S}(\mathbf{R}^n)$ we have $\mu(\alpha) = \lim_j \mu_j(\alpha) \geqslant 0$, and therefore μ is a positive Radon measure on $\mathbf{R}^n$. Since $\mathscr{K}(\mathbf{R}^n) \subset \mathscr{S}'(\mathbf{R}^n)$, the measures μ_j converge vaguely to μ. Furthermore, since $\mu(1) = \sup_{\substack{\alpha \in K(\mathbf{R}^n) \\ 0 \leqslant \alpha \leqslant 1}} \mu(\alpha)$ and $\mu_j(\alpha) \leqslant 1$ for every $\alpha \in \mathscr{K}(\mathbf{R}^n)$ such that $0 \leqslant \alpha \leqslant 1$, we also have $\mu(1) \leqslant 1$. In particular, the measure μ is finite, hence it has a Fourier integral ψ given by $\psi(y) = \int_{\mathbf{R}^n} e^{-2\pi i \langle x, y \rangle} \, d\mu(x)$, and ψ is equal to φ almost everywhere with respect to Lebesgue measure.

If now hypothesis (i) is satisfied, then φ is the Fourier transform of a probability, which is necessarily μ. Thus $\mu(1) = 1$, and it follows from Prop. 5 (§1) that (μ_j) converges to μ narrowly.

Under hypothesis (ii), ψ is equal to φ_0 on a neighborhood of 0, except for a subset of Lebesgue measure zero. If y remains in A and tends to 0, then $\varphi_0(y)$ tends to 1 and $\psi(y)$ tends to $\mu(1)$. Therefore $\mu(1) = 1$, and the proof concludes as before.

Finally, hypothesis (iii) implies that $\varphi_0(0) = 1$ and that φ_0 is continuous in a neighborhood of 0. Now (φ_j) converges to φ in $\mathscr{S}'(\mathbf{R})$, and therefore $\varphi = \varphi_0$ in a neighborhood of 0 with the exception of a set of Lebesgue mesure zero. Thus we are reduced to hypothesis (ii).

REMARK 2. Customarily one considers a *sequence* (μ_j), and it is assumed that the $\varphi_j = \mathscr{F}(\mu_j)$ converge pointwise almost everywhere to a limit φ, so that, by the Lebesgue dominated convergence theorem, (φ_j) converges to φ in $\mathscr{S}'(R^n)$. Then one of the following hypotheses is added:

(i) the limit φ is the Fourier integral of a probability;

(ii) the limit φ is continuous at the origin;

(iii) the convergence is uniform on every compact set.

It is fairly clear that $\varphi = \mathscr{F}(\mu)$ with $\mu(1) \leqslant 1$ but the delicate point is to establish that $\mu(1) = 1$. Since $\varphi_j(0) = 1$ for every j, we have $\varphi(0) = 1$, but without some continuity hypothesis at the origin this does not prove that $\mu(1) = 1$ because $\mathscr{F}(\mu)$ is defined only almost everywhere.

Combining Theorems 5 and 6 we obtain

PROPOSITION 6. *Denote by $P(\mathbf{R}^n)$ the convex cone (Chap. 5, §2) of all continuous functions of positive type, equipped with the topology of uniform convergence on compact sets. Then the Fourier transformation is a homeomorphism from $\mathscr{M}^1_+(\mathbf{R}^n)$ equipped with the narrow topology onto $P(\mathbf{R}^n)$.*

Instead of $\mathbf{R}^n$ and $\mathbf{R}^n$ we could clearly substitute any finite-dimensional vector space and its dual.

5. The space of finite measures on a Suslin space.

THEOREM 7. *If the Hausdorff space X is Polish (or a Lusin space, or a Suslin space), then the same is true for $\mathscr{M}^1_+(X)$ equipped with the narrow topology* (§1, *Def.* 1).

PROOF. (a) Let us first suppose that X is a compact metrizable space. Then the Banach space $\mathscr{C}(X)$ is separable [2, Chap X, 2nd ed., §3, n° 3, Th. 1], the unit ball of its dual is compact and metrizable for the weak*-topology, hence it is a Polish space. Since $1 \in \mathscr{C}(X)$, the set M of positive measures with total mass < 1 is an open subset of this unit ball, hence M is also a Polish space. The map $\mu \longmapsto \mu/(1 + \mu(1))$ is a homeomorphism from $\mathscr{M}^1_+(X)$ onto M, thus $\mathscr{M}^1_+(X)$ is a Polish space.

(b) Let now X be a Polish space. Then X is a G_δ subset of a compact metrizable space K, i.e. $X = \bigcap_n U_n$, where (U_n) is a sequence of open subsets of K. For each n and every $\delta > 0$, the set $S_{n,\delta}$ of all measures $\mu \in \mathscr{M}^1_+(K)$ which satisfy $\mu(K) - \mu(U_n) < \delta$, i.e. $\mu(U_n) > \mu(K) - \delta$ is open. Indeed let $\mu_0 \in S_{n,\delta}$, and $\mu_0(U_n) = \mu_0(K) - \delta + \epsilon$. If μ belongs to the neighborhood $V(\mu_0; \epsilon/3; \chi_{U_n}, -1)$ of μ_0 (§1), then

$$\begin{aligned}\mu(U_n) &\geqslant \mu_0(U_n) - \epsilon/3 = \mu_0(K) - \delta + 2\epsilon/3 \\ &\geqslant \mu(K) - \delta + \epsilon/3 > \mu(K) - \delta,\end{aligned}$$

i.e. $\mu \in S_{n,\delta}$.

The intersection of all the sets $S_{n,\delta}$ is a G_δ set in $\mathscr{M}^1_+(K)$, and it is the set $(\mathscr{M}^1_+)_X(K)$ of all measures belonging to $\mathscr{M}^1_+(K)$ which are concentrated on X. By part (a) of the proof $\mathscr{M}^1_+(K)$ is a Polish space, hence $(\mathscr{M}^1_+)_X(K)$, which is a G_δ set in a Polish space, is itself a Polish space. By Prop. 4 and Remark 1 (§1) the space $(\mathscr{M}^1_+)_X(K)$ is homeomorphic to $\mathscr{M}^1_+(X)$, hence the latter is a Polish space.

(c) Next we assume that X is a Lusin space. Let P be a Polish space, and F a continuous bijective map from P onto X. The map $\mu \longmapsto F(\mu)$ from $\mathscr{M}^1_+(P)$ into $\mathscr{M}^1_+(X)$ is bijective (Part I, Ch. II, Cor. 2 of Th. 10), and by Prop. 3 (§1) it is continuous. Since $\mathscr{M}^1_+(P)$ is a Polish space by part (b), we obtain that $\mathscr{M}^1_+(X)$ is a Lusin space.

(d) Finally, let us suppose that X is a Suslin space. Let P be a Polish space, and F a continuous surjective map from P onto X. We know (Part I, Ch. II, Th. 12) that the map $\mu \longmapsto F(\mu)$ from

$\mathscr{M}^1_+(P)$ into $\mathscr{M}^1_+(X)$ is surjective, and by Prop. 3 (§1) it is continuous. Since by part (b) of the proof $\mathscr{M}^1_+(P)$ is a Polish space, $\mathscr{M}^1_+(X)$ is a Suslin space.

COROLLARY. *If X is a completely regular Suslin space, then $\mathscr{M}^1(X)$ equipped with the narrow topology is a Suslin space.*

Indeed, $(\mu,\nu) \longmapsto \mu-\nu$ is a continuous, surjective map from the Suslin space $\mathscr{M}^1_+(X)\times\mathscr{M}^1_+(X)$ onto $\mathscr{M}^1(X)$.

If X is a compact metrizable space, then $\mathscr{M}^1(X)$ is Lusin space as the weak dual of a separable Banach space. We do not know whether $\mathscr{M}^1(X)$ is a Lusin space if X is a completely regular Lusin space, or even if X is a Polish space.

Let us recall that a map F from a topological space X into a topological space Y is called a Borel map if for every Borel subset B of Y, the set $f^{-1}(B)$ is a Borel subset of X.

THEOREM 8. *For every bounded Borel function f on X, the function $\mu \longmapsto \mu(f)$ on $\mathscr{M}^1_+(X)$ is Borel. If X (and therefore $\mathscr{M}^1_+(X)$) is Suslin, the Borel field on $\mathscr{M}^1_+(X)$ is the smallest one containing all the functions $\mu \longmapsto \mu(f)$.*

PROOF. The set of bounded functions f for which $\mu \longmapsto \mu(f)$ is Borel from $\mathscr{M}^1_+(X)$ into $\mathbf{R}$, is a vector space, stable with respect to the passage to the limit of a bounded sequence (Lebesgue dominated convergence), and it contains the bounded lower semi-continuous functions (in this case $\mu \longmapsto \mu(f)$ is itself lower semi-continuous); therefore it contains all the bounded Borel functions f on X.

Assume now X is Suslin. We know that its Borel field is countably generated (Part I, Ch. II, Cor. of Lemma 18). Therefore there exists a countable family $\mathscr{F}$ of Borel functions on X, such that $\mu(f) = 0$ for every $f \in \mathscr{F}$ implies $\mu = 0$. Then, since X, i.e. $\mathscr{M}^1_+(X)$ is Suslin, the countable family of functions $\mu \longmapsto \mu(f)$ on $\mathscr{M}^1_+(X)$, $f \in \mathscr{F}$, generates its Borel field (Fernique's theorem, Part I, Ch. II, Lemma 18), which proves the second assertion.

COROLLARY 1. *Assume that X is Suslin. Let $\mathscr{F}$ be a countable family of bounded Borel functions on X, stable with respect to multiplication and separating the points of X. The topology of pointwise convergence on the functions $f \in \mathscr{F}$ is a topology τ on $\mathscr{M}^1_+(X)$, which is metrizable and Suslin, and Borel and Radon equivalent to the narrow topology.*

PROOF. The family $\mathscr{F}$ generates the Borel field of X, and separates the points of $\mathscr{M}^1_+(X)$; equipped with this topology τ, $(\mathscr{M}^1_+(X))_\tau$ is a subspace of $\mathbf{R}^{\mathscr{F}}$, therefore metrizable. The bijection $i\,\mathscr{M}^1_+(X) \to (\mathscr{M}^1_+(X))_\tau \subset \mathbf{R}^{\mathscr{F}}$ is Borel, all its components being Borel, and we know that $\mathscr{M}^1_+(X)$ is Suslin; therefore also $(\mathscr{M}^1_+(X))_\tau$ is Suslin and the bijection i is a Borel and Radon equivalence (Part I, Ch. II, Col. 2 of Th. 10).

COROLLARY 2. *If two topologies on the same set X are both Suslin and have the same Borel sets, the corresponding narrow topologies on $\mathscr{M}^1_+(X)$ are Suslin and have the same Borel sets.*

PROOF. In each case, the Borel field on $\mathscr{M}^1_+(X)$ is the smallest one which contains all the maps $\mu \longmapsto \mu(f)$, where f is a bounded Borel function on X.

COROLLARY 3. *Let X, Y be two topological spaces, $H: X \longmapsto Y$ a Borel map. Assume that Y is Suslin. Then the corresponding map $\mathscr{M}^1_+(X) \to \mathscr{M}^1_+(Y)$ is Borel (for the narrow topologies).*

PROOF. The Borel field of $\mathscr{M}^1_+(Y)$ is generated by the maps $\nu \longmapsto \nu(f)$, where f is a bounded Borel function on Y. Therefore, to prove that $\mathscr{M}^1_+(X) \to \mathscr{M}^1_+(Y)$ is Borel, it suffices to prove that, for every such function f, the map $\mu \longmapsto \mu(f \circ H)$ is Borel; this follows from the theorem since $f \circ H$ is Borel on X.

REMARK 3. Let X be a set and $\mathscr{X}$ a field of subsets of X. Denote by $\mathscr{M}^1_+(X, \mathscr{X})$ the set of all positive finite abstract measures defined on $\mathscr{X}$. We shall say that $\mathscr{X}$ is Suslinian, if there exists on X a Suslin topology τ whose field of Borel sets is $\mathscr{X}$. Two Suslin topologies on X having the same Borel sets have the same space $\mathscr{M}^1_+(X) =$

$\mathcal{M}^1_+(X, \mathcal{X})$ of positive finite Radon measures. Let $\mathcal{F}$ be a countable collection of bounded functions of $\mathcal{X}$, stable with respect to multiplication and separating the points of X. It follows from the previous results that, if we equip $\mathcal{M}^1_+(X, \mathcal{X})$ with the pointwise convergence on $\mathcal{F}$, this topology τ is Suslin metrizable, is Borel equivalent to every narrow topology defined by a topology on X having $\mathcal{X}$ as Borel field and therefore is independent up to a Borel equivalence, of the choice of the collection $\mathcal{F}$.

TERMINOLOGICAL INDEX

PART II

1. Absolutely final random function, p. 318.
2. Almost surely continuous random function, p. 255.
3. Brownian motion, p. 360.
4. Canonical family of random variables, p. 246.
5. Canonical random function, p. 254.
6. Completing subset, p. 230.
7. Concentration (scalar; cylindrical), pp. 188, 192.
8. Conuclear maps, conuclear spaces, p. 227.
9. Convergence in probability, p. 247.
10. Convolution of cylindrical measures, p. 183.
11. Cylindrical measures, p. 171.
12. Decomposed map, p. 295.
13. Decomposing map, p. 299.
14. Final random function, p. 308.
15. Fourier transform of a cylindrical measure, p. 184.
16. Fundamental random function, p. 307.
17. Gauss cylindrical measures, p. 325.
18. Hilb (E), p. 331.
19. Hilbert-Schmidt map, p. 207.
20. Hilbert subset, p. 230.
21. Hilbert subspace, p. 228.
22. Image of a cylindrical measure, p. 181.
23. Independent random variables, p. 245.
24. Independent events, p. 244.
25. Independent random functions, p. 254.

26. Initial random function, p. 308.
27. Isonomous random functions, p. 254.
28. Isonomous random variables, p. 245.
29. Linear random function, p. 256.
30. Meas $(\Omega, m; X)$, p. 244.
31. Minlos theorem, p. 233.
32. Narrow convergence, p. 249.
33. Nuclear maps, p. 219.
34. Nuclear spaces, p. 226.
35. Positive type (functions of), p. 187.
36. Prehilbertian neighbourhood, p. 229.
37. Radon-equivalent topologies, p. 155.
38. Radonifying map, p. 299.
39. Random function, p. 253.
40. Random variable, p. 244.
41. Representable random function, p. 306.
42. Reproducing kernel, p. 337.
43. Sample space, p. 244.
44. Saturated collection of subsets, p. 193.
45. Tensor product of cylindrical measures, p. 180.
46. Ultrabornological (vector spaces), p. 157.
47. Universally initial (isonomy class of random functions), p. 318.

REFERENCES

1. N. Bourbaki: *Éléments de mathématique, Livre II, Algèbre*; Hermann, Paris, (1942-1962).
2. N. Bourbaki: *Éléments de mathématique, Livre III, Topologie génerale*; Hermann, Paris, (1953-1961).
3. N. Bourbaki: *Éléments de mathématique, Livre V, Espaces vectoriels topologiques*; Hermann, Paris, (1953, 1955, 1966).
4. N. Bourbaki: *Éléments de mathématique, Livre VI, Intégration*; Hermann, Paris, (1956-1965).
5. M. De Wilde: Réseaux dans les espaces linéaires à semi-normes; *Mém. Soc. Roy. Sci. Liège*, 5e séric, 18, 2 (1969).
6. I. M. Gelfand and N. Ja. Vilenkin : *Generalized functions* IV, *Some applications to harmonic analysis, rigged Hilbert spaces*; Gosudarstvennoe izdatel'stvo fiziko-matematičeskoĭ literatury, Moscow, (1961). English translation: Academic Press, New York–London, (1964).
7. R. Godement : Les fonctions de type positif et la théorie des groupes; *Trans. Amer. Math. Soc.* 63 (1948), 1-84.
8. A. Grothendieck : Sur les applications linéaires faiblement compactes d'espaces du type C(K); *Canad. J. Math.* 5 (1953), 129-173.
9. A. Grothendieck : Produits tensoriels topologiques et espaces nucléaires; *Mem. Amer. Math. Soc.* 16 (1955).
10. J. L. Kelley and I. Namioka : *Linear topological spaces*; D. van Nostrand, New York–London–Toronto, (1963).
11. M. Loéve : *Probability theory, Foundations—Random sequences*; D. van Nostrand, New York–London–Toronto, (1955).
12. L. Schwartz : Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés. (Noyaux reproduisants); *J. Analyse Math.* 13 (1964), 115-256.
13. L. Schwartz : Sur le théorème du graphe fermé; *C. R. Acad. Sci. Paris, Sér. A-B* 263 (1966), A602-A605.
14. L. Schwartz : Extensions du théorème de Sazonov-Minlos à des cas non hilbertiens; *C. R. Acad. Sci. Paris, Sér. A-B* 265 (1967), A832-A834.

15. L. SCHWARTZ : Réciproque du théorème de Sazonov-Minlos dans des cas non hilbertiens; *C. R. Acad. Sci. Paris, Sér. A-B* 266 (1968), A7-A9.

16. L. SCHWARTZ : Démonstration de deux lemmes sur les probabilités cylindriques; *C. R. Acad. Sci. Paris, Sér. A-B* 266 (1968), A50-A52.

17. L. SCHWARTZ: Application radonifiantes Séminaire de l'Ecole Polytechnique, (1969-70), Paris.

18. F. TREVES : *Topological vector spaces, distributions and kernels*; Academic Press, New York and London, (1967).

19. V. V. SAZONOV: Remarks on characteristic functionals; *Theory of probability and its applications*, 3 (1958), 201-205.

20. R. A. MINLOS : *Generalized random processes and their extension in measure*; Selected translations in mathematical statistics and probability, *3, Amer. Math. Soc. Providence.*

21. L. GROSS : *Analysis in Function space*, Massachusetts Institute of Technology, (1963).

22. A. BADRIKIAN : to appear.

23. F. TOPSOE : *Topology and Measure*; Lecture Notes, Springer Verlag, Berlin–Heidelberg–New York, 133, (1970).

24. S. CHEVET : Continuité des trajectoires des processus continus en probabilité; *C. R. Acad. Sc. Paris, Ser. A*, 269, (1969) 593-596.

25. BERNARD MAUREY : Une propriété de la convergence ètroite, *C. R. Ac. Sci. Paris, Sér.* A, 270, 448-451, (1970).

26. L. SCHWARTZ : Mesures cylindriques et applications radonifiantes dans les espaces de suites; *Proceedings of the International Conference on Functional Analysis and related topics, Tokyo, April* (1969) (*Mathematical Society of Japan, University of Tokyo Press*, (1970)).

PRINTED IN INDIA

BY R. SUBBU

AT THE

TATA PRESS LIMITED

BOMBAY

AND

PUBLISHED BY

JOHN BROWN

OXFORD UNIVERSITY PRESS

BOMBAY