Automorphic Forms, Representation Theory and Arithmetic

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Automorphic Forms, Representation Theory and Arithmetic

Papers presented at the Bombay Colloquium 1979, by

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INTERNATIONAL COLLOQUIUM ON AUTOMORPHIC FORMS REPRESENTATION THEORY AND ARITHMETIC

BOMBAY, 8–15 January 1979

REPORT

An International Colloquium on Automorphic forms, Representation theory and Arithmetic was held at the Tata Institute of Fundamental Research, Bombay, from 8 to 15 January 1979. The purpose of the Colloquium was to discuss recent achievements in the theory of automorphic forms of one and several variables, representation theory with special reference to the interplay between these and number theory, e.g. arithmetic automorphic forms, Hecke theory, Representation of GL_2 and GL_n in general, class fields, L-functions, p-adic automorphic forms and p-adic L-functions.

The Colloquium was jointly sponsored by the International Mathematical Union and the Tata Institute of Fundamental Research, and was financially supported by them and the Sir Dorabji Tata Trust.

An Organizing Committee consisting of Professors P. Deligne, M. Kneser, M.S. Narasimhan, S. Raghavan, M.S. Raghunathan and C.S. Seshadri was in charge of the scientific programme. Professors P. Deligne and M. Kneser acted as representatives of the International Mathematical Union on the Organising Committee.

The following mathematicians gave invited addresses at the Colloquium: W. Casselman, P. Deligne, S. Gelbart, G. Harder, K. Iwasawa, H. Jacquet, N.M. Katz, I. Piatetski-Shapiro, S. Raghavan, T. Shintani, H.M. Stark and D. Zagier.

Professor R. Howe was unable to attend the Colloquium but has sent a paper for publication in the Proceedings.

Professors A. Borel and M. Kneser who accepted our invitation, were unable to attend the Colloquium.

The invited lectures were of fifty minutes' duration. These were followed by discussions. In addition to the programme of invited addresses, there were expository and survey lectures by some invited speakers giving more details of their work. Besides the mathematicians at the Tata Institute, there were also mathematicians from other universities in India who were invitees to the Colloquium.

The social programme during the Colloquium included a Tea Party on 8 January; a programme of Western music on 9 January; a programme of Instrumental music on 10 January; a dinner at the Institute to meet the members of the School of Mathematics on 11 January; a performance of classical Indian Dances (Bharata Natyam) on 12 January; a visit to Elephanta on 13 January; a programme of Vocal music on 13 January and a dinner at the Institute on 14 January.

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ON SHIMURA'S CORRESPONDENCE FOR MODULAR FORMS OF HALF-INTEGRAL WEIGHT*

By S. Gelbart and I. Piatetski-Shapiro

Introduction

1

G. Shimura has shown how to attach to each holomorphic cusp form of
half-integral weight a modular form of even integral weight. More pre-
cisely, suppose $f(z)$ is a cusp form of weight $k/2$, level N, and character
χ. Suppose also that f is an eigenfunction of all the Hecke operators 2
$T^N_{k,\chi}(p^2)$, say $T(p^2)f = \omega_p f$. If $k \geq 5$, then the L-function

$$\sum_{n=1}^{\infty} A(n)n^{-s} = \prod_{p<\infty} \left(1 - \omega_p p^{-s} + \chi(p)p^{2k-2-2s}\right)^{-1}$$

is the Mellin transform of a modular cusp form of weight $k-1$, level $N/2$, and character χ^2. for further details, see [Shim] or [Niwa].

Our purpose in this paper is to establish a Shimura correspondence for any (not necessarily holomorphic) cusp form of half-integral weight defined over a global field F(not necessarily $\mathbb{Q}$). Our approach is similar to Shimura's in that we use L-functions. Out point of view is new in that we use the theory of group representations.

[1]Talk presented by S.G.

Roughly speaking, suppose $\overline{\pi} = \bigotimes_v \overline{\pi}_v$ is an automorphic cuspidal representation of the metaplectic group which doesn't factor through GL_2. Then we introduce an L-factor $L(s, \overline{\pi}_v)$ for each v and we prove that the L-function

$$L(s, \overline{\pi}) = \prod_v L(s, \overline{\pi}_v)$$

belongs to an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ in the sense of [Jacquet-Langlands]. Since we characterize those $\overline{\pi}$ which correspond to cuspidal (as opposed to just automorphic) representations of $\mathrm{GL}_2(\mathbb{A}_F)$ we refine as well as generalize Shimura's results.

Let us now describe our correspondence in more detail. Suppose $\overline{\pi}$ is an automorphic cuspidal representation of the metaplectic group. Since $\overline{\pi}$ is determined by its local components $\overline{\pi}_v$, we want to describe its "Shimura image" $S(\overline{\pi})$ in purely *local* terms. Thus we construct a local correspondence

$$S : \overline{\pi}_v \to \pi_v$$

by "squaring" the representation $\overline{\pi}_v$; if $\overline{\pi}_v$ is an induced representation, this means squaring the characters of F_v^x which parametrize $\overline{\pi}_v$. In general, this process of "squaring" tends to smooth out representations, as we shall now explain.

3 Suppose we consider the theta-representations of the metaplectic group. These representations generalize the classical modular forms of half-integral weight given by the theta-series

$$\theta_\chi(z) = \sum_{n=-\infty}^{\infty} \chi(n) e^{2\pi i n^2_z}$$

where χ is an (even) Dirichlet character of Z. Since these representations arise by pasting together a grossencharacter χ of F with the "even or odd" part of the canonical metaplectic representation constructed in [Weil], we denote these representations by r_χ and call them Weil representations. Locally, r_{χ_v} is supercuspidal when $\chi_v(-1) = -1$. Almost everywhere, however, $\chi_v(-1) = 1$, r_{χ_v} is the class 1 quotient of a reducible principal series representation at $s = 1/2$, and the global representation

$$r_\chi = \bigotimes_v r_{\chi_v}$$

is "distinguished" from several different points of view. Most significantly, these r_χ exhaust the automorphic forms of half-integral weight which are determined by just one Fourier coefficient; this is the principal result of [Ge PS2].

Now if $\overline{\pi}_v$ is an even Weil representation r_{χ_v} (i.e. $\chi_v(-1) = 1$), its Shimura image will be the one-dimensional representation χ_v of $\mathrm{GL}_2(F_v)$, whereas if $\overline{\pi}_v$ is an "odd" Weil representation, $S(\overline{\pi}_v)$ will be the special representation $\mathrm{Sp}(\chi_v)$; cf. §7. The Shimura correspondence thus takes cuspidal r_χ to automorphic representations of GL(2) which almost everywhere are one-dimensional and hence *not* cuspidal. The main result of this paper, however, guarantees that these representations are the only cuspidal $\overline{\pi}$ which map to non-cuspidal automorphic forms of GL(2). This explains the restriction $k \geq 5$ in [Shim] and ultimately resolves "Open question (C)" of that paper; cf. §16.

We mention also that the cuspidal representations r_χ contradict the Ramanujan-Petersson conjecture, in complete analogy to the counterexamples of [Ho PS] for Sp(4). In particular, the L-function we attach to a supercuspidal component r_{χ_v} *can have a pole*; cf. §6. Thus these **4**
representations r_χ distinguish themselves in yet another way, and the regularizing nature of the local correspondence S evidence itself (by "lifting" a supercuspidal representation to a non-supercuspidal one).

For a leisurely account of how classical modular forms of half-integral weight can be defined as representations of Weil's metaplectic group the reader is referred to [Ge]. Most of the results described in the present paper were first announced in [Ge PS].

We note Chapter I is purely local: after describing the local metaplectic group, and the notion of Whittaker models for its irreducible admissible representations, we introduce L and ϵ factors and describe the local Shimura correspondence. In Chapters II and III we piece together these notions to obtain a global correspondence. In the process of doing so, we develop a Jacquet-Langlands theory for the metaplectic group. Details and related results are to be found in [Ge], [Ge HPS], and [Ge PS2]. The principal contribution of the present paper is the proof of the global Shimura correspondence in Chapter III.

It is with pleasure that we acknowledge our indebtedness to G. Shimura and R. P. Langlands. Shimura had already suggested the possibility of a representation-theoretic and adelic approach to his results in [Shim]. On the other hand, the concrete suggestions and inspiration of Langlands first brought one of us close to the metaplectic group and got this project started. Langlands also suggested how the Selberg trace formula could be used to obtain (and in fact go beyond) our present results; this suggestion has just recently been developed by Flicker, whose results—improvements of our own—will appear in a forthcoming paper [Flicker].

Chapter I. Local Theory

Throughout this Chapter F will denote a local field of characteristic not equal to two. By Z_2 we shall denote the group of square roots of unity.

1 The Metaplectic Group

1.1 Let $H^2(\mathrm{SL}_2(F).Z_2)$ denote the two-dimensional continuous cohomology group of $\mathrm{SL}_2(F)$ with coefficients in Z_2. From [Weil] and [Moore] it follows that if $F \neq \mathbb{C}$, $H^2(\mathrm{SL}_2(F), Z_2) = Z_2$.

5 If $F = \mathbb{C}$, let $\overline{\mathrm{SL}}_2(F)$ denote the group $\mathrm{SL}_2(F) \times Z_2$. If $F \neq \mathbb{C}$, let $\overline{\mathrm{SL}}_2(F)$ denote the non-trivial central extension of $\mathrm{SL}_2(F)$ by Z_2 determined by the non-trivial element of $H^2(\mathrm{SL}_2(F), Z_2)$.

In all cases, we have an exact sequence of topological groups

$$1 \to Z_2 \to \overline{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F) \to 1.$$

1.2 We want to extend Weil's metaplectic group to GL_2. To do this, we use the fact that any automorphism of $\mathrm{SL}_2(F)$ lifts uniquely to an automorphism of $\overline{SL}_2(F)$.

Let D denote the group

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in F^x \right\}$$

Each element of D operates on $\mathrm{SL}_2(F)$ by conjugation, hence lifts to an automorphism of $\overline{SL_2}$. If $\overline{G}$ denotes the resulting semi-direct product of D and $\overline{\mathrm{SL}_2}(F)$, we obtain an exact sequences of locally compact groups

$$1 \to Z_2 \to \overline{G} \to \mathrm{GL}_2(F) \to 1. \tag{1.2.1}$$

Note $\overline{G}$ is a non-trivial extension of $\mathrm{GL}_2(F)$ unless $F = \mathbb{C}$.

1.3 The sequence (1.2.1) splits over the following subgroups of $\mathrm{GL}_2(F)$:

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \right\}$$

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in F^x \right\}$$

$$Z^2 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in (F^x)^2 \right\}$$

and (if F is non-archimedean, of odd residual characteristic, and O_F is the ring of integers of F),

$$K = \mathrm{GL}_2(O_F).$$

If H is any subgroup of $\mathrm{GL}_2(F)$, let $\overline{H}$ denote its full inverse image in $\overline{G}_F$. If H is such that the sequence (1.2.1) splits over it, then $\overline{H}$ is the direct product of Z_2 with a subgroup of $\overline{G}$ which we again denote by H.

1.4 The *center* of $\overline{G}_F$ is

$$\overline{Z}^2 = Z^2 \times Z_2.$$

6 On the other hand, if

$$Z = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} : \alpha \in F^x \right\},$$

the group $\overline{Z}$ is abelian but *not* central in $\overline{G}$. When convenient, we confuse Z with the group F^x, and Z^2 with the subgroup $(F^x)^2$.

1.5 If $\varphi : \overline{G} \to W$ is any function on $\overline{G}$, with values in a vector space W, we say φ is *genuine* (or *doesn't factor through* GL_2) if

$$\varphi(g\zeta) = \zeta\varphi(g), \quad \text{for all} \quad g \in \overline{G},\ \zeta \in Z_2.$$

Unless specified otherwise, we henceforth deal only with genuine objects on $\overline{G}_F$.

2 Admissible Representations

2.1 By modifying the definitions in [Jacquet-Langlands], we can define, for each local F, the notion of an irreducible admissible representation $\overline{\pi}$ of $\overline{G}_F$.

2.2 If F is archimedean, we shall assume π is actually irreducible unitary, or perhaps the restriction of such a representation to "smooth" vectors. Since $\overline{G}_{\mathbb{C}} = \mathrm{GL}_2(\mathbb{C}) \times Z_2$ we shall have little to say about the case when F is complex.

2.3 Induced Representations. Let B denote the Borel subgroup of $\mathrm{GL}_2(F)$. Although $\overline{B}$ is not abelian, it contains a convenient subgroup of finite index which *is* abelian, and even "splits" in $\overline{G}$. Indeed let B_0 denote the subgroup of B consisting of matrices $\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}$ where a_1 and a_2 have even p-adic order. If F has even residual characteristic we also require that a_1 be a square modulo $1 + 4O_F$. If F is real we simply require that $a_1 > 0$. In any case, $\overline{B}_0 = B_0 \times Z_2$, and the index of $\overline{B}_0$ in $\overline{B}$ is the index of $(F^x)^2$ in F^x.

For any pair of quasi-characters μ_1, μ_2 of F^x, let $\mu_1\mu_2$ denote the (genuine) character of $\overline{B}_0/N$ whose restriction to B_0/N is given by the formula

$$\mu_1\mu_2\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right) = \mu_1(a_1)\mu_2(a_2).$$

The induced representation 7

$$\overline{\rho}(\mu_1, \mu_2) = \mathrm{Ind}(\overline{G}_F, \overline{B}_0, \mu_1\mu_2) \tag{2.3.1}$$

is admissible and

$$\overline{\rho}(\mu_1, \mu_2) \approx \overline{\rho}(\nu_1, \nu_2)$$

not only if $\mu_1 = \nu_2$ and $\mu_2 = \nu_1$, but also if

$$\mu_i^2 = \nu_i^2,\ i = 1, 2. \tag{2.3.2}$$

cf. §2 of [Ge PS2] and §5 of [Ge]. Moreover, $\overline{\rho}(\mu_1, \mu_2)$ is irreducible unless $\mu_1^2\mu_2^{-2}(x) = |x|^1$ or $|x|^{-1}$ (or all integral points in the real case). In any case, the composition series has length at most 2; cf. [Moen] and [Ge Sa].

2.4 Classification of Representations If $\overline{\rho}(\mu_1, \mu_2)$ is irreducible, we denote it by $\overline{\pi}(\mu_1, \mu_2)$ and call it a *principal series* representation. If $\overline{\rho}(\mu_1, \mu_2)$ is reducible, we let $\overline{\pi}(\mu_1, \mu_2)$ denote its unique irreducible subrepresentation. In all cases, $\overline{\pi}(\mu_1, \mu_2)$ defines an infinite-dimensional irreducible admissible representation of $\overline{G}_F$. If $\mu_1^2\mu_2^{-2}(x) = |x|^1$ we call $\overline{\pi}(\mu_1, \mu_2)$ a *special* representation; it is equivalent to the unique quotient of $\overline{\rho}(\mu_2, \mu_1)$.

Suppose $(\overline{\pi}, V)$ is any irreducible admissible (genuine) representation of $\overline{G}_F$. Then $\overline{\pi}$ is automatically infinite-dimensional. If it is not of the form $\overline{\pi}(\mu_1, \mu_2)$ for some pair (μ_1, μ_2), we say $\overline{\pi}$ is *supercuspidal.* If F is archimedean, no such representations exist. On the other hand, if F is non-archimedean, $\overline{\pi}$ is supercuspidal if and only if for every vector v in $V_{\overline{\pi}}$,

$$\int_U \overline{\pi}(u)v\, du = 0$$

for some open compact subgroup U of $N \subset \overline{G}_F$; cf. [Ge], §5.

The construction and analysis of such supercuspidal representations is carried out in [RS] and [Meister].

From [Ge] Section 5, and [Meister], it follows that:

8 **2.4.1** An irreducible admissible representation $\overline{\pi}$ is class 1 if and only if it is of the form $\overline{\pi}(\mu_1, \mu_2)$ with μ_1^2 and μ_2^2 unramified and $\mu_1^2\mu_2^{-2}(x) \neq |x|$, i.e., $\overline{\pi}$ is not special.

2.5 Class 1 Representations Suppose F is non-archimedean and of odd residual characteristic. If $\overline{\pi}$ is an admissible representation of $\overline{G}_F$, recall $\overline{\pi}$ is *class 1*, or spherical, if its restriction to K_F contains the identity representation (at least once). If $\overline{\pi}$ is also irreducible, it can be shown that $\overline{\pi}$ then contains the identity representation *exactly* once; cf. [Ge] and [Meister].

In particular, suppose $\overline{1}_K$ denotes the idempotent of the Hecke algebra of $\overline{G}_F$ belonging to the trivial representation of K_F, i.e.,

$$1_K(g) = \begin{cases} 1 & \text{if } g \in K \\ -1 & \text{if } g \in K \times \{-1\} \\ 0 & \text{if otherwise} \end{cases}$$

Then $\overline{\pi}$ class 1 implies $\overline{\pi}(\overline{1}_K)$ has non-zero range, and $\overline{\pi}$ class 1 irreducible implies the range is one-dimensional.

3 Whittaker Models

Fix once and for all a non-trivial additive character ψ of F.

3.1 Definition Suppose $\overline{\pi}$ is an irreducible admissible representation of $\overline{G}_F$. By a *ψ-Whittaker model for $\overline{\pi}$* we understand a space $W(\overline{\pi}, \psi)$ consisting of continuous functions $W(g)$ on $\overline{G}$ satisfying the following properties:

3.1.1 $W\left(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right) g\right) = \psi(x)W(g)$;

3.1.2 If F is non-archimedean, W is locally constant, and if F is archimedean, W is C^∞;

3.1.3 The space $W(\overline{\pi}, \psi)$ is invariant under the right action of $\overline{G}_F$, and the resulting representation in $W(\overline{\pi}, \psi)$ is equivalent to $\overline{\pi}$.

3.2 In [Ge HPS] we prove that a ψ-Whittaker model always exists. If $W(\overline{\pi}, \psi)$ is unique, we say $\overline{\pi}$ is *distinguished.* Note that if $\overline{\pi}$ is not genuine, i.e., if $\overline{\pi}$ defines an ordinary representation of $\mathrm{GL}_2(F)$, then $\overline{\pi}$ is always distinguished: this is the celebrated "uniqueness of Whittaker models" result of [Jacquet-Langlands].

In general, if $\overline{\pi}$ is genuine (as we are assuming it is), it is not dis- 9
tinguished. To recapture uniqueness, we need to refine our notion of Whittaker model.

3.3 Let $\omega_{\overline{\pi}}$ denote the *central character of* $\overline{\pi}$. This is the genuine character of $(F^x)^2 x Z_2$ determined by the formula

$$\overline{\pi}\begin{pmatrix} a^2 & 0\\ 0 & a^2\end{pmatrix} = \omega_{\overline{\pi}}(a^2)I. \tag{3.3.1}$$

Let $\Omega(\omega_{\overline{\pi}})$ denote the (finite) set of genuine characters of $\overline{Z}$ whose restriction to $\overline{Z}^2$ agrees with $\omega_{\overline{\pi}}$.

3.4 Definition. For each μ in $\Omega(\omega_{\overline{\pi}})$, let $\mathscr{W}(\overline{\pi}, \psi, \mu)$ denote the space of continuous functions $W(g)$ on $\overline{G}_F$ which, in addition to satisfying conditions (3.1.1)-(3.1.3), also satisfy the condition

$$W\left(\overline{z}\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right] g\right) = \mu(\overline{z})\psi(x)W(g), \quad \text{for} \quad \overline{z} \in \overline{Z}. \tag{3.4.1}$$

In [Ge HPS] we prove that *such* a Whittaker model *is* unique. More precisely, there is *at most one* such model, and *for at least one* μ in $\Omega(\omega_{\overline{\pi}})$, a (ψ, μ)-Whittaker model always exists.

3.5 Let $\Omega(\pi) = \Omega(\overline{\pi}, \psi)$ denote the set of μ in $\Omega(\omega_{\overline{\pi}})$ such that $\mathscr{W}(\overline{\pi}, \psi, \mu)$ exists. This set depends on ψ, but its cardinality does not. Indeed if $\lambda \in F^x$, and ψ^λ denotes the character

$$\psi^\lambda(x) = \psi(\lambda x), \tag{3.5.1}$$

then $\mathscr{W}(\overline{\pi}, \psi, \mu)$ is mapped isomorphically to $\mathscr{W}(\overline{\pi}, \psi^\lambda, \mu^\lambda)$ via the map

$$W(g) \to W^\lambda(g) = W\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} g\right). \tag{3.5.2}$$

Here μ^λ denotes the character

$$\mu^\lambda(\overline{z}) = \mu\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} \overline{z} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{3.5.3}$$

with the conjugation carried out in $\overline{G}$. The existence of the isomorphism (3.5.2) means that $\mu \in \Omega(\overline{\pi}, \psi)$ iff $\mu^\lambda \in \Omega(\overline{\pi}, \psi^\lambda)$.

3.6 Remark. $\Omega(\overline{\pi}, \psi)$ is a singleton set if and only if $\overline{\pi}$ is distinguished.

All possible examples of distinguished $\overline{\pi}$ are described in the next Section.

4 The Theta-Representations r_χ

10 These representations are indexed by characters of F^x and treated in complete detail in [Ge PS2]. We simply recall their definition and basic properties.

4.1 In [Weil] there was constructed a genuine admissible representation of $\overline{\mathrm{SL}_2}(F)$. We call this representation the basic Weil representation and denote it by r^ψ; it depends on the non-trivial additive character ψ and splits into two irreducible pieces, one "even", one "odd".

If χ is an even (resp. odd) character of F^x, we can "tensor" χ with the even (resp. odd) piece of r^ψ to obtain a representation r_χ^ψ of $\overline{G}_F^*$, the semi-direct product of $\overline{\mathrm{SL}}_2(F)$ with $\left\{\begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix} : a \in F^x\right\}$. Inducing up to $\overline{G}_F$ produces an irreducible admissible representation which is independent of ψ and denoted r_χ. The restriction of r_χ to $\overline{\mathrm{SL}}_2(F)$ is the direct sum of a finite number of inequivalent representations, namely

$$\{r^{\psi^\lambda}\}_{\lambda\in\Lambda},$$

with Λ an index set for the cosets of $(F^x)^2$ in F^x.

4.2 Each r_χ is a distinguished representation of $\overline{G}_F$. In particular, for each non-trivial character ψ of F, let $\gamma(\psi)$ denote the eighth root of unity introduced in [Weil], Section 14.

Then

$$\Omega(r_\chi, \psi) = \{\chi_{\mu_\psi}\},$$

with μ_ψ the projective character of F^x defined by

$$\mu_\psi(a) = \frac{\gamma(\psi)}{\gamma(\psi^a)} \tag{4.2.1}$$

We note that the restriction of μ_ψ to $(F^x)^2$ is trivial. Moreover, if ψ has conductor O_F, and F is of odd residual characteristic, μ_ψ is also trivial on units.

4.3 *When χ is unramified, and F has odd residue characteristic, r_χ is class 1.* More generally, if χ is an even character, r_χ is the unique irreducible subrepresentation of $\overline{\pi}(\chi^{1/2}|\,|_F^{-1/4}, \chi^{1/2}|\,|_F^{1/4})$.

11

4.4 If χ is an odd character, i.e., $\chi(-1) = -1$, then r_χ is super-cuspidal; cf. [Ge].

4.5 Having observed that each r_χ is distinguished, we conjectured that the family $\{r_\chi\}_\chi$ exhausts the irreducible admissible distinguished representations of $\overline{G}$.

When F is non-archimedean and of odd residue characteristic, the supercuspidal part of this conjecture is established in [Meister]; the non-supercuspidal part is treated in [Ge PS2].

5 A Functional Equation of Shimura Type

As always, F is a local field of characteristic not equal to 2 and ψ is a fixed non-trivial character of F.

5.1 Suppose $\overline{\pi}$ is any irreducible admissible representation of $\overline{G}_F$, and χ is any quasi-character of F^x. Recall the sets $\Omega(\overline{\pi}, \psi)$ and $\Omega(r_\chi, \psi)$ introduced in (3.5). In general, $\Omega(\overline{\pi}, \psi) = \Omega(\omega_{\overline{\pi}})$. However, $\Omega(r_\chi, \psi) = \{\chi\mu_\psi\}$.

To attach an L-factor to $\overline{\pi}$ and χ, we fix some μ in $\Omega(\pi, \psi)$ and introduce the zeta-functions

$$\Psi(s, W, W_\chi, \Phi) = \int_{N\backslash G} W(g)W_\chi(g)|\det(g)|^s \Phi((0.1)g)dg. \tag{5.1.1}$$

Here $W(g)$ is any element of $\mathscr{W}(\pi, \psi, \mu)$, W_χ is any element of $\mathscr{W}(r_\chi, \psi^{-1}, \chi_{\mu_{\psi-1}})$, $\Phi \in \mathscr{S}(F \times F)$, and $s \in \mathbb{C}$. Since W and W_x are genuine, and transform contravariantly under N, their product actually defines a function on $N\backslash G$.

Similarly, we define

$$\widetilde{\Psi}(s, W, W_\chi, \Phi) = \int_{N\backslash G} W(g)W_\chi(g)|\det g|^s \omega_*^{-1}(\det g)\Phi((0, 1)g)dg \tag{5.1.2}$$

with

$$\omega_* = \mu\chi\mu_{\psi^{-1}}. \tag{5.1.3}$$

Note that ω_* is an ordinary character of F^x whose restriction to $(F^x)^2$ is $\chi\omega_{\overline{\pi}}$.

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5.2 For $\mathrm{Re}(s)$ sufficiently large, and g in $\mathrm{GL}_2(F)$, the integrals

$$|\det g|^s \int_{F^x} \Phi((0., t)g)|t|^{2s}\omega_*(t)\,\mathrm{dt} = f_s(g) \tag{5.2.1}$$

and

$$|\det g|^s \omega_*^{-1}(\det g) \int\limits_{F^x} \Phi((0,t)g)|t|^{2s} \omega_*^{-1}(t)\,\mathrm{dt} = h_s(g) \tag{5.2.2}$$

converge and define elements in the space of the induced representations $\rho(s-1/2,(1/2-s)\omega_*^{-1})$ and $\rho(\omega_*^{-1}(s-1/2),1/2-s)$ respectively. Cf. [Ja], 14. Moreover, for such s, the integrals defining Ψ and $\widetilde{\Psi}$ converge.

$$\Psi(s,W,W_\chi,\Phi) = \int\limits_{NZ\backslash G} W(g)W_\chi(g)f(g)dg \tag{5.2.3}$$

and

$$\widetilde{\Psi}(s,W,W_\chi,\Phi) = \int\limits_{NZ\backslash G} W(g)W_\chi(g)h(g)dg$$

Modifying the methods of [Ja] we obtain :

Theorem 5.3. *(a) The functions $\Psi(s,W,W_\chi,\Phi)$ and $\widetilde{\Psi}(s,W,W_\chi,\Phi)$ extend meromorphically to $\mathbb{C}$;*

(b) There exist Euler factors $L(s,\overline{\pi},\chi)$ and $\widetilde{L}(s,\overline{\pi},\chi)$ such that for any W, W_χ, Φ, ψ, and μ, the functions

$$\frac{\Psi(s,W,W_\chi,\Phi)}{\widetilde{L}(s,\overline{\pi},\chi)} \quad \textit{and} \quad \frac{\widetilde{\Psi}(s,W,W_\chi,\Phi)}{\widetilde{L}(s,\overline{\pi},\chi)}$$

are entire;

(c) There is an exponential factor $\epsilon(s,\overline{\pi},\chi,\psi)$ such that for all W, W_χ and Φ as above,

$$\frac{\widetilde{\Psi}(1-s,W,W_\chi,\widehat{\Phi})}{\widetilde{L}(1-s,\overline{\pi},\chi)} = \epsilon(s,\overline{\pi},\chi,\psi)\frac{\Psi(s,W,W_\chi,\Phi)}{L(s,\overline{\pi},\chi)}, \tag{5.3.1}$$

with

$$\widehat{\Phi}(x,y) = \iint \Phi(u,v)\psi(uy-vx)dudv.$$

5.4 The factor $\epsilon(s, \overline{\phi}, \chi, \psi)$ might depend on the choice of μ as well as **13**
ψ. Therefore, to be precise, we should write $\epsilon(s, \overline{\pi}, \chi, \psi, \mu)$ in place of $\epsilon(s, \overline{\pi}, \chi, \psi)$. However, a straightforward computation shows that

$$\epsilon(s, \overline{\pi}, \chi, \mu^{\lambda}) = \omega_{\overline{\pi}}(\lambda^{-2})\chi^{-2}(\lambda)|\lambda|^{2-4s}\epsilon(s, \overline{\pi}, \chi, \psi^{\lambda}, \mu). \tag{5.4.1}$$

Also, as we shall see, *globally* $\epsilon(s, \overline{\pi}, \chi, \psi, \mu)$ is easily seen to be independent of both ψ and μ; cf. Remark 13.4.

5.5 If we introduce the "gamma factor"

$$\gamma(s, \overline{\pi}, \chi, \psi) = \frac{\epsilon(s, \overline{\pi}, \chi, \psi)\widetilde{L}(1 - s, \overline{\pi}, \chi)}{L(s, \overline{\pi}, \chi)}$$

then the functional equation (5.3.1) takes the simpler form

$$\widetilde{\Psi}(1 - s, W, W_\chi, \widehat{\Phi}) = \gamma(s, \overline{\pi}, \chi, \psi)\Psi(s, W_1, W_2, \Phi).$$

6 L and ϵ-Factors

Let $\overline{\pi}, \chi$ and ψ be as in the last section. In this section we collect together the values of $L(s, \overline{\pi}, \chi)$, $\widetilde{L}(s, \overline{\pi}, \chi)$, and $\epsilon(s, \overline{\pi}, \chi, \psi)$ for most representations $\overline{\pi}$. To compute the factors L and $\widetilde{L}$ we need to analyze the possible poles of $\Psi(s, W, W_\chi, \Phi)$ and $\widetilde{\Psi}(s, W, W_\chi, \Phi)$. To compute $\epsilon(s, \overline{\pi}, \chi, \psi)$ we need to compute the functions Ψ and $\widetilde{\Psi}$ explicitly, for judicious choices of W, W_χ and Φ.

Suppose first that F is non-archimedean.

6.1 Suppose $\overline{\pi}$ is a supercuspidal. If $\overline{\pi}$ is not of the form r_ν for any quasi-character ν, then

$$L(s, \pi, \chi) = 1 = \widetilde{L}(s, \overline{\pi}, \chi), \quad \text{for all} \quad \chi.$$

On the other hand, if $\overline{\pi} = r_\nu$, then

$$L(s, \overline{\pi}, \chi) = L(2s, \chi\nu),$$

and

$$\widetilde{L}(s, \overline{\pi}, \chi) = L(2s, \chi^{-1}\nu^{-1})$$

If $\chi\nu$ is unramified,

$$\epsilon(s, \overline{\pi}, \chi, \psi) = \frac{\epsilon(2s, \chi\nu, \psi)\epsilon(2s-1, \chi\nu, \psi)L(1-2s, \nu^{-1}\chi^{-1})}{L(2s-1, \nu\chi)}$$

whereas if $\chi\nu$ is ramified **14**

$$\epsilon(s, \overline{\pi}, \chi, \psi) = \epsilon(2s, \chi\nu, \psi)\epsilon(2s-1, \chi\nu, \psi).$$

Here, as throughout, the factors $L(s, \omega)$ and $\epsilon(s, \omega, \psi)$ are the familiar L and ϵ factors attached to each quasi-character ω of F^x; cf. [Jacquet-Langlands, pp. 108-109].

6.2 Suppose $\overline{\pi}$ is of the form $\overline{\pi}(\mu_1, \mu_2) = \overline{\phi}(\mu_1, \mu_2)$. Then

$$L(s, \overline{\pi}, \chi) = L(2s - 1/2, \mu_1^2\chi)L(2x - 1/2, \mu_2^2\chi),$$

and

$$\widetilde{L}(s, \overline{\pi}, \chi) = L(2s - 1/2, \mu_1^{-2}\chi)L(2s - 1/2, \mu_2^{-2}\chi) \tag{6.2.1}$$

If we set $s' = 2s - 1/2$, then

$$\epsilon(s, \overline{\pi}, \chi, \psi) = \epsilon(s', \mu_1^2\chi, \psi)\epsilon(s', \mu_2^2\chi, \psi) \tag{6.2.2}$$

In particular, suppose F is class 1, χ(also μ) is trivial on units, ψ has conductor O_F, Φ is the characteristic function of $O_F \times O_F$, and W and W_χ are normalized K_F-fixed vectors in $W(\overline{\pi}, \psi, \mu)$ and $W(r_\chi, \psi^{-1})$. Then

$$\begin{cases} \Psi(s, W_1, W_2, \Phi) = L(s, \overline{\pi}, \chi) \\ \widetilde{\Psi}(s, W_1, W_2, \widetilde{\Phi}) = \widetilde{L}(s, \overline{\pi}, \chi) \end{cases} \tag{6.2.3}$$

and

$$\epsilon(s, \overline{\pi}, \chi, \psi) = 1.$$

6.3 Suppose $\overline{\pi}$ is the special representation

$$\overline{\pi} = \overline{\pi}(\mu_1, \mu_2), \text{ with } \mu_1^2\mu_2^{-2}(x) = |x|_F^1, \text{ and } \mu_1(x) = \nu(x)|x|_F^{1/4}$$

Then

$$\begin{cases} L(s, \overline{\pi}, \chi) = L(2s, \chi\nu^2), \\ \widetilde{L}(s, \overline{\pi}, \chi) = L(2s, \nu^{-2}\chi^{-1}), \end{cases}$$

and-if $\pi(\nu^2)$ denotes the special representation $\pi(\nu^2|\ |^{1/2}, \nu^2|\ |^{-1/2})$ of $\mathrm{GL}_2(F)$,

$$\epsilon(s, \overline{\pi}, \chi, \psi) = \epsilon(s', \pi(\nu^2) \otimes \chi, \psi).$$

6.4 If $\overline{\pi}$ is of the form r_ν, with $\nu(-1) = 1$, then

$$\begin{cases} L(s, \overline{\pi}, \chi) = L(2s-1, \chi\nu)L(2s, \chi\nu), \\ \widetilde{L}(s, \overline{\pi}, \chi) = L(2s-1, \chi^{-1}\nu^{-1})L(2s, \chi^{-1}\nu^{-1}), \end{cases}$$

and

$$\epsilon(s, \overline{\pi}, \chi, \psi) = \epsilon(2s-1, \chi\nu, \psi) \in (2s, \chi\nu, \psi)$$

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6.5 Suppose now that F is archimedean. Then each $\overline{\pi}$ occurs as the subrepresentation of some $\overline{\rho}(\mu_1, \mu_2)$, with each μ_i determined up to a character of order 2. Let $S(\overline{\pi})$ denote the unique irreducible admissible representation of $\mathrm{GL}_2(F)$ which appears as a subrepresentation of $\rho(\mu_1^2, \mu_2^2)$. Then

$$\begin{cases} L(s, \overline{\pi}, \chi) = L(s, S(\overline{\pi}) \otimes \chi), \\ \widetilde{L}(s, \overline{\pi}, \chi) = L(s, S(\overline{\pi}) \otimes \chi^{-1}), \end{cases}$$

and

$$\epsilon(s, \overline{\pi}, \chi, \psi) = \epsilon(s, S(\overline{\pi}) \otimes \chi, \psi),$$

the L and ϵ factors on the right being those of [Jacquet-Langlands].

6.6 Stability Given $\overline{\pi}$ and ψ, it can be shown that if F is non-archimedean, and χ is sufficiently highly ramified, the corresponding L and ϵ-factors stabilize. More precisely, for all χ sufficiently highly ramified,

$$L(s,\overline{\pi},\chi) = 1 = \widetilde{L}(s,\overline{\pi},\chi),$$

and

$$\epsilon(s,\overline{\pi},\chi,\psi) = \epsilon(s,\omega_\pi\chi,\psi) \in (s,\chi,\psi) \tag{6.6.1}$$

In (6.6.1), ω_π is the character of F^x defined by the equation

$$\omega_\pi(a) = \omega_{\overline{\pi}}(a^2) \tag{6.6.2}$$

7 A Local Shimura Correspondence

Suppose $\overline{\pi}$ is an irreducible admissible (genuine) representation of $\overline{G}_F$ and $\omega_{\overline{\pi}}$ is its central character.

7.1 Fixing a non-trivial character ψ of F, we call an irreducible admissible representation π of G_F a *Shimura image of* $\overline{\pi}$ if **16**

7.1.1 the central character ω_π of π is such that

$$\omega_\pi(a) = \omega_{\overline{\pi}}(a^2),\ a \in F^x;$$

7.1.2 for any quasi-character χ of F^x,

$$\begin{cases} L(s,\overline{\pi},\chi) = L(s,\pi\otimes\chi), \\ \widetilde{L}(s,\overline{\pi},\chi) = L(s,\widetilde{\pi}\otimes\chi^{-1}), \end{cases}$$

and

$$\epsilon(s,\overline{\pi},\chi,\psi) = \epsilon(s,\pi\otimes\chi,\psi).$$

7.2 If the Shimura image of $\overline{\pi}$ exists, it is unique, and independent of ψ. We denote it by $S(\overline{\pi})$.

7.3 From Section 6 it follows that $S(\overline{\pi})$ exists whenever $\overline{\pi}$ is not a supercuspidal representation (not of the form r_ν). Indeed in this case,

$$\overline{\pi} = \overline{\pi}(\mu_1, \mu_2) \text{ implies } S(\overline{\pi}) = \pi(\mu_1^2, \mu_2^2).$$

In particular,

$$\overline{\pi} = r_\nu(\nu(-1) = -1) \text{ implies } (\overline{\pi}) = \pi(\nu|\,|_F^{1/2}, \nu|\,|_F^{1/2}).$$

On the other hand, as we shall see, if $\overline{\pi}$ is supercuspidal (but not of the form r_ν) its image $S(\overline{\pi})$ must also be supercuspidal.

7.4 In case $F = \mathbb{R}$, and $\overline{\pi}$ corresponds to a discrete series representation of "lowest weight $k/2$", $S(\overline{\pi})$ corresponds to a discrete series representation of lowest weight $k-1$; cf. [Ge], §4.

7.5 Connections with Shimura's theory

The fact that S takes $\overline{\pi}(\mu_1, \mu_2)$ to $\pi(\mu_1^2, \mu_2^2)$ means (in the non-archimedean unramified situation) that eigenvalues for the Hecke algebras are preserved. See §5.3 of [Ge] for a careful analysis of this phenomenon. Keeping in mind (7.4), it follows that our local Shimura correspondence is consistent with the map defined globally (and classically) in [Shim].

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7.6 Summing up, Shimura's correspondence operates locally as follows:

$\overline{\pi}$	$\pi = S(\overline{\pi})$
principal series $\overline{\pi}(\mu_1, \mu_2)$	principal series $\pi(\mu_1^2, \mu_2^2)$
special representation $\overline{\pi}(\nu|\,|^{1/4}, \nu|\,|^{-1/4})$	special rep $\mathrm{Sp}(\nu^2)$
Weil r_ν $(\nu(-1) = -1)$	special rep $\mathrm{Sp}(\nu)$
Weil r_ν $(\nu(-1) = 1)$	one-dimensional rep $\nu \circ \det$

Note *all* special representations arise as Shimura images (whereas a principal series thus arises if it corresponds to even-or squared-characters of F^*); for the supercuspidal representations, see [Flicker] and [Meister].

Chapter II. Global Theory

Throughout this Chapter, F will denote an arbitrary A-field of characteristic not equal to two, $\mathbb{A}$ its ring of adeles, and

$$\psi = \prod_v \psi_v$$

a non-trivial character of $F\backslash\mathbb{A}$.

8 The Metaplectic Group

For each place v of F we defined in §1 a "local" metaplectic group $\overline{G}_v = \overline{G}_{F_v}$. Roughly speaking, the adelic metaplectic group $\overline{G}_A$ is a product of the local groups $\overline{G}_v$.

More precisely, recall that if v is non-archimedean and "odd", $\overline{G}_v$ splits over $K_v = \mathrm{GL}_2(O_{F_v})$. Thus we can consider the restricted direct product

$$\widetilde{G} = \prod_v \overline{G}_v(K_v).$$

The metaplectic group $\overline{G}_A$ is obtained by taking the quotient of $\widetilde{G}$ by 18

$$\widetilde{Z}_e = \left\{ \prod_v \epsilon_v \in \prod_v Z_2 : \epsilon_v = 1 \text{ for all but an even number of } v \right\}.$$

In particular, we can view $\overline{G}_A$ as a group of pairs $\{(h, \zeta) : h \in G_A, \zeta \in Z_2\}$, with multiplication given by

$$(h_1, \zeta_1)(h_2, \zeta_2) = (h_1 h_2, \beta(h_1, h_2)\zeta_1\zeta_2),$$

and β a product of the local two-cocycles defining $\overline{G}_v$. The fact that the exact sequence

$$1 \to Z_2 \to \overline{G}_A \to G_A \to 1$$

splits over the discrete subgroup

$$G_F = \mathrm{GL}_2(F)$$

is equivalent to the quadratic reciprocity law for F; cf. [Weil].

9 Automorphic Representations of Half-Integral Weight

9.1 Recall that $\overline{G}_A$ is the quotient of $\prod_v \overline{G}_v = \widetilde{G}_A$ by the subgroup $\widetilde{Z}_e$.

9.2 Suppose that for each place v of F we are given an irreducible admissible genuine representation $(\overline{\pi}_v, V_v)$ of $\overline{G}_v$. Suppose also that for almost every finite v, $\overline{\pi}_v$ is class 1. Then for almost every v we can choose a K_v-fixed vector e_v in V_v and define a restricted tensor product space

$$V = \bigotimes_v V_v(e_v).$$

The resulting representation of $\widetilde{G}_A$ in V given by

$$\overline{\pi} = \bigotimes_v \overline{\pi}_v \tag{9.2.1}$$

is trivial on $\widetilde{Z}_e$ and defines an irreducible admissible representation of $\overline{G}_A$.

Conversely, suppose $\overline{\pi}$ is an irreducible unitary representation of $\overline{G}_A$. Following step by step the arguments of §9 of [Jacquet-Langlands] we can show that $\overline{\pi}$ must be of the form (9.2.1) with each $\overline{\pi}_v$ determined uniquely by $\overline{\pi}$.

9.3 Let ω denote a character of $(\mathbb{A}^x)^2$ trivial on $(F^x)^2$. Proceeding as in §10 of [Jacquet-Langlands] we can introduce a space $\overline{A}(\omega)$ of *automorphic forms on* $\overline{G}_A$. Each φ in $\overline{A}(\omega)$ is a genuine C^∞ function on $G_F\backslash\overline{G}_A$ which is "slowly increasing" and transforms under the center 19
(of $\overline{G}_A$) according to ω. The group $\overline{G}_A$ acts as expected in $\overline{A}(\omega)$ by right translations.

By $A_0(\omega)$ we denote the subspace of ω-cuspidal functions, those φ in $A_0(\omega)$ such that

(i) the constant term

$$\varphi_0(g) = \int\limits_{F\backslash\mathbb{A}} \varphi\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} g\right) \mathrm{dx} = 0$$

for each g in $\overline{G}_{\mathbb{A}}$;

(ii) the integral

$$\int\limits_{Z_{\mathbb{A}}^2 G_F\backslash\overline{G}_{\mathbb{A}}} |\varphi(g)|^2 dg$$

is finite. This space of cusp forms is clearly stable under the action of $\overline{G}_{\mathbb{A}}$, and each φ in $\overline{A}_0(\omega)$ is rapidly decreasing.

9.4 An irreducible admissible representation $\overline{\pi}$ of $\overline{G}_{\mathbb{A}}$ is called *automorphic* (respectively *cuspidal*) of half-integral weight if it is a constituent of some $\overline{A}(\omega)$ (resp. $\overline{A}_0(\omega)$).

10 Fourier Expansions

Suppose φ is an automorphic form on $\overline{G}_{\mathbb{A}}$, and $\psi = \Pi\psi_v$ is a fixed nontrivial character of $F\backslash\mathbb{A}$.

10.1 Since

$$\varphi\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} g\right)$$

is a C^∞ function on $F\backslash\mathbb{A}$ for each fixed g, $\varphi(g)$ admits a Fourier expansion in terms of the characters of $F\backslash\mathbb{A}$. But each non-trivial character ψ of $F\backslash\mathbb{A}$ is of the form

$$\psi'(x) = \psi^\delta(x) = \psi(\delta x) \tag{10.1.1}$$

for some $\delta \in F^x$. Thus

$$\varphi\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} g\right) = \varphi_0(g) + \sum_{\delta\in F^x} W_\varphi^{\psi^\delta}(g)\psi(\delta x), \tag{10.1.2}$$

with

$$W_\varphi^{\psi^\delta}(g) = \int\limits_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\psi^{-1}(\delta x)\mathrm{dx}. \tag{10.1.3}$$

20 On the other hand, it is easy to check that

$$W_\varphi^{\psi^\delta}(g) = W_\varphi^{\psi}\left(\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix} g\right). \tag{10.1.4}$$

Thus we also have

$$\varphi(g) = \varphi_0(g) + \sum_{\delta\in F^x} W_\varphi^{\psi}\left(\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix} g\right). \tag{10.1.5}$$

In other words—modulo its constant term–$\varphi(g)$ is completely determined by its *first Fourier coefficient*

$$W_\varphi^{\psi}(g) = \int\limits_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\psi(-x)\mathrm{dx}. \tag{10.1.6}$$

We call this function a ψ-Whittaker function since

$$W\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} g\right) = \psi(x)W(g),\ \ x \in \mathbb{A}.$$

Now we must refine this notation to bring into play the local theory of §3.

10.2 Suppose $\overline{\pi} = \otimes\overline{\pi}_v$ is any automorphic representation of half-integral weight. Suppose in addition that $\overline{\pi}$ actually occurs as a sub-representation (as opposed to subquotient) of some $\overline{A}(\omega)$, say in the space $V_{\overline{\pi}}$. Then ω must be the central character $\omega_{\overline{\pi}}$ of $\overline{\pi}$.

Now let $\Omega(\omega_{\overline{\pi}})$ denote the set of (genuine) characters of $Z_F\backslash\overline{Z}_{\mathbb{A}}$ whose restriction to $\overline{Z}^2_{\mathbb{A}}$ agrees with $\omega_{\overline{\pi}}$. Then each φ in $V_{\overline{\pi}}$ has a Fourier expansion of the form

$$\varphi(g) = \varphi_0(g) + \sum_{\mu\in\Omega(\omega_{\overline{\pi}})} \sum_{\delta\in F^x} W_\varphi^{\psi^\delta,\mu}(g) \tag{10.2.1}$$

with

$$W_\varphi^{\psi^\delta,\mu}(g) = \int\limits_{\overline{Z}^2_{\mathbb{A}}\backslash\overline{Z}_{\mathbb{A}}} \int\limits_{F\backslash\mathbb{A}} \varphi\left(\overline{Z}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\psi^{-1}(\delta x)\mu^{-1}(\overline{z})\mathrm{dxdz}. \tag{10.2.2}$$

The (ψ,μ) refinement of (10.1.4) is

$$W_\varphi^{\psi^\delta,\mu}(g) = W_\varphi^{\psi,\mu^\delta}\left(\begin{bmatrix}\delta & 0\\ 0 & 1\end{bmatrix} g\right) \tag{10.2.3}$$

where

$$\mu^\delta(\overline{z}) = \mu\left(\begin{bmatrix}\delta & 0\\ 0 & 1\end{bmatrix}^{-1} \overline{z}\begin{bmatrix}\delta & 0\\ 0 & 1\end{bmatrix}\right),\ z\in\overline{Z}_{\mathbb{A}}.$$

Note that for any μ, 21

$$W_\varphi^{\psi,\mu}\left(\overline{z}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)\mu(\overline{z})W(g),\ z\in\overline{Z}_{\mathbb{A}},\ x\in F. \tag{10.2.4}$$

10.3 For any μ in $\Omega(\omega_{\overline{\pi}})$, let $\mathscr{W}(\overline{\pi},\psi,\mu)$ denote a (ψ,μ)-Whittaker space for $\overline{\pi}$ (analogous to the local definition (3.1); the crucial property of course is (10.2.4)). Let $\Omega(\pi,\psi)$ denote the set of μ in $\Omega(\omega_{\overline{\pi}})$ such that $\mathscr{W}(\overline{\pi},\psi,\mu)$ exists; if $\Omega(\pi,\psi)$ is a singleton set we call $\overline{\pi}$ *distinguished*.

For $\overline{\pi}$ and φ as in (10.2), $W_\varphi^{\psi,\mu}(g)$ is clearly non-zero for at least one μ, and therefore $\mathscr{W}(\overline{\pi},\psi,\mu)$ exists for at least one μ (ψ being supposed fixed). If $W_\varphi^{\psi,\mu}(g)\neq 0$ for exactly one μ in $\Omega(\omega_{\overline{\pi}})$, we say φ is *distinguished.* We note that $\overline{\pi}$ distinguished implies any φ in $V_{\overline{\pi}}$ is distinguished.

Of course if $\overline{\pi} = \otimes \overline{\pi}_v$ is any irreducible admissible representation of $\overline{G}_{\mathbb{A}}$, we might be inclined to call $\overline{\pi}$ distinguished if each $\overline{\pi}_v$ is distinguished in the local sense. Fortunately these notions are compatible. Indeed in [Ge PS2] we prove that an automorphic subrepresentation $\overline{\pi}$ of $\overline{A}$ is distinguished in the above sense if and only if each $\overline{\pi}_v$ is.

If $\overline{\pi}$ is a distinguished subrepresentation of $\overline{A}(\omega_{\overline{\pi}})$ and $\varphi \in V_{\overline{\pi}}$, then (10.2.3) implies

$$\varphi(g) = \varphi_0(g) = \sum_{\delta \in F^x} W_{\varphi}^{\psi,\mu}\left(\begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix} g\right), \tag{10.3.1}$$

a familiar GL_2-type Fourier expansion. In particular, if $\overline{\pi}$ is cuspidal, the first Fourier coefficient $W_{\varphi}^{\psi,\mu}(g)$ completely determines φ through the expansion

$$\varphi(g) = \sum_{\delta \in F^x} W\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} g),$$

and we have:

Theorem 10.3.2. *Every distinguished cuspidal representation of half-integral weight occurs exactly once in $\overline{A}_0$.*

10.4 Let us explain the classical significance of a *distinguished* cusp form. Suppose

$$f(z) = \sum a(n) e^{2\pi i n z}$$

22 is a cusp form of weight $k/2$, and an eigenfunction for all Hecke operators. Since most of these operators act as the zero map, one can't expect their eigenvalues to relate many of the coefficients $a(n)$. In fact, if $T(p^2)f = \omega_p f$, then ω_p serves to relate $a(t)$ only to the coefficients $a(tp^2)$; in particular, the first Fourier coefficient does *not* always determine f. In other words, there is more than "one orbit" of coefficients.

On the other hand, if f is "distinguished", i.e., if there is a t such that $a(n) = 0$ unless $n = tm^2$ for some m, then f *is* determined by just one coefficient (and the knowledge of the ω_p's). This is consistent with (10.3.1).

Note that in our representation theoretic set-up, our φ in $V_{\overline{\pi}}$ is assumed to be an eigenfunction of the Hecke operators. The fact that φ is distinguished means exactly that $W_\varphi^{\psi,\mu} \neq 0$ for exactly "one orbit of characters". In particular, the relation (10.2.3) implies that if $\delta \notin (F^x)^2$, then

$$W_\varphi^{\psi^{\delta},\mu}(g) = W_\varphi^{\psi^\delta,\mu}(g) = W_\varphi^{\psi,\mu^\delta}\left(\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix}g\right) = 0.$$

In classical terms, if φ corresponds to the form $f(z)$, then

$$\begin{aligned} f(z) &= \sum_\delta \sum_n a(\delta n^2) e^{2\pi i \delta n^2 z} \\ &= \sum_n a(\delta_0 n^2) e^{2\pi i \delta_0 n^2 z} \end{aligned}$$

For more details, see [Ge PS2] and [Shim].

Examples of distinguished automorphic representations will now be described.

11 Theta-Representations

11.1 Suppose $\chi = \prod_v \chi_v$ is any character of $F^x \backslash \mathbb{A}^x$. Since almost every χ_v is unramified, we can define an irreducible admissible representation of $\overline{G}_{\mathbb{A}}$ through the formula

$$r_\chi = \otimes r_{\chi_v},$$

where r_{χ_v} is the local theta-representation described in Section 4.

11.2 In [Ge PS2] we show that r_χ occurs in a subspace of χ-automorphic forms on $\overline{G}_{\mathbb{A}}$. In particular, r_χ defines a *distinguished* automorphic representation of half-integral weight.

Our construction 23

$$\chi \to r_\chi$$

generalizes the classical construction of theta-series associated with Dirichlet characters. To wit, suppose $\chi : (\mathbb{Z}/N\mathbb{Z})^x \to \mathbb{C}$ is a primitive Dirichlet character, and $\chi(-1) = 1$ say. Then

$$\theta_\chi(z) = \sum_{n=-\infty}^{\infty} \chi(n) e^{2\pi i n^2 z}$$

defines a "distinguished" modular form of weight $\frac{1}{2}$, level $4N^2$, and character χ.

If $\chi = \Pi\chi_v$ is not *totally even*, i.e., $\chi_v(-1) = -1$ for at least one v, then r_χ is actually cuspidal.

11.3 In [Ge PS] we conjectured that every distinguished cuspidal representation of half-integral weight is of the form r_χ for some χ. In [Ge PS2] we show that this follows from the Shimura correspondence established in this paper; cf. §16,

Chapter III. A Generalized Shimura Correspondence

12 A Shimura-Type Zeta Integral

Suppose $\overline{\pi} = \otimes\overline{\pi}_v$ is an automorphic cuspidal representation of half-integral weight and $\chi = \prod\limits_v \chi_v$ is a grossencharacter of F. Having introduced L and ϵ factors for each $\overline{\pi}_v$ and χ_v, we want to prove that the product

$$L(s, \overline{\pi}, \chi) = \prod_v L(s, \overline{\pi}_v, \chi_v)$$

converges in some half-plane, continues to a meromorphic function in $\mathbb{C}$, and satisfies a functional equation of the form

$$L(s, \overline{\pi}, \chi) = \left(\prod_v \in (s, \overline{\pi}_v, \chi_v \psi_v) \right) \widetilde{L}(1 - s, \overline{\pi}, \chi).$$

To do this, we have to introduce a zeta-integral of Shimura type that essentially equals $L(s, \overline{\pi}, \chi)$.

12.1 Let $\overline{A}_0(\omega_{\overline{\pi}})$ denote the space of cusp forms which transform under $\overline{Z}^2_{\mathbb{A}}$ according to the character $\omega_{\overline{\pi}}$, and suppose $\overline{\pi}$ occurs in the space $V_{\overline{\pi}}$ in $\overline{A}_0(\omega_{\overline{\pi}})$. If $\varphi \in V_{\overline{\pi}}$ then **24**

$$\varphi(g) = \sum_{\mu} \sum_{\delta \in F^x} W_{\varphi}^{\psi,\mu^{\delta}} \left(\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} g \right) \tag{12.1.1}$$

Recall that the first summation extends over all characters μ' of $Z_F \backslash \overline{Z}_{\mathbb{A}}$ whose restriction to $\overline{Z}^2_{\mathbb{A}}$ is $\omega_{\overline{\pi}}$.

Now fix

$$\mu = \bigotimes_{v} \mu_v \quad \text{in} \quad \Omega(\overline{\pi}, \psi),$$

and fix the embedding $V_{\overline{\pi}}$ so that $W_{\varphi}^{\psi,\mu}(g) \neq 0$. Given by character

$$\chi = \Pi \chi_v$$

of $F^x \backslash \mathbb{A}^x$, let μ_χ denote the unique element of the singleton set $\Omega(r_\chi, \psi^{-1})$. These μ, μ_χ determine Whittaker models $\mathscr{W}(\overline{\pi}, \psi, \mu)$ and $\mathscr{W}(r_\chi, \psi, \mu_\chi)$. To define our global analogue of the local zeta functions $\psi(s, W, W_\chi, \Phi)$ we need first to describe some Eisenstein series on $\mathrm{GL}_2(\mathbb{A})$.

12.2 If $\Phi = \prod_v \Phi_v$ is in $S(\mathbb{A} \times \mathbb{A})$, set

$$F_s(g) = F_s^{\Phi}(g) = |\det g|^s \int_{F^x} \Phi((0,t)g)|t|^{2s} \omega_*(t) d^x t, \tag{12.2.1}$$

with ω_* the (ordinary) character of $F^x \backslash A^x$ given by the formula

$$\omega_* = \mu\mu_\chi = \mu\chi\mu_\psi - 1;$$

cf. (5.1.3), (5.2.1), and (5.2.2). The integral in (12.2.1) converges for $\mathrm{Re}(s) \gg 0$ and defines an element

$$F_s = \Pi f_{s,v}$$

in the induced space $\rho\left(s-\frac{1}{2},\left(\frac{1}{2}-s\right)\omega_*^{-1}\right)$. Moreover, the series

$$\sum_{\gamma\in B_F\backslash G_F} F_s(\gamma_g) = E(g,F,s)$$

converges for $\mathrm{Re}(s) \gg 0$, and defines an automorphic form on GL_2, the *Eisenstein series* $E(g,F,s)$; cf. p. 117 of [Ja] (taking $\mu_1 = \alpha^s\frac{1}{2}$, $\mu_2 = \alpha^{\frac{1}{2}-a}\omega_*^{-1}$).

25 **Remark 12.3.** $E(g,F,s)$ extends to a meromorphic function in $\mathbb{C}$ with functional equation

$$E(g,F^{\Phi},s) = E(g,F^{\widehat{\Phi}},1-s); \tag{12.2.1}$$

here, as in the local theory, $\widehat{\Phi}$ is the twisted Fourier transform $\widehat{\Phi}(x,y) = \int\Phi(u,v)\psi(yu-vx)\mathrm{dudv}$, with du and dv the self-dual measure on $\mathbb{A}$; cf. Prop. 19.3 of [Ja]. We also know that the only poles of $E(g,F,s)$ are simple, and occur for $|\ |_{\mathbb{A}}^{2-2s} = \omega_*$ and $|\ |_{\mathbb{A}}^{2s} = \omega_*^{-1}$.

12.4 Given $\overline{\pi},\psi,\mu,\chi$, and F_s as above, we define our zeta integral by the equation

$$\psi^*(s,\varphi,\theta_\chi,F) = \int_{Z_{\mathbb{A}}^2 G_F\backslash G_{\mathbb{A}}} \varphi(g)\theta_\chi(g)E(g,F,s)\mathrm{dg}. \tag{12.4.1}$$

Here $\varphi \in V_{\overline{\pi}}$, and

$$\theta_\chi(g) = \sum_{\delta\in F^x} W^{\psi^{-1},\mu\chi}\left(\begin{bmatrix}\delta & 0\\ 0 & 1\end{bmatrix}g\right) + \theta_0(g) \tag{12.4.2}$$

belongs to the space of the automorphic distinguished representation r_χ. Since the theta-function θ_χ is also slowly increasing, and since $\varphi(g)$ is a cusp form, the integral in (12.4.1) converges in some right half-plane, and its analytic properties in all of $\mathbb{C}$ are reflected by those of $E(g,F,s)$. In particular, we have:

Proposition 12.5. *For any choice of φ, θ_χ, and F_s :*

(i) *the function $\psi^*(s, \varphi, \theta_\chi, F)$ extends to a meromorphic function in $\mathbb{C}$ with functional equation*

$$\psi^*(1-s, \varphi, \theta_\chi, F^{\widehat{\Phi}}) = \psi^*(s, \varphi, \theta_\chi, F^{\Phi}). \tag{12.5.1}$$

(ii) *All poles of ψ^* are simple, with residues proportional to*

$$\int\limits_{Z^2_{\mathbb{A}} G_F \backslash G_{\mathbb{A}}} |\det g|^{s_0} \varphi(g) \theta_\chi(g) dg. \tag{12.5.2}$$

(iii) *ψ^* is bounded at infinity in vertical strips of finite width.*

Corollary 12.6. *If $\overline{\pi}$ is not of the form r_ν for any grossencharacter ν, then $\psi^*(s, \varphi, \theta_\chi, V)$ is actually entire.*

Proof. If $\psi^*(s, \varphi, \theta_\chi, F)$ has a pole, the residue (12.5.2) is non-zero for **26**
some s_0. In other words, the bilinear form on $V_{\overline{\pi}} \times V_{r_\chi}$ defined by

$$(\varphi, \theta_\chi) \to \int\limits_{Z^2_{\mathbb{A}} G_F \backslash G_{\mathbb{A}}} |\det g|^{s_0} \varphi(g) \theta_\chi(g) \mathrm{dg}$$

is not identically zero, and $|\ |^{s_0}_{\mathbb{A}} \otimes \overline{\pi}$ is equivalent to $\widetilde{r}_\chi$. Since $\widetilde{r}_\chi \approx r_{\chi^{-1}}$, this contradicts our hypothesis. □

13 An Euler Product Expansion

13.1 To relate $L(s, \overline{\pi}, \chi)$ to $\psi^*(s, \varphi, \theta_\chi, F)$ we need to express ψ^* as a product of local integrals of the form $\psi(s, W_v, W_{\chi_v}, \Phi_v)$. In greater generality, this Euler product decomposition is sketched in [PS]. To treat the explicit case at hand, we assume that the "first Fourier coefficients"

$W_\varphi^{\psi,\mu}(g)$ and W^{ψ^{-1},μ_χ} of $\varphi(g)$ and θ_χ (cf. (12.1.1) and (12.4.2)) are of the form

$$W_\varphi^{\psi,\mu}(g) = \prod_v W_v(g)$$

and

$$W^{\psi^{-1},\mu_\chi}(g) = \prod W_{\chi_v}(g),$$

with $W_v \in \mathscr{W}(\overline{\pi}_v, \psi_v, \mu_v)$ and $W_{\chi_v}(g) \in \mathscr{W}(r_{\chi_v}, \psi^{-1})$.

Proposition 13.2. *With* φ, θ_χ*, and* F_s^Φ *as above, and* $\mathrm{Re}(s) \gg 0$*,*

$$\psi^*(s, \varphi, \theta_\chi, F^\Phi) = \prod_v \Psi(s, W_v, W_{\chi_v}, \Phi_v).$$

(Recall the local zeta-functions Ψ are defined by (5.1.1).)

Proof. Replacing E by the series defining it, we have

$$\begin{aligned}\psi^*(s, \varphi, \theta_\chi, F^\Phi) &= \int_{Z_{\mathbb{A}}^2 G_F \backslash G_{\mathbb{A}}} \varphi(g)\theta_\chi(g)E(g, F, s)\mathrm{dg}.\\ &= \int_{Z_{\mathbb{A}}^2 B_F \backslash G_{\mathbb{A}}} \varphi(g)\theta_\chi(g)F_s^\Phi(g)\mathrm{dg}\end{aligned}$$

Setting $B^0 = ZN = \{\left[\begin{smallmatrix} a & x \\ 0 & a \end{smallmatrix}\right]\}$, we may write

$$\theta_\chi(g) = \theta_0(g) + \sum_{B_F^0 \backslash B_F} W^{\psi^{-1},\mu\chi}(bg),$$

$$\begin{aligned}\psi^*(s, \varphi, \theta_\chi, F) = &\int_{Z_{\mathbb{A}}^2 B_F \backslash G_{\mathbb{A}}} \varphi(g)\theta_0(g)F_s(g)\mathrm{dg}\\ &+ \int_{Z_{\mathbb{A}}^2 B_F^0 \backslash G_{\mathbb{A}}} \varphi(g)W^{\psi^{-1},\mu\chi}(g)F_s(g)\mathrm{dg}\end{aligned} \tag{13.2.1}$$

We claim now that the first term on the right side of (13.2.1) is zero, i.e., the constant term $\theta_0(g)$ contributes nothing to ψ^*. Indeed $\theta_0(g)F_s(g)$ is left $N_{\mathbb{A}}$-invariant, and $\varphi(g)$ is a cuspidal.

Thus we have

$$\begin{aligned}\psi^*(s,\varphi,\theta_\chi,F) &= \int\limits_{Z_{\mathbb{A}}^2 B_F^0\backslash G_{\mathbb{A}}} \varphi(g)W^{\psi^{-1},\mu\chi}(g)F_s(g)\mathrm{dg}\\ &= \int\limits_{B_{\mathbb{A}}\backslash G_{\mathbb{A}}} I(g)F_s(g)\mathrm{dg} \qquad (13.2.2)\end{aligned}$$

where

$$I(g) = \int\limits_{Z_{\mathbb{A}}^2 B_F^0\backslash B_{\mathbb{A}}} \varphi(bg)W^{\psi^{-1},\mu\chi}(bg)\omega^*(b)db$$

and ω^* is the character of $B_{\mathbb{A}}$ defined by

$$\omega^*\begin{pmatrix} a_1 & x\\ 0 & a_2\end{pmatrix} = \left|\frac{a_1}{a_2}\right|^{2s}\omega_*^{-1}(a_2).$$

To continue, we compute

$$\begin{aligned}I(g) &= \int\limits_{B_{\mathbb{A}}^0\backslash B_{\mathbb{A}}}\left(\int\limits_{Z_{\mathbb{A}}^2\backslash B_F^0 B_{\mathbb{A}}^0} \varphi(b'bg)W^{\psi^{-1},\mu\chi}(b'bg)\omega^*(bb')db'\right)db\\ &= \int\limits_{B_{\mathbb{A}}^0\backslash B_{\mathbb{A}}} \omega^*(b)W^{\psi^{-1},\mu\chi}(bg)\sum_\mu\sum_\delta W^{\psi,\mu'}(bg)\\ &\quad \left(\int\limits_{Z_{\mathbb{A}}^2\backslash Z_{\mathbb{A}}}\int\limits_{F_{\mathbb{A}}} \psi^\delta(x)\psi(-x)(\mu)^{-1}(z)\mu'(z)\mathrm{dxdz}\right)db\end{aligned}$$

But the integral in parenthesis is zero unless $\delta = 1$ and $\mu' = \mu$, in which **28**
case it equals 1. Thus we have

$$I(g) = \int\limits_{B_{\mathbb{A}}^0\backslash B_{\mathbb{A}}} \omega^*(b)W_\varphi^{\psi,\mu}(bg)W^{\psi^{-1},\mu\chi}(bg)db.$$

Plugging this expression into (13.2.2) gives

$$\psi^*(s,\varphi,\theta_\chi,F) = \int\limits_{B_{\mathbb{A}}\backslash G_{\mathbb{A}}} \left(\int\limits_{B^0_{\mathbb{A}}\backslash B_{\mathbb{A}}} W_\varphi^{\psi,\mu}(bg) W^{\psi^{-1},\mu\chi}(bg) F_s^{\Phi}(bg)\,db \right) dg$$

$$= \int\limits_{N_{\mathbb{A}}Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} W_\varphi^{\psi,\mu}(g) W^{\psi^{-1},\mu\chi}(g) F_s(g)\,dg$$

So taking into account the infinite product expression for W_φ, $W^{\psi^{-1},\mu\chi}$, and $F_s = \Pi f_{s,v}$, we obtain the desired Euler product expansion for ψ^*.

□

Theorem 13.3. *Suppose $\overline{\pi}$ is any cuspidal representation of half-integral weight. If $\chi = \Pi\chi_v$ is any character of $F^x\backslash\mathbb{A}^x$ set*

$$L(s,\overline{\pi},\chi) = \prod_v L(s,\overline{\pi}_v,\chi_v)$$

and

$$\widetilde{L}(s,\overline{\pi},\chi) = \prod_v \widetilde{L}(s,\overline{\pi}_v,\chi_v).$$

Then

(i) *these infinite products converge in some half-plane* $\mathrm{Re}(s) > s_0$*;*

(ii) *L and $\widetilde{L}$ extend meromorphically to all of $\mathbb{C}$, are bounded in vertical strips of finite width, and satisfy the functional equation*

$$L(s,\overline{\pi},\chi) = \epsilon(s,\overline{\pi},\chi)\widetilde{L}(1-s,\overline{\pi},\chi)$$

with

$$\epsilon(s,\overline{\pi},\chi) = \prod_v \in (s,\overline{\pi}_v,\chi_v,\psi_v);$$

(iii) *the only poles of $L(s,\overline{\pi},\chi)$ are simple, and these occur only if $\overline{\pi}$ is of the form r_ν for some character ν of $F^x\backslash\mathbb{A}^x$.*

Proof. For almost every v, $\overline{\pi}_v$ is of the form $\overline{\pi}(\nu_v^1,\nu_v^2)$, with $\nu_v^i(x) = |x|^{t_{i,v}}$, and

$$-t_0 \le t_{i,v} \le t_0 (\text{independent of } v)$$

Therefore, since **29**

$$L(s, \overline{\pi}(\nu_v^1, \nu_v^2)) = \left(\frac{1}{1 - q^{-2t_{1,v^{-s'}}}}\right)\left(\frac{1}{1 - q^{-2t_{2,v}-s'}}\right)$$

the infinite products in question converge.

Now fix a set S outside of which everything is unramified, i.e., if $v \notin S$, v is finite and odd, $\overline{\pi}_v$ and χ_v are class 1, ψ_v has conductor O_{F_v}, μ_v is trivial on $O_{F_v}^x$, and W_v, W_{χ_v} and Φ_v are chosen so that

$$\Psi(s, W_v, W_{\chi_v}, \Phi_v) = L(s, \overline{\pi}_v, \chi_v)$$

and

$$\epsilon(s, \overline{\pi}_v, \chi_v, \psi_v) = 1;$$

cf. (6.2.3). For v inside S, choose W_v, W_{χ_v} and Φ_v so that

$$\Psi(s, W_v, W_{\chi_v}, \Phi_v) = L(s, \overline{\pi}_v, \chi_v)$$

modulo a non-vanishing entire factor. Then since $\psi^*(s, \varphi, \theta_\chi, F)$ has the analytic properties asserted in parts (ii) and (iii), so does $L(s, \overline{\pi}_v, \chi_v)$.

To establish the functional equation, we simply compute (using the local functional equations and (12.5.1)):

$$\begin{aligned}
L(s, \overline{\pi}, \chi) &= \prod_{v \in S} L(s, \overline{\pi}_v, \chi_v) \prod_{v \notin S} L(s, \overline{\pi}_v, \chi_v) \\
&= \prod_{v \in S} \frac{L(s, \overline{\pi}_v, \chi_v)}{\Psi(s, W_v, W_{\chi_v}, \Phi_v)} \left(\prod_{\text{all } v} \Psi(s, W_v, W_{\chi_v}, \Phi_v)\right) \\
&= \left(\prod_{v \in S} \in (s, \overline{\pi}_v, \chi_v, \psi_v) \frac{\widetilde{L}(1 - s, \overline{\pi}_v, \chi_v)}{\widetilde{\Psi}(1 - s, W_v, W_{\chi_v}, \Phi_v)}\right) \\
&\qquad x\psi^*(s, \varphi, \theta_\chi, F^\Phi) \\
&= \prod_{v \in S} \frac{\epsilon(s, \overline{\pi}_v, \chi_v, \psi_v) \widetilde{L}(1 - s, \overline{\pi}_v, \chi_v)}{\widetilde{\Psi}(1 - s, W_v, W_{\chi_v}, \Phi_v)} \prod_{\text{all } v} \widetilde{\Psi}(1 - s, W_v, W_{\chi_v}, \widehat{\Phi}_v)
\end{aligned}$$

$$= \prod_{v \in S} \in (s, \overline{\pi}_v, \chi_v, \psi_v) \prod_{\text{all } v} \widetilde{L}(1 - s, \overline{\pi}_v, \chi_v)$$

$$= \epsilon(s, \overline{\pi}, \chi)\widetilde{L}(1 - s, \overline{\pi}, \chi),$$

as was to be shown. □

30 **Remark 13.4.** Since $L(s, \overline{\pi}, \chi)$ doesn't depend on ψ (or μ), neither does the product

$$\epsilon(s, \overline{\pi}, \chi, \psi) = \prod_{v} \epsilon(s, \overline{\pi}_v, \chi_v, \psi_v).$$

14 A Generalized Shimura Correspondence

14.1 Suppose $\overline{\pi} = \otimes \overline{\pi}_v$ is an irreducible admissible (genuine) representation of $\overline{G}_{\mathbb{A}}$ with central character $\omega_{\overline{\pi}}$, and $\pi = \otimes \pi_v$ is an irreducible admissible representation of $G_{\mathbb{A}}$. Then we say π is the *Shimura image* of $\overline{\pi}$, and write $\pi = S(\overline{\pi})$, if each $\pi_v = S(\overline{\pi}_v)$, i.e., each π_v is the local Shimura image of $\overline{\pi}_v$.

Example 14.2. Suppose $\overline{\pi} = r_\chi$, with $\chi = \Pi \chi_v$ a grossencharacter of F. Then for each v, $S(r_{\chi_v})$ is defined, and for almost every v, $S(r_{\chi_v})$ is one-dimensional and class 1. The resulting representation

$$S(\overline{\pi}) = \otimes S(r_{\chi_v})$$

is always automorphic, by the criterion of [Langlands].

Our purpose now is to show that *any* unitary *cuspidal* representation of half-integral weight has an automorphic Shimura image, and this image is actually cuspidal if $\pi \neq r_v$ for any v.

15 The Theorem

Theorem 15.1. *Suppose $\overline{\pi} = \otimes \overline{\pi}_v$ is a unitary cuspidal representation of half-integral weight. Then:*

(i) $S(\overline{\pi}) = \otimes S(\overline{\pi}_v)$ *exists*;

(ii) *$S(\overline{\pi})$ is automorphic, and is cuspidal if and only if $\overline{\pi}$ is not of the form r_ν for any character ν of $F^x\backslash A^x$.*

15.2 Proof. Because of Example 14.2, we may assume $\overline{\pi} \neq r_\nu$ for any ν.

15.3 Fixing ψ, let T be the set of places where $S(\overline{\pi}_v) = \pi_v$ may not be defined. According to Section 7, T is precisely the set of finite places where $\overline{\pi}_v$ is supercuspidal but not a theta-representation.

For almost all $v \notin T$, $\overline{\pi}_v$ is a class 1 representation of the form $\overline{\pi}(\mu_1, \mu_2)$ (possibly of the form $\overline{\pi}(\nu_v^{1/2}|\ |_v^{-1/4}, \nu^{1/2}|\ |_v^{1/4})$. Thus $S(\overline{\pi}_v) =$
$\overline{\pi}(\mu_1^2, \mu_2^2)$ is class 1 (though possibly one-dimensional) for $v \notin T$, and we 31
can define

$$\pi^T = \bigotimes_{v \notin T} \pi_v = \bigotimes_{v \notin T} S(\overline{\pi}_v), \tag{15.3.1}$$

a representation of the restricted product $\overline{G}^T = \bigotimes_{v \notin T} \overline{G}_v$.

If $\chi = \Pi \chi_v$ is any character of $F^x\backslash A^x$, consider the infinite products

$$L(s, \pi^T \otimes \chi) = \prod_{v \notin T} L(s, \pi_v \otimes \chi_v) \tag{15.3.2}$$

and

$$L(s, \widetilde{\pi}^T \otimes \chi^{-1}) = \prod_{v \notin T} L(s, \widetilde{\pi}_v \otimes \chi_v^{-1})$$

These products converge absolutely for $\mathrm{Re}(s) \gg 0$ since for almost all $v \notin T$, $\mu_{i,v}(x) = |x|^{t_{i,v}}$, and for some t_0 independent of v,

$$-t_o \leq t_{i,v} \leq t_o \tag{15.3.3}$$

To conclude that $L(s, \pi^T \otimes \chi)$ extends to the L-function of an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ we need to know that $L(s, \pi^T \otimes \chi)$ satisfies certain analytic properties. In particular, we need to exploit the relation between $L(s, \pi^T \otimes \chi)$ and the Euler product $L(s, \overline{\pi}, \chi)$.

From Theorem **??** (and our assumption on $\overline{\pi}$) we know that for *any* character $\chi = \Pi\chi_v$ of $F^x\backslash A^x$,

$$L(s,\overline{\pi},\chi) = \prod_v L(s,\overline{\pi}_v,\chi_v)$$

and

$$\widetilde{L}(s,\overline{\pi},\chi) = \prod \widetilde{L}(s,\overline{\pi}_v,\chi_v)$$

are entire functions, bounded in vertical strips of finite width, and such that

$$L(s,\overline{\pi},\chi) = \Big(\prod \in (s,\overline{\pi}_v,\chi_v,\psi_v)\Big)\widetilde{L}(1-s,\overline{\pi},\chi). \tag{15.3.4}$$

On the other hand, we also know from 6.6 that if χ_v is sufficiently highly ramified for $v \in T$,

$$\begin{cases} L(s,\overline{\pi},\chi) = \prod\limits_{v\notin T} L(s,\overline{\pi}_v,\chi_v) = L(s,\pi^T \otimes \chi), \\ \widetilde{L}(s,\overline{\pi}\chi) = \prod\limits_{v\notin T} \widetilde{L}(s,\overline{\pi}_v,\chi_v) = L(s,\widetilde{\pi}^T \otimes \chi^{-1}), \end{cases}$$

32 and, for $v \in T$,

$$\epsilon(s,\overline{\pi}_v,\chi_v,\psi_v) = \epsilon(s,\omega_\pi v\chi_v,\psi_v) \in (s,\chi_v,\psi_v).$$

Recall that

$$\omega_{\pi_v}(a) = \omega_{\overline{\pi}_v}(a^2), \tag{15.3.5}$$

and $\omega_\pi = \Pi\omega_{\pi_v}$ defines a grossencharacter of F.

Thus we know that for all $\chi = \Pi\chi_v$ highly ramified inside T, $L(s,\pi^T \otimes \chi)$ and $L(s,\widetilde{\pi}^T \otimes \chi^{-1})$ are entire functions, bounded in vertical strips of finite width, and such that

$$L(s,\pi^T \otimes \chi) = \left(\prod_{v\notin T} \in (s,\pi_v \otimes \chi_v)\right)\Big(\prod \in (s,\omega_{\pi_v}\chi_v,\psi_v) \tag{15.3.6}$$

$$\in (s,\chi_v,\psi_v)\Big) \times L(1-s,\widetilde{\pi}^T \otimes \chi^{-1})$$

Therefore, applying the almost everywhere converse theorem for GL(2) stated in our Appendix (with $\eta = \omega_\pi$) we conclude that either:

(i) $\bigotimes_{v \notin T} \pi_v$ extends to a cuspidal representation π which occurs in $A_0(\omega_\pi)$, or

(ii) there are grossencharacters μ and ν of F such that $\otimes \pi_v$ extends a quotient π of $\rho(\mu, \nu)$ (with every component of π infinite-dimensional).

It remains to show that the v-th component of π equals $S(\overline{\pi}_v)$ for each $v \in T$, and that possibility (ii) can't occur.

15.4 In either case, (i) or (ii), we know that

$$L(s, \pi \otimes \chi) = \prod_v \in (s, \pi_v, \psi_v) L(1 - s, \widetilde{\pi} \otimes \chi^{-1}) \tag{15.4.1}$$

for *all* grossencharacters χ. Therefore, by (15.3.4) and (15.3.6) we conclude that for all χ,

$$\prod_{v \in T} \frac{L(s, \overline{\pi}_v, \chi_v)}{L(s, \pi_v \otimes \chi_v)} = \prod_{v \in T} \frac{\epsilon(s, \overline{\pi}_v, \chi_v, \psi_v)}{\epsilon(s, \pi_v \otimes \chi_v, \psi_v)} \frac{\widetilde{L}(1 - s, \overline{\pi}_v, \chi_v)}{L(1 - s, \widetilde{\pi}_v \otimes \chi_v^{-1})} \tag{15.4.2}$$

Now fix $v_0 \in T$ and let χ_{v_0} denote an arbitrary character of F^x. Choose $\chi = \Pi \chi_v$ so that its v_0-th component is χ_{v_0}, but for each $v \in$
$T - \{v_0\}$, χ_v is so highly ramified so that (6.6.1) holds. Then (15.4.2) **33**
reads

$$\frac{\epsilon(s, \overline{\pi}_{v_0}, \chi_{v_0}, \psi_{v_0}) \widetilde{L}(1 - s, \overline{\pi}_{v_0}, \chi_{v_0})}{L(s, \overline{\pi}_{v_0}, \chi_{v_0})} \tag{15.4.3}$$

$$= \frac{\epsilon(s, \pi_{v_0} \otimes \chi_{v_0}, \psi_{v_0}) L(1 - s, \widetilde{\pi}_{v_0} \otimes \chi_{v_0}^{-1})}{L(s, \pi_{v_0} \otimes \chi_{v_0})}$$

Recall that $v_0 \in T$ implies $\overline{\pi}_{v_0}$ is supercuspidal and not a theta representation. Therefore, by 6.1, $L(s, \overline{\pi}_{v_0}, \chi_{v_0}) = 1 = \widetilde{L}(1 - s, \overline{\pi}_{v_0}, \chi_{v_0})$, and (15.4.3) implies

$$\frac{L(1 - s, \widetilde{\pi}_{v_0} \otimes \chi_{v_0}^{-1})}{L(s, \pi_{v_0} \otimes \chi_{v_0})} = \frac{\epsilon(s, \overline{\pi}_{v_0}, \chi_{v_0}, \psi_{v_0})}{\epsilon(s, \pi_{v_0} \otimes \chi_{v_0}, \psi_{v_0})} \tag{15.4.4}$$

i.e. the quotient $L(1-s,\widetilde{\pi}_{v_0}\otimes\chi_{v_0}^{-1})/L(s,\pi_{v_0}\otimes\chi_{v_0})$ is monomial. Since χ_{v_0} is arbitrary, it is easy to check this implies π_{v_0} must be super-cuspidal. (Indeed for other possible π_{v_0}, we could choose χ_{v_0} so that the quotient would be a rational function of q^{-s}). Thus

$$L(s,\pi_{v_0}\otimes\chi_{v_0}) = 1 = L(1-s,\widetilde{\pi}_{v_0}\otimes\chi_{v_0}^{-1})$$

and (15.4.4) implies $\pi_{v_0} = S(\overline{\pi}_{v_0})$.

Remark 15.5. If the set T is non-empty–as we have assumed it is–our Main Theorem is already proved. Indeed in this case, we have just shown that $S(\overline{\pi}_v)$ still exists for all v. Moreover, we have shown that $\pi_v = S(\overline{\pi}_v)$ must be supercuspidal for $v \in T$. Thus $S(\overline{\pi}) = \otimes S(\overline{\pi}_v)$ must be cuspidal automorphic (since possibility (ii) in §15.3 implies π_v is a quotient of $\rho(\mu_v, \nu_v)$ for all v).

On the other hand, if T is empty, then $S(\overline{\pi}_v)$ exists *a priori*, for all v, but $\pi = \otimes S(\overline{\pi}_v)$ is not *a priori* cuspidal. To complete the proof in this case it remains to note that now

$$L(s,\pi\otimes\chi) = L(s,\overline{\pi},\chi)$$

and

$$L(s,\widetilde{\pi}\otimes\chi^{-1}) = \widetilde{L}(s,\overline{\pi},\chi)$$

34 for *all* grossencharacters χ. Thus, by the well-known GL_2 theory, π must be cuspidal.

15.6 Corollary (Existence of a Generalized Shimura Correspondence)

Given any cuspidal representation $\overline{\pi}$ of half-integral weight over F, there exists an automorphic representation

$$\pi = S(\overline{\pi})$$

of $\mathrm{GL}_2(\mathbb{A}_F)$ such that for any grossencharacter χ of F,

$$L(s,\pi\otimes\chi) = L(s,\overline{\pi},\chi),$$

and $S(\overline{\pi})$ is cuspidal if and only if $\overline{\pi}$ is not of the form r_ν for any ν.

We call $S : \overline{\pi} \to \pi$ the Shimura map. Its "kernel"—those cuspidal $\overline{\pi}$ which map to *non*-cuspidal π—consists precisely of those $\overline{\pi}$ which come from automorphic forms on GL(1). That a similar situation arises with the lifting of cusp forms from GL(2) to GL(3) (cf. [Ge Ja]) cannot be coincidental.

Remark 15.7. Though we have not written down all the details, it seems likely we can prove that the L and ϵ factors of $\overline{\pi}_v$ (and their twists by χ_v) completely determine $\overline{\pi}_v$.

From this it follows that

(i) the Shimura map $S : \overline{\pi} \to \pi$ is 1-to-1; and

(ii) strong multiplicity one holds for $\overline{A}_0(\omega)$.

Apparently similar (and even stronger) results have been obtained by Flicker using the trace formula. Thus we shall not pursue these questions further.

Corollary 15.8. (A Weak Ramanujan-Peterson Theorem for $\overline{G}$). *If $\overline{\pi} = \otimes\overline{\pi}_v$ is a unitary cuspidal representation of half-integral weight, and v is a complex place, then $\overline{\pi}_v$ cannot belong to the "outer half" of the complementary series for $\overline{G}_v = \mathrm{GL}_2(\mathbb{C})xZ_2$; cf. [Ge] Section 4, especially p. 85.*

Proof. Suppose v is complex, and $\overline{\pi}_v = \overline{\pi}_v(\mu_1, \mu_2)$ is as above. Then 35
$S(\overline{\pi}_v) = \pi(\mu_1^2, \mu_2^2)$ is no longer unitary, contradicting the unitarity of the cuspidal representation $S(\overline{\pi}) = \bigotimes_v S(\overline{\pi}_v)$ in $A_0(\omega_\pi)$. □

16 Applications and Concluding Remarks

16.1 In [Ge PS2] we treat the following Corollaries to our Main Theorem 15.1:

Theorem A. *If $\overline{\pi}$ is a distinguished cuspidal representation of half-integral weight then*

$$\overline{\pi} = r_\chi$$

for some grossencharacter χ of F.

The classical interpretation of Theorem A is as follows. Suppose

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

is a cusp form of weight $k/2$ which is "distinguished", i.e, there is a square-free integer t such that $a(n) = 0$ unless $n = tm^2$ for some m. Then f must be of weight $1/2$ or $3/2$ *and* of the form

$$f(z) = \sum_{n=-\infty}^{\infty} \chi(n) n^\nu e^{2\pi i t n^2 z} = \theta_\chi(tz)$$

for some Dirichlet character χ (with $\chi(-1) = (-1)^\nu$). In this form the result was first established in [Vigneras].

Theorem B. *Suppose $\overline{\pi} = \otimes \overline{\pi}_v$ is a cuspidal representation with the property that for at least one place v_0, $\overline{\pi}_v = r_{\chi_{v_0}}$ with χ_{v_0} an even character of $F_{v_0}^x$. Then*

$$\overline{\pi} = r_\chi$$

for some grossencharacter χ of F.

Since this theorem results immediately from the existence (and local properties) of the Shimura correspondence it also appears in the recent work of Flicker's already alluded to.

36 **Corollary.** *Suppose $\overline{\pi} = \otimes \overline{\pi}_v$ is a cuspidal representation "of weight $1/2$", i.e., for at least one archimedean place v_0, $\overline{\pi}_{v_0}$ is the "even" piece of the Weil representation. Then there exists a grossencharacter χ of F such that*

$$\overline{\pi} = r_\chi.$$

In particular, taking $F = \mathbf{Q}$ we obtain an alternate proof of the fact that linear combinations of the theta-series

$$\theta_\chi(tz) = \sum_{n=-\infty}^{\infty} \chi(n) e^{2\pi i n^2 tz}$$

exhaust the modular forms of weight $1/2$ (as χ runs through the set of primitive "even" Dirichlet characters); this result is the principal theorem of [Se-St].

Concluding Remarks.

(i) The Shimura image of any *cusp* form of weight $k/2$, $k \geq 5$, must again be a cusp form. Indeed it cannot be of the form θ_χ, and Theorem 15.1 implies that only the θ_χ's can map to *non*-cusp forms. By the same token, if f is a cusp form of weight $3/2$, it is mapped to a *cuspidal* form of weight 2 iff it is orthogonal to the space spanned by θ_χ's. This settles the first conjecture of problem (C) on p. 478 of [Shim]; the Corollary to Theorem B settles the second.

(ii) In [Flicker] the image of S is characterized and a multiplicity one result is obtained for the *full* cuspidal spectrum of $\overline{G}_{\mathbb{A}}$. This resolves question (A) of [Shim], p. 476, and vastly improves our own Theorem 10.3.2.

Appendix

We reformulate the "almost everywhere converse theorem" of [Jacquet-Langlands] and [Weil 2].

The hypotheses below are slightly stronger than those of [Jacquet-Langlands], but they are quite tractable and seem to suffice for applications; cf. [PS 2] for best possible results.

Hypothesis. Suppose we are given: **37**

(i) a non-trivial character $\psi = \Pi\psi_v$ of $F\backslash\mathbb{A}$, and a character $\eta = \prod_{\eta} \eta_v$ of $F^x\backslash A^x$;

(ii) a finite set of finite places T, and an irreducible admissible representation $\pi^T = \bigotimes_{v \notin T} \pi_v$ of $\bigotimes_{v \notin T} G_v$ satisfying the following conditions:

(a) the central character of π^T is $\bigotimes_{v \notin T} \eta_v$,

(b) whenever $\pi_v = \pi(\mu_v, \nu_v)$ is class 1, and $v \notin T$ is finite,

$$|\widetilde{\omega}_v|^t < |\mu_v(\widetilde{\omega}_v)| < |\widetilde{\omega}_v|^{-t}$$

$$|\widetilde{\omega}_v|^t < |\nu_v(\widetilde{\omega}_v)| < |\widetilde{\omega}_v|^{-t}$$

(here $t > 0$ is a real number independent of v, and $\widetilde{\omega}_v$ is a local uniformizing variable at v); and

(c) for any grossencharacter $\chi = \prod_v \chi_v$, sufficiently highly ramified inside T, the infinite products

$$L(s, \pi, \chi) = \prod_{v \notin T} L(s, \pi_v \otimes \chi_v)$$

and

$$L(s, \widetilde{\pi}^T, \chi^{-1}) = \prod_{v \notin T} L(s, \widetilde{\pi}_v \otimes \chi_v^{-1})$$

continue to entire functions on $\mathbb{C}$, bounded in vertical strips of finite width, such that

$$L(s, \pi^T, \chi) = L(1 - s, \widetilde{\pi}^T, \chi^{-1}) \times \left(\prod_{v \notin T} \in (s, \pi_v \otimes \chi_v, \psi_v) \right)$$
$$\left(\prod_{v \in T} \in (s, \chi_v, \psi_v) \in (s, \chi_v \eta_v, \psi_v). \right)$$

Then:

Conclusion. Either

(i) $\pi^T = \bigotimes_{v \notin T} \pi_v$ extends to a cuspidal representation π in $A_0(\eta)$, or

(ii) there exist grossencharacters μ and ν of F, with $\mu\nu = \eta$, such that π^T extends to an automorphic representation π, with π a quotient of $\rho(\mu, \nu)$.

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PERIOD INTEGRALS OF COHOMOLOGY CLASSES WHICH ARE REPRESENTED BY EISENSTEIN SERIES

By G. Harder

Introduction

41

Our starting point is a very general question. Let Γ be an arithmetic subgroup of a reductive Lie group G_∞. Then the group Γ acts on the symmetric space $X = G_\infty/K_\infty$ where $K_\infty \subset G_\infty$ is a maximal compact subgroup. Since X is contractible one knows that the rational cohomology and homology groups of Γ are isomorphic to the (co) homology groups of the quotient $\Gamma\backslash X$, i.e.

$$H^\nu(\Gamma, \mathbb{Q}) \simeq H^\nu(\Gamma\backslash X, \mathbb{Q})$$

(Comp. [21], 1.6.).

In general the quotient space $\Gamma\backslash X$ is not compact. Borel and Serre have constructed a natural compactification $\Gamma\backslash X \hookrightarrow \Gamma\backslash\overline{X}$ where $\Gamma\backslash\overline{X}$ is a manifold with corners and where the inclusion is a homotopy equivalence. (Comp. [3]). In various papers it has been shown that we can construct cohomology classes on $\Gamma\backslash X$ by starting from cohomology classes on the boundary. Roughly speaking we associate to a cohomology class ψ on the boundary an Eisenstein series $E(\psi, s)$ which is a differential form depending on a complex parameters s. For a special value s_ψ of our complex parameter this form may become a closed form. This closed form represents a cohomology class and its restriction to the boundary is related to our original class ψ([7], [8] and [18]). We look at this as a procedure to construct cohomology classes on $\Gamma\backslash X$.

On the other hand we have another construction which gives us homology classes. To get these homology classes we start from lower dimensional reductive subgroups $M_\infty \hookrightarrow G_\infty$ for which $\Gamma_M = \Gamma \cap M_\infty$ is an arithmetic subgroup. If X_M is the corresponding symmetric space we get a map $\Gamma_M\backslash X_M \to \Gamma\backslash X$. We even can find cases where $\Gamma_M\backslash X_M$ is compact and then the fundamental class of $\Gamma_M\backslash X_M$ gives us a homology class on $\Gamma\backslash X$. Our problem is to find situations where the dimension of $\Gamma_M\backslash X_M$ —which is also the dimension of the homology class–equals **42**
the dimension of an Eisenstein class. If this is the case we can ask for the value of the Eisenstein class on the above homology class which amounts to evaluating the integral

$$\int_{\Gamma_M\backslash X_M} E(\psi, s_\psi)$$

This idea of constructing cycles by means of subgroups $M_\infty \hookrightarrow G_\infty$ appears already in [2] and [16].

In this paper we shall not consider the general problem but only a very special example. We take the group $G_\infty = \mathrm{PGL}_2(\mathbb{C})$ and Γ will be a member of a very specific class of congruence subgroups of $\mathrm{PGL}_2(\mathbf{Z}[i])$. If $\gamma \in \Gamma$ and if γ is not unipotent then it generates a quadratic field extension $E(\gamma)$ in the matrix ring $M_2(\mathbb{Q}(i))$ which defines a reductive subgroup in $\mathrm{PGL}_2(\mathbb{C})$. Then the quotient $\Gamma_M\backslash X_M$ in this case will simply be a circle and we shall compute the integrals of Eisenstein classes over these circles. It will turn out that these period integrals are expressible in terms of values of L-functions with Grossencharaktere of type A_o. The results are stated in section **??**.

Actually we have much more general results. We have a clear picture for those arithmetic groups which come from the group GL_2 over an arbitrary algebraic number field. It is planned to write a paper in which we treat this more general situation. But it is clear that this paper will be very long, very difficult to write and certainly also not easy to read. For instance we shall have to use adeles, we have to introduce coefficient systems and so on. That paper will contain proofs of the results announced in [7] and the results in there have to be generalized. Therefore I made up my mind and decided to write a paper where all this is discussed in a special case. I tried to give many details which will cause some repetition and overlap with older papers and the one planned. But the degree of complexity in the general situation is very high and I think it might be useful to discuss one special case.

During the preparation of this paper here I became aware that also the theory of Eisenstein classes which has been announced in [7] has some interesting arithmetic aspects. We shall devote a large part of this
43 paper to recall the theory of these Eisenstein classes and to discuss these arithmetic aspects which also concern values of some L-series. Therefore the title of the paper is not quite appropriate.

I want to thank D. Zagier for several discussions and for pointing to me how to compute the period integrals by a method that goes back to E. Hecke. ([10], 200).

1.0 Some Notations. If R is any commutative ring with identity we denote its group of invertible elements by R^x.

The field $\mathbb{Q}[i]$ will be denoted by F, throughout this paper we consider F as a subfield of $\mathbb{C}$, i.e. we fix an embedding of F into $\mathbb{C}$. The ring of Gaussian integers $\mathbf{Z}[i] \subset \mathbb{Q}(i)$ will be denoted by $\mathcal{O}$. More general if E is any algebraic number field, we denote by $\mathcal{O}_E$ its ring of algebraic integers.

The finite places of F will be denoted by $\mathsf{p}, \mathsf{q} \ldots$. The finite places of an extension E/F will be denoted by capital letters $\mathsf{P}, \mathsf{Q} \ldots$. We denote by E_P the completion at P, by $\mathcal{O}_{\mathsf{p}} \subset F_{\mathsf{p}}$ the ring of p-adic integers and by $\mathcal{O}_{E,\mathsf{P}} = \mathcal{O}_{\mathsf{P}}$ the ring of P-adic integers. We drop the index E if it is clear which filed we refer to.

We put $U_P = \mathcal{O}_{\mathsf{P}}^x$ and $U_{\mathsf{P}} = \mathcal{O}_{\mathsf{P}}^x$. The place of F at infinity will be denoted by ∞ and the completion F_∞ is canonically identified with $\mathbb{C}$.

The ring of adeles of F is denoted by $\mathbf{A}$ and by the letter we denote the group ideles of F. If we refer to another filed E we write $\mathbf{A}_E$, I_E. Elements of adele rings or idele groups will be denoted by underlined latin letters $\underline{x}, \underline{a}, \underline{u}, \ldots$. If $\underline{x} \in \mathbf{A}$ then we write

$$\underline{x} = (x_\infty, \ldots, x_p, \ldots, x_q, \ldots)$$

i.e. x_p, x_q are the p, q components. By $\mathbf{A}^f$ (resp I^f) we denote the ring (resp. group) of finite adeles (finite ideles) where we drop the component at ∞. Therefore

$$\mathbf{A} = \mathbb{C} \times \mathbb{A}^f, \ I = \mathbb{C}^x \times I^f$$

and for $\underline{x} \in \mathbf{A}$ we write $\underline{x}^f$ for its finite component, so that we have $\underline{x} = (x_\infty, \underline{x}^f)$.

By U^f we denote the maximal compact subgroup of units in I^f, i.e. 44
$\mathsf{U}^f = \Pi_\mathsf{p} U_\mathsf{p}$ and then $\mathsf{U} = U_\infty \times \mathsf{U}^f$ is the maximal compact subgroup in I, where U_∞ is the circle group.

We start from the group $G_o/F = \mathrm{PGL}_2/F$. Then B_o/F, U_o/F and T_o/F will be the standard Borelsubgroup of upper triangular matrices, its unipotent radical and the standard diagonal torus. Sometimes it will be convenient to look at G_o/F as a group over $\mathbb{Q}$, this means we put $G/\mathbb{Q} = R_{F/\mathbb{Q}}(G_o/F)$ where $R_{F/\mathbb{Q}}$ is the functor of restriction of scalars. ([27], 1.3.).

For any group scheme H/A over any ring and any extension $A \to A_1$, we denote the group of points of H with values in A_1 by $H(A_1)$.

1.1 The Cohomology of Γ and the space $\Gamma\backslash X$.

Let us put

$$\Gamma_o = \mathrm{PGL}_2(\mathscr{O}) = \mathrm{PGL}_2(Z[i]) = \mathrm{GL}_2(\mathscr{O})/Z$$

where $Z = \left\{\begin{pmatrix} i^m & 0 \\ 0 & i^m \end{pmatrix} | m \in \mathbf{Z}/4\mathbf{Z}\right\}$. We have $\Gamma_o \subset \mathrm{PGL}_2(\mathbb{C})$ and the group Γ_o acts on the three dimensional hyperbolic space $X = \mathrm{PGL}_2(\mathbb{C})/K_\infty$ where K_∞ is the projective unitary group $\mathrm{SU}(2)/\text{centre} = \mathrm{SO}(3)$. We choose the standard embedding

$$\mathrm{SU}(2)_= \left\{\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \Big| \alpha\beta \in \mathbb{C}, \alpha\overline{\alpha} + \beta\overline{\beta} = 1\right\} \subset \mathrm{SL}_2(\mathbb{C})$$

We choose an ideal $\mathsf{a} \in \mathscr{O}$ which has to satisfy one of the following conditions

$$\begin{aligned} &\mathsf{a} = ((1+i)^3) \qquad\qquad (1.1.1)\\ \text{or}\quad &\mathsf{a} \text{ is an odd prime where } N(\mathsf{a}) = p\\ &\text{is a prime in } \mathbf{Z} \text{ and } p \not\equiv 1 \bmod 8. \end{aligned}$$

This condition (1.1.1) implies that the group $W = \mathscr{O}^x = \{i, i^{-1}, 1, -1\}$ injects into the quotient $(\mathscr{O}/\mathsf{a})^x$ and that i is not a square in $(\mathscr{O}/\mathsf{a})^x$.

Our main object of study are the congruence subgroups

$$\Gamma = \Gamma(\mathsf{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_o \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{Id} \bmod \mathsf{a} \right\}$$

45 this means that Γ is the kernel of the natural homomorphism

$$\Gamma_o = \mathrm{PGL}_2(\mathscr{O}) \xrightarrow{p} \mathrm{PGL}_2(\mathscr{O}/\mathsf{a})$$

Lemma 1.1.2. *The homomorphism p is surjective.*

Proof. It is very easy to see that the map

$$\mathrm{SL}_2(\mathscr{O}) \to \mathrm{SL}_2(\mathscr{O}/\mathsf{a})$$

is surjective. The image of $\mathrm{SL}_2(\mathscr{O}/\mathsf{a})$ in $\mathrm{PGL}_2(\mathscr{O}/\mathsf{a})$ is of index 2 and the factor group is $(\mathscr{O}/\mathsf{a})^x/((\mathscr{O}/\mathsf{a})^x)^2$. Then we see that $p\left(\left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \notin$ image of $\mathrm{SL}_2(\mathscr{O}/\mathsf{a})$ and this proves the lemma.

Let R be any ring in $\mathbb{C}$. We want to assume always that the primes which divide the order of the finite group $\mathrm{PGL}_2(\mathscr{O}/\mathsf{a})$ are invertible in R. We are interested in the cohomology group $H^\nu(\Gamma, R)$ and we can identify

$$H^\nu(\Gamma, R) \simeq H^\nu(\Gamma\backslash X, R)$$

since Γ has no torsion, as one easily checks.

First of all we want to summarize some basic facts and definitions of the cohomology theory. If M is a projective R-module on which we have an action of the finite group $\overline{G} = \Gamma_o/\Gamma = \mathrm{PGL}_2(\mathscr{O}/\mathsf{a})$ we can define the cohomology groups

$$H^\nu(\Gamma_o, M)$$

We will mainly be concerned with $H^1(\Gamma_o, M)$ and we recall the definition in this case:

We write the action of $\overline{G}$ on M by $(\overline{g}, m) \to \overline{g} \cdot m$ and define

$$Z^1(\Gamma_0, M) = \{\Phi : \Gamma_o \to M | \Phi(\gamma_1\gamma_2) = \Phi(\gamma_1) + \gamma_1\Phi(\gamma_2)\}$$

This is the module of 1-cocycles. We have a map

$$M \xrightarrow{\delta_o} Z^1(\Gamma_o, M)$$
$$\delta : m \to \{\gamma \to (m - \gamma m)\}$$

and $H^1(\Gamma_o, M) = Z^1(\Gamma_o, M)/\delta_o(M)$.

There is another way to define these cohomology groups: We look 46
at the projection

$$\overline{\pi} : X \to \Gamma_o\backslash X$$

and we define a sheaf $\widetilde{M}$ on $\Gamma_o\backslash X$ as follows. For any open set $U \subset \Gamma_o\backslash X$ we define

$$\widetilde{M}(U) = \left\{ m : \overline{\pi}^{-1}(U) \to M \;\middle|\; \begin{array}{l} m(\gamma u) = \gamma \cdot m(u) \text{ and} \\ m \text{ is locally constant.} \end{array} \right\}$$

It is well known that under the given assumptions we have ([21], 1.6).

$$H^\nu(\Gamma_o\backslash X, \widetilde{M}) \simeq H^\nu(\Gamma_o, M)$$

Let us look at the special case where $M = R[\overline{G}]$ is the group ring of the finite group $\overline{G}$. In this case we have two actions of $\overline{G}$ on M namely by right and left multiplication

$$(g_1, g_2) : \sum_{\gamma \in G} a_\gamma \gamma \to \sum a_\gamma g_1 \gamma g_2^{-1}$$

We define the cohomology groups

$$H^1(\Gamma_o, R[\overline{G}])$$

by the module structure given by right multiplication; so if $m = \sum\limits_{\gamma \in \overline{G}} a_\gamma \gamma \in R[\overline{G}]$ and $\gamma_1 \in \overline{G}$ we have

$$\gamma_1 m = \sum_\gamma a_\gamma \gamma \gamma_1^{-1} = \sum_\gamma a_{\gamma\gamma_1} \gamma$$

The well known Lemma of Shapiro tells us that

$$H^1(\Gamma, R) \simeq H^1(\Gamma_o, R[\overline{G}]) \tag{1.1.3}$$

and it is very easy to make this isomorphism explicit. If $\Phi : \Gamma_o \to R[G]$ is a 1-cocycle and if we write $\Phi(\gamma) = \sum\limits_{\sigma \in \overline{G}} \Phi_\sigma(\gamma)\sigma$. Then the cocycle relation tells us that $\Phi_\sigma(\gamma_1) + \Phi_{\sigma\gamma_1}(\gamma_2) = \Phi_\sigma(\gamma_1\gamma_2)$ for all γ_1, $\gamma_2 \in \Gamma$ and all $\sigma \in \overline{G}$. If we restrict Φ to the subgroup Γ all the Φ_σ are homomorphisms. It follows from the cocycle relation that for $\gamma \in \Gamma$, $\eta \in \Gamma_o$ and $\overline{\eta} = \eta \bmod \Gamma$

$$\Phi_\sigma(\eta\gamma\eta^{-1}) = \Phi_{\sigma\overline{\eta}}(\gamma)$$

47 This tells us that Φ_1 determines the Φ_σ for $\sigma \neq 1$ and it is easy to see that $\Phi_1 : \Gamma \to R$ is the image of the class represented by Φ and the Shapiro isomorphism (1.1.3).

The group $\overline{G}$ acts on the cohomology groups $H^1(\Gamma, R) = H^1(\Gamma\backslash X, R)$ where the action is induced by conjugation. On the other hand the action of $\overline{G}$ by left multiplication induces an action of $\overline{G}$ on $H^1(\Gamma_o, R[\overline{G}])$ and it is not hard to check that (1.1.3) commutes with these actions.

This isomorphism (1.1.3) allows us to decompose the cohomology, we have

$$R[\overline{G}] = \bigoplus_\theta M_\theta$$

where the M_θ are irreducible $\overline{G} \times \overline{G}$-modules. (Here we use our assumption that $1/|\overline{G}| \in R$). Then we get a decomposition

$$H^1(\Gamma, R) = H^1(\Gamma_o, R[\overline{G}]) = \bigoplus_\theta H^1(\Gamma_o, M_\theta)$$

If we assume in addition that R contains enough roots of unity, then the M_θ will be absolutely irreducible and we get

$$M_\theta = M_{\widehat{\delta}} \otimes M_\delta$$

where δ runs over the irreducible $\overline{G}$-modules and $\widehat{\delta}$ is the contragriedient module. Therefore we get

$$H^1(\Gamma, R) = \bigoplus_{\delta \in \widehat{\overline{G}}} H^1(\Gamma_o, M_\delta) \otimes M_{\widehat{\delta}}$$

and the action of $\overline{G}$ on the right hand side is trivial on the first factor and the given action on $M_{\widehat{\delta}}$. □

1.2 The Compactification of $\Gamma \backslash X$ and the Cohomology at Infinity It is well known that in this case the space $\Gamma \backslash X$ is not compact. It has a finite number of cusps which are in one-to-one correspondence with the Γ-conjugacy classes of Borel subgroups $B \subset G/F$. ([1]) Borel and Serre developed a general theory of compactification of such spaces $\Gamma \backslash X$. They proved in [3] that we have a homotopy equivalence

$$\Gamma \backslash X \hookrightarrow \Gamma \backslash \overline{X}$$

where in this special case $\Gamma \backslash \overline{X}$ is a compact manifold with a boundary.
The boundary components are in one-to-one correspondence with the Γ- **48**
conjugacy classes of Borel subgroups, i.e. they correspond to the cusps. We want to give a precise description of all this in our special situation.

Let B be any Borel subgroup defined over the group field F. Let $U \subset B$ be its unipotent radical. It follows from the Iwasawa decomposition that the group $B(\mathbb{C})$ acts transitively on X. The positive root defines a homomorphism

$$\alpha : B \to G_m$$

and from this we get a homomorphism

$$\alpha : B(\mathbb{C}) \to G_m(\mathbb{C}) = \mathbb{C}^x$$

We put

$$B^{(1)}(\mathbb{C}) = \{b \in B(\mathbb{C}) |\, |\alpha(b)|_{\mathbb{C}} = 1\}$$

where $|z|_{\mathbb{C}} = z\bar{z}$ for $z \in \mathbb{C}^x$. The group

$$B(\mathbb{C}) \cap K_\infty = B^{(1)}(\mathbb{C}) \cap K_\infty = K_\infty^B$$

is a one dimensional circle and it is clear that we have a semidirect product

$$B^{(1)}(\mathbb{C}) = U(\mathbb{C}) \cdot K_\infty^B$$

Therefore we have with $x_o = K_\infty \in G/K_\infty$

$$X_B^{(1)} = B^{(1)}(\mathbb{C}) \cdot x_o = U(\mathbb{C}) \cdot x_o \subset X$$

and $X_B^{(1)} \simeq U(\mathbb{C}) \simeq \mathbb{C}$. If we put $\Gamma_B = B(\mathbb{C}) \cap \Gamma$ then we get a homotopy equivalence

$$\Gamma_B \backslash X_B^{(1)} = \Gamma_B \backslash U(\mathbb{C}) \hookrightarrow \Gamma_B \backslash X$$

and the Borel-Serre theory gives us that $\Gamma_B \backslash X_B^{(1)}$ is diffeomorphic to the boundary component Y_B of $\Gamma \backslash \overline{X}$ which corresponds to B ([3]). Since $\Gamma_B \simeq \mathbf{Z} \oplus \mathbf{Z}$ we get that Y_B is a product of two circles.

Remarks

(1) Our congruence condition guarantees that $\Gamma \cap B(\mathbb{C}) = \Gamma \cap U(\mathbb{C})$ since the image of $\Gamma \cap B(\mathbb{C})$ in $B(\mathbb{C})/U(\mathbb{C})$ has to consist of units in $\mathscr{O}$.

49 (2) To give the reader a better feeling for the Borel-Serre compactification we add a few more comments.

We mentioned already that $B(\mathbb{C})$ acts transitively on X, we use this fact to define the function

$$h_B : X \to \mathbb{R}_t^x$$
$$h_B : x = bx_o \to |\alpha(b)|_{\mathbb{C}}$$

We introduce the sets

$$X^B(c) = \{x \in X | h_B(x) \geq c\}$$

and the reduction theory tells us ([1], and [5], 1.2.) that for c sufficiently large we have an embedding

$$\Gamma_B \backslash X^B(c) \hookrightarrow \Gamma \backslash X$$

and using the geodesic action or the vector field dh^B we find

$$\Gamma_B \backslash X^B(1) = \Gamma_B \backslash X_B^{(1)} \times [1, \infty)$$

The Borel-Serre compactification in this case simply consists of adding ∞ in the second factor

$$\Gamma_B \backslash X^B(1) = \Gamma_B \backslash X_B^{(1)} \times [1, \infty] \hookrightarrow \Gamma_B \backslash X_B^{(1)} \times [1, \infty]$$

and $Y_B = \Gamma_B \backslash X_B^{(1)} \times \{\infty\}$.

The first part of the paper is devoted to the study of the map

$$H^1(\Gamma \backslash X, R) \xrightarrow{\sim} H^1(\Gamma \backslash \overline{X}, R) \to H^1(\partial(\Gamma \backslash \overline{X}), R) \xrightarrow{\sim} \bigoplus_B H^1(Y_B, R)$$

where B runs over a set of representatives for the Γ conjugacy classes of Borel subgroups. The group Γ_B is free abelian of rank 2 and therefore we have

$$H^1(Y_B, R) = \mathrm{Hom}(\Gamma_B, R) = R^2$$

If we want to describe the cohomology of the boundary we have to describe the set of cusps or the set of Γ-conjugacy classes of Borel subgroups. This is very simple in this case since $\mathcal{O}$ has class number one. Actually we shall do a little bit better. We know that $H^1(\mathcal{O}(\Gamma \backslash \overline{X}), R)$ is a $\Gamma_o / \Gamma = \overline{G}$-module and we give a description of this $\overline{G}$-module.

Since $\mathcal{O}$ has class number one it follows that the group Γ_o acts tran- **50**
sitively on the set of boundary components. It is easy to see (and will also follow from considerations in section 1.3) that the stabilizer of the boundary component Y_{B_o} is the group

$$\overline{U}_+ = \overline{U}_o \cdot W = \left\{ \begin{pmatrix} i^m & \overline{u} \\ 0 & 1 \end{pmatrix} \middle| \overline{u} \in \mathcal{O}/\mathsf{a}, i = i \bmod \mathsf{a} \right\}$$

where $W = \left\{ \begin{pmatrix} i^m & 0 \\ 0 & 1 \end{pmatrix} \middle| m \in \mathbf{Z} \right\}$. The group $\overline{U}_+$ acts on $H^1(Y_{B_o}, R)$ and it follows from general principles of representation theory that we have an $\overline{G}$-module isomorphism

$$H^1(\partial(\Gamma\backslash\overline{X}), R) \xrightarrow{\sim} \mathrm{Ind}_{\overline{U}_+}^{\overline{G}} H^1(Y_{B_o}, R)$$

where the induced module is the space of functions

$$\left.\begin{array}{r} \mathrm{Ind}_{\overline{U}_+}^{\overline{G}} H^1(Y_{B_o}, R) = \{h : \overline{G} \to H^1(Y_{B_o}, R) | h(\overline{g}\overline{u}^{-1}) = \overline{u}h(\overline{g}) \\ \text{for } \overline{g} \in \overline{G} \text{ and } \overline{u} \in \overline{U}_+ \end{array}\right\}$$

The group $\overline{G}$ acts on these functions by left translations.

It is easy to decompose this module into irreducible modules. We assume that R contains the $|(\mathscr{O}/\mathfrak{a})^*|$-roots of unity. The group $\overline{U}_+ = \overline{U}_o \cdot W$ and U_o acts trivially on $H^1(Y_{B_o}, R)$. Under the action of W we have a decomposition $H^1(Y_{B_o}, R) = \mathrm{Hom}(\Gamma_{B_o}, R) = L_+ \oplus L_-$ where $\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$ acts on L_+ by multiplication by i and on L_- by multiplication by $-i$.

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} 1_+ = i1_+; \quad \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} 1_- = -i1_-$$

We look at the characters $\phi : (\mathscr{O}/\mathfrak{a})^x \to S^1$ for which $\phi(i) = i$. For each such character we have a subspace

$$\left.\begin{array}{r} M_\phi^* = \{h : \overline{G} \to L_+ | h(\overline{g}\overline{b}^{-1}) = \phi(\overline{b})h(\overline{g}) \\ \text{for } \overline{b} \in \overline{B}_o \text{ and } \overline{g} \in \overline{G} \end{array}\right\}$$

and $\mathrm{Ind}_{\overline{U}_+}^{\overline{G}} L_+$ and analogously we define $M_{\overline{\phi}}^* \subset \mathrm{Ind}_{\overline{U}_+}^{G} L_-$. This gives us a decomposition

$$\mathrm{Ind}_{\overline{U}_+}^{\overline{G}} H^1(Y_{B_o}, R) = H^1(\partial(\Gamma\backslash\overline{X}), R) = \bigoplus_{\substack{\phi:(\mathscr{O}/\mathfrak{a})^x \to S^1 \\ \phi(i)=i}} (M_\phi^* \oplus M_{\overline{\phi}}^*) \quad (1.2.1)$$

51 where the M_ϕ^* and $M_{\overline{\phi}}^*$ are irreducible $\overline{G}$-modules. (1.2.2 and [25], Cor. 4.11.) Here we profit from the fact that ϕ cannot be a trivial or a quadratic character.

1.2.2 At this point I want to give an idea of one of the main questions of this paper. As we have seen already we can study the restriction map

$$H^1(\Gamma\backslash\overline{X}, R) \to H^1(\partial(\Gamma\backslash\overline{X}), R)$$

and we have decomposed the right hand side into irreducible modules (1.2.1). Let us assume that we have selected generators $e_+ \in L_+$ and $e_- \in L_-$ (We shall see later that we have a rather canonical choice, see 1.6.1) then we can identify M_ϕ^* with the induced representation

$$M_\phi = \{\psi : \overline{G} \to R | \psi(\overline{g}\overline{b}^{-1}) = \phi(\overline{b})\psi(\overline{g})\}$$

by mapping $\psi \to \{g \to \psi(g) \cdot g \cdot e_+\}$. One knows that M_ϕ and $M_{\overline{\phi}}$ are irreducible $\overline{G}$-modules and they are isomorphic. The operator

$$\begin{aligned} T_\phi &: M_\phi \to M_{\overline{\phi}} \\ T_\phi &: \psi \to T_\phi\psi(\overline{g}) = \sum_{u \in U_o} \psi(w\overline{ug}) \end{aligned}$$

with $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is a non zero interwining operator ([25], §5).

Since there are no other isomorphisms among these induced representations the decomposition (1.2.1) is isotypical.

Let us denote the quotient field of R by K. For any ϕ we pick the isotypical component of M_ϕ in $H^1(\Gamma\backslash X, K)$ and get a map

$$H^1(\Gamma\backslash X, K)_\phi \to M_\phi \otimes K \oplus M_{\overline{\phi}} \otimes K$$

It follows from topological reasons that the image of the restriction map is of multiplicity one (namely $\frac{1}{2}\times$ the multiplicity of $M_\phi \otimes K \oplus M_{\overline{\phi}} \otimes K$ which is two) (comp. [20] 3.4). Therefore the image is of the form (Schur's lemma)

$$\{(\psi, c_\phi T_\phi \psi) | \psi \in M_\phi \otimes K\} \subset M_\phi \otimes K \oplus M_{\overline{\phi}} \otimes K$$

where $c_\phi \in K$ or $c_\phi = \infty$ in which case the image would be the second **52**
component. What is the value of c_ϕ?

This problem will be attacked by transcendental methods, the theory of Eisenstein series will give us an expression for c_ϕ in terms of values of L-functions.

1.2.3 Before I conclude this section I want to translate the questions and assertions 1.2.1 and 1.2.2 in the language of cohomology groups with coefficients.

We have the isomorphism (1.1.3) and we put $\Gamma_{o,B_o} = B_o(F) \cap \Gamma_o$. Now we want to give a detailed description of the different isomorphisms in the following commutative diagram

$$\begin{tikzcd}
\mathrm{Sh} : H^1(\Gamma_o, R[\overline{G}]) \arrow[r, "\sim"] \arrow[d, "\mathrm{res}"] & H^1(\Gamma, R) \simeq H^1(\Gamma\backslash X, R) \arrow[d] \\
\partial\mathrm{Sh} : H^1(\Gamma_{o,B_o}, R[\overline{G}]) \arrow[r, "\sim"] & H^1(\partial(\Gamma\backslash\overline{X}), R) \arrow[d] \\
 & \bigoplus\limits_{\substack{\phi:(\partial/\mathsf{a})^{x}\to S^1 \\ \phi(i)=i}} (M^*_\phi \oplus M^*_{\overline{\phi}})
\end{tikzcd} \tag{1.2.3.1}$$

In this context it is convenient to identify the group ring $R[\overline{G}]$ with the ring of R valued functions on $\overline{G}$ which is denoted by $\mathsf{C}(\overline{G})$ and

$$\mathsf{C}(\overline{G}) \xrightarrow{\sim} R[\overline{G}]$$

$$\text{by} \qquad f \to \sum_{\sigma\in\overline{G}} f(\sigma)\cdot\sigma$$

Now let us assume that

$$\Phi : \Gamma_o \to \mathsf{C}(\overline{G})$$

is a 1-cocycle. Then for $\gamma \in \Gamma_o$ the value $\Phi(\gamma)$ is a function of $\overline{G}$ and the value of this function at $\sigma \in \overline{G}$ will be denoted by

$$\Phi(\gamma)\ (\sigma)$$

If we restrict Φ to Γ then the map $\gamma \to \Phi(\gamma)(\sigma)$ is a homomorphism for
53 any $\sigma \in \overline{G}$ and we have seen that Sh is given by

$$[\Phi] \to \{\gamma \to \Phi(\gamma)(1)\}$$

where $[\Phi]$ is the class defined by Φ. The class $[\Phi]$ defines a class on the boundary and we get a family of homomorphisms

$$\phi_B : \Gamma_B = \Gamma \cap B(F) \to R$$
$$\phi_B(\gamma_B) = \Phi(\gamma_B)(1)$$

where B runs over a set of Γ conjugacy classes of Borel subgroups. If we write $\Gamma_B = \eta\Gamma_{B_o}\eta^{-1}$ with $\eta \in \Gamma_o$ then we get a homomorphism

$$\Gamma_{B_o} \to R$$
$$\gamma_o \to \phi_B(\eta\gamma_o\eta^{-1}) = \Phi(\eta\gamma_o\eta^{-1})(1)$$

But for $\gamma_o \in \Gamma_{B_o} = B_o(F) \cap \Gamma$ we have

$$\Phi(\eta\gamma_o\eta^{-1}) = \eta\Phi(\gamma_o) = \overline{\eta}\Phi(\gamma_o)$$

where $\overline{\eta}$ is the image of $\eta \in \Gamma_o$ in $\overline{G}$. Therefore we have

$$\Phi(\eta\gamma_o\eta^{-1})(1) = \Phi(\gamma_o)(\overline{\eta})$$

and this tells us that the cocycle Φ defines a map

$$h_\Phi : \overline{G} \to \mathrm{Hom}(\Gamma_{B_o}, R)$$
$$h_\Phi : \sigma \to \{\gamma_o \to (\gamma_o)(\sigma)\}$$

This map is also defined for cocycles on Γ_{o,B_o} with values in $R[\overline{G}]$ and the map $[\Phi] \to h_\phi$ gives us a direct realisation of $\partial\mathrm{Sh}$ and makes the commutativity of the diagram clear.

This means that the study of our restriction map can be reduced to the inverstigation of maps

$$H^1(\Gamma_o, M) \to H^1(\Gamma_{o,B_o}, M)$$

where M is a projective R-module on which we have an irreducible $\overline{G}$-action, i.e. $M \bigotimes_R K$ is an irreducible $\overline{G}$-module. Again we want to assume that R contains enough roots of unity.

We consider $H^1(\Gamma_{o,B_o}, M)$. We have **54**

$$\Gamma_{o,B_o} = \left\{ \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} | t \in \mathscr{O}^x, u \in \mathscr{O} \right\} = \Gamma_{o,U_o} \cdot W$$

where $\Gamma_{o,U_o} = \left\{ \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix} \right) | u \in \mathscr{O} \right\}$ and W is cyclic of order four generated by $\left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix} \right)$.

We always identify $W \subset \Gamma_{o,B_o}$ with its image in

$$\overline{B}_o = \{ \left(\begin{smallmatrix} t & u \\ 0 & 1 \end{smallmatrix} \right) | t \in (\mathscr{O}/\mathrm{a})^x, \overline{u} \in \mathscr{O}/\mathrm{a} \}.$$

Since we assume that $|\overline{G}|$ is invertible in our ring R we see that the action of $\overline{U}_o$ on M is semisimple and it is obvious that

$$H^1(\Gamma_{o,U_o}, M) = \mathrm{Hom}(\Gamma_{o,U_o}, M^{\overline{U}_o})$$

where we have to take into account that $M^{\overline{U}_o} = M^{\Gamma_{o,U_o}}$. (The notation $M^{\overline{U}_o}$ means of course that we take the invariants). Therefore we can restrict our attention to those modules where $M^{\overline{U}_o} \neq (0)$. It is well known that in this case M has to be a submodule of an induced module N_χ where χ is a character $\chi : \overline{B}_o \to \overline{B}_o/\overline{U}_o \to S^1$ and

$$N_\chi = \{ f : \overline{G} \to R | f(\overline{b}\overline{g}) = \chi(\overline{b}) f(\overline{g}) \}$$

The module $N_\chi^{\overline{U}_o}$ is easy to compute. We have the Bruhat decomposition $\overline{G} = \overline{B}_o w \overline{U}_o \cup \overline{B}_o$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ and we put

$$f_o = \begin{cases} \overline{b} w \overline{y} & \to \chi(\overline{b}) \\ \overline{b} & \to 0 \end{cases}$$

$$f_\infty = \begin{cases} \overline{b} w \overline{u} & \to 0 \\ \overline{b} & \to \chi(\overline{b}) \end{cases}$$

Then $N_\chi^{\overline{U}_o} = R f_o \oplus R f_\infty$. The group $\overline{G}$ acts on N_χ by right translations, if we restrict this action to $\overline{B}_o$, then $N_\chi^{\overline{U}_o}$ is an invariant subspace and

$$\overline{b} f_o = \chi(\overline{b})^{-1} f_o, \ \overline{b} f_\infty = \chi(\overline{b}) f_\infty$$

Since $\Gamma_{o,B_o} = \Gamma_{o,U_o} \cdot W$ we have obviously

$$H^1(\Gamma_{o,B_o}, M) = \mathrm{Hom}(\Gamma_{o,U_o}, M^{\overline{U}_o})^W$$

The group W acts on Γ_{o,U_o} by means of the adjoint action and the module $\Gamma_{o,U_o} \otimes \mathscr{O}$ decomposes into two spaces on which $\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \in W$ acts 55
by the eigenvalues i, $-i$. Then it becomes clear, that $H^1(\Gamma_{o,B_o}, M) \neq 0$ if and only if $\chi(i) = \pm i$. We assume $\chi(i) = i$ ans we call this character ϕ again. So $\phi = \chi$, then we have that N_ϕ is irreducible ([25] 4.11.) and $M = N_\phi$.

We find

$$\mathrm{Hom}(\Gamma_{o,U_o}, N_\phi^{\overline{U}_o})^W = \mathrm{Hom}(\Gamma_{o,U_o}, Rf_o)^W \oplus \mathrm{Hom}(\Gamma_{o,U_o}, Rf_\infty)^W$$

and

$$\begin{aligned} \mathrm{Hom}(\Gamma_{o,U_o}, Rf_o) &= \mathrm{Hom}(\Gamma_{o,U_o}, R) = \mathrm{Hom}(\Gamma_{B_o}, R) \\ \mathrm{Hom}(\Gamma_{o,U_o}, Rf_\infty) &= \mathrm{Hom}(\Gamma_{o,U_o}, R) = \mathrm{Hom}(\Gamma_{B_o}, R) \end{aligned}$$

But we have to keep in out mind that W acts *non trivially* on Rf_o, Rf_∞ and it acts trivially on R. If we take up our earlier notations we find

$$\begin{aligned} \mathrm{Hom}(\Gamma_{o,U_o}, Rf_o)^W &= L_+ \subset \mathrm{Hom}(\Gamma_{B_o}, R) \\ \mathrm{Hom}(\Gamma_{o,U_o}, Rf_\infty)^W &= L_- \subset \mathrm{Hom}(\Gamma_{B_o}, R) \end{aligned}$$

We constructed an identification

$$H^1(\Gamma_{o,B_o}, N_\phi) = L_- \oplus L_- = Re_+ \oplus Re_-$$

if we take up the notations in 1.2.2.

We look again at our restriction map

$$H^1(\Gamma_o, N_\phi) \to H^1(\Gamma_{o,B_o}, N_\phi) = Re_+ \oplus Re_-$$

and we want to relate this to 1.2.2.

Let us pick the isotypical component $R[\overline{G}]_\phi$ in $R[\overline{G}]$ then we get

$$\begin{tikzcd}
H^1(\Gamma_o, R[\overline{G}]_\phi) \arrow[d, equal] & \\
H^1(\Gamma, R)_\phi \arrow[r] & M^*_\phi \oplus M^*_{\overline{\phi}} = M_\phi \oplus M_{\overline{\phi}}
\end{tikzcd}$$

On the other hand we realized our given induced representation as a submodule of $R[\overline{G}]_\phi$ namely

$$N_\phi \hookrightarrow R[\overline{G}]_\phi$$

56 Therefore we get a diagram

$$\begin{tikzcd}
H^1(\Gamma_o, N_\phi) \arrow[r] \arrow[d, hook] & L_+ \oplus L_- \arrow[d, hook, "\lambda"] \arrow[r, phantom, "="] & \mathrm{Re}_+ \oplus \mathrm{Re}_- \arrow[d, hook, "\lambda"] \\
H^1(\Gamma_o, R[\overline{G}]_\phi) \arrow[r] & M^*_\phi \oplus M^*_{\overline{\phi}} \arrow[r, phantom, "="] & M_\phi \oplus M_{\overline{\phi}}
\end{tikzcd}$$

and we have to compute the inclusions λ and λ_1.

To get these inclusions we observe that a generator of L_+ is given by the cocycle

$$\Gamma_{U_o} \to Rf_o \subset N_\phi$$
$$\gamma \to e_+(\lambda) \cdot f_o$$

This defines (1.2.3.1) a map

$$h : \sigma \to \{\lambda \to e_+(\lambda) \cdot f_o(\sigma)\}$$
$$h : \overline{G} \to \mathrm{Hom}(\Gamma_{U_o}, R)$$

Now we observe that $f_o \in N_\phi$ is also an element in M_ϕ and we see that $\lambda_1 : e_+ \to f_o \in M_\phi$ and $\lambda_1 : e \to f_\infty \in M_{\overline{\phi}}$. The intertwining operator $T_\phi : M_\phi \to M_{\overline{\phi}}$ maps $T_\phi(f_o) = N(\mathsf{a})f_\infty$ and we get the proposition.

Proposition 1.2.4. *The image of the restriction map*

$$H^1(\Gamma_o, N_\phi \otimes K) \to Ke_+ + Ke_-$$

is spanned by the vector $(e_+, c_\phi N(\mathsf{a})e_-)$ *where* $e_\phi \in K \cup \{\infty\}$.

What is all this good for? If we want to compute explicitely with cocycles it seems to be convenient to work with Γ_o instead of Γ since it has less generators. We pay for it by introducing coefficients. Later on we shall compute $H^1(\Gamma_o, N_\phi)$ in some simple cases and we are then able to compute the number c_ϕ.

1.3 Adeles and the Description of the Set of Cusps In the adele group $G_o(\mathbf{A}) = \mathrm{PGL}_2(\mathbf{A})$ we have the maximal compact subgroup

$$\mathsf{K} = K_\infty \cdot \mathsf{K}^f = K_\infty \times \prod_{\mathsf{p}\text{ finite}} \mathrm{PGL}_2(\mathcal{O}_\mathsf{p})$$

where $K_\infty = PU(2)(1.1)$. The ideal a defines a congruence subgroup 57
$\mathsf{K}^f(\mathsf{a}) \subset \mathsf{K}^f$ namely

$$\mathsf{K}^f(\mathsf{a}) = \{\underline{k}^f \in \mathsf{K}^f | \overline{k}^f \equiv 1 \mathrm{mod}\ \mathsf{a}\}$$

Lemma 1.3.1.. *Every element* $\underline{x} \in G_o(\mathbf{A})$ *can be written*

$$\underline{x} = a \cdot (y_\infty, \underline{k}^f)$$

with $\underline{k}^f \in \mathsf{K}^f(\mathsf{a})$.

Proof. We represent $\underline{x} \in G_o(\mathbf{A})$ by an element $\underline{\widetilde{x}} \in \mathrm{GL}_2(\mathbf{A})$. If $\underline{\widetilde{x}} \in \mathrm{SL}_2(\mathbf{A})$, i.e. $\det(\underline{\widetilde{x}}) = 1$ then the assertion follows from strong approximation for SL_2. We may modify $\underline{\widetilde{x}}$ by an element $\underline{z} \in I$ which we consider as an element of the center of $\mathrm{GL}_2(\mathbf{A})$, then $\det(\underline{\widetilde{x}})$ gets multiplied by z^2. So the obstruction to get the determinant $\underline{\widetilde{x}}$ equal to one sits in I/I^2. We may modify $\underline{\widetilde{x}}$ by an element in $\mathrm{GL}_2(F)$ and by an element in the inverse image of $\mathsf{K}^f(\mathsf{a})$ in $\mathrm{GL}_2(\mathbf{A})$. This means that the obstruction against writing $\underline{x}$ in the above form lies in

$$I/I^2 \cdot F^x \cdot \mathsf{U}^f(\mathsf{a})$$

where $\mathsf{U}^f(\mathsf{a}) = \{\underline{t} \in \mathsf{U}^f | \underline{t} \equiv 1 \bmod \mathsf{a}\}$. Using the fact that F has class number one we find that this group is equal to

$$\mathbb{C}^x \times \mathsf{U}^f/\mathsf{U}^f(\mathsf{a}) \cdot W \cdot (\mathsf{U}^f)^2$$

where $W = \{i\} \subset F^x$. Since nothing is claimed at the infinite place we may drop the infinite component and the obstruction sits in

$$(\mathcal{O}/\mathsf{a})^x/((\mathcal{O}/\mathsf{a})^x)^2 \cdot W = 1.$$

The lemma is proved.

The lemma says simply that

$$G_o(F)\backslash G_o(\mathbf{A})/G_o(\mathbb{C}) \times \mathsf{K}^f(\mathsf{a}) = \{1\}.$$

Now we consider the double coset decomposition

$$B_o(F)\backslash G_o(\mathbf{A}^f)/\mathsf{K}^f(\mathsf{a})$$

Let us write

$$G_o(\mathbf{A}^f) = \bigcup_{\underline{\xi}} B_o(F)\underline{\xi}\mathsf{K}^f(\mathsf{a})$$

We extent $\underline{\xi}$ to an element $\underline{\xi}'$ of $G_o(A)$ by $(\underline{\xi}')_\infty = 1$. According to our
58 previous lemma we may write

$$\underline{\xi}' = a \cdot (a^{-1}, \underline{k}^f).$$

with $\underline{k}^f \in \mathsf{K}^f(\mathsf{a})$. Then $a^{-1}B_o a = B$ is a Borel subgroup over F. We see that the Γ-conjugacy class of this Borel subgroup depends only on $\underline{\xi}$. If we pick a Borel subgroup B/F then we find an $a \in G_o(F)$ such that $B = a^{-1}B_o a$. We choose $\underline{\xi}' = (1, a, \dots, a, \dots)$ and therefore we get a bijection between the set of Γ-conjugacy classes of Borel subgroups B/F of G/F and the set of double cosets

$$B_o(F)\backslash G_o(\mathbf{A}^f)/\mathsf{K}^f(\mathsf{a})$$

We are now able to settle the minor point left open in 1.2 concerning the stabilizer of the boundary component Y_{B_o} under the action of $\overline{G}$. We simply count the number of cusps. We have a map

$$B_o(F)\backslash G_o(\mathbf{A}^f)/\mathsf{K}^f(\mathsf{a}) \to B_o(\mathbf{A}^f)\backslash G_o(\mathbf{A}^f)/\mathsf{K}^f(\mathsf{a})$$

which is surjective.

Since $B_o(\mathbf{A}^f)\cdot\mathsf{K}^f = G_o(\mathbf{A}^f)$ we get

$$B_o(\mathbf{A})\backslash G_o(\mathbf{A}^f)/\mathsf{K}^f(\mathsf{a}) = \mathsf{K}^f/\mathsf{K}^f(\mathsf{a}) \cap B_o(\mathbf{A}^f) = \overline{G}/\overline{B}_o$$

The fibers of this map are equal to

$$B(F)\backslash B(\mathbf{A}^f)/B(\mathbf{A}^f) \cap \mathsf{K}^f(\mathsf{a})$$

where B is any Borel subgroup corresponding to a point in the fiber. Since the unipotent radical has strong approximation we find for these fibers that they are equal to

$$I^f/F^x \cdot \mathsf{U}^f(\mathsf{a}) = \mathsf{U}^f/W\mathsf{U}^f(\mathsf{a}) = (\mathscr{O}/\mathsf{a})^x/W$$

and that proves that the number of cusps is equal to $[\overline{G} : \overline{U}_+]$ □

1.4 Differential Forms and De Rham Cohomology We should look at G_o/F as group over the rationals and therefore we introduce $G/\mathbb{Q} = R_{F\backslash\mathbb{Q}}(G_o/F)$. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G/\mathbb{Q})$ is a $\mathbb{Q}$-vector space and we define $\mathfrak{g}_\infty = \mathfrak{g} \bigotimes_{\mathbb{Q}} \mathbb{R}$. Then $\mathfrak{g}_\infty$ is the Lie algebra of the real group $G_\infty = G(\mathbb{R}) = G_o(\mathbb{C})$. (We shall sometimes denote the group of complex points of groups over F by the subscript ∞ and then we stress the point 59
of view that they may also be considered as real points of a group over ($\mathbb{Q}$). Now

$$\mathfrak{g}_\infty = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \middle| \alpha, \beta, \gamma \in \mathbb{C} \right\}$$

and the Cartan involution obtained from our given maximal compact group is

$$\theta : X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \to -{}^t\overline{X} = \begin{pmatrix} -\overline{\alpha} & -\overline{\gamma} \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

We get the Cartan decomposition

$$\mathfrak{g}_\infty = \mathfrak{k}_\infty + \mathfrak{p}$$

where $\mathfrak{k}_\infty = \mathrm{Lie}(K_\infty)$. The real vector space $\mathfrak{p}$ has the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

The group $K_\infty = PU(2) = SU(2)/\{+\mathrm{Id}\}$ acts on $\mathfrak{p}$ by the adjoint action. If $k_\infty \in PU(2)$ is represented by the matrix $\left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right) \in SU(2)$ then

$$\begin{aligned} \mathrm{ad}(k_\infty)H &= (\alpha\overline{\alpha} - \beta\overline{\beta})H - 2\mathrm{Re}(\alpha\beta)E_1 - 2\mathrm{Im}(\alpha\beta)E_2 \\ \mathrm{ad}(k_\infty)E_1 &= 2\mathrm{Re}(\alpha\beta)H + \mathrm{Re}(\alpha^2 - \beta^2)E_1 + \mathrm{Im}(\alpha^2 + \beta^2)E_2 \\ \mathrm{ad}(k_\infty)E_2 &= 2\mathrm{Im}(\overline{\alpha}\beta)H = \mathrm{Im}(\alpha^2 - \beta^2)E_1 + \mathrm{Re}(\alpha^2 + \beta^2)E_2 \end{aligned} \tag{1.4.1}$$

The normalised Killing form on $\mathfrak{g}_\infty$

$$\langle X, Y\rangle = \frac{1}{16}\,\mathrm{trace}\,(\mathrm{ad}\,X \cdot \mathrm{ad}\,Y)$$

induces a K_∞ invariant, positive definite symmetric quadratic form on $\mathfrak{p}$. With respect to this form our three vectors H, E_1, E_2 form an orthonormal basis. We shall use this form to identify the space $\mathfrak{p}$ with its dual space.

The projection map

$$\pi : G_\infty \to G_\infty/K_\infty = X$$

defines an isomorphism

$$(d\pi)_e : \mathfrak{p} \to T_{x_o} = \check{T}_{x_o}$$

between $\mathfrak{p}$ and the tangent space of X at the point x_o.

60 This allows us to identify the space of differential p-forms on X with a certain space of $\Lambda^p\mathfrak{P}$-valued functions on the group G_∞. To be more

precise we can identify the space $\Omega^p(X)$ of C^∞-p-forms on X and the space of C^∞-functions

$$C^p(G_\infty, \Lambda^p \,\mathrm{ad}, \Lambda^p \mathsf{p}) = \{\omega : G_\infty \to \Lambda^p \mathsf{p} | \omega(g_\infty k_\infty) = \Lambda^p \,\mathrm{ad}(k_\infty^{-1})\omega(g_\infty)\}$$

([8], 1.3). We want to make this identification perfect in the sense that we do not distinguish between the p-form and the function on G_∞. The identification goes as follows: Let $\omega : G_\infty \to \Lambda^p \mathsf{p}$ which satisfies $\omega(g_\infty k_\infty) = \Lambda^p \,\mathrm{ad}(k_\infty^{-1})\omega(g_\infty)$. If $x \in X$ and $g_\infty \in G_\infty$ satisfies $g_\infty x_o = x$ then the left translation $y \to g_\infty y$ on X induces an isomorphism of tangent spaces

$$dL_{g_\infty} : T_{x_o} \xrightarrow{\sim} T_x$$

If $t_x \in \Lambda^p T_x$ then ω considered as a p-form has to have a value on t_x

$$\omega(x)(t_x) = \langle \omega(g_\infty), \Lambda^p dL_{g_{\infty^{-1}}}(t_x) \rangle \tag{1.4.2}$$

This identification is compatible with the action of G_∞ from the left on X, so we may divide by Γ and get

$$\Omega^p(\Gamma\backslash X) = C^p(\Gamma\backslash G_\infty, \Lambda^p \,\mathrm{ad}, \Lambda^p \mathsf{p})$$

It is important that we can write this also as a space of function on the adele group. Using lemma 1.3.1 we find

$$\begin{aligned}
&C^p(\Gamma\backslash G_\infty, \Lambda^p \,\mathrm{ad}, \Lambda^p \mathsf{p}) = \\
&C^p(G_o(F)\backslash G_o(\mathbf{A})/\mathsf{K}^f(\mathsf{a}), \Lambda^p \,\mathrm{ad}, \Lambda^p \mathsf{p}) = \\
&\left\{ \begin{array}{l} \omega : G_o(F)\backslash G_o(\mathbf{A}) \to \Lambda^p \mathsf{p} | \omega \text{ is } C^\infty \text{ in the infinite component} \\ \text{and } \omega(g\underline{k}) = \Lambda^p \,\mathrm{ad}(k_\infty^{-1})\omega(g) \text{ where } \underline{k} = (k_\infty, \underline{k}^f) \text{ and} \\ \mathsf{k}^f \in \mathsf{K}^f(\mathsf{a}) \end{array} \right\}
\end{aligned}$$

1.5 De Rham Cohomology at Infinity : Let B/F be any Borel subgroup of G_o/F. This Borel subgroup defines a boundary component $Y_B \subset \partial(\Gamma\backslash X)$ and we want to describe the cohomology of this boundary component in terms of differential forms.

We still fix our base point $x_o \in X$. We have seen that the boundary component associated to B is diffeomorphic to **61**

$$\Gamma_B\backslash X_B^{(1)} = \Gamma_B\backslash U(\mathbb{C})x_o = \Gamma_B\backslash U(\mathbb{C})$$

and we have homotopy equivalences

$$\Gamma_B\backslash X_B^{(1)} \hookrightarrow \Gamma_B\backslash X \hookrightarrow \Gamma_B\backslash X \cup Y_B$$

(1.2 Remark 2). The group $B_\infty = B(\mathbb{C})$ acts transitively on X and we put

$$K_\infty^b = B_\infty \cap K_\infty$$

Then K_∞^B is a circle. This allows us another description of the space of C^∞-p-forms on $\Gamma_B\backslash X$:

$$\Omega^p(\Gamma_B\backslash X) = C^p(\Gamma_B\backslash B_\infty; \Lambda^p \operatorname{ad}, \Lambda^p \mathsf{p}) =$$
$$\left\{ \begin{array}{l} \omega|\omega : \Gamma_B\backslash B_\infty \to \Lambda^p\mathsf{p}; \omega \text{ is } C^\infty \text{ and} \\ \qquad\qquad \omega(b_\infty k_\infty) = \Lambda^p \operatorname{ad}(k_\infty^{-1})\omega(b_\infty) \\ \qquad\qquad \text{for } k_\infty \in K_\infty^B. \end{array} \right\}$$

Under the action of K_∞^B we have a canonical decomposition of

$$\mathsf{p} = \mathsf{p}_{o,B} \oplus \mathsf{p}_{1,B} = \mathsf{p}_o \oplus \mathsf{p}_1$$

where p_o is of dimension 1 and K_∞^B acts trivially and $\mathsf{p}_{1,B}$ is two dimensional irreducible. In the case of $B = B_o$ this decomposition becomes

$$\mathsf{p} = \mathbb{R}H \oplus (\mathbb{R}E_1 \oplus \mathbb{R}E_2)$$

Therefore we get for any 1-form on $\Gamma_B\backslash X$ a decomposition

$$\omega = \omega_{o,B} + \omega_{1,B} = \omega_o + \omega_1$$

It is clear that the ω_o component vanishes if we restrict it to the "slices"

$$\Gamma_B\backslash X_B^{(1)} \to \Gamma_B\backslash X$$

and this tells us that our decomposition does not depend on the choice of the base point x_o.

Let us assume that $\omega \in \Omega^1(\Gamma_B\backslash X)$ is a closed 1-form. Then ω defines a cohomology class $[\omega] \in H^1(\Gamma_B\backslash X;\mathbb{R}) = H^1(\Gamma_B\backslash X_B^{(1)};\mathbb{R})$. We want to compute this class. The group U_∞ acts on $\Gamma_B\backslash X$ by translations and
62 from this it follows that the cohomology class $[\omega]$ is also represented by the form

$$\omega^{(0)}(b_\infty) = \int\limits_{\Gamma_B\backslash U_\infty} (u_\infty b_\infty) du_\infty$$

where the volume $\Gamma_B\backslash U_\infty$ is normalized to be equal to 1. If we restrict this 1-form $\omega^{(0)}$ to $\Gamma_B\backslash X^{(1)}$ we get

$$\omega^{(0)}\Big|\Gamma_B\backslash X^{(1)} = \omega_B^{(0)}\Big|\,\Gamma_B\backslash X^{(1)}$$

and $\omega_{1,B}^{(0)}$ is translation invariant and therefore constant. This means

$$\omega_{1,B}^{(0)}(u_\infty) = \omega_{1,B}^{(0)}(1) \in \mathfrak{p}_{1,B}$$

This element $\omega_{1,B}^{(0)}$ defines a homomorphism from Γ_B into R:

Every element $\gamma \in \Gamma_B$ can be written in the form $\gamma = \exp\log\gamma$ where $\log\gamma = \mathrm{Id} - \gamma \in \mathrm{Lie}(R_{F/\mathbb{Q}}(U/F))$ and the homomorphism is

$$\begin{aligned}\gamma \to \langle \log\gamma, \omega_{1,B}^{(0)}(1)\rangle &= \\ = \langle \log\gamma, \omega^{(0)}(1)\rangle&\end{aligned}$$

Since we have $H^1(\Gamma_B,\mathbb{R}) = \mathrm{Hom}(\Gamma_B,\mathbb{R})$ we find the formula

$$[\omega](\gamma) = \langle \log\gamma, \omega^{(0)}(1)\rangle = \langle \log\gamma, \omega_{1,B}^{(0)}(1)\rangle \tag{1.5.1}$$

We consider the group B_∞ as a real algebraic subgroup of $\mathrm{PGL}_2(\mathbb{C}) = G(\mathbb{R})$ where $G = R_{F/\mathbb{Q}}(G_o)$. The centralizer of K_∞^B is a real torus T_∞ which is of dimension 2 and decomposes into a one dimensional split torus and a one dimensional anisotropic torus. Therefore we have

$$\begin{aligned}T_\infty &\xrightarrow{\sim} \mathbb{C}^x = \mathbb{R}^x \times S^1 \\ t_\infty &\xrightarrow{\sim} (t'_\infty, k(t_\infty))\end{aligned}$$

If $\omega(1) \in \mathfrak{p}_{1,B}$ we construct for any complex number $s \in \mathbb{C}$ a form

$$\omega_s : \Gamma_B \backslash B_\infty \to \mathfrak{p}_{1,B} \otimes C$$

by

$$\omega_s(b_\infty) = \omega_S(u_\infty t_\infty) = |t_\infty|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}} \operatorname{ad}(k(t_\infty)^{-1})\omega(1)$$

where as before $|z|_{\mathbb{C}} = z\bar{z}$ for $z \in \mathbb{C}$.

63 **Lemma 1.5.2.** *The 1-form ω_s is closed if and only if $s = 0$.*

This is an easy computation (see also [8], Lemma 3.1).

This lemma allows us to go back and forth from forms on $\Gamma_B \backslash X_B^{(1)}$ to forms on $\Gamma_B \backslash X_B$.

1.6 The Adelic Description of the Cohomology at the Boundary In the last section we gave a discussion of the de Rham cohomology of an individual boundary component. Now we want to look at all the boundary components and to describe the cohomology in terms of differential forms which depend on adelic variables.

We start from our standard Borel subgroup B_o and as in 1.5 we decompose $\mathfrak{p} = \mathfrak{p}_o \oplus \mathfrak{p}_1 = \mathfrak{p}_{o,B_o} \oplus \mathfrak{p}_{1,B_o}$. We define $B_{o,\infty}^{(1)} = \{b_\infty \in B_{o,\infty} |\ |\alpha(b_\infty)| = 1\}$. We introduce the space of maps

$$\mathsf{H}_\infty = \left\{ \begin{array}{l} \omega : U_o(\mathbf{A})B_o(F)\backslash B_{o,\infty}^{(1)} \cdot G_o(\mathbf{A}^f) \to \mathfrak{p}_1 \otimes \mathbb{C} | \\ \omega(\underline{gk}) = \operatorname{ad}(k_\infty^{-1})\omega(g) \text{ for } \underline{k} = (k_\infty, \underline{k}^f) \\ \text{and } k_\infty \in K_\infty^{B,o}, \underline{k}^f \in \mathsf{K}^f(\mathfrak{a}) \end{array} \right\}$$

We want to show that we have a natural identification

$$\mathsf{H}_\infty \xrightarrow{\sim} H^1(\partial(\Gamma\backslash\overline{X}), \mathbb{C})$$

To get this identification we start from a computation which is heuristical at the moment, but will also be used later.

Let us assume we have a 1-form 1.4

$$\omega : G_o(F)\backslash G_o(\mathbf{A})/\mathsf{K}^f(\mathfrak{a}) \to \mathfrak{p}$$

We recall the double coset decomposition (1.3)

$$G_o(\mathbf{A}^f) = \cup B_o(F) \cdot \underline{\xi} \mathsf{K}^f(\mathsf{a})$$

where the double cosets are in $1-1$ correspondence to the cusps. Let us pick an element $b_\infty \in B_{o,\infty}$ and we compute $\omega(b_\infty \underline{\xi})$. As in (1.3) we write $\underline{\xi}' = (1, \underline{\xi}) = a \cdot (a^{-1}, \underline{k}^f)$ and get

$$\omega(b_\infty \underline{\xi}) = \omega(\underline{b}_\infty \cdot a \cdot (a^{-1}, 1)) \text{ where}$$

$\underline{b}_\infty = (b_\infty, 1, \ldots, 1, \ldots)$. Then $b'_\infty = a^{-1} b_\infty a \in B_\infty$ where B is a representative for the Γ-conjugacy class of Borel subgroups corresponding to **64**
$\underline{\xi}$.

Then

$$\omega(b_\infty \underline{\xi}) = \omega(\underline{b}'_\infty \cdot (a^{-1}, 1) = \omega(b'_\infty \cdot a^{-1})$$

where we observe that the adele $\underline{b}'_\infty \cdot (a^{-1}, 1)$ is 1 at the finite components. We write $a^{-1} = b_{a^{-1}} \cdot k_{a^{-1}}$ with $b_{a^{-1}} \in B_\infty$ and $k_{a^{-1}} \in K_\infty$ and find

$$\omega(b_\infty \underline{\xi}) = \mathrm{ad}(k_{a^{-1}}^{-1}) \omega(b'_\infty b_{a^{-1}})$$

We substitute $b'_\infty b_{a^{-1}} = b''_\infty$ and get

$$\omega(b''_\infty) = \mathrm{ad}(k_{a^{-1}}) \cdot \omega(a b''_\infty b_{a^{-1}}^{-1} a^{-1} \cdot \underline{\xi})$$

Our forms in H_∞ are not defined on all of G_∞ but only on $B_{o,\infty}^{(1)}$. Therefore we do the following:

We write

$$G_o(\mathbf{A}^f) = \bigcup_{\underline{\xi}} B_o(F) \underline{\xi} \mathsf{K}^f(\mathsf{a})$$

and

$$\underline{\xi}' = (1, \underline{\xi}) = a(a^{-1}, \underline{k}_f)$$

and

$$B = a^{-1} B_o a$$

and

$$a^{-1} = b_a \cdot k_a \qquad b_a \in B_\infty,\ k_a \in K_\infty$$

then we put

$$\omega^B : B_\infty \to \mathsf{p}_{1,B}$$
$$\omega^B : b''_\infty \to \omega(ab''_\infty b^{-1}_{a^{-1}} a^{-1} \cdot \underline{\xi})$$

One checks that

$$\omega^B(b''_\infty k_\infty) = \mathrm{ad}(k_\infty^{-1})\omega^B(b''_\infty)$$

for $k_\infty \in B_\infty \cap K_\infty = K_\infty^B$ and that

$$\omega^B(u''_\infty b''_\infty) = \omega^B(b''_\infty)$$

65 Therefore we get for any $\omega \in \mathsf{H}_\infty$ a collection of differential forms $\omega^B \in \Omega^1(\Gamma_B \backslash X_B^{(1)})$ which are U_∞ invariant and represent cohomology classes of the corresponding boundary component at ∞. (1.5.1). This gives us a map

$$\mathsf{H}_\infty \to \bigoplus_B H^1(Y_B, \mathbb{C})$$

which is obviously an isomorphism and does not depend on any choice. Let us assume that we have a 1-form

$$\omega : G(F)\backslash G(\mathbf{A})/\mathsf{K}^f(\mathsf{a}) \to \mathsf{p}$$

which is closed (1.4). So it represents a cohomology class $[\omega]$. We know that the restriction of $[\omega]$ to the boundary is given by an element in H_∞, we want to compute that element. On the adele group $U_o(\mathbf{A})$ we choose a Haar measure $d\underline{u}$ so that the volume $U_o(F)\backslash U_o(\mathbf{A})$ becomes equal to 1. Then we compute

$$\omega^{(0)}(g) = \int\limits_{U_p(F)\backslash U_o(\mathbf{A})} \omega(\underline{u}\underline{g})d\underline{u}$$

If we restrict $\omega^{(0)}$ to $B^{(1)}_{o,\infty} \cdot G_o(\mathbf{A}^f)$ we can project the values to $\mathsf{p}_1 = \mathsf{p}_{1,B_o}$ and get

$$\omega_1^{(0)} : U_o(\mathbf{A}) \cdot B_o(F)\backslash B^{(1)}_{o,\infty} G_o(\mathbf{A}^f) \to \mathsf{p}_1$$

which is an element in H_∞.

Proposition 1.6.1. *Under the natural identification constructed above the element* $\omega_1^{(0)}$ *corresponds to the restriction of* $[\omega]$ *to the boundary.*

This follows from 1.5 where we did the corresponding thing for the individual cusps and the computation at the beginning of this section. The normalisation of the measure corresponds exactly to the one in 1.5. We have the decomposition (1.2.1) for the cohomology of the boundary. For the rest of this section we want to analyse our identification

$$\mathsf{H}_\infty \xrightarrow{\sim} H^1(\partial(\Gamma\backslash X);\mathbb{C})$$

from the point of view of (1.2.1). Actually we shall very explicitely associate to any element $\psi \in M^*_\phi$ or $M^*_{\overline{\phi}}$ an element $\omega(\psi)$ of H_∞.

We have 66

$$K_\infty^{B_o} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \middle| \theta \in \mathbb{R} \bmod 2\pi \right\}$$

The group acts on $\mathsf{p}_1 \otimes \mathbb{C} = \mathbb{C}E_1 \oplus E_2$ and the vectors

$$\begin{aligned} e_{+1} &= E_1 - i \otimes E_2 \\ e_{-1} &= E_1 + i \otimes E_2 \end{aligned}$$

are eigenvectors with respect to this action:

$$\operatorname{ad}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} e_{+1} = e^{i\theta} \cdot e_{+1}, \ \operatorname{ad}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} e_{-1} = e^{-i\theta} e_{-1}$$

The two elements e_{+1}, e_{-1} define homomorphisms from Γ_{B_o} to R (1.5) and we shall use them as canonical generators of the two modules L_+ and L_- (1.2.2).

Therefore we have now established the identification

$$M^*_\phi = M_\phi; \ M^*_{\overline{\phi}} = M_{\overline{\phi}}$$

in 1.2.2. Now we shall give an explicit formula for the identification maps

$$H^1(\partial(\Gamma\backslash\overline{X}), \mathbb{C}) \xrightarrow{\sim} \bigoplus_{\substack{\phi:(\mathscr{O}/\mathsf{a})^x \to S^1 \\ \phi(i)=i}} (M_\phi \otimes \mathbb{C} \oplus M_\phi \otimes \mathbb{C}) \to \mathsf{H}_\infty$$

The crucial point is the following simple

Lemma 1.6.2. *To any $\phi : (\mathscr{O}/\mathsf{a})^x \to S^1$ which satisfies $\phi(i) = i^{\pm 1}$ there exists exactly one character*

$$\widetilde{\phi} : I/F^x \mathsf{U}^f(\mathsf{a}) \to S^1$$

for which

$$\widetilde{\phi}|\mathsf{U}^f/\mathsf{U}^f(\mathsf{a}) = \widetilde{\phi}|(\mathscr{O}/\mathsf{a})^x = \phi$$

any for $z \in \mathbb{C}^x$

$$\widetilde{\phi}((z, 1, \ldots, 1)) = \left(\frac{z}{|z|}\right)^{\mp 1}.$$

Proof. As in 1.3 we start from

$$I/F^x \mathsf{U}^f(\mathsf{a}) \xrightarrow{\sim} \mathbb{C}^x x(\mathscr{O}/\mathsf{a})^x/W$$

67 Since we have to have

$$\widetilde{\phi}((i, \ldots, i, \ldots)) = 1$$

and $\phi(i) = i^{\pm 1}$ we get existence and uniqueness easily.

To any of our characters $\phi : (\mathscr{O}/\mathsf{a})^x \to S^1$ for which $\phi(i) = i^{\pm 1}$ we introduce the number $\epsilon(\phi) = \pm 1$ such that $\phi(i) = i^{\epsilon(\phi)}$.

We have

$$U_o(\mathbf{A}) \cdot B_o(F)\backslash B_o(\mathbf{A}) \simeq T_o(\mathbf{A})/T_o(F) = I/F^x$$

and therefore we may also look at $\widetilde{\phi}$ as a character

$$\widetilde{\phi} : B_o(F)\backslash B_o(\mathbf{A}) \to S^1$$

which is trivial on $U_o(\mathbf{A})$.

To any $\psi \in M_\phi$ where $\phi(i) = i^{\pm 1}$ we associate an element $\omega(\ , \phi, \psi) \in \mathsf{H}_\infty$ by the formula

$$\omega(b_\infty \underline{g}^f, \phi, \psi) = \omega(b_\infty \underline{b}^f \underline{k}^f, \phi, \psi) =$$
$$\widetilde{\phi}(b_\infty \underline{b}^f) \cdot \psi((\underline{k}^f)^{-1}) \cdot e_{\epsilon(\phi)}$$

where we identify $\mathsf{K}^f/\mathsf{K}^f(\mathsf{a}) = \overline{G}$. It's of course pure routine but we want to check whether this is well defined and the signs are correct.

If $b_\infty = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} = h(\theta)$ then we should have

$$\begin{aligned}
&\omega(h(\theta)\underline{g}^f, \phi, \psi) = \mathrm{ad}(h(\theta)^{-1}) \cdot \omega((1, \underline{g}^f, \phi, \psi) = \\
&\mathrm{ad}(h(\theta)^{-1}) \cdot \phi(\underline{b}^f) \cdot \psi(\underline{k}^f)^{-1}) \cdot e_{\epsilon(\phi)} = \\
&\widetilde{\phi}(\underline{b}^f) \cdot \psi(\underline{k}^f)^{-1}) \cdot e^{-\epsilon(\phi)\theta} \cdot e_{\epsilon(\phi)}
\end{aligned}$$

and on the other hand we have

$$\widetilde{\phi}(h(\theta) \cdot \underline{b}^f) = e^{-\epsilon(\phi)} \cdot \widetilde{\phi}(\underline{b}^f)$$

so the component at infinity is ok. To prove that it is well defined we have to write

$$\underline{g}^f = \underline{b}^f \cdot \underline{k}^f = \underline{b}^f \cdot \underline{b}_1^f \cdot (\underline{b}_1^f)^{-1} \cdot \underline{k}^f$$

and get from the finite places **68**

$$\begin{aligned}
&\widetilde{\phi}(\underline{b}^f \underline{b}_1^f) \cdot \psi(((\underline{b}_1^f)^{-1} \underline{k}^f)^{-1}) = \\
&\widetilde{\phi}(\underline{b}^f) \cdot \widetilde{\phi}(\underline{b}_1^f) \cdot \psi((\underline{k}^f)^{-1} \cdot \underline{b}_1^f) = \\
&\widetilde{\phi}(\underline{b}^f) \cdot \phi(\underline{b}_1^f) \cdot \widetilde{\phi}(\underline{b}_1^F)^{-1} \cdot \psi((\underline{k}^f)^{-1}) = \\
&\widetilde{\phi}(\underline{b}^f) \cdot \psi((\underline{k}^f)^{-1}).
\end{aligned}$$

and this proves that $\omega(\quad, \phi, \psi)$ is well defined.

The map

$$\bigoplus_{\phi(i)=i} (M_\phi \oplus M_{\overline{\phi}}) \otimes \mathbb{C} \to \mathsf{H}_\infty$$

which maps $\psi \in M_\phi$ and $\psi' \in M_{\overline{\phi}}$ to $\omega(\quad, \phi, \psi)$ and $\omega(\quad, \overline{\phi}, \psi')$ is equal to the identification between $H^1(\partial(\Gamma\backslash\overline{X}), \mathbb{C})$ and H_∞ if we take (1.2.1) and (1.2.2) into account.

One remark concerning the notation. The ψ is always an element in M_ϕ where $\phi(i) = \pm i$ so ϕ is determined by ψ. But I think it is better always to keep in mind from which space the ψ has been taken, so therefore we keep the ϕ in the notation. $\square$

2 The Eisenstein Series

We start from a cohomology class at infinity. We have the identifications 1.2.1 and 1.2.2 and we have seen how to associate to a class $\psi \in M_\phi$ a map

$$\omega(\quad, \phi, \psi) : U_o(\mathbf{A}) \cdot B_o(F)\backslash B^{(1)}_{o,\infty} G_o(\mathbf{A}^f) \to \mathbb{C}e_{\epsilon(\phi)} \subset \mathsf{p}_1 \otimes \mathbb{C}$$

We extend this to a map from $G(\mathbf{A})$ to $\mathsf{p} \otimes \mathbb{C}$. To get this extension we choose a complex number $s \in \mathbb{C}$. We have seen that $B_{o,\infty} = B^{(1)}_{o,\infty} \cdot \mathbb{R}^x$ (1.5 and 1.5.2) and $G_\infty = B_{o,\infty} K_\infty$. We write an element $g_\infty \in G_\infty$ as $g_\infty = b_\infty \cdot t_\infty \cdot k_\infty$ where $t_\infty \in (\mathbb{R}_+)^x$, $b_\infty \in B^{(1)}_\infty$ and $k_\infty \in K_\infty$ and put

$$\omega_s((g_\infty, \underline{g}^f), \phi, \psi) = \omega_s((b_\infty t_\infty k_\infty, \underline{g}^f), \phi, \psi) =$$
$$|t_\infty|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}} \cdot \mathrm{ad}(k_\infty^{-1}) \cdot \omega((b_\infty, \underline{g}^f), \phi, \psi)$$

Now we are in the position to define the Eisenstein series. For $\mathrm{Re}(s) > 1$ the series

$$E(\underline{g}, \phi, \psi, s) = \sum_{a \in B_o(F)\backslash G_o(F)} \omega_s(a\underline{g}, \phi, \psi)$$

69 is absolutely and locally uniformly convergent. Moreover it is known that our series has a meromorphic continuation into the entire s-plane ([9], Thm. 7., [13], Chap. 6). We can interpret $E(\underline{g}, \phi, \psi, s)$ as a 1-form on $\Gamma\backslash X(1.4)$ and this 1-form is closed for $s = 0$. ([8], 4.3). It is well known that $E(\underline{g}, \phi, \psi)$ is holomorphic at $s = 0$.

If we want to know the restriction of the Eisenstein class

$$[E(g, \phi, \psi, 0)]$$

to the boundary we have to compute the constant term (Prop. 1.6.1).

$$\int_{U_o(F)\backslash U_o(A)} E(\underline{u}\underline{g}, \phi, \psi, 0) d\underline{u} = E^{(0)}(\underline{g}, \phi, \psi, 0)$$

We do this by analytic continuation and compute for $\underline{g} = (b_\infty, \underline{g}^f)$ with $b_\infty \in B_\infty$ and $\mathrm{Re}(s) > 1$

$$\int_{U_o(F)\backslash U_o(\mathbf{A})} E(\underline{u}\underline{g}, \phi, \psi, s) d\underline{u}$$

This computation has been carried out at several places ([6], 1.6., [11], 6, and [13]). So we recall only the main steps in the computation. We start from the Bruhat decomposition $G_0(F) = B_0(F) \cup B_0(F)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) U_0(F)$ and substitute the definition of the Eisenstein series into the integral. Then we get two terms

$$\int\limits_{U_o(F)\backslash U_o(\mathbf{A})} \omega_s(\underline{u}\underline{g}, \phi, \psi) d\underline{u} + \int\limits_{U_o(\mathbf{A})} \omega(w\underline{u}\underline{g}, \phi, \psi) d\underline{u}$$

where $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The first integral is constant and therefore we find

$$\omega_s(\underline{g}, \phi, \psi) + \int\limits_{U_o(\mathbf{A})} \omega_s(w\underline{u}\underline{g}, \phi, \psi) d\underline{u}$$

We have a map $B_o(\mathbf{A}) \xrightarrow{\alpha} I$ defined by the positive root and for $b \in \underline{B}_o(\mathbf{A})$ we define $|\underline{b}| = |\alpha(\underline{b})|$ where $|\underline{x}| =$ idelenorm of $\underline{x} \in I$. We write $\underline{g} =$

$$(b_\infty, \underline{g}^f) = (b_\infty, \underline{b}^f) \cdot (1, \underline{k}^f) = \underline{b} \cdot \underline{k} \text{ and get}$$

$$\omega_s(\underline{g}, \phi, \psi) = |\underline{b}|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}} \cdot \widetilde{\phi}(\underline{b}) \cdot \psi((\underline{k}^f)^{-1}) \cdot e_{\epsilon(\phi)}$$

and

$$\int\limits_{U_o(\mathbf{A})} \omega_s(w\underline{ubk}, \phi, \psi) du$$

$$= |b|_{\mathbb{C}}^{\frac{1}{2}-\frac{s}{2}} \widetilde{\phi}(\underline{b})^{-1} \int\limits_{U_o(\mathbf{A})} \omega_s(\underline{uk}, \phi, \psi) d\underline{u}$$

The functions $\omega_S(\underline{g}, \phi, \psi)$ are product of local functions **70**

$$\omega_s(g, \phi, \psi) = \omega_s^{(\infty)}(g_\infty, \phi, \psi) \prod_{\mathsf{p}\text{ finite}} \omega_s^{(\mathsf{p})}(g_{\mathsf{p}}, \phi, \psi)$$

This is so since they are defined by $\widetilde{\phi}$ and ψ which are both products of local functions

$$\psi(\underline{k}^f) = \psi(k_{\mathsf{p}_o}) \text{ where } \{\mathsf{p}_o\} = \text{supp } (\mathsf{a})$$

$$\widetilde{\phi}(\underline{x}) = \widetilde{\phi}_\infty(x_\infty) \cdot \prod_{\mathsf{p}} \widetilde{\phi}_{\mathsf{p}}(x_{\mathsf{p}}).$$

We have for $\mathsf{p} \nmid \mathsf{a}$

$$\omega_s^{(\mathsf{p})}(g_{\mathsf{p}}, \phi, \psi) = \omega_s^{(\mathsf{p})}(b_{\mathsf{p}} k_{\mathsf{p}}, \phi, \psi) = \widetilde{\phi}_{\mathsf{p}}(b_{\mathsf{p}}) |b_{\mathsf{p}}|_{\mathsf{p}}^{\frac{1}{2}+\frac{s}{2}}$$

for $\mathsf{p}|\mathsf{a}$

$$\omega_s^{(\mathsf{p})}(g_{\mathsf{p}}, \phi, \psi) = \widetilde{\phi}_{\mathsf{p}}(b_{\mathsf{p}}) |b_{\mathsf{p}}|_{\mathsf{p}}^{\frac{1}{2}+\frac{s}{2}} \cdot \psi(k_{\mathsf{p}}^{-1})$$

and

$$\omega_s^{(\infty)}(g_\infty, \phi, \psi) = \omega_s^{(\infty)}(b_\infty \cdot t_\infty k_\infty, \phi, \psi) =$$
$$|t_\infty|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}} \cdot \widetilde{\phi}_\infty(b_\infty) \cdot \mathrm{ad}(k_\infty^{-1}) e_{\epsilon(\phi)}$$

Therefore the integral decomposes into a product of local integrals. We have to write the measure as a product of local measures and we are in the fortunate case that we can take

$$d\underline{u} = d\underline{u}_\infty \prod_{\mathsf{p}} du_{\mathsf{p}}$$

where $\mathrm{vol}_{du_{\mathsf{p}}}(\mathscr{O}_{\mathsf{p}}) = 1$ for all p and $du_\infty = \mathrm{dx\ dy}$ (Actually there should be a $\frac{1}{2}$ at $(1+i)$ and a 2 at infinity but they cancel).

For those p which do not divide a (these are all except one) we find

$$\int\limits_{U_o(F_{\mathsf{p}})} \omega_s^{(\mathsf{p})}(wu_{\mathsf{p}}, \phi, \psi) du_{\mathsf{p}} = \frac{1 - \widetilde{\phi}_{\mathsf{p}}(\pi_{\mathsf{p}})^2 |\pi_{\mathsf{p}}|_{\mathsf{p}}^{1+s}}{1 - \widetilde{\phi}_{\mathsf{p}}(\pi_{\mathsf{p}})^2 |\pi_{\mathsf{p}}|_{\mathsf{p}}^{s}}$$

and this follows from a standard computation ([11], §6).

What happens at p_o where p_o is the prime dividing a? In this case
71 we note that $U_o(F_{\mathsf{p}_o}) = F_{\mathsf{p}_o}$ and our integral is a sum

$$\int\limits_{\mathscr{O}_{\mathsf{p}_o}} \omega_s^{(\mathsf{p}_o)}(wu_{\mathsf{p}_o}, k_{\mathsf{p}_o}\phi, \psi) du_{\mathsf{p}_o} +$$

$$\sum_{n=1}^{\infty} \int_{\mathscr{O}_{\mathfrak{p}_o}^x} \omega_s^{(\mathfrak{p}_o)} \left(w \begin{pmatrix} 1 & \pi_{\mathfrak{p}_o}^{-n} \epsilon_{\mathfrak{p}_o} \\ 0 & 1 \end{pmatrix} k_{\mathfrak{p}_o}, \phi, \psi \right) d\epsilon_{\mathfrak{p}_o}$$

and it is for $n > 0$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \pi_{\mathfrak{p}_o}^{-n} \epsilon_{\mathfrak{p}_o} \\ 0 & 1 \end{pmatrix} k_{\mathfrak{p}_o} = \begin{pmatrix} \pi_{\mathfrak{p}_o}^{n} \epsilon_{\mathfrak{p}_o}^{-1} & 1 \\ 0 & \pi_{\mathfrak{p}_o}^{-n} \epsilon_{\mathfrak{p}_o} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ \pi_{\mathfrak{p}_o} \epsilon_{\mathfrak{p}_o}^{-1} & -1 \end{pmatrix} k_{\mathfrak{p}_o}$$

We substitute this into the integrals of the infinite sum. We get that each integral in the infinite sum has value zero since ϕ^2 is not a trivial character. We have only the first term and get

$$\int_{\mathscr{O}_{\mathfrak{p}_o}} \omega_s^{(\mathfrak{p}_o)} (w, u_{\mathfrak{p}_o} k_{\mathfrak{p}_o}, \phi, \psi) du_{\mathfrak{p}_o} =$$

$$\frac{1}{N(\mathfrak{a})} \sum_{\overline{u} \in \mathscr{O}_{\mathfrak{p}_o}/\mathfrak{p}_o} \psi(k_{\mathfrak{p}_o}^{-1} \overline{u}^{-1} w) = \frac{1}{N(\mathfrak{a})} T_\phi \psi(k_{\mathfrak{p}_o}^{-1})$$

where T_ϕ is the intertwining operator constructed in 1.2.2.

At the infinite place we have to compute

$$\int_{\mathbb{C}} \omega_s^{(\infty)} \left(w \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right), \phi, \psi) \mathrm{dxdy}$$

where $z = x + iy$. We introduce polar coordinates and get

$$\int_0^{\infty} \int_0^{2w} \omega_s^{(\infty)} \left(w \begin{pmatrix} 1x & e^{i\theta} \\ 0 & 1 \end{pmatrix}, \phi, \psi \right) x\mathrm{dxd}\theta$$

and we have

$$\begin{pmatrix} 1 & xe^{i\theta} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{1\theta} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix}$$

This gives us

$$\int_0^{\infty} \int_0^{2\pi} e^{+\epsilon(\phi) i\theta} \cdot \mathrm{ad} \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \right) \omega_s^{(\infty)} \left(w \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \phi, \psi \right) x\mathrm{dx}d\theta$$

Let us write **72**

$$\omega_s^{(\infty)}\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\phi,\psi\right) = A(x)\cdot e_{\epsilon(\phi)} + B(x)\cdot H + C(x)\cdot e_{-\epsilon(\phi)}$$

Integrating the first two terms over θ we find zero, so we are left with

$$\int\limits_0^\infty\int\limits_0^{2\pi}\omega_s^{(\infty)}\left(w\begin{pmatrix}1 & x\cdot e^{i\theta}\\ 0 & 1\end{pmatrix}\right), \phi,\psi, x\mathrm{dx}d\theta =$$

$$2\pi\left(\int\limits_0^\infty |b(x)|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}}\cdot C(x)x\mathrm{dx}\right)e_{-\epsilon(\phi)}$$

We have to start from the Iwasawa decomposition

$$w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}(1+x^2)^{-1/2} & x\\ 0 & (1+x^2)^{1/2}\end{pmatrix}\cdot\begin{pmatrix}-x(1+x^2)^{-1/2} & -(1-x^2)^{-1/2}\\ (1+x^2)^{-1/2} & -x(1+x^2)^{-1/2}\end{pmatrix}$$

Then $b(x) = (1+x^2)^{-1}$ and a simple computation using (1.4.1) yields $C(x) = -(1+x^2)^{-1}$.

We get for our integral

$$\left(-2\pi\int\limits_0^\infty (1+x^2)^{-1-s}(1+x^2)^{-1}x\mathrm{dx}\right)e_{-\epsilon(\phi)} = -\frac{\pi}{s+1}e_{-\epsilon(\phi)}$$

Multiplying all this together we find for $\mathrm{Re}(s) > 1$ and $g = (b_\infty, \underline{g}^f)$

$$\int\limits_{U_o(F)\backslash U_o(\mathbf{A})} E(\underline{u}\underline{g},\phi,\psi,s)d\underline{u} =$$

$$\omega_s(\underline{g},\phi,\psi) - \frac{\pi}{s+1}\cdot\frac{L(\widetilde{\phi}^2,s)}{L(\widetilde{\phi}^2,s+1)}\cdot\omega_{-s}(\underline{g},\overline{\phi},T_\phi\psi)$$

where the L-function is defined as

$$L(\widetilde{\phi}^2,s) = \prod_{\mathsf{p}\neq\mathsf{p}_o}(1-\widetilde{\phi}^2_{\mathsf{p}}(\pi_{\mathsf{p}})|\pi_{\mathsf{p}}|^{+s})^{-1}$$

([12], X/V, §8) Since both sides have meromorphic continuation into the entire s-plane we find that the equality holds for all s.

Before stating our main result we look at the expression

$$-\frac{\pi}{s+1}\frac{L(\widetilde{\phi}^2, s)}{L(\widetilde{\phi}^2, s+1)}\bigg|_{s=0}$$

a little bit more closely. The first crucial fact is that $L(\widetilde{\phi}^2, 1) \neq 0$ ([12], XV, §4).

So we have to compute 73

$$-\pi\frac{L(\widetilde{\phi}^2, 0)}{L(\widetilde{\phi}^2, 1)}$$

Now we exploit the functional equation. Let us assume that $\mathsf{a} = \mathsf{p}_o$ is an odd prime and $N(\mathsf{p}_o) = p$. If we follow the instructions in [12], p. 299 carefully we find

$$L(\widetilde{\phi}^2, 0) = +\overline{W(\widetilde{\phi}^2)}\sqrt{p}\cdot\pi^{-1}L(\widetilde{\phi}^2, 1)$$

and therefore

$$-\pi\frac{L(\widetilde{\phi}^2, 0)}{L(\widetilde{\phi}^2, 1)} = -\overline{W(\widetilde{\phi}^2)}\cdot\sqrt{p}\frac{L(\overline{\widetilde{\phi}}^2, 1)}{L(\widetilde{\phi}^2, 1)}$$

We apply the formula for the number $W(\widetilde{\phi}^2)$ given in [12], p. 300 and get $W(\widetilde{\phi}^2) = i^2\cdot\tau(\widetilde{\phi}^2)\dfrac{1}{\sqrt{p}}\cdot\widetilde{\phi}^2(\mathsf{D}^{-1}_{(1+i)})$ where $\tau(\widetilde{\phi}^2)$ is a Gaussain sum.

([12], XIV, §4). Now $i^2 = -1$ and $\widetilde{\phi}^2(\mathsf{D}^{-1}_{(1)}) = \widetilde{\phi}^2((1, \frac{i}{2}, 1, \ldots,))$ where the $i/2$ stands at the $(1+i)$th componente of the idele. Then this is

$$\widetilde{\phi}^2((-2i, 1, -2i, \ldots)) = (-1)\cdot\phi^2(-2i)$$

where the last $-2i$ is the residue class of $-2i$ in $\mathscr{O}/\mathsf{p}_o$. Therefore we find

$$-\pi\frac{L(\overline{\widetilde{\phi}}^2, 0)}{L(\widetilde{\phi}^2, 1)} = \overline{-\tau(\widetilde{\phi}^2)}\cdot\overline{\phi^2(-2i)}\frac{L(\overline{\widetilde{\phi}}^2, 1)}{L(\widetilde{\phi}^2, 1)}$$

If we have $\mathsf{a} = (1+i)^3$ we find

$$-\pi \frac{L(\overline{\overline{\widetilde{\phi}}}^{2}, 0)}{L(\widetilde{\phi}^2, 1)} = \overline{-W(\widetilde{\phi}^2)\cdot 2} \frac{L(\overline{\overline{\widetilde{\phi}}}^{2}, 1)}{L(\widetilde{\phi}^2, 1)} =$$

$$\overline{\tau(\widetilde{\phi}^2)} \frac{L(\overline{\overline{\widetilde{\phi}}}^{2}, 1)}{L(\widetilde{\phi}^2, 1)} = -2 \frac{L(\overline{\overline{\widetilde{\phi}}}^{2}, 1)}{L(\widetilde{\phi}^2, 1)}$$

Now we can state the first main theorem of the paper. In the statement we refer to the different identifications made before.

74 **Theorem 2.1.** *For $\psi \in M_\phi \otimes \mathbb{C}$ the Eisenstein series $E(\underline{g}, \phi, \psi, 0)$ is a closed 1-form and the cohomology class $[E(\underline{g}, \phi, \psi, 0)]$ restricted to the boundary is equal to*

$$[E(\underline{g}, \phi, \psi, 0)]_{\partial(\Gamma\backslash\overline{X})} = \psi - \overline{\widetilde{\phi}^2(-2i)\tau(\widetilde{\phi}^2)} \cdot \frac{L(\overline{\overline{\widetilde{\phi}}}^{2}, 1)}{L(\widetilde{\phi}^2, 1)} \cdot \frac{1}{p} T_\phi \psi$$

if $\mathsf{a} = \mathsf{p}_o$ is prime and $N(\mathsf{p}_o) = p$ and equal to

$$\psi - \frac{L(\overline{\overline{\widetilde{\phi}}}^{2}, 1)}{L(\widetilde{\phi}^2, 1) 4} \frac{1}{4} T_\phi \psi$$

if $\mathsf{a} = (1+i)^3$.

2.2 Arithmetic Applications In this section we assume that $\mathsf{a} = \mathsf{p}_o$ is an odd prime. The theorem gives us the value of the number c_ϕ in 1.2.2, we get with $p = N(\mathsf{p}_o)$

$$c_\phi = -\frac{\overline{\phi^2(-2i)\tau(\widetilde{\phi}^2)}}{p} \cdot \frac{L(\overline{\overline{\widetilde{\phi}}}^{2}, 1)}{L(\widetilde{\phi}^2, 1)} \tag{2.2.1}$$

Corollary 2.2.1. *We have*

$$|c_\phi| = \frac{1}{\sqrt{p}}$$

and in particular $c_\phi \neq 0, \infty$.

This is a consequence of the properties of the Gaussian sums.

To give another interpretation of the Corollary 2.2.1 we recall that we have a scalar product on M_ϕ

$$\langle \psi, \psi \rangle = \int\limits_{\overline{G}/\overline{B}_o} \psi(\overline{g})\overline{\psi(\overline{g})}$$

and the norm of the operator T_ϕ is obviously $\sqrt{p}$. So the $\sqrt{p}$ cancels and we find that the Corollary says that

$$c_\phi T_\phi : M_\phi \otimes \mathbb{C} \to M_{\overline{\phi}} \otimes \mathbb{C}$$

is an unitary operator.

I was unable to see this form a topological point of view and shall **75**
come back to this kind of questions later[1].

But we can also reverse the argument. We had the identifications (1.2.1)

$$H^1(\partial(\Gamma\backslash\overline{X}), R) \xrightarrow{\sim} \bigoplus_{\substack{\phi \\ \phi(i)=i}} (M^*_\phi \oplus M^*_\phi) \xrightarrow{\sim} \bigoplus_{\substack{\phi \\ \phi(i)=i}} (M_\phi \oplus M_\phi)$$

where the last identification has been by means of the elements e_{+1}, $e_{-1} \in \mathfrak{p} \otimes \mathbb{C}$ (see 1.2.2 and 1.6). Of course we must have $c_\phi \in R \bigotimes\limits_{Z} \mathbb{Q}$ and therefore we get the information that

$$\overline{\tau(\widetilde{\phi}^2)} \frac{L(\overline{\widetilde{\phi}}^{2}, 1)}{L(\widetilde{\phi}^2, 1)} \in R \otimes \mathbb{Q}$$

But we can do better. The cohomology

$$H^1(\partial(\Gamma\backslash\overline{X}), R) = H^1(\partial(\Gamma\backslash\overline{X}), Z) \otimes R$$

and we have an action of the Galois group Gal $(K/\mathbb{Q})$ on the cohomology where K is the filed of fractions of R. But this galois group is also acting

[1] Added in Proof: This is actually very easy to see.

on

$$\bigoplus_{\substack{\phi \\ \phi(i)=i}} (M^*_\phi + M^*_{\bar{\phi}})$$

in an obvious way and the action is compatible with the identification, moreover we see that e_{+1} and e_{-1} are both defined over $\mathbb{Q}(i)$ and the complex conjugation interchanges these two homomorphisms. Therefore we can say that also the last identification is compatible with the action of the galois group. The galois group Gal $(K/\mathbb{Q})$ acts on our character ϕ simply by acting on the values

$$\phi^\sigma(x) = \phi(x)^\sigma$$

and $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ maps M_ϕ into $M_\phi\sigma$. It is clear that $T^\sigma_\phi = T_{\phi\sigma}$ and all this tells us

Corollary 2.2.3. *We have*

$$\phi^2(-2i)\overline{\tau(\widetilde{\phi}^2)} \cdot \frac{L(\overline{\overline{\widetilde{\phi}}}^2, 1)}{L(\phi^2, 1)} \in R \otimes \mathbb{Q}$$

and if $\sigma \in \mathrm{Gal}(K/Q)$ *and* $\phi_1 = \phi^\sigma$ *then*

$$\left(\overline{\phi^2(-2i)\tau(\widetilde{\phi}^2)} \frac{L(\overline{\overline{\widetilde{\phi}}}^2, 1)}{L(\widetilde{\phi}^2, 1)} \right)^\sigma = \overline{\phi_1^2(-2i)\tau(\phi_1^2)} \cdot \frac{L(\overline{\overline{\widetilde{\phi}}}^2_1), 1}{L(\widetilde{\phi}_1^2, 1)}$$

76 This follows of course from the observation that the image

$$H^1(\Gamma\backslash\overline{X}, R) \to H^1(\partial(\Gamma\backslash\overline{X}), R)$$

has to be invariant under the action of the galois group.

This corollary is related to results of Damerell, Shimura and Razar. Damerell's result is to some extent much stronger since it says that

$$L(\widetilde{\phi}^2, 1) = \omega^2 \cdot \alpha$$

where $\omega = \int\limits_0^1 \frac{dx}{\sqrt{x - x^3}}$ and where α is an algebraic number whose denominator can be bounded in terms of our data ([4], II, Thm. 2). But on the other hand it seems to be so that our information concerning the ratio $L(\overset{=2}{\widetilde{\phi}}, 1)/L(\widetilde{\phi}^2, 1)$ is much more precise and I do not know whether this can be deduced from his methods. There is also a certain relation to the results of Shimura and Razar. Shimura considers Dirichlet L-series corresponding to modular forms ([23])

$$\sum_{n=1}^{\infty} a_n n^{-s} = L(f, s)$$

and twists them by Dirichlet characters

$$D(f, \phi, s) = \sum a_n \phi(n) n^{-s}, \ \sum a_n \psi(n) n^{-s} = D(f, \psi, s)$$

Then he is able to say something about the values

$$\frac{D(f, \phi, s)}{D(f, \psi, s)}$$

at special values of s and then his results becomes very similar to ours. But I do not see whether his result implies Corollary **??** or whether it can be obtained from his methods.

To conclude this section I want to discuss a few examples very explicitely. We start from the following general remark: The cohomology $H^1(\Gamma\backslash X, R) = H^1(\Gamma_o, R[G])$ can be computed in principle in an effective way once our data-this means a-are given. This will be discussed in the thesis of E. Mendoza ([15]). This means we are also able to compute the **77**
number c_ϕ in a given case and this gives an effective way of computing the ratios $L(\overset{=2}{\widetilde{\phi}}, 1)/L(\phi^2, 1)$. (Comp. also [23], Intr.). We want to discuss this computation in a couple of cases where we chose a slightly different method than the one suggested by [15]. We compute $H^1(\Gamma_o, N_\phi)(1.2.3)$ by starting from the cochain complex. We look at

$$\begin{tikzcd}
 & N_\phi \arrow[d, equal] & & \\
0 \arrow[r] & C^o(\Gamma_o, N_\phi) \arrow[r] \arrow[d, equal] & C^1(\Gamma_o, N_\phi) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & C^o(\Gamma_{o,B_o}, N_\phi) \arrow[r] & C^1(\Gamma_{o,B_o}, N_\phi) \arrow[r] & {}
\end{tikzcd}$$

We computed

$$H^1(\Gamma_{o,B_o}, N_\phi) = \mathrm{Hom}(\Gamma_{o,U_o}, N_\phi^{\overline{U}_o})^W = \mathrm{Hom}(\Gamma_{o,U_o}, \mathrm{Rf}_o \oplus \mathrm{Rf}_\infty)^W.$$

Let $\Phi \in \mathrm{Hom}(\Gamma_0, B_0, N_\phi)$ and let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Phi : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A \to af_o + bf_\infty$$

$$\Phi : \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} = CAC^{-1} \to cf_o + df_\infty.$$

Such a Φ is invariant under W if and only if $c = ia$ and $d = -ib$. This means that a cohomology class in $H^1(\Gamma_{o,B_o}, N_\phi)$ is canonically represented by a cocycle

$$\Phi : A \to af_o + bf_\infty$$
$$\Phi : C \to 0$$
$$\Phi : \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \to +iaf_o \to ibf_\infty$$

In 1.6 we introduce e_{+1}, $e_{-1} \in \mathfrak{p}_1 \otimes \mathbb{C}$ and they define homomorphisms (1.5)

$$e_{+1} : \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \frac{1}{2} \\ \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \to -i/2 \end{cases}$$

$$e_{-1} \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \frac{1}{2} \\ \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \to i/2 \end{cases}$$

78 Therefore we have in the notations of 1.2.3, if we put $e_+ = e_{+1}$ and $e_- = e_{-1}$ (what we did all the time) that

$$\Phi = 2ae_+ + 2be_-$$

and $c_\phi = ba^{-1} \cdot N(\mathfrak{p}_o)^{-1}$ and our problem to compute c_ϕ amounts to: When can we extend the cocycle

$$\Phi : A \to af_o + bf_\infty ; \Phi : C \to 0$$

to a cocycle on Γ_o with values im N_ϕ? The only thing we have to do is we have to give the value $\Phi(B) \in N_\phi$. But we have certain restrictions for this value. These restrictions come from the relations

$$B^2 = 1 \quad BC - C^{-1}B \quad (AB)^3 = 1$$

which imply

$$\Phi(B) = C\Phi(B), \quad \Phi(B) + B\Phi(B) = 0 \text{ and}$$
$$\Phi(AB) + AB\Phi(AB) + (AB)^2\Phi(AB) = 0$$

(These are not all relations, but they are sufficient in our special cases)

We stick to the case $\mathfrak{a} = \mathfrak{p}_o$ is an odd prime and introduce a basis in N_ϕ. The basis consists of the functions $\delta_{\overline{u}}$ where $\overline{u} \in \mathscr{O}/\mathfrak{p}_o = F_p$ and δ_∞ and

$$\delta_{\overline{u}} : \begin{cases} w \cdot \begin{pmatrix} 1 & \overline{u} \\ 0 & 1 \end{pmatrix} & \to 1 \\ w \cdot \begin{pmatrix} 1 & \overline{v} \\ 0 & 1 \end{pmatrix} & \to 0 \qquad \text{for } \overline{v} \neq \overline{u} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \to 0 \end{cases}$$

$$\delta_\infty : \begin{cases} w \begin{pmatrix} 1 & \overline{u} \\ 0 & 1 \end{pmatrix} & \to 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \to 1 \end{cases}$$

The group acts as follows

$$A\delta_{\overline{u}} = \delta_{\overline{u}-1},\ A\delta_\infty = \delta_\infty;$$

$$B\delta_o = \delta_\infty, B\delta_\infty = \delta_o,\ B\delta_{\overline{u}} = \phi(\overline{u}^2)\delta_{-\overline{u}^{-1}} \quad (u \neq 0),$$

$$C\delta_{\overline{u}} = \phi(i)^{-1}\cdot\delta_{i\overline{u}},\ C\delta_\infty = \phi(i)\delta_\infty$$

The cocycles we are looking for are

$$\Phi : A \to a\left(\sum_{\overline{u}\in F_p}\delta_{\overline{u}}\right) + b\delta_\infty$$

$$\Phi : C \to 0$$

$$\Phi : B \to ?$$

79 We write $\Phi(B) = \Sigma x_{\overline{u}}\delta_{\overline{u}} + x_\infty\delta_\infty$ and we get from $C\Phi(B) = \Phi(B)$ that $x_o = x_\infty = 0$ and $x_{-i\overline{u}} = \phi(i)\cdot x_{\overline{u}}$. The dihedral group generated by B and C acts on $\overline{B}\backslash\overline{G}$ and we have one orbit of length 2 namely $\{0,\infty\}$, one orbit of length 4 namely $\{1, i, -i, -1\}$ and the other $\dfrac{p-5}{8}$ orbits are of length 8 and those consist of two orbits under $\{C\}$ which are flipped by B. Therefore we see that the relations $C\Phi(B) = \Phi(B)$ and $\Phi(B)+B\Phi(B) = 0$ restrict the possible values for $\Phi(B)$ to an $\left(\dfrac{p-5}{8}+1\right)$-dimensional vector space.

Now we look at the relation

$$\Phi(AB) + AB\Phi(AB) + (AB)^2\Phi(AB) = 0$$

To see what this means it is convenient to look also at the space $N_{\overline{\phi}}$. We have a natural pairing $N_\phi \times N_{\overline{\phi}} \to R$ and the $\widehat{\delta_{\overline{u}}}, \widehat{\delta_\infty}$ form a dual basis with respect to this pairing. Then this last relation says

$$\langle\Phi(AB), l\rangle = 0$$

for all $l \in N_{\overline{\phi}}$ for which $ABl = l$. This means

$$\langle\Phi(A) + A\cdot\Phi(B), l\rangle = 0$$

for all such l and this is equivalent to

$$\langle \Phi(A) + \Phi(B),\ \mathrm{Ker}(\widehat{\mathrm{Id} - BA}) \rangle = 0$$

where $\widehat{\mathrm{Id} - BA} : N_{\overline{\phi}} \to N_{\overline{\phi}}$. We have a 2-dimensional space of choices for $\Phi(A)$ and $a\left(\frac{p-5}{8}+1\right)$-dimensional space of choices for $\Phi(B)$ and on this $\left(\frac{p-1}{8}+3\right)$-dimensional space we get $\frac{p+1}{3}$ $\left(\text{resp. } \frac{p-1}{3}+2\right)$ linear equations if $p \equiv -1 \bmod 3$ (resp $p \equiv 1 \bmod 3$) for the possible cocycles. This gives us some kind of vague feeling the occurence of cohomology is something accidental. But we know that there has to be at least a one dimensional space of solutions.

We consider special cases:

I. $\mathsf{p}_o = (2-i)$, then we have exactly one character ϕ with $\phi(i) = i$. We have the residue classes $0, 1, 2, 3, 4 \bmod 5$ **80**

$$\Phi(A) = a(\delta_o + \delta_1 + \delta_2 + \delta_3 + \delta_4) + b \cdot \delta_\infty$$
$$\Phi(C) = 0$$
$$\Phi(B) = x \cdot (\delta_1 - i\delta_2 - \delta_4 + i\delta_3)$$

The elements $\widehat{\delta_o} + \widehat{\delta_1} + \widehat{\delta_\infty}$ and $\widehat{\delta_2} + \widehat{\delta_4} - \widehat{\delta_3}$ form a basis for $\mathrm{Ker}(\widehat{\mathrm{Id} - BA})$ in the dual module and have to be orthogonal to $\Phi(A) + \Phi(B)$ and we get the linear equations

$$2a + b + x = 0$$
$$a + (-2i - 1)x = 0$$

We put $a = 1$, then $x = \frac{1}{2i+1}$ and $b = -\frac{3+4i}{2i+1} = \frac{(1-2i)^2}{1+2i}$. Therefore we have constructed a cocycle representing an Eisenstein class and this Eisenstein cocycle is

$$\Phi_E : A \to (\delta_o + \delta_1 + \delta_2 + \delta_3 + \delta_4) + \frac{(1-2i)^2}{(1-2i)} \cdot \delta_\infty$$

$$\Phi_E : C \to 0$$
$$\Phi_E : B \to \frac{1}{2i+1}(\delta_1 - i\delta_2 - \delta_4 + i\delta_3)$$

This gives in view of 1.2.4.

$$c_\phi = \frac{(1-2i)^2}{(1+2i)}\frac{1}{N(\mathsf{p}_o)} = \frac{(1-2i)^2}{(1+2i)}\frac{1}{5} = \frac{(1-2i)}{(1+2i)^2}$$

If we take (2.2.1) into account we find

$$\frac{L(\overline{\widetilde{\phi}}^2,1)}{L(\widetilde{\phi}^2,1)} = -\phi^2(-2i)\overline{\tau(\widetilde{\phi}^2)^{-1}}\frac{(1-2i)^2}{1+2i} = -\overline{\tau(\widetilde{\phi}^2)^{-1}} \cdot \frac{(1-2i)^2}{1+2i}$$

The definition of $\tau(\widetilde{\phi}^2)$ is given in [12], and we get

$$\tau(\widetilde{\phi}^2) = \frac{2+i}{2-i}\sum_{x \bmod 5}\phi^2(x)\cdot e^{\frac{2\pi i x}{5}} = \frac{2+i}{2-i}\sqrt{5} = -\frac{1-2i}{1+2i}\sqrt{5}$$

81 so we end up with

$$\frac{L(\overline{\widetilde{\phi}}^2,1)}{L(\widetilde{\phi}^2,1)} = \frac{(1-2i)^3}{(1+2i)^2}\frac{1}{\sqrt{5}}$$

II. $\mathsf{p}_o = (3+2i)$, then $\mathscr{O}/\mathsf{p}_o = \mathbb{Z}/13\mathbb{Z}$ and $i \bmod \mathsf{p}_o = 5 \bmod 13$. Our cocycle has to look like

$$\Phi(A) = a\left(\sum_{\overline{u}\in\mathscr{O}/\mathsf{p}_o}\delta_{\overline{u}}\right) + b\delta_\infty, \ \Phi(C) = 0$$
$$\Phi(B) = x_1(\delta_1 - i\delta_s - \delta_{12} + i\delta_8) +$$
$$x_2(\delta_2 - i\delta_{10} - \delta_{11} + i\delta_3 - \phi(4)\delta_6 + i\phi(9)\cdot\delta_9 + \phi(9)\delta_7 - i\phi(9)\delta_4)$$

The vector $\Phi(A) + \Phi(B)$ has to be orthogonal to the following vectors in the dual space $\widehat{\delta_1} + \widehat{\delta_o} - \widehat{\delta_\infty}$, $\widehat{\delta_5} + \phi(9)\widehat{\delta_3} - \widehat{\delta_6}$, $\widehat{\delta_{12}} + \phi(10)\widehat{\delta_7} + \widehat{\delta_2}$, $\widehat{\delta_8} - \phi(4)\widehat{\delta_{11}} - \delta_9$ and if $\phi(3) = 1$ then also to $\widehat{\delta_{10}}$ and $\widehat{\delta_4}$.

(α) $\phi(3) = 1$. Since $(Z/13Z)^x = \{5\} \times \{3\}$ this fixes ϕ since we have $\phi(i) = \phi(5) = i$. Then its easy to solve the system of linear equations and we find the only solution $\mathsf{a} = 1$, $x_2 = -1$, $x_1 = 1 - 2i$, $b = -3 + 2i$.
The Eisenstein cocycle is

$$\Phi_E(A) = \left(\sum_{\overline{u}\in\mathscr{O}\mathsf{p}_o} \delta_{\overline{u}}\right) + (-3+2i)\delta_\infty,\ \Phi_E(C) = 0$$

$$\Phi_E(B) = (1-2i)(\phi_1 - i\delta_5 - \delta_{12} + i\delta_8)$$

$$-(i\delta_2 - i\delta_{10} - \delta_{11} + i\delta_3 + \delta_6 - i\delta_9 - \delta_7 - i\delta_4)$$

Therefore we find $c_\phi = \dfrac{(-3+2i)}{13}$ and this gives us

$$\frac{L(\widetilde{\overline{\phi}}^{2}, 1)}{L(\widetilde{\phi}^2, 1)} = -\frac{(3-2i)^2}{(3+2i)}\frac{1}{\sqrt{13}}$$

(β) $\phi(3) = \rho = -\dfrac{1}{2} + \dfrac{1}{2}i\sqrt{3}$ where $\sqrt{3}$ is the positive root. In this case our computations gives again one solution and we get

$$\Phi_E(A) = \left(\sum_{\overline{u}\in\mathscr{O}/\mathsf{p}}\right) - i\rho^2\frac{(1+i-\rho^2)(1+i-\rho)^2}{(1-i-\rho^2)}\delta_\infty$$

$$\Phi_E(C) = 0$$

$$\Phi_E(B) = \frac{-3\rho^2 - \rho^2 i + 1}{1 - i - \rho^2}(\delta_1 - i\delta_5 - \delta_{12} + i\delta_8)$$

$$-\frac{i}{1-i}\frac{\rho - i\rho^2 - 2}{1 - i - \rho^2}\cdot$$

$$(\delta_2 + i\delta_{10} - \delta_{11} + i\delta_3 + \delta_6 + i\delta_9 - \delta_7 - i\delta_4)$$

and we get **82**

$$c_\phi = i\rho^2\frac{(1+i-\rho^2)(1+i-\rho)^2}{1 - i\rho^2}\cdot\frac{1}{13}$$

We introduce the Gaussian sum

$$
\begin{aligned}
G(\widetilde{\phi}^2) = G(\phi^2) = 2\bigg(&\cos\frac{2\pi}{13} - \cos 5\cdot\frac{2}{13} - \cos 3\cdot\frac{2\pi}{13} + \cos 2\frac{2\pi}{13}\\
&+ 2\rho\bigg(\cos 4\cdot\frac{2\pi}{13} - \cos 6\cdot\frac{2\pi}{13} - \cos 6\cdot\frac{2\pi}{13} - \cos 3\frac{2\pi}{13} + \cos 2\frac{2\pi}{13}\bigg)
\end{aligned}
$$

and get

$$\frac{L(\overline{\widetilde{\phi}}^2, 1)}{L(\phi^2, 1)} = i\frac{(1+i-\rho)^3\cdot(1+i-\rho^2)^2}{(1-i-\rho)^3\cdot(1-i-\rho^2)}\cdot\frac{1}{G(\phi^2)}$$

We want to conclude this section by mentioning an interesting question. One of the consequences of our theory is the non vanishing of c_ϕ. We could now also look at cohomology with torsion coefficients say in the ring $R/\mathsf{p} = F_q$. We have basically the same situation as in characteristic zero if we stay away from some bad characteristics. So we may again ask whether this number $c_\phi \neq 0, \infty$. Of course it is clear that this question is closely related to the question, what is the prime number decomposition of $L(\overline{\widetilde{\phi}}, 1)/(L(\widetilde{\phi}^2, 1)$? So one might ask for instance whether c_ϕ is always a unit in R, once we inverted the divisors of $|G|^2$.

3.1 The Period Integrals

In the last section we constructed the cohomology classes

$$[E(\underline{g}, \phi, \psi, 0)] \in H^1(\Gamma\backslash X, \mathbb{C}).$$

Actually it can be shown that these classes live already in $H^1(\Gamma\backslash X, K)$
83 where K is the fraction field of R. This will be done in a subsequent paper and we have to use the Hecke algebra and the strong version of the multiplicity one theorem.

I ask the reader to accept this fact here, it will be of importance only at the end of this paper. So we have homomorphisms

$$[E(\underline{g}, \phi, \psi, 0)] : \Gamma \to K$$

[2] Added in Proof: Further computations show that this idea is much too naive.

and we may ask for a formula for the value of $[E(\underline{g}, \phi, \psi, 0)]$ on a given element $\gamma \in \Gamma$. If γ is unipotent, then the answer is given by theorem 2.1. Therefore we are left with the non unipotent elements γ. In this case the centralizer T_γ of γ is an anisotropic torus over F. The aim of this section is to show that we have a canonical way of associating a cycle z_γ to γ and that

$$[E(\underline{g}, \phi, \psi, 0)](\gamma) = \int_{z_\gamma} E(\quad, \phi, \psi, 0)$$

We want to give an explicit expression for that integral.

Our given element γ defines a torus T_γ and this torus defines a quadratic extension E/F, the splitting field of T_γ/F. We have

$$T_\gamma(F) = E^x/F^x$$

and it follows from Dirichlet's theorem on units that $T_\gamma(F) \cap \Gamma =$ infinite cyclic group which contains γ. We shall say that γ is primitive if it generates this infinite cyclic group.

The group of complex points of our torus T_γ decomposes in a canonical way

$$T_\gamma(\mathbb{C}) = R_+^x x S^1 = T_\gamma^{(s)} x T_\gamma^{(c)}$$

where $S^1 = \{z \in \mathbb{C} |\ |z| = 1\}$ and the superscripts s and c stand for spit and compact. We pick an element $x_\gamma \in X = \mathrm{PGL}_2(\mathbb{C})/K_\infty$ for which $T_\gamma^{(c)} x_\gamma = x_\gamma$. These elements form a real line in X. We get a map

$$j_{x_\gamma} : T_\gamma^{(s)} = T_\gamma(\mathbb{C})/T_\gamma^{(s)} \to X$$
$$j_{x_\gamma} : t \to t x_\gamma$$

and with $\Gamma_\gamma = \Gamma \cap T_\gamma(F)$ we get a map

$$\bar{j}_{x_\gamma} : \Gamma_\gamma \backslash T_\gamma(\mathbb{C})/T_\gamma^{(c)} \to \Gamma \backslash X$$

84 and this gives us our one cycle z_γ since the left hand side is obviously a circle. Now it is obvious that

$$[E(\underline{g}, \phi, \psi, 0)](\gamma) = \int\limits_{z_\gamma} E(\quad, \phi, \psi, 0)$$

As I said already we want to have this formula a little bit more explicit. So let us assume that ω is any 1-form on $\Gamma\backslash X$. We get the identification

$$T_\gamma(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^x$$

from the selection of a generator in the character group of our torus $T_\gamma \underset{F}{\times} \mathbb{C}$. This selection is unique up to a sign. Therefore we have

$$T_\gamma^{(s)} \xrightarrow{\sim} R_+^x$$

On R_+^x we have the canonical vector field $t\dfrac{\partial}{\partial t}$ which we lift back to a vector field Y on $T_\gamma^{(s)}$ by means of that identification. We have $T_\gamma^{(s)} = T_\gamma(\mathbb{C})/T_\gamma^{(c)}$ and let $\overline{\gamma}$ be the image of γ in $T_\gamma^{(s)}$. Then

$$\int\limits_{z_\gamma} \omega = \int\limits_1^{\overline{\gamma}} \langle \omega(tx_\gamma), (dj_{x_\gamma}(Y)) \rangle \frac{\mathrm{dt}}{t}$$

where we consider $T_\gamma^{(s)} = R_+^x$, and dj_{x_γ} is of course the derivative of our map j_{x_γ}. We write $x_\gamma = g_\gamma x_o$ with $g_\gamma \in \mathrm{PGL}_2(\mathbb{C})$. We agreed already to look at ω as a function

$$\omega : \Gamma\backslash G_\infty \to \mathsf{p}$$

such that $\omega(g_\infty k_\infty) = \mathrm{ad}(k_\infty^{-1})\omega(g_\infty)$. We apply (1.4.2) and get for the integral

$$\int\limits_1^{\overline{\gamma}} \langle \omega(tg_\gamma),\ dL_{(tg_\gamma)^{-1}} \circ dj_{x_\gamma}(Y) \rangle \frac{\mathrm{dt}}{t}$$

Since our vector field is invariant under translations we find

$$dL_{(tg_\gamma)^{-1}} \cdot dj_{x_\gamma}(Y) = \mathrm{ad}(g_\gamma^{-1})(Y)$$

where we consider $Y \in \mathrm{Lie}(T_\gamma^{(s)}) \subset \mathrm{Lie}(G_\infty) = \mathfrak{g}_\infty$. It follows from our construction that $\mathrm{ad}(g_\gamma^{-1})(Y) \in \mathfrak{p}$ and the first 'explicit' form of our integral is

$$\int_1^{\bar{\gamma}} \langle \omega(tg_\gamma),\ \mathrm{ad}(g_\gamma^{-1})(Y)\rangle \frac{\mathrm{dt}}{t} \tag{3.1.1}$$

85

3.1.2 Summation over the Classes in the Genus

Our final goal is the evaluation of the integrals (3.1.1) for $\omega = E(g, \phi, \psi, 0)$. But we shall not be able to do this directly since we run into some trouble with class numbers. So we have to discuss these class number problems first. We say that two elements γ, $\gamma_1 \in \Gamma$ are in the same class if they are conjugate under the action of Γ. If ω is any closed form we shall certainly have $\int\limits_{z_{\gamma_1}} \omega = \int\limits_{z_\gamma} \omega$ if γ and γ_1 are in the same class. We say that γ, γ_1 are in the same genus if they are conjugate under the action of $\mathrm{PGL}_2(F)$ and if we can find an element $\underline{k}^f \in \mathsf{K}^f(\mathsf{a})$ which conjugates γ into γ_1. We shall see that the set of classes in a given genus is a finite set which has a natural structure of an abelian group. If $\gamma \in \Gamma$ and if I_γ is the set of classes in the genus of γ then we shall compute

$$\sum_{\gamma_1 \in 1_\gamma} \chi(\gamma_1) \int\limits_{z_{\gamma_1}} E(\quad, \phi, \psi, 0)$$

for all characters of I_γ and for this expression we shall write down a rather explicit formula.

Now we shall describe the set of classes in a given genus. Let us pick a non unipotent element γ. If we have

$$\gamma_1 = a^{-1}\gamma a \text{ with } a \in G_o(F),\ \gamma_1 = \underline{k}^f \gamma (\underline{k}^f)^{-1};\ \underline{k}^f \in \mathsf{K}^f(\mathsf{a})$$

then $\underline{t} = a \cdot \mathsf{k}^f \in T_\gamma(\mathbf{A}^f)$. It is very easy to check that the correspondence $\underline{t} \leftrightarrow \gamma_1$ induces a bijection

$$I_\gamma \xrightarrow{\sim} T_\gamma(F)\backslash T_\gamma(\mathbf{A}^f)/\mathsf{K}^f(\mathsf{a}) \cap T_\gamma(\mathbf{A}^f)$$

To see this we use 1.3.1 and use the same arguments as in 1.3 for the set of Γ-conjugacy classes of Borel subgroups. This of course defines the group structure on I_γ.

Our next step is to transform the sum

$$\sum_{\gamma_1 \in I_\gamma} \chi(\gamma_1) \int\limits_{z_{\gamma_1}} \omega$$

into an adelic integral over $T_\gamma(\mathbf{A})/T_\gamma(F)$. We start from the observation that we can view ω as a function

$$\omega : G_o(F)\backslash G_o(A) \to \mathsf{p}$$

which satisfies $\omega((\underline{g}\underline{k}) = ad(k_\infty^{-1})\omega(\underline{g})$ of $\underline{k} = (k_\infty, \underline{k}^f)$ and $\underline{k}^f \in \mathsf{K}^f(\mathsf{a})$ (1.4).

86 We have

$$T_\gamma(F)\backslash T_\gamma(\mathbf{A}^f)/\mathsf{K}^f(\mathsf{a}) \cap T_\gamma(\mathbf{A}^f) = T_\gamma(F)\backslash T_\gamma(\mathbf{A})/T(\mathbb{C}) \cdot (\mathsf{K}^f(\mathsf{a}) \cap T_\gamma(\mathbf{A}))$$

and we write

$$T_\gamma(\mathbf{A}) = \bigcup_{\xi} T_\gamma(F) \cdot T_\gamma(\mathbb{C}) \cdot \underline{\xi} \cdot (\mathsf{K}^f(\mathsf{a}) \cap T_\gamma(\mathbf{A}^f))$$

where $\underline{\xi} \in T_\gamma(\mathbf{A}^f)$ and where we embed $T_\gamma(\mathbf{A}^f) \hookrightarrow T_\gamma(\mathbf{A})$ by putting the component at infinity equal to one. Then we use again **??** and write

$$\underline{\xi} = a_{\underline{\xi}} \underline{k}^f_{\underline{\xi}} a \in G_o(F),\ \underline{k}^f \in \mathsf{K}^f(\mathsf{a})$$

and extending this to $T_\gamma(\mathbf{A})$ we obtain

$$\underline{\xi} = (1, \underline{\xi}) = a_{\underline{\xi}}(a_{\underline{\xi}}^{-1}, \underline{k}^f_{\underline{\xi}})$$

The element $\underline{\xi}$ defines a class in the genus of γ and $\gamma_{\underline{\xi}} = a_{\underline{\xi}}^{-1}\gamma a_{\underline{\xi}} \in \Gamma$ is a representative of this class. We want to compute

$$\sum_{\underline{\xi}} \chi(\gamma_{\underline{\xi}}) \int_{z_{\gamma_{\underline{\xi}}}} \omega$$

We observe that χ is a character on $T_\gamma(F)\backslash T_\gamma(\mathbf{A})/T_\gamma(\mathbb{C})(\mathsf{K}^f(\mathsf{a}) \cap T_\gamma(\mathbf{A}^f)$ and the above expression becomes

$$\sum_{\underline{\xi}} \chi(\underline{\xi}) \int_{z_{\gamma_{\underline{\gamma}}}} \omega = \sum_{\underline{\xi}} \chi(\underline{\xi}) \int_1^{\overline{\gamma}_{\underline{\xi}}} \langle \omega(t g_{\gamma_{\underline{\xi}}}), \mathrm{ad}(g_{\gamma_{\underline{\xi}}}^{-1})(Y) \rangle$$

Now we can choose $x_{\gamma_{\underline{\xi}}} = a_{\underline{\xi}}^{-1} x_\gamma$ and $g_{\gamma_{\underline{\xi}}} = a_{\underline{\xi}}^{-1} g$.

Then we get for our sum

$$\sum \chi(\underline{\xi}) \cdot \int_1^{\gamma_{\underline{\xi}}} \langle \omega(t a_{\underline{\xi}}^{-1} g_\gamma), \mathrm{ad}\, g_\gamma^{-1} \circ \mathrm{ad}\, a_{\underline{\xi}}(Y_{\underline{\xi}}) \rangle \frac{\mathrm{dt}}{t}$$

In the adele group we have

$$a_{\underline{\xi}} \cdot (a_{\underline{\xi}}^{-1}, \underline{k}^f) = \underline{\xi}'$$

and therefore

$$a_{\underline{\xi}} \cdot (a_{\underline{\xi}}^{-1}, 1, \ldots, 1, \ldots, 1) = \underline{\xi}' \cdot (\underline{k}^f)^{-1}$$

We put $t' = a_{\underline{\xi}} t a_{\underline{\xi}}^{-1}$ and note that $\mathrm{ad}\, a_{\underline{\xi}}(Y_{\underline{\xi}}) = Y$ then the contribution coming from the class $\underline{\xi}$ is

$$\int_1^{\gamma} \langle \omega(a_{\underline{\xi}}^{-1} t' g_\gamma), \mathrm{ad}(g_\gamma^{-1})(Y) \rangle \frac{\mathrm{dt}'}{t'}$$

87 Now we observe that this integration takes place at the component at infinity. We have

$$(a_{\underline{\xi}}^{-1}, 1, 1, \ldots) = a_{\underline{\xi}}^{-1} \cdot \underline{\xi}' \cdot (\underline{k}^f)^{-1}$$

and we find for the above integral the value

$$\int\limits_{T_\gamma(F)\backslash T_\gamma(F)\cdot T_\gamma(\mathbb{C})\cdot \underline{\xi}} \langle \omega(\underline{t\xi}), \mathrm{ad}(g_\gamma^{-1})(Y)\rangle d^x\underline{t}$$

where the measure $d^x\underline{t}$ on the adele group has to be normalized as follows.

We have

$$d^x\underline{t} = \Pi d^x t_v$$

and $d^x t_\infty = d^x t_\infty^{(s)} \times d^x t_\infty^{(c)}$; the $d^x t_\infty^{(s)}$ is the Lebesgue measure $\frac{\mathrm{dt}}{t}$ on $R_+^x = T_\gamma^{(s)}$ and $d^x t_\infty^{(c)}$ gives volume one to the circle. At the finite places we require

$$\mathrm{vol}_{\mathrm{dt}_\mathrm{p}}(T(F_\mathrm{p}) \cap \mathsf{K}_\mathrm{p}^f(\mathsf{a})) = 1.$$

At this place we used the fact that γ is primitive. This gives us the final formula

$$\sum_{\gamma_1 \in I_\gamma} \chi(\gamma_1) \int\limits_{z_\gamma} \omega = \int\limits_{T_\gamma(F)\backslash T_\gamma(\mathbf{A})} \omega(\underline{t}\underline{g}_\gamma, \mathrm{ad}(\underline{g}_\gamma^{-1})(Y) d^x\underline{t} \tag{3.1.1.1}$$

where $d^x\underline{t}$ is normalized as above and where $\underline{g}_\gamma = (g, 1, 1, \ldots, 1)$.

3.1.3 Now we apply this formula in the case where $\omega = E(\underline{g}, \phi, \psi, 0)$. What we shall do is to evaluate the right hand side in the case where $\omega = E(\underline{g}, \phi, \psi, s)$ with $\mathrm{Re}(s) > 1$. Then the result will be dependent on the different choices we made. If we continue analytically and evaluate at $s = 0$ then the result is intrinsic and we get a formula for the left hand side.

We have for $\mathrm{Re}(s) > 1$

$$\int\limits_{T_\gamma(F)\backslash T_\gamma(\mathbf{A})} \chi(\underline{t}) \sum_{a\in B_o(F)\backslash G_o(F)} \omega_s(a\underline{t}\underline{g}_\gamma, \phi, \psi))d^x\underline{t}$$

The map $B_o(F)\times T_\gamma(F) \to G_o(F)$ given by multiplication is easily seen to be bijective, so we find that the last term is equal to 88

$$\int\limits_{T_\gamma(F)\backslash T_\gamma(\mathbf{A})} \chi(\underline{t}) \sum_{t\in T_\gamma(F)} \omega_s(t\underline{t}\underline{g}_\gamma, \phi, \psi))d^x\underline{t} =$$

$$\int\limits_{T_\gamma(\mathbf{A})} \chi(\underline{t})\omega_s(\underline{t}\underline{g}_\gamma, \phi, \psi)d^x\underline{t}$$

As in section 2 we write $\omega_s(\underline{t}\underline{g},\quad,)$ as a product of local functions

$$\omega_S(\underline{t}\underline{g}_\gamma, \phi, \psi) = \omega_s^{(\infty)}(t_\infty g_\gamma, \widetilde{\phi}_\infty, \psi)\prod_{\mathfrak{p}} \omega_s^{(\mathfrak{p})}(t_{\mathfrak{p}}, \widetilde{\phi}_{\mathfrak{p}}, \psi)$$

and our integral becomes an infinite product of integrals

$$\int\limits_{T_\gamma(\mathbb{C})} \omega_s^{(\infty)}(t_\infty g_\gamma, \widetilde{\phi}_\infty, \psi)d^x t_\infty \prod_{\mathfrak{p}} \int \chi_{\mathfrak{p}}(t_{\mathfrak{p}})\omega_{\mathfrak{p}}^{(s)}(t_{\mathfrak{p}}, \widetilde{\phi}_{\mathfrak{p}}, \psi)d^x t_{\mathfrak{p}}$$

(We should perhaps mention that the ψ enters only in the factor belonging to $\mathfrak{p}_o$).

Before we enter into the computation of the individual local terms we want to state the result. Let us look at a finite prime $\mathfrak{p}$. We shall call a finite sum

$$\sum_{\nu=-r}^{\nu=r} a_\nu |\pi_{\mathfrak{p}}|^{(1/2+s/2)\nu} = G_{\mathfrak{p}}(s)$$

where $a_\nu \in K[\chi]$ (i.e. we adjoin the values of χ to the quotient field K of R) an elementary factor. We shall call such a factor an exponential factor if it is of the form $a_\nu|\pi|^{(\frac{1}{2}+\frac{1}{2}s)^\nu}$ for some $\nu \in \mathbf{Z}$ and $a \neq 0$.

If we have any character η of the idele class group of any field L/k and if p is a prime of k then we write

$$L(\eta, s)_{\mathsf{p}} = \begin{cases} \frac{1}{1-\eta_{\mathsf{p}}(\pi_{\mathsf{p}})|\pi_{\mathsf{p}}|^s} & \text{if } \eta \text{ is unramified} \\ 1 & \text{if } \eta \text{ is ramified} \end{cases}$$

We know that the character χ above can be identified with a character on our quadratic extension E/F. We have the norm mapping $N : I_E \to I_F$, so $\phi \circ N$ is also a character on the idele class group of E and we put

$$L_E(\chi \cdot \widetilde{\phi} \circ N, s) = \prod_{\mathsf{P} \backslash \mathsf{p}} L_E(\chi \cdot \widetilde{\phi} \circ N, s)_{\mathsf{P}}$$

89 With these conventions we can state: We have for all p

$$\int \omega_s^{(\mathsf{p})}(t_{\mathsf{p}}, \widetilde{\phi}_{\mathsf{p}}, \psi) d^x t_{\mathsf{p}} = G_{\mathsf{p}}(\widetilde{\phi}, \psi, \chi, \gamma, s) \cdot \frac{L_E\left(\chi \cdot \widetilde{\phi} \circ N, \frac{s+1}{2}\right)_{\mathsf{p}}}{L_F(\widetilde{\phi}^2, s+1)_{\mathsf{p}}} \qquad (*)$$

where $G_{\mathsf{p}}(\widetilde{\phi}, \psi, \chi, \gamma, s) = G_{\mathsf{p}}(s)$ is an elementary factor. This elementary factor is equal to one for almost all p.

At the infinite place we shall get an expression

$$G_\infty(s) \cdot \frac{\Gamma\left(\frac{s}{2}+1\right)^2}{\Gamma(s+2)} = G_\infty(s) \frac{L_E\left(\chi \cdot \widetilde{\phi} \circ N, \frac{s+1}{2}\right)_\infty}{L_F(\widetilde{\phi}^2, s+1)_\infty}$$

where $G_\infty(s)$ is an exponential factor and where the Γ-factors are exactly the ones one expects ([12], XIV, §8).

We are now ready to enter the long and tedious computations which shall give us (*). We have to be quite careful since we want to get as much information as possible on the local elementary factors. Especially we would like to know whether they can vanish at $s = 0$ for reasons which become clear later.

A last remark concerns our character χ. The formula makes sense for any character on the idele class group which is trivial at infinity. But for our purpose we have only to look at those characters which are trivial on $T_\gamma(\mathbf{A}^f) \cap \mathsf{K}^f(\mathsf{a})$, since only those enter on the left hand side. So we shall always assume that

3.1.4 We look, at the finite places first and we begin our computations by discussing an integral representation of our function $\omega_s(g_{\mathsf{p}}, \widetilde{\phi}, \psi)$.

We write the action of $\mathrm{GL}_2(F_{\mathsf{p}})$ on F_{p}^2 as

$$(x, y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

and for any locally constant function with compact support

$$\Phi : F_{\mathsf{p}}^2 \rightarrow \mathbb{C}$$

we write 90

$$L_{\phi}(g_{\mathsf{p}}, \widetilde{\phi}, s) = L_{\Phi}(g_{\mathsf{p}}) = \int_{F_{\mathsf{p}}^x} \Phi[(0, a_{\mathsf{p}})g_{\mathsf{p}}]|a_{\mathsf{p}}^2 \cdot \det g_{\mathsf{p}}|_{\mathsf{p}}^{\frac{1}{2}+\frac{s}{2}} \widetilde{\phi}(a_{\mathsf{p}}^2 \det g_{\mathsf{p}}) d^x a_{\mathsf{p}}$$

where the measure $d^x a_{\mathsf{p}}$ is normalized in such a way that the units get volume 1. One checks very easily that

$$L_{\Phi}\left(\begin{pmatrix} t_{1,\mathsf{p}} & x \\ 0 & t_{2,\mathsf{p}} \end{pmatrix} g_{\mathsf{p}}\right) = \left|\frac{t_{1,\mathsf{p}}}{t_{2,\mathsf{p}}}\right|_{\mathsf{p}}^{\frac{1}{2}+\frac{s}{2}} \cdot \widetilde{\phi}\left(\frac{t_{1,\mathsf{p}}}{t_{2,\mathsf{p}}}\right) \cdot L_{\Phi}(g_{\mathsf{p}})$$

This means that our function L_{Φ} transforms under the left action of $B_o(F_{\mathsf{p}})$ the same way as $\omega_s^{(\mathsf{p})}(g_{\mathsf{p}}, \phi, \psi)$ does, so we can expect to find a Φ such that

$$L_{\Phi}(g_{\mathsf{p}}) = \omega_s(g_{\mathsf{p}}, \widetilde{\phi}, \psi)$$

This is easy:

If $\mathsf{p} \neq \mathsf{p}_o$ then our character $\widetilde{\phi}$ is unramified at p, we choose for $\Phi = (1 - \widetilde{\phi}^2(\pi_{\mathsf{p}})|\pi_{\mathsf{p}}|_{\mathsf{p}}^{1-s}) \times$ characteristic function of $\mathcal{O}_{\mathsf{p}} \oplus \mathcal{O}_{\mathsf{p}}$. The one checks easily that

$$L_{\Phi}(g_{\mathsf{p}}, \widetilde{\phi}, s) = \omega_s^{(\mathsf{p})}(g_{\mathsf{p}}, \phi, \psi) \tag{3.1.3.1}$$

If $\mathsf{p} = \mathsf{p}_o$ then we consider the space of functions

$$\{\Phi | \Phi : \mathcal{O}/\mathsf{p}_o \oplus \mathcal{O}/\mathsf{p}_o \rightarrow \mathbb{C} | \Phi(\lambda(x, y)) = \overline{\phi^2(\lambda)} \Phi(x, y)\}$$

We look at such a function as a function on $\mathscr{O}_{\mathfrak{p}_o} \oplus \mathscr{O}_{\mathfrak{p}_o}$ and extend it outside of $\mathscr{O}_{\mathfrak{p}_o} \oplus \mathscr{O}_{\mathfrak{p}_o}$ by zero. Then for $g_{\mathfrak{p}} = k_{\mathfrak{p}} \in K_{\mathfrak{p}}$

$$L_\Phi(g_{\mathfrak{p}}, \widetilde{\phi}, s) = \Phi[(0,1)k_{\mathfrak{p}}] \cdot \widetilde{\phi}(\det(k_{\mathfrak{p}}))$$

and therefore if we put

$$\psi(k_{\mathfrak{p}}) = \Phi[(0,1)k_{\mathfrak{p}}]\widetilde{\phi}(\det(k_{\mathfrak{p}})) \tag{3.1.3.2}$$

we find

$$L_\Phi(g_{\mathfrak{p}}, \widetilde{\phi}, s) = \omega_{\mathfrak{p}}(g_{\mathfrak{p}}, \widetilde{\phi}, \psi)$$

91 Here we identify $M_\phi = \{\psi : K^f \to R | \psi(\underline{b}\underline{k}) = \phi(\underline{b})\psi(\underline{k})$ for all $\underline{b} \in K^f \cap B(A^f)\}$. Now we compute the local integrals by starting from the integral representation of our functions ω_s. We observed that our element γ generates a quadratic extension E/F in the matrix ring $M_2(F)$. Then we have $T_\gamma(F) = E^x/F^x$ and we put $E^x_{\mathfrak{p}} = (E \otimes F_{\mathfrak{p}})^x$. If $\mathfrak{p}$ splits in E then $E^x_{\mathfrak{p}} = E^x_{\mathfrak{P}} \times E^x_{\mathfrak{P}}$. Our character χ is in a canonical way identified with a character χ on $I_E/I \cdot F^x$. We have to compute

$$\int\limits_{E^x_{\mathfrak{p}}/F^x_{\mathfrak{p}}} \chi(t_{\mathfrak{p}})\omega_s^{(\mathfrak{p})}(t_{\mathfrak{p}}, \widetilde{\phi}, \psi)\overline{d^x t_{\mathfrak{p}}} =$$

$$\int\limits_{E^x_{\mathfrak{p}}/F^x_{\mathfrak{p}}} \int\limits_{F^x_{\mathfrak{p}}} \Phi[(0, a_{\mathfrak{p}})t_{\mathfrak{p}}]|a^2_{\mathfrak{p}} \det t_{\mathfrak{p}}|_{\mathfrak{p}}^{\frac{1}{2}+\frac{s}{2}}$$

$$\widetilde{\phi}(a^2_{\mathfrak{p}} \det t_{\mathfrak{p}}) \cdot \chi(t_{\mathfrak{p}}) d^x a_{\mathfrak{p}} \overline{d^x t_{\mathfrak{p}}} =$$

$$\int\limits_{E^x_{\mathfrak{p}}} \Phi[(0,1)t_{\mathfrak{p}}]|\det t_{\mathfrak{p}}|_{\mathfrak{p}}^{\frac{1}{2}+\frac{s}{2}} \widetilde{\phi}(\det t_{\mathfrak{p}})\chi(t_{\mathfrak{p}}) d^x t_{\mathfrak{p}}$$

We have to spend a moment of thinking about the normalization of the measures. The measure $d^x t_{\mathfrak{p}}$ will be the product measure of the measure $d^x a_{\mathfrak{p}}$ which gives the volume 1 to the units and the measure $\overline{d^x t_{\mathfrak{p}}}$ which gives volume one to $T_\gamma(F_{\mathfrak{p}}) \cap K_{\mathfrak{p}} \subset E^x_{\mathfrak{p}}/F^x_{\mathfrak{p}}$.

Let us put

$$I^{(\gamma)}_{\mathfrak{p}} = (E_{\mathfrak{p}} \cap M_2(\mathscr{O}_{\mathfrak{p}}))^x$$

then $\mathsf{U}_{\mathsf{p}}^{(\gamma)}$ is an open subgroup of the group of units $\mathsf{U}_{\mathsf{p}} \subset E_{\mathsf{p}}^{x}$. If p is ramified then the image of $\mathsf{U}_{\mathsf{p}}^{(\gamma)}$ in $T_{\gamma}(\mathscr{O}_{\mathsf{p}})$ is of index one or two, so $\mathrm{vol}_{dt_{\mathsf{p}}} \mathsf{U}_{\mathsf{p}}^{(\gamma)}) = 1$ or $1/2$.

Remark. At this point we can derive already the statement (*) very easily, it is almost the definition ([12], XIV, 8). We have to check that $t_{\mathsf{p}} \to \Phi[(0,1)t_{\mathsf{p}}]$ is a Schwartz-Bruhat function, which is not quite clear if p splits in E. But we said already that we are interested in very specific informations on the local factors so we have to work a little more.

Case I. Let us start with the case where our extension E/F splits at p. At the moment we do not assume $\mathsf{p} \neq \mathsf{p}_{o}$. In this case we can find two eigenvectors $e_1, e_2 \in \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}$ such that for $t \in E_{\mathsf{p}}$

$$e_1 t = t_1 e_1;\ e_2 t = t_2 e_2$$

where $t = (t_1, t_2)$ with respect of the decomposition $E_{\mathsf{p}} = E_{\mathsf{p}} \oplus E_{\mathsf{p}}$. We **92**
find a constant f_{p} such that

$$\pi_{\mathsf{p}}^{f\mathsf{p}}(\mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}) \subset \mathscr{O}_{\mathsf{p}} e_1 \oplus \mathscr{O}_{\mathsf{p}} e_2 \subset \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}$$

and we call the embedding *regular* if we can choose e_1, e_2 in such a way that $\mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}} = \mathscr{O}_{\mathsf{p}} e_1 \oplus \mathscr{O}_{\mathsf{p}} e_2$, i.e. $f_{\mathsf{p}} = 0$.

Let us write

$$(0,1) = \alpha e_1 + \beta e_2 \qquad \alpha\beta \neq 0$$

where $\alpha, \beta \in E_{\mathsf{p}}$ and $\pi_{\mathsf{p}}^{f_{\mathsf{p}}}\alpha$, $\pi_{\mathsf{p}}^{f_{\mathsf{p}}}\beta \in \mathscr{O}_{\mathsf{p}}$. Any element $t_{\mathsf{p}} \in E_{\mathsf{p}}$ and be written

$$t_{\mathsf{p}} = \left(\prod_{\mathsf{P}}^{\nu} \epsilon \mathsf{P}, \prod_{\mathsf{P}}^{\nu} \epsilon \mathsf{P} \right).$$

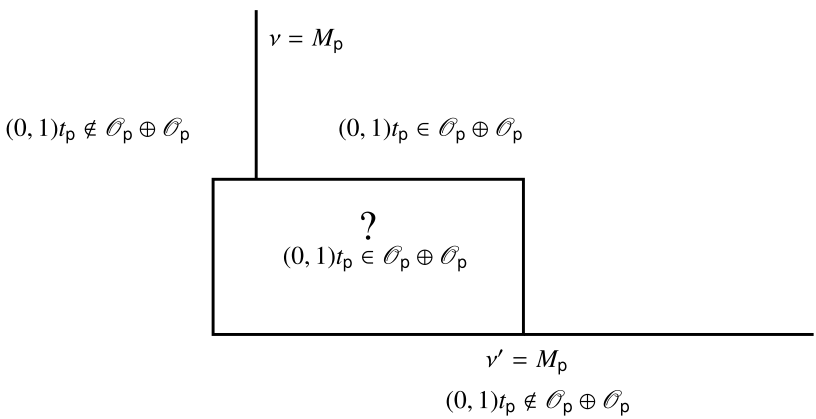
and we put $\deg(t_{\mathsf{p}}) = (\nu, \nu')$. We have to discuss the question: When do we get $(0,t)t_{\mathsf{p}} \in \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}$? We get a rough picture as follows: We have integer constants $M_{\mathsf{p}}, N_{\mathsf{p}}$ and $M'_{\mathsf{p}}, N'_{\mathsf{p}}$ such that

(i) If $\deg(t_{\mathsf{p}}) = (\nu, \nu')$ and $\nu \geq M_{\mathsf{p}}$ then $(0,1)t_{\mathsf{p}} \in \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}$ if and only if $\nu' \geq N_{\mathsf{p}}$.

(i′) If $\deg(t_{\mathsf{p}}) = (\nu, \nu')$ and $\nu' \geq M'_{\mathsf{p}}$ then $(0, 1)t_{\mathsf{p}} \in \mathcal{O}_{\mathsf{p}} \oplus \mathcal{O}_{\mathsf{p}}$ if and only if $\nu \geq N'_{\mathsf{p}}$.

(ii) there exists only a finite number of pairs (ν, ν') with $\nu < M_{\mathsf{p}}$ and $\nu' < M'_{\mathsf{p}}$ such that we have a number t_{p} with $\deg(t_{\mathsf{p}}) = (\nu, \nu')$ and $(0, 1)t_{\mathsf{p}} \in \mathcal{O}_{\mathsf{p}} \oplus \mathcal{O}_{\mathsf{p}}$.

So the picture looks like that



93 So if $\nu \geq M_{\mathsf{p}}$ or $\nu' \geq M'_{\mathsf{p}}$ then $(0, 1)t_{\mathsf{p}} \in \mathcal{O}_{\mathsf{p}} \oplus \mathcal{O}_{\mathsf{p}}$ depends only on the degree. Let us call the finite set of points described in (ii) simply S_{p}.

If we have a regular embedding the situation becomes nicer. In this case we write $(0, 1) = \alpha e_1 + \beta e_2$ with $\alpha, \beta \in \mathcal{O}_{\mathsf{p}}$ and if we put $+M_{\mathsf{p}} = -\operatorname{ord}_{\mathsf{p}}(\alpha)$, $+M'_{\mathsf{p}} = -\operatorname{ord}_{\mathsf{p}}(\beta)$ then $(0, 1)t_{\mathsf{p}} \in \mathcal{O}_{\mathsf{p}} \oplus \mathcal{O}_{\mathsf{p}}$ if and only if we have for $\deg(t_{\mathsf{p}}) = (\nu, \nu')$ that $\nu \geq M_{\mathsf{p}}$ and $\nu' \geq M'_{\mathsf{p}}$. We call the embedding *strongly regular* if $\operatorname{ord}_{\mathsf{p}}(\alpha) = \operatorname{ord}_{\mathsf{p}}(\beta) = 0$, this is the nicest case, we have $(0, 1)t_{\mathsf{p}} \in \mathcal{O}_{\mathsf{p}} \oplus \mathcal{O}_{\mathsf{p}}$ if and only if $\nu, \nu' \geq 0$.

Now we can evaluate our integral and we write

$$\int\limits_{E^x_{\mathsf{p}}} \Phi[(0, 1)t_{\mathsf{p}}] |\det t_{\mathsf{p}}|_{\mathsf{p}}^{\frac{1}{2}+\frac{s}{2}} \widetilde{\phi}(\det t_{\mathsf{p}}) \chi(t_{\mathsf{p}}) d^x t_{\mathsf{p}} =$$

$$\sum_{\nu,\nu'=-\infty}^{\infty} \int\limits_{\deg(t_{\mathsf{p}})=(\nu,\nu')} \Phi[(0,1)t_{\mathsf{p}}]|\det t_{\mathsf{p}}|_{\mathsf{p}}^{\frac{1}{2}+\frac{s}{2}}\widetilde{\phi}(\det t_{\mathsf{p}})\chi(t_{\mathsf{p}})d^x t_{\mathsf{p}} =$$

$$\sum_{\nu,\nu'=-\infty}^{\nu,\nu'=+\infty} |\pi_{\mathsf{p}}|^{\left(\frac{1}{2}+\frac{s}{2}\right)(\nu+\nu')}\widetilde{\phi}(\pi_{\mathsf{p}})^{\nu+\nu'} \cdot \chi(\Pi_{\mathsf{p}})^{\nu} \cdot \chi(\Pi_{\mathsf{p}'})^{\nu'} \times$$

$$\int\limits_{\mathsf{U}_{\mathsf{p}}} \Phi[(0,1)(\Pi_{\mathsf{p}}^{\nu},\Pi_{\mathsf{p}'}^{\nu'})\epsilon]\phi(\det(\epsilon))\chi(\epsilon)d^x\epsilon \qquad (3.1.3.3)$$

Now we should distinguish several cases :

(α) $\mathsf{p} \neq \mathsf{p}_o$ and χ is unramified. Then our considerations above tells us that our sum decomposes

$$\sum_{(\nu,\nu')\in S} + \sum_{\substack{\nu\geq M_{\mathsf{p}} \\ N_{\mathsf{p}}\leq\nu'\leq M'_{\mathsf{p}-1}}} + \sum_{\substack{\nu'\geq M'_{\mathsf{p}} \\ N_{\mathsf{p}}\leq\nu'\leq M'_{\mathsf{p}-1}}} + \sum_{\substack{\nu\geq M_{\mathsf{p}} \\ \nu'\geq M'_{\mathsf{p}}}}$$

The first sum is finite and in the other sums the value of the integral is always equal to $\mathrm{vol}_{d^x t_{\mathsf{p}}}(\mathsf{U}_{\mathsf{p}})$. Then these sums can be evaluated easily and we get

$$G_{\mathsf{p}}(\widetilde{\phi},\psi,\chi,\gamma,s)\times$$
$$\times\frac{(1-\widetilde{\phi}(\pi_{\mathsf{p}})^2\cdot|\pi_{\mathsf{p}}|_{\mathsf{p}}^{1+s})}{\left(1-\chi(\Pi_{\mathsf{p}})\widetilde{\phi}(\pi_{\mathsf{P}})|\Pi_{\mathsf{P}}|_{\mathsf{P}}^{\frac{1}{2}+\frac{s}{2}}\right)\left(1-\chi(\Pi_{\mathsf{P}})\widetilde{\phi}(\pi_{\mathsf{P}})|\Pi_{\mathsf{P}}|_{\mathsf{P}}^{\frac{1}{2}+\frac{s}{2}}\right)}$$

where we recall that $|\Pi_{\mathsf{P}}|_{\mathsf{P}} = |\Pi_{\mathsf{P}'}|_{\mathsf{P}'} = |\pi_{\mathsf{P}}|_{\mathsf{P}}$ and the numerator **94**
stems from the normalization of the characteristic function.

If our embedding is regular then we have

$$G_{\mathsf{P}}(\widetilde{\phi},\psi,\chi,\gamma,s) = |\pi_{\mathsf{P}}|^{\left(\frac{1}{2}+\frac{s}{2}\right)(M_{\mathsf{P}}+M_{\mathsf{P}'})}\widetilde{\phi}(\pi_{\mathsf{P}})^{M_{\mathsf{P}}+M_{\mathsf{P}'}}\cdot\chi(\Pi_{\mathsf{P}})^{M_{\mathsf{P}}}\chi(\Pi_{\mathsf{P}'})^{M_{\mathsf{P}}}$$

so we get an exponential factor in this case. It is equal to one in case of a strongly regular embedding.

(β) $\mathsf{p} \neq \mathsf{p}_o$ and χ is ramified. Our character $\widetilde{\phi}$ is unramified in this case and it is clear that χ has to be ramified at P and P' since it is one on I. Therefore most of the integrals disappear and we are left with a finite sum, which gives us the desired elementary factor. In this case we can't have a regular embedding because of the earlier remark.

(γ) $\mathsf{P} = \mathsf{P}_o$. In this case we have in addition that Φ vanishes on $(\mathscr{O}_{\mathsf{P}} \oplus \mathscr{O}_{\mathsf{P}})\pi_{\mathsf{P}}$ and transforms under scalar multiplication as follows

$$\Phi[\lambda(x,y)] = \overline{\phi^2(\lambda)}\Phi(x,y)$$

If we restrict to a specific degree $(\nu\nu')$ we see that

(i) $\Phi[(0,1)(\Pi^{\nu}_{\mathsf{P}}, \Pi^{\nu'}_{\mathsf{P}'})\epsilon] = 0$ if ν and ν' are large.

(ii) If ν is large and $\epsilon = (\epsilon_{\mathsf{P}}, \epsilon_{\mathsf{P}'})$

$$\Phi[(0,1)(\Pi^{\nu}_{\mathsf{P}}, \Pi^{\nu'}_{\mathsf{P}'})\epsilon] = \overline{\widetilde{\phi}^2(\epsilon_{\mathsf{P}'})}\Phi[(0,1)(\Pi^{\nu}_{\mathsf{P}}, \Pi^{\nu'}_{\mathsf{P}})]$$

(ii)$'$ If ν' is large then we have

$$\Phi[(0,1)(\Pi^{\nu}_{\mathsf{P}}, \Pi^{\nu'}_{\mathsf{P}'})\epsilon] = \overline{\widetilde{\phi}^2(\epsilon_{\mathsf{P}})}\Phi[(0,1)(\Pi^{\nu}_{\mathsf{P}}\ \Pi^{\nu'}_{\mathsf{P}'})]$$

If we decompose our integral according to degree we find (**??**)

$$\sum_{\nu,\nu' \text{ small}} + \sum_{\substack{\nu \text{ small} \\ \nu' \text{ large}}} + \sum_{\nu \text{ large} \nu' \text{ small}}$$

where the first sum is finite. Let us consider a term

$$\int_{\mathsf{U}_{\mathsf{P}}} \Phi[(0,1)(\Pi^{\nu}_{\mathsf{P}}, \Pi^{\nu}_{\mathsf{P}'})\epsilon]\widetilde{\phi}(\det \epsilon) \cdot \chi(\epsilon) d^x \epsilon$$

95 where ν is large. We have $\mathsf{U}_{\mathsf{P}} = \mathsf{U}_{\mathsf{P}} \times \mathsf{U}_{\mathsf{P}'}$, and get

$$\int_{\mathsf{U}_{\mathsf{P}}} \int_{\mathsf{U}_{\mathsf{P}'}} \Phi[(0,1)(\Pi^{\nu}_{\mathsf{P}}, \Pi^{\nu}_{\mathsf{P}'})]\widetilde{\phi}(\epsilon_{\mathsf{P}}/\epsilon_{\mathsf{P}'})\chi(\epsilon_{\mathsf{P}})\chi(\epsilon_{\mathsf{P}'}) d^x \epsilon_{\mathsf{P}} d^x \epsilon_{\mathsf{P}'}$$

Here we should be careful enough to say that of course

$$\chi(\epsilon_{\mathsf{P}}) = \chi(1, \ldots, \underset{\substack{\uparrow \\ \mathsf{P}\text{-th component}}}{\epsilon_{\mathsf{P}}}, 1, \ldots) = \chi_{\mathsf{P}}(\epsilon_{\mathsf{P}})$$

$$\chi(\epsilon_{\mathsf{P}'}) = \chi(1, \ldots, 1, \underset{\substack{\uparrow \\ \mathsf{P}'\text{-th component}}}{\epsilon_{\mathsf{P}'}}, 1, \ldots 1) = \chi_{\mathsf{P}},(\epsilon_{\mathsf{P}'})$$

and then the condition $\chi|I_F = 1$ implies $\chi_{\mathsf{P}}(\epsilon) = \chi_{\mathsf{P}'}(\epsilon)$ for $\epsilon \in \mathsf{U}_{\mathsf{P}} \subset F^x_{\mathsf{P}}$. Therefore our integral turns out to be

$$\int_{\mathsf{U}_{\mathsf{P}}} \int_{\mathsf{U}_{\mathsf{P}'}} \Phi[(0,1)(\Pi^\nu_{\mathsf{P}}, \Pi^\nu_{\mathsf{P}})]\widetilde{\phi}(\eta)\chi_{\mathsf{P}}(\eta) d^x\eta d^x\epsilon$$

This integral vanishes if $\widetilde{\phi}\chi_{\mathsf{P}}$ is non trivial on $\mathsf{U}_{\mathsf{P}} \subset F^x_{\mathsf{P}}$. But if $\widetilde{\phi}\chi_{\mathsf{P}}$ is trivial on U_{P} then the value of the integral is equal to

$$\mathrm{vol}_{d^x\epsilon}(\mathsf{U}_{\mathsf{P}}) \cdot \Phi[(0,1)(\Pi^\nu_{\mathsf{P}}\ \Pi^\nu_{\mathsf{P}'})]$$

and it is clear that this value does not depend on ν but only on ν' provided ν is large.

A similar assertion holds if ν' is large. Now we interpret our conditions on $\chi_{\mathsf{P}} \cdot \widetilde{\phi}$ and $\chi_{\mathsf{P}'} \cdot \phi$ to be trivial or non trivial on U_{P} resp. $\mathsf{U}_{\mathsf{P}'}$. Of course we have $\chi_{\mathsf{P}} \cdot \widetilde{\phi}$ is trivial on U_{P} if and only if the character $\chi\widetilde{\phi} \circ N$ is unramified at P and the same holds for P'. And we observe that $\chi\widetilde{\phi} \circ N$ can be unramified at most one of these place. Therefore we get:

If $\chi\widetilde{\phi} \circ N$ is ramified at P and at P' then in our decomposition

$$\sum_{\nu,\nu' \text{ small}} - \sum_{\substack{\nu \text{ large} \\ \nu' \text{ small}}} + \sum_{\substack{\nu \text{ small} \\ \nu' \text{ large}}}$$

the second and third terms contribute zero. Then our integral is given by the first sum which is an elementary factor if ψ takes values in R. If

$\chi\widetilde{\phi}\circ N$ is unramified at P then we get a contribution from the second sum and this gives obviously again 96

$$G_{\mathsf{p}}(\widetilde{\phi},\psi,\chi,\gamma,s)\frac{1}{1-\chi\widetilde{\phi}\circ N(\Pi_{\mathsf{P}})|\Pi_{\mathsf{P}'}|_{\mathsf{P}}^{\frac{1}{2}+\frac{s}{2}}}$$

and this is again the term we want.

The same thing happens at P', if $\chi\widetilde{\phi}\circ N$ turns out to be unramified at P'.

Now we discuss the case of a regular embedding and we want to give explicit expressions for the elementary factor. We write

$$(0,1)=\alpha e_1+\beta e_2$$

and we put $\mathrm{ord}_{\mathsf{p}}(\alpha)=-M_{\mathsf{p}}$, $\mathrm{ord}_{\mathsf{p}}(\beta)=-M'_{\mathsf{p}}$. One of these two numbers has to be zero, if both are zero then we are in the case of a strongly regular embedding.

We recall

$$\overline{G}=\mathrm{PGL}_2(\mathscr{O}/\mathsf{P})=\mathsf{K}^f/\mathsf{K}^f(\mathsf{P})$$

and we recall from 1.2.2.

$$M_\phi=\{\psi:\overline{G}\to R|\psi(\overline{g}\overline{b}^{-1})=\phi(\overline{b})\psi(\overline{g})\}=$$
$$\left\{\begin{array}{r}\psi:\mathsf{K}^f/\mathsf{K}^f(\mathsf{P})\to R|\psi(\underline{b}^f\underline{k}^f)=\widetilde{\phi}(\underline{b}^f)\cdot\psi(\underline{k}^f)\\ \text{for all }\underline{b}^f\in B_o(\mathbf{A}^f)\cap\mathsf{K}^f\end{array}\right\}$$

(1.6.2) Moreover we identified the two spaces

$$\{\Phi:\mathscr{O}/\mathsf{P}\oplus/\mathsf{p}\to R|\Phi[\lambda(x,y)]=\widetilde{\phi}^2(\lambda)\Phi[(x,y)]\}$$

and M_ϕ by (3.1.3.2)

$$\psi(\underline{k}^f)=\Phi[(0,1)k_{\mathsf{p}}^f]\cdot\widetilde{\phi}(\det k_{\mathsf{p}}^f)$$

Now our infinite sum over ν, ν' specializes to

$$\text{one term for }\nu=M_{\mathsf{p}},\nu'=M_{\mathsf{p}'}+\sum_{\substack{\nu=M_{\mathsf{p}}\\ \nu'>M'_{\mathsf{p}}}}+\sum_{\substack{\nu>M_{\mathsf{p}}\\ \nu'=M'_{\mathsf{p}}}}$$

97 We choose a uniformizing element π_{p} at P and choose $\Pi_{\mathsf{P}}, \Pi_{\mathsf{P}'}$ to be the projections of π_{p} to the components in $E \otimes F_{\mathsf{p}} = E_{\mathsf{P}} \oplus E_{\mathsf{p}'}$.

Then the first term is equal to

$$|\pi_{\mathsf{p}}|^{(\frac{1}{2}+\frac{s}{2})(M_{\mathsf{p}}+M'_{\mathsf{p}})}\widetilde{\phi}(\pi_{\mathsf{p}})^{M_{\mathsf{p}}+M'_{\mathsf{p}}}\chi(\Pi_{\mathsf{P}})^{M_{\mathsf{p}}} \cdot \chi(\Pi_{\mathsf{p}'})^{M_{\mathsf{p}'}}$$

$$\int\limits_{\mathsf{U}_{\mathsf{p}}} \Phi[(0,1)(\Pi_{\mathsf{P}}^{M_{\mathsf{p}}}, \Pi_{\mathsf{P}'}^{M_{\mathsf{p}}})\epsilon]\widetilde{\phi}(\det(\epsilon))\chi(\epsilon)d^x\epsilon$$

The vector $\xi_o = (0,1)(\Pi^{M_{\mathsf{p}}},_{\mathsf{P}}, \Pi_{\mathsf{P}'}^{M'}) = \alpha' e_1 + \beta' e_2$ has both components $\alpha', \beta' \neq 0$. We choose an element $k_o \in \mathsf{K}_{\mathsf{p}}$ with $\det(k_o) = 1$ such that $(0,1)k_o = \xi_o$. Then our integral becomes in view of (3.1.3.2)

$$\int\limits_{\mathsf{U}_{\mathsf{p}}} \psi(k_o\epsilon)\chi(\epsilon)d^x\epsilon = (p-1)P^{\overline{\chi}}\psi(k_o)$$

Here $P^{\overline{\chi}}$ is the following projection operator: The basis e_1, e_2 defines a torus $\widetilde{T}_\gamma \times F_{\mathsf{p}} \subset \mathrm{GL}_2/F_{\mathsf{p}}$ and the image of $\widetilde{T}_\gamma \times F_{\mathsf{p}}$ in $\mathrm{PGL}_2 \times F_{\mathsf{p}}$ is $T_\gamma \times F_{\mathsf{p}}$. Let $\overline{T}_\gamma$ be the reduction of $T_\gamma \bmod \mathsf{p}$. Then this is a split torus in G defined by the reduction of the basis $e_1, e_2 \bmod \mathsf{p}$. Our character χ can be considered as a character on the group $\widetilde{T}_\gamma$ because it vanishes by assumption on the center. Then we define

$$M_\phi^{\overline{\chi}} = \left.\begin{array}{c} \psi \in M_\phi | \psi(\overline{g}\overline{t}) = \phi(\overline{g})\chi(\overline{t})^{-1} \\ \text{for } \overline{t} \in \overline{T}_\gamma \end{array}\right\}$$

and $P^{\overline{\chi}}$ is the projection from M_ϕ to $M_\phi^{\overline{\chi}}$. We get the factor $p-1$ in front because of the normalization of the measures.

The other terms vanish by the same argument as before unless $\chi\widetilde{\phi} \circ N$ is unramified at P or at P'. If for instance it is unramified at P we have to consider

$$\int\limits_{\mathsf{U}_{\mathsf{p}}} \Phi[(0,1)(\Pi_{\mathsf{p}}^{\nu}, \Pi_{\mathsf{p}'}^{M'_{\mathsf{p}'}})(\epsilon_{\mathsf{p}}, \epsilon'_{\mathsf{p}})] \times$$

$$\widetilde{\phi}(\epsilon_{\mathsf{p}}\epsilon_{\mathsf{p}'})\chi_{\mathsf{p}}(\epsilon_{\mathsf{p}})\chi_{\mathsf{p}'}(\epsilon_{\mathsf{p}'})d^x\epsilon_{\mathsf{p}}d^x\epsilon_{\mathsf{p}'},$$

for $\nu > M_{\mathsf{p}}$. And as before we find the value for this integral is

$$(p-1)\Phi[(0,1)(\Pi_{\mathsf{p}}^{\nu}, \Pi_{\mathsf{P}'}^{M_{\mathsf{p}}})] = \Phi[e_2](p-1)$$

If we treat the integral the same way we did before in the case $\nu = M_{\mathsf{p}}$, $\nu' = M'_{\mathsf{p}}$ we get : If $k_1 \in \mathsf{K}_{\mathsf{p}}$, such that $\det(k_1) = 1$ and $(0,1)k_1 = e_2$ then the integral turns out to be

$$(p-1)P^{\widetilde{\chi}}\psi(k_1) = (p-1)\psi(k_1) = \Phi[e_2].$$

98 Therefore we find: If we put

$$W_{\mathsf{p}}(\phi, \chi, s) = |\pi_{\mathsf{p}}|^{\left(\frac{1}{2}+\frac{s}{2}\right)(M_{\mathsf{P}}+M'_{\mathsf{P}})}\widetilde{\phi}(\pi_{\mathsf{p}})^{M_{\mathsf{P}}+M'_{\mathsf{P}}}\chi(\Pi_{\mathsf{P}})^{M_{\mathsf{P}}}\chi(\Pi_{\mathsf{P}'})^{M'_{\mathsf{P}}}$$

then the value of the local contribution is equal to

$$(p-1)W_{\mathsf{p}}(\phi, \psi, s) \cdot P^{\chi}\psi(k_o)$$

if $\chi \cdot \phi \circ N$ is ramified at P and P'

$$(p-1)\frac{W_{\mathsf{p}}(\phi, \chi, s) \cdot (P^{\overline{\chi}}\psi(k_o) + \widetilde{\phi}(\pi_{\mathsf{p}})\chi(\Pi_{\mathsf{p}})|\pi_{\mathsf{p}}|^{\frac{1}{2}+\frac{s}{2}}(P^{\overline{\chi}}\psi(k_o) - P^{\overline{\chi}}(k_o))}{(1-\widetilde{\phi}(\pi_{\mathsf{p}})\chi(\Pi_{\mathsf{p}})|\pi_{\mathsf{p}}|_{\mathsf{p}}^{\frac{1}{2}+\frac{s}{2}}}$$

if $\chi \cdot \widetilde{\phi} \circ N$ is unramified at P (3.1.3.2)

and we get a corresponding expression in case of non ramification at P'.

Remark. In the first case the expression W_{p} does depend on the choice of π_{p} but the second factor does so too and the two factors cancel. We did all this since we want to know whether this elementary factor may vanish at $s = 0$. We ignore the exponential factor and therefore we have to look at the expressions

$$P^{\overline{\chi}}\psi(k_o) \qquad \text{if } \chi \cdot \phi\circ \text{ Nisramified at } \mathsf{P} \text{ and } \mathsf{P}'.$$

$$P^{\overline{\chi}}\psi(k_o) + \widetilde{\phi}(\pi_{\mathsf{p}})\chi(\Pi_{\mathsf{p}})|\pi_{\mathsf{p}}|_{\mathsf{p}}^{\frac{1}{2}}(P^{\overline{\chi}}\psi(k_1) - P^{\overline{\chi}}\psi(k_o))$$

if $\chi \cdot \widetilde{\phi} \circ N$ is unramified at P. This means that χ considered as a character on $(\mathscr{O}/\mathsf{P})^x$ is equal to $\phi^{\pm 1}$. The group $\overline{T}_\gamma$ acts on the projective line

$\overline{B}_o \backslash \overline{G}$ and we have exactly two fixed points therefore we get an orbit decomposition

$$\overline{G} = \overline{B}_o g_o \cdot \overline{T}_\gamma \cup B_o x_1 \cup B_o x_2$$

Here we have to choose for g_o and element in $\overline{G}$ which does not conjugate T_γ into B_o and x_1, x_2 do conjugate $\overline{T}_\gamma$ into $\overline{B}_o$.

We observe that we can for instance choose $\overline{g}_o$ simply the reduction of $k_o$mod p and the reduction of k_1 will be x_1 or $x_2$mod $\overline{B}_o$. Therefore we find if $\chi \neq \phi^{\pm 1}$

$$M_\phi^{\overline{\chi}} = \{h : G \to R | h(\overline{b}\overline{g}_o \cdot \overline{t}) = \phi(\overline{b}) \cdot \chi(\overline{t})^{\pm 1} h | Bx_1 = 0, h \cdot Bx_2 = 0\}$$

and we get. **99**

If $P^{\overline{\chi}}\psi \neq 0$ then $P^{\overline{chi}}\psi(k_o) \neq 0$ and we know exactly when our local elementary factor does not vanish.

If $\chi = \phi^{\pm 1}$ then $\dim M_\phi^{\chi} = 2$ and we find a basis for $M_\phi^{\overline{\chi}}$ by constructing h_1

$$h_1(\overline{b}\overline{g}_o \overline{t}) = \phi(\overline{b}) \cdot \chi^{-1}(\overline{t})$$

and $h_2 \neq 0$ concentrated on Bx_1 or Bx_2. Then we have

$$\begin{aligned} h_1(k_o) \neq 0, &\quad h_1(k_1) = 0 \\ h_2(k_o) = 0, &\quad h_2(k_1) \neq 0 \end{aligned}$$

and again we see that for suitable $\psi \in M_\phi^{\overline{\chi}}$ the local expression will not be zero, for instance we can choose $\psi = \lambda h_2$ with $\lambda \neq 0$ and then the local factor is $\neq 0$.

Case II. We assume that E is non split at p. We keep the notation $E_{\mathsf{p}} = E \otimes F_{\mathsf{p}}$ and we choose a uniformizing element Π_{p} in E_{p}.

Then we can find two constants $M_1 < N_1$ such that for $t_{\mathsf{p}} = \epsilon \cdot \Pi_{\mathsf{p}}^{\nu}$ we have

$$\begin{aligned} (0,1)t_{\mathsf{p}} \notin \mathcal{O}_{\mathsf{p}} \oplus \mathcal{O}_{\mathsf{p}} &\quad \text{if } \nu < M_1 \\ (0,1)t_{\mathsf{p}} \in \mathcal{O}_{\mathsf{p}} \oplus \mathcal{O}_{\mathsf{p}} &\quad \text{if } \nu > N_1 \end{aligned}$$

The local contribution is equal to

$$\sum_{\nu=M_1}^{\infty} |\det \Pi_{\mathsf{P}}|^{(\frac{1}{2}+\frac{s}{2})\nu} \widetilde{\phi}(\det \Pi_{\mathsf{P}})^{\nu} \cdot \chi(\Pi_{\mathsf{P}})^{\nu}$$

$$\int\limits_{\mathsf{U}_{\mathsf{p}}} \Phi[(0,1)\Pi_{\mathsf{P}}^{\nu}\epsilon]\widetilde{\phi}(\det \epsilon)\chi(\epsilon)d^{x}\epsilon$$

Then it is obvious that this local factor is of the form (*). We simply have to observe that for $\mathsf{p} \neq \mathsf{p}_o$ the integral does not depend on ϵ if $\nu < N_1$. If $\mathsf{p} = \mathsf{p}_o$ then the function $\Phi[(0,1)\pi_{\mathsf{p}}^{\nu}\epsilon] = 0$ if ν is large and we get a finite sum. Here we have to take into account that the character ϕ is ramified at p_o and that in this case $\chi \cdot \phi \circ N$ is ramified at P.

Again we discuss the case of a regular embedding more closely

(α) $\mathsf{p} \neq \mathsf{p}_o$ and E/F is unramified at p. Then we choose of course $\Pi_{\mathsf{p}} = \pi_{\mathsf{p}}$ and we have $\mathsf{U}_{\mathsf{p}} = E_{\mathsf{p}}^{x} \cap \mathrm{GL}_2(\mathscr{O}_{\mathsf{p}})$. Then for $t_{\mathsf{p}} = (0,1)\pi^{\nu}\epsilon \in \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}$ if and only if $\nu \geq 0$. We recall that it fol-
100 lows from our assumptions that $\chi|\mathsf{U}_{\mathsf{p}} = 1$, only these χ are of interest for us. Then the local factor is equal to

$$\frac{1 - \widetilde{\phi} \circ N(\pi_{\mathsf{p}}) \cdot \chi(\pi_{\mathsf{p}})|\det \pi_{\mathsf{p}}|_{\mathsf{p}}^{\frac{1}{2}+\frac{s}{2}}}{1 - \widetilde{\phi}^{2}(\pi_{\mathsf{p}})|\pi_{\mathsf{p}}|^{1+s}} = 1$$

(β) The extension is ramified at p. We may have two cases, namely

$$(0,1)\Pi_{\mathsf{P}}^{\nu}\epsilon \in \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}} \qquad \text{if and only if } \nu \geq -1$$

or

$$(0,1)\Pi_{\mathsf{P}}^{\nu}\epsilon \in \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}} \qquad \text{if and only if } \nu \geq 0.$$

If $M_o = -1$ in the first case and $M_o = 0$ in the second case, we find as local contribution

$$\sum_{\nu=M_o}^{\infty} |\det \Pi_{\mathsf{P}}|_{\mathsf{p}}^{(\frac{1}{2}+\frac{s}{2})\nu} \cdot \widetilde{\phi}(\det \Pi_{\mathsf{P}})^{\nu} \cdot \chi(\Pi_{\mathsf{P}})^{\nu}\cdot$$

$$\int_{\mathsf{U}_\mathsf{p}} \Phi[(0,1)\pi_\mathsf{p}^{\nu}\epsilon]\widetilde{\phi}(\det\epsilon)\chi(\epsilon)d^x\epsilon$$

The function under the integral sign is constant since $\widetilde{\phi}$ and χ are unramified. We map U_p into $E_\mathsf{p}^x/F_\mathsf{p}^x$ then the image is of index 2. Since we normalized vol $(E_\mathsf{p}^x/F_\mathsf{p}^x) = 1$ we get as local contribution

$$\frac{1}{2}|\det\Pi_\mathsf{p}|_\mathsf{p}^{(\frac{1}{2}+\frac{s}{2})M_o}\widetilde{\phi}(N(\Pi_\mathsf{p}))^{M_o}\cdot\chi(\Pi_\mathsf{p})^{M_o}$$
$$\frac{1-\widetilde{\phi}^2(\pi_\mathsf{p})|\pi_\mathsf{p}|^{1+s}}{1-\chi(\Pi_\mathsf{P})\widetilde{\phi}(N(\Pi_\mathsf{P}))|\Pi_\mathsf{P}|_\mathsf{P}^{\frac{1}{2}+\frac{s}{2}}}$$

and the elementary factor is exponential.

(γ) $\mathsf{p}\neq\mathsf{p}_o$ and the extension is unramified at p. This is the nicest case the local integral is equal to

$$\int_{\mathsf{U}_\mathsf{p}} \Phi[(0,1)\epsilon]\widetilde{\phi}^2(\det\epsilon)\chi(\epsilon)d^*\epsilon = (p+1)\cdot P^{\overline{\chi}}\psi(1)$$

Here we observe that the reduction mod p of T_γ is an anisotropic torus $\overline{T}_\gamma$ in $\overline{G}$ and $\overline{P}$ is the projection to the space

$$M_\phi^{\overline{\chi}} = \{\psi : G\to R|\psi(\overline{b}\overline{g}\overline{t}) = \phi(\overline{b})\psi(\overline{g})\chi(\overline{t})^{-1} \text{ for } \overline{b}\epsilon\overline{B}_o, \overline{t}\in\overline{T}_\gamma\}$$

Since $\overline{G} = \overline{B}_o\overline{T}_\gamma$ this space is of dimension 1 and generated by a function ψ_1 which satisfies $\psi_1(1) = 1$, the local factor is $\neq 0$.

(δ) $\mathsf{p} = \mathsf{p}_o$ and the extension E/F is ramified at the place p. The **101**
group of units injects into $\mathrm{GL}_2(\mathscr{O}_\mathsf{p})$ and this induces an injection

$$\mathsf{U}_\mathsf{p}/\mathsf{U}_\mathsf{p}^{(2)} \hookrightarrow \mathrm{GL}_2((\mathscr{O}/\mathsf{p}))$$

where $\mathsf{U}_\mathsf{p}^{(2)} = \{\epsilon|\epsilon\equiv 1 \mathrm{mod}\ \mathsf{P}^2\}$. The units of F_p mapped into $\mathrm{GL}_2(\mathscr{O}/\mathsf{p})$ fill up the center and we get

$$\overline{T}_\gamma = \mathrm{Im}(\mathsf{U}_\mathsf{p}/\mathsf{U}_\mathsf{p}^{(2)}\to\overline{G})$$

is cyclic of order p, this means that T_γ is the group of $\mathscr{O}/\mathsf{p}$-valued points of a Borel subgroup of $\overline{G}$. The element π_{p} induces via multiplication an endomorphism of $\mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}$. The reduction of this endomorphism to $\mathscr{O}/\mathsf{p} \oplus \mathscr{O}/\mathsf{p}$ has obviously a one dimensional kernel, which is generated by a vector $\xi_1 \epsilon \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}$. It is clear that the reduction of $\xi_1 \mathrm{mod}\ \mathsf{p}$ generates the stabilizer of $\overline{T}_\gamma$. This tells us that we have to consider two cases

$$\delta 1) \overline{T}_\gamma \subset \overline{B}_o \langle = \rangle (0,1)\Pi_{\mathsf{p}}^{-1} \in \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}$$
$$\delta 2) \overline{T}_\gamma \not\subset \overline{B}_o \langle = \rangle (0,1)\Pi_{\mathsf{p}}^{-1} \in \mathscr{O}_{\mathsf{p}} \oplus \mathscr{O}_{\mathsf{p}}$$

In the case δ) we get the local contribution

$$\begin{aligned} &|\det \Pi_{\mathsf{P}}|_{\mathsf{P}}^{-(\frac{1}{2}+\frac{s}{2})} \widetilde{\phi}(\det \Pi_{\mathsf{P}})^{-1} \chi(\Pi_{\mathsf{P}})^{-1} \cdot \\ &\int_{\mathsf{U}_{\mathsf{p}}} \Phi[(0,1)\pi_{\mathsf{p}}^{-1}\epsilon]\widetilde{\phi}(\det \epsilon) \cdot \chi(\epsilon) d^x \epsilon + \\ &\quad + \int_{\mathsf{U}_{\mathsf{p}}} \widetilde{\Phi}[(0,1)\epsilon]\widetilde{\phi}(\det \epsilon)\chi(\epsilon) d^x \epsilon \end{aligned}$$

Now the situation is quite similar to the split case. We choose $k_o \in \mathsf{K}_{\mathsf{p}}$ with $\det(k_o) = 1$ such that $(0,1)\pi_{\mathsf{P}}^{-1} = (0,1)k_o$ then we get

$$\frac{\pi}{2}(|\Pi_{\mathsf{P}}|_{\mathsf{P}}^{\frac{1}{2}+\frac{s}{2}} \widetilde{\Phi}(N(\Pi_{\mathsf{P}}))^{-1} \chi(\Pi_{\mathsf{P}})^{-1} P^{\overline{\chi}} \psi(k_o) + P^{\chi} \psi(1))$$

The vector $(0,1)\Pi_{\mathsf{p}}^{-1} \mathrm{mod}\ \mathsf{p}$ is not stabilized by $\overline{B}_o$ and therefore the image $\overline{k}_o$ of k_o in $\overline{G}$ is not contained in $\overline{B}_o$. We have

$$\overline{G} = \overline{B}_o \cdot \overline{k}_o \cdot \overline{T}_\gamma \cup \overline{B}_o$$

and we get that $\dim M_\phi^{\overline{\chi}} = 1$ (resp. 2) if $\overline{\chi} \neq 1$ (resp $\overline{\chi} = 1$). In both cases we can construct a $\psi \in M_\phi^{\overline{\chi}}$ such that $\psi(k_o) \neq 0$ and $\psi(1) = 0$, and this implies that with this choice the elementary factor is non zero.

102 The case $\delta 2)$ is quite similar. In this case we have also to sum over two terms namely $(0,1)\epsilon$, $(0,1)\Pi_{\mathsf{p}}\epsilon$. Then we get

$$\frac{p}{2}P^{\overline{\chi}}\psi(1) + |\Pi_{\mathsf{P}}|_{\mathsf{P}}^{\frac{1}{2}+s/s}\widetilde{\phi}(N(\Pi_{\mathsf{P}})) \cdot \chi(\Pi_{\mathsf{P}})P^{\overline{\chi}}\psi(k_1))$$

where $(0,1)\pi_{\mathsf{P}} = (0,1)k_1$. The same argument as above shows the non vanishing if ψ is suitably chosen.

3.1.5 The Infinite Place We have to evaluate the integral

$$\int\limits_{T_\gamma(\mathbb{C})} \omega_s^{(\infty)}(t_\infty g_\gamma, \widetilde{\phi})dj_{x_\gamma}(Y))d^x t_\infty$$

We recall that (1.6)

$$\omega_s^{(\infty)}(g_\infty, \phi) = \omega_s^{(\infty)}(b_\infty k_\infty, \widetilde{\phi}) =$$
$$|b_\infty|_{\mathbb{C}}^{+\frac{1}{2}+\frac{s}{2}}\widetilde{\phi}(b_\infty)(\mathrm{ad}\, k_\infty^{-1}) \cdot e_{\epsilon(\phi)}$$

where $b_\infty = \left(\begin{smallmatrix} x_\infty & u \\ 0 & 1 \end{smallmatrix}\right) \in B_o(\mathbb{C})$ and

$$|b_\infty|_{\mathbb{C}}^{\frac{1}{2}+\frac{1}{2}s}\widetilde{\phi}(b_\infty) = |x_\infty|_{\mathbb{C}}^{\frac{1}{2}+\frac{1}{2}s} \cdot \widetilde{\phi}(x_\infty)$$

We recall that $T_\gamma(\mathbb{C}) = T_\gamma^{(s)} x T_\gamma^{(c)} = R_+^x x S^1$ (3.1) and we have selected g_γ in such a way that

$$\langle \omega_s^\infty(t_\infty g_\gamma, \phi), dj_{x_\gamma}(Y)\rangle$$

does not depend on the circular variable. (3.1).

Now we have of course that $E_\infty = F_\infty \otimes E = \mathbb{C} \otimes E = \mathbb{C} \oplus \mathbb{C}$, this defines a split torus in $\mathrm{PGL}_2(\mathbb{C})$ and this torus is certainly not contained in $B_o(\mathbb{C})$.

We can find a matrix

$$x = \begin{pmatrix} y & u \\ 0 & 1 \end{pmatrix} \in B_o(\mathbb{C})$$

such that

$$T_\gamma(\mathbb{C}) = xT_1x^{-1}$$

where **103**

$$T_1 = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{C}, a^2 - b^2 \neq 0 \right\} \text{mod center of } \mathrm{GL}_2(\mathbb{C})$$

and one checks that y is unique up to a sign. The maximal compact subgroup of T_1 is contained in the maximal compact subgroup K_∞ and therefore we can take $g_\gamma = x$. Our integral becomes

$$\int_{T_1} \langle \omega_s^{(\infty)}(xt_1, \phi), \mathrm{ad}(x^{-1}) \cdot (Y) \rangle d^x t_1$$

The generator $Y \in \mathrm{Lie}(T_\gamma^{(s)})$ is selected in a canonical way up to a sign since the character module of the torus T_γ has a canonical generator λ_o up to a sign and $d\lambda(Y) = 1$. Therefore we get that

$$\mathrm{ad}(x^{-1})(Y) = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{p}$$

and we choose y in such a way that

$$\mathrm{ad}(x^{-1})(Y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E_1$$

Observing that the integral does not depend on the circular variable we find for the value of the integral

$$|y|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}} \widetilde{\phi}(y) \int_{T_1^{(s)}} \langle \omega_s^{(\infty)}(t, \widetilde{\phi}), E_1 \rangle d^x t$$

and

$$T_1^{(s)} = \left\{ \exp x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\} =$$

$$\left\{ \begin{pmatrix} \cos hx \sin hx \\ \sin hx \cos nx \end{pmatrix} = t(x) \middle| x \in \mathbb{R} \right\}$$

and the measure was normalized such that we have to compute

$$|y|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}}\widetilde{\phi}(y)\int_{-\infty}^{+\infty}\langle\omega_s^{(\infty)}(t(x),\widetilde{\phi}),E_1\rangle\mathrm{dx}$$

We put $h(x)=(\cos h^2x+\sin h^2x)^{\frac{1}{2}}$ and get

$$t(x)=\begin{pmatrix}h(x)^{-1} & * \\ 0 & h(x)\end{pmatrix}\begin{pmatrix}\frac{\cos hx}{h(x)} & \frac{\sin hx}{h(x)} \\ -\frac{\sin hx}{h(x)} & \frac{\cos hx}{h(x)}\end{pmatrix}=b(x)\cdot k(x)$$

Then the integral is equal to **104**

$$|y|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}}\widetilde{\phi}(y)\int_{-\infty}^{+\infty}h(x)^{-2-2s}\langle\operatorname{ad}k(x)^{-1}e_{\epsilon(\phi)},E_1\rangle\mathrm{dx}$$

But $\langle\operatorname{ad}k(x)^{-1}e_{\epsilon(\phi)},E_1\rangle=\langle e_{\epsilon(\phi)},\operatorname{ad}k(x)E_1\rangle$.

From (1.4.1) we get that

$$\langle e_{\epsilon(\phi)},\operatorname{ad}k(x)E_1\rangle=h(x)^{-2}\cdot\langle E_1,E_1\rangle=h(x)^{-2}$$

and we wind up with

$$|y|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}}\widetilde{\phi}(y)\int_{-\infty}^{+\infty}(\cos h^2x+\sin h^2x)^{-2-s}\mathrm{dx}$$

and this integral turns out to be equal to

$$\frac{1}{2}\cdot|y|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}}\cdot\widetilde{\phi}(y)\frac{\Gamma(s/2+1)\Gamma(1/2)}{\Gamma\left(\frac{s+1}{2}+1\right)}$$

and if we exploit Legendre's duplication formula ([28], 12.15) we find

$$2^s|y|_{\mathbb{C}}^{\frac{1}{2}+\frac{s}{2}}\widetilde{\phi}(y)\cdot\frac{\Gamma\left(\frac{s}{2}+1\right)^2}{\Gamma(s+2)}$$

for the local contribution.

Now we can evaluate at $s = 0$. Then the value of the integral is the value of the period of the class $[E(\phi, \psi, 0)]$ on the cycle $\sum_{\xi \in I_\gamma} \chi(\xi) z_{\gamma_\xi}$ and this is an intrinsic value which does not depend on the different choices we made. We get our second main theorem.

Theorem 3.1.6. *The value of the Eisenstein class $[E(\phi, \psi, 0)]$ on the cycle*

$$\sum_{\xi \in I_\gamma} \chi(\xi) Z_{\gamma_\xi}$$

for a primitive $\gamma \in \Gamma$ is given by

$$|y|_{\mathbb{C}}^{\frac{1}{2}} \widetilde{\Phi}(y) \cdot \Pi_{\mathsf{p}} G_{\mathsf{p}}(\widetilde{\phi}, \psi, \chi, \gamma, 0) \cdot \frac{L_E\left(\chi\widetilde{\phi} \circ N, \frac{1}{2}\right)}{L_F(\widetilde{\phi}^2, 1)}$$

105 *We have that almost all local elementary factors $G_{\mathsf{p}}(\phi, \psi, \chi, \gamma, 0) = 1$. If our embedding is regular at all places, then we can choose ψ so that all local factors $\neq 0$.*

Remark. The theorem as it is stated is somewhat weak because we do not say very much about the elementary factors. So we should understand it in connection with our results on these factors which have been obtained in course of the computations. It seems to be difficult to incorporate these computational results into the statement of the theorem.

4 Arithmetic Applications

In the beginning of 3.1 we stated without proof that the classes

$$[E(\underline{g}, \phi, \psi, 0)]$$

are cohomology classes in $H^1(\Gamma \backslash X, K)$, where K is the field of fraction of R. We want to accept this fact form now on, in any case we have given several examples in 2.2 in the cases $\mathsf{P}_o = (2 - i)$, $\mathsf{P}_o = (3 + 2i)$ where we checked this assumption directly. We abbreviate

$$[E(\underline{g}, \phi, \psi, 0)] = \Phi_E(\psi)$$

and then we get a homomorphism

$$\Phi_E(\psi) : \Gamma \to K$$

Then our main theorem 3.1.6 may also be stated as follows:

Let $\gamma \in \Gamma$ be primitive, let I_γ the group of classes in the genus of γ, for any character $\chi : I_\gamma \to S^1$ we consider

$$\sum_{\xi \in I_\gamma} \chi(\xi)\gamma_\xi \in \Gamma/[\Gamma,\Gamma] \otimes Z[\chi]$$

where $(Z[\chi]$ is the ring of integers of the field generated by the values of $\chi)$. Then

$$\Phi_E(\psi)(\sum \chi(\xi)\gamma_\xi) = |y|_{\mathbb{C}}^{\frac{1}{2}}\widetilde{\phi}(y) \cdot \Pi_{\mathsf{p}} G_{\mathsf{p}}(\widetilde{\phi}, \psi, \chi, \gamma, 0) \frac{L_E\left(\chi \cdot \widetilde{\phi} \circ N, \frac{1}{2}\right)}{L_F(\widetilde{\phi}^2, 1)}$$

The first consequence of this formula is

Corollary 4.1. *The number* **106**

$$|y|_{\mathbb{C}}^{\frac{1}{2}}\widetilde{\phi}(y) \cdot (\Pi_{\mathsf{p}} G_{\mathsf{p}}(\widetilde{\phi}, \psi, \chi, \gamma, 0)) \frac{L_E\left(\chi \cdot \widetilde{\phi} \circ N, \frac{1}{2}\right)}{L_F(\widetilde{\phi}^2, 1)}$$

is in $K[\chi]$

We have an action of the Galois group $\mathrm{Gal}(K(\chi)/Q)$ on the group

$$\mathrm{Hom}(\Gamma, K(\chi))$$

simply given by the action on the group of values. This induces an action of this Galois group on the characters ϕ, χ and on the functions ψ. Since it follows from the above theorem, that $\Phi_E(\psi)^\sigma = \Phi(\psi^\sigma)$ we get even information concerning the galois action on the above numbers

$$\left(|y|^{\frac{1}{2}} \cdot \widetilde{\phi}(y)(\Pi_{\mathsf{p}} G_{\mathsf{p}}(\widetilde{\phi}, \psi, \chi, \gamma, 0)) \frac{L_E\left(\chi \cdot \widetilde{\phi} \circ N, \frac{1}{2}\right)}{L(\widetilde{\phi}^2, 1)}\right)^\sigma =$$

$$|y|^{\frac{1}{2}}\widetilde{\phi}^{\sigma}(y)(\Pi_{\mathsf{p}}G_{\mathsf{p}}(\widetilde{\phi}^{\sigma},\psi^{\sigma},\chi^{\sigma},\gamma,0))\frac{L\left(\chi^{\sigma}\circ N,\frac{1}{2}\right)}{L((\widetilde{\phi}^{\sigma})^2,1)}$$

But we can say a little bit more: The Eisenstein classes $\Phi_E(\psi) : \Gamma \to K$ have of course to satisfy certain integrality conditions. This means that in any given case we find a number $d \in Z$, such that $\Phi_E(\psi)$ takes its values already in $R' = R\left[\chi, \frac{1}{d}\right]$. Then we get of course the same estimates for the denominators of the right hand side. In any case these questions about the denominators in the Eisenstein classes seem to be very interesting. We discussed already some of the aspects at the end of 2.2. We should certainly expect those primes in the denominator which occur in the torsion of the cohomology of Γ. We should also expect the primes dividing c_ϕ. But we can say that if there are other primes in the denominator of an Eisenstein class, then they will create congruences between the Fourier coefficients of cusp forms and Eisenstein series. We hope to come back to these questions later.

107 Of course the main object of interest are the values $L_E(\chi, \widetilde{\phi}\circ N, \frac{1}{2})$ themselves. If we want to understand these values we have to get hold of the local factors, especially we have to prove that they are non zero. We have collected some informations concerning this question, but I do not want to discuss these problems in this paper. Instead of trying to give a general statement I will treat a very special example where one can see how our results can be used to get informations on these special values of L-functions. Before I come to this example I want to say one more word about the relationship of our result to Shimura's results in [24]. He considers special values of L-functions $L_E(\eta, s)$ where E is a CM field and η a Grossencharakter of type A_o. He proves that for certain special values of s the value of the L-function divided by a suitable power of a period is an algebraic number.

Our method here gives some information on the ratios

$$\frac{L_E\left(\chi\cdot\widetilde{\phi}\circ N,\frac{1}{2}\right)}{L_F(\widetilde{\phi}^2,1)}$$

where the period ω^2 cancels out. So our information is weaker to some extent, but we get informations for an infinite number of fields $E/\mathbb{Q}$, which are not necessarily CM-fields. We get informations on the Galois action and on the denominators. In some cases we get even an effective procedure to compute these ratios, and I want to conclude this paper by describing this procedure and doing the computation in one specific case.

We identified the space of R-valued function $\mathsf{C}(\overline{G})$ with the group ring

$$\mathsf{C}(\overline{G}) \xrightarrow{\sim} R[\overline{G}]$$
$$f \to \sum_{\sigma\in\overline{G}} f(\sigma)\sigma$$

This is an isomorphism of $\overline{G}\times\overline{G}$-modules, the actions on the group ring are given by

$$L_\tau : m = \sum_{\sigma\in G} a_\sigma\sigma \to \sum_{\sigma\in G} a_\sigma\tau\sigma = \sum_{\sigma\in G} a_{\tau^{-1}\sigma}\sigma$$
$$R_\tau : m = \sum_{\sigma\in G} a_\sigma\sigma \to \sum_{\sigma\in G} a_\sigma\sigma\tau^{-1} = \sum_{\sigma\in G} a_{\sigma\tau}\sigma$$

We consider $R[\overline{G}]$ as a Γ_o-module with respect to the action induced by **108**
right multiplication and with respect to this action we defined

$$H^1(\Gamma_o, R[\overline{G}])$$

and we have (1.1)

$$H^1(\Gamma_o, R[\overline{G}] = H^1(\Gamma, R)$$

In $\mathsf{C}[\overline{G}]$ we considered the submodule (1.2.3)

$$N_\phi = \{f : \overline{G} \to R | f(\overline{b}\overline{g}) = \phi(\overline{b})\cdot f(\overline{g})\}$$

and in some cases we have explicitely computed the cohomology groups (2.2)

$$H^1(\Gamma_o, N_\phi) \hookrightarrow H^1(\Gamma_o, R[\overline{G}])$$

In those case which we considered we found that $\dim_R H^1(\Gamma_o, N_\phi) = 1$ and we did even better namely we constructed explicitely cocycles

$$\Phi : \Gamma_o \to N_\phi$$

whose cohomology class generates the cohomology. Here we observe, that such a cocycle

$$\Phi : \Gamma_o \to N_\phi$$

is uniquely determined by its class, provided we know that $\Phi : \Gamma_{oB_o} \to N_\phi^{\mathsf{U}_o}$. Therefore we can say that the cocycle Φ is a canonical representative of the given class.

Now we got from our construction (1.2.3) that the class Φ which satisfied (2.2)

$$\Phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \left(\sum_{u \in U_o} \delta_{\overline{u}}\right) + b \cdot \delta_\infty$$

has to be equal to the Eisenstein class, i.e.

$$[\Phi] = [E(\phi, \psi_o, 0)]$$

where we have $\psi_o \in M_\phi$ and

$$\psi_o(\overline{g}) = \begin{cases} \phi(\overline{b})^{-1} & \text{for } \overline{g} = \overline{u}w\overline{b} \\ 0 & \text{for } \overline{g} = \overline{b} \end{cases}$$

109 This tells us that the cocycles which we computed in the examples in (2.2) actually are equal to the canonical representatives of a very specific Eisenstein class.

Remark. This argument of course breaks down if we do not know that $H^1(\Gamma_o, N_\phi)$ is of rank 1. In that case we have to separate the Eisenstein class from the other classes by using the action of the Hecke algebra.

Now it is clear how we get an "explicit" formula for the value

$$[E(\phi, \psi_o, 0)](\gamma)$$

for $\gamma \in \Gamma$. Let us assume we have computed the value $\Phi(\gamma)$ of the representing cocycle on γ. Then

$$\Phi(\gamma) = \sum_{\sigma \in \overline{G}} \phi_\sigma(\gamma)\sigma$$

and according to (1.1) we have

$$[E(\phi, \psi_o, 0)](\gamma) = \Phi_1(\gamma)$$

Now we have a set of generators for Γ_o namely the matrices

$$u(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \qquad \alpha \in \mathscr{O}$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

We know in our examples the value of Φ on each of the generators and if $\gamma = \gamma_1, \ldots, \gamma_t$ is written as a product of generators then we have

$$\Phi(\gamma) = \sum_{\nu=1}^{t} R\overline{\gamma_1, \ldots, \gamma_{\nu-1}}\Phi(\gamma_\nu) = \sum_{\nu=1}^{t} \sum_{\sigma \in \overline{G}} \Phi_\sigma \overline{\gamma_1, \ldots, \gamma_{\nu-1}}(\gamma_\nu)\sigma$$

where $\overline{\gamma_1, \ldots, \gamma_{\nu-1}}$ is the image of $\gamma_1, \ldots, \gamma_{\nu-1}$ in $\overline{G}$. We have

$$\Phi_1(\gamma) = \sum_{\nu=1}^{t} \Phi\overline{\gamma_1 \quad \gamma_{\nu-1}}(\gamma_\nu)$$

and if we interpret $\Phi(\gamma) \in R[\overline{G}]$ as an R-valued function on $\overline{G}$, then we find

$$\Phi_1(\gamma) = [E(\phi, \psi_o, 0)](\gamma) = \sum_{\nu=1}^{t} \Phi(\gamma_1)\overline{(\gamma_1, \ldots, \gamma_{\nu-1})}$$

We want to generalize this formula slightly, we are interested in **110**

$$[E(\phi, \psi, 0)](\gamma) \quad \text{for all} \quad \psi \in M_\phi.$$

We observe that M_ϕ is an irreducible $\overline{G}$-module with respect to the action induced by multiplication from the left. Therefore it suffices to compute these numbers in the special case that $\psi = L_{\sigma_o}(\phi_o)$ for $\sigma_o \in \overline{G}$. Then we get of course the representing cocycle

$$L_{\sigma_o}\Phi(\gamma) = \sum_{\sigma} \Phi_\sigma(\gamma)\sigma_o\sigma$$

and form that we get the formula

$$[E(\phi, L_{\sigma_o}\phi_o, 0)](\gamma) = \sum_{\nu=1}^{t} \Phi(\gamma_\nu)\sigma_o^{-1}\gamma_1, \ldots, \gamma_{\nu-1} \tag{4.1}$$

if $\gamma = \gamma_1, \ldots, \gamma_t$ is a presentation of γ as a word in the generators of Γ_o we gave above.

Now we want to evaluate the formula in one special case. We take $\mathsf{P}_o = (2 - i)$ and E/F shall be the field of eight roots of unity.

If $\zeta = \sqrt{i} = e^{\frac{\pi i}{4}}$ then we have

$$\mathcal{O}_E = \mathcal{O}_F[\zeta]$$

([12], IV, Thm. 3). This field contains the maximal totally real subfield $L = \mathbb{Q}(\sqrt{2})$ and the fundamental unit in L is $1 + \sqrt{2} = \epsilon$. We embed $E \hookrightarrow M_2(F)$ by means of the identification

$$a + b\zeta \rightarrow (a, b)$$

and then we have

$$E = \left\{ \begin{pmatrix} a & b \\ bi & a \end{pmatrix} \middle| a, b \in F \right\} \subset M_2(F)$$

Since we have $\mathcal{O}_E = \mathcal{O}_F[\zeta]$ this embedding is everywhere strongly regular as one checks easily. Now the element $\eta = \epsilon^3$ is a primitive element in the group Γ and it is given by the matrix

$$\eta = \begin{pmatrix} 7 & 5 - 5i \\ 5 + 5i & 7 \end{pmatrix} = \gamma$$

The class number of E is one and starting from this one checks that there is only one class in the genus of γ, this is so since $(\mathscr{O}_E/\mathsf{p}_o\mathscr{O}_E)^x/(\mathscr{O}_F/\mathsf{p}_o)^x$
111 global units $= 1$. This means there is only the trivial character $\chi_o = 1$ and our main theorem says-

$$[E(\phi, \psi, 0)](\gamma) = \text{ product of local factors } \times \frac{L_E\left(\widetilde{\phi} \circ N, \frac{1}{2}\right)}{L_F(\widetilde{\phi}^2, 1)}$$

We have to determine the local factors. They are certainly equal to one at all places except $(1-i)$, p_o and infinity. So we compute these factors explicitely at these places.

We look at $(1+i)$ first. In this case we have the uniformizing element $\pi_2 = (1-\zeta)$. Then

$$\pi_2^{-1} = \frac{1}{1-\zeta} = \frac{1+\zeta}{(1-\zeta)(1+\zeta)} = \frac{1+\zeta}{1-i}$$

The corresponding matrix is

$$\frac{1}{1-i}\begin{pmatrix} 1 & 1 \\ i & 1 \end{pmatrix}$$

and

$$(0,1)\cdot\pi_2^{-1} = \frac{1}{1-i}(0,1)\begin{pmatrix} 1 & 1 \\ i & 1 \end{pmatrix} = \frac{1}{1-i}(i,1)$$

and hence $(0,1)\pi_2^{-1} \notin \mathscr{O}_2 \oplus \mathscr{O}_2$. We are in case II, β) and find the local elementary factor

$$G_2(\widetilde{\phi}, \psi_o, \chi, \gamma, 0) = \frac{1}{2}$$

Now we look at the local factor at $\mathsf{p} = \mathsf{p}_o = (2-i)$. We are in case II, γ) and we have to compute

$$(p+1)P^{\chi_o}\psi_o(1) = \sum_{t\in\overline{T}_\gamma}\psi_o(t)$$

The group $\overline{T}_\psi$ is cyclic of order 6 and generated by $\eta = \left(\begin{smallmatrix} 3 & 1 \\ 2 & 3 \end{smallmatrix}\right)$.

Recalling the definition of ψ_o and we find

$$G_{\mathsf{p}_o}(\widetilde{\phi}, \psi_o, \chi_o, \gamma, 0) = 6 \cdot P^{\chi_o}\psi_o(1) = 2 + i$$

And at infinity our torus is given by **112**

$$T_\gamma(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ bi & a \end{pmatrix} \middle| a, b \in \mathbb{C} \right\} \Big/ \text{center}$$

If we choose our matrix

$$x = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

then $T_\gamma(\mathbb{C}) = xT_1x^{-1}$ in our previous notations. We recall that $\zeta = e^{\frac{\pi i}{4}}$. Now we see that the factor at infinity is

$$\cdot e^{-\frac{\pi i}{4}}$$

Our formula becomes

$$[E(\widetilde{\phi}, \psi_o,)](\gamma) = \cdot e^{-\frac{\pi i}{4}} \cdot \frac{1}{2} \cdot (2 + i) \frac{L_E\left(\widetilde{\phi} \circ N, \frac{1}{2}\right)}{L_F(\widetilde{\phi}^2, 1)}$$

Now we compute the left hand side by using (4.1). We have to write down γ in terms of the generators and this is easily done by using the euclidian algorithm.

$$\gamma = \begin{pmatrix} 1 & 1 - i \\ 0 & 1 \end{pmatrix} \cdot C^2 \cdot B \begin{pmatrix} 1 & -2 - 2i \\ 0 & 1 \end{pmatrix} B \cdot \begin{pmatrix} 1 & -2 + 2i \\ 0 & 1 \end{pmatrix} \cdot B \begin{pmatrix} 1 & -i - 1 \\ 0 & 1 \end{pmatrix} B$$

Now it is a question of sitting down and to compute the value of $\Phi(\gamma)$, using (4.1) and (2.2). Case I we found

$$\Phi_1(\gamma) = \frac{8 + 15i}{1 + 2i}$$

Therefore we obtain the formula

$$\frac{L_E\left(\widetilde{\phi} \circ N, \frac{1}{2}\right)}{L_F(\widetilde{\phi}^2, 1)} = 2 \cdot e^{-\frac{\pi i}{4}} \frac{8 + 15i}{(1 + 2i)(2 + 1)} = 2 \cdot e^{-\frac{\pi i}{4}} \frac{i(15 - 8i)}{i(2 - i)(2 + i)} =$$

$$= 2 \cdot e^{-\frac{\pi i}{4}} \frac{(4-i)^2}{5}$$

113 **Remark 1.** This last computation has been done by hand and has not been checked by a numerical computation. But if one believes in the Birch-Swinnerton-Dyer conjecture ([26]) then the value

$$L_E\left(\widetilde{\phi} \circ N, \frac{1}{2}\right)$$

should have something to do with an order of a Tate-Shafarewic group.

Since it is not more than five minutes ago that I computed the value above I must confess that I am still pleased by the occurrence of the square.

Remark 2. For this particular character the value

$$\frac{L(\overline{\widetilde{\phi}}^2, 1)}{L(\widetilde{\phi}^2, 1)} = \frac{(1-2i)^3}{(1+2i)^2} \frac{1}{\sqrt{5}}$$

has been numerically checked. In this case I also computed numerically the value

$$L(\widetilde{\phi}^2, 1) = -\omega^2 \frac{2^2}{5} \cdot \frac{1+2i}{(1-2i)^2} \sqrt{1-2i}$$

where $\omega = \frac{1}{2} \int\limits_0^1 \frac{\mathrm{dx}}{\sqrt{x-x^3}}$ and

$$-\frac{\pi}{2} \langle \arg \sqrt{1-2i} \rangle 0$$

But this has to be taken with caution since we have not really proved this. The numerical values are equal up to 8 digits. But if we believe that this value is correct then we find

$$L_E\left(\widetilde{\phi} \circ N, \frac{1}{2}\right) = -\omega^2 2^3 \cdot \frac{1}{5^2} e^{-\frac{\pi i}{4}} (4-i)^2 \frac{1+2i}{(1-2i)^2} \sqrt{1-2i}$$

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WAVE FRONT SETS OF REPRESENTATIONS OF LIE GROUPS

By Roger Howe[1]

Introduction

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In the past few years the concept of wave front set [D] has proved fruitful for the theory of distributions and P.D.E. It seems it might also be of use in the representation theory of Lie groups. Its close relative, the singular spectrum of a hyperfunction, has already been discussed in a special context in [K-V], which served as the catalyst for this note. The purpose here is to define and discuss general properties of wave front sets of representations, and to give some examples.

I would like to thank Nolan Wallach for very valuable discussions regarding this paper. Especially, the principle of proof of proposition **??** comes from him. Also I thank Richard Beals for valuable technical discussions.

1 Generalities

Let ρ be a representation of the Lie group G. For convenience we shall assume ρ is unitary, although this is not strictly necessary. Let H be

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the Hilbert space on which ρ acts, and let $J_1(H) = J_1$ be the trace class operators on H. Given $T \in J_1$, put

$$\operatorname{tr}_\rho(T)(g) = \operatorname{tr}(\rho(g)T) \quad g \in G \tag{1.1}$$

where tr is the usual trace functional on J_1. Then

$$\operatorname{tr}_\rho : J_1 \to \mathbb{C}_b(G) \tag{1.2}$$

where $\mathbb{C}_b(G)$ is the space of bounded functions on G, is a norm-decreasing map. The image of tr_ρ is called the *space of (continuous) matrix coefficients of ρ.*

We may also regard $\operatorname{tr}_\rho(T)$ as a distribution on G by integration, in the usual fashion

$$\operatorname{tr}_\rho(T)(f) = \int_G f(g) \operatorname{tr}_\rho(T) dg = \operatorname{tr}(\rho(f)T) \quad f \in \mathbb{C}_c^\infty(G). \tag{1.3}$$

118 Here dg is Haar maasure on G. Since $\operatorname{tr}_\rho(T)$ is a distribution on G, we may consider its wave front set $WF(\operatorname{tr}_\rho(T))$. Our basic reference for wave front sets is [D] and we shall recall their basic definitions and properties as they are needed. For now, recall $WF(\operatorname{tr}_\rho(T))$ is a closed, conical (i.e., closed under positive dilations in the fibers) set in T^*G, the cotangent bundle of G.

Definition. *WF_ρ is the closure of the union of $WF(\operatorname{tr}_\rho(T))$ as T varies over J_1.*

Thus WF_ρ is also a closed conical set of T^*G.

Remark. This is not the same as the wave front set defined in [H1], which is sort of a dual notion to the present one.

Proposition 1.1. *WF_ρ is invariant under left and right translations of G on T^*G.*

Proof. Define as usual left and right translations on functions and distributions:

$$\begin{cases} L_g(f)(g') = f(g^{-1}g') : R_g(f)(g') = f(g'g) \quad f \in \mathbb{C}_c^\infty(G) \\ L_g(D)(d) = D(L_{g^{-1}}f);\ R_g(D)(f) = D(R_{g^{-1}}f) D \in D(G). \end{cases} \tag{1.4}$$

Then we have the well-known relations

$$L_g \operatorname{tr}_\rho(T) = \operatorname{tr}_\rho(T_\rho(g)^{-1});\ R_g \operatorname{tr}_\rho(T) = \operatorname{tr}_\rho(\rho(g)T) \tag{1.5}$$

Left and right translations of G also induce in the usual way transformations L_g^* and R_g^* on T^*G. By the naturality of the wave front set ([D], proposition 1.3.3.) one has, for a distribution D on G.

$$WF(L_g D) = L_g^*(WF(D)) \quad \text{and} \quad WF(R_g(D)) = R_g^*(WF(D)). \tag{1.6}$$

The proposition follows directly from the definition and equations (1.5) and (1.6).

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let Ad be the adjoint action of G on $\mathfrak{g}$, and let Ad^* be the contragredient action on $\mathfrak{g}^*$. We can identify $\mathfrak{g}^*$ with the left invariant exterior 1-forms on G. This leads to an identification

$$T^*G \simeq G \times \mathfrak{g}^* \tag{1.7}$$

Thus if $\psi \in \mathbb{C}_c^\infty(G)$, we can regard $d\psi$, the differential of ψ, as a g^*- **119**
valued function on G. Doing so, we have the following behaviour under right and left translations

$$d(L_g\psi) = L_g d\psi \quad d(R_g\psi) = \mathrm{Ad}\, g(R_g d\psi) \tag{1.8}$$

One sees from (1.8) that a bi-invariant set in $T^*(G)$ is identified via (1.7) with $G \times X$ where $X \subseteq \mathfrak{g}^*$ is an $\mathrm{Ad}^*\, G$ invariant set. Thus we can associate to WF_ρ a closed conical $\mathrm{Ad}^*\, G$-invariant subset of $\mathfrak{g}^*$, to be denoted WF_ρ^o. The set WF_ρ^o then determines WF_ρ via (1.7).

It is conceivable that WF_ρ could be very uninteresting–it might always be all of $\mathfrak{g}^*$ for example. Thus it may be instructive to point out at the beginning that for irreducible ρ at least, WF_ρ is limited to a certain characteristic and non-trivial behavior.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. It is well known that there is a canonical linear isomorphism, the symmetrization map

$$\sigma : U(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{g}) \simeq P(\mathfrak{g}^*) \tag{1.9}$$

where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$, and $P(\mathfrak{g}^*)$ the polynomial algebra on $\mathfrak{g}^*$, the two algebras being identified in the standard way. The symmetrization σ is an intertwining map for the adjoint actions of G on $U(\mathfrak{g})$ and on $P(\mathfrak{g}^*)$. Thus σ restricts to a linear isomorphism between $ZU(\mathfrak{g})$, the center of $U(\mathfrak{g})$, and $IP(\mathfrak{g}^*)$, the $\mathrm{Ad}^*\,G$ invariants in $P(\mathfrak{g}^*)$.

The map σ has a natural interpretation in terms of P.D.E. We can identify each $u \in U(\mathfrak{g})$ to a left invariant differential operator R_u on G. If R_u has order m, then the leading symbol of R_u, in the sense of P.D.E [D], will be a left-invariant section of $S^mT(G)$, the m-th symmetric power of the tangent bundle of G. Thus the symbol of R_u is determined by its value at the identity, which will be an element of $S^m\mathfrak{g} \simeq P^m(\mathfrak{g}^*)$. It is known and easy to check from the definitions that the symbol of R is just the m-th homogeneous part of $\sigma(u)$.

Let $V(\mathfrak{g}^*)$ denote the set of common zeroes of the homogeneous elements of positive degree of $IP(\mathfrak{g}^*)$. We call V the *characteristic variety* of $\mathfrak{g}^*$ (or of G). □

120 Let ρ be as above a unitary representation of G.

Proposition 1.2. *Let ρ be irreducible. Then*

$$WF_\rho^o \subseteq V(\mathfrak{g}^*) \tag{1.10}$$

Proof. Since ρ is irreducible, the action of $ZU(\mathfrak{g})$ on the smooth vectors of ρ is by scalars [Se]. Say $\rho(z)x = \mu(z)x$ for x a smooth vector and $z \in ZU(\mathfrak{g})$, where $\mu : ZU(\mathfrak{g}) \to \mathbb{C}$ is the infinitesimal character of ρ. Thus let x, y be smooth vectors in H, the space of ρ. Let $E_{x,y}$ be the dyad

$$E_{x,y}(u) = (u, x)y \qquad u \in H \tag{1.11}$$

Then

$$\mathrm{tr}_\rho(E_{x,y})(g) = (\rho(g)y, x) \tag{1.12}$$

It follows by differentiating (1.5) that

$$R_z\,\mathrm{tr}_\rho(E_{x,y}) = \mathrm{tr}_\rho(E_{x,\phi(z)y}) = \mu(z)\,\mathrm{tr}_\rho(E_{x,y}) \tag{1.13}$$

Here R_z is as above, the right convolution operator on G corresponding to z.

Since every element in J_1 is a limit in the trace norm of sums of smooth dyads, and tr_ρ is norm-decreasing, we find that

$$R_Z\,\mathrm{tr}_\rho(T) = \mu(z)\,\mathrm{tr}_\rho(T) \quad T \in J_1 \tag{1.14}$$

That is, the $\mathrm{tr}_\rho(T)$ are all eigendistributions for $ZU(\mathfrak{g})$. Since as z varies in $ZU(\mathfrak{g})$, the symbol in the sense of P.D.E. will vary through all homogeneous elements of $IP(\mathfrak{g}^*)$, we see that $V(\mathfrak{g}^*)$ is just the intersection of all the characteristic directions of the R_z, $z \in ZU(\mathfrak{g})$. Hence by [D], proposition 5.1.1, we have the inclusions $WF_{\mathrm{tr}_\rho}(T) \subseteq G \times V(\mathfrak{g}^*)$ for all T in J_1. By definition of WF^o_ρ, the inclusion (1.10) follows. □

Remark. We can formulate a relative version of this also. Let $N \subseteq G$ be a normal subgroup. Let $ZU(\mathsf{N})^G$ be the Ad G invariants in $ZU(\mathsf{N})$, where N is the Lie algebra of N. The corresponding sub-algebra of $P(\mathsf{N}^*)$ is clearly $IP(\mathsf{N}^*)^G$, the $\mathrm{Ad}^*\,G$ invariants in $IP(\mathsf{N}^*)$. Let $V(N^*;G)$ be the intersection of the zeroes of the homogeneous elements of positive degree in $IP(\mathsf{N}^*)$. Then by the same proof as for the above propositions, we may assert: If ρ is an irreducible representation of G, and $\rho|_H$ is the
restriction of ρ to N, then **121**

$$WF^o(\rho|H) \subseteq V(\mathsf{N}^*;G) \tag{1.15}$$

Next we observe that WF_ρ behaves very simply under direct sums. If ρ is a representation of G, let $n\rho$, where n is a natural number or ∞, denote the n-fold direct sum of ρ with itself. If ρ_1 and ρ_2 are two representations, recall that ρ_1 and ρ_2 are called quasi-equivalent if $\infty\rho_1$ and $\infty\rho_2$ are equivalent.

Proposition 1.3.

(a) *If ρ_1 and ρ_2 are quasi-equivalent, then $WF^o_{\rho_1} = WF^o_{\rho_2}$.*

(b) *In general $WF^o(\rho_1 \oplus \rho_2) = WF^o_{\rho_1} \cup WF^o_{\rho_2}$*

Proof. To prove (a), it is enough to show that $WF^o_\rho = WF^o(\infty\rho)$; but this is clear because ρ and $\infty\rho$ have the same matrix coefficients. Similarly, the general matrix coefficient of $\rho_1 \oplus \rho_2$ is easily seen to have the form

$$\mathrm{tr}_{\rho_1}(T_1) + \mathrm{tr}_{\rho_2}(T_2), \quad T_i \in J_1(H_i)$$

where H_1 is the space of ρ_i. Setting $T_1 = 0$ and letting T_2 vary, then vice-versa, we see $WF^o_{\rho_i}$ is contained in $WF^o(\rho_1 \oplus \rho_2)$. On the other hand, [D], definition 1.3.1 assures us of the other inclusion necessary for statement (b). This concludes the proposition.

We will now give a technical result offering various descriptions of WF^o_ρ.

Recall that if f is a function of a positive real variable t, then f is *rapidly decreasing* as $t \to \infty$ if

$$\sup\{|f(t)|t^n : t \geq 1\} = \gamma_n(f) < \infty, \quad \text{all } n \text{ in } \mathbf{Z}$$

Let e denote the identity element of G. Let supp(φ) denote the support of $\varphi \in \mathbb{C}^\infty_c(G)$. □

Theorem 1.4. *Let $U \subseteq \mathfrak{g}^*$ be an open set. The following conditions on U are all equivalent*

(i) *$U \cap WF^o_\rho$ is empty*

122 (ii) *For any T in $J_1(H)$, and every real-valued $\psi \in \mathbb{C}^\infty(G)$ such that $d\psi(e) \in U$, there is an open neighborhood V of e such that for any $\varphi \in \mathbb{C}^\infty_c(V)$ the integral*

$$I(\varphi, \psi, T)(t) = \int_G \mathrm{tr}_\rho(T)(g)\varphi(g)e^{it\psi(g)}dg \tag{1.17}$$

is rapidly decreasing as $t \to \infty$. *Furthermore, if* $\psi = \psi_\alpha$ *and* $\varphi = \varphi_\alpha$ *depend smoothly on a parameter* α *varying in a neighborhood of* 0 *in* $\mathbb{R}^k$, *then for some perhaps smaller neighborhood* Y *of* 0 *in* $\mathbb{R}^k$, *the neighborhood* V *and the quantities* $\gamma_n(I(\varphi_\alpha, \psi_\alpha, T))$ *can be chosen independently of* α *in* Y.

(iii) *For all* T *in* J_1, *for all* $\varphi \in \mathbb{C}_c^\infty(G)$ *and for all real-valued* $\psi \in \mathbb{C}_c^\infty(G)$ *such that* $d\psi(\operatorname{supp}\varphi) \subseteq U$, *the integral* $I(\varphi, \psi, T)$ *is rapidly decreasing as* $t \to \infty$. *If* φ *and* ψ *depend on a parameter* α *as in* (ii), *then there is uniformity in* α *as described there.*

(iv) *The same as* (iii), *but is enough to choose an open neighborhood* V *of* e *and choose* $\varphi \in \mathbb{C}_c^\infty(V)$.

(v) *The same as* (iii), *but we have the estimates*

$$\gamma_n(I(\varphi, \psi, T)) \leq c_n(\varphi, \psi)\|T\|_1 \tag{1.18}$$

for some number $c_n(\varphi, \psi)$. *If* α *is an auxiliary parameter as described in* (ii), *then the numbers* $c_n(\varphi_\alpha, \psi_\alpha)$ *may be bounded uniformly on compact sets of* α*'s.*

(vi) *For* $\varphi \in \mathbb{C}_c^\infty(G)$ *and real-valued* $\psi \in \mathbb{C}^\infty(G)$ *such that* $d\psi(\operatorname{supp}\varphi) \subseteq U$, *the norm of the operator* $\rho(\varphi e^{it\phi})$ *is rapidly decreasing as* $t \to \infty$. *If* φ *and* ψ *depend on a parameter* α *as in* (ii), *then the quantities* $\gamma_n(\|\rho(\varphi e^{it\psi}\|)$ *can be bounded uniformly on compact sets of* α.

(vii) *Same as* (vi), *except it is enough to choose a neighborhood* V *of* e *and varify* (vi) *for* $\varphi \in \mathbb{C}_c^\infty(V)$.

Proof. First we will check that statements (ii) though (vii) are equivalent, then we will compare them with (i). It is immediate that (v) implies (iii) and that (iii) implies (iv). Likewise (vi) clearly implies (vii). Also, in view of formula (1.3) and the duality between $J_1(H)$ and the space $L(H)$ of all bounded operators on H, we see that (v) and (vi) are equivalent. If $\psi \in \mathbb{C}^\infty(G)$, and $d\psi(e) \in U$, then $d\psi^{-1}(U) = V_1$ is a neighborhood of e in G. If V is as in (iv), then $V \cap V_1$ will be a neighborhood **123**
that works for (ii). Hence (iv) implies (ii).

Fix $\varphi \in \mathbb{C}_c^\infty(G)$ and $\psi \in \mathbb{C}^\infty(G)$. Suppose for any $T \in J_1$, the integral $I(\varphi, \psi, T)(t)$ is rapidly decreasing as $t \to \infty$. For any n, and number $a > 0$, the set X_a of T such that $\gamma_n(I(\varphi, \psi, T)) \leq a$ is convex and symmetric around 0. Since $I(\varphi, \psi, T)(t)$ is continuous on J_1, we see that X_a is also closed. Since $X_{ab} = bX_a$, and $\bigcup_{a \geq 0} X_a = J_1$ by assumption, we see X_a contains a neighborhood of the origin in J_1. Thus we see that (iii) implies (v).

We will show that (ii) implies (iii) by a partition of unity argument. Observe the identity

$$I(\varphi, \psi, T) = I(L_g\varphi, L_g\psi, T\rho(g)^{-1}). \tag{1.19}$$

This follows from the definition of $I(\varphi, \psi, T)$ and formula (1.5). Suppose that $d\psi(\operatorname{supp}\varphi) \subseteq U$, so that φ and ψ satisfy the hypotheses of (iii). By formula (1.8), we see $d(L_{g-1}\psi)(e) \in U$ if $g \in \operatorname{supp}\varphi$. Then (ii) tells us that given $T \in J_1$, there is a neighborhood $V = V(L_{g-1}\psi, T\rho(g))$ such that if $\varphi' \in \mathbb{C}_c^\infty(V)$, then $I(\varphi', L_{g-1}\psi, T\rho(g))$ is rapidly decreasing. From (1.19) we can conclude that for $g \in \operatorname{supp}\varphi$, there is a neighborhood V_g of g such that if $\varphi'' \in \mathbb{C}_c^\infty(V_g)$, then $I(\varphi'', \psi, T)$ is rapidly decreasing. We can cover $\operatorname{supp}\varphi$ with a finite number of the neighborhoods V_g, and construct a partition of unity subordinate to this cover of $\operatorname{supp}\varphi$. That is, we can find g_i such that the V_{g_i} cover $\operatorname{supp}\varphi$, and we can find $\varphi_i'' \in \mathbb{C}_c^\infty(V_{g_i})$ such that $\sum_i \varphi_i'' = 1$ on $\operatorname{supp}\varphi$. Then

$$I(\varphi, \psi, T) = \sum_i I(\varphi\varphi_i'', \psi, T)$$

so $I(\varphi, \psi, T)$ is rapidly decreasing. Clearly we can do this uniformly in some auxiliary parameter α. Thus we see that (ii) implies (iii). A completely analogous, slightly simpler argument shows that (vii) implies (vi). Hence all conditions (ii) through (vii) are equivalent.

Finally we observe that by [D], proposition 1.3.2, for any point $\lambda \in U$, the condition that the point $(e, \lambda) \in T^*G$ not belong to $WF(\operatorname{tr}_\rho T)$ for $T \in J_1$, is just statement (ii) restricted to those ψ such that $d\psi(e) = \lambda$ (and with parameter α). Thus we see that (i) implies (ii). Conversely,

(ii) certainly implies that $(e, \lambda) \notin WF(\mathrm{tr}_\rho T)$ for any $T \in J_1$ and any
$\lambda \in U$. Since U is open and WF_ρ is G-biinvariant, we find that also (ii) **124**
implies (i). Thus the theorem is proved.

Using theorem 1.4 we can establish a relation between the wave front set of a representation and that of its restriction to a subgroup. Let $H \subseteq G$ be a Lie subgroup of H, with Lie algebra h. We have the restriction map

$$q : \mathfrak{g}^* \to \mathsf{h}^*$$

Note that q is $\mathrm{Ad}^*\, H$-equivariant, and if H is normal in G, then q is $\mathrm{Ad}^*\, G$-equivariant. □

Proposition 1.5. *We have the inclusion*

$$q(WF^o_\rho) \subseteq WF^o(\rho\backslash H) \tag{1.21}$$

Proof. Let U be an open subset of h^* not intersecting $WF^o(\rho|H)$. Choose a small neighborhood V of the identity in G so that in VH there is a smooth cross-section Y to H, so we can write uniquely

$$v = yh \quad v \in V,\ y \in Y,\ h \in H.$$

Choose $\varphi \in \mathbb{C}^\infty_c(V)$ and let $\psi \in \mathbb{C}^\infty(V)$ be such that $d\psi \subseteq q^{-1}(U)$. We can compute

$$\begin{aligned} \rho(\varphi e^{it\psi}) &= \int_G \varphi(g) e^{it\psi(g)} \rho(g) dg \qquad (1.22)\\ &= \int_Y \int_H \varphi(yh) e^{it\psi(yh)} \rho(yh) dy dh \\ &= \int_Y \rho(y) \left(\rho|H(\varphi_y e^{it\psi} y)\right) dy \end{aligned}$$

where we have written

$$\varphi_y(h) = \varphi(yh) \quad \psi_y(h) = \psi(yh)$$

As y varies, φ_y varies smoothly in $\mathbb{C}_c^\infty(H)$ and ψ varies smoothly in $\mathbb{C}^\infty(H)$, with $d\psi_y(\operatorname{supp}\varphi_y) \subseteq U$. Hence by theorem 1.4, part (vi), the norms of the operators

$$\rho|H(\varphi_y e^{it\psi_y})$$

125 are rapidly decreasing as $t \to \infty$, with uniform estimates at least locally in y. Since $\varphi_y \equiv 0$ for y outside a compact set we see from (22) that $\rho(\varphi e^{it\psi})$ is also rapidly decreasing at $t \to \infty$, whence $q^{-1}(U)$ is disjoint from WF^o_ρ, by theorem 4, part (vii).

An interesting aspect of proposition 1.5 is that it proceeds in the opposite direction from the standard results ([D] proposition 1.33, see also [**?**], section 1X.9) concerning restrictions of distributions and wave front sets. This contrast allows us to prove a partial converse to proposition 1.5.

Let $\mathsf{h}^\perp$ be the kernel of the projection map q of (**??**). We will say H is *crosswise to* ρ if $\mathsf{h}^\perp \cap WF^o_\rho = \{0\}$. When H is crosswise to ρ we can, according to [D], proposition 1.3.3, restrict $\operatorname{tr}_\rho T$ (or any of its derivatives) to H. The wave front set of $(\operatorname{tr}_\rho T)|_H$, which will be the same as the wave front set of $\operatorname{tr}_\rho |_H(T)$, will then be contained in $q(WF(\operatorname{tr}_\rho T))$. Combining this with proposition 1.5 we may assert. □

Proposition 1.6. *If H is crosswise to WF^o_ρ, then*

$$WF^o(\rho|H) = q(WF^o_\rho) \tag{1.23}$$

Note that if H is crosswise to F, the projection $q(WF_\rho)$ will be closed.

In particular if $\mathsf{h}^\perp \cap V(\mathfrak{g}^*) = \{0\}$, then H will be crosswise to all irreducible ρ.

Proposition 1.5 also implies a restriction on the wave front set of (outer) tensor products. Let G_1 and G_2 be two Lie groups, and ρ_i unitary representations of G_i on spaces H_i. We can form the tensor product representation $\rho_1 \otimes \rho_2$ of $G_1 \times G_2$ on $H_1 \times H_2$.

Proposition 1.7. *We have the inclusion*

$$WF^o(\rho_1 \otimes \rho_2) \subseteq WF^o_{\rho_1} \times WF^o_{\rho_2} \subseteq \mathfrak{g}_1 \times \mathfrak{g}_2 \tag{1.24}$$

Proof. We have $(\rho_1 \otimes \rho_2)|_{G_1} \simeq (\dim \rho_2)\rho_1$. Hence by propositions 1.5 and 1.3, we see

$$WF^o(\rho_1 \otimes \rho_2) \subseteq WF^o_{\rho_1} \times \mathfrak{g}_2^*.$$

Interchanging G_1 and G_2, repeating and intersecting gives (24). **126**

We remark that the inclusion **??** can be strict. An example of this will be found in part II.

For certain representations there is a plausible alternate definition of wave front set. We consider this and compare it with our first notion given above. Recall that there is an antiautomorphism* on $U(\mathfrak{g})$ defined property that it is -1 on $\mathfrak{g}$:

$$x^* = -x \quad x \in \mathfrak{g}.$$

If ρ is a unitary representation of G, then

$$\rho(u^*) = \rho(u)^* \quad u \in U(\mathfrak{g}) \tag{1.26}$$

where the* on the right-hand side indicates the restriction of the adjoint of $\rho(u)$ to the space of smooth vectors of ρ. Thus if $u = u^*$, then $\rho(u)$ is a symmetric operator, and elements of the form u^*u are mapped to non-negative symmetric operators, and so are sums of such elements. We call sums $\sum u_i^* u_i$ in $U(\mathfrak{g})$ *formally positive*. Evidently the formally positive elements form a cone in $U(\mathfrak{g})$, invariant by*.

In the following discussion we take G to be unimodular for convenience.

We will say that ρ is of *strong trace class* if there is some formally positive element v of $U(\mathfrak{g})$ such that $\rho(v)$ (with domain understood to the the smooth vectors of ρ) is essentially self-adjoint, and invertible with trace class inverse. We note irreducible representations are often of strong trace class.

If ρ is of strong trace class, then for all φ in $\mathbb{C}_c^\infty(G)$, the operator $\rho(\varphi)$ will be trace class, with trace norm satisfying

$$\|\rho(\varphi)\|_1 \le \|\rho(v)^{-1}\|_1 \|\rho(R_v\varphi)\| \le \|\rho(v)^{-1}\|_1 \|R_v\varphi\|_1 \tag{1.27}$$

where $\|\rho(\varphi)\|_1$ indicates the trace norm on $J_1(H)$, and $v \in U(\mathfrak{g})$ is a formally positive element which makes ρ strongly trace class, and R_v is the left-invariant operator on G corresponding to v, and $\|\rho(R_v\varphi)\|$ is the usual operator norm of $\rho(R_v\varphi)$ and $\|R_v\varphi\|_1$ is the L^1-norm of $R_v\varphi$ as a function on G. It is clear from (1.27) that the trace linear functional

$$\chi_\rho(\varphi) = \mathrm{tr}_\rho(\varphi) \tag{1.28}$$

127 is a distribution on G. We of course call it the *character* of ρ. We note that χ_ρ is a conjugation invariant distribution, in the sense that

$$\chi_\rho(\mathrm{Ad}\, g(\varphi)) = \chi_\rho(\varphi) \tag{1.29}$$

where $\mathrm{Ad}\, g(\varphi) = L_g R_g(\varphi)$.

If ρ is of strong trace class, so that its character χ_ρ is well-defined as a distribution, then in the context of this paper, an obvious thing to do is to consider the wave front set $WF(\chi_\rho)$. This will be a conjugation invariant set in T^*G. In particular the intersection of $WF(\chi_\rho)$ with the cotangent space at the identity, which is canonically identifiable with $\mathfrak{g}^*$, defines a closed, $\mathrm{Ad}^*\, G$-invariant, conical set in $\mathfrak{g}^*$. Denote this set by $WF^o(\chi_\rho)$. It is natural to compare this with our WF^o_ρ defined earlier. □

Theorem 1.8. *When ρ is of strong trace class with distributional character χ_ρ, we have*

$$WF^o(\chi_\rho) = WF^o_\rho. \tag{1.30}$$

Proof. Let v be a formally positive element of $U(\mathfrak{g})$ with respect to which ρ is strongly trace class. Write $\rho(v)^{-1} = T \in J_1(H)$. Then for φ in $\mathbb{C}_c^\infty(G)$ we have

$$\begin{aligned}\chi_\rho(\varphi) &= \mathrm{tr}(\rho(\varphi)) = \mathrm{tr}(T_\rho(v)\rho(\varphi))\\ &= \mathrm{tr}(\rho(L_v(\varphi))T) = \mathrm{tr}_\rho(T)(L_v\varphi)\\ &= L_{v^*}(\mathrm{tr}_\rho(T))(\varphi).\end{aligned}$$

In other words

$$\chi_\rho = L_{v^*}(\mathrm{tr}_\rho(T)) \tag{1.31}$$

Since action by differential operators does not increase the wave front set, we see

$$WF(\chi_\rho) \subseteq WF(\mathrm{tr}_\rho(T)) \subseteq WF_\rho.$$

Hence, looking at the fibre of T^*G over the identity of G we see that the left side of (1.30) is contained in the right side.

To prove the reverse inclusion, consider a point p in $\mathfrak{g}^* - WF^o(\chi_\rho)$.

Let U be a neighborhood of p with compact closure disjoint from 128
$WF^o(\chi_\rho)$. Since $WF(\chi_\rho)$ is closed in T^*G, there is a neighborhood V of the identity e in G such that $V \times U \subseteq T^*G$ is disjoint from $WF(\chi_\rho)$. It follows that for φ in $\mathbb{C}_c^\infty(V)$ and real-valued ψ in $\mathbb{C}^\infty(V)$ such that $d\psi$ $(\operatorname{supp}\varphi) \subseteq U$, one has that $\chi_\rho(\varphi e^{it\psi})$ is rapidly decreasing as $t \to \infty$, with estimates uniform in smooth parametrized families of φ's and ψ's. Let V_1 be a symmetric neighborhood of e such that $V_1^2 \subseteq V$. Then if $\varphi \in \mathbb{C}_c^\infty(V_1)$, we see that $\chi_\rho(L_g(\varphi e^{it\psi}))$ is rapidly decreasing in t, uniformly in g in V_1 and in any other auxiliary parameter of interest. Set

$$\varphi e^{it\psi} = \varphi_t \quad \text{and} \quad \varphi_t^*(g) = \overline{\varphi}_t(g^{-1})$$

where —— indicates complex conjugation.

Integrating, we find

$$\int\limits_G \overline{\varphi}_t(g^{-1})\chi_\rho(L_g\varphi_1)dg = \int\limits_G \chi_\rho(\overline{\varphi}_t(g^{-1})(L_g\varphi_t)dg \tag{1.32}$$

$$= \chi_\rho(\varphi_t^* * \varphi_t) = \mathrm{tr}(\rho(\varphi_t^* * \varphi_t)) = \mathrm{tr}(\rho(\varphi_t)^*\rho(\varphi_t))$$

is rapidly decreasing as $t \to \infty$. Here $\varphi_t^* * \varphi_t$ indicates the convolution of these functions. But the final expression in (1.32) is just the Hilbert-Schmidt norm of $\rho(\varphi_t)$. Since it is rapidly decreasing, the operator norm of $\rho(\varphi_t)$ is also. Hence criterion (vii) of Theorem 1.4 tells us U is disjoint from WF_ρ^o, and Theorem 1.8 is established. □

Before concluding this section, let us mention two plausible general properties of wave front sets not established here. First, is it true that $WF^o(\rho_1 \otimes \rho_2) \subseteq \overline{(WF_{\rho_1}^o + WF_{\rho_2}^o)}$ (the — here denoting closure) for an inner tensor product? Second, is it true that $WF^o(\mathrm{ind}_H^G \sigma) \supseteq \mathfrak{h}^\perp$?

2 Examples

Here we will show how to compute WF^o_ρ for various familiar classes of groups, and examine the possibilities for WF^o_ρ in some interesting cases.

A. Abelian Groups.

If G is abelian, then G is a homomorphic image of a vector space V, so we may as well assume $G = V$. Then we may identify V with its
129 Lie algebra. Also the dual vector space V^* can be identified with $\widehat{V}$, the Pontrjagin dual of V, by the usual method. Define

$$\alpha : V^* \to \widehat{V}$$

by

$$\alpha(\lambda)(v) = e^{2\pi i\lambda(v)} \quad \lambda \in V^*, v \in V. \tag{2.1}$$

Define Fourier transform from $L_1(V)$ to $\mathbb{C}_0(V^*)$ by the usual recipe:

$$\widehat{\varphi}(\lambda) = \int_V \varphi(v)e^{-2\pi i\lambda(v)}dv \quad \varphi \in L_1(V),\ \lambda \in V^* \tag{2.2}$$

Then the inverse Fourier transform is

$$\widehat{f}^{-1}(v) = \int_{V^*} f(\lambda)e^{2\pi i\lambda(v)}dv \quad f \in L_1(V^*),\ v \in V \tag{2.3}$$

Let ρ be a unitary representation of V on the Hilbert space H. Take $T \in J_1(H)$, and consider the matrix coefficient $\mathrm{tr}_\rho(T)$. According to Bochner's Theorem [R-S], $\mathrm{tr}_\rho(T)^\vee$ exits as a finite measure on V^*, positive if T is. Moreover from our formulas (1.5) and (2.2) we can compute that

$$\mathrm{tr}_\rho(\rho(\varphi)T)\widehat{\ } = (\check{\varphi})\widehat{\ }\,(\mathrm{tr}_\rho\, T)\widehat{\ } \tag{2.4}$$

where

$$\varphi(v) = \check{\varphi}(-v) \tag{2.5}$$

We define $\mathrm{supp}\,\rho$ to be the closure of the union of the supports of the measures $(\mathrm{tr}(T))\widehat{\ }$. It is clear from (2.4) that

$$\|\rho(\varphi)\| = \sup\{(\check{\varphi})\widehat{\ }(\lambda) : \lambda \in \mathrm{supp}\,\rho\} \tag{2.6}$$

Given a set S in a vector space U, define $AC(S)$, the *asymptotic cone* of S as follows. Given u in U, if any cone containing a neighborhood of u intersects S in an unbounded set, then u is in $AC(S)$.

In terms of these objects we can give the not unexpected description of WF_ρ.

Proposition 2.1. *For a unitary representation ρ of a vector space V, one has*

$$WF_\rho^o = -AC(\operatorname{supp}\varphi) \tag{2.7}$$

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Remark. The minus sign in (2.7) is an artifact of our conventions and could be eliminated by appropriate juggling.

Proof. We will apply criterion (vi) of Theorem 1.4, with $\psi = 2\pi\lambda$, $\lambda \in V^*$. (We will actually use the definition 1.3.1 of [D] rather than proposition 1.3.2 used for Theorem 4). Take φ in $\mathbb{C}_c^\infty(V)$. Then one sees from (2.2) that

$$\left((\varphi e^{2\pi i\lambda})^\vee\right)^\frown(\lambda') = (\check{\varphi})^\frown(\lambda' + t\lambda) \tag{2.8}$$

Suppose that $\lambda_0 \notin -AC(\operatorname{supp}\rho)$. Then we can choose a small neighborhood U of λ_0 such that the distance between $-t\lambda$ and $\operatorname{supp}\rho$ (in any convenient norm) increases linearly in t. Therefore $-t\lambda$ has a ball around it of size $\geq \gamma t$, γ being some constant independent of λ, disjoint from $\operatorname{supp}\rho$. Since $(\check{\varphi})^\frown$is rapidly decreasing for $\varphi \in \mathbb{C}_c^\infty(V)$, we see from formulas (2.6) and (2.8) that $\|\rho(\varphi e^{2\pi it\lambda})\|$ decreases rapidly as $t \to \infty$. This shows the left side of (2.7) is contained in the right side. The reverse inclusion is equally easy. If $-\lambda_0 \in AC(\operatorname{supp}\rho)$, then no matter how small a neighborhood U of λ_0 we choose, the cone on $-U$ will intersect $\operatorname{supp}\rho$ in a non-bounded set. This means we can choose t arbitrarily large, and λ_1 in U, such that $-t\lambda_1$, is in $\operatorname{supp}\rho$. We may assume for convenience that φ is positive-definite, so that $\widehat{\varphi}(0) = \|\widehat{\varphi}\|_\infty$. Then we see that $\|\rho(\varphi e^{2\pi it\lambda_1})\| = \|\widehat{\varphi}\|_\infty$ by formulas (2.6) and (2.8), so that U violates condition (vi) of Theorem 1.4. Hence the right side of (2.7) is contained in the left side, and the proposition is proved. $\square$

We will use proposition 2.1 to give an example of strict inclusion in proposition 1.7. Let $V = \mathbb{R}$, and let N be the direct sum of the characters

$$t \to e^{2\pi i n! t} \quad n \geq 1$$

Then

$$\operatorname{supp}\rho = \{n!, n \in \mathbb{Z}^+\}$$

Hence by proposition 2.1, we have

$$WF^o(\rho) = -AC(\operatorname{supp}\rho) = \mathbb{R}^- = \{t \in \mathbb{R}, t \leq 0\}.$$

131 Consider the tensor product $\rho \otimes \rho$ as a representation of $\mathbb{R}^2$. Then clearly

$$\operatorname{supp}(\rho \otimes \rho) = \{(n!, m!) : n, m \in \mathbb{Z}^+\}.$$

It is easy to see that $AC(\operatorname{supp}(\rho \otimes \rho))$ consists of the positive x-axis, the positive y-axis, and the positive ray of the 45° line $x = y$. Thus $WF^o(\rho \otimes \rho)$, being the negatives of these 3 rays, is properly contained in $WF^o_\rho \times WF^o_\rho$, which is the whole southwest quadrant.

B: Nilpotent Groups.

We will discuss only irreducible representations of general nilpotent groups. Let N be a nilpotent Lie group, assumed to be connected and simply connected for simplicity. Let N be its Lie algebra, and $\exp : \mathsf{N} \to N$ the exponential map. Let ρ be an irreducible representation of N. It is known that ρ is of strong trace class, and according to the orbit theory of Kirillov [K], there is an $\mathrm{Ad}^* N$ orbit $O(\rho) = O$ in N^*, such that

$$\chi_\rho(\varphi) = \int_O (\varphi O \exp)^\frown (\lambda) do(\lambda) \quad \varphi \in \mathbb{C}_c^\infty(N) \tag{2.9}$$

where χ_ρ is the character of ρ, as in (1.27), and $\frown$ is as in (2.2), and do is a properly normalized $\mathrm{Ad}^* N$ invariant measure on O. Given formula (2.9) and theorem 1.8, it is an easy matter to establish the following result. We omit the details.

Proposition 2.2. *If ρ is an irreducible representation of N, and $O \subseteq \mathsf{N}^*$ is the associated orbit, then*

$$WF^o_\rho = -AC(O). \tag{2.10}$$

C. Compact Groups.

Let K be a compact connected Lie group, and let $T \subseteq K$ be a maximal torus. Let W be the Weyl group of T, the normalizer of T modulo the centralizer of T. Let t and k be the Lie algebras of T and K. If K is semi-simple we can identify k with k^* via the Killing form. In general, we will suppose given some $\mathrm{Ad}\, K$-invariant, negative definite, bilinear form on k allowing us to identify k and k^*. Then we can also identify t and t^*, and may regard t^* as a subspace of k^*, and we will have

$$\mathrm{Ad}^*\, K(\mathsf{t}^*) = \mathsf{k}^* \tag{2.11}$$

Thus any $\mathrm{Ad}^*\, K$ invariant set in k^* is determined by its intersection with **132**
t^*, and this intersection will be a Weyl group invariant set. Fix a Weyl chamber C^+ in t^*, and fix an ordering of the roots of t by letting this chosen Weyl chamber be positive. We have

$$\mathrm{Ad}^*\, W(C^+) = \mathsf{t}^* \tag{2.12}$$

so that an $\mathrm{Ad}^*\, K$ invariant set in k^* is determined by its intersection with C^+.

The irreducible representations of K are described by the celebrated highest weight theory of Cartan and Weyl. Let $\widehat{T}$ be the character group of T. Since T is a quotient of t via the exponential map, we can as described in paragraph IIA identify $\widehat{T}$ with a lattice in t^*, the so-called lattice of weights. The intersection

$$\widehat{T}^+ = \widehat{T} \cap C^+$$

is called the set of dominant weights. The dominant weights parametrize the set $\widehat{K}$ of irreducible unitary representations of K. We recall how.

Let $\mathsf{k}_{\mathbb{C}}$ be the complexification of k. We can write

$$\mathsf{k}_{\mathbb{C}} = \mathsf{t}_{\mathbb{C}} \otimes \sum_{\alpha} L_{\alpha} \tag{2.13}$$

where the L_{α} are the root spaces, that is, the non-trivial eigenspaces of $\mathrm{Ad}\,T$ acting on $\mathsf{k}_{\mathbb{C}}$. We parametrize L_{α} by the character α it defines, and we regard α as an element of t^* as explained above. We call a root α positive if $(\alpha, c) \leq 0$ for all $c \in C^+$, where $(,)$ is the posited bilinear form by means of which we identified k and k^*.

Denote the set of positive roots by $\sum^+$. Put

$$\mathsf{N}^+ = \bigoplus_{\alpha \in \sum^+} L_{\alpha} \tag{2.14}$$

Then N^+ is a nilpotent subalgebra of $\mathsf{k}_{\mathbb{C}}$, and it is known that

$$\mathsf{k}_{\mathbb{C}} = \mathsf{t}_{\mathbb{C}} \oplus \mathsf{N}^+ \oplus \mathsf{N}^- \tag{2.15}$$

where N^- is the image of N^+ under complex conjugation in $\mathsf{k}_{\mathbb{C}}$. Let ρ be a representation of K on a Hilbert space H. Denote by H^+ the subspace of H annihilated by all elements of N^+. The space H^+ is the space of highest weight vectors for ρ. Clearly H is invariant by $\rho(T)$, so it may
133 be decomposed into a direct sum

$$H^+ = \sum_{\gamma} H^+_{\gamma} \tag{2.16}$$

where H^+_{γ} is the eigenspace of T on which T acts by the character $\gamma \in \widehat{T}$. The highest weight theory asserts the following facts:

(i) Each γ is in $\widehat{T}^+$

(ii) If ρ is irreducible, then $\dim H^+ = 1$, so that $H^+ = H^+_{\gamma}$ for some well-defined γ.

(iii) The map from $\widehat{K}$ to $\widehat{T}^+$ implied by (ii) is a bijection.

Now consider an arbitrary unitary representation ρ of K. Denote the set of highest weights of ρ by $\operatorname{supp}\rho$. Thus

$$\operatorname{supp}\rho \subseteq \widehat{T}^{+} \subseteq C^{+}$$

The following result is very closely akin to results in [K-V].

Proposition 2.3. *For a unitary representation ρ of K, we have*

$$-WF^{o}_{\rho} \cap C^{+} = AC(\operatorname{supp}\rho), \quad \text{or} \tag{2.17}$$
$$WF^{o}_{\rho} = \mathrm{Ad}^{*}K(-AC(\operatorname{supp}\rho))$$

Proof. By proposition 1.3, it suffices to prove this when ρ is multiplicity free, that is, when ρ contains only one copy of each of its irreducible constituents. Then H^{+} (the space of ρ being H as usual) will be multiplicity free under the action of T. Let σ denote the representation of T on H^{+}. Then by definition $\operatorname{supp}\sigma = \operatorname{supp}\rho$.

Let x and y be two vectors in H^{+}. Consider the matrix coefficient $\mathrm{tr}_{\rho}(E_{x,y})$. Since the intersection of the characteristics of the elements of N^{+} is just $\mathfrak{t}^{*}$, and since $\mathrm{tr}_{\rho}(E_{x,y})$ is annihilated by N^{+} (acting either on the right or the left), we see by [D], proposition 5.1.1, that the wave-front set of $\mathrm{tr}_{\rho}(E_{x,y})$ at the identity of K is contained in $\mathfrak{t}^{*}$. This also implies by [D], proposition 1.3.3, that $\mathrm{tr}_{\rho}(E_{x,y})$ restricts to T; this restriction must of course just equal to $\mathrm{tr}_{\sigma}(E_{x,y})$. One then has again by [D], proposition 1.3.3.

$$WF^{o}(\mathrm{tr}_{\sigma}(E_{x,y})) \subseteq WF^{o}(\mathrm{tr}_{\rho}(E_{x,y})) \subseteq WF^{o}_{\rho} \tag{2.18}$$

where in the first two expressions the ^{o} in WF^{o} mean we are looking at the fibre over the identity in K. From (2.18) we immediately have **134**

$$WF^{o}\sigma \subseteq WF^{o}_{\rho} \tag{2.19}$$

Since WF^{o} is $\mathrm{Ad}^{*}K$ invariant, we see by proposition 2.1 that the left side of (2.17) contains the right side.

On the other hand, since ρ is multiplicity free, it is of strong trace class, so to compute WF^{o}_{ρ}, it is enough by Theorem 1.8 to compute

$WF^o(\chi_\rho)$. Let Δ be the element in $U(\mathsf{k})$ corresponding to our given bilinear form. Then R_Δ is elliptic, and $\rho(1+\Delta)$ is positive definite, and some power of $\rho(1+\Delta)$ has trace class inverse. Standard and straightforward arguments allow us to find $x \in H^+$ such that for some sufficiently large l we have

$$\chi_\rho = R_{(1+\Delta)}l \int_K \mathrm{Ad}\, K(\mathrm{tr}_\rho(E_{x,x}))dk \tag{2.20}$$

Since R_Δ is elliptic, we have

$$\begin{aligned} WF(\chi_\rho) &= WF\left(\int_K \mathrm{Ad}\, K(\mathrm{tr}_\rho(E_{x,x}))dk\right) \\ &\subseteq \mathrm{Ad}\, K(WF(\mathrm{tr}_\rho(E_{x,x}))) \end{aligned} \tag{2.21}$$

Hence if we can show

$$WF^o(\mathrm{tr}_\rho(E_{x,x})) \subseteq WF^o\sigma \tag{2.22}$$

we will be done. In fact (2.22) is proven in just the same manner as proposition 1.5. The reasoning is exactly the same as in equation (1.22), except instead of considering simply $\rho(\varphi e^{it\psi})$, one looks at the product $\rho(\varphi e^{it\psi})E_{x,x}$. □

D: Semisimple Groups.

We come now to the motivating examples of this paper. Let G be a semisimple Lie group with finite center and with Iwasawa decomposition

$$G = KAN \quad \mathfrak{g} = \mathsf{k} \oplus \mathsf{a} \oplus \mathsf{N} \tag{2.23}$$

One also has the Cartan decomposition

$$\mathfrak{g} = \mathsf{k} \oplus \mathsf{p} = \mathsf{k} \oplus \mathrm{Ad}\, K(\mathsf{a}) \tag{2.24}$$

where p is the orthogonal complement of k with respect to the Killing 135
form of $\mathfrak{g}$. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form. Thus in what follows we will speak of $\mathfrak{g}$ when strictly we should say $\mathfrak{g}^*$.

Let N be the nilpotent set of $\mathfrak{g}$. It is well known that $N = V(\mathfrak{g})$ is the characteristic variety of $\mathfrak{g}$, in the sense of proposition 1.2. It is also known that there are only finitely many conjugacy classes of nilpotent elements. Thus from proposition 1.2 we have the following result.

Proposition 2.4. *If ρ is an irreducible unitary representation of G, then*

$$WF^o_\rho \subseteq N. \tag{2.25}$$

In particular, there are only finitely many possibilities for WF^o_ρ.

Let ρ be irreducible, and consider ρ/K. It is a classic result of Harish-Chandra (see [W]) that ρ/K contains each irreducible representation of K a finite number of times, and that in fact ρ/K is of strong trace class. These facts are also reflected in the behavior of wave front sets. From the Cartan decomposition (2.24), noting that p consists of semisimple elements, we see that k is crosswise to WF^o_ρ in the sense of proposition 1.6. Thus we have the following immediate consequence of that result.

Proposition 2.5.

(a) *For irreducible ρ we have*

$$WF^o(\rho/K) = q(WF^o\rho) \tag{2.26}$$

where q is orthogonal projection of $\mathfrak{g}$ onto k (with kernel p).

(b) *In particular $WF^o(\rho/K)$ is the orthogonal projection on K of certain nilpotent orbits in $\mathfrak{g}$, and is one of only finitely many possibilities.*

Remark. Part (b) of this proposition is very similar to proposition of [K-V]. However, the proof is substantially different from the proof in [K-V]. Also, Kashiwara-Vergne do not relate (their version of) $WF^o(\rho/K)$ to an object on G attached intrinsically to ρ. We note that WF^o_ρ is a finer

invariant than $WF^o(\rho/K)$, as simple examples already on $SL_3(\mathbb{R})$ show. (However WF is not finer than ρ/K, when multiplicities are taken into account. It seems to be roughly equivalent to $WF^o(\rho/K)$ plus some rough information on multiplicities. See the discussion below of the analogy with the *op*-adic case). Furthermore WF^o_ρ provides a link be-
136 tween the N-spectrum and the K-spectrum of ρ, empirical observation of which was the original motivation of Kashiwara-Vergne. Indeed applying proposition 1.5 to N, and using proposition 2.5 we arrive at the following fact. Note that the Killing form induces the identification

$$\mathsf{N}^* \simeq \mathfrak{g}/(\mathsf{a} \oplus \mathsf{N}) \tag{2.27}$$

Let

$$q' : \mathfrak{g} \to \mathfrak{g}/(\mathsf{a} \oplus \mathsf{N}) \tag{2.28}$$

be the natural quotient map.

Proposition 2.6. *We have the inclusion*

$$WF^o(\rho/K) \subseteq q(q'^{-1}(WF^o(\rho/N)) \tag{2.29}$$

Acutally, this proposition is true with N replaced by any subgroup of G.

To illustrate the above results, we offer some observations about the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R}) = \mathrm{Sp}$. This is the subgroup of GL_{2n} preserving the standard symplectic form on $\mathbb{R}^{2n}$. Similar ideas apply to other classical groups. Each element of sp may be regarded as a linear transformation on $\mathbb{R}^{2n}$ in the obvious way, and as such may be assigned its rank, a positive integer. Given an irreducible representation ρ of Sp, we define the *singular rank* of ρ to be the maximum of the rank of the elements of WF^o_ρ.

There are other useful notions of rank for ρ also. Let X be a maximal isotropic subspace of $\mathbb{R}^{2n}$, and let N_1 be the subgroup of Sp that leaves X fixed pointwise. It is well known that $N_1 \simeq S^2(X)$, the second symmetric power of X. Hence N_1^* is identifiable to the space of symmetric bilinear forms on X, and each of its elements has a well-defined rank.

We will say a representation ρ of Sp has N_1-*rank* j if $WF^o(\rho/N_1)$ contains elements of rank j, but none of rank greater than j. By means of Proposition 2.1 and a little symplectic geometry, the notion of N_1-rank can be made considerably more concrete. It is discussed at more length in [H2].

The singular rank of ρ can vary from zero to $2n - 1$, while the N_1-rank can vary only from zero to n. However, within their common range, they are closely related.

Proposition 2.7. *Given irreducible ρ with N_1 rank less than n, one has* 137
the inequality

$$\text{singular rank } (\rho) \leq N_1 \text{ rank } (\rho). \tag{2.30}$$

Probably (2.30) should be an equality.

Proof. By applying proposition 1.5, we reduce the proof to an exercize in symplectic geometry. Let Y be an isotropic subspace of $\mathbb{R}^{2n}$ complementary to X. Let N_1^- be the subalgebra of sp annihilating Y, and let m be the subalgebra of sp stabilizing X and Y. Then

$$\mathsf{sp} = N_1^- \oplus \mathsf{m} \oplus \mathsf{N}_1. \tag{2.31}$$

Also the orthogonal complement of N_1 in sp with respect to the Killing form is just $\mathsf{m} \oplus \mathsf{N}_1$, so we may identify N_1^- with N_1^*. Thus what we want to show is that in a nilpotent Ad Sp orbit in sp consisting of elements of rank $l < n$, there are elements whose N_1^- component in the decomposition (2.31) has rank l also. The rank of the N_1^- component of $s \in \mathsf{sp}$ is easily seen to be

$$\text{rank}(s/X) - \dim(s(X) \cap X) \tag{2.32}$$

Hence, reversing the roles of s and X, it will suffice, given s of rank $l < n$ to find a maximal isotropic X such that (2.32) also equals l. This entails

$$\text{rank}(s/X) = l \quad s(X) \cap X = 0. \tag{2.33}$$

Consider the action of s. It is elementary that

$$\text{im } s = (\ker s)^{\perp}$$

where $^\perp$ indicates orthogonal complement for the standard symplectic form on $\mathbb{R}^{2n}$. Hence $Z = \mathrm{im}\ s \cap \ker s$ is isotropic, and

$$\mathrm{im}\ s + \ker s = Z^\perp.$$

We can write

$$\mathbb{R}^{2n} = Z \oplus V_1 \oplus V_2 \oplus \widetilde{Z}$$

where V_1 is a complement to Z in im s, and V_2 is a complement to Z in $\ker s$, and $\widetilde{Z}$ is an isotropic complement to Z in $(V_1 \oplus V_2)^\perp$. The
138 assumption that rank $s < n$ implies $\dim V_1 < \dim V_2$. Hence we may choose an embedding $\alpha : V_1 \to V_2$ such that $\langle \alpha(v_1), \alpha(v_1') \rangle = -\langle v_1, v_1' \rangle$, where $\langle , \rangle$ denotes the symplectic form on $\mathbb{R}^2$. The space U_1 of points

$$U_1 = \{v + \alpha(v) : v \in V_1\}$$

is then isotropic. Let U_2 be a maximal isotropic subspace of $\alpha(V_1)^\perp \cap V_2$. Then $X = U_1 \oplus U_2 \oplus \widetilde{Z}$ is a maximal isotropic subspace of $\mathbb{R}^{2n}$, and it is easy to check it satisfies the conditions (2.33).

To illustrate proposition 2.7, consider the two components of the oscillator representation [H3]. It is an easy matter to compute their N_1 spectrum, and in particular to see they have N_1 rank equal to one, hence by the proposition, singular rank equal to one. (Singular rank zero would imply only a finite number of K-types, hence finite dimensionality). There are only two conjugacy classes of rank one nilpotent elements, the transvections

$$ty : x \to \langle x, y \rangle y \quad x,\ y \in \mathbb{R}^{2n}$$

where $\langle , \rangle$ denotes the symplectic form, forming one and their negatives forming the other. The two holomorphic oscillator representations have the class of transvections for their wave front set, and the antiholomorphic oscillator representations have the negatives of the transvections for their wave front set. More generally the representation of Sp_{2n} coming from its pairing with $O_{p,q}$ inside $\mathrm{Sp}_{2n(p+q)}$ (see [H3]) has N_1 rank equal to $p + q$, with obvious consequences for the wave-front sets of its irreducible components if $p + q < n$.

We will conclude the paper with a few remarks. First, the analogy of the present discussion with the results of [H4] and [HC] should be pointed out. In those papers it is shown that for an irreducible representation ρ of a reductive p-adic group G, over a field of characteristic zero, the character χ_ρ of ρ has an "asympotic expansion", valid in a neighborhood of the identity. This expansion expresses χ_ρ as a linear combination of distributions attached in a direct way to nilpotent conjugacy classes in the Lie algebra $\mathfrak{g}$ of G. From this expansion, one can read off directly information about the asymptotics of the K-spectrum, or the N-spectrum, where K is a maximal compact subgroup of G, and
N a unipotent subgroup. This expansion is thus comparable to the wave- **139**
front set, but it is more precise in two ways. First, it permits more precise description of the K – or N – spectra than does the wave front set. Secondly, it attaches to ρ not simply a closed set of nilpotent orbits, but a collection of individual orbits with numbers, which might be thought of as multiplicities, attached. It would clearly be desirable to have an analogue of this expansion for groups over $\mathbb{R}$. It seems that Barbasch and Vogan [BV] have established the existence of such an analogue.

A second analogy that might be made is with the characteristic variety of a primitive ideal of a semisimple Lie algebra, as discussed by Borho and Kraft in [B-K]. It would seem the wave front set is the analytical analogue of their construction. □

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ON P-ADIC REPRESENTATIONS ASSOCIATED WITH Z_p-EXTENSIONS

By Kenkichi Iwasawa

In the present paper, we shall discuss some results on the p-adic 141
representations of Galois groups, associated with so-called cyclotomic Z_p-extensions of finite algebraic number fields.

1. Let p be a prime number which will be fixed throughout the following, and let Z_p and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers respectively. A Galois extension K/k is called a Z_p-extension if its Galois group is isomorphic to the additive group of the compact ring Z_p[1]. Let Ω denote the field of all algebraic numbers, i.e., the algebraic closure of the rational field $\mathbb{Q}$ in the field $\mathbb{C}$ of all complex numbers, and let W_∞ be the group of all p^n-th roots of unity in Ω for all $n \geq 0$. Then the field $\mathbb{Q}(W_\infty)$ contains a unique subfield $\mathbb{Q}_\infty$ which is a Z_p-extension over $\mathbb{Q}$. In fact, $\mathbb{Q}_\infty$ is the unique Z_p-extension over $\mathbb{Q}$ contained in Ω, and the degree of the extension $\mathbb{Q}(W_\infty)/\mathbb{Q}_\infty$ is either $p-1$ or 2 according as $p > 2$ or $p = 2$. For any finite extension k of $\mathbb{Q}$, the composite $k_\infty = k\mathbb{Q}_\infty$ is then a Z_p-extension over k and it is called the cyclotomic Z_p-extension over k. For each integer $n \geq 0$, there then exists a unique intermediate field k_n with $[k_n K] = p^n$, and

$$k = k_0 \subset k_1 \subset \ldots \subset k_n \subset \ldots \subset k_\infty = \bigcup_{n \geq 0} k_n.$$

[1]For various definitions and results on Z_p-extensions referred to throughout the following, see Iwasawa [5] or Lang [6].

Let C_n denote the Sylow p-subgroup of the ideal class group of k_n. For $n \leq m$, $k_n \subseteq k_m$, there exists a natural homomorphism $C_n \to C_m$, and these homomorphisms define the direct limit

$$C_\infty = \varinjlim C_n.$$

Clearly C_∞ is a p-primary abelian group and its Tate module $T(C_\infty)$ is a Z_p-module. It is known that

$$T(C_\infty) \simeq Z_p^\lambda$$

142 where $\lambda = \lambda_p(k)$ is a non-negative integer, called the λ-invariant of k for the prime number p. Hence

$$V = T(C_\infty) \underset{Z_p}{\otimes} \mathbb{Q}_p$$

is a λ-dimensional vector space over $\mathbb{Q}_p$. Let

$$\Gamma = \mathrm{Gal}(k_\infty/k) = \varprojlim \mathrm{Gal}(k_n/k)$$

so that $\Gamma \simeq Z_p$. Clearly $\mathrm{Gal}(k_n/k)$ acts on C_n for each $n \geq 0$ and hence Γ acts on $C_\infty = \varinjlim C_n$ in the natural manner. Therefore Γ acts also on $T(C_\infty)$ and V. Thus we have a natural continuous finite dimensional p-adic representation of the Galois group $\Gamma = \mathrm{Gal}(k_\infty/k)$ on the λ-dimensional vector space V over $\mathbb{Q}_p$. We shall next investigate the properties of the p-adic representation space V for Γ.

2. Let us first consider the special case where $p > 2$ and where $k = \mathbb{Q}(\sqrt[p]{1})$ = the cyclotomic field of p-th roots of unity.

Let

$$K = k_\infty = k\mathbb{Q}_\infty = \mathbb{Q}(W_\infty).$$

In this case, $K/\mathbb{Q}$ is an abelian extension and

$$G = \mathrm{Gal}(K/\mathbb{Q}) = \Gamma \times \Delta$$

where $\Gamma = \mathrm{Gal}(K/k) \simeq \mathbb{Z}_p$ and where $\Delta = \mathrm{Gal}(K/\mathbb{Q}_\infty) = \mathrm{Gal}(k/\mathbb{Q})$ is a cyclic group of order $p - 1$. Let $\widehat{\Delta}$ denote the character group of Δ; we may identify $\widehat{\Delta}$ with $\mathrm{Hom}(\Delta, \mathbb{Z}_p^\times)$ where $\mathbb{Z}_p^\times$ denotes the multiplicative group of all p-adic units in $\mathbb{Q}_p$. It is well known that $\widehat{\Delta}$ may be identified also with the group of all Dirichlet characters to the modulus p and that it is generated by a special character ω called the Teichmuller character for p. A character χ in $\widehat{\Delta}$ is called even or odd according as $\chi(-1) = 1$ or $\chi(-1) = -1$ respectively.

As one sees immediately, in this special case, not only $\Gamma = \mathrm{Gal}(K/k)$ but $G = \mathrm{Gal}(K/\mathbb{Q})$ also acts on C_∞, $T(C_\infty)$, and $V = T(C_\infty) \underset{Z_p}{\otimes} \mathbb{Q}_p$ naturally. Hence V is again a p-adic representation space for G. For each χ in $\widetilde{\Delta}$, let

$$V_\chi = \{v | v \in V, \delta \cdot v = \chi(\delta) v \text{ for all } \delta \text{ in } \Delta\}.$$

Since $G = \Gamma \times \Delta$, V_χ is then a Γ-subspace of V and 143

$$V = \underset{\chi}{\otimes} V_\chi, \quad \chi \in \widehat{\Delta}.$$

Let γ_o denote the element of Γ such that $\gamma_0(\zeta) = \zeta^{1+p}$ for all ζ in W_∞. γ_o is a topological generator of Γ; namely, the cyclic subgroup generated by γ_0 is dense in Γ. For each χ in $\widehat{\Delta}$, let

$$g_\chi(X) = \text{ the characteristic polynomial of } \gamma_0 - 1 \text{ acting on } V_\chi$$

and let

$$\begin{aligned} g(X) &= \text{the characteristic polynomial of } \gamma_0 - 1 \text{ acting on } V \\ &= \prod_\chi g_\chi(X). \end{aligned}$$

On the other hand, let $L_p(s;\chi)$ denote the p-adic L-function for the Dirichlet character χ in $\widehat{\Delta}$. It is known in the theory of p-adic L-functions[2] that for each such χ, there exists a power series $\xi_\chi(T)$ in the ring $Z_p[[T]]$ of all formal power series in an indeterminate T with coefficients in Z_p such that

$$L_p(s;\chi) \begin{array}{ll} = \xi_{\omega_{\chi^{-1}}}((1+p)^s - 1) & , \text{ for } \chi \neq 1, s \in Z_p, \\ = \xi_\omega((1+p)^s - 1)/((1+p)^{1-s} - 1), & , \text{ for } \chi = 1, s \in Z_p, s \neq 1. \end{array}$$

[2]See Iwasawa [4] or Lang [6].

Since $L_p(s;\chi) \not\equiv 0$ if χ is even but $L_p(s;\chi) \equiv 0$ if χ is odd, $\xi_\chi(T) \equiv 0$ if χ is even and $\xi_\chi(T) \not\equiv 0$ if χ is odd. By Weierstrass' preparation theorem, $\xi_\chi(T)$ for add χ can be uniquely written in the form

$$\xi_\chi(T) = \eta_\chi(T) p^{e_\chi} f_\chi(T)$$

where $\eta_\chi(T)$ is an invertible power series in the ring $Z_p[[T]]$, e_χ is a non-negative integer[3], and $f_\chi(T)$ is a so-called distinguished polynomial in $Z_p[T]$. The next theorem tells us that there exists a relation between the p-adic representation of $\Gamma = \mathrm{Gal}(K/k)$ on V and the p-adic L-functions $L_p(s;\chi)$ for the characters χ in $\widehat{\Delta}$, or, more precisely, between the polynomials $g_\chi(X)$ and $f_\chi(T)$ defined above. Namely, we have the following result[4]:

144 **Theorem 1.** *Let k^+ denote the maximal real subfield of the cyclotomic field $k = \mathbb{Q}(\sqrt[p]{\sqrt{1}})$ and let h^+ be the class number of k^+. Assume that h^+ is not divisible by p. Then*

$$g_\chi(X) = 1, V_\chi = 0, \quad \text{for all even } \chi \text{ in } \widehat{\Delta},$$

$$g_\chi(X) = f_\chi(X), \qquad \text{for all odd } \chi \text{ in } \widehat{\Delta}.$$

The assumption $p \nmid h^+$ in the theorem is known as Vandiver's conjecture. It has been verified by numerical computation for all primes $p < 125,000$, and no counter example is yet found. On the other hand, if we define, following Leopoldt, the p-adic zeta function $\zeta_p(s;k^+)$ of the totally real field k^+ by

$$\zeta_p(s;k^+) = \prod_\chi{}^{+} \; L_p(s;\chi), \quad \chi \in \widehat{\Delta}, \chi(-1) = 1,$$

then the theorem implies that under the assumption $p \nmid h^+$, $\zeta_p(s;k^+)$ is essentially equal to the characteristic polynomial $g(X)$ of $\gamma_0 - 1$ acting on the representation space V over $\mathbb{Q}_p$, up to the change of variables $s \to (1+p)^s$. The result is mysteriously analogous to a well known

[3] A recent theorem of B. Ferrero and L. Washington implies that $e_x = 0$ for all odd χ.
[4] See Iwasawa [3].

theorem of A. Weil which states that a similar relation exists between the zeta function of an algebraic curve defined over a finite field and the characteristic polynomial of the Frobenius endomorphism acting on the p-adic representation space defined by the Jacobian variety of that curve.

Now, although the above theorem is proved only for a very special case (and even that under the assumption $p \nmid h^+$), we feel that it is not just an isolated fact for $k = \mathbb{Q}(\sqrt[p]{1})$, but is rather a part of a much more general result on teh cyclotomic Z_p-extensions over finite algebraic number fields. In fact, Greenberg [2] generalizes Theorem 1 to the case where the ground field k is a certain type of finite abelian extension over the rational field $\mathbb{Q}$, and Coates [1] also discusses such a generalization for an abelian extension k of an arbitrary totally real field. In the following, we shall report some results on cyclotomic Z_p-extensions, related to some further generalization of the above Theorem 1.

3. We now assume that p is an odd prime, $p > 2$[5], and consider
as our ground field a finite algebraic number field k with the following **145**
properties:

(i) k is a Galois extension of the rational field,

(ii) k contains primitive p-th roots of unity so that it is a totally imaginary field,

(iii) k also contains a totally real subfield k^+ with $[k : k^+] = 2$; namely, k is a number field of C-M type.

In general, let J denote the automorphism of the complex field $\mathbb{C}$ which maps each complex number α to its complex conjugate $\overline{\alpha}$. For simplicity, the restriction of J on any subfield of $\mathbb{C}$, invariant under J, will be denoted again by J. Let

$$\Delta = \mathrm{Gal}(k/\mathbb{Q})$$

for the field k mentioned above. Then by (ii) and (iii), J is an element in the center of Δ and $J \neq 1$, $J^2 = 1$. As in §1, let $K = k_\infty = k\mathbb{Q}_\infty$

[5]The case $p = 2$ can be treated similarly but with some modifications.

denote the cyclotomic Z_p-extension over k. Since k contains p-th roots of unity, $K = k(W_\infty)$. Similarly, let $K^+ = k^+_\infty = k^+\mathbb{Q}_\infty$ be the cyclotomic Z_p-extension over k^+. Then K^+ is a totally real subfield of the totally imaginary field K with $[K; K^+] = 2$. Clearly $K/\mathbb{Q}$ is a Galois extension because both $k/\mathbb{Q}$ and $\mathbb{Q}_\infty/\mathbb{Q}$ are Galois extensions. Hence, let

$$G = \mathrm{Gal}(K/\mathbb{Q}), \quad \Gamma = \mathrm{Gal}(K/k) \simeq Z_p.$$

Then we see immediately that Γ is a central subgroup of G and

$$\Delta = \mathrm{Gal}(k/\mathbb{Q}) = G/\Gamma.$$

As in the special case of §2, the Galois group G acts on C_∞, $T(C_\infty)$, and $V = T(C_\infty)\underset{Z_p}{\otimes}\mathbb{Q}_p$ so that V provides us with a finite dimensional p-adic representation space for $G = \mathrm{Gal}(K/\mathbb{Q})$.

Theorem 2. *Assume that $\lambda_p(k^+) = 0$ and that the so-called Leopoldt's conjecture holds for all intermediate fields k^+_n, $n \geq 0$, of the extension K^+/k^+. Then $V = T(C_\infty)\underset{Z_p}{\otimes}\mathbb{Q}_p$ is cyclic over $G = \mathrm{Gal}(K/\mathbb{Q})$; namely, there exists a vector v_o in V such that the whole space V is spanned over $\mathbb{Q}_p$ by the vectors $\sigma \cdot v_0$, $\sigma \in G$.*

Recall that $\lambda_p(k^+)$ denotes the λ-invariant of the totally real field k^+ for the prime p and that Leopoldt's conjecture for k^+_n states that any
146 set of units in k^+_n, multiplicatively linearly independent over the ring of rational integers Z, remains multiplicatively linearly independent over Z_p when these units are imbedded in the multiplicative group of the algebra $k^+\underset{\mathbb{Q}}{\otimes}\mathbb{Q}_p$. We note that both these assumptions are conjectured to be true for any totally real number field k^+. Note also that since $T(C_\infty) \simeq Z_p^\lambda$, the conclusion of the theorem is equivalent to say that there exists an element v_0 in $T(C_\infty)$ such that the elements of the form $\sigma \cdot v_0$, $\sigma \in G$, generate over Z_p a submodule of finite index in $T(C_\infty)$. The proof of the theorem will be briefly indicated in the next section.

In general, let G be any profinite group and let $G = \varprojlim G_i$ with a family of finite groups $\{G_i\}$. The homomorphisms $G_j \to G_i$, $i \leq j$, which define the inverse limit, induce the homomorphisms $Z_p[G_j] \to Z_p[G_i]$ of the group rings of finite groups over Z_p, and they in turn define

$$Z_p[[G]] = \varprojlim Z_p[G_i].$$

$Z_p[[G]]$ is a compact topological algebra over Z_p and it depends only upon G and is independent of the family $\{G_i\}$ such that $G = \varprojlim G_i$.

We apply the above general remark for $G = \mathrm{Gal}(K/\mathbb{Q})$ in Theorem 2 and define

$$R = Z_p[[G]], \quad R' = R \underset{Z_p}{\otimes} \mathbb{Q}_p.$$

Let $G_n = \mathrm{Gal}(k_n/\mathbb{Q})$, $R_n = Z_p[G_n]$, $n \geq 0$. Since $G = \varinjlim G_n$, we then have

$$R = \varprojlim R_n.$$

Since C_n is an R_n-module in the obvious manner, $C_\infty = \varprojlim C_n$ is an R-module. Hence $T(C_\infty)$ also is an R-module and $V = T(C_\infty) \underset{Z_p}{\otimes} \mathbb{Q}_p$ is an R'-module. We next define a subset A_n of R_n by

$$A_n = \{\alpha | \alpha \in (1 - J)R_n,\ \alpha \cdot C_n = 0\}.$$

Note that $J = J|k_n$ is contained in the center of G_n so that A_n is a two-sided ideal of R_n, contained in $(1 - J)R_n$. Furthermore, if n is large enough and $m \geq n$, then the homomorphism $R_m \to R_n$ maps A_m into A_n. Therefore

$$A = \varprojlim A_n$$

is defined and it is a two-sided ideal of R, contained in $(1 - J)R$. Let **147**

$$A' = A \underset{Z_p}{\otimes} \mathbb{Q}_p.$$

Clearly A' is a two-sided ideal of $R' = R \underset{Z_p}{\otimes} \mathbb{Q}_p$, contained in $(1 - J)R'$. Moreover, it can also be proved that

$$A' = \{\alpha' | \alpha' \in (1 - J)R',\ \alpha' \cdot V = 0\},$$

namely, that A' is the annihilator of the R'-module V in $(1 - J)R'$. Let

$$d = [k : \mathbb{Q}].$$

Using Theorem 2, we can then easily prove the following

Theorem 3. *Let*

$$V' = (1 - J)R'/A'.$$

Under the same assumptions as in Theorem 2, there exist exact sequences of R'-modules

$$V' \to V \to 0, \quad 0 \to V' \to V^d.$$

In particular, V' is a finite dimensional vector space over $\mathbb{Q}_p$, and as p-adic representation spaces for $G = \mathrm{Gal}(K/\mathbb{Q})$, V and V' have the same composition factors.

At this point, let us consider again the special case where $k = \mathbb{Q}(\sqrt[p]{1})$, $p > 2$; the field $\mathbb{Q}(\sqrt[p]{1})$ certainly satisfies the conditions (i), (ii), and (iii) stated at the beginning of this section. In this case, $K = k_\infty$, $K^+ = k_\infty^+$, and k_n^+, $n \geq 0$, are all abelian extensions over $\mathbb{Q}$, and Leopoldt's conjecture for k_n^+ is known to be true by a theorem of Brumer. On the other hand, it is easy to deduce $\lambda_p(k^+) = 0$ from Vandiver's conjecture $p \nmid h^+$ for the class number h^+ of k^+. Therefore we know by Theorem 2 that under the assumption $p \nmid h^+$, V is cyclic over $G = \mathrm{Gal}(K/\mathbb{Q})$, namely,

$$V = R'v_0$$

with some vector v_0 in V. Now, $\lambda_p(k^+) = 0$ also implies $V = (1 - J)V$ so that $V = (1 - J)R'v_0$. Since $G = \mathrm{Gal}(K/\mathbb{Q})$ is an abelian group in this case, both R and R' are commutative rings. Hence it follows from the above that the map $\alpha' \to \alpha' v_0$, $\alpha' \in (1 - J)R'$, induces an
148 R'-isomorphism

$$V' = (1 - J)R'/A' \xrightarrow{\sim} V.$$

Furthermore, we know in this special case that there are many explicitly described elements in the ideal A_n of R_n, $n \geq 0$, called Stickelberger operators for k_n, and that the p-adic L-functions $L_p(s; \chi)$ for χ

in $\widehat{\Delta} = \mathrm{Hom}(\Delta, \Omega_p^+)$ can be constructed by means of such Stickelberger operators[6]. Thus we obtain a relation between the p-adic representation space V' and the p-adic L-functions $L_p(s;\chi)$, and hence between V and $L_p(s;\chi)$ through the above isomorphism. This is the way how Theorem 1 is proved, and the proof is similar for Greenberg's generalization.

We now consider again the general case where k is any finite algebraic number field satisfying the conditions (i), (ii), and (iii). For each C-M sub-field k' of k such that k/k' is abelian, Stickelberger operators for k_n/k' are still defined, and it is proved by Deligne and Ribet that such Stickelberger operators are related to abelian p-adic L-functions for $k' \cap k^+$ in much the same way as in the special case mentioned above. However, it is not known whether such general Stickelberger operators belong to the ideal A_n and provide us with any essential part of A_n defined above[7]. This prevents us from obtaining any nice relation between the p-adic representation space V' and p-adic L-functions. On the other hand, we can find examples of $k/\mathbb{Q}$, satisfying (i), (ii), (iii) and also the assumptions in Theorem 2, such that the representation spaces V and V' for $G = \mathrm{Gal}(K/\mathbb{Q})$ in Theorem 3 are not isomorphic to each other. Thus we see that the results of Theorems 2, 3 tell us much less on the nature of the p-adic representation space V for $G = \mathrm{Gal}(K/\mathbb{Q})$ than Theorem 1 for the special case $k = \mathbb{Q}(\sqrt[p]{1})$. Nevertheless, we still feel and hope that those theorems would be of some use in the future investigations to obtain a full generalization of Theorem 1 in §2.

We also note in this connection that in such a generalization of Theorem 1, one has certainly to consider p-adic (non-abelian) Artin L-functions. Given any Galois extension L/K of totally real finite algebraic number fields, it is not difficult to define p-adic Artin L-function $L_p(s;\chi)$ for each character χ of the Galois group $\mathrm{Gal}(L/K)$ so that $L_p(s;\chi)$ **149**
is related to the classical Artin L-function $L(s;\chi)$ in the usual manner and that those $L_p(s;\chi)$ share with the classical functions $L(s;\chi)$ all essential formal properties such as the formula concerning induced characters. One can even formulate the p-adic Artin conjecture for such

[6]See Iwasawa [4] or Lang [6].
[7]See the discussions in Coates [1].

L-functions; the conjecture is not yet verified and, in fact, it is closely related to the above mentioned problem of generalizing Theorem 1. For all these, we refer the reader to forth-coming papers by R. Greenberg and B. Gross, noting here only that Weil's solution of Artin's conjecture for L-functions of algebraic curves defined over finite fields is based upon the study of the representations of Galois groups on the spaces similar to V mentioned above.

4. We shall next briefly indicate an outline of the proof of Theorem 2[8]. Following the general definition in §3, let

$$\Lambda = Z_p[[\Gamma]]$$

for the profinite group $\Gamma = \mathrm{Gal}(K/k)$, and let γ_0 be any topological generator of $\Gamma \simeq Z_p$. Let $Z_p[[T]]$ denote as in §2 the ring of all formal power series in T with coefficients in Z_p. Then it is known that there is a unique isomorphism of compact algebras over Z_p:

$$\Lambda = Z_p[[\Gamma]] \xrightarrow{\sim} Z_p[[T]]$$

such that $\gamma_0 \to 1 + T$. Hence fixing a topological generator γ_0, we may identify $\Lambda = Z_p[[\Gamma]]$ with $Z_p[[T]]$ so that $\gamma_0 = 1 + T$. Then

$$\Lambda' = \Lambda \underset{Z_p}{\otimes} \mathbb{Q}_p = Z_p[[T]] \underset{Z_p}{\otimes} \mathbb{Q}_p$$

and it is easy to see that Λ' is a principal ideal domain. One also proves immediately that $\Lambda = Z_p[[\Gamma]]$ is a central subalgebra of $R - Z_p[[G]]$ and that the latter is a free Λ-module of rank $d = [k : \mathbb{Q}] = [G : \Gamma]$. Hence $R' = R \underset{Z_p}{\otimes} \mathbb{Q}_p$ is an algebra over $\Lambda' = \Lambda \underset{Z_p}{\otimes} \mathbb{Q}_p$ and it is a free module of rank d over the principal ideal domain Λ'.

Now, let L denote the maximal unramified abelian p-extension (i.e., Hilbert's p-class field) over K, and M the maximal p-ramified abelian
150 p-extension over K. Then

$$\mathbb{Q} \subseteq k \subseteq K \subseteq L \subseteq M$$

[8]Cf. the proof of Theorem **??** in Greenberg [2].

and both $L/\mathbb{Q}$ and $M/\mathbb{Q}$ are Galois extensions. Let

$$X = \mathrm{Gal}(L/K), \quad Y = \mathrm{Gal}(M/K).$$

These are abelian pro-p-groups and, hence, are $\mathbf{Z}_p$-modules in the natural manner. Since $G = \mathrm{Gal}(K/\mathbb{Q})$ acts on X and Y in the obvious way, we see that both X and Y are R-modules and, consequently, also Λ-modules. It is know that X is a torsion Λ-module so that there is an element $\xi \neq 0$ in Λ such that

$$\xi \cdot X = 0.$$

Let

$$X' = X \underset{\mathbf{Z}_p}{\otimes} \mathbb{Q}_p.$$

It is clear that X' is an R'-module. However it is also known in the theory of $\mathbf{Z}_p$-extensions that

$$V = T(C_\infty) \underset{\mathbf{Z}_p}{\otimes} \mathbb{Q}_p \xrightarrow{\sim} X'$$

as modules over R'. Hence, in order to prove Theorem 2, we have only to show that X' is cyclic over R' under the assumptions of that theorem.

Let Y^- denote the submodule of all y in Y satisfying $(1 + J)y = 0$. Since $p > 2$, $Y^- = (1 - J)Y$ and since $J = J|K$ is contained in the center of $G = \mathrm{Gal}(K/\mathbb{Q})$, Y^- is an R-submodule of Y. Let $t(Y^-)$ denote the torsion Λ-submodule of the Λ-module Y^- and let

$$Z = Y^-/t(Y^-), \quad Z' = Z \underset{\mathbf{Z}_p}{\otimes} \mathbb{Q}_p.$$

Then Z is an R-module, and Z' an R'-module. Furthermore, we can prove by using the assumptions of Theorem 2 that there is an exact sequence of R'-modules

$$Z'/\xi Z' \to X' \to 0.$$

Therefore the proof is now reduced to show that $Z'/\xi Z'$ is cyclic over R'.

Let **151**

$$R'^{-} = (1 - J)R' = R'(1 - J).$$

Then we have the following two lemmas:

Lemma 1. *Both Z' and R'^{-} are free Λ'-modules with the same rank $\frac{d}{2}$ and*

$$Z'/TZ' \simeq R'^{-}/TR'^{-}$$

as modules over R'.

Lemma 2. *Let A and B be R'-modules which are free and of the same finite rank over Λ', and let*

$$A/TA \simeq B/TB$$

as R'-modules. Then, as modules over R',

$$A/\mathfrak{p}A \simeq B/\mathfrak{p}B$$

for any non-zero prime ideal $\mathfrak{p}$ of the principal ideal domain Λ'.

That Z' is a free Λ'-module of rank $\frac{d}{2}$, where $d = [k : Q]$, is a known fact in the theory of Z_p-extensions. The rest of Lemma 1 can be proved by considering the Galois group of the maximal p-ramified abelian p-extension over k. To see the proof of Lemma 2, let us first assume for simplicity that

$$k \cap \mathbb{Q}_\infty = \mathbb{Q}.$$

In this case, $G = \Gamma \times \Delta$ where $\Delta = \mathrm{Gal}(k/\mathbb{Q}) = \mathrm{Gal}(K/\mathbb{Q}_\infty)$, and $R = Z_p[[G]]$ is nothing but the group ring of the finite group Δ over $\Lambda = Z_p[[\Gamma]]$:

$$R = \Lambda[\Delta].$$

Hence

$$R' = \Lambda[\Delta]$$

where $\Lambda' = \Lambda \underset{Z_p}{\otimes} \mathbb{Q}_p$ is a principal ideal domain. The lemma then follows
152 easily from the results of Swan on the group ring of finite groups over

Dedekind domains[9]. The case $k \cap \mathbb{Q}_\infty \neq \mathbb{Q}$ can be proved similarly by reducing it to the above mentioned special case.

Now, since $R'^- = R'(1 - J)$ is cyclic over R', we see from the above two lemmas that $Z'/\mathfrak{p}Z'$ is cyclic over R' for all $\mathfrak{p}$ as stated in Lemma 2. As ξ is a non-zero element of the principal ideal domain Λ', it follows that $Z'/\xi Z'$ also is cyclic over R'. This completes the proof of Theorem 2.

Finally, we would like to mention here also the following result which can be proved by similar arguments as described above. Namely, changing the notations from the above, let

$$\begin{aligned} k &= \text{an arbitrary (totally) real finite Galois extension over } \mathbb{Q}, \\ K &= k_\infty - k\mathbb{Q}_\infty = \text{the cyclotomic } \mathbb{Z}_p\text{-extension over } k, \\ L' &= \text{the maximal unramified abelian } p\text{-extension over } K \text{ in} \\ &\quad\ \text{which every } p\text{-spot of } K \text{ is completely decomposed}, \\ M &= \text{the maximal } p\text{-ramified abelian } p\text{-extension over } K. \end{aligned}$$

Then, again,

$$\mathbb{Q} \subseteq k \subseteq K \subseteq L' \subseteq M$$

and $K/\mathbb{Q}$, $L'/\mathbb{Q}$, and M/Q are Galois extensions. Let

$$G = \mathrm{Gal}(K/\mathbb{Q}), \quad R = \mathbb{Z}_p[[G]], \quad R' = R \underset{Z_p}{\otimes} \mathbb{Q}_p.$$

As in the case discussed above, the Galois groups

$$\mathrm{Gal}(M/K) \quad \text{and} \quad \mathrm{Gal}(M/L')$$

are modules over R so that $\mathrm{Gal}(M/K)\underset{Z_p}{\otimes}\mathbb{Q}_p$ and $\mathrm{Gal}(M/L')\underset{Z_p}{\otimes}\mathbb{Q}_p$ are R'-modules.

Theorem 4. $\mathrm{Gal}(M/K')\underset{Z_p}{\times}\mathbb{Q}_p$ is cyclic over R'. If in particular $\lambda_p(k) = 0$, then $\mathrm{Gal}(M/K)\underset{Z_p}{\otimes}\mathbb{Q}_p$ also is cyclic over R'.

The first part of the theorem is proved without any assumption, and the second part without assuming Leopoldt's conjecture. Hence the theorem might be more useful in some applications than Theorem 2.

[9] See Swan [7].

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DIRICHLET SERIES FOR THE GROUP GL(N).

By Herve Jacquet

1 Introduction

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Suppose φ is a modular cusp form with Fourier expansion:

$$\varphi(z) = \sum_{n \geq 1} a_n \exp(2i\,\pi n z). \tag{1.1}$$

The Mellin transform of φ is the integral

$$\int_0^{+\infty} \varphi(iy) y^{s-1} dy. \tag{1.2}$$

If we replace φ by its Fourier expansion then we see that (1.2) is equal to

$$\sum_{n \geq 1} a_n n^{-s} \int_0^{+\infty} \exp(-2\pi y) y^{s-1} dy. \tag{1.3}$$

Since

$$\int_0^{+\infty} \exp(-2\pi y) y^{s-1} dy = (2\pi)^{-s} \Gamma(s). \tag{1.4}$$

this integral representation gives the analytic continuation of the series

$$\sum_{n\geq 1} a_n n^{-s}, \tag{1.5}$$

as a meromorphic function of s in the whole complex plane. Furthermore it shows that the analytic continuation satisfies a simple functional equation. Finally if φ is an eigen function of the Hecke operators, then the series (1.5) has an infinite euler product.

If φ' is another form then one can also consider the "convolution" of the Dirichlet series attached to φ and φ', namely the Dirichlet series

$$\sum_{n\geq 1} \frac{a_n a_n'}{n^s} \tag{1.6}$$

156 It has a simple integral representation and analytic properties similar to that of (1.5). Furthermore, if both φ and φ' are eigen functions of the Hecke operators then it has an Euler product.

Classically, it is just as easy to pursue the theory for other types of forms: holomorphic forms for congruence sub-groups, Maass forms, Hilbert modular forms... The theory can also be generalied to the groups GL(r) with $r > 2$. It is still incomplete but, as an introduction, we shall discuss the case of the "Maass forms" for the group

$$\Gamma_r = \mathrm{GL}(r, \mathbb{Z}), \tag{1.7}$$

also noted simply Γ. Naturally the discussion of the most general case would entail the use of adèles and group representations.

This report is based directly on the work of the authors of [J-S-P 1,2,3,]. That work in turn owes much to the results and ideas, published or not, of the authors of [G-K].

2 Maass forms

Let φ be a function on

$$G_r = \mathrm{GL}(r, R), \tag{2.1}$$

invariant on the left under Γ_r, on the right under the orthogonal group, and, on both sides, under the center Z_r of G_r. The function φ will be said to be a cusp form if it satisfies some additional conditions that we now describe. It will be assumed to be C^∞ and an eigen function of the algebra Z of bi-invariant differential differential operators. The corresponding algebra morphism from Z to C will be denoted by λ. We will also assume φ cuspidal. This means that for every group of the form

$$V = \left\{ \begin{pmatrix} I_{r_1} & & & & * \\ & \ddots & & & \\ & & I_{r_2} & & \\ & & & \ddots & \\ 0 & & & & I_{r_s} \end{pmatrix} \right\} \tag{2.2}$$

the "constant term of φ along V", that is, the integral

$$\int\limits_{\Gamma \cap V \backslash V} \varphi(ug)du, \tag{2.3}$$

vanishes for all g. It is perhaps unnecessary to recall that $V \cap \Gamma$ is a discrete cocompact subgroup of V.

There is also a condition of growth at infinity which, because we 157
are considering only cuspidal functions, amounts to demand that φ be square integrable on the quotient $Z_r\Gamma\backslash G_r$. Actually, for a given λ, the functions φ satisfying the above conditions make up a finite dimensional Hilbert space V_λ. It is invariant under the action of the Hecke algebra; the corresponding algebra of operators on V_λ is diagonalizable and so we may, and do, demand that our forms be eigen vectors of the Hecke algebra.

3 Fourier expansions

Let N_r be the group of upper triangular matrices with unit diagonal. For every $(r-1)$-tuple of non zero integers $(n_1, n_2, \ldots, n_{r-1})$ define a character $\theta_{n_1,n_2,\ldots,n_{r-1}}$ of N_r by

$$\theta_{n_1,n_2,\ldots,n_{r-1}}(x) = \prod_{1\leq j\leq r-1} \exp(2i\pi n_j x_{j.j+1}). \tag{3.1}$$

It is clearly trivial on $N_r \cap \Gamma$. Set

$$\varphi_{n_2,n_2,\ldots,n_{r-1}}(g) = \int_{N_r\cap\Gamma\backslash N_r} \varphi(ug)\bar{\theta}(u)du \tag{3.2}$$

where θ stands for $\theta_{n_1,n_1,\ldots,n_{r-1}}$. Then φ has the following expansion:

$$\varphi(g) = \sum \varphi_{n_1,n_2,\ldots,n_{r-1}} \left[\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right] \tag{3.3}$$

where we sum for all $(r-1)$-tuples with $n_i \geq 1$ and γ in a set of representatives for $N_{r-1} \cap \Gamma_{r-1}\backslash\Gamma_{r-1}$. Actually we will need to introduce also, for $0 \leq j \leq r-1$, the subgroup V_r^j of matrices $u \in N_r$ of the form

$$u = \begin{pmatrix}1_{n-j} & *\\ 0 & *\end{pmatrix}.$$

For $j = r-1$ this is the group N_r itself. We will set:

$$\varphi_{n_{r-j},n_{r-j+1},\ldots,n_{r-1}}(g) = \int_{\Gamma\cap V_r^j\backslash V_r^j} \varphi(ug)\bar{\theta}(u)du$$

where $\theta = \theta_{n_1,n_2,\ldots,n_{r-1}}$; the right hand side does not depend on n_1, $n_2, \ldots, n_{r-j-1}$ which justifies the notation. Then we have the more general expansion:

$$\varphi_{n_{r-j+1},\ldots,n_{r-1}}(g) = \sum \varphi_{n_1,n_2,\ldots,n_{r-1}} \left[\begin{pmatrix}\gamma & 0\\ 0 & 1_j\end{pmatrix} g\right] \tag{3.4}$$

where We sum for all $r-j$ tuples $(n_1, n_2, \ldots, n_{r-j})$ with $n_i \geq 1$ and all γ in a set of representatives for $N_{r-j} \cap \Gamma_{r-j}\backslash\Gamma_{r-j}$.

It is not simple to explain the ideas involved in these expansions. We **158**
will point out however that our assertions are a mere reformulation of the expansions given in [P1] or [Sha].

So far our assertions do not depend on the assumption that φ be an eigen function of the Hecke algebra. If this assumption is taken in account, then it is found that

$$\varphi_{n_1,n_2,\dots,n_{r-1}}(g) = a_{n_1,n_2,\dots,n_{r-1}} W(\zeta g) \tag{3.5}$$

where we have denoted by W the function

$$\underbrace{\varphi_{1,1,\dots,1}}_{r-1}(g) \tag{3.6}$$

and by ζ the diagonal matrix

$$\operatorname{diag}(n_1\, n_2 \dots n_{r-1}, n_2 \dots n_{r-1}, \dots, n_{r-1}, 1). \tag{3.7}$$

The constants $a_{n_1,n_2,\dots,n_{r-1}}$ which appear can be computed solely in terms of the homomorphisms of the Hecke algebra into C determined by φ. The reader will note that both sides of (3-5) transform on the left under the character $\theta_{n_1,n_2,\dots,n_{r-1}}$ of the group N_r. As for W, within a scalar factor, it is determined solely by the morphism λ of Z into C. Again, our assertions are mere reformulation of the results of [C-S], [Sha], [Shi].

4 The Mellin Transform

Let us first simplify our notations. For $0 \leq j \leq r-1$ we set

$$\varphi^j = \underbrace{\varphi_{1,1,\dots,1}} \tag{4.1}$$

so that $\varphi^0 = \varphi$ and $\varphi^{r-1} = W$. We set also, for $1 \leq j \leq r-1$,

$$a_{n_1,n_2,\dots,n_j} = a_{n_1,n_2,\dots,n_j,\underbrace{1,1,\dots,1}_{r-j-1}}.$$

Combining (3.4) [with $j = r - 1$] with (3.5) we get

$$\varphi^{r-2}(g) = \sum_{n\geq 1, \epsilon=\pm 1} a_n W\left[\begin{pmatrix} n\epsilon & 0 \\ 0 & 1_{r-1}\end{pmatrix} g\right]. \tag{4.2}$$

In view of this formula it is entirely reasonable to define the Mellin
159 transform of φ to be the integral

$$\int_{R^\times/\{\pm 1\}} \varphi^{r-2}\begin{pmatrix} a & 0 \\ 0 & 1_{r-1}\end{pmatrix} |a|^{s-1} da. \tag{4.3}$$

It is equal to

$$\sum_{n\geq 1} a_n n^{-s} \int_{R^\times} W\begin{pmatrix} a & 0 \\ 0 & 1_{r-1}\end{pmatrix} |a|^{s-1} da. \tag{4.4}$$

If we knew that the integral in (4.4) were a product of Γ -factors–as it should be–then the previous computation would give the analytic continuation of the Dirichlet Series

$$\sum_{n\geq 1} a_n n^{-s}. \tag{4.5}$$

On the other hand, just as in the case $r = 2$, the Dirichlet Series has an infinite Euler product:

$$\sum_{n\geq 1} a_n n^{-s+\frac{1}{2}(r-1)} = \prod_p \det(1 - p^{-s} X_p)^{-1},$$

where X_p is a semi-simple conjugacy class in $\mathrm{GL}(r, C)$.

5 The convolution

The convolution (1.6) also generalizes. Namely let φ' be another cusp-form on G_r, with $r' \leq r$. Let us denote with a prime the objects attached to φ.

Suppose first $r' \leq r-1$. Consider the integral

$$\int\limits_{\Gamma_{r'}\backslash G_{r'}} \varphi^{r-1-r'}\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'} \end{pmatrix} \varphi'(g) |\det g|^s d^\times g, \tag{5.1}$$

where $d^\times g$ is an invariant measure on the quotient $\Gamma_{r'}\backslash G_{r'}$.

Combining (3.4) with (3.5) we have the following expansion:

$$\varphi^{r-1-r'}(g) = \sum a_{n_1,n_2,\ldots,n_{r'}} W\left[\begin{pmatrix} \gamma & 0 \\ 0 & 1_{r-r'} \end{pmatrix} g\right]. \tag{5.2}$$

Replacing $\varphi^{r-1-r'}$ by this expression in (5.1) we get, after a "few" formal manipulations,

$$\sum_{n_1\geq 1, n_2 \geq 1, \ldots, n_r \geq 1} a_{n_1,n_2,\ldots,n_{r'}} a'_{n_1,n_2,\ldots,n_{r'-1}} |n_1 n_2^2 \ldots n_{r'}^{r'}|^{-s} \tag{5.3}$$
$$\int\limits_{N_{r'}\backslash G_{r'}} W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-r'} \end{pmatrix}\right] W'[\epsilon g] |\det g|^s d^\times g,$$

where $d^\times g$ is now an invariant measure on the quotient $N_{r'}, \backslash G_{r'}$, and ϵ **160**
is the r' by r' diagonal matrix

$$\operatorname{diag}(-1, 1, -1, \ldots).$$

The multiple series which appears in (5.3) may be regarded as a Dirichlet series in the usual sense. Again if we knew that the integral in (5.3) were a product of Γ-factors, our computations would give the analytic continuation of this series.

Just as in the previous case, the series has an Euler product:

$$\sum a_{n_1,n_2,\ldots,n_{r'}} a'_{n_1,n_2,\ldots,n_{r'-1}} |n_1 n_2 \ldots n_{r'}|^{s-\frac{1}{2}(r-r')} \tag{5.4}$$
$$= \prod_p \det(1 - p^{-s} X_p \otimes X'_p)^{-1}.$$

When $r = r'$, the previous construction needs to be modified. We denote by Φ the Schwartz-function on the space of row matrices with r entries defined by

$$\Phi(x) = \exp(-\pi x \cdot {}^t x) \tag{5.5}$$

and we introduce an "Epstein zeta function":

$$E(g,s) = \sum_{\xi \in Z^r - \{0\}} \int_{-\infty}^{+\infty} \Phi(t\xi g)|t|^{rs-1} dt |\det g|^s \tag{5.6}$$

[Here ξg is the product of the row matrix ξ by the square matrix g; t is a scalar]. It can also be written as an "Eisenstein series":

$$E(g,s) = \zeta(rs) \sum_{\gamma \in \Gamma \cap P_r \backslash \Gamma} \int \Phi[(0,0,\ldots,0,t)\gamma g]|t|^{rs-1}\,\mathrm{dt}\,|\det g|^s, \tag{5.7}$$

where P_r is the standard parabolic subgroup of type $(r-1,1)$.

Then, instead of (5.2), we have to consider the integral

$$\int_{Z_r \Gamma \backslash G_r} \varphi(g)\varphi'(g)E(g,s)d^\times g, \tag{5.8}$$

where $d^\times g$ is an invariant measure on the quotient $Z_r\Gamma\backslash G_r$. It turns out to be equal to

$$\begin{aligned} &\zeta(rs) \sum a_{n_1,n_2,\ldots,n_{r-1}} a'_{n_1,n_2,\ldots,n_{r-1}} |n_1 n_2^2 \ldots n_{r-1}^{r-1}|^{-s} \\ &\int_{N_r \backslash G_r} W(g)W'(\epsilon g)\Phi[(0,0,\ldots,0,1)g]|\det g|^s d^\times g. \end{aligned} \tag{5.9}$$

161 Moreover:

$$\begin{aligned} \zeta(rs) \sum a_{n_1,n_2,\ldots,n_{r-1}} a'_{n_1,n_2,\ldots,n_{r-1}} |n_1 n_2^2 \ldots n_{r-1}^{r-1}|^{-s} \\ = \prod_p \det(1 - p^{-s} X_p \otimes X'_p)^{-1}. \end{aligned} \tag{5.10}$$

Remark 5.11. If we take $r = 1$ then $\varphi = \varphi' = \varphi_0$, the constant function equal to one on $G_1 = R^\times$; moreover $X_p = X_{p'} = 1$, and (5.10) reduces to the Euler product for the ζ-function. Similarly, we may regard the theory of §4 as a special case of the theory of §5 where $r' = 1$ and $\varphi' = \varphi_0$. This remark will be used without further warning.

6 Functional Equations

We have already pointed out that we do not have enough information on the integrals of (4.4), (5.3), and (5.9). If we assume the missing information then we can address ourselves to the question of the functional equation satisfied by these Euler products. The functional equation should state that the analytic continuation of

$$\prod_p (1 - p^{-s} X_p \otimes X'_p)^{-1},$$

times the appropriate Γ-factor, is equal to the analytic continuation of

$$\prod_p (1 - p^{-1+s} X_p^{-1} \otimes X'^{-1}_p)^{-1},$$

times the appropriate Γ-factor.

To see this we introduce the function

$$\widetilde{\varphi}(g) = \varphi({}^t g^{-1}).$$

It is also a Maass cusp form. We denote by a tilda the objects attached to $\widetilde{\varphi}$. Then:

$$\widetilde{W}(g) = W(w_r {}^t g^{-1}), \quad \text{where} \quad w_r = \begin{pmatrix} 0 & & & -1 \\ & & 1 & \\ & -1 & & \\ \cdot^{\cdot^{\cdot}} & & & 0 \end{pmatrix},$$

$$\widetilde{a}_{n_1, n_2, \ldots, n_{r-1}} = a_{n_{r-1}, n_{r-2} \ldots n_t}, \widetilde{X}_p = X_p^{-1}.$$

If $r = r'$ our starting point is the functional equation of the Epstein zeta-function:

$$E(g, s) = E({}^t g^{-1}, 1 - s);$$

from which we get

$$\int \varphi(g) \varphi'(g) E(g, s) dg = \int \widetilde{\varphi}(g) \widetilde{\varphi}'(g) E(g, 1 - s) dg.$$

The functional equation follows readily. 162

If $r' = r-1$ then $\varphi^{r-1-r'}$ is just φ. Clearly

$$\int \varphi\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}\varphi'(g)|\det g|^{s-\frac{1}{2}}d^{\times}g = \int \widetilde{\varphi}\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}\widetilde{\varphi}'(g)|\det g|^{\frac{1}{2}-s}d^{\times}g$$

and again the functional equation follows readily.

However if $r' \leq r-2$ (which includes the case $r' = 1$) we have to take in account a somewhat unexpected relation between $\widetilde{\varphi}^{r-r'-1}$ and $\varphi^{r-r'-1}$. Namely

$$\int \varphi^{r-r'-1}\left[\begin{pmatrix} 1_{r'} & 0 & 0\\ x & 1_{r-r'-1} & 0\\ 0 & 0 & 1\end{pmatrix}{}^{t}g^{-1}\right]\mathrm{dx}$$

is actually a left-translate of $\widetilde{\varphi}^{r-r'-1}(g)$; the integral is on the full space of matrices with r' columns and $r-r'-1$ rows. Rather than trying to explain the details, we refer the reader to [J-S-P1] where the case $r' = 1$, $r = 3$ is discussed.

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CRYSTALLINE COHOMOLOGY, DIEUDONNÉ MODULES, AND JACOBI SUMS

By Nicholoas M. Katz

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Introduction

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Hasse [20] and Hasse-Davenport [21] were the first to realize the connection between exponential sums over finite fields and the theory of zeta and L-functions of algebraic varieties over finite fields. This connection was exploited to Weil; one of the very first applications that Weil gave of the then newly proven "Riemann Hypothesis" for curves over finite fields was the estimation of the absolute value of Kloosterman sums (cf [46]). The basic idea (cf [20]) is that by using the theory of L-functions, one can express the negative of such an exponential sum as the sum of certain of the reciprocal zeroes of the zeta function itself; because the magnitude of these zeroes is given by the "Riemann Hypothesis," one gets an estimate. In a fixed characteristic p, the estimate one gets in this way for all the fintie fields $\mathbf{F}_{p^n}$ is best possible. On the other hand, very little is known about the variation with p of the absolute values, even for Kloosterman sums, though in this case there is a conjecture, of Sato-Tate type, which seems inaccessible at present.

One case in which the problem of unknown variation with p does not arise is when the expression of the exponential sum as a sum of reciprocal roots of zeta reduces to a sum consisting of a *single* reciprocal root; then the Riemann Hypothesis tells us the exact magnitude of the exponential sum. Conversely, an elementary argument shows that in a certain sense, this is the only case in which such exact knowledge of the magnitude of exponential sums can arise, and it shows further that a theorem of Hasse-Davenport type always results from such exact knowledge. Examples of exponential sums of this sort are Gauss sums and Jacobi sums.

Honda was the first to suggest that the identification of say, Jaboci sums, with reciprocal zeroes of zeta functions could also lead to significant non-archimedean information about Jacobi sums. A few years before his untimely death, Honda conjectured a p-adic limit formula for Jacobi sums in terms of ratios of binomail coefficients ([23]). I gave an over-complicated proof (in a letter to Honda of Nov. 1971) which managed to shed no light whatever on the meaning of the formula. Recently, B. H. Gross and N. Koblitz [14] showed that Honda's limit formula was really an *exact* p-adic formula for Jacobi sums in terms of products of values of Morita's p-adic Γ-function; as such, it constituted the first improvement in this century over Stickelberger's formula which gave the
p-adic valuation and the *first* non-vanishing p-adic digit in the p-adic **167**
expansion of a Jacobi sum!

In this paper, I will discuss the cohomological genesis of formulas of the sort discovered by Honda. The basic idea is that the reciprocal zeroes of zeta are the eigenvalues of the Frobenius endomorphism of a suitable cohomology group; if this group, together with the action of Frobenius upon it, can be made sufficiently explicit, one obtains the desired "explicit formulas".

There are two approaches to the question, which differ more in style than in substance. The first and longer is based on Honda's explicit construction of the Dieudonné module of a formal group in terms of "formal de Rham cohomology". The second, less elementary but more efficient, is grounded in crystalline cohomology, particularly in the theory of the de Rham-Witt complex. I hope the reader will share my belief that there is something to be gained from each of the approaches, and pardon my decision to discuss both of them.

I would like to thank B. Dwork for many helpful discussions concerning the original proof of Honda's conjecture. Whatever I know of

the Grothendieck-Mazur-Messing approach to Dieudonné theory through exotic Ext's, I was taught by Bill Messing. I would also like to thank Spencer Bloch for his encouragement when I was trying to understand Honda's explicit Dieudonné theory, and Luc Illusie for gently correcting some extravagent assertions I made at the Colloquium.

Finally, I would like to dedicate this paper to the memory of T. Honda.

I. Elementary Axiomatics, and the Hasse-Davenport Theorem. Consider a projective, smooth and geometrically connected variety X, say of dimension d, over a finite field $\mathbf{F}_q$. For each integer $n \geq 1$, we denote by $X(\mathbf{F}_{q^n})$ the finite set of points of X with values in $\mathbf{F}_{q^n}$, and by $\sharp X(\mathbf{F}_{q^n})$ the cardinality of this set. The zeta function $Z(X/\mathbf{F}_q, T)$ of X over $\mathbf{F}_q$ is the formal power series in T with Q-coefficients defined as

$$Z(X/\mathbf{F}_q, T) = \exp\left(\sum_{n\geq 1} \frac{T^n}{n} \sharp X(\mathbf{F}_{q^n})\right).$$

168 Thanks to Deligne [6], we know that this zeta function has a unique expression as a finite alternating product of polynomials $P_i(T) \in \mathbf{Z}[T]$, $i = 0, \ldots, 2d$:

$$Z(X/\mathbf{F}_q, T) = \prod_{i=0}^{2d} P_i(T)^{(-1)^{i+1}} = \frac{P_1 P_3 \ldots P_{2d-1}}{P_0 P_2 \ldots P_{2s}}$$

in which each polynomial $P_i(T) \in \mathbf{Z}[T]$ is of the form

$$P_i(T) = \prod_{j=1}^{\deg P_i} (1 - \alpha_{i,j} T)$$

with $\alpha_{i,j}$ algebraic integers such that

$$|\alpha_{i,j}| = \sqrt{q}^{i}$$

for *any* archimedean absolute value $|\ |$ on the field $\bar{\mathbf{Q}}$ of all algebraic numbers. The extreme polynomials P_0, P_{2d} are given explicitly:

$$P_0(T) = (1 - T), P_{2d}(T) = (1 - q^d \cdot T)$$

Despite this apparently "elementary" characterization of the polynomials $P_i(T)$, their true genesis is cohomological. Let us recall this briefly.

For each prime number l different from the characteristic p of $\mathbf{F}_q$, let us denote by $H^i_l(X)$ the finitely generated $\mathbf{Z}_l$-module defined as

$$H^i_l(X) = \varprojlim_n H^i_{\text{etale}}(X \otimes \bar{\mathbf{F}}_q, \mathbf{Z}/l^n\mathbf{Z}).$$

Corresponding to the prime p itself, we denote by $W(\mathbf{F}_q)$ the ring of p-Witt vectors of $\mathbf{F}_q$, and by $H^i_{\text{cris}}(X)$ the finitely generated $W(\mathbf{F}_q)$-module defined as

$$H^i_{\text{cris}}(X) = \varprojlim_n H^i_{\text{cris}}(X/W_n(\mathbf{F}_q)).$$

The Frobenius endomorphism F of X relative to $\mathbf{F}_q$ acts, by functoriality, on these various cohomology groups $H^i_l(X)$ for $l \neq p$, and $H^i_{\text{cris}}(X)$; and F induces automorphisms of the corresponding vector spaces

$$H^i_l(X) \bigotimes_{Z_l} Q_l,\ H^i_{\text{cris}}(X) \bigotimes_{W(\mathbf{F}_q)} K$$

(K denoting the fraction field of $W(\mathbf{F}_q)$). The polynomial $P_i(T) \in \mathbf{Z}[T]$ which occurs in the factorization of the zeta function is then given cohomologically by the formulas

$$P_i(T) = \det(1 - TF \big| H^i_l(X) \otimes \mathbf{Q}_l) \text{ for } l \neq p$$
$$P_i(T) = \det(1 - TF \big| H^i_{\text{cris}}(X) \otimes K).$$

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The resulting formula for zeta as the alternating product of characteristic polynomials of F on the H^i, in each of the cohomology theories $H^i_l(X) \otimes Q_l$ for $l \neq p$, $H^i_{\text{cris}}(X) \otimes K$, is equivalent, via logarithmic differentiation, to the identities in those theories

$$\neq X(\mathbf{F}_{q^n}) = \sum (-1)^i \text{ trace } (F^n \big| H^i). \text{ for all } n \geq 1.$$

By viewing the set $X(\mathbf{F}_{q^n})$ as the set of fixed points of F^n acting on $X(\bar{\mathbf{F}}_q)$, this identity becomes a Lefschetz trace formula

$$\#\text{ Fix }(F^n) = \sum(-1)^i \text{ trace }(F^n\big|H^i) \text{ all } n \geq 1$$

for F and its iterates in each of our cohomology theories. If we take as *given* these Lefschetz trace formulas, then the identification of P_i with $\det(1 - FT\big|H^i)$ is equivalent to the assertion:

> On any of the groups $H^i_l(X) \otimes Q_l$ with $l \neq p$,
> $H^i_{\text{cris}}(X) \otimes K$, the eigenvalues of F are algebraic integers all of whose archimedean absolute values are $\sqrt{q}^i$.

In fact, there is not a great deal more that is known about the action of F on the $H^i_l(X)\otimes Q_l$ for $l \neq p$, and on $H^i_{\text{cris}}(X)\otimes K$. It is still *not* known, for example, whether the action of F on these cohomology groups is always semi-simple when $i > 1$. (That it is when $i = 1$ results from the theory of abelian varieties).

Suppose that a finite group G operates on X by $\mathbf{F}_q$-automorphisms. Let us choose a number field E big enough that all complex representations of G are realizable over E, and whose residue fields at all p-adic places contain $\mathbf{F}_q$. (For example, the field $Q(\zeta_{q-1}, \zeta_N)$, where N is the l.c.m. of the orders of elements of G, is such an E). We denote by λ an l-adic place of E, $l \neq p$, and by P a p-adic place of E. Thus E_λ is a finite extension of Q_l, and E_P is a finite extension of K.

Let M be a finite dimensional E-vector space given with an action of G, say $\rho : G \to \text{Aut}_E(M)$. The associated L-function $L(X/\mathscr{F}_q, \rho, T)$ is the formal power series with E-coefficients defined as

$$L(X/\mathscr{F}_q, \rho, T) = \exp\left(\sum_{n\geq 1} \frac{T^n}{n} \cdot \frac{1}{\#G} \sum_{g\in G} tr(\rho(g^{-1}))\#Fix(F^n g)\right)$$

where Fix $(F^n g)$ denotes the finite set of fixed points of $F^n g$ acting on $X(\bar{\mathbf{F}}_q)$. We recover the zeta function of $X/\mathbf{F}_q$ by taking for ρ the regular representation of G. The usual formalism of zeta and L-functions gives

$$Z(X/\mathbf{F}_q, T) = \prod_{\rho \text{ irred}} L(X/\mathbf{F}_q, \rho, T)^{\deg(\rho)}$$

It follows from Deligne's results that for any representation ρ, we have a unique expression for the corresponding L-function as an alternating product of polynomials $P_{i,\rho}(T) \in E[T]$,

$$L(X/\mathbf{F}_q, \rho, T) = \prod_{i=0}^{2d} P_{i,\rho}(T)^{(-1)^{i+1}},$$

which are of the form

$$P_{i,\rho}(T) = \prod_{j=1}^{\deg P_{i,\rho}} (1 - \alpha_{i,j,\rho} T)$$

with algebraic integers $\alpha_{i,j,\rho}$ such that

$$|\alpha_{i,j,\rho}| = \sqrt{q}^i$$

for any archimedean absolute value $|\ |$ on the field $\bar{Q}$ of all algebraic numbers.

The cohomological expression of there $P_{i,\rho}$ is straighforward (cf. [18]). Because the action of G is "defined over $\mathbf{F}_q$" it commutes with F, and therefore the induced action of G on the cohomology commutes with the action of F. Therefore G, acting by composition, induces automorphisms of the E_λ-vector spaces,$l \neq \rho$,

$$\mathrm{Hom}_{E_\lambda[G]}(M \bigotimes_E E_\lambda, H^i_l(X) \bigotimes_{Z_l} E_\lambda).$$

and of the E_{P}-vector spaces

$$\mathrm{Hom}_{E_{\mathsf{P}}[G]}(M \bigotimes_E E_{\mathsf{P}}, H^i_{\mathrm{cris}}(X) \bigotimes_{W(\mathscr{F}_q)} E_{\mathsf{P}}).$$

The polynomials $P_{i,\rho}(T) \in E[T]$ are given by the formulas

$$P_{i,\rho}(T) = \det(1 - TF \big| \operatorname{Hom}_{E_\lambda[G]}(M \bigotimes_E E_\lambda, H^i_l(X) \bigotimes_{Z_l} E_\lambda)) \text{ for } l \neq \rho$$

$$P_{i,\rho}(T) = \det(1 - TF \big| \operatorname{Hom}_{E_{\mathsf{P}}[G]}(M \bigotimes_E E_{\mathsf{P}}, H^i_{\mathrm{cris}}(X) \bigotimes_{W(\mathscr{F}_q)} E_{\mathsf{P}})).$$

171 Let us recall the derivation of these formulas. We first observe that the characteristic polynomial of F on $\operatorname{Hom}_G(M, H^i) \simeq (\check{M} \otimes H^i)^G \subset \check{M} \otimes H^i$ *divides* $\det(1 - FT \big| H^i)^{\dim \check{M}}$, and hence the eigenvalues of F on $\operatorname{Hom}_G(M, H^i)$ are algebraic integers, all of whose archimedean absolute values are $\sqrt{q}^i$. So it remains only to verify that the alternating product of those characteristic polynomials is indeed the L-function, i.e. . that

$$L(X \backslash \mathbf{F}_q, \rho, T) = \prod \det(1 - FT \big| (\check{M} \otimes H^i)^G)^{(-1)^{i+1}},$$

Equivalently, we must check that

$$\begin{aligned}\frac{1}{\#G} \sum \operatorname{trace} \rho(g^{-1}) \,\#\, \operatorname{Fix}\,(F^n g) &\\ &= \sum (-1)^i \operatorname{trace}\,(1 \otimes F^n \big| (\check{M} \otimes H^i)^G) \\ &= \sum (-1)^i \frac{1}{\#G} \sum_{g \in G} \operatorname{trace}\,(g \otimes F^n g \big| \check{M} \otimes H^i) \\ &= \sum (-1)^i \frac{1}{\#G} \sum_{g \in G} \operatorname{trace} \check{\rho}(g) \cdot \operatorname{trace}\,(F^n g \big| H^i) \\ &= \frac{1}{\#G} \sum_{g \in G} \operatorname{trace} \rho(g^{-1}) \sum (-1)^i \operatorname{trace}\,(F^n g \big| H^i).\end{aligned}$$

To check this last equality, we would like to invoke the Lefschetz trace formula, not for F^n, but for $F^n g$, with g an automorphism of *finite order* which commutes with F; this amounts to invoking the Lefschetz trace formula for Fg on X and on all its "extensions of scalars" $X \otimes \mathbf{F}_{q^n}$. But an elementary descent argument shows that given an automorphism g of finite order which commutes with F, there is *another* variety $X'/\mathbf{F}_q$

together with an isomorphism $X\otimes\bar{\mathbf{F}}_q \simeq X'\otimes\bar{\mathbf{F}}_q$ under which $Fg\otimes 1$ corresponds to $F\otimes 1$. Because this isomorphism also induces isomorphisms of cohomology groups

$$H_l^i(X)^{\text{dfn}} H_{\text{et}}^i(X'\otimes\bar{\mathbf{F}}_q,\mathbf{Z}_l) \simeq H^i(X\otimes\bar{\mathbf{F}}_q,\mathbf{Z}_l)^{\text{dfn}} H_l^i(X),$$
$$H_{\text{cris}}^i(X')\otimes W(\bar{\mathbf{F}}_q) \simeq H_{\text{cris}}^i(X'\otimes\bar{\mathbf{F}}_q) \simeq H_{\text{cris}}^i(X\otimes\bar{\mathbf{F}}_q) \simeq$$
$$\simeq H_{\text{cris}}^i(X)\otimes W(\bar{\mathbf{F}}_q),$$

the truth of the Lefschetz formula for Fg on X results from its truth for 172
F on X'.

Let us now consider in greater detail the case of an irreducible ρ. Then $P_{i,\rho}$ is a polynomial whose degree is the common *multiplicity* of ρ in any of the $H_l^i(X)\otimes E_\lambda$, $l\neq\rho$, or in $H_{\text{cris}}^i(X)\otimes E_{\mathsf{P}}$. Decomposing the regular representation leads to the factorization

$$P_i(T) = \prod_{\rho\text{ irred}} P_{i,\rho}(T)^{\deg(\rho)}$$

The coarser factorization

$$P_i(T) = \prod_{\rho\text{irred}} (P_{i,\rho}(T)^{\deg(\rho)})$$

corresponds to the decomposition of $H_l^i(X)\otimes E_\lambda$, *resp.* $H_{\text{cris}}^i(X)\otimes E_{\mathsf{P}}$, into ρ-isotypical components

$$H_l^i(X)\otimes E_\lambda \simeq \bigotimes_{\text{irred}\rho}\left(H_l^i(X)\times E_\lambda\right)^\rho$$
$$H_{\text{cris}}^i(X)\otimes E_{\mathsf{P}} \simeq \bigotimes_{\text{irred}\rho}\left(H_{\text{cris}}^i(X)\otimes E_{\mathsf{P}}\right)^\rho$$

Indeed the corresponding identities, for ρ irreducible, are

$$P_{i,\rho}(T)^{\deg(\rho)} = \det(1-TF\big|(H_l^i(X)\otimes E_\lambda)^\rho)l\neq p$$
$$P_{i,\rho}(T)^{\deg(\rho)} = \det(1-TF\big|(H_{\text{cris}}^i(X)\otimes E_{\mathsf{P}})^\rho).$$

Let us denote by $S(X/\mathbf{F}_q, \rho, n)$ the exponential sums used to define the L-function:

$$S(X/\mathscr{F}_q, \rho, n) = \frac{1}{\#G} \sum_{g \in G} tr(\rho(g)) \# \text{ Fix } (F^n g^{-1}).$$

The following lemma gives the cohomological meaning of theorems of Hasse-Davenport type (cf. [20]).

Lemma 1.1: *Let $X/\mathbf{F}_q$ be projective and smooth. Let a finite group G operate on X by $\mathbf{F}_q$-automorphisms, and let p be an irreducible complex representation of G. Fix an integer $i_\circ$, and denote by $H^{i_\circ}$ any one of the cohomology groups $H_l^{i_\circ}(X) \bigotimes_{Z_l} E_\lambda$ with $l \neq p$, or $H_{cris}^{i_\circ}(X) \bigotimes_{W(\mathbf{F}_q)} E_{\mathsf{P}}$. Let $|\ |$ be any archimedean absolute value on the filed $\bar{Q}$ of all algebraic numbers. The following conditions are equivalent:*

173 (1) *The multiplicity of ρ in $H^{i_\circ}$ is one, and the multiplicity of ρ in H^i is zero if $i \neq i_\circ$.*

(2) *For all $n \geq 1$, we have*

$$(-1)^{i_\circ} S(X/\mathbf{F}_q, \rho, n) = ((-1)^{i_\circ} S(X/\mathbf{F}_q, \rho, 1))^n,$$

and
$$|S(X/\mathbf{F}_q, \rho, 1)| = \sqrt{q}^{i_\circ}$$

(3) *For all $n \geq 1$, we have*

$$|S(X/\mathbf{F}_q, \rho, n)| = \sqrt{q}^{i_\circ n}$$

(4) *For all $n \geq 1$, we have*

$$|S(X/\mathbf{F}_q, \rho, n)| = |S(X/\mathbf{F}_q, \rho, 1)|^n$$

and
$$\sqrt{q}^{i_\circ} \leq |S(X/\mathbf{F}_q, \rho, 1)| < \sqrt{q}^{1+i_\circ}$$

(5) *The polynomial $P_{i_\circ,\rho}(T)$ is given by*

$$P_{i_\circ,\rho}(T) = 1 - (-1)^{i_\circ} S(X/\mathbf{F}_q, \rho, 1)T$$

and for $i \neq i_\circ$, we have $P_{i,\rho}(T) = 1$.

(6) *The ρ-isotypical component $(H^i)^\rho = 0$ for $i \neq i_\circ$, $(H^{i_\circ})^\rho$ has dimension* = $\deg(\rho)$, *and F operates on $(H^{i_\circ})^\rho$ as the scalar*

$$(-1)^{i_\circ} S(X/\mathbf{F}_q, \rho, 1).$$

Proof. This is an easy exercise, using the basic identities:

$$\begin{cases} \exp\left(\sum \dfrac{T^n}{n} S(X/\mathbf{F}_q, \rho, n)\right) = L(X/\mathbf{F}_q, \rho, T) = \prod\limits_i P_{i,\rho}(T)^{(-1)^{i+1}} \\ P_{i,\rho}(T) = \prod\limits_j (1 - \alpha_{i,j,\rho}T), |\alpha_{i,j,\rho}| = \sqrt{q}^i \\ \deg P_{i,\rho} = \textit{ multiplicity of } \rho \textit{ in } H^i = \dfrac{1}{\deg(\rho)} \cdot \dim((H^i)\rho). \end{cases}$$

Suppose, first, that (1) holds, or equivalently that for $i \neq i_\circ$, $P_{i,\rho}(T) = 1$, while $P_{i_\circ,\rho}$ is a linear polynomial $P_{i_\circ,\rho}(T) = (1 - AT)$ with $|A| = \sqrt{q}^{i_\circ}$. The cohomological expression for L then becomes

$$\exp\left(\sum \frac{T^n}{n} S(X/\mathbf{F}_q, \rho, n)\right) = \left(\frac{1}{1 - AT}\right)^{(-1)^{i_\circ}}.$$

Taking logarithms and equating coefficients, we find **174**

$$(-1)^{i_\circ} S(X/\mathbf{F}_q, \rho, n) = A^n \quad \text{for all } n \geq 1.$$

In particular (2) and (5) hold.

The implications (5) ⇒ (1), (6) ⇒ (1) are obvious. Also (5) ⇒ (6), for if $P_{i_\circ,\rho}$ is linear, then ρ has multiplicity one in $H^{i_\circ}$, so that $(H^{i_\circ})^\rho$ is G-irreducible, and hence F must operate on $(H^{i_\circ})^\rho$ as a *scalar*, which we compute by the formula

$$P_{i_\circ,\rho}(T)^{\deg(\rho)} = \det(1 - TF\big|(H^{i_\circ})^\rho).$$

Clearly we have (2) ⇒ (3) ⇒ (4). We must show that if (4) holds, then exactly *one* of the $P_{i,\rho}$ is $\neq 1$, and that one is linear. Logarithmically differentiating the cohomological formula for L, we find

$$S(X/\mathbf{F}_q, \rho, n) = \sum_i (-1)^i \sum_{j=1}^{\deg P_{i,\rho}} (\alpha_{i,j,\rho})^n, \quad |\alpha_{i,j,\rho}| = \sqrt{q}^i.$$

We must show that if (4) holds, then the double sum has only a single term in it. Separating the $\alpha_{i,j,\rho}$ according to the *parity* of i, we get two disjoint sets of non-zero complex numbers (disjoint because their absolute values are disjoint), to which we apply the following lemma. □

Lemma 1.2: *Let $N \geq 0$ and $M \geq 0$ be non-negative integers. Let $\{A_i\}$ be a family of N not-necessarily distinct elements of C^x, and $\{B_i\}$ a family of M not-necessarily distinct elements of C^x. Suppose that for all i, j, $A_i \neq B_j$. If, for some real number $R > 0$, we have*

$$\left|\sum A_j^n - \sum B_j^n\right| = R^n \quad \textit{for all } n \geq 1,$$

then $N + M = 1$, i.e. either there is just one A and no B's, or just one B and no A's.

Proof. Suppose first that either $N = 0$ or $M = 0$, say $M = 0$. Then we have

$$\left|\sum A_i^n\right| = R^n.$$

Squaring, we get

$$\sum_{ij} (A_i \overline{A}_j)^n = (R^2)^n \quad \text{for} \quad n \geq 1$$

175 whence

$$\prod_{ij} (1 - A_i \overline{A}_j T) = (1 - R^2 T),$$

and hence $N = 1$.

In case both $N \geq 1$ and $M \geq 1$, squaring leads to

$$\sum (A_i \overline{A}_k)^n + \sum (B_j \overline{B}_l)^n = (R^2)^n + \sum (A_i \overline{B}_j)^n + \sum (\overline{A}_i B_j)^n$$

or equivalently,

$$\frac{1}{(1 - R^2 T)} = \frac{\prod (1 - A_i \overline{B}_j T) \prod (1 - \overline{A}_i B_j T)}{\prod (1 - A_i \overline{A}_k T) \prod (1 - B_j \overline{B}_l T)}$$

Let $R_{\max}$ be $\max(|A_i|, |B_j|)$, and consider the order of pole at $T = R_{\max}^{-2}$. The numerator's factors $1 - A_i\overline{B}_jT$, $1 - \overline{A}_iB_jT$ are all non-zero there (for if $A_i\overline{B}_j = R_{\max}^2$, by maximality we must have $A_i = B_j = R_{\max}$, in which case we see, using polar coordinates, that $A_i - B_j$, which is forbidden). In the denominator, each of the terms $(1-|A_i|^2T)$, $(1-|B_j|^2T)$ with $|A_i| = R_{\max}$ and $|B_j| = R_{\max}$ vanishes at $T = R_{\max}^{-2}$. Therefore we may conclude that in fact $R = R_{\max}$, and that precisely *one* among all the A_i and B_j has this absolute value. A similar argument shows that $R_{\min} = R$. □

In a similar but lighter vein, we have the following variant, whose proof is left to the reader.

Lemma 1.3. *Let X/F_q be projective and smooth. Let a finite group G operate on X by F_q-automorphisms, and let ρ be an irreducible complex representation of G. Denote by H^i any of the cohomology groups $H_l^i(X)\underset{Z_l}{\otimes}E_\lambda$ with $l \neq p$, or $H_{cirs}^i(X)\underset{W}{\otimes}E_P$. The following conditions are equivalent.*

(1) *For all i, ρ does not occur in H^i, i.e. we have $(H^i)^\rho = 0$.*

(2) *For all $n \geq 1$, we have*

$$S(X/F_q, \rho, n) = 0.$$

II. Gauss and Jacobi Sums as exponential sums, and as eigenvalues of Frobenius 176

We begin by discussing Gauss sums. Let us fix an integer $N \geq 2$ prime to p, and a number field E containing the Np'th roots of unity. Given an additive character ψ of F_p, i.e. a homomorphism

$$\psi : (F_p, +) \to E^\times,$$

we define an additive character ψ_q of each finite extension F_q by composing ψ with the trace map:

$$\begin{tikzcd} F_q \arrow[r, "\mathrm{trace}_{F_q/F_p}"] \arrow[rr, bend right, "\psi_q"'] & F_p \arrow[r, "\psi"] & E^\times \end{tikzcd}$$

Given a character of μ_N, i.e. a homomorphism

$$\chi : \mu_N(E) \to E^\times,$$

a p-adic place P of E, with residue field $F_{N(P)}$, and a finite extension F_q of this residue field, the map "reduction mod P" induces an isomorphism

$$\mu_N(E) \xrightarrow{\sim} \mu_N(F_{N(P)}) = \mu_N(F_q)$$

Because $F_q^\times$ is cyclic, we know that $q \equiv 1 \bmod N$, and that the map $x \to x^{\frac{q-1}{N}}$ defines a surjection

$$F_q^\times \twoheadrightarrow \mu_N(F_q) = \mu_N(F_{N(P)}) \xrightarrow{\sim} \mu_N(E)$$

We define the character χ_q of $F_q^\times$ as the composite

$$\begin{tikzcd} F_q^\times \arrow[r, two heads] \arrow[rr, bend right, "\chi_q"'] & \mu_N(F_q) = \mu_N(F_{N(P)}) \simeq \mu_N(E) \arrow[r] & E^\times. \end{tikzcd}$$

The Gauss sum $g_q(\psi, \chi, \mathsf{P})$ attached to this situation is defined by the formual

$$g_q(\psi, \chi, \mathsf{P}) = \sum_{x \in F_q^\times} \psi_q(x)\chi_q(x)$$

177 An elementary computation shows that

$$g_q(\psi, \chi, \mathsf{P}) = \begin{cases} q-1 & \text{if } \psi, \chi \text{ both trivial} \\ 0 & \text{if } \psi \text{ trivial, } \chi \text{ non-trivial} \\ -1 & \text{if } \psi \text{ non-trivial, } \chi \text{ trivial} \end{cases}$$

while

$$|g_q(\psi,\chi,\mathsf{P})| = \sqrt{q} \text{ if } \psi,\chi \text{ both non-trivial}$$

for any archimedean absolute value on E (cf [47]).

Now consider the Artin-Schreier curve X/F_q, defined to be the complete non-singular model of the affine smooth geometrically connected curve over F_q with equation

$$T^P - T = X^N.$$

Set theoretically, X consists of this affine curve plus a single rational point at ∞. The group $F_q \times \mu_N(F_q)$ operates on X/F_q curve by the affine formulas

$$(a,\zeta) : (T,X) \to (T+a, \zeta X),$$

fixing the point at ∞. Via the "reduction mod P" isomorphism

$$\mu_N(E) \xrightarrow{\sim} \mu_N(F_{N(\mathsf{P})}) = \mu_N(F_q),$$

we may view (ψ,χ) as a character of the group $F_p \times \mu_N(F_q)$:

$$(\psi,\chi)(a,\zeta) = \psi(a)\chi(\zeta).$$

Thus we may speak of the sums

$$S(X/F_q,(\psi,\chi),n) = \frac{1}{pN} \sum_{(a,\zeta)\in F_p\times\mu_N} \psi(a)\chi(\zeta)\sharp \text{ Fix } (F^n \cdot (a,\zeta)^{-1})$$

attached to this situation.

Lemma 2.1. *If χ is non-trivial and ψ is arbitrary, then we have*

$$S(X/F_q,(\psi,\chi),n) = g_{q^n}(\psi,\chi,\mathsf{P}). \tag{2.1.1}$$

Proof. It suffices to treat the case $n = 1$, for we have **178**

$$S(X/F_{q^n},(\psi,\chi),1) = S(X/F_q,(\psi,\chi),n).$$

We can rewrite $S(X/F_q, (\psi, \chi), 1)$ as

$$\sum_{x \in X(\overline{F}q)} \frac{1}{pN} \sum_{\substack{(a,\zeta) s.t. \\ F(x)=(a,\zeta)(x)}} \psi(a)\chi(\zeta)$$

Given any point $x \in X(\overline{F}_q)$, the set of $(a, \zeta) \in F_p \times \mu_N$ which satisfy $F(x) = (a, \zeta)(x)$ is either empty or principal homogeneous under the inertia *subgroup* I_x of $F_p \times \mu_N$ which *fixes* x; therefore if the restriction of (ψ, χ) to this subgroup is *non-trivial*, the inner sum above *vanishes*. Because χ is assumed *non-trivial*, this vanishing applies to the point at ∞ (for which I_x is all of $F_p \times \mu_N$) and to any finite point $(T, 0)$ whose X-coordinate is zero (then $I_{(T,0)} = \{0\} \times \mu_N$).

Given a point (T, X) with $X \neq 0$, we have

$$F(T, X) = (T^q, X^q)$$

and the inertia subgroup $I_{(T,X)}$ is trival. If there is an element $(a, \zeta) \in F_p \times \mu_N$ satisfying $F(T, X) = (T + a, \zeta X)$, then it is given by the formulas

$$a = T^q - T, \ \zeta = X^{q-1}$$

Since the point (T, X) is subject to the defining equation

$$T^p - T = X^N$$

we see that

$$(X^N)^{q-1} = (X^{q-1})^N = \zeta^N = 1, \text{ hence } X^N \in F_q^\times, \ \zeta = (X^N)^{\frac{q-1}{N}}$$
$$T^p - T = X^N \in F_q^\times,$$
$$a = T^q - T = \text{ trace}_{F_q/F_p}(T^p - T) = \text{trace}_{F_q/F_p}(X^N).$$

For each $u \in F_q^\times$, the equations $(T^P - T = u, X^N = u)$ have pN solutions (T, X) over $\overline{F}_q$, all of which satisfy

$$F(T, X) = (a, \zeta)(T, X)$$

179 with the *same* (a, ζ), namely $(\mathrm{trace}_{F_q/F_p}(u), u^{\frac{q-1}{N}})$, and *every* point (T, X) which contributes to our sum lies over some $u \in F_q^\times$. Thus our sum becomes

$$\sum_{u \in F_q^\times} \psi(\mathrm{trace}_{F_q/F_p}(u))\chi(u^{\frac{q-1}{N}})^{\mathrm{dfn}} g_q(\psi, \chi, \mathsf{P}).$$

$\square$

Corollary 2.2. *Let H^i denote any of the cohomology groups $H^i_l(X) \otimes E_\lambda$ with $l \neq p$, or $H^i_{cris}(X) \underset{W}{\otimes} E_P$ of the Artin Schreier curve X/F_q.*

(1) *If ψ and χ are both non-trivial, then the eigenspace $(H^1)^{\psi \cdot \chi}$ is one-dimensional, and we have a direct sum decomposition*

$$H^i = \oplus (H^1)^{\psi \cdot \chi}$$

indexed by the $(p-1(N-1)$ pairs $(\psi, \chi))$ of non-trivial characters.

(2) *The eigenvalue of F on $(H^1)^{\psi \cdot \chi}$ is $-g_q(\psi, \chi, \mathsf{P})$, and for each $n \geq 1$ we have the Hasse-Davenport formula*

$$-g_{q^n}(\psi, \chi, \mathsf{P}) = (-g_q(\psi, \chi, \mathsf{P}))^n.$$

(3) *The group $F_q \times \mu_N$ acts trivially on both H^0 and H^2.*

Proof. That the group acts trivially on both H^0 and H^2 follows from the fact that these are one-dimensional spaces on which F always acts as 1 and q respectively. The descent argument shows that for any automorphism of *finite order* g which commutes with F, Fg also acts as 1 and q on H^0 and H^2 respectively, and hence that g itself acts trivially on H^0 and H^2.

That the multiplicity of (ψ, χ) in H^1 is *one* when both ψ and χ are non-trivial follows from the lemma of the previous section, given the identity (2.1.1) and the known absolute value of gauss sums; and assertion (2) above is just a repetition of part of that lemma in this particular case. To see that no other characters occurs in H^1, we recall that the dimension of H^1 is known to be $2g$, g = genus of X, and so it suffices to verify that $2g = (p-1)(N-1)$. This formula, whose elementary verification we leave to the reader, is in fact valid in any characteristic prime to $N(p-1)$. (Hing: view $T^P - T = X^N$ an an N-fold covering of the T-line!) □

180 We now turn to the consideration of Jacobi sums. We fix an integer $N \geq 2$ prime to p, and a number field E containing the N'th roots of unity. Given a p-adic place P of E, a character χ of μ_N

$$\chi : \mu_N(E) \to E^\times$$

and a finite extension F_q of the residue field $F_{N(\mathsf{P})}$ at P, we obtain the character χ_q

$$\chi_q : F_q^\times \to E^\times$$

in the manner explained above. Given two characters χ, χ' of μ_N, the Jacobi sum $J_q(\chi, \chi', \mathsf{P})$ is defined by the formula

$$J_q(\chi, \chi', \mathsf{P}) \overset{\text{dfn}}{=} \sum_{\substack{x \in Fq \\ x \neq 0,1}} \chi_q(x)\chi'_q(1-x).$$

An elementary computation (cf [14]) shows that if the *product* $\chi\chi'$ is non-trivial, then for any non-trivial additive character ψ of F_p, we have the formula

$$g_q(\psi, \chi, \mathsf{P}) g_q(\psi, \chi', \mathsf{P}) = J_q(\chi, \chi', \mathsf{P}) g_q(\psi, \chi\chi', \mathsf{P})$$

In particular, from the known absolute values of Gauss sums we obtain

$$|J_q(\chi, \chi', \mathsf{p})| = \sqrt{q}$$

for all archimedean absolute values of E, provided that χ, χ', and $\chi\chi'$ are *all* non-trivial.

Now consider the Fermat curve Y/F_q, defined by the homogeneous equation

$$X^N + Y^N = Z^N$$

The group $\mu_N \times \mu_N$ operates on this curve by the formula

$$(\zeta_1, \zeta_2) : (X, Y, Z) \to (\zeta_1 X, \zeta_2 Y, Z).$$

Viewing (χ, χ') as a character of this group

$$(\chi, \chi')(\zeta_1, \zeta_2) \overset{\text{dfn}}{=} \chi(\zeta_1)\chi'(\zeta_2),$$

we may speak of the sums $S(Y/F_q, (\chi, \chi')n)$ attached to this situation.

In complete analogy with the situation for the Artin-Schreier curve, **181**
we have the following lemma and corollary, whose analogous proofs are left to the reader.

Lemma 2.3. *If χ and χ' are non-trivial characters of μ_N such that $\chi\chi'$ is also non-trivial, then we have, for all $n \geq 1$,*

$$S(Y/F_q, (\chi, \chi'), n) = J_{q^n}(\chi, \chi', \mathsf{p}). \tag{2.3.1}$$

Corollary 2.4. *Let H^i denote any of the cohomology groups $H^i_l(Y) \otimes E_\lambda$ with $l \neq p$, or $H^i_{cris}(X) \underset{W}{\otimes} E_\mathsf{p}$ of the Fermat curve Y/F_q.*

(1) *If χ, χ' and $\chi\chi'$ are all non-trivial, then the eigenspace $(H^1)^{(\chi,\chi')}$ is one-dimensional, and we have a direct sum decomposition*

$$H^1 = \oplus (H^1)^{(\chi,\chi')}$$

indexed by the $(N-1)(N-2)$ pairs (χ, χ') of non-trivial characters of μ_N whose product $\chi\chi'$ is also non-trivial.

(2) *The eigenvalue of F on $(H^1)^{(\chi,\chi')}$ is $-J_q(\chi, \chi', \mathsf{P})$, and for each integer $n \geq 1$ we have the Hasse-Davenport formula*

$$-J_{q^n}(\chi, \chi', \mathsf{P}) = (-J_q(\chi, \chi', \mathsf{P}))^n.$$

(3) *The group $\mu_N \times \mu_N$ operates trivially on both H^0 and H^2.*

III. The problem of "explicitly" computing Frobenius. We return now to the general setting of a projective, smooth, and geometrically connected variety X/F_q of dimension d. A tantalizing feature of all the cohomology theories that we have been discussing is that when the variety X "lifts" to characteristic zero, then the corresponding cohomology groups $H^i(X)$ have an "elementary" description in terms of standard algebro-geometric and topological invariants of the lifting.

More precisely, suppose we are given a projective smooth scheme X over $W(F_q)$, together with an F_q-isomorphism of its special fibre with X. (This is a rather strong notion of what a "lifting" of X should mean, but it is adequate for our purposes, and it avoids certain technical problems related to ramification). Then there is a canonical isomorphism

$$H^i_{\text{cris}} \to H^i_{\mathbf{DR}}(X/W(F_q))$$

182 of H^i_{cris} with the algebraic de Rham cohomology of the lifting (cf [19], [27]).

To discuss $H^i_l(X)$, we must in addition choose (!) a complex embedding

$$W(F_q) \hookrightarrow C.$$

By means of such an embedding, we may "extend scalars" to obtain from X/W a projective smooth complex variety X_C, and an associated complex manifold X_C^{an}. For each prime number $l \neq p$, there is a canonical isomorphism

$$H^i_l(X) \to H^i_{\text{top}}(X_C^{\text{an}}, Z) \underset{Z}{\times} Z_l,$$

where H^i_{top} denotes the usual "topological" cohomology.

To emphasize the similarity between these two sorts of isomorphisms, recall that by GAGA and the holomorphic Poincaré lemma, we have a canonical isomorphism

$$\begin{array}{ccccc} H^i_{\mathbf{DR}}(X/W)\underset{W}{\otimes}C & \xrightarrow{\sim} & H^i_{\mathbf{DR}}(X/C) & \xrightarrow{\sim} & H^i_{\top}(X_C^{an}, C) \\ & & & & \Big\uparrow \wr \\ & & & & H^i_{\top}(X^{an}, Z)\underset{Z}{\otimes}C \end{array}$$

Unfortunately, these rather concrete descriptions of the various cohomology *groups* $H^i(X)$ shed little light on their *functoriality*. In the rather unusual case of an F_q-endomorphism $f : X \to X$ which happens to admit a lifting to a W-endomorphism

$$f : X \to X,$$

we have the simple formulas

$$\begin{cases} f^* \text{ on } H^i_{\text{cris}}(X) = f^* \text{ on } H^i_{\mathbf{DR}}(X/W) \\ f^* \text{ on } H^i_l(X) = (f^{an}_C)^* \otimes 1 \text{ on } H^i_{\top}(X^{an}_C, Z) \underset{Z}{\otimes} Z_l,\, l \neq p \end{cases}$$

But for those f which do not lift, we are left somewhat in the dark as to an explicit description of the map f^* on cohomology.

Suppose for example that a finite group G operates on X by F_q-automorphisms, and that this action can be lifted to an action of G on X by W-automorphisms. Then our canonical isomorphisms

$$\begin{cases} H^i_{\text{cris}}(X) \xrightarrow{\sim} H^i_{\mathbf{DR}}(X/W) \\ H^i_l(X) \xrightarrow{\sim} H^i_{\top}(X^{an}_C, Z) \otimes Z_l \text{ for } l \neq p \end{cases}$$

are G-equivariant. In particular, we can "explicitly compute" the mul- 183
tiplicities of the various complex irreducible representations ρ of G in the cohomology of X, and we can "explicitly compute" the various isotypical components of the cohomology. If it turns out that a given irreducible representation ρ occurs in a given H^i with multiplicity *one*, then we know a priori that F must operate on the corresponding isotypical component $(H^i)^\rho$ as a *scalar*, and we know this even when F itself does not lift.

For example, we could recover the isotypical decomposition of H^1 of the Fermat curve Y under the action of $\mu_N \times \mu_N$ by lifting the curve and the group action (use the "same" equations) and making an explicit algebro-geometric or topological calculation of the corresponding isotypical decomposition in characteristic zero. In terms of, say, the crystalline cohomology, we obtain an F-stable decomposition

$$H^1_{\text{cris}}(Y) \xrightarrow{\sim} H^1_{\mathbf{DR}}(Y/W)^{(\chi,\chi')};$$

in a basis of $H^1_{\mathbf{DR}}(Y/W)$ adapted to this decomposition, the matrix of F is the diagonal matrix

$$\begin{pmatrix} \ddots & & O \\ & -J_q(\chi,\chi',\mathsf{P}) & \\ O & & \ddots \end{pmatrix}$$

However, it must be borne in mind that the Fermat curve is atypically susceptible to this sort of analysis; it is unusual for a group action, even on a curve, to be liftable to characteristic zero. For example, the action of F_p on an Artin-Schreier covering of A^1 doesn't lift to characteristic zero. To get around this non-liftability, we will be led to consider the Washnitzer-Monsky cohomology as well, in Chapter VII.

IV. H^1 and abelian varieties; preliminaries. Consider an abelian variety A/F_q, say of dimension g. We denote by End(A) the ring of all F_q-endomorphisms of A, and by End$(A)^0$ the opposite ring. As Z-modules,
184 they are free and finitely generated. For each prime $l \neq p$, the cohomology group $H^1_l(A)$ is a free Z_l-module of rank $2g$, and is an End$(A)^0$-module. (It is also the case that $H^1_{\text{cris}}(A)$ is a free W-module of rank $2g$, and is an End$(A)^0$-module, but we will not make use of this fact for the moment).

Lemma 4.1. *If E is a number field, and λ is a place of E lying over a prime $l \neq p$, the natural maps*

$$\begin{tikzcd} \mathrm{End}(A)^0 \underset{Z}{\otimes} E \arrow[r] & \mathrm{End}(A)^0 \underset{Z}{\otimes} E_\lambda \arrow[r] & \mathrm{End}_{Z_l}(H^1_l(A)) \underset{Z_l}{\otimes} E_\lambda \\ & & \mathrm{End}_{E_\lambda}(H^1_l(A) \underset{Z_l}{\otimes} E_\lambda) \arrow[u, "\wr"'] \end{tikzcd}$$

are all injective.

Proof. The first map is injective simply because $E \subset E_\lambda$, and because End$(A)^0$ is flat over Z. The second map is obtained from the map

$$\mathrm{End}(A)^0 \underset{Z}{\otimes} Z_l \to \mathrm{End}_{Z_l}(H^1_l(A))$$

by tensoring over Z_l with the flat Z_l-module E_λ. In fact this flatness is irrelevant, for the above map is injective and has Z_l-flat cokernel. To see this, recall that (by the Kummer sequence in etale cohomology) we have a canonical isomorphism

$$H^1_l(A) \simeq T_l(\mathrm{Pic}^0(A))(-1) \simeq \mathrm{Hom}(T_l(A), Z_l),$$

under which the map considered above is the "opposite" of the map

$$\mathrm{End}(A) \underset{Z}{\otimes} Z_l \to \mathrm{End}_{Z_l}(T_l(A))$$

Our assertion of its injectivity with Z_l-flat cokernel is equivalent to the injectivity of (any one of) the maps

$$\mathrm{End}(A)/l^n\,\mathrm{End}(A) \to \mathrm{End}(A_{l^n}),$$

and this injectivity follows from the exactness of the sequence

$$0 \to A_{l^n} \to A \xrightarrow{l^n} A \to 0$$

in the etale topology. □

Now consider a projective, smooth and geometrically connected variety X/F_q. Its Albanese variety $\mathrm{Alb}(X)$ is an abelian variety over F_q which for our purposes is best viewed as the *dual* of the Picard va- **185**
riety $\mathrm{Pic}(X)$, itself defined in terms of the Picard scheme Pic_{X/F_q} as $(\mathrm{Pic}^0_{X/F_q})^{\mathrm{red}}$. The Kummer sequence in etale cohomology together with the duality of abelian varieties gives isomorphisms for each $l \neq p$

$$H^1_l(X) \xrightarrow{\sim} T_l(\mathrm{Pic}(X))(-1) \tag{4.1.1}$$

$$H^1(\mathrm{Alb}(X)) \xrightarrow{\sim} T_l(\mathrm{Pic}(\mathrm{Alb}(X))(-1) = T_l(\mathrm{Pic}(X))(-1) \tag{4.1.2}$$

which combine to give a canonical isomorphism

$$H^1_l(X) \simeq H^1_l(\mathrm{Alb}(X)) \quad \text{for} \quad l \neq p \tag{4.1.3}$$

Suppose now that a finite group G operates on X by F_q-automorphisms. Let ρ be an absolutely irreducible representation of G defined over a number field E, which occurs in $H^1(X)$ with multiplicity r. Denote by

$$P_{1,\rho}(T) = 1 + a_1(\rho)T + \cdots + a_r(\rho)T^r \in \mathscr{O}_E[T]$$

the reversed characteristic polynomial of F acting on the space

$$\mathrm{Hom}_G(\rho, H^1(X))$$

of occurrences of ρ in H^1;

$$P_{1,\rho}(T) = \det(1 - TF|\,\mathrm{Hom}_G(\rho, H^1(X)).$$

Let us denote by $\mathrm{Proj}(\rho) \in \mathscr{O}_E[1/\sharp G][G]$ the projector

$$\mathrm{Proj}(\rho) = \frac{\deg(\rho)}{\sharp G} \sum_{g \in G} \mathrm{tr}(\rho(q^{-1})) \cdot [g].$$

By functoriality, G also operates on $\mathrm{Alb}(X)$ by F_q-automorphisms, so we may view $\mathrm{Proj}(\rho)$, or indeed any element of the $\mathscr{O}_E[1/\sharp G]$-group ring of G, as defining an element of $\mathrm{End}(\mathrm{Alb}(X)) \otimes \mathscr{O}_E[1/\sharp G]$.

Proposition 4.2. *In the above situation, we have the formula*

$$(F^r + a_1(\rho)F^{r-1} + \cdots + a_r(\rho)) \cdot \mathrm{Proj}(\rho) = 0$$
$$\mathrm{Proj}(\rho) \cdot (F^r + a_1(\rho)F^{r-1} + \cdots + a_r(\rho)) = 0$$

in $\mathrm{End}(\mathrm{Alb}(X)) \otimes \mathscr{O}_E[1/\sharp G]$. *(N.B. since F and G commute, these formulas are equivalent).*

Proof. Since $\mathrm{End}(\mathrm{Alb}(X)) \otimes \mathscr{O}_E[1/\sharp G]$ is contained in $\mathrm{End}(\mathrm{Alb}(X)) \otimes E$,
186 which is in turn contained in $\mathrm{End}(H^1_l(\mathrm{Alb}(X)) \underset{Z}{\otimes} E_\lambda)$ for any $l \neq p$, it suffices to verify that $F^r + a_1(\rho)F^{r-1} + \cdots + a_r(\rho)$ annihilates $(H^1(\mathrm{Alb}(X))^\rho$. But this space is isomorphic to $(H^1(X))^\rho$, which is in turn isomorphic to $\rho \otimes \mathrm{Hom}_G(\rho, H^1(X))$, with F acting through the second factor, so we need the above polynomial in F to annihilate $\mathrm{Hom}_G(\rho, H^1(X))$. This follows from the Cayley-Hamilton theorem. $\square$

Corollary 4.3. *Let D be any contravariant additive functor from the category of abelian varieties over F_q to the category of $\mathscr{O}_E[1/\sharp G]$-modules. For any element $m \in (D(\mathrm{Alb}(X)))^\rho$, we have*

$$F^r(m) + a_1(\rho)F^{r-1}(m) + \cdots + a_r(\rho) \cdot m = 0$$

in $D(\mathrm{Alb}(X))$.

We will apply this to the functor "Dieudonne module of the formal group of A," constructed a la Honda.

V. Explicit Dieudonné Theory à la Honda; generalities

5.1. Basic Constructions. We being by recalling the notions of formal Lie variety and formal Lie groups. Over any ring R, an n-dimensional formal Lie variety V is a set-valued functor on the category of adic R-algebras which is isomorphic to the functor.

$$R' \to n\text{-tuples of topologically nilpotent elements of } R'.$$

A system of coordinates $X_1, \ldots, X_n$ for V is the choice of such an isomorphism. The coordinate ring $A(V)$ is the R-algebra of all maps of set-functors from V to the "identical functor" $R' \mapsto R'$; in coordinates, $A(V)$ is just the power series ring $R[[X_1, \ldots, X_n]]$. Although the ideal $(X_1, \ldots, X_n)$ in $A(V)$ is *not* intrinsic, the adic topology it defines on $A(V)$ is intrinsic, and $A(V)$, viewed as an adic R-algebra, represents the functor V.

The de Rham cohomology groups $H^i_{\mathbf{DR}}(V/R)$ are the R-modules obtained by taking the cohomology groups of the formal de Rham complex $\Omega^\bullet_{V/R}$ (the separated completion of the "literal" de Rham complex of $A(V)$ as R-algebra); in terms of coordinates $X_1, \ldots, X_n$ for V, $\Omega^\bullet_{V/R}$ is the exterior algebra over $A(V)$ on $dX_1, \ldots, dX_n$, with exterior differentiation $d : \Omega^i \to \Omega^{i+1}$ given by the customary formulas.

A pointed formal Lie variety $(V, 0)$ over R is a formal Lie variety V **187**
over R together with a marked point "0" $\epsilon V(R)$. A formal Lie group G over R is a "group-object" in the category of formal Lie varieties over R.

We denote by $CFG(R)$ the additive category of commutative formal Lie groups over R. The "sum" map

$$\mathrm{sum} : G \times G \to G$$

as well as the two projections

$$\mathrm{pr}_1, \mathrm{pr}_2 : G \times G \to G$$

are morphisms in this category. For $G \in CFG(R)$, we define $D(G/R)$ to be the R-submodule of $H^1_{\mathbf{DR}}(G/R)$ consisting of the primitive elements, i.e. the elements $a \in H^1_{\mathbf{DR}}(G/R)$ such that

$$\mathrm{sum}^*(a) = \mathrm{pr}_1^*(a) + \mathrm{pr}_2^*(a) \text{ in } H^1_{\mathbf{DR}}((G \times G)/R).$$

Lemma 5.1.1. *Over any ring R, the construction $G \to D(G/R)$ defines a (contravariant) additive functor from $CFG(R)$ to R-modules.*

Proof. This is a completely "categorical" result. To begin, let G, $G' \in CFG(R)$, and let $f : G' \to G$ be a homomorphism. Then the diagram

$$\begin{tikzcd} G' \times G' \arrow[r, "\mathrm{sum}"] \arrow[d, "f \times f"] & G' \arrow[d, "f"] \\ G \times G \arrow[r, "\mathrm{sum}"] & G \end{tikzcd}$$

commutes, as do the analogous diagrams with "sum" replaced by pr_1 or pr_2. Therefore given any element $a \in H^1_{\mathbf{DR}}(G/R)$, we have

$$\mathrm{sum}^*(f^*(a)) - \mathrm{pr}_1^*(f^*(a)) - \mathrm{pr}_2(f^*(a)) =$$
$$(f \times f)^*(\mathrm{sum}^*(a) - \mathrm{pr}_1^*(a) - \mathrm{pr}_2^*(a)).$$

In particular, if $a \in D(G/R)$ then $f^*(a) \in D(G'/R)$.

Given f_1, f_2 homomorphisms $G' \to G$, let f_3 be their sum. Then we have a commutative diagram

$$\begin{tikzcd} G' \arrow[r, "f_1 \times f_2"] \arrow[rr, bend right, "f_3"'] & G \times G \arrow[r, "\text{sum}"] & G \end{tikzcd}$$

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as well as a commutative diagram

$$\begin{tikzcd} G' \arrow[r, "f_1 \times f_2"] \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] & G \times G \arrow[r, shift left, "\text{pr}_1"] \arrow[r, shift right, "\text{pr}_2"'] & G \end{tikzcd}$$

Therefore for any $a \in H^1_{\mathbf{DR}}(G/R)$, we have

$$f_3^*(a) - f_1^*(a) - f_2^*(a) = (f_1 \times f_2)^*(\text{sum}^*(a) - \text{pr}_1^*(a) - \text{pr}_2^*(a)).$$

In particular, if $a \in D(G/R)$, then $f_3^*(a) = f_1^*(a) + f_2^*(a)$. □

For the remainder of this section, we will consider a ring R which is flat over Z, and an ideal $I \subset R$ which has divided powers. The flatness means that if we denote by K the Q-algebra $R \otimes Q$, then $R \subset K$. That the ideal $I \subset R$ has divided powers means that for any integer $n \geq 1$, and any element $i \in I$, the element $i^n/n!$ of K actually lies in I.

Given a formal Lie variety V over R, we denote by $V \otimes K$ the formal Lie variety over K obtained by extension of scalars. In terms of coordinates $X_1, \ldots, X_n$ for V, $A(V \otimes K)$ is the power-series ring $K[[X_1, \ldots, X_n]]$. We say that an element of $A(V \otimes K)$ is *integral* if it lies in the subring $A(V)$; similarly, an element of the de Rham complex $\Omega_{V\otimes K/K}$ is said to be *integral* if it lies in the subcomplex $\Omega_{V/R}$.

Lemma 5.1.2. *Let* $(V, 0)$ *be a pointed Lie variety over a Z-flat ring R. Then exterior differentiation induces an isomorphism of R-modules*

$$\frac{\{f \in A(V \otimes K) | f(0) = 0, df \text{ integral}\}}{\{f \in A(V) | f(0) = 0\}} \xrightarrow{\sim} H^1_{\mathbf{DR}}(V/R)$$

which is compatible with morphisms of pointed Lie varieties.

189 *Proof.* Because K is a Q-algebra, the formal Poincare lemma gives $H^0_{\mathbf{DR}}(V \otimes K/K) = K$, $H^i_{\mathbf{DR}}(V \otimes K/K) = 0$ for $i \geq 1$. Therefore any closed one-form on V/R can be written as df with $f \in A(V \otimes K)$, and this f is unique up to a constant. If we normalize f by the condition $f(0) = 0$, we get the asserted isomorphism. □

Key Lemma 5.1.3. *Let* $(V, 0)$ *and* $(V', 0)$ *be pointed formal Lie varieties over a Z-flat ring R, and let* $I \subset R$ *be an ideal with divided powers. If* f_1, f_2 *are two pointed morphisms* $V' \to V$ *such that* $f_1 = f_2$ *mod I, then the induced maps*

$$f_1^*, f_2^* : H^1_{\mathbf{DR}}(V/R) \to H^1_{\mathbf{DR}}(V'/R)$$

are equal.

Proof. Let φ_1, φ_2 denote the algebra homomorphisms $A(V) \to A(V')$ corresponding to f_1 and f_2. By the previous lemma, we must show that for every element $f \in A(V \otimes K)$ with $f(0) = 0$ and df integral, the difference $\varphi_1(f) - \varphi_2(f)$ lies in $A(V')$, i.e. is itself integral. (Because f_1 and f_2 were assumed pointed, this difference automatically has constant term zero).

In terms of pointed coordinates $X_1, \ldots, X_n$ for V' and $Y_1, \ldots, Y_m$ for V, the maps φ_1 and φ_2 are given by substitutions

$$\varphi_1(f(Y)) = f(\varphi_1(X))$$
$$\varphi_2(f(Y)) = f(\varphi_2(X))$$

where $\varphi_1(X)$, $\varphi_2(X)$ are m-tuples of series in $X = (X_1, \ldots, X_n)$ without constant term. The hypothesis $f_1 = f_2$ mod I means that the component-by-component difference $\Delta = \varphi_2(X) - \varphi_1(X)$ satisfies

$$\Delta(0) = 0, \Delta \text{ has all coefficients in } I.$$

We now compute using Taylor's formula, and usual multi-index notations:

$$\varphi_2(f) - \varphi_1(f) = f(\varphi_2(X)) - f(\varphi_1(X))$$

$$
\begin{aligned}
&= f(\varphi_1(X) + \Delta) - f(\varphi_1(X)) \\
&= \sum_{|\underline{n}| \geq 1} \frac{\Delta^{\underline{n}}}{(\underline{n})!} \left(\frac{\partial^{\underline{n}}}{\partial Y^{\underline{n}}} f \right) (\varphi_1(X)).
\end{aligned}
$$

This last sum is X-adically convergent (because Δ has no constant term), **190**
and its individual terms are integral (because Δ has coefficients in the divided power ideal I, the terms $\Delta^{\underline{n}}/(\underline{n})!$ all have coefficients in I, and hence in R; because df is integral, all the first partials $\partial f/\partial Y_i$ are integral, and a fortiori all the higher partials are integral). □

Theorem 5.1.4. *Let R be a $\mathbf{Z}$-flat ring, and $I \subset R$ a divided power ideal. Let G, G' be commutative formal Lie groups over R, and denote by G_0, G'_0 the commutative formal Lie groups over $R_0 = R/I$ obtained by reduction mod I.*

(1) *If $f : G' \to G$ is any morphism of pointed formal Lie varieties whose reduction mod I, $f_0 : G'_0 \to G_0$, is a group homomorphism, then the induced map $f^* : H^1_{\mathbf{DR}}(G/R) \to H^1_{\mathbf{DR}}(G'/R)$ maps $D(G/R)$ to $D(G'/R)$.*

(2) *If f_1, f_2, f_3 are three maps $G' \to G$ of pointed formal Lie varieties whose reductions mod I are group homomorphisms which satisfy $(f_3)_0 = (f_1)_0 + (f_2)_0$ in $\mathrm{Hom}(G'_0, G_0)$, then for any element $a \in D(G/R)$ we have*

$$f_1^*(a) + f_2^*(a) = f_3^*(a).$$

Proof. If $f : G' \to G$ is a pointed map which reduces mod I to a group homomorphism, the diagram

$$
\begin{array}{ccc}
G' \times G' & \xrightarrow{\text{sum}} & G' \\
\downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} \\
G \times G & \xrightarrow{\text{sum}} & G
\end{array}
$$

commutes mod I, i.e.

$$\text{sum}\,(f \times f) \equiv f\ \text{sum} \quad \text{mod } I.$$

and hence for any $a \in H^1_{\mathbf{DR}}(G/R)$ we have, by the previous lemma,

$$(f \times f)^*(\mathrm{sum}^*(a)) = \mathrm{sum}^*(f^*(a))$$

The analogous diagrams with "sum" replaced by pr_1 or pr_2 commute, hence

$$(f \times f)^*(\mathrm{pr}_i^*(a)) = \mathrm{pr}_i^*(f^*(a)) \quad \text{for } i = 1, 2.$$

191 Combining these, we find

$$(f \times f)^*(\mathrm{sum}^*(a) - \mathrm{pr}_1^*(a) - \mathrm{pr}_2^*(a)) =$$
$$\mathrm{sum}^*(f^*(a)) - \mathrm{pr}_1^*(f^*(a)) - \mathrm{pr}_2^*(f^*(a)).$$

In particular, if $a \in D(G/R)$ then $f^*(a) \in D(G'/R)$.

Similarly, if f_1, f_2 and f_3 are as in the assertion of the theorem, the diagram

$$\begin{tikzcd} G' \arrow[r, "f_1 \times f_2"] \arrow[rr, bend right, "f_3"'] & G \times G \arrow[r, "\mathrm{sum}"] & G \end{tikzcd}$$

commutes mod I, and the diagram

$$\begin{tikzcd} G' \arrow[r, "f_1 \times f_2"] \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_2"'] & G \times G \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & G \end{tikzcd}$$

commutes. So again using the preceding lemma, we see that for any $a \in H^1_{\mathbf{DR}}(G/R)$, we have

$$f_3^*(a) - f_1^*(a) - f_2^*(a) = (f_1 \times f_2)^*(\mathrm{Sum}^* a) - \mathrm{pr}_1^*(a) - \mathrm{pr}_2^*(a)).$$

In particular, for $a \in D(G/R)$, we obtain the asserted formula

$$f_3^*(a) = f_1^*(a) + f_2^*(a).$$

□

Let $CFG(R;R_0)$ denote the additive category whose *objects* are the commutative formal Lie groups over R, but in which the morphisms are the homomorphisms between their reductions mod I:

$$\mathrm{Hom}_{CFG(R,R_0)}(G', G) = \mathrm{Hom}(G'_0, G_0).$$

Given a homomorphism $f_0 : G'_0 \to G_0$, it always lifts to a pointed morphism $f : G' \to G$ of formal Lie varieties (just lift its power-series 192
coefficients one-by-one, and keep the constant terms zero).

According to the theorem, the induced map

$$f^* : D(G/R) \to D(G'/R)$$

is *independent* of the choice of pointed lifting f of f_0. So it makes sense to denote the induced map

$$(f_0)^* : D(G/R) \to D(G'/R).$$

Theorem 5.1.5. *Let R be a Z-flat ring, and $I \subset R$ a divided power ideal. Then the construction $G \mapsto D(G/R)$, $f_0 \mapsto (f_0)^* =$ (any pointed lifting)* defines a contravariant additive functor from the category $CFG(R;R_0)$ to the category of R-modules.*

Proof. This is just a restatement of the previous theorem. □

Remarks. (1) Thanks to Lazard [33], we know that every commutative formal Lie group G_0 over R_0 lifts to a commutative formal Lie group G over R. If G' is another lifting of G_0, then the identity endomorphism of G_0 is an isomorphism of G' with G in the category $CFG(R;R_0)$. Formation of the induced isomorphism $D(G/R) \xrightarrow{\sim} D(G'/R)$ provides a transitive system of identifications between the D's of all possible liftings. In this way, it is possible to view the construction

$$G_0 \mapsto D(G/R), \text{ where } G \text{ is some lifting of } G_0$$

as providing a contravariant additive functor from $CFG(R_0)$ to the category of R-modules. We will not pursue that point of view here.

(2) Even without appealing to Lazard, one can proceed in an elementary fashion by observing that any commutative formal Lie group G_0 over R_0 can certainly be lifted to a formal Lie "monoid with unit" M over R (simply lift the individual coefficients of the group law, and always lift 0 to 0). For a monoid, one can still define $D(M/R)$ as the primitive elements of $H^1_{\mathbf{DR}}(M/R)$, and one can still show exactly as before that the construction

$$G_0 \to D(M/R), \quad M \text{ any monoid lifting of } G_0$$

193 defines a contravariant additive functor from $CFG(R_0)$ to R-modules.

A variant. The reader cannot have failed to notice the purely formal nature of most of our arguments. We might as well have begun with *any* contravariant functor H from formal Lie varieties over a Z-flat ring R to R-modules for which the key lemma (5.1.3) holds. One such H, which we will denote $H^1_{\mathbf{DR}}(V/R; I)$, is defined as H^1 of the *subcomplex* of the de Rham complex of V/R

$$\text{“}IA(V)\text{''} \to \Omega^1_{V/R} \to \Omega^2_{V/R} \to \dots$$

where "$IA(V)$" denotes the kernel of reduction mod I:

$$\text{“}IA(V)\text{''} = \operatorname{Ker}(A(V) \twoheadrightarrow A(V_0)).$$

In terms of coordinates for V, "$IA(V)$" is the ideal consisting of those series all of whose coefficients lie in I. The analogue of lemma (5.1.2) becomes

$$\frac{\{f \in A(V \otimes K) | f(0) = 0, dt \text{ integral}\}}{\{f \in \text{“}IA(V)\text{''} | f(0) = 0\}} \xrightarrow[\sim]{d} H^1_{\mathbf{DR}}(V/R; I).$$

This much makes sense for *any* ideal $I \subset R$. If I has divided powers, then the proof of the key lemma (**??**) is almost word-for-word the same. (It works because the terms $\Delta^{\underline{n}}/(\underline{n})!$ all have coefficients in I.)

The corresponding theory, "primitive elements in $H^1_{\mathbf{DR}}(G/R; I)$," is denoted $D_1(G/R)$. In terms of coordinates $X = (X_1, \ldots, X_n)$ for G, we have the explicit description

$$D_1(G/R) = \\ = \frac{\{f \in K[[X]] | f(0) = 0, df \text{ integral}, f(X \underset{G}{+} Y) - f(X) - f(Y) \in I[[X, Y]]\}}{\{f \in I[[X]] | f(0) = 0\}}$$

as compared with the explicit description

$$D(G/R) = \\ = \frac{\{f \in K[[X]] | f(0) = 0, df \text{ integral}, f(X \underset{G}{+} Y) - f(X) - f(Y) \text{ integral}\}}{\{f \in R[[X]] | f(0) = 0\}}$$

For ease of later reference we summarize the above discussion in a the- 194
orem.

Theorem 5.1.6. *Let R be a Z-flat ring, and $I \subset R$ a divided power ideal. The key lemma (**??**) holds for $H^1_{\mathbf{DR}}(V/R; I)$, and theorems* (5.1.4) *and* (5.1.5) *hold for $D_1(G/R)$.*

The natural map $D_1 \to D$ is not an isomorphism, but its kernel and cokernel are visibly killed by I. In the work of Honda and Fontaine, it is D_1 rather than D which occurs; in the work of Grothendieck and Mazur-Messing ([17], [35]), it is D which arises more naturally.

Let us denote by $\underline{\omega}_{G/R}$ the R-module of translation-invariant, or what is the same, primitive, one-forms on G/R. Because G is commutative, every element $w \in \underline{\omega}_{G/R}$ is a closed form, so we have natural maps

$$\begin{tikzcd} \underline{\omega}_{G/R} \arrow[r] & D_1(G/R) \arrow[d] \\ & D(G/R) \end{tikzcd}$$

(Notice that in the extreme case $I = (0)$, the map $\underline{\omega} \to D_1$ is an isomorphism!)

Lemma 5.1.7. *Suppose R flat over Z, and $I \subset R$ an ideal. We have exact sequences*

$$0 \to \mathrm{Hom}_{R\text{-groups}}(G, G_a) \xrightarrow{d} \underline{\omega}_{G/R} \to D(G/R)$$
$$0 \to |Hom_{R/I\text{-groups}}(G \underset{R}{\otimes} (R/I), (G_a)_{R/I}) \to D_1(G/R) \to D(G/R)$$

Proof. The first is the special case $I = 0$ of the second; the second is clear from the explicit description of D_1 and D given above. □

Corollary 5.1.8. *If* $\mathrm{Hom}_{R\text{-groups}}(G, G_a) = 0$*, then the natural maps*

$$\underline{\omega}_G \to D_1(G/R) \quad \text{and} \quad \underline{\omega}_G \to D(G/R)$$

are injective.

The reader interested in obtaining the limit formula for Jacobi sums
195 conjectured by Honda may skip the rest of this chapter! Others may also be tempted.

5.2 Interpretation via Ext a La Mazur-Messing We denote by

$$\mathrm{Ext}(G, G_a)$$

the group of isomorphism classes of extensions of G by G_a, i.e. of short exact sequences

$$0 \to G_a \to E \to G \to 0$$

of abelian f.p.p.f. sheaves on (Schemes/R). We denote by $\mathrm{Ext}^{\mathrm{rigid}}(G, G_a)$ the group of isomorphism classes of "rigidified extensions," i.e. pairs consisting of an extension of G by G_a together with a *splitting* of the corresponding extension of Lie algebras:

$$\begin{tikzcd} 0 \arrow[r] & R = \mathrm{Lie}(G_a) \arrow[r] & \mathrm{Lie}(E) \arrow[r] & \mathrm{Lie}(G) \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$

Because $\mathrm{Lie}(G)$ is a free R-module of rank $n = \dim(G)$, any extension of G by G_a admits such a rigidification, which is indeterminate up to an element of $\mathrm{Hom}(\mathrm{Lie}(G), \mathrm{Lie}(G_a)) = \underline{\omega}_{G/R}$. Passing to isomorphism classes and remembering that the set of splittings of a trivial extension of G by G_a is itself principal homogeneous under $\mathrm{Hom}(G, G_a)$, we obtain a four-term exact sequence (valid over *any* ring R)

$$\mathrm{Hom}(G, G_a) \xrightarrow{d} \underline{\omega}_G \to \mathrm{Ext}^{\mathrm{rigid}}(G, G_a) \to \mathrm{Ext}(G, G_a) \to 0$$

Theorem 5.2.1. *If R is flat over Z, there is a natural isomorphism*

$$D(G/R) \xleftarrow{\sim} \mathrm{Ext}^{\mathrm{rigid}}(G, G_1)$$

in terms of which the resulting four term exact sequence

$$0 \to \mathrm{Hom}(G, G_a) \to \underline{\omega}_G \to D(G/R) \to \mathrm{Ext}(G, G_a) \to 0$$

is the concatenation of the three-term sequence of (5.1.3) *and the map* $D(G/R) \to \mathrm{Ext}(G, G_a)$ defined by

$$f \to \text{ the class of the symmetric 2-cocycle}$$

$$\partial f = f(X \underset{G}{+}) - f(X) - f(Y)$$

Proof. We begin by constructing the isomorphism. Given a rigidified extension **196**

$$\begin{tikzcd}
0 \arrow[r] & G_a \arrow[r] & E \arrow[r] & G \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Lie}(G_a) \arrow[r] & \mathrm{Lie}(E) \arrow[r] & \mathrm{Lie}(G) \arrow[r] \arrow[l, bend right, "s"'] & 0,
\end{tikzcd}$$

extend scalars from R to $K = R \otimes Q$. Because K is a Q-algebra, the Lie functor defines an equivalence of categories between commutative formal Lie groups over K and free finitely generated K-modules.

Therefore there is a unique splitting as K-groups

$$\begin{tikzcd}
0 \arrow[r] & G_a \underset{R}{\otimes} K \arrow[r] & E \underset{R}{\otimes} K \arrow[r] & G \underset{R}{\otimes} K \arrow[r] \arrow[l, bend right, "\exp(s)"'] & 0
\end{tikzcd}$$

whose differential is the given splitting S on Lie algebras.

At the same time, we may choose a cross section S in the category of *pointed* f.p.p.f. sheaves over R

$$\begin{tikzcd} 0 \arrow[r] & G_a \arrow[r] & E \arrow[r] & G \arrow[r] \arrow[l, bend right, "S"'] & 0. \end{tikzcd}$$

The difference $f = S - \exp(s)$ is a pointed map from $G \otimes K$ to $(G_a) \otimes K$, i.e. an element $f \in A(G \otimes K)$, and it satisfied $f(0) = 0$. We have $df = dS - s$, so df is integral, and the formula

$$f(X \underset{G}{+} Y) - f(X) - f(Y) = S(X \underset{G}{+} Y) - S(X) - s(Y),$$

valid because $\exp(s)$ is a homomorphism, shows that $f(X \underset{G}{+} Y) - f(X) - f(Y)$ is integral.

Because the initial choice of S is indeterminate up to addition of a pointed map from G to G_a, the class of $f = S - \exp(s)$ in $D(G/R)$ is well-defined independently of the choice of S, and it vanishes if and only if $\exp(s)$ is itself integral, i.e. if and only if the original rigidified extension is trivial as a rigidified extension. Thus we obtain an injective map

$$\mathrm{Ext}^{\mathrm{rigid}}(G, G_a) \to D(G/R).$$

197 To see that it is an isomorphism, note that in any case the map $D(G/R) \to \mathrm{Ext}(G, G_a)$ defined by $f \to$ the class of ∂f sits in an exact sequence

$$0 \to \mathrm{Hom}(G, G_a) \to \underline{\omega}_G \to D(G/R) \to \mathrm{Ext}(G, G_a),$$

which receives the $\mathrm{Ext}^{\mathrm{rigid}}$ exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(G, G_a) \arrow[r] & \underline{\omega}_G \arrow[r] & D(G/R) \arrow[r] & \mathrm{Ext}(G, G_a) \\ 0 \arrow[r] & \mathrm{Hom}(G, G_a) \arrow[r] \arrow[u, equal] & \underline{\omega}_G \arrow[r] \arrow[u, equal] & \mathrm{Ext}^{\mathrm{rigid}}(G, G_a) \arrow[r] \arrow[u, hook] & \mathrm{Ext}(G, G_a) \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

The result is now visible. $\square$

Given an ideal $I \subset R$, we denote by $\mathrm{Ext}(G, G_a; I)$ the group of isomorphism classes of pairs consisting of an extension of G by G_a together with a splitting of its reduction modulo I. We denote by $\mathrm{Ext}^{\mathrm{rigid}}(G, G_a; I)$ the group of isomorphism classes of pairs consisting of a rigidified extension and a splitting of the reduction mod I of the underlying extension. Analogously to the previous theorem, we have

Theorem 5.2.2. *If R is flat over Z, and $I \subset R$ an ideal, there is a natural isomorphism*

$$\mathrm{Ext}^{\mathrm{rigid}}(G, G_a; I) \xrightarrow{\sim} D_1(G/R)$$

and a four-term exact sequence

$$0 \to \mathrm{Hom}(G, G_a) \to \underline{\omega}_G \to D_1(G/R) \xrightarrow{\partial} \mathrm{Ext}(G, G_a; I) \to 0$$

in which the map ∂, given by

$$f \to \text{ the class of the symmetric 2-cocycle}$$
$$\partial f = f(X \underset{G}{+} Y) - f(X) - f(Y),$$

corresponds to the map "forget the rigidification" on Ext*'s.*

5.3 The Case of p-Divisible Formal Groups Let p be a prime number. A ring R is said to be p-adic if it is complete and separated in its p-adic topology, i.e., if

$$R \xrightarrow{\sim} \varprojlim R/p^n R.$$

A commutative formal Lie group G over a p-adic ring R is said to be 198
p-divisible of height h if the map 'multiplication by p" makes $A(G)$ into a finite locally free module over itself of rank p^h.

If we denote by G^v the dual of G in the sense of p-divisible groups, it makes sense to speak of the tangent space of G^v at the origin, noted $\underline{t}_{G^v}$; it is known that $\underline{t}_{G^v}$ is a locally free R-module of rank $h - \dim(G)$, and that there is a canonical isomorphism

$$\mathrm{Ext}(G, G_a) \xrightarrow{\sim} \underline{t}_{G^v}. \tag{5.3.1}$$

Because G is p-divisible and R is p-adic, $\operatorname{Hom}(G, G_a) = 0$, and the four-term exact sequence becomes a Hodge-like exact sequence

$$0 \to \underline{\omega}_G \to D(G/R) \to \underline{t}_{G^\vee} \to 0 \tag{5.3.2}$$

Thus we find

Theorem 5.3.3.

(1) *If R is a p-adic ring which is flat over Z, then for a p-divisible commutative formal Lie group G over R, the R-module D(G/R) is locally free of rank h = height (G), and its formation commutes with arbitrary extension of scalars of Z-flat p-adic rings.*

If an addition $I \subset R$ is an ideal which is closed in the p-adic topology, then R/I is again a p-adic ring, $G \otimes (R/I)$ is still p-divisible, and therefore admits no non-trivial homomorphisms to G_a over R/I. It follows that

$$\begin{cases} D_1(G/R) \subset D(G/R) \\ \operatorname{Ext}(G, G_a; I) \xrightarrow{I} I \operatorname{Ext}(G, G_a) \simeq I \cdot \underline{t}_{G^\vee} \end{cases} \tag{5.3.4}$$

and we have a short exact sequence

$$0 \to \underline{\omega}_G \to D_1(G/R) \to I \cdot \underline{t}_{G^\vee} \to 0. \tag{5.3.5}$$

5.5 Relation to the Classical Theory Let k be a perfect field of characteristic $p > 0$, and take $R = W(k)$, $I = (p)$. Let CW denote the k-group-functor "Witt covectors" (in the notations of Fontaine ([13]), with its structure of $W(k)$-module. According to Fontaine, for any formal Lie variety V over $W(k)$, we obtain a $W(k)$-linear isomorphism

$$w : CW(A(V \otimes k)) \xrightarrow{\sim} H^1_{\mathbf{DR}}(V/W(k); (p)) \tag{5.5.1}$$

199 by defining

$$w(\ldots, a_{-a}, \ldots, a_0) = d\left(\sum_{n \geq 0} \frac{(\widetilde{a}_{-a})^{p^n}}{p^n}\right) \tag{5.5.2}$$

where $\widetilde{a}_{-n}$ denotes an arbitrary lifting to $A(V)$ of $a_{-n} \in A(V \otimes k)$. Similarly, we can define, following Grothendieck, Mazur-Messing ([35]), a σ-linear isomorphism

$$\psi : CW(A(V \otimes k)) \xrightarrow{\sim} H^1_{\mathbf{DR}}(V/W(k)) \tag{5.5.3}$$

by the formula

$$\psi(\ldots, a_{-n}, \ldots, a_0) = d\left(\sum_{n\geq 0} \frac{(\widetilde{a}_{-n})^{p^{n+1}}}{p^{n+1}}\right). \tag{5.5.4}$$

These isomorphisms sit in a commutative diagram

$$\begin{tikzcd} & H^1_{DR}(V/W(k);(p)) \arrow[dd, "\wr"', "\frac{1}{p}F"] \\ CW(A(V \otimes k)) \arrow[ur, "\sim", "w"'] \arrow[dr, "\sim", "\psi"'] & \\ & H^1_{\mathbf{DR}}(V/W(k)). \end{tikzcd} \tag{5.5.5}$$

When G is a commutative formal Lie group over $W(k)$ which is *p-divisible*, the "classical" Dieudonne module of $G_0 = G \otimes k$ is *defined* as

$$\begin{tikzcd} M(G_0) \arrow[r, equal, "\mathrm{dfn}"] & \mathrm{Hom}_{k-gp}(G_0, CW) \arrow[d, equal] \\ & \text{the primitive elements in } CW(A(G_0)). \end{tikzcd} \tag{5.5.6}$$

Combining this definition with the previous isomorphisms, we find a commutative diagram of isomorphisms

$$\begin{tikzcd} & D_p(G/W(k)) \arrow[dd, "\wr"', "\frac{1}{p}F"] \\ M(G_0) \arrow[ur, "\sim", "w"'] \arrow[dr, "\sim", "\psi"'] & \\ & D(G/W(k)). \end{tikzcd} \tag{5.5.7}$$

200 5.6 Relation with Abelian Schemes and with the General Theory

In this section, we recall without proofs some of the main results and compatibilities of the general D-theory of Grothendieck and Mazur-Messing.

Given an abelian scheme A over an arbitrary ring R, there are canonical isomorphisms

$$\begin{cases} \mathrm{Ext}^{\mathrm{rigid}}(A, G_a) \xrightarrow{\sim} H^1_{\mathbf{DR}}(A/R) \\ \mathrm{Ext}(A, G_a) \xrightarrow{\sim} H^1(A, O_A) = \mathrm{Lie}(A^\nu) \end{cases} \tag{5.6.1}$$

in terms of which the $\mathrm{Ext}^{\mathrm{rigid}}$-exact sequence "becomes" the Hodge exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \underline{\omega}_A \arrow[r] \arrow[d, equal] & \mathrm{Ext}^{\mathrm{rigid}}(A, G_a) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Ext}(A, G_a) \arrow[r] \arrow[d, "\wr"'] & 0 \\ 0 \arrow[r] & \underline{\omega}_A \arrow[r] & H^1_{\mathbf{DR}}(A/R) \arrow[r] & H^1(A, O_A) \arrow[r] \arrow[d, equal] & 0 \\ & & & \mathrm{Lie}(A^\nu) & \end{tikzcd} \tag{5.6.2}$$

Given a p-divisible (Barsotti-Tate) group $G = \varinjlim G_n$ over a ring R in which p is nilpotent, the exact sequence

$$0 \to G_n \to G \xrightarrow{p^n} G \to 0 \tag{5.6.3}$$

for any n sufficiently large that $p^n = 0$ in R, leads to a canonical isomorphism

$$\mathrm{Lie}(G^\nu) = \mathrm{Lie}(G_n^\nu) = \mathrm{Hom}(G_n, G_a) \xrightarrow{\sim} \mathrm{Ext}(G, G_a). \tag{5.6.4}$$

The $\mathrm{Ext}^{\mathrm{rigid}}$-exact sequence can thus be written

$$0 \to \underline{\omega}_G \to \mathrm{Ext}^{\mathrm{rigid}}(G, G_a) \to \mathrm{Lie}(G^\nu) \to 0, \tag{5.6.5}$$

where $\underline{\omega}_G$ is the R-linear dual of $\mathrm{Lie}(G)$.

Given an abelian scheme A over a ring R in which p is nilpotent, the exact sequence

$$0 \to A_{p^n} \to A \xrightarrow{p^n} A \to 0 \tag{5.6.6}$$

for any n sufficiently large that $p^n = 0$ in R leads to a canonical isomorphism 201

$$\mathrm{Lie}(A^\vee) = \mathrm{Lie}(A^\vee_{p^n}) = \mathrm{Hom}(A_{p^n}, G_a) \xrightarrow{\sim} \mathrm{Ext}(A, G_a). \tag{5.6.7}$$

Therefore the inclusion $A_{p^\infty} \hookrightarrow A$ induces an isomorphism

$$\mathrm{Ext}(A, G_a) \xrightarrow{\sim} \mathrm{Ext}(A_{p^\infty}, G_a) \tag{5.6.8}$$

(the identity on $\mathrm{Hom}(A_{p^n}, G_a)$!), and consequently we obtain a commutative diagram of isomorphisms

$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega}_A \arrow[r] \arrow[d, equal] & \mathrm{Ext}^{\mathrm{rigid}}(A, G_a) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Ext}(A, G_a) \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & \underline{\omega}_{A_{p^\infty}} \arrow[r] & \mathrm{Ext}^{\mathrm{rigid}}(A_{p^\infty}, G_a) \arrow[r] & \mathrm{Ext}(A_{p^\infty}, G_a) \arrow[r] & 0,
\end{tikzcd} \tag{5.6.9}$$

i.e., an isomorphism

$$H^1_{\mathbf{DR}}(A/R) \xrightarrow{\sim} D(A_{p^\infty}/R) \tag{5.6.10}$$

compatible with the Hodge filtration.

For variable $B-T$ groups G over a fixed ring R in which p is nilpotent, the functors $\underline{\omega}_G$, $\mathrm{Lie}(G^\vee)$, and consequently $\mathrm{Ext}^{\mathrm{rigid}}(G, G_a)$, are exact functors whose values are locally free R-modules of finite rank; their formation commutes with arbitrary extension of scalars of rings in which p is nilpotent.

Following Grothendieck and Mazur-Messing we *define*

$$D(G/R) \overset{\mathrm{dfn}}{=\!=\!=} \mathrm{Ext}^{\mathrm{rigid}}(G, G_a) \tag{5.6.11}$$

when G is a $B-T$ group over a ring R in which p is nilpotent.

When R is a p-adic ring, and G is a $B-T$ group over R, we *define*

$$\begin{cases} D(G/R) = \varprojlim_n D(G \otimes (R/p^n R)/(R/p^n R)) \\ \mathrm{Lie}(G) = \varprojlim \mathrm{Lie}(G \otimes (R/p^n R)) \\ \underline{\omega}_G = \varprojlim \underline{\omega}_G \otimes (R/p^n R) \end{cases} \tag{5.6.12}$$

202 Thus for variable $B-T$ groups G over a p-adic ring R, the functors $\underline{\omega}_G$, $\mathrm{Lie}(G^v)$ and $D(G/R)$ are all exact functors in locally free R-modules of finite rank, sitting in an exact sequence

$$0 \to \underline{\omega}_G \to D(G/R) \to \mathrm{Lie}(G^v) \to 0 \tag{5.6.13}$$

whose formation commutes with arbitrary extension of scalars of p-adic rings. When A is an abelian scheme over a p-adic ring R, we obtain an isomorphism

$$H^1_{\mathbf{DR}}(A/R) \xrightarrow{\sim} D(A(p^\infty)/R),$$

compatible with Hodge filtrations, by passage to the limit.

As we have seen in the previous section, this general $\mathrm{Ext}^{\mathrm{rigid}}$ notion of $D(G/R)$ agrees with our more explicit one in the case that both are defined, namely when G is a p-divisible formal group over a Z-flat p-adic ring R.

5.7 Relation with Cohomology

Theorem 5.7.1. *Let A be an abelian scheme over the Witt vectors $W(k)$ of an algebraically closed field k of characteristic $p > 0$. There is a short exact sequence of W-modules*

$$0 \to H^1_{\mathrm{et}}(A \otimes k, Z_p) \otimes W \xrightarrow{\alpha} H^1_{\mathrm{cris}}(A \otimes k/W) \xrightarrow{\beta} D(\widehat{A}/W) \to 0$$

which is functorial in $A \otimes k$.

Proof. We begin by defining the maps α and β. They will be defined by passage to the limit from maps α_n, β_n in an exact sequence

$$0 \to H^1_{\mathrm{et}}(A \otimes k, Z/p^n Z) \otimes W_n \xrightarrow{\alpha_n} H^1_{\mathrm{cris}}(A \otimes k/W_n) \tag{5.7.2}$$

$$\xrightarrow{\beta_n} D(\widehat{A} \otimes W_n/W_n) \to 0.$$

of W_n-modules.

An element of $H^1(A \otimes k, Z/p^nZ)$ is (the isomorphism class of) a Z/p^nZ-torsor over $A \otimes k$. An element of $H^1_{\text{cris}}(A \otimes k/W_n)$ is (the isomorphism class of) a rule which assigns to every test situation $Y \hookrightarrow Y_n$ **203**
consisting an $A \otimes k$ scheme Y and a divided-power thickening of Y to a W_n-scheme Y_n a G_a-torsor on Y_n in a way which is compatible with inverse image whenever we have a morphism $(Y, Y_n) \to (Y', Y'_n)$ of such test situations (cf. [35] for more details).

Given a Z/p^nZ-torsor T on $A \otimes k$, we must define for every test situation $Y \hookrightarrow Y_n$, a G-torsor $\alpha_n(T)_{(Y,Y_n)}$ on Y_n. Because Y is given as an $A \otimes k$ scheme, we can pull back T to obtain a Z/p^nZ-torsor T_Y on Y. Because Y_n is a W_n-scheme which is a divided-power thickening, its ideal of definition is necessarily a nil-ideal; therefore the etale Y-scheme T_Y extends uniquely to an etale Y_n-scheme $T_{(Y,Y_n)}$, and its structure of Z/p^nZ-torsor extends uniquely as well. Because Y_n is a W_n-scheme, the natural map

$$Z/p^nZ \to W_n$$

gives rise to a morphism of algebraic groups on Y_n

$$(Z/p^nZ)_{Y_n} \xrightarrow{\alpha_n} (G_a)_{Y_n};$$

the required G_a-torsor $\alpha_n(T)_{(Y,Y_n)}$ is obtained by "extension of structural groups via α_n" from the $Z/p^n\ Z$-torsor $T_{(Y,Y_n)}$.

To define β_n, we begin with an element Z of $H^1_{\text{cris}}(A \otimes k/W_n)$. We must define an element $\beta_n(Z)$ in $\text{Ext}^{\text{rigid}}(\widehat{A} \otimes W_n, (G_a) \otimes W_n) = D(\widehat{A} \otimes W_n/W_n)$. Its value on the test object $A \otimes k \hookrightarrow A \otimes W_n$ is a G_a-torsor on $A \otimes W_n$ which is endowed with an integrable connection (cf. [2], [3]), i.e., it is an element of $H^1_{\mathbf{DR}}(A \otimes W_n/W_n)$. [This interpretation provides the canonical isomorphism

$$H^1_{\text{cris}}(A \otimes k/W_n) \xrightarrow{\sim} H^1_{\mathbf{DR}}(A \otimes W_n/W_n).]$$

Composing with the isomorphism

$$H^1_{\mathbf{DR}}(A \otimes W_n/W_n) \xrightarrow{\sim} \text{Ext}^{\text{rigid}}(A \otimes W_n, G_a \otimes W_n),$$

we obtain an element of $\mathrm{Ext}^{\mathrm{rigid}}(A \otimes W_n, G_a \otimes W_n)$, whose restriction to the formal group $\widehat{A} \otimes W_n$ is the required element $\beta_n(Z)$.

To see that the map β obtained from these β_n by passage to the limit
204 is in fact functorial in $A \otimes k$, we first note that it sits in the commutative diagram

$$\begin{tikzcd} H^1_{\mathrm{cris}}(A \otimes k/W) \arrow[r, "\beta"] \arrow[d, "\wr"', "\text{canonical isom}"] & D(\widehat{A}/W) \arrow[d, hook, "\substack{\text{inclusion of}\\ \text{primitive}\\ \text{elements}}"] \\ H^1_{\mathbf{DR}}(A/W) \arrow[r, "\text{natural map}", "\text{``restriction to } \widehat{A}\text{''}"'] & H^1_{\mathbf{DR}}(\widehat{A}/W). \end{tikzcd} \tag{5.7.3}$$

What must be shown is that if we are given a second abelian scheme B over W, and a homomorphism

$$f_0 : B \otimes k \to A \otimes k$$

then the diagram

$$\begin{tikzcd} H^1_{\mathrm{cris}}(A \otimes k/W) \arrow[r, "\beta"] \arrow[d, "(f_0)^*"] & D(\widehat{A}/W) \arrow[d, "\substack{\text{(any pointed}\\ \text{lifting of } \widehat{f_0})^*}"] \\ H^1_{\mathrm{cris}}(B \otimes k/W) \arrow[r, "\beta"] & D(\widehat{B}/W) \end{tikzcd} \tag{5.7.4}$$

is *commutative*.

But in virtue of the commutativity of the previous diagram (5.7.3), it is enough to show the commutativity of the diagram

$$\begin{tikzcd} H^1_{\mathrm{cris}}(A \otimes k/W) \simeq H^1_{\mathbf{DR}}(A/W) \arrow[r, "\text{restriction}"] \arrow[d, "(f_0)^*"] & H^1_{\mathbf{DR}}(\widehat{A}/W) \arrow[d, "\substack{\text{(any pointed}\\ \text{lifting of } \widehat{f_0})^*}"] \\ H^1_{\mathrm{cris}}(B \otimes k/W) \simeq H^1_{\mathbf{DR}}(B/W) \arrow[r, "\text{restriction}"] & H^1_{\mathbf{DR}}(\widehat{B}/W). \end{tikzcd} \tag{5.7.5}$$

205 This last commutativity has nothing to do with abelian schemes, nor does it require pointed liftings. It is an instance of the following general fact, whose proof we defer for a moment.

General Fact 5.7.6. *For any two pointed W-schemes A, B which are both proper and smooth, any pointed map $f_0 : B \otimes k \to A \otimes k$, and any integer $i \geq 0$, we have a commutative diagram*

$$\begin{tikzcd}
H^1_{\mathrm{cris}}(A \otimes k/W) \sim H^i_{\mathbf{DR}}(A/W) \arrow[r, "\text{restriction}"] \arrow[d, "(f_0)^*"] & H^i_{\mathbf{DR}}(\widehat{A}/W) \arrow[d, "\text{(any lifting of } \widehat{f_0})^*"] \\
H^1_{\mathrm{cris}}(B \otimes k/W) \simeq H^i_{\mathbf{DR}}(B/W) \arrow[r, "\text{restriction}"] & H^i_{\mathbf{DR}}(\widehat{B}/W)
\end{tikzcd}$$

To conclude the proof of the theorem (!), it remains to see that our marvelously functorial maps α, β really do form an exact sequence. To do this, we will use the abelian scheme A over W. Its formal group $\widehat{A}$ is p-divisible, and sits in an exact sequence of p-divisible groups over W,

$$0 \to \widehat{A}_{p^\infty} \to A_{p^\infty} \to E \to 0,$$

in which $E = \varinjlim E_n$ denote the etale quotient of A_{p^∞}. Because k is algebraically closed, E is a *constant* p-divisible group, namely the abstract p-divisible group $\varinjlim A_{p^n}(k)$ of all p-power torsion points of $A(k)$.

We will identify the exact sequence of the proposition with the exact sequence

$$0 \to D(E/W) \xrightarrow{\alpha'} D(A_{p^\infty}/W) \xrightarrow{\beta'} D(\widehat{A}/W) \to 0,$$

and we will identify the (α_n, β_n)-sequence with the exact sequence

$$0 \to D(E \otimes W_n/W_n) \xrightarrow{\alpha'_n} D(A_{p^\infty} \otimes W_n/W_n) \xrightarrow{\beta'_n} D(\widehat{A} \otimes W_n/W_n) \to 0.$$

206 It is clear from the construction of β_n that we have a commutative diagram

$$
\begin{tikzcd}
D(A_{p^\infty} \otimes W_n/W_n) \arrow[r, "\beta'_n"] \arrow[d, equal] & D(\widehat{A} \otimes W_n/W_n) \arrow[d, equal, "\mathrm{dfn}"] \\
\mathrm{Ext}^{\mathrm{rigid}}(A_{p^\infty} \otimes W_n, G_a) \arrow[r, "\text{restriction}"] & \mathrm{Ext}^{\mathrm{rigid}}(\widehat{A} \otimes W_n, G_a) \\
\mathrm{Ext}^{\mathrm{rigid}}(A \otimes W_n, G_a) \arrow[u, "\wr"] \arrow[d, "\wr"'] \arrow[ur, "\text{restriction}"] & \\
H^1_{\mathbf{DR}}(A \otimes W_n/W_n) & \\
H^1_{\mathrm{cris}}(A \otimes k/W_n). \arrow[u, "\wr"] \arrow[uuur, "\beta_n"'] &
\end{tikzcd}
$$

To relate the map α_n to the D-maps, use the exact sequence

$$0 \to E_n \otimes W_n \to E \otimes W_n \xrightarrow{p^n} E \otimes W_n \to 0$$

to compute

$$
\begin{tikzcd}
D(E \otimes W_n/W_n) \arrow[r, "\sim"] & \mathrm{Ext}(E \otimes W_n, G_a) \arrow[r, "\sim"] & \mathrm{Hom}(E_n \otimes W_n, G_a) \arrow[d, "\wr"', "(E_n \text{ is constant})"] \\
& & \mathrm{Hom}(E_n(W_n), G_a(W_n)) \arrow[d, equal] \\
& & \mathrm{Hom}(E_n(k), W_n) \arrow[d, equal] \\
& & \mathrm{Hom}(E_n(k), Z/p^n Z) \otimes W_n.
\end{tikzcd}
$$

Next use the sequence

$$0 \to A_{p^n} \otimes W_n \to A \otimes W_n \xrightarrow{p^n} A \otimes W_n \to 0$$

to compute **207**

$$\begin{tikzcd}
\mathrm{Ext}(A \otimes W_n, Z/p^nZ) \arrow[r, "\sim"] & \mathrm{Hom}(A_{p^n} \otimes W_n, Z/p^nZ) \\
& \mathrm{Hom}(E_n \otimes W_n, Z/p^nZ) \arrow[u, "\wr"', "(E_n = \text{etale quotient of } A_{p^n})"] \arrow[d, "\wr"'] \\
H^1_{\mathrm{et}}(A \otimes k, Z/p^nZ) & \mathrm{Hom}(E_n(k), Z/p^nZ). \arrow[l, "\sim"']
\end{tikzcd}$$

Combining these isomorphisms, and remembering that

$$\mathrm{Ext} = \mathrm{Ext}^{\mathrm{rigid}}$$

when either of the arguments is etale, we find a commutative diagram

$$\begin{tikzcd}
D(E \otimes W_n/W_n) \arrow[r, "\alpha'_n"] & D(A_{p^\infty} \otimes W_n/W_n) \\
\mathrm{Hom}(E_n(k), Z/p^nZ) \otimes W_n \arrow[u, "\wr"] & D(A \otimes W_n/W_n) \arrow[u, "\wr"] \\
\mathrm{Ext}^{\mathrm{rigid}}(A \otimes W_n, Z/p^nZ) \otimes W_n \arrow[u, "\wr"] \arrow[r, "***"] & \mathrm{Ext}^{\mathrm{rigid}}(A \otimes W_n, G_a) \arrow[u, equal] \\
H^1_{\mathrm{et}}(A \otimes k, Z/p^nZ) \otimes W_n \arrow[u, "\wr"] \arrow[r, "\alpha_n"] & H^1_{\mathrm{cris}}(A \otimes k/W_n) \arrow[u, equal]
\end{tikzcd}$$

in which the arrow $***$ is "push-out" along the homomorphism

$$Z/p^nZ \to W_n \to (G_a)_{W_n}.$$

□

Corollary 5.7.7. *Let A be an abelian scheme over the Witt vectors $W(k)$ of a perfect field k of characteristic $p > 0$. Then we have a short exact sequence of $W(k)$-modules*

$$0 \to \Big((H^1_{\mathrm{et}}(A \otimes \bar{k}, Z_p) \otimes W(\bar{k})\Big)^{\mathrm{Gal}(\bar{k}/k)} \to$$

$$H^1_{\text{cris}}(A \otimes k/W(k)) \to D(\widehat{A}/W(k)) \to 0,$$

in which $\overline{k}$ *denotes an algebraic closure of* k*, and in which the galois group* $\mathrm{Gal}(\overline{k}/k)$ *acts simultaneously on* $H^1_{et}(A \otimes \overline{k}, Z_p)$ *and on* $W(\overline{k})$ *by "transport of structure".*

208 *Proof.* One can obtain this sequence either by passing to $\mathrm{Gal}(\overline{k}/k)$-invariants in the already-established analogous sequence for $A \otimes W(\overline{k})$, or by repeating the *proof* given for the proposition. In the latter case, one finds, in the notations of the proof,

$$\begin{aligned} D(E \otimes W_n(k)/W_n(k)) &\simeq \mathrm{Hom}(E_n \otimes W_n(k), (G_a)_{W_n(k)}) \\ &\simeq \mathrm{Hom}(E_n(\overline{k}), W_n(\overline{k}))^{\mathrm{Gal}(\overline{k}/k)} \\ &\simeq \mathrm{Hom}(A_{p^n}(\overline{k}), W_n(\overline{k}))^{\mathrm{Gal}(\overline{k}/k)} \\ &= \left(H^1_{\text{et}}(A \otimes \overline{k}, Z/p^n Z) \otimes W_n(\overline{k})\right)^{\mathrm{Gal}(\overline{k}/k)} \end{aligned}$$

and the rest of the proof remains unchanged. □

Corollary 5.7.8. *Let* A *be an abelian scheme over the Witt vectors of a perfect field* k *of characteristic* $p > 0$*. The above exact sequence is the Newton-Hodge filtration*

$$0 \to (\text{slope } 0) \to H^1_{\text{cris}}(A \otimes k/W) \to (\text{slope } > 0) \to 0$$

of $H^1_{cris}(A \otimes k/W)$*) as an* F*-crystal.*

Proof. Since F induces a σ-linear automorphism of

$$\begin{aligned} &(H^1_{\text{et}}(A \otimes \overline{k}, Z_p) \otimes W(\overline{k}))^{\mathrm{Gal}} \\ &\simeq \left(\mathrm{Hom}(T_p(A \otimes \overline{k}), W(\overline{k}))\right)^{\mathrm{Gal}(\overline{k}/k)}, \end{aligned}$$

it remains only to see that F is topologically nilpotent on $D(\widehat{A}/W(k))$, for its p-adic topology. Because $D(\widehat{A}/W(k))$ is a *finitely generated* $W(k)$

sub-module of $H^1_{\mathbf{DR}}(\widehat{A}/W(k))$, the topology *induced* on $D(\widehat{A}/W(k))$ by the *inverse limit* topology on $H^1_{\mathbf{DR}}$ through the isomorphism (cf. lemma 5.8.1. ahead)

$$H^1_{\mathbf{DR}}(\widehat{A}/W(k)) \xrightarrow{\sim} \varprojlim H^1_{\mathbf{DR}}(\widehat{A} \otimes W_n(k)/W_n(k)) \tag{5.7.9}$$

must be *equivalent* to the p-adic topology in $D(\widehat{A}/W(k))$. So it suffices to remark that F^n annihilates $H^1_{\mathbf{DR}}(\widehat{A} \otimes W_n/W_n)$ (indeed F^n annihilates $\Omega^i_{\widehat{A}\otimes W_n/W_n}$ for $i \geq 1$, since for any pointed lifting of $X \mapsto X^p$, $F(dX) = d(F(X)) = d(X^p + pY) \in p\Omega^1$) to establish the required topological nilpotence of F on $D(\widehat{A}/W)$. □

209

5.8 The Missing Lemmas It remains for us to establish the "general fact" (5.7.7), and to establish the isomorphism (5.7.9). In fact, the two questions are intimately related. We begin with the second.

Lemma 5.8.1. *Let R be a Z-flat p-adic ring, and let $R_n = R/p^nR$. For any formal Lie variety V over R, we have isomorphisms*

$$H^i_{\mathbf{DR}}(V/R) \xrightarrow{\sim} \varprojlim H^i_{\mathbf{DR}}(V \otimes R_n/R_n).$$

Proof. Pick coordinates $X_1, \ldots, X_N$ for V. Over any ring R, we can define a Z^N-grading of the de Rham complex of $R[[X_1, \ldots, X_N]]/R$, by attributing the weight $(a_1, \ldots, a_N) \in Z^N$ to each "monomial"

$$\left(\prod X_i^{a_i}\right)\left(\prod_{j\in S} \frac{dX_j}{X_j}\right) \qquad S \text{ any subset of } \{1, \ldots, N\}.$$

Exterior differentiation is homogeneous of degree zero, and the de Rham complex is the *product* of all its homogeneous graded pieces

$$\Omega^\bullet = \Pi\Omega^\bullet(a_1, \ldots, a_N).$$

Because both cohomology and inverse limits commute with products, we are reduced to proving the lemma homogeneous component by homogeneous component.

The individual complexes $\Omega^\bullet(a_1, \ldots, a_N)$ are quite simple. They vanish except when all $a_i \geq 0$. The complex $\Omega^\bullet(0, \ldots, 0)$ is

$$R \to 0 \to 0 \to \ldots$$

If some $a_i \geq 1$, and all $a_i \geq 0$, the complex $\Omega^\bullet(a_1, \ldots, a_N)$ is the tensor product complex

$$\bigotimes_{i \text{ with } a_i \geq 1} \left(R \xrightarrow{a_i} R\right).$$

What is important for us is that each of these complexes is obtained from a complex of free finitely generated Z-modules (!) by extension of scalars to R.

Thus let K denote any complex of free finitely-generated Z_p-modules. We must show that for a Z-flat p-adic ring R we have

$$H^i(K^\bullet \otimes R) \xrightarrow{\sim} \varprojlim H^i(K^\bullet \otimes R_n).$$

210 The exact sequence of complexes

$$0 \to K^\bullet \otimes R \xrightarrow{p^n} K^\bullet \otimes R \to K^\bullet \otimes R_n \to 0$$

gives a "universal coefficients" exact sequence

$$0 \to H^i(K^\bullet \otimes R) \otimes R_n \to H^i(K^\bullet \otimes R_n) \to p^n\text{-Torsion } (H^{i+1}(K^\bullet \otimes R)) \to 0.$$

Passing to the inverse limit over n leads to an exact sequence

$$0 \to \varprojlim H^i(K^\bullet \otimes R) \otimes R_n \to \varprojlim H^i(K^\bullet \otimes R^n) \to T_p(H^{i+1}(K^\bullet \otimes R)) \to 0.$$

To see that $T_p(H^{i+1}(K^\bullet \otimes R))$ vanishes, notice that an element of this T_p is represented by a system of elements $a_n \in K^{i+1} \otimes R$ with $d(a_n) = 0$, $pa_{n+1} = a_n - d(b_n)$, $a_0 = 0$; because both $K^i \otimes R$ and $K^{i+1} \otimes R$ are p-adically complete and separated, we may infer

$$\begin{aligned} a_n &= pa_{n+1} + d(b_n) \\ &= p\,(pa_{n+2} + d(b_{n+1})) + d(b_n) \end{aligned}$$

$$\begin{aligned} &= \dots \\ &= d\left(\sum_{i\geq 0} p^i b_{n+i}\right). \end{aligned}$$

To see that the natural map

$$H^i(K^\bullet \otimes R) \to \varprojlim H^i(K^\bullet \otimes R) \otimes R_n$$

is an isomorphism, use the Z-flatness of R and the Z-finite generation of the K^i to write

$$\begin{aligned} H^i(K^\bullet \otimes R) \overset{\sim}{\leftarrow} H^i(K^\bullet) \otimes R &= (\text{fin. gen. } Z\text{-module}) \otimes R \\ &= \left(Z^n \oplus (\oplus Z/p^{n_i}) \oplus \begin{pmatrix}\text{prime-to-}p \\ \text{torsion}\end{pmatrix}\right) \otimes R \\ &= R^n \oplus (\oplus R_{n_i}). \end{aligned}$$

□

We now turn to the proof of the "general fact".

Lemma 5.8.2. *Let k be a perfect field of characteristic $p > 0$, A and B two proper, smooth pointed $W(k)$-schemes, $f_0 : B \otimes k \to A \otimes k$ a pointed k-morphism and $\widehat{f} : \widehat{B} \to \widehat{A}$ a W-lifting of f_0 to the formal completions* 211
viewed as functors only on p-adic W-algebras. Then the diagram

$$\begin{tikzcd} H^i_{\text{cris}}(A \otimes k/W) \arrow[r, "\sim"] \arrow[d, "(f_0)^*"] & H^i_{\mathbf{DR}}(A/W) \arrow[r, "\text{restriction}"] & H^i_{\mathbf{DR}}(\widehat{A}/W) \arrow[d, "(\widehat{f})^*"] \\ H^i_{\text{cris}}(B_0 \otimes k/W) \arrow[r, "\sim"] & H^i_{\mathbf{DR}}(B/W) \arrow[r, "\text{restriction}"] & H^i_{\mathbf{DR}}(\widehat{B}/W) \end{tikzcd}$$

is commutative.

Proof. If f_0 lifted, this would be obvious. But it does lift locally, which is enough for us. More precisely, let $U \subset A$ and $V \subset B$ be affine open neighborhoods of the marked W-valued points of A and B respectively such that f_0 maps $V \otimes k$ to $U \otimes k$. Because V is affine and U is smooth over W, we may successively construct a compatible system of W_n-maps $f_n : V \otimes W_n \to U \otimes W_n$ with $f_{n+1} \equiv f_n \bmod p^n$. The f_n induce compatible maps $\widehat{f_n} : \widehat{B} \otimes W_n \to \widehat{A} \otimes W_n$ of formal completions, but these $\widehat{f_n}$ need *not* be *pointed* morphisms.

We denote by $\widehat{f_\infty} : \widehat{B} \to \widehat{A}$ the limit of these $\widehat{f_n}$. (Strictly speaking, $\widehat{f_\infty}$ only makes sense as a map of functors when we *restrict* $\widehat{B}$ and $\widehat{A}$ to the category of p-adic W-algebras).

For each n, we have a commutative diagram

$$\begin{tikzcd}
H^i_{\mathrm{DR}}(A \otimes W_n/W_n) \arrow[r, dash, "\sim"] \arrow[rrr, bend left, "\mathrm{restr}"] & H^i_{\mathrm{cris}}(A \otimes k/W_n) \arrow[r, "\mathrm{restr.}"] \arrow[d] & H^i_{\mathrm{cris}}(U \otimes k/W_n) \arrow[r, dash, "\sim"] & H^i_{\mathrm{DR}}(U \otimes W_n/W_n) \arrow[r, "\mathrm{restr.}"] \arrow[d] & H^i_{\mathrm{DR}}(\widehat{A} \otimes W_n/W_n) \arrow[d] \\
H^i_{\mathrm{DR}}(B \otimes W_n/W_n) \arrow[r, dash, "\sim"] \arrow[rrr, bend right, "\mathrm{restr}"'] & H^i_{\mathrm{cris}}(B \otimes k/W_n) \arrow[r, "\mathrm{restr.}"] & H^i_{\mathrm{cris}}(V \otimes k/W_n) \arrow[r, dash, "\sim"] & H^i_{\mathrm{DR}}(V \otimes W_n/W_n) \arrow[r, "\mathrm{restr.}"] & H^i_{\mathrm{DR}}(\widehat{B} \otimes W_n/W_n)
\end{tikzcd}$$

Passing to the inverse limit over n, and using the previous lemma to identify the right-hand inverse limits, we obtain a commutative diagram

$$\begin{tikzcd}
H^i_{\mathrm{cris}}(A \otimes k/W) \arrow[r, dash, "\sim"] \arrow[d, "(f_0)^*"] & H^i_{\mathbf{DR}}(A/W) \arrow[r, "\text{restriction}"] & H^i_{\mathbf{DR}}(\widehat{A}/W) \arrow[d, "(\widehat{f_\infty})^*"] \\
H^i_{\mathrm{cris}}(B \otimes k/W) \arrow[r, dash, "\sim"] & H^i_{\mathbf{DR}}(\widehat{B}/W) \arrow[r, "\text{restriction}"] & H^i_{\mathbf{DR}}(\widehat{B}/W).
\end{tikzcd}$$

212 To conclude the proof, we need to know that the induced map

$$(\widehat{f_\infty})^* : H^i_{\mathbf{DR}}(\widehat{A}/W) \to H^i_{\mathbf{DR}}(\widehat{B}/W)$$

depends only on the underlying map $\widehat{f_0} : \widehat{B} \otimes k \to \widehat{A} \otimes k$, and not on the particular choice of lifting. In fact this is true for the individual $\widehat{f_n}$ as well! $\square$

Lemma 5.8.3. *Let R be a p-adic ring. Let V and V' be formal Lie varieties over R, and let f_1 and f_2 be morphisms of functors $V' \to V$ of the* *restrictions of V', V to the category of p-adic R-algebras. If f_1 f_2*mod p*, then for each i, the induced maps*

$$f_1^*, f_2^* : H^i_{\mathbf{DR}}(V/R) \to H^i_{\mathbf{DR}}(V'/R)$$

are equal.

Proof. (compare Monsky [39]). In terms of coordinates $X_1, \ldots, X_n$ for V', $Y_1, \ldots Y_m$ for V, the corresponding R-algebra homomorphisms

$$\varphi_1, \varphi_2 : R[[Y_1, \ldots, Y_m]] \to R[[X_1, \ldots, X_n]]$$

are related by

$$\varphi_2(Y) = \varphi_1(Y) + p\Delta(Y).$$

Introduce a new variable T, and consider the map

$$\varphi : R[[Y_1, \ldots, Y_m]] \to R[[X_1, \ldots, X_n, T]]$$
$$\varphi(Y) = \varphi_1(Y) + T \cdot \Delta(Y).$$

We have a commutative diagram of algebraic homomorphisms

$$\begin{tikzcd}
R[[Y]] \arrow[r, "\varphi"'] \arrow[rr, bend left, "\varphi_1"] \arrow[rr, bend right, "\varphi_2"'] & R[[X,T]] \arrow[r, shift left, "T \to 0"] \arrow[r, shift right, "T \to p"'] & R[[X]].
\end{tikzcd}$$

So it suffices to consider the situation **213**

$$R[[X,T]] \underset{T \to p}{\overset{T \to 0}{\rightrightarrows}} R[[X]]$$

and show that these two maps have the same effect on $H_{\mathbf{DR}}$.

A form ω on $R[[X,T]]$ may be written uniquely

$$\omega = \sum_{n \geq 0} a_n \cdot T^n + \sum_{n \geq 1} b_n T^n \frac{dT}{T}$$

with a_n, b_n's forms on $R[[X]]$. This form is *closed* if and only if

$$d(a_n) = 0 \quad \text{for} \quad n \geq 0, \quad n \cdot a_n + d(b_n) = 0 \quad \text{for} \quad n \geq 1.$$

Its images under $T \to 0$ and $T \to p$ are

$$a_0, \quad \sum_{n \geq 0} a_n p^n$$

respectively. Their *difference*, if ω is closed, is exact, namely

$$\omega|_{T=0} - \omega|_{T=p} = \sum_{b \geq 1} a_n p^n = d\left(\sum_{n \geq 1} \frac{p^n}{n} \cdot b_n \right).$$

□

It seems worthwile to point out that this last lemma can be considerably strengthened.

Lemma 5.8.4. *Let R be a p-adic ring, $I \subset R$ a divided power ideal, V and V' two formal Lie varieties over R, and f_1, f_2 two morphisms of functors $V' \to V$ of the restrictions of V, V' to the category of p-adic R-algebras. If $f_1 \equiv f_2 \bmod I$, then for all i the induced maps*

$$f_1^*, f_2^* : H^i_{\mathbf{DR}}(V/R) \to H^i_{\mathbf{DR}}(V'/R)$$

are equal.

Proof. If we had $f_1 \equiv f_2 \bmod I'$ with $I' \subset I$ a *finitely generated* ideal, then we could repeat the proof of the previous lemma, introducing several new variables T_i, one for each generator of I'. In particular, the
214 lemma is true if f_1 and f_2 are *polynomial* maps in some coordinate system. But we easily reduce to this situation, for in terms of coordinates $X_1, \ldots, X_n$ for V', we have a Z^n-graduation of its de Rham complex and a corresponding product decomposition

$$H^i_{\mathbf{DR}}(V'/R) = \prod_{(a_1,\ldots,a_n)} H^i_{\mathbf{DR}}(V'/R)(a_1, \ldots, a_n).$$

Therefore it suffices to show that the composite maps

$$H^i_{\mathbf{DR}}(V/R) \underset{f_2^*}{\overset{f_1^*}{\rightrightarrows}} H^i_{\mathbf{DR}}(V'/R) \xrightarrow{\text{projection}} H^i_{\mathbf{DR}}(V'/R)(a_1,\dots,a_n)$$

agree, for every $(a_1,\dots,a_n) \in Z^n$. But for fixed $(a_1,\dots,a_n)$, these composites depend only on the terms of total degree $\leq \sum a_i$ in the power series formulas for the maps f_1, f_2. Thus we are reduced to the case when f_1 and f_2 are each *polynomial* maps. □

Remark 5.8.5. If the ideal I is closed, the proof gives the same invariance property for the groups $H^i_{\mathbf{DR}}(V/R;I)$ defined as the cohomology of

$$\text{“}I\Omega^{i-1}_{V/R}\text{”} \xrightarrow{d} \Omega^i_{V/R} \xrightarrow{d} \Omega^{i+1}_{V/R}.$$

5.9 Application to the Cohomology of Curves Throughout this section we work over a mixed-characteristic valuation ring R of residue characteristic p, which is complete for a rank-one (i.e., real-valued) valuation. Let C be a projective smooth curve over R, with geometrically connected fibres of genus g. Its Jacobian $J = \mathrm{Pic}^0(C/R)$ is a g-dimensional autodual abelian scheme over R. For each rational point $x \in C(R)$, we denote by φ_x the corresponding Albanese mapping

$$\varphi_x : C \to J$$

given on S-valued points, S any R-scheme, by

$$\varphi_x(y) = \text{ the class of the invertible sheaf } I(y)^{-1} \otimes I(x),$$

where $I(y)$ denotes the invertible ideal sheaf of $y \in C(S)$ viewed as a Cartier divisor in $C \underset{R}{\times} S$. As is well-known (cf. [44], [45]), this morphism 215
induces isomorphisms

$$\begin{cases} H^1(J, O_j) \xrightarrow{\sim} H^1(C, O_C) \\ H^0(J, \Omega^1_{J/R}) = \underline{\omega}_J \xrightarrow{\sim} H^0(C, \Omega^1_{C/R}) \\ H^1_{\mathbf{DR}}(J/R) \xrightarrow{\sim} H^1_{\mathbf{DR}}(C/R) \end{cases} \tag{5.9.1}$$

which are independent of the choice of the rational point x.

Let $\widehat{C}_x$ denote the formal completion of C along x; it is a pointed formal Lie variety of dimension one over R. Because $\varphi_X(0) = 0$, φ_x induces a map of pointed formal Lie varieties

$$\widehat{\varphi}_x : \widehat{C}_x \to \widehat{J},$$

whence an induced map on cohomology

$$D(\widehat{J}/R) \subset H^1_{\mathbf{DR}}(\widehat{J}/R) \xrightarrow{(\widehat{\varphi}_x)^*} H^1_{\mathbf{DR}}(\widehat{C}_x/R).$$

Theorem 5.9.2. *The composite map*

$$D(\widehat{J}/R) \xrightarrow{(\widehat{\varphi}_x)^*} H^1_{\mathbf{DR}}(\widehat{C}_x/R)$$

is injective.

Corollary 5.9.3. *The natural map*

$$H^0(C, \Omega^1_{C/R}) \to H^1_{\mathbf{DR}}(\widehat{C}_x/R)$$

is injective, i.e., a non-zero differential of the first kind cannot be formally exact.

Proof. Because $\widehat{J}$ is p-divisible, the natural map $\underline{\omega}_J \to D(J/R)$ is injective.

The corollary then follows immediately from the theorem and the commutativity of the diagram

$$\begin{tikzcd} D(\widehat{J}/R) \arrow[r, hook, "(\widehat{\varphi}_x)^*"] \arrow[d, phantom, "\cup"] & H^1(\widehat{C}_x/R) \\ \underline{\omega}_J \arrow[r, "\sim"] & H^0(C, \Omega^1_{C/R}). \arrow[u] \end{tikzcd} \tag{5.9.4}$$

216 To prove the theorem, we choose an integer $n \geq 2g - 1$, and consider the mapping

$$\varphi_x^{(n)} : C^n \to J$$

defined by

$$\varphi_x^{(n)}(y_1, \ldots, y_n) = \sum_{i=1}^{n} \varphi_x(y_i),$$

the summation taking place in J. Passing to formal completions, we obtain

$$\widehat{\varphi}_x^{(n)} : (\widehat{C}_x)^n \to \widehat{J}$$

defined by

$$\widehat{\varphi}_x^{(n)}(y_1, \ldots, y_n) = \sum \varphi_x(y_i).$$

In terms of the projections

$$\widehat{\mathrm{pr}}_i : (\widehat{C}_x)^n \to \widehat{C}_x$$

onto the various factors, we can rewrite this as

$$\widehat{\varphi}_x^{(n)} = \sum_{i=1}^{n} \widehat{\varphi}_x \circ \widehat{\mathrm{pr}}_i,$$

the summation taking place in the abelian group of pointed maps to $\widehat{J}$. Because $D(\widehat{J}/R)$ is defined to consist precisely of the *primitive* elements in $H^1_{\mathbf{DR}}(\widehat{J}/R)$, we have, for any $a \in D(\widehat{J}/R)$,

$$(\widehat{\varphi}_x^{(n)})^*(a) = \sum_{i=1}^{n} (\widehat{\varphi}_x \circ \widehat{\mathrm{pr}}_i)^*(a) = \sum_{i=1}^{n} (\widehat{\mathrm{pr}}_i)^*(\widehat{\varphi}_x)^*(a).$$

Therefore the theorem would follow from the injectivity of the map

$$(\widehat{\varphi}_x^{(n)})^* : D(\widehat{J}/R) \to H^1_{\mathbf{DR}}((\widehat{C}_x)^n/R).$$

Because $D(\widehat{J}/R)$ is a flat R-module contained in $H^1_{\mathbf{DR}}(\widehat{J}/R)$, it suffices to show that the kernel of the map

$$(\widehat{\varphi}_x^{(n)})^* : H^1_{\mathbf{DR}}(\widehat{J}/R) \to H^1_{\mathbf{DR}}((\widehat{C}_x)^n/R)$$

consists entirely of torsion elements. In fact, we will show that this kernel is annihilated by $n!$. To do this, we observe that the map

$$\widehat{\varphi}_x^{(n)} : C^n \to J$$

is obviously invariant under the action of the symmetric group $\mathfrak{S}_n$ on C^n by permutation of the factors. Therefore we can factor it **217**

$$\begin{tikzcd} C^n \arrow[r, "\pi"] \arrow[rr, bend right, "\varphi_x^{(n)}"'] & \mathrm{Symm}^n(C) \arrow[r, "\psi"] & J. \end{tikzcd}$$

Passing to formal completions, we get a factorization

$$\begin{tikzcd} (\widehat{C}_x)^n \arrow[r, "\widehat{\pi}"] \arrow[rr, bend right, "\widehat{\varphi}_x^{(n)}"'] & \mathrm{Symm}^n(\widehat{C}_x) \arrow[r, "\widehat{\psi}"] & \widehat{J} \end{tikzcd}$$

We will first show that $(\widehat{\psi})^*$ is injective on $H^1_{\mathbf{DR}}$, by showing that the map $\widehat{\psi}$ has a cross-section. This in turn follows from the global fact that ψ is a P^{n-g}-bundle over J which is locally trivial on J for the Zariski topology. To see this last point, take a Poincare line bundle $\mathscr{C}$ on $C \times J$. Because $n \geq 2g - 1$, the Riemann-Roch theorem and standard base-changing results show that the sheaf on J given by $(\mathrm{pr}_2)_*(\mathscr{C} \otimes \mathrm{pr}_1^*(I^{-1}(x)^{\otimes n}))$ is locally free of rank $n+1-g$. The associated projective bundle is naturally isomorphic to ψ.

It remains only to show that the kernel of the map

$$(\widehat{\pi})^* : H^1_{\mathbf{DR}}(\mathrm{Symm}^n(\widehat{C}_x)/R) \to H^1_{\mathbf{DR}}((\widehat{C}_x)^n/R)$$

is annihilated by $n!$. But if a one-form ω on $\mathrm{Symm}^n(\widehat{C}_x)$ becomes exact when pulled back to $(\widehat{C}_x)^n$, say $\omega = \mathrm{df}$ with $f \in A((\widehat{C}_x)^n)$, then

$$n!\omega = \sum_{\sigma \in \mathfrak{S}_n} \sigma(\omega) = d\left(\sum_{\sigma \in \mathfrak{S}_n} \sigma(f)\right)$$

is exact on $\mathrm{Symm}^n(\widehat{C}_x)$. □

Remark. The fact that for n large the symmetric product $\mathrm{Symm}^n(C)$ is a projective bundle over J may be used to give a direct proof that C and J have isomorphic H^1's in any of the usual theories (e.g., coherent, Hodge, De Rham, etale, crystalline...).

Theorem 5.9.5. *Let k be a perfect field of characteristic $p > 0$, $\bar{k}$ its algebraic closure, C a projective smooth curve over $W(k)$ with geometrically connected fibre, $J = \mathrm{Pic}^0(C/W(k))$ its jacobian, $x \in C(W(k))$ a rational point of C, and $\varphi_x : C \to J$ the corresponding Albanese* **218**
mapping. There is an exact sequence of W-modules

$$0 \to (H^1_{\mathrm{et}}(C \otimes \bar{k}, Z_p) \otimes W(\bar{k}))^{\mathrm{Gal}(\bar{k}/k)} \xrightarrow{\alpha} H^1_{\mathrm{cris}}(C \otimes k/W(k)) \xrightarrow{\beta} H^1\mathrm{DR}(\widehat{C}_x/W(k)),$$

the maps in which are functorial in $(C, x) \otimes k$ as pointed k-scheme.

Proof. The map α is defined exactly as was its abelian variety analogue (cf. 5.7.1); the map β is defined as the composite

$$\begin{tikzcd} H^1_{\mathrm{cris}}(C \otimes k/W(k)) \arrow[r, "\sim"] \arrow[rr, bend right] & H^1_{\mathrm{DR}}(C/W(k)) \arrow[r, "\mathrm{restr.}"] & H^1_{\mathrm{DR}}(\widehat{C}_x/W(k)). \end{tikzcd}$$

By construction, α is functorial in $(C, x) \otimes k$. By lemma (5.8.2), β is similarly functorial. To see that the sequence is *exact*, use the fact that the Albanese map induces isomorphisms on both crystalline (or de Rham!) and etale H^1's, (cf. SGAI, Exp. XI, last page, for the etale case), i.e., we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & (H^1_{\mathrm{et}}(J \otimes \bar{k}, Z_p) \otimes W(\bar{k}))^{\mathrm{Gal}} \arrow[r, "\alpha."] \arrow[d, "\wr"', "(\varphi, \otimes k)^*"] & H^1_{\mathrm{cris}}(J \otimes k/W(k)) \arrow[r] \arrow[d, "\wr"', "(\varphi, \otimes k)^*"] & D(\widehat{J}/W(k)) \arrow[r] \arrow[d, hook, "(\widehat{\varphi}_x)^*"] & 0 \\ & (H^1_{\mathrm{et}}(C \otimes \bar{k}, Z_p) \otimes W(\bar{k}))^{(\mathrm{Gal}(\bar{k}/k))} \arrow[r, "\alpha"] & H^1_{\mathrm{cris}}(C \otimes k/W(k)) \arrow[r, "\beta"] & H^1_{\mathbf{DR}}(\widehat{C}_x/W(k)). & \end{tikzcd}$$

□

Corollary 5.9.6. (1) *The kernel of the "formal expansion at a point" map*

$$H^1_{\mathbf{DR}}(C/W(k)) \to H^1_{\mathbf{DR}}(\widehat{C}_x/W(k))$$

in $H^1_{\mathbf{DR}}(C/W(k)) \simeq H^1_{\mathrm{cris}}(C \otimes k/W(k))$ *is the "slope-zero" part of the* F*-crystal* $H^1_{\mathrm{cris}}(C \otimes k/W(k))$*, i.e., we have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & (H^1_{\mathrm{et}}(C \otimes \bar{k}, Z_p) \otimes W(\bar{k}))^{(\mathrm{Gal}(\bar{k}/k))} \arrow[r] \arrow[d, equal] & H^1_{\mathbf{DR}}(C/W) \arrow[r] & (\text{image of } H^1_{\mathbf{DR}}(C/W) \text{ in } H^1_{\mathbf{DR}}(\widehat{C}_x/W(k)) \arrow[r] & 0 \\
0 \arrow[r] & (\text{slope } 0) \arrow[r] & H^1_{\mathrm{cris}}(C \otimes k/W(k)) \arrow[r] \arrow[u, "\wr"'] & (\text{slope} > 0) \arrow[r] \arrow[u, "\wr"'] & 0.
\end{tikzcd}$$

(2) *The image of the "formal expansion at a point" map is the "slope* > 0*" quotient of* $H^1_{\mathrm{cris}}(C \otimes k/W(k))$*; this quotient is isomorphic, via the Albanese map* φ_x*, to* $D(\widehat{J}/W(k))$*.*

219 **VI. Applications to congruences and to Honda's conjecture.** Let C be a projective smooth curve over $W(F_q)$ with geometrically connected fibres. Let G be a finite group of order prime to p, all of whose absolutely irreducible complex representations are realizable over $W(F_q)$ (e.g., if the exponent of G divides $q-1$, this is automatic). Suppose that G operates on C by $W(F_q)$-automorphisms. Then G operates also on $C \otimes F_q$ by F_q-automorphisms. For each absolutely irreducible representation ρ of G, let $P_{1,\rho}(T) \in W(F_q)[T]$ be the numerator of the associated L-function $L(C \otimes F_q/F_q, G, \rho; T)$;

$$P_{1,\rho}(T) = 1 + a_1(\rho)T + \cdots + a_r(\rho)T^r.$$

Let $\omega \in H^0(C, \Omega^1_{C/W})^\rho$ be a differential of the first kind on C which lies in the ρ-isotypical component of $H^0(C, \Omega^4_{C/W})$. Let $x \in C(W(F_q))$ be a rational point on C, and let X be a parameter at x (i.e., X is a coordinate for the one-dimensional pointed formal Lie variety $\widehat{C}_x$ over $W(F_q)$). Consider the formal expansion of ω around x:

$$\omega = \sum_{n \geq 1} b(n) \cdot X^n \frac{dX}{X} \quad b(n) \in W(F_q).$$

We extend the definition of $b(n)$ to rational numbers $n > 0$ by decreeing that $b(n) = 0$ unless n is an integer.

Theorem 6.1. *In the above situation, the coefficients $b(n)$ satisfy the congruences*

$$\frac{b(n)}{n} + a_1(\rho) \cdot \frac{b(nq)}{nq} + \cdots + a_r(\rho) \frac{b(nq^r)}{nq^r} \in pW(F_q)$$

for every rational $n > 0$.

Proof. Let J denote the Jacobian of $C/W(F_q)$, and denote by $\widehat{\omega} \in \underline{\omega}_J$ the unique invariant one-form on J which pulls back to give ω under the Albanese mapping φ_x. The group G operates, by functoriality, on J and on $\underline{\omega}_J$, and the isomorphism $\underline{\omega}_J \xrightarrow{\sim} H^0(C, \Omega^1_{C/W})$ is G-equivariant. Therefore $\widetilde{\omega}$ lies in $(\underline{\omega}_J)^\rho$. Via the G-equivariant inclusion

$$\underline{\omega}_J \subset D_{(p)}(\widehat{J}/W)$$

we have **220**

$$\widetilde{\omega} \in (D_{(p)}(\widehat{J}/W))^\rho$$

Now let F denote the Frobenius endomorphism of $J \otimes F_q$ relative to F_q. Then both F and the group G act on $J \otimes F_q$. By (4.2), we know that

$$(F^r + a_1(\rho)F^{r-1} + \cdots + a_r(\rho)) \cdot \mathrm{Proj}(\rho) = 0$$

in $\mathrm{End}(J \otimes F_q) \underset{Z}{\otimes} W(F_q)$. Because $D(\widehat{J}/W)$ is an additive functor of $J \otimes F_q$ with values in $W(F_q)$-modules, and $\widetilde{\omega}$ lies in its ρ-isotypical component, it follows that

$$F^r(\widetilde{\omega}) + a_1(\rho)F^{r-1}(\widetilde{\omega}) + \cdots + a_r(\rho) \cdot \widetilde{\omega} = 0 \qquad (6.1.1)$$

in $D_{(p)}(\widehat{J}/W)$.

The Albanese map $\varphi_x : C \to J$ induces a map

$$\widehat{\varphi}_x : \widehat{C}_x \to \widehat{J},$$

whence a map

$$D_{(p)}(\widehat{J}/W) \subset H^1_{\mathbf{DR}}(\widehat{J}/W;(p)) \xrightarrow{(\widehat{\varphi}_x)^*} H^1_{\mathbf{DR}}(\widehat{C}_x;(p))$$

which is functorial in the pointed schemes $(\widehat{J},0)\otimes F_q$ and $(\widehat{C}_x,x)\otimes F_q$. So if we denote also by F the q-th power Frobenius endomorphism of $\widehat{C}_x \otimes F_q$, we have

$$(\widehat{\varphi}_x)^* \circ F = F \circ (\widehat{\varphi}_x)^*,$$

whence a relation

$$F^r(\omega) + a_1(\rho)F^{r-1}(\omega) + \cdots + a_r(\rho)\cdot\omega = 0 \qquad (6.1.2)$$

in $H^1_{\mathbf{DR}}(\widehat{C}_x/W;(p))$.

The asserted congruences on the $b(n)$'s are simply the spelling out of this relation. Explicitly, in terms of the chosen coordinate X for $\widehat{C}_x$, a particularly convenient pointed lifting of F on $\widehat{C}_x \otimes F_q$ is provided by

$$F : X \mapsto X^q.$$

221 In terms of the isomorphism

$$H^1_{\mathbf{DR}}(\widehat{C}_x/W;(p)) \xleftarrow{\sim} \frac{\{f \in K[[X]] | f(0) = 0, \mathrm{df}\ \text{integral}\}}{\{f \in pW[[X]] | f(0) = 0\}}$$

the cohomology class of ω is represented by the series

$$f(X) = \sum_{n>0} \frac{b(n)}{n} X^n,$$

and the cohomology class of $F^i(\omega)$ is represented by

$$f(X^{q^i}) = \sum \frac{b(n)}{n} X^{nq^i}.$$

The relation (**??**) thus asserts that

$$f(X^{q^r}) + a_1(\rho) f(X^{q^{r-1}}) + \cdots + a_r(\rho) f(X)$$

is a series whose coefficients all lie in $pW(F_q)$. The congruence asserted in the statement of the theorem is precisely that the coefficient of X^{nq^r} in this series lies in $pW(F_q)$. □

Remark. In the special case $G = \{e\}$, ρ trivial, the polynomial $P_{1,\rho}(T)$ is the numerator of the zeta function of $C \otimes F_q$, and every differential of the first kind $\omega \in H^i(C, \Omega^1_{C/W})$ is ρ-isotypical. The resulting congruences on the coefficients of differentials of the first kind were discovered independently by Cartier and by Honda in the case of elliptic curves, and seem by now to be "well-known" for curves of any genus. ([1], [5], [8], [22]).

Theorem 6.2. *Hypothesis and notation as above, suppose that the polynomial $P_{1,\rho}(T)$ is linear*

$$P_{1,\rho}(T) = 1 + a_1(\rho)T,$$

i.e., that ρ occurs in H^1 with multiplicity one. Then

(1) *$a_1(\rho)$ is equal to the exponential sum $S(C \otimes F_q/F_q, \rho, 1)$ and for every $n \geq 1$ we have*

$$(-a_1(\rho))^n = -S(C \otimes F_q/F_q, \rho, n).$$

(2) *If ρ occurs in $H^0(C, \Omega^1_{C/W})$, then $\operatorname{ord}_p(a_1(\rho)) > 0$, i.e., $a_1(\rho)$ is not* **222**
a unit in $W(F_q)$.

(3) *If ρ occurs in $H^0(C, \Omega^1_{C/W})$, choose $\omega \in H^0(C, \Omega^1_{C/W})^\rho$ to be non-zero, and such that at least one of coefficients $b(n)$ is a unit in $W(F_q)$. For any n such that $b(n)$ is a unit, the coefficients $b(nq)$, $b(nq^2), \ldots$ are all non-zero, and we have the limit formulas (in which $\overline{\rho}$ denotes the contragradient representation)*

$$-S(C \otimes F_q/F_q, \rho, 1) = -a_1(\rho) = \lim_{N\to\infty} \frac{q \cdot b(nq^N)}{b(nq^{N+1})}$$

$$-S(C \otimes F_q/F_q, \overline{\rho}, 1) = -a_1(\overline{\rho}) = \frac{-q}{a_1(\rho)} = \lim_{N\to\infty} \frac{b(nq^{N+1})}{b(nq^N)}.$$

Proof. If ρ occurs in H^1 with multiplicity *one*, then ρ must be a non-trivial representation of G (for if ρ were the trivial representation, G

would have a one-dimensional space of invariants in H^1; but the space of invariants in H^1 of the quotient curve $C \otimes F_q$ modulo G, so is *even-dimensional*!). Therefore ρ does *not* occurs in H^0 or H^2, as both of these are the trivial representation of G. The first assertion now results from (1.1).

If ρ also occurs in $H^0(C, \Omega^1_{C/W})$, pick any non-zero ω in

$$H^0(C, \Omega^1_{C/W})^\rho$$

and look at its formal expansion around x:

$$\omega = \sum b(n) X^n \frac{dX}{X}.$$

An elementary "q-expansion principle"-argument (cf. [28]) shows that if all $b(n)$ are divisible by p, then ω is itself divisible by p in $H^0(C, \Omega^1_{C/W})$. So after dividing ω by the highest power of p which divides all $b(n)$, we obtain an element $\omega \in H^0(C, \Omega^1_{C/W})^\rho$ which has some coefficient a unit.

Consider the congruences satisfied by the $b(n)$:

$$\frac{b(n)}{n} + a_1(\rho)\frac{b(nq)}{nq} \in pW(F_q).$$

223 If $a_1(\rho)$ were a *unit*, we could infer (by induction on the precise power of p dividing n) that

$$\text{for } all\ n \geq 1,\ \frac{q}{p} \cdot \frac{b(n)}{n} \in W(F_q).$$

In particular, we would find that $\frac{q}{p} \cdot \omega$ is *formally* exact at x, which by (5.9.3) is impossible.

Given that $a_1(\rho)$ is a non-unit, choose n such that $b(n)$ *is* a unit. Then

$$\operatorname{ord}(b(n)/n) \leq 0.$$

From the congruences

$$\frac{b(n)}{n} \equiv -a_1(\rho)\frac{b(nq)}{nq} \mod pW$$

$$\vdots$$

$$\frac{b(nq^N)}{nq^N} \equiv -a_1(\rho)\frac{b(nq^{N+1})}{nq^{N+1}} \text{ mod } pW$$

and the fact that $\text{ord}(a_1(\rho)) > 0$, it follows easily by induction on N that

$$\text{ord}\left(\frac{b(nq^N)}{nq^N}\right) = \text{ord}(b(n)/n) - N\,\text{ord}(a_1(\rho)).$$

Therefore we may *divide* the congruences, and obtain

$$\text{ord}\left(\frac{qb(nq^N)}{b(nq^{N+1})} + a_1(\rho)\right) \geq 1 + (N+1)\,\text{ord}(a_1(\rho)) - \text{ord}(b(n)/n)$$

$$\text{ord}\left(\frac{b(nq^{N+1})}{b(nq^N)} + \frac{q}{a_1(\rho)}\right) \geq 1 + \text{ord}\left(\frac{q}{a_1(\rho)}\right) + N\,\text{ord}(a_1(\rho)) - \text{ord}\left(\frac{b(n)}{n}\right).$$

Letting $N \to \infty$, we get the asserted limit formulas for $-a$, (ρ) and for $-q/a_1(\rho)$. By the Riemann Hypothesis for curves over finite fields, we know that $-q/a_1(\rho)$ is the complex conjugate $\overline{a_1(\rho)}$. Let $\overline{\rho}$ denote the contragradient representation of ρ; because the definition of the L-series $L(C \otimes F_q/F_q, G, \rho; T)$ is purely algebraic, the L-series for $\overline{\rho}$ is obtained by applying (any) complex conjugation to the coefficients of the L-series for ρ. Therefore $\overline{a_1(\rho)} = a_1(\overline{\rho})$, and $\overline{\rho}$ also occurs in H^1 with multiplicity 224
one. $\square$

Example 6.3. Consider the Fermat curve of degree N over $W(F_q)$, with $q \equiv 1 \text{mod } N$. For each integer $0 \leq r \leq N-1$, denote by χ_r the character of μ_N given by

$$\chi_r(\zeta) = \zeta^r.$$

We know that under the action of $\mu_N \times \mu_N$ (acting as $(x, y) \to (\zeta x, \zeta' y)$ in the affine model $x^N + y^N = 1$), the characters which occurs in H^1 are precisely

$$\chi_r \times \chi_s \qquad 1 \leq r, s \leq N-1, r+s \neq N,$$

each with multiplicity one. Those which occur in $H^0(\Omega^1)$ are precisely the

$$\chi_r \times \chi_s \qquad 1 \leq r, s \leq N-1, r+s < N,$$

the corresponding eigen-differential $\omega_{r,s}$ is given by

$$\omega_{r,s} = x^r y^s \frac{dx}{xy^N}.$$

If we expand $\omega_{r,s}$ at the point $(x = 0, y = 1)$, in the parameter x, we obtain

$$\begin{aligned}
\omega_{r,s} &= x^r(1 - x^N)\frac{s}{N} - 1 \cdot \frac{dx}{x} \\
&= \sum_{j\geq 0}(-1)^j \binom{\frac{s}{N}-1}{j} x^{r+Nj}\frac{dx}{x} \\
&= \sum_{n\geq 1} b(n)x^n \frac{dx}{x}.
\end{aligned}$$

Conveniently, the first non-vanishing coefficient $b(r)$ is 1. The successive coefficients $b(rq^n)$ are given by

$$b(rq^n) = (-1)^{\frac{r}{N}(q^n-1)} \cdot \binom{\frac{s}{N} - 1}{\frac{r}{N}(q^n - 1)}.$$

225 The eigenvalue of F on the $\chi_r \times \chi_s$-isotypical component of H^1 is the *negative* of the Jacobi sum $J_q(\chi_r, \chi_s)$. There we obtain the limit formulas

$$-J_q(\chi_r, \chi_s) = \lim_{n\to\infty} \frac{(-1)^{\frac{r}{N}(q-1)\cdot q^n} \binom{\frac{s}{N} - 1}{\frac{r}{N}(q^n - 1)}}{\binom{\frac{s}{N} - 1}{\frac{r}{N}(q^{n+1} - 1)}}$$

$$-J_q(\chi_{N-r}, \chi_{N-s}) = \lim_{n\to\infty} \frac{(-1)^{\frac{r}{N}(q-1)\cdot q^n} \cdot \binom{\frac{s}{N} - 1}{\frac{r}{N}(q^{n+1} - 1)}}{\binom{\frac{s}{N} - 1}{\frac{r}{N}(q^n - 1)}}$$

valid for $1 \leq r, s \leq N-1$, $r+s \neq N$. These formulas are the ones originally conjectured by Honda, and recently interpreted by Gross-Koblitz [14] in terms of Morita's p-adic gamma function.

VII. Application of Gauss sums. In this chapter we will analyze the cohomology of certain Artin-Schreier curves, and then obtain a limit formula for Gauss sums in the style of the preceding section.

We fix a prime p, an integer $N \geq 2$ prime to p, and consider the smooth affine curve U over $Z[1/N(p-1)]$ defined by the equation

$$T^p - T = X^N.$$

It may be compactified to a projective smooth curve C over $Z[1/N(p-1)]$ with geometrically connected fibres by adding a single "point at infinity", along which $T^{-1/N}$ is a uniformizing parameter.

The group-scheme $\mu_{N(p-1)}$ operates on U, by

$$\zeta : (T, X) \to (\zeta^N T, \zeta X).$$

This action extends to C, and fixes the point at infinity. **226**

A straightforward computation gives the following lemma.

Lemma 7.1. (1) *The genus of C is $\frac{1}{2}(N-1)(p-1)$, and a basis of everywhere holomorphic differentials on C is given by the forms*

$$X^a T^b \frac{dT}{X^{N-1}}$$

with $0 \leq a \leq N-2$, $0 \leq b \leq p-2$, *and* $pa + Nb < (p-1)(N-1) - 1$.

(2) *The space* $H^1_{\mathbf{DR}}(C \otimes Q/Q) \xrightarrow{\sim} H^1_{\mathbf{DR}}(U \otimes Q/Q)$ *has dimension* $(N-1)(p-1)$, *any d basis is given by the cohomology classes of the forms*

$$X^q T^b \frac{dT}{X^{N-1}} \qquad 0 \leq a \leq N-2,\ 0 \leq b \leq p-2.$$

(3) *The characters of* $\mu_{N(p-1)}$ *which occur in* $H^1_{\mathbf{DR}}(C \otimes Q/Q)$ *are precisely those whose restrictions to* μ_N *is non-trivial, and each of these occurs with multiplicity one.*

In characteristic p, there are new automorphisms. The additive group F_p operates on $C \otimes F_p$ by

$$a : (T, X) \to (T + a, X).$$

This action does not commute with the action of $\mu_{N(p-1)}$. However, the two together define an action of the semi-direct product

$$F_p \ltimes \mu_{N(p-1)}$$

formed via the homomorphism

$$\mu_{N(p-1)} \xrightarrow{-N} \mu_{p-1} \simeq F_p^{\times} = \mathrm{Aut}(F_p)$$

Explicitly, the multiplication is

$$(a, \zeta)(b, \zeta_1) = (a + \zeta^{-N} b, \zeta\zeta_1),$$

and the action is

$$(a, \zeta) : (T, X) \to (\zeta^N T + \zeta^N a, \zeta X).$$

227 The group $F_p \ltimes \mu_{N(p-1)}$ contains $F_p \times \mu_N$ as a normal subgroup, acting on $C \otimes F_p$ in the usual manner.

Remark. This action of a group of order $p(p-1)N$ on a curve of genus $g = \frac{1}{2}(p-1)(N-1)$ provides a nice example of how "wrong" the characteristic zero estimate $84(g-1)$ can become in the presence of wild ramification!

Let E be a number field containing the $N(p-1)$'st roots of unity, P a p-adic place of E, F_q a finite extension of the residue field $F_{N(\mathsf{P})}$, of P, G the abstract group $F_p \ltimes \mu_{N(p-1)}(F_q)$. Let H^1 denote any of the vector spaces $H_l^1(C \otimes F_q) \underset{Z_l}{\otimes} E_\lambda$ for $l \neq p$, or $H^1_{\mathrm{cris}}(C \otimes F_q / W(F_q)) \otimes K$.

By functoriality, the group G operates on H^1. Because the center of G is $\mu_N(F_q)$, the decomposition

$$H^1 = \otimes (H^1)^\chi$$

of H^1 according to the characters of μ_N is G-stable.

Proposition 7.2. *For each of the $N-1$ non-trivial E-valued characters χ of $\mu_N(E) \xrightarrow{\sim} \mu_N(F_{N(\mathsf{P})}) = \mu_N(F_q)$, the corresponding eigenspace $(H^1)^\chi$ is a $p-1$ dimensional absolutely irreducible representation of G; the restriction to F_p of $(H^1)^\chi$ is the augmentation representation of F_p; the restriction to $\mu_{N(p-1)}(F_q)$ of $(H^1)^\chi$ is the induction, from μ_N to $\mu_{N(p-1)}$, of χ.*

Proof. All assertions except for the G-irreducibility of $(H^1)^\chi$ follow immediately from the preceding lemma, giving the action of $\mu_{N(p-1)}$, and from Corollary (2.2), giving the action of $F_p \times \mu_N$. The irreducibility follows from these facts together with the fact that in *any* complex representation of G, the set of characters of F_p which occur is stable under the action of $\mu_{N(p-1)}$ in F_p by conjugation; because this action has only the two orbits $F_p^\times$ and 0, as soon as any one non-trivial character of F_p occurs, *all* non-trivial characters must also occur. □

Corollary 7.3. (1) *Over any finite extension F_q of F_p which contains all the $N(p-1)$'st roots of unity (i.e., $q \equiv 1 \bmod N(p-1)$), the Frobenius F relative to F_q operates as a scalar on each of the spaces $(H^1)^\chi$, χ a non-trivial character of μ_N. This scalar is the common value*

$$-g_q(\psi, \chi; \mathsf{P})$$

of the Gauss sums attached to any of the non-trivial additive characters 228
ψ of F_{P}.

Proof. Over such an F_q, Frobenius commutes with the action of G on H^1, so it acts on each $(H^1)^\chi$ as a G-morphism. Because $(H^1)^\chi$ is G-irreducible, this G-morphism must be a scalar, and this scalar is equal to *any* eigenvalue of F on $(H^1)^\chi$. As we have already seen (2.1), these eigenvalues are precisely the asserted Gauss sums, corresponding to the decomposition of $(H^1)^\chi$ under F_p. □

The common value of these Gauss sums over a sufficiently large F_q is itself a Jacobi sum, in consequence of the fact that universally, i.e., over $Z[1/N(p-1)]$, the curve C is the *quotient* of the Fermat curve

Fermat $(N(p-1))$ of degree $N(p-1)$ by the *subgroup* H of $\mu_{N(p-1)} \times \mu_{N(p-1)}$ consisting of all (ζ_1, η_2) satisfying

$$\zeta_1^{p-1} = \zeta_2^p$$

Explicitly, the map is given rationally by the formulas

$$\begin{aligned} &(W, V) \text{ on } W^{N(p-1)} + V^{N(p-1)} = 1 \\ &\quad\downarrow \\ &(T, X) \text{ on } T^p - T = X^N \\ &T = 1/V^N, X = W^{p-1}/V^p. \end{aligned}$$

Lemma 7.4. *Let χ_1 be a character of $\mu_{N(p-1)}$ whose restriction to μ_N is non-trivial. Under the map*

$$H^1_{\mathbf{DR}}(C \otimes Q/Q) \xrightarrow{\sim} H^1(\text{Fermar } (N(p-1)) \otimes Q/Q)^H$$

we have

$$H^1_{\mathbf{DR}}(C \otimes Q/Q)^{\chi_1} \xrightarrow{\sim} H^1_{\mathbf{DR}}(\text{Fermat } (N(p-1)) \otimes Q/Q)^{\chi_1^{p-1} \times \chi_1^{-p}}$$

Proof. That $H^1(C) \xrightarrow{\sim} H^1(\text{Fermat})^H$ in rational cohomology results from the Hochschild-Serre spectral sequence. Since the characters of $\mu_{N(p-1)}$ (resp of $\mu_{N(p-1)} \times \mu_{N(p-1)}$) occur, if at all, with multiplicity one in $H^1(C)$ (resp H^1 (Fermar)), it suffices to check that the χ_1-eigenspace of $H^1(C)$ is mapped to the $(\chi_1^{p-1}, \chi_1^{-p})$-eigenspace of H^1(Fermat). This
229 we do by inspection:

$$\begin{aligned} &X^a T^b \frac{dT}{X^{N-1}} = \\ &\qquad = X^{a+1-N} T^{b+1} \frac{dT}{T} \mapsto \left(\frac{W^{p-1}}{V^p}\right)^{a+1-N} \left(Z^{-N}\right)^{b+1} \left(\frac{-NdZ}{Z}\right). \end{aligned}$$

□

Corollary 7.5. *If F_q contains the $N(p-1)$'st roots of unity, then for any non-trivial character χ of μ_N, and extension χ_1 of χ to $\mu_{N(p-1)}$ and any non-trivial additive character ψ of F_p, the scalar by which F acts on $H^1(C \otimes F_q)^\chi$ is given by*

$$\begin{cases} F|H^1(C\otimes F_q)^\chi = F|H^1(C\otimes F_q)^{\psi\times\chi} = -g_q(\psi,\chi;\mathsf{P}) \\ \qquad \Big\| \\ F|H^1(C\otimes F_q)^{\chi_1} = F|H^1(\text{Fermat}\otimes F_q)^{\chi_1^{p-1}\times\chi_1^{-p}} = -J_q(\chi_1^{p-1},\chi_1^{-p};\mathsf{P}) \end{cases}$$

We now turn to the "determination" of the Gauss sum $-g_q(\psi,\chi;\mathsf{P})$ over an F_q which is merely required to contain the N'th roots of unity. Unless $p-1$ and N are relatively prime, such an F_q need *not* contain the $N(p-1)$'st roots of unity! Moreover, the Gauss sum does *not* in general lie in the Witt vectors $W(F_q)$, as it does when F_q contains the $N(p-1)$'st roots of unity!

Let π denote any solution of

$$\pi^{p-1} = -p.$$

We recall without proof the following standard lemma (cf. [31] or [32]).

Lemma 7.6. *The fields $Q_p(\zeta_p)$ and $Q_p(\pi)$ coincide. There is a bijective correspondence*

$$\textit{primitive } p\textit{'th roots of } 1 \longleftrightarrow \textit{solutions } \pi \textit{ of } \pi^{p-1} = -p$$

under which $\zeta \longleftrightarrow \pi$ if and only if

$$\zeta \equiv 1 + \pi \bmod \pi^2.$$

For each solution π of $\pi^{p-1} = -p$, we denote by

$$\psi_\pi : F_p \to Q_p(\zeta_p)^\times$$

the unique non-trivial additive character which satisfies **230**

$$\psi_\pi(1) \equiv 1 + \pi \bmod \pi^2.$$

If we fix a $W(F_q)$-valued point x on C, we have the map "formal expansion at x"

$$H^1_{\mathrm{cris}}(C \otimes F_q/W(F_q)) \to H^1_{\mathbf{DR}}(\widehat{C}_x \otimes W(F_q)/W(F_q)).$$

If we denote by R the ring

$$R = W(F_q)[\pi]$$

which is a free W-module of finite rank $(p-1)$, we may tensor with R to obtain

$$\begin{tikzcd}
H^1_{\mathrm{cris}}(C \otimes F_q/W(F_q)) \underset{W}{\otimes} R \arrow[rr] \arrow[d, dash, "\wr"] & & H^1_{\mathbf{DR}}(\widehat{C}_x \otimes R/R). \\
H^1_{\mathbf{DR}}(C \otimes R/R) \arrow[urr] & &
\end{tikzcd}$$

Theorem 7.7. (1) *For any $W(F_q)$-valued point x on C, the "formal expansion" map is injective :*

$$H^1_{\mathrm{cris}}(C \otimes F_q/W(F_q)) \hookrightarrow H^1_{\mathbf{DR}}(\widehat{C}_x \otimes W(F_q)/W(F_q))$$

(2) *Let π be any solution of $\pi^{p-1} = -p$, ψ_π the corresponding additive character, a an integer $1 \leq a \leq N-1$ and χ_a the corresponding nontrivial character of $\mu_N(\chi_a(\zeta) = \zeta^a)$. If we take for x the point $(T = 0, X = 0)$ on C, with parameter X, then the image of*

$$(H^1_{\mathrm{cris}}(C \otimes F_q/W(F_q)) \otimes Q_p(\pi))^{\psi_w \times \chi_a} \to H^1_{\mathbf{DR}}(\widehat{C}_x \otimes R/R) \underset{R}{\otimes} Q_p(\pi)$$

is the one-dimensional $Q_p(\pi)$-space spanned by the cohomology class of

$$\exp(-\pi X^N) X^a \frac{dX}{X} = \sum b(n) X^n \frac{dX}{X}.$$

Corollary 7.8. *Notations as above, let $f(X)$ denote the power series*

$$f(X) = \sum_{n \geq 1} b(n) \frac{X^n}{n} = \sum_{n \geq 0} \frac{(-\pi)^n}{n!} \frac{X^{nN+a}}{nN+a}$$

231 *Then the series*

$$f(X^q) + g_q(\psi_\pi, \chi_a; \mathsf{P}) \cdot f(X)$$

has coefficients with bounded denominators, and we have a limit formula

$$\begin{cases} -g_q(\psi_\pi, \chi_a; \mathsf{P}) = \lim\limits_{r\to\infty} \frac{q\cdot b(aq^r)}{b(aq^{r+1})} \\ \text{with} \quad b(aq^r) = \frac{(-\pi)^{(q^r-1)\frac{a}{N}}}{((q^r-1)\frac{a}{N})!} \end{cases} \tag{7.8.1}$$

We first deduce the corollary from the theorem. We know that F has eigenvalue $-g_q(\psi_\pi, \chi_a; \mathsf{P})$ on the $\psi_\pi \times \chi_a$-eigenspace of $H^1_{\text{cris}} \otimes Q_p(\pi)$, hence F has the same eigenvalue on the image of this one-dimensional eigenspace in $H^1_{\mathbf{DR}}(\widehat{C}_x \otimes R/R) \otimes Q_p(\pi)$. This image is spanned by the cohomology class of df : therefore $F + g_q(\psi_\pi, \chi_a; \mathsf{p})$ annihilates the class of df mod torsion, whence

$$f(X^q) + g_q(\psi_\pi, \chi_a; \mathsf{P}) \cdot f(X)$$

has bounded denominators. The final limit formula comes from looking successively at the coefficients of $X^{aq^{r+1}}$ in the above expression; one has

$$\operatorname{ord}\left(\frac{b(aq^r)}{aq^r} + g_q(\psi_\pi, \chi_a; \mathsf{p}) \cdot \frac{b(aq^{r+1})}{aq^{r+1}}\right) \geq -A$$

for some constant A independent of r. An explicit elementary calculation shows that

$$\operatorname{ord}\left(\frac{b(aq^r)}{aq^r}\right) \to -\infty \quad \text{as} \quad r \to +\infty,$$

and this allows us to "divide" the additive congruence and obtain the asserted limit formula.

It remains to prove the theorem. In view of the exact sequence of (5.9.5), the injectivity of

$$H^1_{\text{cris}}(C \otimes F_q/W(F_q)) \to H^1_{\mathbf{DR}}(\widehat{C}_x \otimes W/W)$$

is equivalent to the *absence* of any p-adic unit eigenvalues of F in H^1_{cris}. 232

But these eigenvalues are the Gauss sums

$$-g_q(\psi,\chi) \equiv -\sum \psi_q(x)\chi_q(x).$$

Because $\psi_q(x) \equiv 1(\pi)$ for all x, while χ_q is a *non-trivial* character of F_q^x, we have

$$-g(\psi,\chi) \equiv -\sum \chi_q(x) = 0 \bmod \pi.$$

(Alternately, one could observe that each non-trivial character χ of μ_N has at least one extension χ_1 to $\mu_{N(p-1)}$ which occurs in $H^0(C\otimes Q, \Omega^1_{C\otimes Q})$; the eigenvalue of F^{p-1} on this eigenspace is then a non-unit by (**??**); as F^{p-1} is a scalar on $(H^1)^\chi$, this scalar is non-unit.)

It remains to verify that the image of the $\psi_\pi \times \chi_a$-eigenspace is indeed spanned by

$$\exp(-\pi X^N) X^a \frac{dX}{X}$$

This seems to require the full strength of the Washnitzer-Monsky "dagger" cohomology, as follows. Let A^t denote the "weak completion" of the coordinate ring $R[T,X]/(T^p - T - X^N)$ of $U \otimes R$. Because $U \otimes F_q$ is a "special affine variety" with coordinate X, there are *unique* liftings to A^t of the actions of F and of the group $F_p \otimes \mu_N$ whose effect on X is given by

$$\begin{cases} F(X) = X^q \\ (a,\zeta)(X) = \zeta X. \end{cases}$$

Thanks to Dwork, we know that the power series in T

$$\exp(\pi T - \pi T^p)$$

actually lies in $R[T]^t$, and hence in A^t for any π satisfying $\pi^{p-1} = -p$. As Monsky pointed out, under the action of F_p and A^t, this series transforms by the character ψ_π. It follows that for $1 \le a \le N-1$ the differential form

$$\exp(\pi T - \pi T^p) X^a \frac{dX}{X}$$

transforms by $\psi_\pi \times \chi_a$ under the action of $F_p \times \mu_N$. Therefore its cohomology class in **233**

$$H^1_{W-M}(U \otimes F_q; R) \otimes Q \overset{\text{dfn}}{=} H^1(\Omega^{\bullet}_{U \otimes R/R} \otimes A^t) \otimes Q$$

lies in the $\psi_\pi \times \chi_a$ eigenspace of H^1_{W-M}. A direct computation ([31], [32]) shows that each of these eigenspaces is one-dimensional, and is spanned by the above-specified form.

Furthermore, there is a natural "formal expansion map" attached to any R-valued point x of U;

$$H^1_{W-M}(U \otimes F_q; R) \to H^1_{\mathbf{DR}}(\widehat{U}_x \otimes R/R).$$

For the particular choice of point $(T = 0, X = 0)$, the formal expansion map carries

$$\exp(\pi T - \pi T^p) X^a \frac{dX}{X} \mapsto \exp(-\pi X^N) X^a \frac{dX}{X}.$$

To conclude the proof, we need to *identify* $H^1_{WM}(U \otimes F_q; R) \otimes Q$ with $H^1_{\text{cris}}(C \otimes F_q/R) \otimes Q$ in a way compatible with the formal expansion map and with the action of F and of $F_p \times \mu_N$. We will do this with a somewhat ad hoc argument.

Because U is the complement of a single point in C, it follows from the theory of residues for both $H_{\mathbf{DR}}$ and H_{W-M} that we have isomorphisms

$$H^1_{\mathbf{DR}}(C \otimes R/R) \otimes Q \xrightarrow{\sim} H^1_{\mathbf{DR}}(U \otimes R/R) \otimes Q \xrightarrow{\sim} H^1_{W-M}(U \otimes F_q; R) \otimes Q.$$

These sit in a commutative diagram

$$\begin{tikzcd}
H^1_{\text{cris}}(C \otimes F_q/R) \otimes Q \arrow[r] \arrow[d, "2"] \arrow[dr] \arrow[rr, bend left, "8"] \arrow[dddr, bend right=60, "1"'] & H^1_{\text{DR}}(U \otimes R/R) \otimes Q \arrow[r] \arrow[d] & H^1_{WM}(U \otimes F_q; R) \otimes Q \arrow[dl, "6"] \arrow[dddl, bend left=60, "7"] \\
(\varprojlim H^1_{\text{cris}}(U \otimes F_q/R_n) \otimes Q \arrow[r, "\sim", "5"'] \arrow[ddr, "3"] & (\varprojlim H^1_{\text{DR}}(U \otimes R_n/R_n)) \otimes Q \arrow[d] & \\
 & (\varprojlim H^1_{\text{DR}}(\widehat{U}_x \otimes R_n/R_n)) \otimes Q \arrow[d, "\wr"'] & \\
 & H^1_{\text{DR}}(\widehat{U}_x \otimes R/R) \otimes Q &
\end{tikzcd}$$

234 In this diagram, the maps ② , ⑤ and ⑥ are each compatible with the actions of F and of $F_p \times \mu_N$ imposed by crystalline and by $W - M$ theory (simply because these actions *lift* to the $U \otimes W_n$). Therefore the compatibility of the isomorphism ⑧ with the actions of F and of $F_p \times \mu_N$ would follow from the *injectivity* of arrows ② and ⑥ . The injectivity of these arrows follows from the commutativity of the diagram and the already noted injectivity of arrow ① (which is injective exactly because F has no p-adic unit eigenvalues in H^1_{cris} of our particular C).

A Question 7.8.2. *Let U be a smooth affine W-scheme which is the complement of a divisor with normal crossings in a proper and smooth W-scheme.*

Are the maps

$$H^1_{\mathbf{DR}}(U/W) \otimes Q \to (\varprojlim H^1_{\mathbf{DR}}(U \otimes W_n/W_n)) \otimes Q$$

always injective?

7.9 The Gross-Koblitz Formula In this section we will derive the Gross-Koblitz formula from our limit formulas.

Morita's p-adic gamma function is the unique continuous function

$$\Gamma_p : Z_p \to Z_p^\times$$

whose values on the strictly positive integers are given by the formula

$$\Gamma_p(1+n) = (-1)^{n+1} \cdot \prod_{\substack{1 \le i \le n \\ p \nmid n}} i = \frac{(-1)^{n+1} \cdot n!}{[n/p]!p^{[n/p]}} \tag{7.9.1}$$

where [ ] denotes "integral part."

Lemma 7.9.2. *For any integer $n \ge 0$, and any π satisfying $\pi^{p-1} = -p$, we have the identity*

$$\frac{(-\pi)^n/n!}{(-\pi)^{[n/p]}/[n/p]!} = (-1) \cdot \frac{(\pi)^{n-p[n/p]}}{\Gamma_p(1+n)}. \tag{7.9.3}$$

Proof. This is just a rearrangement of (7.9.1). □

Corollary 7.9.4. *Let $q = p^f$ with $f \geq 1$, π any solution of $\pi^{p-1} = -p$ and $n \geq 0$ any integer. Let* 235

$$n = n_0 + n_1 p + \cdots \qquad 0 \leq n_i \leq p-1$$

be the p-adic expansion of n. Then we have

$$\frac{(-\pi)^n/n!}{(-\pi)^{[n/q]}/[n/q]!} = \frac{(-1)^f \cdot (\pi)^{n_0+n_1+\cdots+n_{f-1}}}{\prod_{i=0}^{f-1} \Gamma_p(1+[n/p^i])} \tag{7.9.5}$$

Proof. Simply apply (7.9.3) successively to n, $[n/p], \ldots [n/p^{f-1}]$. □

For a fixed integer $i \geq 0$, the map on positive integers

$$n \mapsto [n/p^i]$$

extends to a continuous function $Z_p \to Z_p$ which we denote

$$n \mapsto [n/p^i]_p.$$

In terms of the p-adic "digits" of n, this map is just the i-fold shift:

$$n = \sum n_j p^j \mapsto \sum_{j>0} n_{j+i} p^j = [n/p^i] \tag{7.9.6}$$

Lemma 7.9.7. *Let $0 < \alpha < 1$ be a rational number with a prime-to-p denominator. If $p^f = 1 \bmod$ denom (α) for some $f \geq 1$, then we have the identity*

$$-\langle p^{f-1}\alpha\rangle = [-\alpha/p^i]_p \quad \text{in} \quad Z \tag{7.9.8}$$

for $i = 0, 1, \ldots, f-1$ (where $\langle \quad \rangle$ denotes the "fractional part" of a rational number).

Proof. Write $(p^f - 1)\alpha = A$. Then A is an integer, $0 < A < p^f - 1$, so we may write its p-adic expansion as

$$A = a_0 + a_1 p + \cdots + a_{f-1} p^{f-1}; \qquad \begin{array}{l} 0 \le a_i \le p - 1 \\ a_i < p - 1 \text{ for } \textit{some } i. \end{array}$$

We now *extend* the definition of a_n to *all* $n \in Z$ by requiring **236**

$$a_n = a_{n+f} \qquad \forall\, n \in Z.$$

Then

$$p^{f-i}\alpha = p^{f-i}\frac{A}{p^f - 1} = \frac{\sum_{j=0}^{f-1} a_j p^{f+j-i}}{p^f - 1}$$

$$\equiv \frac{\sum_{j=0}^{f-1} a_{j+i} p^j}{p^f - 1} \bmod Z$$

whence

$$-\langle p^{f-i}\alpha\rangle = \frac{\sum_{j=0}^{f-1} a_{j+i} p^j}{1 - p^f} = \sum_{j\ge 0} a_{j+i} p^j$$

$$= \left[\frac{\sum_{j\ge 0} a_j p^j}{p^i}\right]_p$$

But we readily calculate

$$-\alpha = \frac{A}{1 - p^f} = \sum_{j\ge 0} a_j p^j.$$

□

Corollary 7.9.9. *Let $q = p^f$ with $f \geq 1$, π any solution of $\pi^{p-1} = -p$, and α any rational number satisfying*

$$\begin{cases} 0 \leq \alpha \leq 1 \\ (q-1)\alpha \in Z. \end{cases}$$

Let

$$A = (q-1)\alpha = a_0 + a_1 p + \cdots + a_{f-1}p^{f-1}, \quad 0 \leq a_i \leq p-1$$

be the p-adic expansion of $(q-1)\alpha$, and let 237

$$S((q-1)\alpha) = a_0 + a_1 + \cdots + a_{f-1}$$

be the sum of the p-adic digits of $(q-1)\alpha$. Then we have the formula

$$\lim_{n \to -\alpha} \frac{(-\pi)^n/n!}{(-\pi)^{[n/q]!}} = \frac{(-1)^f \cdot (\pi)^{S((q-1)\alpha)}}{\prod_{i=0}^{f-1} \Gamma_p(1 - \langle p^i \alpha \rangle)} \tag{7.9.10}$$

in which the limit is taken over positive integers n which approach $-\alpha$ p-adically.

Proof. Simply combine (7.9.5) and (7.9.8), and use the p-adic continuity of both Γ_p and of $n \to [n/p^i]$. □

Combining this last formula with our limit formula for Gauss sums, we obtain the Gross-Koblitz formulas.

Theorem 7.10 (Gross-Koblitz). *Let $N \geq 2$ prime to p, E a number field containing the Np'th roots of unity, P a p-adic place of E, $\pi \in E_p$ a solution of $\pi^{p-1} = -p, \psi_\pi$ the corresponding additive character of F_p, a an integer $1 \leq a \leq N-1$, χ_a the corresponding character $\zeta \mapsto \zeta^a$ of μ_N, and F_q, $q = p^f$, a finite extension of the residue field $F_{N(\mathsf{P})}$ of E at P. We have the formulas, in E_{P},*

$$-g_q(\psi_\pi, \chi_a; \mathsf{P}) = \frac{(-1)^f \cdot q \cdot \prod_{i \bmod f} \Gamma_p\left(i - \langle \frac{p^i a}{N} \rangle\right)}{(\pi)^{S((q-1)\frac{a}{N})}} \tag{7.10.1}$$

$$-g_q(\psi_\pi, \overline{\chi}_a; \mathsf{P}) = (\pi)^{S((q-1)\frac{a}{N})} \prod_{i \bmod f} \Gamma_p\left(\langle \frac{p^i a}{N} \rangle\right) \tag{7.10.2}$$

Proof. The sequence $n_r = (q^r - 1)(a/N)$ tends to $-a/N$ as r grows, and satisfies $[n_r/q] = n_{r-1}$ for $r \geq 1$. Therefore the first formula follows from the limit formula (7.8.1) and from the preceding formula (7.9.10) with $\alpha = a/N$. The second formula is obtained from the first by replacing a by $N - a$. □

238 **VIII. Interpretation via the De Rham-Witt Complex.** Throughout this chapter, we fix an algebraically closed field k of characteristic p, and a proper smooth connected scheme X over its Witt vectors $W = W(k)$. For each $n \geq 1$, we denote by X_n the W_n-scheme $X \underset{W}{\otimes} W_n$.

The "second spectral sequence" of de Rham cohomology of X_n/W_n

$$E_2^{p,q}(n) = H^p(X_n, \mathscr{H}^q_{\mathbf{DR}}(X_n/W_n)) \Rightarrow H^{p+q}(X_n/W_n)$$

has an intrinsic interpretation in terms of $X \otimes k$ as the Leray spectral sequence for the "forget the thickening" map

$$(X \otimes k/W_n)_{\text{cris}} \to (X \otimes k)_{\text{Zar}}.$$

As such, it may be rewritten

$$E_2^{p,q}(n) = H^p(X \otimes k, \mathscr{H}^q_{\text{cris}}(X \otimes k/W_n)) \Rightarrow H^{p+q}_{\text{cris}}(X \otimes k/W_n).$$

An explicit construction of this spectral sequence may be given in terms of the De Rham-Witt pro-complex on $X \otimes k$

$$\{W_n\Omega^{\bullet}\}_n$$

of Deligne and Illusie; it is simply the second spectral sequence of this complex:

$$E_2^{p,q}(n) = H^p(X \otimes k, \mathscr{H}^q(W_n\Omega^{\bullet})) \Rightarrow H^{p+q}(X \otimes k, W_n\Omega^{\bullet}).$$

It is known that the E_2 terms of this spectral sequence are finitely generated $W_n(k)$-modules. Therefore we may pass to the inverse limit and obtain a spectral sequence

$$E_2^{p,q} = \varprojlim_n E_2^{p,q}(n) \Rightarrow H_{\mathrm{cris}}^{p+q}(X \otimes k/W).$$

Let x be a W-valued point of X, and assume X connected. The formal expansion map we have exploited

$$H_{\mathrm{cris}}^i(X \otimes k/W) \simeq H_{\mathbf{DR}}^i(X/W) \to H_{\mathbf{DR}}^i(\widehat{X}_x/W)$$

is the composition of the edge-homomorphism

$$H_{\mathrm{cris}}^i(X/W) \twoheadrightarrow E_\infty^{0,i} \hookrightarrow E_2^{0,i}$$

with the natural map

$$E_2^{0,i} = \varprojlim_n H^0(X_n, \mathscr{H}_{\mathbf{DR}}^i(X_n/W_n)) \to \varprojlim H_{\mathbf{DR}}^i(\widehat{X}_x \otimes W_n/W_n).$$

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Lemma 8.1. *This map is in fact injective; indeed, the induced maps*

$$H^0(X_n, \mathscr{H}_{\mathbf{DR}}^i(X_n/W_n)) \to H_{\mathbf{DR}}^i(\widehat{X}_x \otimes W_n/W_n)$$

as injective.

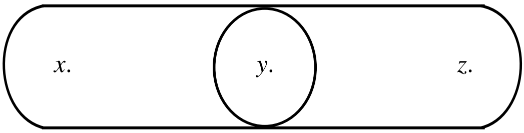
Proof. Because X_n is irreducible, it suffices to show

(*) for any closed point y of X_n, and any affine open $V \ni y$ which is étale over standard affine space $A = \mathrm{Spec}(W_n[T_1, \ldots, T_d])$, the natural map

$$H^0(V, \mathscr{H}_{\mathbf{DR}}^i(X_n/W_n)) \to \mathscr{H}_{\mathbf{DR}}^i(\widehat{V}_y/W_n).$$

is injective.

For once (*) is established we argue as follows. Let ξ be a global section over X_n of $\mathscr{H}_{\mathbf{DR}}^i$ which dies formally at x. We must show that for any closed point z in X_n, there is an open set $V \ni z$ such that ξ dies on V. Let U be an affine open neighborhood of x étale over A, and V an affine open neighborhood of z étale over A. Because X_n is irreducible, $U \cap V$ is non-empty. Let y be a closed point of X_n contained in $U \cap V$.



Then (*) for $U \ni x$ shows that ξ dies on U. Therefore ξ dies formally at y. Applying (*) to $V \ni y$, we find that ξ dies on V, as required.

We now prove (*). Let $F : A \to A^{(\sigma)}$ be *any* σ-linear map lifting absolute Frobenius (e.g. $T_i \to T_i^p$). Because V is étale over A, F extends uniquely to a σ-linear map $F : V \to V^{(\sigma)}$ which lifts absolute Frobenius.

Because all iterates of F, especially $F^n : V \to V^{(\sigma^n)}$, are homeomorphisms, the functor $(F^n)_*$ is exact. Therefore we have

$$\begin{cases} H^0(V, \mathscr{H}^i_{\mathbf{DR}}(V/W_n)) = H^0(V^{(\sigma^n)}, (F_n)_*(\mathscr{H}^i_{\mathbf{DR}}(V/W_n))) \\ (F^n)_*\mathscr{H}^i_{\mathbf{DR}}(V/W_n) = \mathscr{H}^i((F^n)_*(\Omega^*_{V/W_n})) \end{cases}$$

But the complex $(F^n)_*(\Omega^\bullet_{V/W_n})$ on $V^{(\sigma^n)}$ is a complex of locally free
240 sheaves of finite rank on $V^{(\sigma^n)}$, with $\mathscr{O}$-linear differential. For any closed point y V, the formal stalk at $y^{(\sigma^n)}$ is

$$(F^n)_*(\Omega^\bullet_{V/W_n}) \bigotimes_{\mathscr{O}_{V(\sigma^n)}} \widehat{\mathscr{O}}_{V(\sigma^n),y(\sigma^n)} \simeq (F^n)_*\Omega^\bullet_{\widehat{V}_y/W_n}.$$

Therefore the sheaves on $V^{(\sigma^n)}$

$$\mathscr{F}^i = \mathfrak{F}^i_{V/W_n} \overset{\text{dfn}}{=} (F^n)_*(\mathscr{H}^i_{\mathbf{DR}}(V/W_n)) = \mathscr{H}^i((F^n)_*\Omega^\bullet_{V/W_n})$$

are *coherent*, and (by flatness of the completion) their formal stalks are given by

$$(\widehat{\mathscr{F}^i})_{y^{(\sigma^n)}} = H^i_{\mathbf{DR}}(\widehat{V}_y/W_n)$$

We must show that

$$H^\circ(V^{(\sigma^n)}, \mathscr{F}^i) \hookrightarrow (\widehat{F^i})_{y^{(\sigma^n)}}.$$

For this, it suffices to explicit a finite filtration

$$\mathscr{F}^i \supset \mathrm{Fil}^1 \mathscr{F}^i \supset \dots$$

whose associated graded sheaves are *locally free sheaves* on $V^{(\sigma^n)} \otimes k$. We claim that the filtration induced by the p-adic filtration on $\Omega^{\bullet} V/W_n$ has this property.

To see this, we first reduce to the case $V = A$, as follows. The diagram

$$\begin{tikzcd} V \arrow[r, "F^n"] \arrow[d] & V^{(\sigma^n)} \arrow[d] \\ A \arrow[r, "F^n"] & A^{(\sigma^n)} \end{tikzcd}$$

is cartesian (because V is étale over A). Therefore we have an isomorphism

$$(F^n)_*\Omega^{\bullet}_{V/W_n} \xleftarrow{\sim} ((F^n)_*\Omega^{\bullet}_{A/W_n}) \bigotimes_{\mathscr{O}_{A^{(\sigma^n)}}} \mathscr{O}_{V^{(\sigma^n)}}.$$

Because $\mathscr{O}_{V^{(\sigma^n)}}$ is *flat* over $\mathscr{O}_{A^{(\sigma^n)}}$, this isomorphism is a filtered isomorphism (for the p-adic filtrations of $\Omega^{\bullet}_{V/W_n}$ and of $\Omega^{\bullet}_{A/W_n}$).

By flatness again, this filtered isomorphism induces isomorphisms **241**

$$\mathrm{gr}^j_{\mathrm{Fil}}(\mathscr{F}^i_{V/W_n}) \simeq (\mathrm{gr}^j_{\mathrm{Fil}}\, \mathscr{F}^i_{A/W_n}) \bigotimes_{\mathscr{O}_{A^{(\sigma^n)}}} \mathscr{O}_{V^{(\sigma^n)}}$$

It remains to show that $\mathrm{gr}^j_{\mathrm{Fil}}(\mathscr{F}^i_{A/W_n})$ is a locally free sheaf on $A^{(\sigma^n)} \otimes k$. It is certainly a *coherent* sheaf on $A^{(\sigma^n)}$ (because the p-adic filtration on $(F^n)_*\Omega^{\bullet}_{A/W_n}$ is $\mathscr{O}_{A^{(\sigma^n)}}$-linear), and it is killed by p; therefore it is a coherent sheaf on $A^{(\sigma^n)} \otimes k$. Because it is coherent, it is locally free on a non-void open set; if we knew that it were translation-invariant, i.e. isomorphic to all it translates by k-valued points of $A^{(\sigma^n)} \otimes k$, we would conclude that it is locally free everywhere.

As a sheaf of abelian groups, it is visibly translation-invariant. It's $\mathscr{O}_{A^{(\sigma^n)} \otimes k}$-module structure is the composite of its natural module-structure over the sheaf of rings

$$\mathrm{gr}^{\circ}_{\mathrm{Fil}}\, \mathscr{H}^{\circ}_{\mathbf{DR}}(A/W_n)$$

with the σ^n-linear isomorphism

$$\mathscr{O}_{A\otimes k} \xrightarrow{\sim} \mathrm{gr}^{\circ}_{\mathrm{Fil}}\, \mathscr{H}^{\circ}(A/W_n)$$
$$f \mapsto (F^n)^*(\widetilde{f}),$$

where $\widetilde{f}$ denotes any local section of $\mathscr{O}_A$ lifting f.

To conclude the proof, we must verify that this isomorphism is translation-invariant. For this, it suffices to show that it is independent of the *particular choice* of F lifting Frobenius which figures in its definition. For this independence, we simply notice that an "intrinsic" description of the same σ^n-linear isomorphism

$$\mathscr{O}_{A\otimes k} \xrightarrow{\sim} \mathrm{gr}^{\circ}_{\mathrm{Fil}}\, \mathscr{H}^{\circ}(A/W_n)$$

is provided by

$$f \mapsto (\widetilde{f})^{p^n}$$

where again $\widetilde{f} \in \mathscr{O}_A$ denotes any lifting of f. □

Lemma 8.2. *The $E_2^{i,0}$ terms of the spectral sequence are given by*

$$E_2^{i,0} \simeq H^i_{\mathrm{et}}(X \otimes k, Z_p) \otimes W(k)$$

Proof. For each integer $n \geq 1$, there is an isomorphism (cf. [24], [25])

$$W_n(O_{X\otimes k}) \xrightarrow{\sim} \mathscr{H}^0_{\mathbf{DR}}(X_n/W_n)$$

242 defined by

$$(g_0, \dots, g_{n-1}) \mapsto \sum_{i=0}^{n-1} p^i (\widetilde{g}_i)^{p^{n-i}}$$

where $\widetilde{g}_i$ is a local lifting of $g_i \in O_{X\otimes k}$ to O_{x_n} (Compare (**??**)).

For variable n, these isomorphisms sit in a commutative diagram

$$\begin{tikzcd}
W_{n+r}(O_{X\otimes k}) \arrow[r, "\sim"] \arrow[d, "\text{usual projection}"] & \mathscr{H}^0_{\mathbf{DR}}(X_{n+r}/W_{n+r}) \arrow[dd, "\text{reduction mod } p^n"] \\
W_n(O_{X\otimes k}) \arrow[d, "F^r"] & \\
W_n(O_{X\otimes k}) \arrow[r, "\sim"] & \mathscr{H}^0_{\mathbf{DR}}(X_n/W_n).
\end{tikzcd}$$

Therefore we may calculate

$$\begin{aligned}
E_2^{i,0} &= \varprojlim_n H^i(X\otimes k, \mathscr{H}^0_{\mathbf{DR}}(X_n/W_n)) \\
&\xrightarrow{\sim} \varprojlim_n \left(\bigcap_r (\text{image of } F^r \text{ or } H^i(X\otimes k, W_n(O_{X\otimes k}))\right). \\
&\simeq \varprojlim(\text{fixed points of } F \text{ in } H^i(X\otimes k, W_n(O_{X\otimes k})) \bigotimes_{Z/p^nZ} W_n(k) \\
&\simeq \varprojlim_n H^i_{\text{et}}(X\otimes k, Z/p^nZ)\otimes W_n(k).
\end{aligned}$$

□

Consider now the exact sequence of terms of low degree

$$0 \to E_2^{1,0} \to H^1_{\text{cris}}\quad (X\otimes k/W) \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0}$$

Lemma 8.3. *The map $d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$ vanishes.*

Proof. Because both $H^1_{\text{cris}}(X\otimes k/W)$ and $E_2^{2,0} = H^2_{\text{et}}(X\otimes k, Z_p)\otimes W_j$ are finitely generated W-modules, we see that $E_2^{0,1}$ is a finitely generated W-module. Therefore its inverse limit topology (as $\varprojlim E_2^{0,1}(n)$) is equivalent to its p-adic topology. Because F^n annihilates the sheaf $\mathscr{H}^1_{\text{cris}}(X\otimes k/W_n)$, it annihilates its global sections $E_2^{0,1}(n)$, and hence F is topologically nilpotent on $E_2^{0,1}$. But F is an *automorphism* of the finitely generated W-module $E_2^{2,0}$; as d_2 commutes with F, this forces $d_2^{0,1}$ to *vanish.* □

Thus we obtain the following theorem.

243 **Theorem 8.4.** *The exact sequence of terms of low degree*

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathrm{et}}(X\otimes k, Z_p)\otimes \arrow[r] & H^1_{\mathrm{cris}}(X\otimes k/W) \arrow[r] \arrow[d, "\wr"'] & E_2^{0,1} \arrow[r] \arrow[d, hook] & 0 \\
 & & H^1_{\mathbf{DR}}(X/W) \arrow[r, "\text{formal}", "\text{expansion}"'] & H^1_{\mathbf{DR}}(\widehat{X}_x/W) &
\end{tikzcd}$$

defines the Newton-Hodge filtration on H^1_{cris}

$$0 \to (\text{slope } 0) \to H^1_{\mathrm{cris}}(X\otimes k/W) \to (\text{slope } > 0) \to 0.$$

[When X/W is a curve, or an abelian scheme, this exact sequence coincides with the exact sequence ((5.7.2) or (5.9.5)!]

Illusie and Raynaud have recently been able to generalize these results to H^i_{cris} for all i. Their remarkable result is the following.

Theorem 8.5. (Illusie-Raynaud). *Let X_0 be proper and smooth over an algebraically closed field of characteristic $p > 0$. The second spectral sequence of the De Rham-Witt complex*

$$E_2^{p,q} = \varprojlim_n H^p(X_0, \mathscr{H}^q(W_n\Omega^\bullet)) \Rightarrow H^{p+q}_{\mathrm{cris}}(X_0/W)$$

degenerates at E_2 after tensoring with Q:

$$E_2^{p,q} \underset{Z}{\times} Q \simeq E_\infty^{p,q} \underset{Z}{\otimes} Q, d_r\otimes Q = 0 \text{ for } r \geq 2,$$

and defines the Newton-Hodge filtration on $H_{\mathrm{cris}}(X_0/W)\otimes Q$:

$$q - 1 < \text{slopes of } E_2^{p,q}\otimes Q \leq q.$$

Corollary 8.6. *If X_0/k lifts to X/W, then for any W-valued point x of X, and any integer i, the image of the formal expansion map*

$$H^i_{\mathrm{cris}}(X\otimes k/W)\otimes Q \simeq H^i_{\mathbf{DR}}(X/W)\otimes Q \to H^i_{\mathbf{DR}}(\widehat{X}_x/W)\otimes Q$$

is precisely the quotient "slopes $> i-1$" of $H^i_{\mathrm{cris}}\otimes Q$.

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ESTIMATES OF COEFFICIENTS OF MODULAR FORMS AND GENERALIZED MODULAR RELATIONS

By S. Raghavan

247 We shall be concerned here with two questions, motivated by arithmetic, from the theory of modular forms. The first one deals with the estimation of the magnitude of the Fourier coefficients of Siegel modular forms, while the second pertains to certain generalized modular relations (which may also be called Poisson formulae of Hecke type and) which appear to provide some kind of a link between automorphic forms (of one variable), representation theory and arithmetic.

§Modular forms of degree n

Let $r_m(t)$ denote the number of ways in which a natural number t can be written as a sum of m squares of integers. We have the well-known Hardy-Ramanujan asymptotic formula [H-R] for $m > 4$:

$$r_m(t) = \pi^{m/2}\sigma_m(t)t^{(m/2)-1}/\Gamma(m/2) + O(t^{m/4}) \tag{1}$$

with $\sigma_m(t)$ denoting the 'singular series'. Arithmetical functions such as $r_m(t)$ or, more generally, the number $A(S, t)$ of m-rowed integral columns x with ${}^t xSx = t$ for a given m-rowed integral positive-definite matrix S (where ${}^t X$ = transpose of x) occur as Fourier coefficients of modular forms. While Hardy and Ramanujan used the 'circle method' to prove (1), the approach of Hecke [H1] to (1) was via the decomposition of the space of (entire) modular forms into the subspace generated by Eisenstein seris and the subspace of cusp forms, the explicit determination of the Fourier expansion of Eisentein series and the estimation of the Fourier coefficients $c(t)$ of cusp forms of weight k as $c(t) = O(t^{k/2})$.

More generally, let $A(S,T)$ be the number of integral matrices G such that ${}^tGSG = T$ for n-rowed integral T (For any matirx B, let ${}^tB$ denote its transpose and for a square matrix C, let $tr(C)$ and det C denote its trace and determinant respectively). For $A(S,T)$, we have, as a 'generating function', the theta series $\vartheta(S,Z) = \sum\limits_G \exp(2\pi\sqrt{-1}tr({}^tGSGZ))$
where G runs over all (m,n) integral matrices and Z is in the Siegel 248
half-plane 'H_n' of n-rowed complex symmetric matrices $Z = (z_{ij})$ with $Y = (y_{ij})$ positive definite and $y_{ij} = \operatorname{Im} z_{ij}$; further, the theta series is a modular form of degree n, weight $m/2$ and stufe 4 det S. Let $\Gamma_n(s)$ denote the principal congruence subgroup of stufe s in the Siegel modular group of degree n and $\{\Gamma_n(s), k\}$ denote the space of modular forms of degree n, weight k and stufe s. Pursuing the approach of Hecke and Petersson and using Siegel's generalized Farey dissection [S], the following result was proved in [R]: namely, if $k > n+1$ and $f(Z) = \sum\limits_{T\geq 0} a(T)\exp(2\pi\sqrt{-1}tr(TZ)/s)\ \epsilon\{\Gamma_n(s), k\}$, there exists a linear combination $g(z) = \sum\limits_{T\geq 0} b(T)\exp(2\pi\times\sqrt{-1}tr(TZ)/s)$ of Eisenstein series in $\{\Gamma_n(s), k\}$ such that for positive-definite

$$T, a(T) = b(T) + O((\min T^{-1})^{n(n+1-2k)/2}(\min T)^{(n+1-k)/2} \tag{2}$$

(For positive definite R, $\min R$ is the first minimum in the sense of Minkowski). Specialising f to be $\varepsilon(s,z)$ above, (2) implies the formula:

$$A(s,T) = \lambda\prod_p \alpha_p(S,T)(\det T)^{(m-n-1)/2} + O((\det T)^{(m(2n-1)-2(n^2-1))/4n}) \tag{3}$$

where $m > 2n+2$,

$$= \pi^{n(2m-n+1)/4}(\det S)^{-n/2}\{\Gamma(m/2)\ldots\Gamma((m-n+1)/2)\}^{-1},$$

$\prod_p \alpha_p(s, T)$ is the product (over all primes p) of the p-adic densities $\alpha_p(S, T)$ of representation of T by S; further, in (3), T tends to infinity such that for a fixed constant c, $\min T \geqslant c(\det T)^{1/n}$. From (3), an analogue of a theorem of Tartakowsky resulted for $n = 2$ [R]: namely, under the conditions above, for larde $\det T, A(S, T) \neq 0$ for every matrix in the 'genus' of S or for none at all, depending on certain congruence classes to which T belongs. It should be mentioned that, without using Siegel's generalized Farey dissection, only estimates of the type $a(T) = O((\det T)^k)$ could be derived, in general, earlier; for improving upon (2), it was felt that the decomposition of the space of modular forms of degree n into $n + 1$ subspaces through Maass' Poincaré series should be invoked.

Hsia, Kitaoke and Kneser [H-K-K] obtained, using an arithmetic aproach, a very elegant proof of the analogue of Tartakowsky's theorm for any $n \geqslant 1$ and $m \geqslant 2n + 3$. Quite recently, Kitaoka [KI] gave an analytic proof of the same result in the case when S is an even positive
249 definite m-rowed unimodular matrix with $m \geqslant 4n+4$. By considering for even $k \geqslant n+r+2$, $Z \epsilon$ 'H_n' and $0 < r < n$, the Eisenstein series $E(Z, h) = E^k_{n,r}(Z, h)$ which have been studied by Klingen [KL] and which arise by 'lifting' a cusp form h in $\{\Gamma_r(1), k\}$ to $\{\Gamma_n(1), k\}$, Kitaoka has obtained, in the same paper, the estimate $a(T, h) = O((\det T)^{k-(n+1)/2} \times (\det T_1)^{(r+1-k)/2})$ for the Fourier coefficients $a(T, h)$ of $E(Z, h)$ with $T = \begin{pmatrix} T_1 & * \\ * & * \end{pmatrix}$ and r-rowed symmetric T_1. If f is in $\{\Gamma_n(1), k\}$ with even $k \geqslant 2n + 2$ and $\Phi^n f = 0$ for the Siegel operator Φ, then for the Fourier coefficients a (T) of f with positive definite T, Kitaoka derived, as a consequence, the estimate

$$a(T) = O((\det T)^{k-(n+1)/2}(\min T)^{1-k/2}) \tag{4}$$

From [C], it can be seen that any f in $\{\Gamma_n(s), k\}$ for $k > 2n + 1$ is a finite linear combination of Poincaré series $G_k(Z; \Gamma_n(s); T)$ and their transforms under coset representatives of Γ_n (1) modulo $\Gamma_n(s)$ for non-negative definite T. Following Kitaoka's method with appropriate modifications (e.g. of Lemma 7, §2, [KI]), it is not hard to prove the following

Theorem. *If* $f(Z) = \sum_{T \geqslant 0} a(T)\exp(2\pi\sqrt{-1}tr(TZ)/z)\epsilon\{\Gamma_m(s), k\}$ *with* $k >$ $2n+1$ *is such that for every* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *in* Γ_n *(1), the constant term in the Fourier expansion of* $f((AZ+B)(CZ+D)^{-1})\det(CZ+D)^{-k}$ *is* 0, *then we have* $a(T) = O((\delta T)^{k-(n+1)/2}(\min T)^{1-k/2})$*, for positive definite* T.

Kitaoka [KI] has conjectured that the above theorem is true even for $2k \geqslant 2n+3$. One can also consider the analogues of the theorem above her hermitian and Hilbert-Siegel modular forms.

§Poisson formulae of Hecke type.

Arithmetical identities have played a useful role in the estimation of the order or the average order of arithmetical functions. For Ramanujan's function $\tau(n)$, we have an interesting identity

$$\sum_{1 \leqslant n < \infty} \tau(n)\exp(-s\sqrt{n}) = 2^{36}\pi^{23/2}\Gamma(25/2)s \sum_{1 \leqslant n < \infty} \tau(n)(s^2 + 16\pi^2 n)^{-2\,5/2}$$

for $s > 0$, which looks more involved than the 'theta-relation'

$$\sum_{1 \leqslant n < \infty} \tau(n)\exp(-ny) = (2\pi/y)^{1\,2} \sum_{1 \leqslant n < \infty} \tau(n)\exp(-4\pi^2 n/y) \quad (y > 0).$$

Such identities (or modular relations as they are referred to in the litera- **250**
ture) seem to be included by "Poisson formulae of Hecke type" considered by Igusa [I], which may thus be called *generalized modular relations.*

Let $\mathscr{F}$ be the space of complex-valued C^∞ functions F on the space $\mathbb{R}_+^x$ of positive real numbers which behave like Schwartz functions at infinity and which have, as t tends to 0, an asymptotic expansion $F(t) \approx \sum_{r \geqslant 0} a_r t^r$ which is termwise differentiable (infinitely often). Let $\mathscr{Z}$ be the space of complex-valued functions Z on the complex plane such that $Z(s)/\Gamma(s)$ is entire in s and further, for every polynomial P, the functions ZP is bounded in any vertical strip $\{s|\alpha \operatorname{Re} s \leqslant \beta\}$ with neighbourhoods of $0, -1, -2, \ldots$ removed therefrom. The usual Mellin transform $F \mapsto MF$ established a one-one correspondence between $\mathscr{F}$ and $\mathscr{Z}$. On the other hand, for any real $\kappa > 0$, there exists in $\mathscr{Z}$, an involution $Z \mapsto Z^{\times}$ with $Z^{\times}(s) = Z(\kappa - s)\Gamma(s)/\Gamma(\kappa - s)$ and this carries over to a unitary operator $F \mapsto \mathbb{W}F$ in $\mathscr{F}$. If $\varphi(s) = \sum\limits_{1 \leqslant n < \infty} a_n n^{-s}$ is a Dirichlet series (absolutely convergent in a half-plane and) of signature $\{\lambda, \kappa, \gamma\}$ in the sense of Hecke [H2] so that $(s - \kappa)\varphi(s)$ is entire and of finite genus and further $\xi(s) = (\lambda/2\pi)^s \Gamma(s)\varphi(s) = \gamma\xi(\kappa - s)$, then the Poisson formula established by Igusa in [I] reads:

$$\sum_{0 \leqslant n < \infty} a_n (\mathbb{W}F)(2\pi n/\lambda) = \gamma \sum_{0 \leqslant n < \infty} a_n F(2\pi n/\lambda) \tag{5}$$

for every F in $\mathscr{F}$, where $a_o = \gamma(\lambda/2\pi)^\kappa \Gamma(\kappa)$. Residue $\varphi(s)$. This includes a result of Yamazaki.

Let $G(s) = \prod\limits_{1 \leqslant j \leqslant r} (\Gamma(\alpha_j s + \beta_j))^{m_j}$ with $\alpha_j > 0$, $\operatorname{Re}\beta_j \geqslant 0$, $m_j \geqslant 1$ and further, for $i \neq j$, $\alpha_i\beta_j - \alpha_j\beta_i$ is not of the form $m\alpha_j - n\alpha_i$ with integers $m, n \geqslant 0$.

Let $\{\varphi_j(s) = \sum\limits_{n \neq 0} a_n^{(j)} |n|^{-s}; 1 \leqslant j \leqslant N\}$ and

$\{\psi_j(s) = \sum\limits_{n \neq 0} b_n^{(j)} |n|^{-s}; 1 \leqslant j \leqslant N\}$ be two sets of N Dirichlet series (each converging in some right half-plane absolutely) so that if we write

$$\xi_j(s) = \lambda^s G(s)\varphi_j(s), \eta_j(s) = \lambda^s G(s)\psi_j(s) \qquad (1 \leqslant j \leqslant N)$$

251 for some fixed $\lambda > 0$, then we have the functional equations

$$\xi_j(\kappa - s) = \sum_{1 \leqslant k \leqslant N} c_{jk}\eta_k(s) \quad (1 \leqslant j \leqslant N) \tag{6}$$

with real c_{jk}; we may suppose that $(c_{jk})^2$ is the identity matrix and also that ξ_k, η_1 have only finitely many poles. Following Igusa [I], the spaces $\mathscr{F}$, $\mathscr{Z}$ may be redefined so that $\mathscr{Z}$ consists, for example, only of meromorphic functions Z on the complex plane such that Z/G is entire and PZ is bounded in 'vertical strips' (with neighbourhoods of poles removed) for every polynomial P. In the space $\mathscr{F}$, we have a unitary operator $\mathbb{W}$ such that for every F in $\mathscr{F}$, $(M(\mathbb{W}F))(s)/G(s) = (MF)(\kappa - s)/G(\kappa - s)$ for a $\kappa > 0$, M being the Mellin transform. Let no ξ_k have a pole on $\operatorname{Re} s = \kappa/2$, for simplicity and let $u_1, \ldots, u_p$ be all the poles of ξ_k's. Then we have a Poisson formula of Hecke type [R-R] given by the following

Theorem. *For any function $F : \mathbb{R}_+^x \to \mathbb{C}$ whose Mellin transform MF is such that MF/G is entire and $P.MF$ is bounded in vertical strips (with neighbourhoods of poles removed) for every polynomial P and for $\xi_1, \ldots, \xi_N, \eta_1, \ldots \eta_N$ satisfying* (6), *we have*

$$\sum_{n \neq 0} a_n^{(k)} F(|n|/\lambda) - \sum_{\operatorname{Re} u_j < \kappa/2} \underset{s=u_j}{Residue} \frac{MF(s)}{G(s)} \xi_k(s) = \tag{7}$$

$$= \sum_{1 \leqslant 1 \leqslant N} c_{kl} \left(\sum_{n \neq 0} b_n^{(k)} (\mathbb{W}F)(|n|/\lambda) - \right.$$

$$\left. - \sum_{\operatorname{Re}\ u_j < \kappa/2} \underset{s=u_j}{Residue} \frac{(M(\mathbb{W}F))(s)}{G(s)} \eta_1(s) \right)$$

Formula (7) generalizes some well-known relations of a similar nature considered, for example, by Maass [M1] in the Hecke theory of non-analytic automorphic forms and by B.C. Berndt. The proof of (7) is on the same lines as in Hecke [H2]; the sum over residues has to be interpreted suitably in terms of the coefficients in the asymptotic expansions of F and $\mathbb{W}F$ at 0 and the residue of the Dirichlet series involved and sometimes, it takes a simple form as in (5). A Poisson summation 252
formula for a generalized Fourier transformation due to Kubota can also be treated with arguments similar to those for (7). In the study of non-analytic automorphic forms, Maass [M2] has considered (for Dirichlet series) functional equations in matrix form involving a generalized Γ-function $\Gamma(s; \alpha, \beta)$ which is the Mellin transform of the standard Whittaker function $W(y; \alpha, \beta)$; in this case again, a general Poisson formula

like (7) for pairs (F_1, F_2) of C^∞ functions on $\mathbb{R}_+^x$ with prescribed behaviour at infinity and at 0 can be obtained. Specialising $F_1(t)$, $F_2(t)$ to be $W(ty; \alpha, \beta)$, $W(ty; \beta, \alpha)$ respectively (with $y > 0$), one gets the corresponding formula in [M2]; in the light of a recent paper of Ranga Rao, it turns out that there are quite a few pairs (F_1, F_2) for which our Poisson formula holds.

In the context of formula (5) proved in the lectures [I], one comes across the natural question as to whether a p-adic analogue of the operator $\mathbb{W}$ exists. One may consider, instead of $\mathscr{F}$ above, the space $\mathscr{F}(\mathbf{Q}_p^x)$ of complex-valued F on $\mathbb{Q}_p^x$ which are locally constant, with

$$F(t) = \left\{\left\{ \begin{array}{l} 0 \text{ for all } t \text{ with valuation } |t|_p \text{ large} \\ a\mu_1(t)|t|_p^{\frac{1}{2}} + b\mu_2(t)|t|_p^{1/2} \text{ for all } t \text{ with } |t|_p small \end{array} \right\}\right. \tag{8}$$

constants a, b and quasicharacters μ_1, μ_2. This is a so-called Kirillov model for irreducible admissible representations π_p of $GL_2(\mathbb{Q}_p)$. In this case, if $L(s, \pi_p) = \{(1 - \mu_1(p)p^{\frac{1}{2}-s})(1 - \mu_2(p)p^{\frac{1}{2}-s})\}^{-1}$, then the $\mathbb{W}$-operator is given again via the Mellin transform M:

$$\frac{(M(\mathbb{W}F))(1-s)}{L(1-s, \pi_p)} = \epsilon(s, \pi_p)\frac{(MF)(s)}{L(s, \pi_p)} \tag{9}$$

with a certain function $\epsilon(s, \pi_p)$ for which $\epsilon(s, \pi_p)$. $\epsilon(1 - s, \pi_p) = 1$. Let W_p^0 be the Whittaker function on $GL_2(\mathbb{Q}_p)$ whose Mellin transform (over $\mathbb{Q}_p^x$) is $L(s, \pi_p)$, for every prime p and further let $\{\pi_p\}_p$ be such that together with a representation π_∞ of $GL_2(\mathbb{R})$, the tensor product $\pi_\infty \otimes_p \pi_p$ gives an irreducible unitary representation of $GL_2(\mathbb{Q}_A)$ and moreover, let $\prod\limits_p L(s, \pi_p)$ be a Dirichlet series $\sum\limits_{n \neq 0} a_n |n|^{-s}$ converging absolutely in a right s-half plane, with a functional equation $s \to 1 - s$, involving
253 $L(s, \pi_\infty) = (2\pi)^{-s-(p+1)/2}\Gamma(s + (p+1)/2)$ for $p \geqslant 0$ in $\mathbb{Z}$ or $\pi^{-s-\nu} \times \Gamma((s + \nu)/2)\Gamma((s - \nu)/2)$ with ν in $\mathbb{C}$. Then for F on $\mathbb{Q}_A^\times$ built from $\mathscr{F}$ and the various W_p^0, we have an adelic analogue of our Poisson formula. Under specialization, a formula of this kind constitutes an important step in the Jacquet-Langlands' theory, for showing that a global representation of $GL_2(\mathbb{Q}_A)$ occurs in the space of cusp forms. Further details may be found in [R-R].

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A REMARK ON ZETA FUNCTIONS OF ALGEBRAIC NUMBER FIELDS[1]

By Takuro Shintani

Introduction

For a totally real algebraic number field k, it is known that every (partial) 255
zeta function of k is a finite sum of Dirichlet series which are regarded as natural generalizations of the Hurwits zeta function (see [1] and [2]). In this note we show that the similar result holds for arbitrary (not necessarily totally real) algebraic number field. At the time of the Bombay Colloquium (1979), H. M. Stark orally communicated to the author that he has obtained such a result for non-real cubic fields. His oral communication was an initial impetus to the present work. The author wishes to express his gratitude to Stark.

Notation. We denote by **Z**, **Q**, **R** and **C** the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers respectively. The set of positive real numbers is denoted by $\mathbf{R}_+$. For an algebraic number field k, we denote by $E(k)$ and $O(k)$ the group of units of k and the ring of integers of k respectively.

[1]Results presented at the time of the Colloquium were relevant to automorphic forms on unitary groups of order 3. However, later the author found several gaps in the proof of those results. Here, another result obtained after the Colloquium is exposed.

[2]Takuro Shintani suddenly passed away on November 14, 1980. Ed.

1. Let V be an n-dimensional real vector space. For $\mathbf{R}$-linearly independent vectors $v_1, v_2, \ldots, v_t \epsilon V (1 \leqslant t \leqslant n)$, we denote by $C(v_1, \ldots, v_t)$ the set of all *positive* linear combinations of $v_1, \ldots, v_t$. We call

$$C(v_1, \ldots, v_t)$$

a *t-dimensional open simplicial cone with generators* $v_1, \ldots, v_t$. Note that generators of a given open simplicial cone are unique up to permutations and multiplications by positive scalars. We call a disjoint union of a finite number of open simplicial cones in V a *general polyhedral cone*. Thus a general polyhedral cone is not necessarily convex. Now assume that V has a $\mathbf{Q}$-structure. Thus, an n-dimensional $\mathbf{Q}$-vector sub-
256 space $V_{\mathbf{Q}}$ such that one has $V = V_{\mathbf{Q}} \bigotimes_{\mathbf{Q}} \mathbf{R}$ is identified in V. An open simplicial cone is said to be $\mathbf{Q}$-*rational* if, for a suitable choice of generators, all generators are in $V_{\mathbf{Q}}$. A disjoint union of a finite number of $\mathbf{Q}$-rational open simplicial cones is said to be a $\mathbf{Q}$-*rational general polyhedral cone*.

A linear form on V is said to be $\mathbf{Q}$-rational if it is $\mathbf{Q}$-valued on $V_{\mathbf{Q}}$.

Lemma 1. *Let $C^{(1)}$ and $C^{(2)}$ be two $\mathbf{Q}$-rational general polyhedral cones. Then $C^{(1)} - C^{(2)}$ is again a $\mathbf{Q}$-rational general polyhedral cone.*

Proof. If is sufficient to prove the Lemma assuming that both $C^{(1)}$ and $C^{(2)}$ are Q-rational simplicial cones. Let t be the dimension of $C^{(2)}$. There are n $\mathbf{R}$-linearly independent Q-rational linear forms $L_1, \ldots L_t$; $M_1, \ldots, M_{n-t}$ on V such that

$$\begin{aligned} C^{(2)} = \{v \epsilon V; L_a(v) > 0,\ a = 1, \ldots, t, \\ M_b(v) = 0,\ b = 1, \ldots, n-1\}. \end{aligned}$$

For each $b (1 \leqslant b \leqslant n - t)$, set

$$\begin{aligned} C^{(1)}(b, \pm) = \Big\{v \epsilon C^{(1)}; M_1(v) = \ldots = M_{b-1}(v) = 0, \\ \pm M_b(v) > 0\}. \end{aligned}$$

For each a ($1 \leqslant a \leqslant t$), set

$$C^{(1)}(n-t+1,a) = \Big\{v\epsilon C^{(1)}; M_b(v) = 0 \text{ for } b = 1,\ldots,n-t, \\ L_1(v) > 0,\ldots,L_{a-1}(v) > 0, L_a(v) \leqslant 0\Big\}.$$

Then it is immediate to see that $C^{(1)} - C^{(2)}$ is a disjoint union of sets: $C^{(1)}(b,+)(1 \leqslant b \leqslant n-t)$, $C^{(1)}(b,-)(1 \leqslant b \leqslant n-t)$ and $C^{(1)}(n-t+1,a)(1 \leqslant a \leqslant t)$. It follows from Lemma 2 of [1] and its corollary that $C^{(1)}(b,\pm)(1 \leqslant b \leqslant n-t)$ and $C^{(1)}(n-t+1,a)(1 \leqslant a \leqslant t)$ are all disjoint unions of finite number of **Q**-rational open simplicial cones. □

2. Let k be an algebraic number field of degree n with r_1 real and r_2 complex infinite primes ($n = r_1 + 2r_2$). Let $x \mapsto x^{(i)}$ $(1 \leqslant i \leqslant n)$ be n mutually distinct embeddings of k into the field of complex numbers $\mathbf{C}$. We may assume that $x^{(1)},\ldots,x^{(r_1)}$ are all real and that $x^{(r_1+i)} = x^{-(r_1+r_2+i)}$
($1 \leqslant i \leqslant r_2$). We embed k into an n-dimensional real vector space **257**
$V = \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ via the map: $x \longmapsto (x^{(1)},\ldots,x^{(r_1)},x^{(r_1+1)},\ldots,x^{(r_1+r_2)})$. We identify k with an n-dimensional **Q**-vector subspace of V by means of the embedding. Fix a **Q**-structure of V by setting $V_{\mathbf{Q}} = k$. Set $V_+ = \mathbf{R}_+^{r_1} \times (\mathbf{C})^{r_2}$, $k_+ = V_+ \cap k$ and $E(k)_+ = E(k) \cap k_+$. Thus $E(k_+)$ is the group of totally positive units of k. By componentwise multiplications, the group $E(k)_+$ acts on V_+.

Proposition 2. *There exists a finite system* $\{C_j; j \in J\}(|J| < \infty)$ *of open simplicial cones with generators all in* k_+ *such that* $V_+ = \bigcup_{j\in J}\bigcup_{u\in E(k)_+} uC_j$ *(disjoint union).*

Proof. For each $x \in V$, we denote by $N(x)$ the "norm" of x given by $N(x) = x^{(1)}\ldots x^{(r_1)}|x^{(r_1+1)}\ldots x^{(r_1+r_2)}|^2$. Let V_+^1 be the subset of V_+ consisting of all vectors with norm 1:

$$V_+^1 = \{x \in V_+; N(x) = 1\}.$$

Note that each vector in V_+ is uniquely expressed as a positive scalar multiple of a vector in V_+^1 : $x = \{N(x)^{1/n}\}\{N(x)^{-1/n}.x\}$.

If follows from the Dirichlet unit theorem that the group $E(k)_+$ acts on V_+^1 properly discontinuously and that $E(k)_+/V_+^1$ is compact. Thus, there exists a compact subset F of V_+^1 such that

$$V_+^1 = \bigcup_{u \in E(k)_+} uF. \tag{1}$$

Note that the subset of V_+^1 gives as $\{N(x)^{-1/n}x; x \in k_+\}$ is dense in V_+^1.

Hence for each $X \in F$, there exists an n-dimensional open simplicial cone C with generators all in k_+ such that $x \in C \cap V_+^1$ and that $C \cap uC =$ for any $1 \neq u \in E(k)_+$. Thus, there exists a finite system $C_1, \ldots, C_s$ of n-dimensional open simplicial cones with generators all in k_+ such that

$$F = \bigcup_{i=1}^{s} (C_i \cap V_+^1) \tag{2}$$

and that

$$C_i \cap uC_i = \text{ for any } 1 \neq u \in E(k)_+ (1 \leqslant i \leqslant s). \tag{3}$$

258 If follows from (1) and (2) that

$$V_+ = \bigcup_{i=1}^{s} \bigcup_{u \in E(k)} uC_i.$$

Set $C_1^{(1)} = C_1$ and set

$$C_i^{(1)} = C_i - \bigcup_{u \in E(k)_+} uC_1 (2 \leqslant i \leqslant s).$$

Note that uC_1 is disjoint to C_i except for a finite number of u. Hence Lemma 1. implies that $C_i^{(1)}$ is a $\mathbf{Q}$-rational general polyhedral cone. Taking (3) into account, we have

$$V_+ = \bigcup_{i=1}^{s} \bigcup_{u \in E(k)_+} uC_i^{(1)} \text{ and}$$

$$uC_1^{(1)} \cap C_i^{(1)} = \text{ for any } u \in E(k)_+ \text{ if } i \geqslant 2.$$

Now assume that a finite system of **Q**-rational general polyhedral cones $C_1^{(a)}, \dots, C_s^{(a)} (1 \leqslant a \leqslant s-2)$ with the following three properties is given:

$$C_i^{(a)} \subset C_i \tag{$4)_{(a)}$}$$

$$V_+ = \bigcup_{i=1}^{s} \bigcup_{u \in E(k)_+} uC_i^{(a)}, \tag{$5)_{(a)}$}$$

$$uC_i^{(a)} \cap C_j^{(a)} = \text{ for any } u \in E(k)_+ \text{ if } i \leqslant a \text{ and } i \neq j. \tag{$6)_{(a)}$}$$

Then set $C_i^{(a+1)} = C_i^{(a)}$ for $i \leqslant a+1$ and set

$$C_i^{(a+1)} = C_i^{(a)} - \bigcup_{u \in E(k)} uC_{a+1}^{(a)} \text{ for } i \geqslant a+2.$$

Then $\{C_1^{(a+1)}, \dots, C_s^{(a+1)}\}$ is a finite system of **Q**-rational general polyhedral cones with properties $(4)_{a+1}$, $(5)_{a+1}$ and $(6)_{a+1}$.

It is easy to see that $\{C_1^{(s-1)}, \dots, C_s^{(s-1)}\}$ is a finite system of **Q**-rational general polyhedral cones such that

$$V_+ = \bigcup_{i=1}^{s} \bigcup_{u \in E(k)_+} uC_i^{(s-1)} \qquad \text{(disjoint union).}$$

□

Remark. For totally real fields k, Proposition 2 is obtained in [1] by a different method (cf. Proposition 4 of [1]).

3. We choose and fix a finite system $\{C_j; j \in J(|J| < \infty)\}$ of open simplicial cones with generators all in k_+ such that

$$V_+ = \bigcup_{j \in J} \bigcup_{u \in E(k)_+} uC_j \quad \text{(disjoint union).} \tag{7}$$

The existence of such a system is guaranteed by Proposition 2.. For each **259**
C_j, we denote by t_j the dimension of C_j and choose and fix generators $v_{j1,\dots,}V_{jt_j}$ of C_j *so that they are all in* $O(k)_+ = O(k) \cap k_+$.

Furthermore, we choose and fix integral ideals $a_1, a_2, \ldots, a_{h_0}$ so that they form a complete set of representatives for *narrow* ideal classes of k. Lef f be an integral ideal of k and let $H_k(f)$ be the group of narrow ideal classes modulo f. There is a natural homomorphism from the group $H_k(f)$ onto the group of narrow ideal classes of k. Fro each $c \in H_k(f)$ there uniquely exists an index $i(c)(1 \leqslant i(c) \leqslant h_0)$ such that c is mapped to the class represented by $f a_{i(c)}$.

Set

$$C_j^1 = \left\{ s_1 v_{j1} + s_2 v_{j2} + \ldots + s_{t_j} v_{jt_j}; 0 < s_1, s_2, \ldots, s_{t_j} \leqslant 1 \right\}$$

and

$$R(c, C_j) = \{x \in C_j^1 \cap f^{-1} a_{i(c)}^{-1}; (x) f a_{i(c)} \in c\}.$$

Then $R(c, C_j)$ is *finite*.

Let C be a t-dimensional open simplicial cone with a prescribed system of generators $v_1, \ldots, v_t$.

For each $x \in C$, we denote by $\zeta(s, C, x)$ the Dirichlet series given by

$$\zeta(s, C, x) = \sum_z N(x + z_1 v_1 + \ldots + z_t v_t)^{-s}, \tag{8}$$

where $z = (z_1, \ldots, z_t)$ ranges over the set of all t-tuples of non- negative integers (the notation N is introduced at the beginning of the proof of Proposition 2).

Let $\zeta_k(s, c)$ be the zeta functions of k corresponding to the ray class c given by

$$\zeta_k(s, c) = \sum_{\mathfrak{g}} N(\mathfrak{g})^{-s}, \tag{9}$$

where $\mathfrak{g}$ ranges over the set of all integral ideals of k in the ray class c.

Proposition 3. *The notation and assumptions being as above.*

$$\zeta_k(s, c) = N(\mathfrak{f}\mathfrak{a}_{i(c)})^{-s} \sum_{j \in J} \sum_{x \in R(c, C_j)} \zeta(s, C_j, x).$$

260 *Proof.* Let $\mathfrak{g}$ be an integral ideal in the ray class c. Then $\mathfrak{g}$ and $\mathfrak{f}\mathfrak{a}_{i(c)}$ are in the same narrow ideal class of k. Thus, for a suitable $w \in k_+$,

$\mathfrak{g} = \mathfrak{f}\mathfrak{a}_{i(c)}(w)$. In view of (7), we may assume that $w \in C_j \cap k_+$ for a suitable $j \in J$.

Set $w = y_1 v_{j1} + \ldots + y_{t_j} v_{jt_j}$.

Then $y_1, \ldots, y_{t_j}$ are all positive rational numbers. Let the integer part of y_a be $z_a (a = 1, \ldots, t_j)$.

Then $x = w - (z_1 v_{j1} + \ldots + z_{t_j} v_{jt_j}) \in C_j^1 \cap (\mathfrak{f}\mathfrak{a}_{i(c)})^{-1}$.

Furthermore $(x)\mathfrak{f}\mathfrak{a}_{i(c)}$ is in the ray class c.

Thus $x \in R(c, C_j)$. A simple consideration shows that j, $z_1, \ldots, z_{t_j}$ and x are uniquely determined by $\mathfrak{g}$.

On the other hand, for an $x \in R(c, C_j)$ and a t_j-tuple of non-negative integers $z = (z_1, \ldots, z_{t_j})$, $\mathfrak{a}_{i(c)}\mathfrak{f}(x + z_1 v_{j1} + \ldots + z_{t_j} v_{jt_j})$ is an integral ideal in the ray class c.

We denote by $\mathbf{Z}_+$ the set of non-negative integers. We have seen that the following map establishes a bijection from the set $\bigcup_{j \in J} \{R(c, C_j) \times \mathbf{Z}_+^{t_j}\}$ onto the set of integral ideals of k in the ray class c:

$$(x, z) \in R(c, C_j) \times \mathbf{Z}_+^{t_j} \longmapsto \mathfrak{a}_{i(c)}\mathfrak{f}(x + \sum_{a=1}^{t_j} z_a v_{ja}).$$

Thus Proposition 3. now follow immediately from (9). □

Remark. For totally real field k, Proposition 3. is given in the proof of Theorem 1 of [1] (see also [2]).

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DERIVATIVES OF L-SERIES AT $S = 0$

By H. M. Stark

1 Introduction

261 In 1970, I introduced [5] a rather vague general conjecture on values of Artin L-series at $s = 1$. Since then the conjecture has been considerably refined, especially for certain types of characters [6, II, III, IV]. It is appropriate to present a paper on this subject here since it was at the Tata Institute that the complex quadratic case was treated in the lectures of Siegel [4] and later work of Ramachandra [3]. It has become clear in recent years that the formulas at $s = 0$, although equivalent to formulas at $s = 1$ via the functional equation, are considerably simpler. In this paper, we will concentrate on the case of Artin L-series with first order zeros at $s = 0$. Included in this category of L-series are the abelian L-series over complex quadratic ground fields studied by Ramachandra. Since his results have been improved, this is a good place to begin.

2 Complex quadratic ground fields

Let k be a complex quadratic field, $\mathfrak{f}$ an integral ideal of k, $\mathfrak{f} \neq (1)$. Suppose $G(\mathfrak{f})$ is the ray class group of $k(\text{mod } \mathfrak{f})$ and let J be a subgroup of $G(\mathfrak{f})$ and K the class field corresponding to $H = G(\mathfrak{f})/J$. The characters χ of H are precisely those ray class characters of $k(\text{mod } \mathfrak{f})$ which are identically 1 on J. We let $L(s,\chi)$ denote the L-series corresponding to the primitive version of χ and $L(s,\chi,\mathsf{f})$ denote the L-series corresponding to the (possibly imprimitive) character $\chi(\bmod\ \mathsf{f})$. This is the series that results from $L(s,\chi)$ by deleting the p-factors from the Euler product of $L(s,\chi)$ for each p/f.

Our improvement of Ramachandra's result is the following theorem which is proved in [6, IV].

Theorem. *For each coset c of J in $G(\mathsf{f})$, there is an algebraic integer $\varepsilon(\mathsf{c})$ such that the following three properties hold:*

i). For each character χ of H,

$$L'(0\chi,\mathsf{f}) = -\frac{1}{W}\sum_{c\in H}\chi(\mathsf{c})\log(|\varepsilon(\mathsf{c})|^2)$$

where W is the number of roots of unity in K.

ii). The explicit reciprocity law is given by 262

$$\varepsilon(J)^{N(\rho)} \equiv \varepsilon(\mathsf{c})(\bmod\ \mathsf{p})$$

where p is a prime ideal in c. Further, $\varepsilon(\mathsf{c})/\varepsilon(J)^{N(\mathsf{p})}$ is a W^{th} power of a number in K and the $\varepsilon(\mathsf{c})$ are all associates.

iii). If $\mathsf{f} = \mathsf{p}^a$ where p is a prime ideal, then

$$N_{K/\mathbb{Q}}(\varepsilon(\mathsf{c})) = N_{k/\mathbb{Q}}(\mathsf{p})^b$$

where

$$b = \frac{Wh}{w}$$

and h is the class-number of k, w the number of roots of unity in k. In all other cases, $\epsilon(\mathsf{c})$ is a unit.

Actually, part iii) is a simple corollary of part i) with χ being the (imprimitive) trivial character of **H** since by part ii) the $\varepsilon(\mathsf{c})$ are the conjugates of $\varepsilon(J)$.

As an example, suppose $k = \mathbf{Q}(\sqrt{d})$ has class-number one and $\mathfrak{f} = \mathfrak{p}^a$ for a first degree prime ideal $\mathfrak{p}$ of norm p relatively prime to 6d. Here $W = w$ and for each character χ of $G(\mathfrak{f})$, we have

$$L'(0, \chi, \mathfrak{f}) = \frac{-1}{W} \sum_{c} \chi(\mathfrak{c}) \log(|\varepsilon(\mathfrak{c})^2)$$

where $\epsilon(\mathfrak{c})$ is in the ray class field $K(\mathfrak{f})$ of $k(\text{mod } \mathfrak{f})$. The norm of $\epsilon(\mathfrak{c})$ from $K(\mathfrak{f})$ to $\mathbb{Q}$ is p. By our Theorem,

$$\frac{\varepsilon(\mathfrak{c})}{\varepsilon(\mathfrak{c}_0)} = \varepsilon_c^w$$

where $\mathfrak{c}_0$ is the principal ray class (mod $\mathfrak{f}$), ε_c is in $\mathbf{K}(\mathfrak{f})$ and is a unit. As we show in [6, IV] by the theory of group determinants as discussed by Siegel [4], the units $\varepsilon_{\mathfrak{c}}$, $\mathfrak{c} \neq \mathfrak{c}_0$, together with the w^{th} roots of unity generate a subgroup of the unit group of $\mathbf{K}(\mathfrak{f})$ of index precisely the class-number of $K(\mathfrak{f})$. All previous results in this direction have had a much larger index. The situation in this example figured strongly in the work of Coates and Wiles [2].

For the rest of this section, we will suppose that $\mathfrak{f}^\tau = \mathfrak{f}$ and$J^\tau = J$ where τ denotes complex conjugation. Thus the field K is normal over $\mathbb{Q}$. We identity H with the Galois group of K/k via our Theorem and
263 now write

$$L'(0, \chi, \mathfrak{f}) = \frac{-1}{W} \sum_{h \in H} \chi(h) \log(|\varepsilon^h|^2)$$

where $\varepsilon = \varepsilon(J)$. We let G denote the Galois group of $K/\mathbb{Q}$. We will denote the characters of G by the Greek letter ψ while continuing to denote the characters of H by χ. In particular, if ψ is the character of G induced by χ then for any h in H,

$$\psi(h) = \chi(h) + \chi(\tau h \tau^{-1}),$$
$$\psi(h\tau) = 0.$$

It turns out that ε was constructed so that some power of ε is real. Therefore,

$$|\varepsilon^{\tau h \tau^{-1}}| = |\varepsilon^h|$$

and it follows that

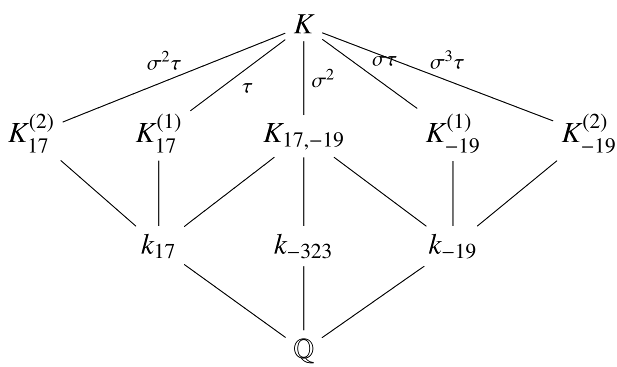
$$L'(0,\psi,N(\mathfrak{f})) = L'(0,\chi,\mathfrak{f}) = \frac{-1}{2W}\sum_{g\in G}\psi(g)\log(|\varepsilon^g|^2)$$

Although different in appearance, this is equivalent to the general conjecture in [6, II] for this case with “fudge constant” $-1/(2W)$.

To illustrate some of the possibilities that occur, we will take as an example the case where G is the dihedral group of order 8 with generators σ, τ and relations $\sigma^4 = \tau^2 = 1$, $\sigma\tau = \tau\sigma^3$. This group arises over $\mathbf{Q}(\sqrt{-19})$ with τ being complex conjugation and

$$H = H_{-19} = \{1,\sigma^2,\sigma\tau,\sigma^3\tau\},$$

the Klein four group.



There is a pair of prime ideals of norm 17 in $Q(\sqrt{-19})$, $\mathfrak{p}_{17}^{(1)} =$ **264**
$\left(\frac{7+\sqrt{-19}}{2}\right)$ and $\mathfrak{p}_{17}^{(2)} = \mathfrak{p}_{17}^{(1)}\tau = \left(\frac{7-\sqrt{-19}}{2}\right)$. For $j = 1, 2$, there are unique ray class characters $\chi^{(i)}(\mathrm{mod}\ \mathfrak{p}_{17}^{(i)})$ of order two. They are primitive characters and give rise to ray class fields $K_{-19}^{(1)}$ and $K_{-19}^{(2)} = K_{-19}^{(1)\tau}$. The composite field K comes from the ray class group (mod 17) modulo a subgroup of index 4 where both $\chi^{(1)}$ and $\chi^{(2)}$ are defined. Further, $G(K/\mathbb{Q}) =$

G, the dihedral group of order 8. The product character $\chi^{(1)}\chi^{(2)}$ of order two is a primitive character (mod 17) and corresponds to the class field $K_{17,-19} = \mathbf{Q}(\sqrt{17}, \sqrt{-19})$. Of the five possibilities, we see that $\{1, \sigma\tau\}$ or $\{1, \sigma^3\tau\} = \tau^{-1}\{1, \sigma\tau\}\tau$ must be $G(K_{-19}^{(1)}/k_{-19})$ and we assume σ has been picked so that $G(K_{-19}^{(1)}/k_{-19}) = \{1, \sigma\tau\}$. This makes H as claimed. There are two other quadratic subfields of K : $k_{17} = \mathbb{Q}(\sqrt{17})$ and $k_{-323} = \mathbb{Q}(\sqrt{-323})$. We see that $G(K/k_{-323}) = \{1, \sigma, \sigma^2, \sigma^3\}$ is cyclic while $G(K/k_{17}) = \{1, \tau, \sigma^2, \sigma^2\tau\}$ is the other Klein four group in G. (The real subfield of K is fixed by $\{1, \tau\}$ and is not normal over $\mathbb{Q}$. This is what allows us to decide which group goes to which field.) The remaining two quartic subfields of K are quadratic extensions of k_{17} : $K_{17}^{(1)}$ fixed by $\{1, \tau\}$ and $K_{17}^{(2)}$ fixed by $\{1, \sigma^2\tau\}$

	1	σ^2	$\sigma\tau$	$\sigma^3\tau$
χ_1	1	1	1	1
$\chi^{(1)}$	1	-1	1	-1
$\chi^{(2)}$	1	-1	-1	1
$\chi^{(1)}\chi^{(2)}$	1	1	-1	-1

Character table of H_{-19}

By our Theorem, there is a number π in K_{-19} of norm 17 such that

$$L'(\mathsf{o}, \chi^{(1)}, \mathsf{p}_{17}^{(1)}) = \frac{-1}{2}[\log(|\pi|^2) - \log(|\pi^{\sigma^2}|^2)].$$

There is also such a number in $K_{-19}^{(2)}$ for $L'(0, \chi^{(2)}, \mathsf{p}_{17}^{(2)})$ but it is just $\bar{\pi}$ and the formula is the same. It is no surprise that the formula should give the same answer since $L(s, \chi^{(1)}, \mathsf{p}_{17}^{(1)})$ and $L(s, \chi^{(2)}, \mathsf{p}_{17}^{(2)})$ are the same
265 Dirichlet series. Indeed this series arises in several different ways. From the character table of G (whose characters have been

1	σ^2	$\tau, \sigma^2\tau$	$\sigma\tau, \sigma^3\tau$	σ, σ^3	
ψ_1	1	1	1	1	1
ψ_{17}	1	1	1	1	-1
ψ_{-19}	1	1	-1	1	-1
ψ_{-323}	1	1	-1	-1	1
ψ_2	2	-2	0	0	0

Character table of G

given suggestive names), we see that $\chi^{(1)}$ and $\chi^{(2)}$ both give the same induced character of G, namely ψ_2. But ψ_2 also arises as an induced character from $G(K/k_{17})$ and $G(K/k_{-323})$. In particular, there is a primitive ray class character of k_{17} modulo a prime ideal of norm 19 which corresponds to $K_{17}^{(1)}$. It takes the values $1, 1, -1, -1$, at $1, \tau, \sigma^2, \sigma^2\tau$ respectively and also induces ψ_2 on G. Further, by our Theorem there is a unit E in K_{-19}^1 such that

$$\frac{\pi}{\pi^{\sigma^2}} = E^2$$

With

$$\eta = |E|^2 = EE^\tau$$

we have

$$L'(0, \psi_2) = \frac{-1}{2}\log(|E^2|^2) = -\log(\eta)$$

where η is in $K_{17}^{(1)}$. Also $E^{\sigma^2} = \pm 1/E$ so that $(EE^\tau)^{\sigma^2} = 1/(EE^\tau)$ and hence, $\eta^{\sigma^2} = \eta^{-1}$. The unit η is precisely what is called for in my conjecture for real quadratic L-series. However, I have proved my conjecture for relative quadratic extensions, such as $K_{17}^{(1)}/k_{17}$ without aid of complex multiplication.

We return to K/k_{-19} again and now consider $\chi^{(1)}$ and $\chi^{(2)}$ as imprim- **266**
itive characters (mod 17). According to our Theorem, there is a unit ε of K such that for any of the four characters χ of H,

$$L'(0, \chi, 17) = -\frac{1}{2}\sum_{h \in H} \chi(h)\log(|\varepsilon^h|^2).$$

In fact, ϵ is real and so is also in $K_{17}^{(1)}$. The question then arises if $\varepsilon = \eta$. The answer to this question is related to the question as to why we bother with the imprimitive version of $L(s,\chi^{(1)})$ since $\chi^{(1)}(\mathsf{p}_{17}^{(2)}) = -1$ and so

$$L'(0,\chi^{(1)},17) = 2L'(0,\chi^{(1)}).$$

It turns out that we get new units this way. For instance, since ε is real, $\varepsilon^{\sigma 2\tau} = \varepsilon^{\tau\sigma^2} = \varepsilon^{\sigma^2} = \varepsilon^{\sigma^2}$ so that ε^{σ^2} is real and

$$|\varepsilon^{\sigma\tau}| = |\varepsilon^{\tau\sigma\tau}| = |\varepsilon^{\sigma^3}| = |\varepsilon^{\sigma^3\tau}|$$

Hence

$$\begin{aligned} L'(0,\chi^{(1)},17) &= -\log\left(\left|\frac{\varepsilon}{\varepsilon^{\sigma^2}}\right|\right) \\ &= -2\log(\eta). \end{aligned}$$

Of course $\eta/\eta^{\sigma^2} = \eta^2$ and so $\varepsilon = \eta$ is still possible. However,

$$\begin{aligned} L'(0,\chi^{(1)}\chi^{(2)}) &= -\log\left(\left|\frac{(\varepsilon\varepsilon^{\sigma 2})^2}{N_{K/k_{-19}}}(\varepsilon)\right|\right) \\ &= -2\log(|\varepsilon\varepsilon^{\sigma 2}|), \end{aligned}$$

while by Dirichlet's class-number formula,

$$\begin{aligned} L'(0,\chi^{(1)}\chi^{(2)}) &= L(0,\psi_{-323})L'(0,\psi_{17}) \\ &= h(k_{-323})h(k_{17})\log(\varepsilon_{17}) \\ &= 4\log(\varepsilon_{17}) \end{aligned}$$

Thus $\varepsilon\varepsilon^{\sigma^2} = \pm\bar{\varepsilon}_{17}^2$ while $\eta\eta^{\sigma^2} = 1$ and so $\varepsilon \neq \eta$. Since

$$\frac{\epsilon}{\varepsilon^{\sigma^2}} = \frac{\varepsilon^2}{\varepsilon\varepsilon^{\sigma^2}}$$

we also have a confirmation of the fact that $\varepsilon/\varepsilon^{\sigma^2}$ is a square in K.

267 Thus far, we have looked at $L'(0,\psi_2)$ in three different ways (twice over k_{-19} and once over k_{17}) and found the three different numbers π, η, ϵ

all leading to the same result. But we can also look at $L'(0, \psi_2)$ viewed over k_{-323}. In the table below, the two characters χ' and $\bar{\chi}'$ of order four of $H_{-323} = G(K/k_{-323})$ induce ψ_2. Here K is actually the Hilbert class field of k_{-323}. This has the unfortunate consequence that the conductor of $L(s, \psi_2)$ viewed over k_{-323} is (1) and our Theorem does not apply directly. However, we may make all four characters of H_{-323} imprimitive by raising the conductor. It is tempting to use the

	1	σ	σ^2	σ^3
χ'_1	1	1	1	1
χ'	1	i	-1	$-i$
χ'^2	1	-1	1	-1
$\bar{\chi}' = \chi'^3$	1	$-i$	-1	i

Character table of H_{-323}

unique ideal p'_{17} of k_{-323} of norm 17 as our conductor. Since p'_{17} is in the class of order two, the corresponding Frobenius automorphism of H_{-323} is σ^2. (Note 17 ramifies from $\mathbf{Q}$ to K so we must be very careful in going from H_{-323} to G with Frobenius automorphisms.)

Hence,

$$L'(0, \chi', \mathsf{p}_{17}) = 2L'(0, \chi'),$$

(the same is true of $\bar{\chi}'$) and we are once again evaluating

$$L'(0, \psi_2, 17) = 2L'(0, \psi_2).$$

We see from our Theorem that instead of getting ε again, there is a number π', in K such that for all four characters χ of H_{-323},

$$L'(0, \chi, p'_{17}) = -\frac{1}{2} \sum_{h \epsilon H_{-323}} \chi(h) \log(|\pi'^h|^2)$$

where 268

$$N_{K/\mathbb{Q}}(\pi') = 17^4$$

Further π' is not just π^2 or even π^2 times a unit since

$$(\pi') = \mathsf{p}'_{17}$$

so that $(\pi')^2 = (17) = \mathfrak{p}_{17}^{(1)}\mathfrak{p}_{17}^{(2)}$ while $(\pi^2) = \mathfrak{p}_{17}^{(1)}$. Thus we have found still another number of K. Here again, π' is real and so $L'(0,\chi',\mathfrak{p}'_{17})$ simplifies to

$$-2\log(\eta) = L'(0,\chi',\mathfrak{p}'_{17}) = -\log\left(\left|\frac{\pi'}{\pi'^{\sigma^2}}\right|\right).$$

where π'/π'^{σ^2} is real and is a square in K.

The difficulty in using conductor (1) is that the trivial character gives $\zeta_{k_{-323}}(s)$ whose first derivative at zero is rather horrible. However, for the three non-trivial characters χ of H_{-323}, one can write all three $L'(0,\chi)$ simultaneously in terms of a nice number given by quotients of Dedekind eta-functions. But this simultaneous expression of all three L-series would appear to require a worse coefficient than $-1/2$ on the right side of the equation. It does seem possible to express any one of three $L'(0,\chi)'s$ in a nice manner. For instance, there is a number α in K (non-integral) given by

$$\alpha = 3\frac{\eta(\omega)^2}{\eta(\omega/9)^2}, \quad \omega = \frac{1+\sqrt{-323}}{2},$$

where we have used the eta-function on the right, and

$$L'(0,\chi') = -\log(|\alpha|^2)$$

from which we see that

$$\eta = N_{K/K_{17}^{(1)}}(\alpha)$$

3 L-series considered over Q

In this section, K is a normal extension of $\mathbf{Q}$ with Galois group G whose characters will again be denoted by ψ. We have seen in the last section that if K has a complex quadratic subfield k such that $G(K/k) = H$ is
269 abelian with conductor $\mathfrak{f} = \mathfrak{f}^\tau \neq (1)$ and ψ_2 is a of G induced by a character of H, then there is an integer ε in K such that

$$L'(0,\psi_2,N(\mathfrak{f})) = \frac{-1}{2W}\sum_{g\in G}\psi_2(g)\log(|\varepsilon^g|^2).$$

This tempts us to try and relate every $L(s, \psi)$ to $\sum \psi(g)\log(|\varepsilon^g|^2)$. To see the difficulties that we face, let us momentarily return to the dihedral group example of the previous section. We recall that each time we considered $L(s, \psi_2)$ from a new perspective, we came up with a new number in K related to $L'(0, \psi_2)$. From the point of view of characters of G, it is not at all clear why so many different numbers of K should arise or which number we should use. However, for illustrative purposes, let us take the real unit ε in K from the last section which satisfied,

$$L'(0, \psi_2, 17) = \frac{-1}{4}\sum_{g\in G}\psi_2(g)\log(|\varepsilon^g|^2).$$

Further, for $\psi = \psi_1, \psi_{-19}$ or ψ_{-323},

$$\sum_{g\in G}\psi(g)\log(|\varepsilon^g|^2) = 0.$$

For $\psi = \psi_1$, this is because ε is a unit while for $\psi = \psi_{-19}$ or ψ_{-323}, it is because $\psi(g\tau) = -\psi(g)$ for all g in G which allows a pairing of terms. For $\psi = \psi_{17}$, the situation is even more intriguing since

$$L'(0, \psi_{17}, 17) = L'(0, \psi_{17}) = \log(\varepsilon_{17})$$

and so we expect some relation between

$$L'(0, \psi_{17}) \quad \text{and} \quad \sum \psi_{17}(g)\log(|\varepsilon^g|^2).$$

We found earlier that

$$\begin{aligned}\sum \psi_{17}(g)\log(|\varepsilon^g|^2) &= \sum [\psi_{17}(g) + \psi_{-323}(g)]\log(|\varepsilon^g|^2)\\ &= -4L'(0, \chi^{(1)}\chi^{(2)})\\ &= -16\log(\varepsilon_{17}) = -16L'(0, \psi_{17}).\end{aligned}$$

The factor of 16 is rather hard to guess beforehand. Worse still, there are primes p which don't split in k_{-19} with $\psi_{17}(p) = -1$. For these primes, $L(s, \chi^{(1)}\chi^{(2)})$ has a p-factor $(1 - p^{-2s})^{-1}$ and so $L(s, \psi_{17} + \psi_{-323}, 17p)$ **270** has a second order zero at $s = 0$. This means that we come up with a unit such that

$$\sum_{g\in G} \psi_{17}(g)\log(|\mathrm{unit}^g|^2) = \sum [\psi_{17}(g) + \psi_{323}(g)]\log(|\mathrm{unit}^g|^2) = 0,$$

even though

$$L'(0, \psi_{17}, 17p) = 2L'(0, \psi_{17}) \neq 0.$$

Thus it appears that is we wish a common factor such as $-1/(2W)$ in front, we must give up looking simultaneously at all characters ψ of G such that $L(s, \psi)$ has a first order zero at $s = 0$. For second degree characters, we may still ask if this is possible. Precisely, we ask the following.

Question. *Suppose that K is a complex normal extension of $\mathbb{Q}$ with Galois group G containing W roots of unity. Suppose that f is divisible by the conductor of every irreducible second degree character ψ of G with $\psi(\tau) = 0$ where τ in G represents complex conjugation. Is there an integer π in K such that*

i). π^g is an associate of π for all g in G and some power of π is real.

ii). π^g/π^p is a W^{th} power in K where p is a prime not dividing Wf times the discriminant of K and whose associated Frobenius automorphisms are conjugate to g in G.

iii). For every irreducible second degree character ψ of G with $\psi(\tau) = 0$,

$$L'(0, \psi, \mathfrak{f}) = \frac{-1}{2W}\sum_{g\in G}\psi(g)\log(|\pi^g|^2).$$

This question is probably most safely asked when at least one of the characters ψ under consideration is not a character of any quotient group of G. The extra difficulties that arise otherwise can be illustrated by taking K to be the 36^{th} degree field generated by the Hilbert class fields of $\mathbf{Q}(\sqrt{-23})$ and $\mathbf{Q}(\sqrt{-31})$. Also, a study of inertial groups should enable us to replace $\mathfrak{f}$ by a smaller number in many cases.

Suppose we have a set of n irreducible characters ψ satisfying the hypotheses of our Question such that if ψ is in the set of n characters, so is every algebraic conjugate of ψ. Then we can expect to isolate n pieces of information about units from the numerical values of the $L'(0, \psi\mathfrak{f})$. 271
We do this by imitating the orthogonality relations for G. Consider the n-dimensional $\mathbb{Z}$ lattice in $\mathbb{C}^n$ generated by column vectors of the form $v_g = (\psi(g))$ where ψ runs through the n characters under consideration in some fixed order. The dual lattice consists of those n-dimensional vectors u such that $< u, v_g >$ is in $\mathbb{Z}$ for all g in G. Without the hypothesis on algebraic conjugates of ψ being present, we needn't have a lattice and then there may not be any non-zero u such that $< u, v_g >$ is in $\mathbb{Z}$ for all g. We now have

$$
\begin{aligned}
< u, L'(0, \psi, \mathfrak{f}) > &= \frac{-1}{2W} \sum_{g \in G} < u, v_g > \log\left(\left|\pi^g\right|^2\right) \\
&= \frac{-1}{2W} \log\left(|\varepsilon_u|^2\right).
\end{aligned}
$$

Here

$$\varepsilon_u = \prod_{g \in G} (\pi^g)^{< u, v_g >}$$

is a unit since the π^g are associates and $\sum_g < u, v_g >= 0$ by the orthogonality relations. In fact, since $\pi^\tau = \zeta\pi$ for a root of unity ζ,

$$\varepsilon_u^\tau = \prod_g (\zeta^g)^{< u, v_g >} \prod_g (\pi^g)^{< u, v_g >},$$

and so up to a root of unity ε_u is real. It seems likely that π can be chosen so that this root of unity is one (for example, if π itself is real) and ε_u is positive. We would then expect that

$$< u, L'(0, \psi, \mathfrak{f}) >= \frac{-1}{W} \log(\varepsilon_u), \tag{1}$$

where ε_u is a positive real unit in K.

Further, ε_u is already a W^{th} power in K. To see this, let M be the field of W^{th} roots of unity and $H = G(K/M)$. If χ_1 is the trivial character of H, then by the definition of M, the induced character χ_1^* is the sum of all the one dimensional characters of G. It follows from the Frobenius reciprocity law that for any of our n characters ψ, the restriction of ψ to H does not contain χ_1. If ρ is a representation of G with character ψ, then for any g in G,

$$\sum_{h\in H} \rho(gh) = \rho(g) \sum_{h\in H} \rho(h) = 0,$$

272 and hence

$$\sum_{h\in H} \psi(gh) = 0.$$

therefore

$$\sum_{h\in H} v_{gh} = 0.$$

For each g in G, let p_g be chosen according to part ii) of our Question so that π^g/π^{p_g} is a W^{th} power in K. For any h in H, $p_{gh} \equiv p_g (\text{mod } W)$ and hence

$$\sum_{h\in H} p_{gh} < u, v_{gh} > \equiv p_g < u, \sum_{h\in H} v_{gh} > \equiv 0 (\text{mod } W)$$

Therefore,

$$\varepsilon_u = \prod_{g\in G} \left(\frac{\pi^g}{\pi^{p_g}}\right)^{<u,v_g>} \prod_{g\in G} \pi^{p_g <u,v_g>}$$

is a W^{th} power in K as claimed.

I have shown numerically in several instances that the Question has an affirmative answer in cases where K is a class field of a real quadratic field [6, III, IV]. Just as this Colloquium was taking place, Ted Chinburg [1] formulated the Conjecture on Artin L-series with first order zeros at $s = 0$ in terms of (1) and investigated (1) in the case that K is the 48^{th} degree field corresponding to the non-abelian modular form of conductor 133 found by Tate. He found a unit ε_u in K which is a W^{th} power and which satisfies (1) to 13 decimal places. In fact he found ε_u by using the numerical values of the $L'(0, \psi)$ in a manner similar to [6, II] but with a nice improvement in the method that avoids the small searches that I had to make.

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EISENSTEIN SERIES AND THE RIEMANN ZETA-FUNCTION

By D. Zagier[1]

275 In this paper we will consider the functions $E(z,\rho)$ obtained by setting the complex variable s in the Eisenstein series $E(z, s)$ equal to a zero of the Riemann zeta-function and will show that these functions satisfy a number of remarkable relations. Although many of these relations are consequences of more or less well known identities, the interpretation given here seems to be new and of some interest. In particular, looking at the functions $E(z,\rho)$ leads naturally to the definition of a certain representation of $SL_2(R)$ whose spectrum is related to the set of zeroes of the zeta-function.

We recall that the Eisenstein series $E(z, s)$ is defined for $z = x + iy \in \mathsf{H}$ (upper half-plane) and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by

$$E(z, s) = \sum_{\gamma\in\Gamma_\infty/\Gamma} \mathrm{Im}(\gamma z)^s = \frac{1}{2} \sum_{\substack{c,d\varepsilon Z \\ (c,d)=1}} \frac{y^s}{|cz + d|^{2s}} \tag{1}$$

where $\Gamma = PSL_2(\mathbf{Z})$, $\Gamma_\infty = \left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\middle| n \in \mathbf{Z}\right\} \subset \Gamma$. If we multiply both

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sides of (1) by $\zeta(2s) = \sum_{r=1}^{\infty} r^{-2s}$ and write $m = rc$, $n = rd$, we obtain

$$\zeta(2s)E(z, s) = \frac{1}{2}\sum_{m,n}{}' \frac{y^2}{|mz + n|^{2s}}, \tag{2}$$

where $\sum'$ indicates summation over all $(m, n) \in \mathbf{Z}^2/\{(0, 0)\}$. The function $\zeta(2s)E(z, s)$ has better analytic properties than $E(z, s)$; in particular, it has a holomorphic continuation to all s except for a simple pole at $s = 1$.

There is thus an immediate connection between the Eisenstein series at s and the Riemann zeta-function at $2s$. This relationship has been made use of by many authors and has several nice consequences, two of
which will be mentioned in §1. Our main theme, however, is that there **276**
is also a relationship between the Eisenstein series and the zeta function at the *same* argument. We will give several examples of this in §2. Each takes the form that a certain linear operator on the space of functions on Γ/H, when applied to $E(\cdot, s)$, yields a function of s which is divisible by $\zeta(s)$. Then this operator annihilates all the $E(\cdot, \rho)$, and it is natural to look for a space $\mathscr{E}$ of functions of Γ/H which contains all the $E(\cdot, \rho)$ and which is annihilated by the operators in question. Such a space is defined in §3. In §4 we show that $\mathscr{E}$ is the set of K-fixed vectors of a certain G-invariant subspace $\mathscr{V}$ of the space of functions on Γ/G (where $G = PSL_2(\mathbf{R})$, $K = PSO(2)$). Then $\mathscr{V}$ is a representation of G whose spectrum with respect to the Casimir operator contains $\rho(1-\rho)$ discretely with multiplicity (at least) n if ρ is an n-fold zero of $\zeta(s)$. In particular, if (as seems very unlikely) one could show that $\mathscr{V}$ is unitarizable, i.e. if one could construct a positive definite G-invariant scalar product on $\mathscr{V}$, then the Riemann hypothesis would follow.

The paper ends with a discussion of some other representations of G related to $\mathscr{V}$ and reformulation in the language of adeles.

§ 1. We begin by reviewing the most important properties of Eisenstein series.

a) *Analytic continuation and functional equation.*

The function $E(z,s)$ has a meromorphic continuation to all s, the only singularity for $\mathrm{Re}(s) > \frac{1}{2}$ being a simple pole at $s = 1$ whose residue is independent of z:

$$\mathrm{res}_{s=1} E(z,s) = \frac{3}{\pi} (\forall z \in \mathsf{H}). \tag{3}$$

The modified function

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) \tag{4}$$

is regular except for simple poles at $s = 0$ and $s = 1$ and satisfies the functional equation

$$E^*(z,s) = E^*(z,1-s). \tag{5}$$

These statements are proved in a way analogous to Riemann's proof
277 of the analytic continuation and functional equation of $\zeta(s)$; we rewrite (2) as

$$E^*(z,s) = \frac{1}{2}\pi^{-s}\Gamma(s)\sum_{m,n}{}' Q_z(m,n)^{-s} = \frac{1}{2}\int_o^\infty (\Theta_z(t)-1)t^{s-1}dt, \tag{6}$$

where $Q_z(m,n) (z \in \mathsf{H})$ denotes the quadratic form

$$Q_z(m,n) = \frac{|mz+n|^2}{y} \tag{7}$$

of discriminant -4 and $\Theta_z(t) = \sum\limits_{m,n\in z} e^{-\pi t Q_z(m,n)}$ the corresponding theta-series; then the Poisson summation formula implies $\Theta_z(\frac{1}{t}) = t\Theta_z(t)$ and the functional equation and other properties of $E(z,s)$ follow from this and equation (6).

b) "*Rankin-Selberg method*".

Let $F : \mathsf{H} \to \mathbb{C}$ be a Γ-invariant function which is of rapid decay as $y \to \infty$ (i.e. . $F(x+iy) = 0(y^{-N})$ for all N). Let

$$C(F;y) = \int_0^1 F(x+iy)dx \qquad (y > 0) \tag{8}$$

be the constant term of its Fourier expansion and

$$I(F;s) = \int_{o}^{\infty} C(F;y)y^{s-2}dy \ (\mathrm{Re}(s) > 1) \tag{9}$$

the Mellin transform of $C(F;y)$. From (1) we obtain

$$I(F;s) = \int_{\Gamma_\infty/\mathsf{H}} F(z)y^s dz = \int_{\Gamma/\mathsf{H}} F(z)E(z,s)dz, \tag{10}$$

where dz denotes the invariant volume element $\frac{dxdy}{y^2}$. Therefore the properties of $E(z,s)$ given in a) imply the corresponding properties of $I(F;s)$: it can be meromorphically continued, has a simple pole at $s = 1$ with

$$\mathrm{res}_{s=1} I(F;s) = \frac{3}{\pi} \int_{\Gamma/\mathsf{H}} F(z)dz, \tag{11}$$

and the function

$$I^*(F;s) = \pi^{-s}\Gamma(s)\zeta(2s)I(F;s) \tag{12}$$

is regular for $s \neq 0, 1$ and satisfies **278**

$$I^*(F;s) = I^*(F;1-s) \tag{13}$$

c) *Fourier development.*

The function $E^*(z,s)$ defined by (4) has the Fourier expansion

$$E^*(z,s) = \zeta^*(2s)y^s + \zeta^*(2s-1)y^{1-s} \tag{14}$$
$$+2\sqrt{y}\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)K_{s-1/2}(2\pi ny)\cos 2\pi nx,$$

where

$$\zeta^*(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) \qquad (s \in \mathbb{C}), \tag{15}$$

$$\sigma_\nu(n) = \sum_{d|n} d^\nu \qquad (n \in N,\ \nu \in \mathbb{C}),$$

$$K_\nu(t) = \int_o^\infty e^{-t\cosh u} \cosh \nu u \, du \qquad (\nu \in \mathbb{C},\ t > 0). \tag{16}$$

The expansion (14), which can be derived without difficulty from (2), gives another proof of the statements in a); in particular, the functional equation (5) follows from (14) and the functional equations

$$\zeta^*(s) = \zeta^*(1-s), \sigma_\nu(n) = n^\nu \sigma_{-\nu}(n), K_\nu(t) = K_{-\nu}(t).$$

Because of the rapid decay of the K-Bessel functions (16), equation (14) also implies the estimates

$$\frac{\partial^n}{\partial s^n} E(z,s) = O(y^{\max(\sigma,1-\sigma)} \log^n y)(n = 0, 1, 2, \ldots, \sigma = \mathrm{Re}(s), \qquad y = \mathrm{Im}(z) \to \infty) \tag{17}$$

for the growth of the Eisenstein series and its derivatives. Finally, it follows from (14) or directly from (1) or (2) that the Eisenstein series $E(z, s)$ are eigenfunctions of both the Laplace operator

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and the Hecke operators

$$T(n) : F(n) \to \sum_{\substack{ad=n \\ a,d>0}} \sum_{b (\mathrm{mod}\ d)} F(\frac{az+b}{d}) \quad (n > 0),$$

279 namely

$$\Delta E(z,s) = s(s-1)E(z,s), \quad T(n)E(z,s) = n^s \sigma_{1-2s}(n) E(z,s). \tag{18}$$

We now come to the promised applications of the relationship between $E(z, s)$ and $\zeta(2s)$. The first (which has been observed by several authors and greatly generalized by Jacquet and Shalika [3]) is a simple proof of the non-vanishing of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$. Indeed, if $\zeta(1 + it) = 0$, the (14) implies that the function $F(z) = E(z, \frac{1}{2} + \frac{1}{2}$ it) is of rapid decay, does not vanish identically, and has constant term $C(F; y)$ identically equal to 0. But then $I(F; s) = 0$ for all s, and takings $s = \dfrac{1}{2} - \dfrac{1}{2}$ it in (10) we find $\int_{\Gamma/\mathsf{H}} |F(z)|^2 dz = 0$, a contradiction.

The second "application" is a direct but striking consequence of the Rankin-Selberg method. Let $\mathscr{C}_y \subset \Gamma/\mathsf{H}$ be the horocycle $\Gamma_\infty/(\mathbf{R} + iy)$; it is a closed curve of (hyperbolic) length $\dfrac{1}{y}$. The claim is that, as $y \to 0$, the curye $\mathscr{C}_y$ "fills up" Γ/H in a very uniform way: not only does $\mathscr{C}_y$ meet any open set $U \subset \Gamma/\mathsf{H}$ for y sufficiently small, but the fraction of $\mathscr{C}_y$ contained in U tends to $\mathrm{vol}(U)/\mathrm{vol}(\Gamma/\mathsf{H})$ as $y \to 0$ and in fact

$$\frac{\text{length }(\mathscr{C}_y \cap U)}{\text{length}(\mathscr{C}_y)} = \frac{\mathrm{vol}(U)}{\mathrm{vol}(\Gamma/\mathsf{H})} + 0(y^{\frac{1}{2}-\varepsilon}) \quad (y \to 0);$$

moreover, if the error term in this formula can be replaced by $0(y^{3/4-\varepsilon})$ for all U, then the Riemann hypothesis is true! To see this, take $F(z)$ in b) to be the characteristic function χ_U of U. Then

$$C(F; y) = \frac{\text{length}(\mathscr{C}_y \cap U)}{\text{length }(\mathscr{C}_y)}$$

and the Mellin transform $I(F, s)$ of this is holomorphic in $\mathrm{Re}(s) > \frac{1}{2}\Theta$ (where Θ is the supremum of the real parts of the zeroes of $\zeta(s)$) except for a simple pole of residue $\kappa = \frac{3}{4} \int_{\Gamma/\mathsf{H}} F(z)dz = \dfrac{\mathrm{vol}(U)}{\mathrm{vol}(\Gamma/\mathsf{H})}$ at $s = 1$. If F were sufficiently smooth (say twice differentiable) we could deduce that $I(F; \sigma + it) = 0(t^{-2})$ on any vertical strip $\mathrm{Re}(s) = \sigma > \frac{1}{2}\Theta$, and the Mellin inversion formula would give $C(F; y) = \kappa + 0(y^{1-\frac{1}{2}\Theta-\varepsilon})$. For $F = \chi_U$ we can prove only $C(F; y) = \kappa + 0(y^{\frac{1}{2}-\varepsilon})$; conversely, however, if 280
$C(F; y) = \kappa + 0(y^\alpha)$ then $I(F; s) - \dfrac{\kappa}{s-1}$ is holomorphic for $\mathrm{Re}(s) > 1 - \alpha$, and if this holds for all $F = \chi_U$ we obtain $\Theta \leqslant 2(1 - \alpha)$.

§2. In this section we give examples of special properties of the functions $E^*(z,\rho)$ or, more generally, of the functions

$$F(z) = F_{\rho,m}(z) = \frac{\partial^m}{\partial s^m} E^*(z,s)\Big|_{s=\rho} \qquad (0 \le m \le n_\rho - 1), \qquad (19)$$

where ρ is a non-trivial zero of $\zeta(s)$ of order n_ρ.

Example 1:

Let $D < 0$ be the discriminant of an imaginary quadratic field K. To each positive definite binary quadratic form $Q(m,n) = am^2 + bmn + cn^2$ of discriminant D we associate the root $z_Q = \dfrac{-b+\sqrt{D}}{2a} \in \mathsf{H}$. The Γ-equivalence class of Q determines uniquely an ideal class A of K such that the norms of the integral ideals of A are precisely the integers represented by Q. Also, the form Q_{z_Q} defined by (7) equals $\dfrac{2}{\sqrt{|D|}}Q$. Therefore (6) gives

$$\begin{aligned} E^*(z_Q,s) &= \frac{1}{2}\left(\frac{|D|}{4}\right)^{s/2} \pi^{-s}\Gamma(s) \sum_{m,n}{}' Q(m,n)^{-s} \\ &= \frac{w}{2}\left(\frac{|D|}{4}\right)^{s/2} \pi^{-s}\Gamma(s)\zeta(A,s), \end{aligned}$$

where $w(=2,4 \text{ or } 6)$ is the number of roots of unity in K and $\zeta(A,s) = \sum\limits_{\mathfrak{a}}^{-s}$ is the zeta-function of A. (Note that this equation makes sense because $E^*(z_Q,s)$ depends only on the Γ-equivalence class of z_Q and hence of **Q**.) Thus if $Q_1,\dots,Q_{h(D)}$ are representatives for the equivalence classes of forms of discriminant D, we have

$$\begin{aligned} \sum_{i=1}^{h(D)} E^*(z_{Q_i},s) &= \frac{w}{2}\left(\frac{|D|}{4}\right)^{s/2} \quad \pi^{-s}\Gamma(s) \sum_{i=1}^{h(D)} \zeta(A_i,s) \\ &= \frac{w}{2}\left(\frac{|D|}{4}\right)^{s/2} \quad \pi^{-s}\Gamma(s)\zeta_K(s) \\ &= \frac{w}{2}\left(\frac{|D|}{4}\right)^{s/2} \quad \pi^{-s}\Gamma(s)\zeta(s)L(s,D), \end{aligned}$$

where $\zeta_K(s)$ is the Dedekind zeta-function of K and $L(s, D)$ the L-series **281**
$\sum_{n=1}^{\infty} \left(\frac{D}{n}\right) n^{-s}$. Since the latter is holomorphic, we deduce that the function $\sum_{i=1}^{h(D)} E^*(z_{Q_i}, s)$ is divisible by $\Gamma(s)\zeta(s)$, i.e. that it vanishes with multiplicity $\geq n_\rho$ at a non-trivial zero ρ of $\zeta(s)$. A similar statement holds for any negative integer D congruent to 0 or 1 modulo 4 (not necessarily the discriminant of a quadratic field) if we replace $\zeta_K(s)$ in the equation above by the function

$$\zeta(s, D) = \sum_{i=1}^{h(D)} \sum_{\substack{(m,n)\varepsilon Z^2/\Gamma_{Q_i} \\ Q_i(m,n)>0}} \frac{1}{Q_i(m,n)^s} \tag{20}$$

where $Q_i(i = 1, \ldots, h(D))$ are representatives for the Γ-equivalence classes of binary quadratic forms of discriminant D and Γ_{Q_i} denotes the stabilizer of Q_i in Γ. Again the quotient $L(s, D) = \zeta(s, D)/\zeta(s)$ is entire ([11], Prop. 3, ii), p. 130). This proves

Proposition 1: *Each of the functions* (19) *satisfies*

$$\sum_{i=1}^{h(D)} F(z_{Q_i}) = 0 \tag{21}$$

for all integers $D < 0$, where $z_{Q_1}, \ldots, z_{Q_{j(D)}}$ are the points in Γ/H *which satisfy a quadratic equation with integral coefficients and discriminant D.*

Notice how strong condition (21) is: the points satisfying some quadratic equation over $\mathbf{Z}$ ("points with complex multiplication") lie dense in Γ/H, so that it is not at all clear *a* priori that there exists any non-zero continuous function $F : \Gamma/\mathsf{H} \to \mathbb{C}$ satisfying eq. (21) for all $D < 0$.

Example 2:

This is the analogue of Example 1 for positive discriminants. Let $D > 0$ be the discriminant of a real quadratic field K and $Q_1, \ldots, Q_{h(D)}$

representatives for the Γ-equivalence classes of quadratic forms of discriminant D. To each Q_i we associate, not a point $z_{Q_i} \in \Gamma/\mathsf{H}$ as before, but a closed curve $C_{Q_i} \subset \Gamma/\mathsf{H}$ as follows: Let $w_i, w_i' \in \mathbf{R}$ be the roots of
282 the quadratic equation $Q_i(x,1) = a_i x^2 + b_i x + c_i = 0$ and let Ω_i be the semicircle in H with endpoints w_i and w_i'. The subgroup

$$\Gamma_{Q_i} = \left\{ \pm \begin{pmatrix} \frac{1}{2}(t - b_i u) & -c_i u \\ a_i u & \frac{1}{2}(t + b_i u) \end{pmatrix} \middle| \, t, u \in \mathbf{Z}, t^2 - Du^2 = 4 \right\} \tag{22}$$

of Γ, which is isomorphic to {units of K}/{± 1} and hence to $\mathbf{Z}$, maps Ω_i to itself, and C_{Q_i} is the image Γ_{Q_i}/Ω_i of Ω_i in Γ/H. On C_{Q_i} we have a measure $|d_{Q_i} z|$, unique up to a scalar factor, which is invariant under the operation of the group $\Gamma_{Q_i} \otimes \mathbf{R}$ obtained by replacing $\mathbf{Z}$ by $\mathbf{R}$ in (22); if we parametize Ω_i by

$$z = \frac{w_i i p + w_i'}{ip + 1} \qquad (0 < p < \infty),$$

then Γ_{Q_i} acts by $p \to \varepsilon^2 p$ ($\varepsilon = \dfrac{t + u\sqrt{D}}{2}$ a unit of K) and $|d_{Q_i} z| = \dfrac{dp}{p}$. A theorem of Hecke ([2], p. 201) asserts that the zeta-function of the ideal class A_i of K corresponding to Q_i is given by

$$\zeta(A_i, s) = \frac{\pi^s}{\Gamma(\frac{s}{2})^2} D^{-s/2} \int\limits_{C_{Q_i}} E^*(z,s)|d_{Q_i} z|$$

(cf. [10], §3 for a sketch of the proof). Thus

$$\sum_{i=1}^{h(D)} \int\limits_{C_{Q_i}} E^*(z,s)|d_{Q_i} z| = \pi^{-s} D^{s/2} \Gamma(\frac{s}{2})^2 \zeta_K(s),$$

which again is divisible by $\zeta(s)$, and as before we can take for D any positive non-square congruent to 0 or 1 modulo 4 and get a similar identity with $\zeta_K(s)$ replaced by the function (20). Thus we obtain

Proposition 2: *Each of the functions* (19) *satisfies*

$$\sum_{i=1}^{h(D)} \int_{C_{Q_i}} F(z)|d_{Q_i}z| = 0 \tag{23}$$

for all non-square integers $D < 0$, where $Q_i(m,n) = a_i m^2 + b_i mn + c_i n^2$ $(i = 1, \ldots, h(D))$ are representatives for the Γ-equivalence classes of **283**
binary quadratic forms of discriminant D, C_{Q_i} is the image of

$$\{z = x + iy \in \mathsf{H} |\, a_i|z|^2 + b_i x + c_i = 0\} \text{ in } \Gamma/\mathsf{H},$$

and

$$|d_{Q_i}z| = \frac{\sqrt{D}}{|a_i z^2 + b_i z + c_i|}((dx)^2 + (dy)^2)^{1/2}.$$

Example 3: The third example comes from the theory of modular forms. Let $f(z)$ be a cusp form of weight k on $SL_2(\mathbf{Z})$ which is a normalized eigenfunction of the Hecke operators, i.e. f satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \left(\forall z \in \mathsf{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})\right)$$

and has a Fourier development of the form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

with $a_1 = 1$ and $a_{nm} = a_n a_m$ if $(n,m) = 1$. Define $D_f(s)$ by

$$D_f(s) = \prod_p \frac{1}{(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s})} (\mathrm{Re}(s) >> 0),$$

where α_p and β_p are the roots of $X^2 - a_p X + p^{k-1} = 0$; it is easily checked that

$$D_f(s) = \frac{\zeta(2s-2k+2)}{\zeta(s-k+1)} \sum_{n=1}^{\infty} |a_n|^2 n^{-s}.$$

Applying the Rankin-Selberg method (10) to the Γ-invariant function $F(z) = y^k|f(z)|^2$ with constant term $C(F : y^k) = \int\limits_{n=1}^{\infty} |a_n|^2 e^{-4\pi n y}$, we find $\int\limits_{\Gamma/\mathsf{H}} y^k|f(z)|^2 E^*(z,s)dz = I^*(F;s)$

$$
\begin{aligned}
&= \pi^{-s}\Gamma(s)\zeta(2s)\cdot(4\pi)^{-s-k+1}\Gamma(s+k-1)\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{s+k-1}} \\
&= 4^{-s-k+1}\pi^{-2s-k+1}\Gamma(s)\Gamma(s+k-1)\zeta(s)D_f(s+k-1).
\end{aligned}
$$

284 This formula, which was the original application of the Rankin-Selberg method ([5], [6]), shows that the product $\zeta(s)D_f(s+k-1)$ is holomorphic except for a simple pole at $s = 1$. It was proved by Shimura [7] and also by the author [11] that in fact $D_f(s)$ is an entire function of s. Thus the above integral is divisible by $\Gamma(s)\Gamma(s+k-1)\zeta(s)$, and we obtain

Proposition 3: *Each of the functions* (19) *satisfies*

$$\int\limits_{\Gamma/\mathsf{H}} y^k|f(z)|^2 F(z)dz = 0$$

for every normalized Hecke eigenform f of level 1 and weight k.

The statement of Proposition 3 remains true if f is allowed to be a non-holomorphic modular form (Maass wave-form); the proof for $k = 0$ is given in [12] in this volume and the general case is included in the results of [1] or [4].

Finally, we can extend our list of special properties of the functions (19) by observing that each of these functions is an eigenfunction of the Laplace and Hecke operator (eq. (18)) and hence (trivially) has the property that

$$\Delta^i F(z) \text{ and } T(n)F(z) \text{ satisfy Proposition 1-3 for all } i \geqslant 0,\ n \geqslant 1. \quad (25)$$

Note that for general functions $F : \Gamma/\mathsf{H} \to \mathbb{C}$ (not eigenfunctions), eq. (25) expresses a property no contained in Proposition 1 to 3: for example, eq. (21) for $D = -4$ says that $F(i) = 0$, and this does not imply $\Delta F(i) = 0$.

§ 3. In § 2 we proved that the functions $E^*(z, \rho)$, and more generally the functions (19), satisfy a number of special properties. In this section we will both explain and generalize these results by defining in a natural way a space $\mathscr{E}$ of functions in Γ/H which contains the functions (19) and has the same special properties.

Let D be an integer congruent to 0 or 1 modulo 4. For $\Phi : \mathbf{R} \to \mathbb{C}$ a function satisfying certain restrictions (e.g. (27) and (29) below) we define a new function $\mathscr{L}_D\Phi : \mathsf{H} \to \mathbb{C}$ by

$$\mathscr{L}_D\Phi(z) = \frac{1}{2} \sum_{\substack{a,b,c \in Z \\ b^2-4ac=D}} \Phi\left(\frac{a|z|^2 + bz + c}{y}\right) \quad (z = x + iy \in \mathsf{H}), \tag{26}$$

where the summation extends over all integral binary quadratic forms 285
$Q(m, n) = am^2 + bmn + cn^2$ of discriminant D. Since Q and $-Q$ occur together in the sum, we may assume that ϕ is an even function; the factor $\frac{1}{2}$ has then been included in the definition to avoid counting each term twice.

The sum (26) converges absolutely for all $z \in \mathsf{H}$ if we assume that

$$\Phi(X) = O(|X|^{-1-\varepsilon}) \quad (|X| \to \infty) \tag{27}$$

for some $\varepsilon > 0$. Moreover, the expression $\dfrac{a|z|^2 + bx + c}{y}$ is unchanged if one acts simultaneously on $x + iy \in \mathsf{H}$ and $Q(m, n) = am^2 + bmn + cn^2$ by an element $\gamma \in \Gamma$. Hence $\mathscr{L}_D\Phi(\gamma z) = \mathscr{L}_D\Phi(z)$, so $\mathscr{L}_D$ is an operator from functions on $\mathbf{R}$ satisfying (27) to functions on Γ/H.

Before going on, we need to know something about the growth of $\mathscr{L}_D\Phi$ in Γ/H. If D is not a perfect square, then $a \neq 0$ in (26), so (for Φ even)

$$\mathscr{L}_D\Phi(z) = \sum_{a=1}^{\infty} \sum_{\substack{b=-\infty \\ b^2 \equiv D (\mathrm{mod}\ 4a)}}^{\infty} \Phi\left(ay + \frac{a(x + b/2a)^2 - D/4a}{y}\right)$$

$$= \sum_{a=1}^{\infty} \sum_{\substack{b=-\infty \\ b^2 \equiv D (\mathrm{mod}\ 4a)}}^{\infty} O\left(\left(ay + \frac{a(x+b/2a)^2 - D/4a}{y}\right)^{-1-\varepsilon}\right)$$

$$= \sum_{a=1}^{\infty} O\left(n_D(a) \int_{-\infty}^{\infty} \left(ay + \frac{ax^2}{y}\right)^{-1-\varepsilon} dx\right)$$

as $y = \mathrm{Im}(z) \to \infty$, where

$$n_D(a) = \neq \left\{ b (\mathrm{mod}\ 2a) | \, b^2 \equiv D (\mathrm{mod}\ 4a) \right\}. \tag{28}$$

Since the integral is $O(a^{-1-\varepsilon} y^{-\varepsilon})$ and $\sum\limits_{a=1}^{\infty} \frac{n_D(a)}{a^{1+\varepsilon}}$ converges, we find

$$\mathscr{L}_D \Phi(z) = O(y^{-\varepsilon}).$$

286 If D is a square, then the same argument applies to the terms in (26) with $a \neq 0$ and we are left with the sum

$$\frac{1}{2} \sum_{b^2 = D} \sum_{c=-\infty}^{\infty} \Phi(\frac{bx+c}{y})$$

to estimate. If Φ is sufficiently smooth (say twice differentiable), then the inner sum differs by a small amount from the corresponding integral

$$\int_{-\infty}^{\infty} \Phi(\frac{c}{y}) dc = y \cdot \frac{\infty}{-\infty} \Phi(X) dX,$$

and this will be small as $y \to \infty$ only if $\int\limits_{-\infty}^{\infty} \Phi(X) dX$ vanishes. Thus with the requirement

$$\Phi \text{ is } C^2 \text{ and } \int_{-\infty}^{\infty} \Phi(X) dX = 0 \text{ if } D \text{ is a square} \tag{29}$$

as well as (27) we have $\mathscr{L}_D \Phi(z) = O(y^{-\varepsilon})$ as $y \to \infty$ for all D, and so the scalar product of $\mathscr{L}_D \Phi$ with F in Γ/H converges for any $F : \Gamma/\mathsf{H} \to \mathbb{C}$ satisfying $F(z) = O(y^{1-\varepsilon})$ for some $\varepsilon > 0$ (or even $F(z) - O(y)$). Therefore the definition of $\mathscr{E}$ in the following theorem makes sense.

Theorem. *For each integer* $D \in Z$, $D \equiv 0$ *or* 1 $(\text{mod } 4)$*, let*

$$\mathscr{L}_D \begin{Bmatrix} \textit{even functions } \Phi : \mathbf{R} \to \mathbb{C} \\ \textit{satisfying } (27) \textit{ and } (29) \end{Bmatrix} \longrightarrow \{\textit{functions } \Gamma/\mathsf{H} \to \mathbb{C}\}$$

be the operator defined by (26). Let $\mathscr{E}$ be the set of functions $F : \Gamma/\mathsf{H} \to \mathbb{C}$ such that

a) $F(z) = O(y^{1-\varepsilon})$ *for some* $\varepsilon > 0$

b) $F(z)$ *is orthogonal to* $\sum_D \operatorname{Im}(\mathscr{L}_D)$*, i.e.* $\int_{\Gamma/\mathsf{H}} \mathscr{L}_D\Phi(z)F(z)dz = 0$ *for all* $D \in \mathbf{Z}$ *and all* Φ *satisfying* (27) *and* (29).

Then

i) $\mathscr{E}$ *contains the functions* (19)*;*

ii) $\mathscr{E}$ *is closed under the action of the Laplace and Hecke operators;*

iii) *Any* $F \in \mathscr{E}$ *satisfies* (21)*,* (23) *(for all* D*) and* (284) *(for all* f*).*

Proof. i) The functions (19) satisfy a) because of equation (17), since $0 < \operatorname{Re}(\rho) < 1$. To prove b), we must show that the integral of any function $\mathscr{L}_D\Phi(z)$ against $E^*(z, s)$ is divisible by $\zeta(s)$. Consider first the **287**
case when D is not a square. Let Φ be any function satisfying (27) and $F : \Gamma/\mathsf{H} \to \mathbb{C}$ a function which is $O(y^\alpha)$ as $y \to \infty$ for some $\alpha \leqslant 1 + \varepsilon$. If $Q_i(m, n) = a_i m^2 + b_i mn + c_i n^2 (i = 1, \ldots, h(D))$ are representatives for the classes of binary quadratic forms of discriminant D, then any form of discriminant D equals $Q_i \circ \gamma$ for a unique i and $\gamma \in \Gamma_{Q_i}/\Gamma$ (Γ_{Q_i} = stabilizer of Q_i in Γ). Hence

$$\mathscr{L}_D\Phi(z) = \sum_{i=1}^{h(D)} \sum_{\gamma \in \Gamma_{Q_i}/\Gamma} \Phi\left(\frac{a_i|\gamma z|^2 + b_i \operatorname{Re}(\gamma z) + c_i}{\operatorname{Im}(\gamma z)}\right)$$

and so

$$\int_{\Gamma/\mathsf{H}} \mathscr{L}_D\Phi(z)F(z)dz = \sum_{i=1}^{h(D)} \int_{\Gamma_{Q_i}/\mathsf{H}} \Phi\left(\frac{a_i|z|^2 + b_i x + c_i}{y}\right) F(z)dz.$$

Taking $F(z) = \zeta(2s)E(z,s)$ with $1 \leqslant \operatorname{Re}(s) < 1 + \varepsilon$ and using (2), we find that the right-hand side of this equations equals

$$\frac{1}{2}\sum_{i=1}^{h(D)} \sum_{(m,n)\in Z^2/\Gamma_{Q_i}} \int_{\mathsf{H}} \Phi\left(a_i|z|^2 + b_i x + c_i\right) \frac{y^s}{|mz+n|^2 2s} dz.$$

Since D is not a square, $Q_i(n,-m)$ is different from 0 for all $(m,n) \neq (0,0)$, so, since Φ is an even function, we can restrict the sum to $(m,n) \in Z^2$ with $Q_i(n,-m) > 0$ if we drop the factor $\frac{1}{2}$. Then the substitution $z \to \dfrac{nz - \frac{1}{2}b_i n + c_i m}{-mz + a_i n - \frac{1}{2}b_i m}$ introduced in [11], p. 127, maps H to H and gives

$$\int_H \Phi\left(\frac{a_i|z|^2 + b_i x + c_i}{y}\right) \frac{y}{|mz+n|^{2s}} dz$$
$$= Q_i(n,-m)^{-s} \int_{\mathsf{H}} \Phi\left(\frac{|z|^2 - D/4}{y}\right) y^s dz.$$

Therefore we have

$$\zeta(2s) \int_{\Gamma/\mathsf{H}} \mathscr{L}_D\Phi(z)E(z,s)dz = \zeta(s,D) \int_{\mathsf{H}} \Phi\left(\frac{|z|^2 - D/4}{y}\right) y^s dz \tag{30}$$

for $1 < \operatorname{Re}(s) < 1 + \varepsilon$, with $\zeta(s,D)$ defined as in (20). Since $\zeta(2s)$ and
288 $\zeta(s,D)$ have meromorphic continuations to all s and both integrals in (30) converge for $0 < \operatorname{Re}(s) < 1 + \varepsilon$, we deduce that the identity is valid in this larger range; the divisibility of $\zeta(s;D)$ by $\zeta(s)$ now implies the orthogonality of the functions (19) with $\mathscr{L}_D\Phi(z)$.

If D is a square, we would have to treat the terms with $Q_i(n,-m) = 0$ in the above sum separately (as in [11], pp. 127-128). We prefer a different method, which in fact works for all D. By the Rankin-Selberg method, we know that $\int \mathscr{L}_D\Phi(z)E(z,s)dz$ equals the Mellin transform of the constant term of $\mathscr{L}_D\Phi$, and writing $\mathscr{L}_D\Phi(z)$ as

$$\sum_{a=1}^{\infty} \sum_{\substack{b (\mathrm{mod}\ 2a) \\ b^2 \equiv D (\mathrm{mod}\ 4a)}} \sum_{n=-\infty}^{\infty} \Phi\left(\frac{a\left|z + \frac{b}{2a} + n\right|^2 - D/4a}{y}\right)$$

$$+ \frac{1}{2} \sum_{b^2 = D} \sum_{c=-\infty}^{\infty} \Phi\left(\frac{bx + c}{y}\right),$$

we see that this constant term is given by

$$C(\mathscr{L}_D \Phi; y) = \sum_{a=1}^{\infty} n_D(a) \int_{-\infty}^{\infty} \Phi\left(\frac{ax^2 + ay^2 - D/4a}{y}\right) dx$$

$$+ \begin{cases} 0 \text{ if } D \neq m^2, \\ y. \displaystyle\int_{-\infty}^{\infty} \Phi(X) dX \text{ if } D = m^2 > 0, \\ \displaystyle\frac{1}{2} \sum_{c=-\infty}^{\infty} \Phi\left(\frac{c}{y}\right) \text{ if } D = 0, \end{cases}$$

where $n_D(a)$ is defined by (28). The Mellin transform of the first term is

$$\left(\sum_{a=1}^{\infty} \frac{n_D(a)}{a^s}\right) \cdot \int_0^{\infty} \int_{-\infty}^{\infty} \Phi\left(\frac{x^2 + y^2 - D/4}{y}\right) y^{s-2} dx dy,$$

and since $\sum n_D(a) a^{-s} = \zeta(s, D)/\zeta(2s)$ ([11], Prop. 3, i), p. 130) we recover eq. (30) if D is not a square. The second term vanishes if $D = m^2 \neq 0$ because of the assumption (29), so eq. (30) remains valid in this case. If $D = 0$, then, using equation (29) and the Poisson summation formula, we see that the second term in the formula for $C(\mathscr{L}_0 \Phi; y)$ **289**
equals

$$\frac{1}{2} \sum_{c=-\infty}^{\infty} \Phi(\frac{c}{y}) - \frac{1}{2} y \int_{-\infty}^{\infty} \Phi(X) dX = y \sum_{n=1}^{\infty} \tilde{\Phi}(ny),$$

where

$$\tilde{\Phi}(y) = \int_{-\infty}^{\infty} \Phi(X) e^{2\pi i X y} dX$$

is the Fourier transform of Φ. The Mellin transform of this is $\zeta(s)$ times the Mellin transform of $\tilde{\Phi}$, so we obtain

$$\zeta(2s)\int\limits_{\Gamma/\mathsf{H}} \mathscr{L}_0\Phi(z)E(z,s)dz = \zeta(s,0)\int\limits_{\mathsf{H}} \Phi\left(\frac{|z|^2}{y}\right)y^s dz + \zeta(s)\zeta(2s)\int\limits_0^\infty \tilde{\Phi}(y)y^{s-1}dy \tag{31}$$

for $1 < \mathrm{Re}(s) < 1+\varepsilon$. Again both sides extend meromorphically to the critical strip and, since $\zeta(s,0) = \zeta(s)\zeta(2s-1)$, we again find that the integral on the left is divisible by $\zeta(s)$, i.e. that the functions (19) are orthogonal to the image of $\mathscr{L}_0$. This completes the proof of i).

We observe that the same calculations as in [12], §4, allow us to perform one of the integrations in the double integral on the right-hand side of (30), obtaining

$$\int\limits_{\Gamma/\mathsf{H}} \mathscr{L}_D\Phi(z)E^*(z,s)dz \tag{32}$$
$$= \begin{cases} (2\pi)^{1-s}|D|^{s/2}\Gamma(s)\zeta(s,D)\displaystyle\int\limits_1^\infty P_{s-1}(t)\phi(|D|^{\frac{1}{2}}t)dt & \text{if } D<0, \\ \dfrac{1}{2}\pi^{-s}D^{s/2}\Gamma\left(\dfrac{s}{2}\right)^2\zeta(s,D)\displaystyle\int\limits_0^\infty F\left(\frac{s}{2},\frac{1-s}{2};\frac{1}{2};-t^2\right)\Phi(D^{\frac{1}{2}}t)dt & \text{if } D>0, \end{cases}$$

where $P_{s-1}(t)$ and $F\left(\frac{s}{2},\frac{1-s}{2};\frac{1}{2};-t^2\right)$ denote Legendre and hypergeometric functions, respectively; since both of these functions are invariant under $s \to 1-s$, we see that (30) is compatible with (and indeed gives another proof of) the functional equation of $\zeta(s,D)$ for $D \neq 0$ ([11],
290 Prop. 3, ii), p. 130). We can also make the functional equation apparent in the case $D = 0$ by substituting $-\frac{1}{z}$ for z in the first integral on the

right-hand side of (31) and using the identity

$$\int_0^\infty y^{s-1}\cos 2\pi Xy\,dy = \frac{1}{2}\pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}|X|^{-s} \quad (0 < \mathrm{Re}(s) < 1)$$

in the second; this gives

$$\int_{\Gamma/\mathrm{H}} \mathscr{L}_0\Phi(z)E^*(z,s)dz \tag{33}$$

$$= \zeta(s)\zeta^*(2-2s)\int_0^\infty \Phi(X)X^{s-1}dX + \zeta(1-s)\zeta^*(2s)\int_0^\infty \Phi(X)X^{-s}dX$$

for $0 < \mathrm{Re}(s) < 1$, with $\zeta^*(s)$ as in eq. (15).

A calculation similar to the one given here can be found in §2 of Shintani [8].

ii) Since both the Laplace and the Hecke operators are self-adjoint, it is sufficient to show that the space $\sum_D \mathrm{Im}(\mathscr{L}_D)$, or a dense subspace of it, is closed under the action of these operators. An elementary calculation shows that

$$\Delta\Phi\left(\frac{a|z|^2+bx+c}{y}\right) = \Phi_1\left(\frac{a|z|^2+bx+c}{y}\right) \tag{34}$$

with

$$\Phi_1(X) = 2X\Phi'(X) + (X^2+D)\Phi''(X). \tag{35}$$

Hence $\Delta\mathscr{L}_D\phi(z) = \mathscr{L}_D\Phi_1(z)$. If Φ is C^∞ and of rapid decay, then Φ_1 also is and satisfies conditions (27) and (29), and since such Φ form a dense subspace the first assertion of ii) is proved.

The calculation for the Hecke operators is harder. It suffices to treat the operators $T(p)$ with p prime, since these generate the Hecke algebra. We claim that

$$T(p)\circ\mathscr{L}_D = \mathscr{L}_{Dp^2}\circ\alpha_p + \left(\frac{D}{p}\right)\mathscr{L}_D + p\mathscr{L}_{D/p^2}\circ\beta_p \tag{36}$$

where α_p and β_p denote the operators 291

$$\alpha_p \Phi(X) = \Phi(X/p), \beta_p \Phi(X) = \Phi(pX),$$

$\left(\frac{D}{p}\right)$ is the Legendre symbol, and $\mathscr{L}_{D/p^2}$ is to be interpreted as 0 if $p^2 \nmid D$. To prove this write

$$\begin{aligned}
T(p)\mathscr{L}_D\Phi(z) &= \mathscr{L}_D\Phi(pz) + \sum_{j=1}^{p} \mathscr{L}_D\phi\left(\frac{z+j}{p}\right) \\
&= \sum_{b^2-4ac=D} \left\{ \Phi\left(\frac{ap^2|z|^2 + bpx + c}{py}\right) \right. \\
&\quad \left. + \sum_{j=1}^{p} \Phi\left(\frac{a|z|^2 + (2aj+bp)x + (aj^2+bjp+cp^2)}{py}\right) \right\} \\
&= \sum_{b^2-4ac=Dp^2} n(a,b,c)\Phi\left(\frac{a|z|^2 + bx + c}{py}\right)
\end{aligned}$$

with

$$n(a,b,c) = \varepsilon\left(\frac{a}{p^2}, \frac{b}{p}, c\right) + \sum_{j=1}^{p} \varepsilon\left(a, \frac{b-2aj}{p}, \frac{c-bj+aj^2}{p^2}\right)$$

(where $\varepsilon(a,b,c)$ equals 1 if a,b,c are integral, 0 otherwise). To prove (36) we must show that

$$n(a,b,c) = 1 + \left(\frac{D}{p}\right)\varepsilon\left(\frac{a}{p}, \frac{b}{p}, \frac{c}{p}\right) + p\varepsilon\left(\frac{a}{p^2}, \frac{b}{p^2}, \frac{c}{p^2}\right)$$
$$(a,b,c \in \mathbf{Z},\ b^2 - 4ac = Dp^2).$$

292 For p odd, this follows from the following table, in which $v_{p^1}(m)$ denotes the exact power of p dividing an integer m.

$v_{p^1}(a)$	$v_{p^1}(b)$	$v_{p'}(c)$	$\varepsilon\left(\frac{a}{p^2}, \frac{b}{p}, c\right) \sum_{j=1}^{p}$	$\varepsilon\left(a, \frac{b-2aj}{p}, \frac{c-bj+aj^2}{p^2}\right)$	$\varepsilon\left(\frac{a}{p}, \frac{b}{p}, \frac{c}{p}\right)$	$\varepsilon\left(\frac{a}{p^2}, \frac{b}{p^2}, \frac{c}{p^2}\right)$
0	$\geqslant 0$	$\geqslant 0$	0	1	0	0
1	$\geqslant 1$	$\geqslant 1$	0	$1 + \left(\frac{D}{p}\right)$	1	0
$\geqslant 2$	$\geqslant 1$	0	1	0	0	0
$\geqslant 2$	1	$\geqslant 1$	1	1	1	0
$\geqslant 2$	$\geqslant 2$	1	1	0	1	0
$\geqslant 2$	$\geqslant 2$	$\geqslant 2$	1	p	1	1

The proof for $p = 2$ is similar but there are more cases to be considered.

iii) We will show that each of the properties in question is implied by the orthogonality of F with $\mathscr{L}_D\Phi$ for special choices of D and Φ.

For (21) we choose

$$\Phi(X) = \delta(X^2 + D),$$

where δ is the Dirac delta-function. From the identity

$$\left(\frac{a|z|^2 + bx + c}{y}\right)^2 + D = \frac{|az^2 + bz + c|^2}{y^2} \tag{37}$$

we see that the support of $\mathscr{L}_D\Phi$ is the set of points in H satisfying some quadratic equation of discriminant D, and an easy calculation shows that

$$\int_{\Gamma/\mathsf{H}} F(z)\mathscr{L}_D\Phi(z)dz = \frac{\pi}{2\sqrt{|D|}} \sum_{i=1}^{h(D)} F(z_{Q_i}) \tag{38}$$

for any continuous $F : \Gamma/\mathsf{H} \to \mathbb{C}$. (Of course, $\delta(X^2 + D)$ is not a function, and equation (38) must be interpreted in the sense that it holds in the limit $n \to \infty$ if we choose $\Phi(X) = \delta_n(X^2 + D)$ where $\{\delta_n\}$ is a sequence of smooth, even functions with integral 1 and support tending to $\{0\}$.) Hence any $F \in (\mathrm{Im}\,\mathscr{L}_D)^\perp$ satisfies (21).

The case $D > 0$, D not a square, is similar; here we choose $\Phi(X) =$ **293**
$\delta(X)$. so that $\mathscr{L}_D\Phi(z)$ is supported on the semicircles $a|z|^2 + bx + c = 0$

$(a, b, c \in \mathbf{Z}, b^2 - 4ac = D)$, and find

$$\int\limits_{\Gamma/\mathsf{H}} F(z)\mathscr{L}_D\Phi(z)dz = \frac{1}{\sqrt{D}}\sum_{i=1}^{h(D)}\int\limits_{C_{Q_i}} F(z)|d_{Q_i}z|, \tag{39}$$

where the equation is to be interpreted in the same way as (38). Thus $F \in (\operatorname{Im}\mathscr{L}_D)^{\perp}$ implies (23).

It remains to prove that any $F \in \mathscr{E}$ satisfies equation (284). We follow the proof of the divisibility of $\sum\limits_{n=1}^{\infty}|a_n|^2/n^{s+k-1}$ by $\zeta(s)$ given in [11]. Equations (37) and (**??**) of that paper give the identity

$$\begin{aligned}&\sum_{i=1}^{r}\frac{a_i(m)}{(f_i, f_i)}y^k|f_i(z)|^2 \\ &= \frac{(-1)^{k/2}}{\pi}2^{k-4}m^{k-1}(k-1)\sum_{t=-\infty}^{\infty}\mathscr{L}_{t^2-4m}\Phi_{k,t}(z) \quad (z \in \mathsf{H})\end{aligned} \tag{40}$$

for all integers $m > 0$, where

$$r = \dim S_k(SL_2(Z)), f_i(z) = \sum_{m=1}^{\infty}a_i(m)e^{2\pi imz}(i = 1, \ldots, r)$$

are the normalized Hecke eigenforms of weight

$$k,\ (f_i, f_i) = \int\limits_{\Gamma/\mathsf{H}} y^k|f_i(z)|^2dz,$$

and $\Phi_{k,t}(X) = (X - it)^{-k} + (X + it)^{-k}$. Thus any function in $\mathscr{E}$ is orthogonal to the sum on the left-hand side of (**??**) and therefore, since the Fourier coefficients $a_i(m)$ are linearly independent, to each of the functions $y^k|f_i(z)|^2$.

Using the computations of [12] and an extension of the Rankin-Selberg method [13] , it seems to be possible to prove the orthogonality of $F \in \mathscr{E}$ with $|f(z)|^2$ also for Maass eigenforms (= non-holomorphic cusp forms which are eigenvalues of the Laplace and Hecke operators) of weight 0. □

§4. Let $G = PSL_2(\mathbf{R})$ and $K = SO(2)/\{\pm 1\}$ its maximal compact subgroup, and identity the symmetric space G/K with H by $gK = \begin{pmatrix} a & b \\ c & d \end{pmatrix} K \leftrightarrow g \cdot i = \dfrac{ai+b}{ci+b}$. In this section we will construct a representation $\mathscr{V}$ of G in the space of functions of Γ/G whose space of K-fixed vectors $\mathscr{V}^K$ is $\mathscr{E}$.

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$$\mathscr{X}_R = \{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \bigg| a, b, c \in \mathbf{R}\}$$

be the 3-dimensional vector space of symmetric real 2×2 matrices and $\mathscr{X}_Z \subset \mathscr{X}_R$ the lattice consisting of matrices with $a, b, c \in \mathbf{Z}$. The group G acts on $\mathscr{X}_R$ by $g \circ M = g^t M g (g \in G, M \in \mathscr{X}_R)$, and $\mathscr{X}_Z$ is stable under the action of the subgroup Γ. For $M \in \mathscr{X}_R$ and $g \in G$, the expression $tr(g^t M g)$ depends only on the right coset gK (since $k^t = k^{-1}$ for $k \in K$), i.e. only on $g \cdot i \in \mathsf{H}$. An easy calculation shows that

$$tr(g^t M g) = \frac{a|z|^2 + bx + c}{y} (M = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in \mathscr{X}_R, z = g \cdot i \in \mathsf{H}). \quad (41)$$

This explains where the strange expression $\dfrac{a|z|^2 + bx + c}{y}$ in the definition of $\mathscr{L}_D$ comes from and also why this expression is invariant under the simultaneous operation of Γ on the upper half-place $(gK \to \gamma gK)$ and on binary quadratic forms $(M \to (\gamma^{-1})^t M \gamma^{-1})$..

Using (41), we can rewrite the definition of $\mathscr{L}_D$ as

$$\mathscr{L}_D \Phi(gK) = \sum_{\substack{M \in \mathscr{X}_Z \\ \det M = -D/4}} \Phi(tr(g^t M g)).$$

To pass from functions on H to functions on G, we replace the special function $M \to \Phi(tr(M))$ by an arbitrary function Φ on the 2-dimensional submanifold

$$\mathscr{X}_R(D) = \{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in \mathscr{X}_R \bigg| b^2 - 4ac = D\}$$

of $\mathscr{X}_R$. Thus we extend $\mathscr{L}_D$ to an operator (still denoted $\mathscr{L}_D$) from the space of nice functions on $\mathscr{X}_R(D)$ to the space of functions on Γ/G by setting

$$\mathscr{L}_D\Phi(g) = \sum_{M\in\mathscr{X}_Z(D)} \Phi(g^t Mg) \quad (g \in G), \tag{42}$$

where $\mathscr{X}_Z(D) = \mathscr{X}_R(D) \cap \mathscr{X}_Z$. Here "nice" means that Φ satisfies the obvious extensions of (27) and (**??**), i.e. it must be of sufficiently rapid decay in $\mathscr{X}_R(D)$ and, if D is a square, must be smooth and have zero integral along each of the lines $l_{g,\varepsilon} = g^t \begin{pmatrix} 0 & \frac{1}{2}\varepsilon\sqrt{D} \\ \frac{1}{2}\varepsilon\sqrt{D} & \mathbb{R} \end{pmatrix} g (g \in G, \varepsilon =$
295 $\pm 1)$ on the ruled surface $\mathscr{X}_R(D)$.

It is clear that $\mathscr{L}_D\Phi(g)$ is left Γ-invariant, since $\mathscr{X}_Z(D)$ is stable under Γ and the sum (42) is absolutely convergent. Also, the image of $\mathscr{L}_D$ is stable under the representation π of G given by right translation, since

$$\pi(g) \circ \mathscr{L}_D = \mathscr{L}_D \circ \pi'(g) \quad (g \in G),$$

where $\pi'(g)\Phi(M) = \Phi(g^t Mg)$. Hence the space

$$\mathscr{V} = \bigcap_{D\in Z} (\operatorname{Im} \mathscr{L}_D)^\perp \tag{43}$$

of functions $F : \Gamma/G \to \mathbb{C}$ satisfying an appropriate growth condition and such that

$$\int_{\Gamma/G} \mathscr{L}_D\Phi(g)F(g)dg = 0 \quad (dg = \text{Haar measure}) \tag{44}$$

for all $D \in Z$ and all nice functions Φ on $\mathscr{X}_R(D)$, is also stable under G. Also, it is clear that $\mathscr{V}^K$ coincides with the space $\mathscr{E}$ defined in §3. In particular, $\mathscr{V}$ contains the vectors $v_\rho : g \to E^*(g \cdot i, \rho)$ (ρ a non-trivial zero of $\zeta(s)$) and more generally $v_{\rho,m} \cdot g \to F_{\rho,m}(g \cdot i)$ ($F_{\rho,m}$ as in (19)).

[1]Since the various manifolds $\mathscr{X}_R(D) \subset \mathscr{X}_R$ are disjoint, we can also define $\mathscr{V}$ as $(\operatorname{Im} \mathscr{L})^\perp$, where $\mathscr{L}$ is the operator from nice functions on $\mathscr{X}_R$ to functions on Γ/G defined by

$$\mathscr{L}\Phi(g) = \sum_{M\in\mathscr{X}_Z} \Phi(g^1 Mg).$$

On the other hand, because the function $z \to E(z, s)$ is an eigenfunction of the Laplace operator on H, the representation theory of G tells us that (at least for $s \notin Z$) the smallest G-invariant space of functions on Γ/G containing the function $g \to E(g \cdot i, s)$ is an irreducible representation isomorphic to the principal series representation $\mathscr{P}_s$. (Recall that $\mathscr{P}_s$ is the representation of G by right translations on the set of functions $f\colon G \to \mathbb{C}$ satisfying $f\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g\right) = |a|^{2s} f(g)$ and $f\big|_K \in L^2(K)$). Thus *$\mathscr{V}$ contains the principal series representation $\mathscr{P}_\rho$ for every non-trivial zero ρ of the Riemann-zeta-function.*

On the other hand, $\mathscr{P}_s$ is unitarizable if and only if $s(1-s) > 0$, **296**
i.e. $s \in (0, 1)$ or $\mathrm{Re}(s) = \frac{1}{2}$. Thus the existence of a unitary structure on $\mathscr{V}$ would imply the Riemann hypothesis.

However, the following argument suggests that it may be unlikely that such a unitary structure can be defined in a natural way. If ρ is a zero of $\zeta(s)$ of order $n > 1$, then the functions $F_{\rho,m} (m = 0, \ldots, n-1)$ belong to $\mathscr{V}^K$, and these functions are not eigenfunctions of Δ, through the space they generate is stable under Δ (for example, differentiating (18) with respect to s we find $\Delta F_{\rho,1} = \rho(\rho-1) F_{\rho,1} + (2\rho - 1) F_{\rho,0}$. Therefore $\mathscr{V}$ contains a G-invariant subspace $\mathscr{V}_{\rho,n}$ corresponding to the eigenvalues $\rho(1-\rho)$ which is reducible but is not a direct sum of irreducible representations (we have $\dim \mathscr{V}^K_{\rho,n} = n$ and $\mathscr{V}_{\rho,n} \supset \mathscr{V}_{\rho,n-1} \supset \ldots \supset \mathscr{V}_{\rho,1} \supset \mathscr{V}_{\rho,0} = \{0\}$ with $\mathscr{V}_{\rho,m}/\mathscr{V}_{\rho,m-1} \cong \mathscr{P}_\rho$), and such a representation cannot have a unitary structure. Thus the unitarizability of $\mathscr{V}$ would imply not only the Riemann hypothesis, but also the simplicity of the zeroes of $\zeta(s)$. Since an analogue of $\mathscr{V}$ can be defined for any number field or function field (cf. . §5), and since there are examples of such fields whose zeta-functions are known to have multiple zeroes, there cannot be any generally applicable way of putting a unitary structure on $\mathscr{V}$. Of course,

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We may also identify $\mathscr{X}_R$ with the Lie algebra $i_R = \{\begin{pmatrix} -\frac{1}{2}b & -c \\ a & \frac{1}{2}b \end{pmatrix} \Big| a, b, c \in R\}$ of G by $M \to M\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; then the operation $M \to g' M g$ of G on $\mathscr{X}_R$ becomes the adjoint representation $X \to g^{-1} X g$ of G on i_R, and $\mathscr{V}$ is the set of functions on Γ/G orthogonal to all functions of the form $g \to \sum\limits_{X \in i_Z} \Psi(Ad(g)X)$, where Ψ is a nice function on i_R.

this does not preclude the possibility that our particular $\mathscr{V}$ (for the filed $\mathbf{Q}$) has a unitary structure defined in some special way, and indeed, if the zeros of $\zeta(s)$ are simple and lie on the critical line and if (as seems likely) $\mathscr{E}$ is spanned by the V_ρ, then $\mathscr{V}$ is in fact unitarizable, indeed in infinitely many ways, since we are essentially free to choose the norm of v_ρ. For various reasons, a natural choice seems to be $\|v_\rho\| = |\zeta^*(2\rho)|$.

Finally, we should mention that the construction of $\mathscr{V}$ is closely related to the Weil representation. The functions $\mathscr{L}_D\Phi(g)$ are essentially the Fourier coefficients for a "lifting" operator from functions on Γ/G to autormorphic forms on the metaplectic group, in analogy with the construction of Shintani [9] in the holomorphic case; thus the space $\mathscr{V}$ can be interpreted as the kernel of the lifting.

§5. The proof of the theorem in § 3 shows that almost every statement of the theorem can be strengthened to a statement about the individual spaces

$$\mathscr{E}_D = (\operatorname{Im}\mathscr{L}_D)^\perp \quad (D \in \mathbf{Z})$$

297 rather than just their intersection $\mathscr{E}$. Thus in part iii) of the theorem, to prove that a function F satisfies (21) or (23) we needed only $F \in \mathscr{E}_D$ for the value of D in question, and it was only for (284) that $F \in \bigcap_D \mathscr{E}_D$ was needed. Similarly in part ii), equation (34) shows that each space $\mathscr{E}_D$ is stable under the Laplace operator. The same is not true for the Hecke operators, since $T(p)$ maps $\operatorname{Im}(\mathscr{L}_D)$ to $\operatorname{Im}(\mathscr{L}_{Dp^2}) + \operatorname{Im}(\mathscr{L}_D) + \operatorname{Im}(\mathscr{L}_{D/p^2})$ but the intersection of the spaces $\mathscr{E}_D$ for all D with a common squarefree part *is* stable under the Hecke algebra. Finally—and most interesting—from equation (30) or (32) we see that $\mathscr{E}_D (D \neq 0)$ contains $E^*(z,\rho)$ whenever ρ is a zero of $\zeta(s,D)$ (resp. $\left.\dfrac{\partial^i}{\partial s^i}E^*(z,s)\right|_{s=\rho}$ for $0 \leqslant i \leqslant n-1$ if ρ is a zero of multiplicity n). Since $\zeta(s,D) = \zeta(s)L(s,D)$ and $L(s,D)$ has infinitely many zeroes in the critical strip, this shows that $\mathscr{E}_D$ contains many more Eisenstein series than just the functions (19). (This conclusion holds also when D is a square; in this case $L(s,D)$ is equal to $\zeta(s)^2$ up to an elementary factor and we do not get any new zeroes, but they all occur with twice the multiplicity and so we get twice as many functions as in (19). For $D = 0$, however, we get only the functions

(19), since the expression on the right-hand side of (31) or (33) is a linear combination of $\zeta(s)\zeta(2s)$ and $\zeta(s)\zeta(2s-1)$ rather than a multiple of $\zeta(s,0)$.)

The functions $\zeta(s,D)$ for two discriminants D withthe same square-free part differ by a finite Euler product and have the same non-trivial zeroes ρ. This, together with eq. (36), suggests that the most natural thing to do is to put together the corresponding spaces $\mathscr{E}_D$. Thus we let E denote either a quadratic extension of $\mathbf{Q}$, or $\mathbf{Q}+\mathbf{Q}$, or $\mathbf{Q}$, and define

$$\mathscr{E}(E) = \bigcap_{f=1}^{\infty} \mathscr{E}_{df^2},$$

where d denotes the discriminant of E, or 1, or 0, respectively. Then the above discussion can be summarized as follows:

i) Each of the spaces $\mathscr{E}(E)$ is stable under the Laplace and Hecke operators;

ii) $\bigcap_E \mathscr{E}(E) = \mathscr{E}$;

iii) $\mathscr{E}(E)$ contains $\left.\dfrac{\partial^i}{\partial s^i} E^*(z,s)\right|_{s=\rho}$ for $0 \leqslant i \leqslant n-1$ if ρ is an n-fold **298**
zero of $\zeta_E(s)$, where $\zeta_E(s)$ denotes the Dedekind zeta-function of E if $E = Q$ or E is q quadratic field and $\zeta_{\mathbf{Q}+\mathbf{Q}}(s) = \zeta(s)^2$.

Of course, we can also define representations $\mathscr{V}_D = (\operatorname{Im}\mathscr{L}_D)^{\perp}$ and $\mathscr{V}(E) = \cap\mathscr{V}_{df^2}$ similarly; then $\mathscr{V}(E)^K = \mathscr{E}(E)$ and $\mathscr{V}(E)$ is a representation of $PSL_2(\mathscr{R})$ whose spectrum is related to the zeroes of $\zeta_E(s)$ in the same way as that of $\mathscr{V}$ to those of $\zeta(s)$.

The representations $\mathscr{V}(E)$ have a very nice interpretation in the language of adeles; we end the paper by describing this. As motivation, we recall that our starting point for the definition of $\mathscr{E}$ was the fact that the zeta-function of a quadratic field E can be written as the integral of $E(z,s)$ over a certain set $\mathscr{S}_E \subset \Gamma/\mathsf{H}$ consisting of a finite number of points if E is imaginary and of a finite number of closed curves if E is real (Proposition 1 and 2). Hence the functions $E(z,\rho)$, ρ a zero of $\zeta_E(s)$, belong to the space of functions whose integral over $\mathscr{S}_E$ vanishes.

Now let G denote the algebraic group $GL(2)$, Z its center, and $\mathbf{A}$ the ring of adeles of $\mathbf{Q}$. Choosing a basis of E over $\mathbf{Q}$ gives an embedding of $E^\times$ in $GL(2,\mathbf{Q})$ and a non-split torus $T \subset G$ with $T(\mathbf{Q}) = E^\times$. There is a projection $G(\mathbf{Q})Z(\mathbf{A})/G(\mathbf{A}) \to \Gamma/\mathsf{H}$ and under this projection $T(\mathbf{Q})Z(\mathbf{A})/T(\mathbf{A})$ maps to $\mathscr{S}_E$. The adelic analogue of Proposition 1 and 2 is the fact that the integral of an Eisenstein series over $T(\mathbf{Q})Z(\mathbf{A})/T(\mathbf{A})$ is a multiple of the zeta-function of E. To prove it, we must recall the definition of the Eisenstein series. Let Φ be a Schwartz-Bruhat function on $\mathbf{A}^2$; then the Eisenstein series $E(g,\Phi,s)$ is defined for $g \in G(\mathbf{A})$ and $s \in \mathbb{C}$ with sufficiently large real part by

$$E(g,\Phi,s) = \int\limits_{Z(Q)/Z(A)} \sum_{\xi\in Q^2\{0\}} \Phi[\xi z g] |\det zg|_Q^s dz, \tag{45}$$

where $|\ |_Q$ denote the idele norm and dz the Haar measure on Z. (This definition is the analogue of equation (2). The more usual definition of $E(g,\Phi,s)$, analogous to eq. (1), is

$$E(g,\Phi,s) = \sum_{\gamma\in P(Q)/G(Q)} f(\gamma g,\Phi,s),$$

299 where $P = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$ and

$$F(G,\Phi,s) = |\det g|_Q^s \int\limits_{A^\times} \Phi[(0,a)g]|a|^{2s} d^\times a,$$

which is easily seen to agree with (45); note that $f(g,\Phi,s)$ equals $\zeta_Q(2s)$ times an elementary function of s by Tate theory, and that

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g,\Phi,s\right) = \left|\frac{a}{b}\right|^s f(g,\Phi,s),$$

so the analogue of $f(g,\Phi,s)$ in the upper half-plane is the function $\zeta(2s)\Im(g\cdot i)^s$.) Identifying $\mathbf{Q}^2\{0\}$ with $E^\times$ and observing that the $\mathbf{Q}$-idele norm of $\det t (t \in T(\mathbf{A}))$ equals the E-idele norm of t under the identification of $T(\mathbf{A})$ with $\mathbf{A}_E^\times$, we find

$$\begin{aligned}\int\limits_{T(Q)Z(A)/T(A)} E(tg,\Phi,s)dt &= \int\limits_{T(Q)/T(A)} \sum_{\xi\in Q^2\{0\}} \Phi[\xi tg]|\det tg|_Q^s dt \\ &= |\det g|_Q^s \int\limits_{E^\times/A_E^\times} \sum_{\xi\in E^\times} \Phi[\xi tg]|t|_E^s dt \\ &= |\det g|_Q^s \int\limits_{A_E^\times} \Phi[eg]|e|_E^s d^\times e.\end{aligned}$$

Since $e \to \Phi[eg]$ is a Schwartz-Bruhat function on $\mathbf{A}_E$, this is precisely the Tate integral for $\zeta_E(s)$. (The computation just given is the basis for Harder's computations of period integrals in this volume as well as for the generalization of the Selberg trace formula in [4].) In particular, it follows that the integral of

$$F(g) = \frac{\partial^i}{\partial s^i} E(g,\Phi,s)\Big|_{s=\rho}$$

over $T(\mathbf{Q})Z(\mathbf{A})/T(\mathbf{A})g$ vanishes if ρ is a zero of $\zeta_E(s)$ of multiplicity $\geqslant i+1$. The natural adelic definition of $\mathscr{V}(E)$ is thus as the space of functions $F : G(\mathbf{Q})Z(\mathbf{A})/G(\mathbf{Q}) \to \mathbb{C}$ satisfying

$$\int\limits_{T(Q)Z(A)/T(A)} F(tg)dt = 0 \qquad (\forall g \in G(A)) \tag{46}$$

as well as some appropriate growth condition. The space $\mathscr{V}(E)$ then contains irreducible principal series representations corresponding to the zeroes of $\zeta_E(s)$. Condition (46) is similar to the condition

$$\int\limits_{N(Q)/N(A)} F(ng)dn = 0 \qquad (\forall g \in G(A))$$

defining cusp forms (where N is the unipotent radical of a parabolic **300**
subgroup of G), so the space $\mathscr{V}(E)$ can be thought of as an analogue of

the space $L_0^2(G(\mathbf{Q})Z(\mathbf{A})/G(\mathbf{A}))$ of cusp forms. Like L_0^2, it probably has a discrete spectrum. The difference is that, whereas L_0^2 has a unitary structure, the corresponding statement for $\mathscr{V}(E)$ would imply the Riemann hypothesis and the simplicity of the zeroes of $\zeta_Q(s)$. We call functions F satisfying (46) *toroidal forms* (in analogy with the French terminology of "formes paraboliques" for cusp forms) and the $\mathscr{V}(E)$ *toroidal representations*.

The calculation given above is unchanged if we replace $\mathbf{Q}$ by any global field F and take E to be a quadratic extension of F. In the case where F is the functional field of a curve X over a finite field, there are only finitely many zeroes of $\zeta_F(s)$, their number being equal to the first Betti number of X. Then the K-finite part of our representation $\mathscr{V} = \bigcap_E \mathscr{V}(E)$ is a complex vector space of dimension at least (and hopefully exactly) equal to this Betti number, and Barry Mazur pointed out that this space might have a natural interpretation as a complex cohomology group $H^1(X;\mathbb{C})$. It is not yet clear whether this point of view is tenable. At any rate, however, from conversations with Harder and Deligne it appears that it will at least be possible to show that the dimension of the space in question is finite.

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EISENSTEIN SERIES AND THE SELBERG TRACE FORMULA I

By Don Zagier*

0 Introduction

303 The integral $\int K_\circ(g,g)E(g,s)dg$. Let $G = SL_2(R)$ and Γ be an arithmetic subgroup of G for which Γ/G has finite volume but is not compact. The space $L^2(\Gamma/G)$ has the spectral decomposition (with respect to the Casimir operator)

$$L^2(\Gamma/G) = L^2_\circ(\Gamma/G) \bigoplus L^2_{sp}(\Gamma/G) \bigoplus L^2_{\text{cont}}(\Gamma/G),$$

where $L^2_\circ(\Gamma/G)$ is the space of cusp forms and is discrete, $L^2_{sp}(\Gamma/G)$ is the discrete part of $(L^2_\circ)^\perp$, given by residues of Eisenstein series, and L^2_{cont} is the continuous part of the spectrum, given by integrals of Eisenstein series. If φ is a function of compact support or of sufficiently rapid decay on G, then convolution with φ defines an endomorphism T_φ of $L^2(\Gamma/G)$, and the kernel function

$$K(g,g') = \sum_{\gamma\in\Gamma} \varphi(g^{-1}\gamma g') \qquad (g,g' \in G) \tag{0.1}$$

of T_φ has a corresponding decomposition as $K_\circ + K_{\text{sp}} + K_{\text{cont}}$, where K_{sp} and K_{cont} can be described explicitly using the theory of Eisenstein series. The restriction of T_φ to $L^2_\circ(\Gamma/G)$ is of trace class; its trace is given by

$$Tr(T_\varphi, L^2_\circ) = \int\limits_{\Gamma/G} K_\circ(g,g)dg. \tag{0.2}$$

The Selberg trace formula is the formula obtained by substituting

$$K(g,g) - K_{sp}(g,g) - K_{\text{cont}}(g,g) \quad \text{for} \quad K_\circ(g,g)$$

and computing the integral. However, although $K_\circ(g,g)$ is of rapid decay in Γ/G, the individual terms $K(g,g)$, $K_{sp}(g,g)$ and $K_{\text{cont}}(g,g)$ are not, so that to carry out the integration one has to either delete small neighbourhoods of the cusps form a fundamental domain or else "truncate" the kernel functions by subtracting off their constant terms in such neighbourhoods, and then to compute the limit as these neighbourhoods shrink to points. This procedure is perhaps somewhat unsatisfactory, both from an aesthetic point of view and because of the analytical difficulties it involves.

To get around these difficulties we introduce the integral **304**

$$I(s) = \int\limits_{\Gamma/G} K_\circ(g,g)E(g,s)dg, \tag{0.3}$$

where $E(g,s)$ $(g \in G, s \in C)$ denotes an Eisenstein series. The idea of integrating a Γ-invariant function $F(g)$ against an Eisenstein series was introduced by Rankin [5] and Selberg [6], who observed that in the region of absolute convergence of the Eisenstein series this integral equals the Mellin transform of the constant term in the Fourier expansion of F (see §2 for a more precise formulation). Applying this principle to $F(g) = K_\circ(g,g)$ we can calculate $I(s)$ for $\mathrm{Re}(s) > 1$ as a Mellin transform, obtaining a representation of $I(s)$ as an infinite series of terms. Each of these terms can be continued meromorphically to $\mathrm{Re}(s) \leqslant 1$; in particular, the contribution of a hyperbolic or elliptic conjugacy class of γ's in (0.1) is the product of a certain integral transform of φ with the Dedekind zeta-functio of the corresponding real or imaginary quadratic field. Since the residue of $E(g,s)$ at $s = 1$ (resp. the value of $E(g,s)$ at

$s = 0$) is a constant function, we recover the Selberg trace formula by computing $\mathrm{res}_{s=1}(I(s))$ (resp. . $I(0)$). This proof of the trace formula is more invariant and in some respects computationally simpler than the proofs involving truncation. It also gives more insight into the origin of the various terms in the trace formula; for instance, the class numbers occurring there now appear as residues of zeta-functions.

However, the formula for $I(s)$ has other consequences than the trace formula. The most striking is that $I(s)$ (and in fact each of the infinitely many terms in the final formula for $I(s)$) is divisible by the Riemann zeta-function, i.e. the quotient $I(s)/\zeta(s)$ is an entire function of s. Interpreting this as the statement that the Eisenstein series $E(g, \rho)$ is orthogonal to $K_\circ(g, g)$ (in fact, to each of infinitely many functions whose sum equals $K_\circ(g, g)$) whenever $\zeta(\rho) = 0$, one is led to the construction of a representation of G whose spectrum is related to the set of zeros of the Riemann zeta-function (cf. [11] in this volume).

On the other hand, the formula for $I(s)$ can be used to get information about cusp forms. The function $K_\circ(g, g')$ is a linear combination of terms $f_j(g)f_j(g')$, where $\{f_j\}$ is an orthogonal basis for $L^2_\circ(\Gamma/G)$ and
305 where the coefficients depend on the function φ and on the eigenvalues of f_j ("Selberg transform"). Moreover, applying the Rankin-Selberg method to the function $F(g) = |f_j(g)|^2$ one finds that the integral of this function against $E(g, s)$ equals the "Rankin zeta-function" $R_{f_j}(s)$ (roughly speaking, the Dirichlet series $\sum\limits_{n=1}^{\infty} |a_n|^2 n^{-s}$, where the a_n are the Fourier coefficients off); indeed, this is the situation for which the Rankin-Selberg method was introduced. Thus $I(s)$ is a linear combination of the functions $R_{f_j}(s)$, and so one can get information about the latter from a knowledge of $I(s)$. In particular, using a "multiplicity one" argument one can deduce from the divisibility of $I(s)$ by $\zeta(s)$ that in fact each $R_{f_j(s)}$ is so divisible (this result had been proved by another method by Shimura [8] for holomorphic cusp forms and by Gelbart and Jacquet [2] in the general case). Other applications of the results proved here might arise by comparing them with the work of Goldfeld [1]. It does not seem impossible that the formula of $I(s)$ can be used to obtain information about the Fourier coefficients of cusp forms.

The idea we have described can be applied in several different situations:

1. By working with an appropriate kernel function, we can isolate the contribution coming from holomorphic cusp forms of a given weight k (discrete series representations in $L^2(\Gamma/G)$). This case was treated in [10]. The computation of $I(s)$ here is considerably easier than in the general case because there is no continuous spectrum and only finitely many cusp forms f_j are involved. We can therefore represent each Rankin zeta-function $R_{f_j}(s)$ as an infinite linear combination of zeta-functions of real and imaginary quadratic fields. Moreover, for certain odd positive values of s the contributions of the hyperbolic conjugacy classes in Γ to $I(s)$ vanish and one is left with an identity expressing $R_{f_j}(s)$ as a *finite* linear combination of special values of zeta-functions of imaginary quadratic extensions of $\mathbf{Q}$. As a corollary of this identity one obtains the algebraicity (and behaviour under $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$) of $\dfrac{1}{(f_j, f_j)} R_{f_j}(s)/\pi^{k-1}\zeta(s)$ for the values of s in question ([10], Corollary to Theorem 2, p. 115), a result proved independently by Sturm [9] by a different method.

2. The first case involving the continuous spectrum is that of Maass **306**
wave forms of weight zero, i.e. cusp forms in
$$L^2(\Gamma/G/K) = L^2(\Gamma/\mathsf{H}),$$
where K denotes $SO(2)$ and $\mathsf{H} = G/K$ the upper half-plane. This is the case treated in the present paper (with $\Gamma = SL_2(\mathbf{Z})$).

3. Next, one can replace $SL_2(\mathbf{R})$ and $SL_2(\mathbf{Z})$ by $GL_2(2, \mathbf{A})$ and $GL(2, F)$, respectively, where F is a global field and $\mathbf{A}$ the ring of adeles of F. This case, which is the most general one as far as $GL(2)$ is concerned, will be treated in a joint paper with Jacquet [3]. It includes as special cases 1 and 2, as well as their generalizations to holomorphic and non-holomorphic modular forms of

arbitrary weight and level, Hilbert modular forms, and automorphic forms over function fields.

4. Finally, the definition of $I(s)$ makes sense in any context where Eisenstein series can be defined, so it may be possible to apply the method sketched in this introduction to discrete subgroups of algebraic groups other than $GL(2)$.

1 Statement of the main theorem

In this section we describe the main result of this paper, namely a formula for $I(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$. In order to reduce the amount of notation and preliminaries needed, we will state the formula in terms of a certain holomorphic function $h(r)$; the relationship of $h(r)$ to the function $\varphi(g)$ of the introduction (Selberg transform) is well-known and will be reviewed in § 2. Except at the end of § 5, we will always consider only forms of weight 0 on the full modular group $\Gamma = SL_2(\mathbf{Z})/\{\pm 1\}$. The results for congruence subgroups are similar but messier to state and in any case will be subsumed by the results of [3].

Any continuous Γ-invariant function $f : \mathsf{H} \to \mathbf{C}$ has a Fourier expansion of the form

$$f(z) = \sum^{\infty} 2(\_n = -\infty A_n(f;y)e^{2\pi inx} \quad (z \in \mathsf{H}) \tag{1.1}$$

(here and in future we use x and y to denote the real and imaginary parts of $z \in \mathsf{H}$). We denote by $L^2(\Gamma/\mathsf{H})$ the Hilbert space of Γ-invariant functions $f : \mathsf{H} \to \mathbf{C}$ such that $(f, f) = \int_{\Gamma/\mathsf{H}} |f(z)|^2 dz$ is finite $\left(dz = \frac{dxdy}{y^2}\right)$
307 and by $L^2_\circ(\Gamma/\mathsf{H})$ the subspace of functions with $A_\circ(f;y) \equiv 0$. The space $L^2_\circ(\Gamma/\mathsf{H})$ is stable under the Laplace operator

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

and has a basis $\{f_j\}_{j\geqslant 1}$ consisting of eigenforms of Δ(see [4], § 5.2).

We write

$$\Delta f_j = -\left(\frac{1}{4} + r_j^2\right) f_j \quad (j = 1, 2, \ldots) \tag{1.2}$$

where $r_j \in \mathbf{C}$. Since Δ is negative definite, we have $r_j^2 + \dfrac{1}{4} \geqslant 0$, i.e. r_j is either real or else pure imaginary of absolute value $\leqslant \frac{1}{2}$. In fact it is known that the r_j are real for $SL_2(\mathbf{Z})$, but the corresponding statement for congruence subgroups is not known and we will use only $r_j^2 \geqslant -\dfrac{1}{4}$. From (1.2) we find that the n^{th} Fourier coefficient $A_n(f_j, Y)$ satisfies the second order differential equation

$$y^2 \frac{d^2}{dy^2} A_n(f_j, y) - 4\pi^2 n^2 y^2 A_n(f; y) = -\left(\frac{1}{4} + r_j^2\right) A_n(f_j; y).$$

The only solution of this equation which is bounded as $y \to \infty$ is $\sqrt{y} K_{ir_j}(2\pi|n|y)$, where $K_\nu(z)$ is the K-Bessel function, defined (for example) by

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t} \cosh \nu t dt \quad (\nu, z \in \mathbf{C}, \mathrm{Re}(z) > 0). \tag{1.3}$$

Hence the f_j have Fourier expansions of the form

$$f_j(z) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} a_j(n) \sqrt{y} K_{ir_j}(2\pi|n|y) e^{2\pi i n x} \tag{1.4}$$

with $a_j(n) \in \mathbf{C}$. We can choose the f_j to be normalized eigenfunctions of the Hecke operators

$$\begin{cases} T(n) : f(z) \longrightarrow \dfrac{1}{n} \displaystyle\sum_{\substack{a,d>0 \\ ad=n}} \sum_{b (\mathrm{mod}\ d)} f\left(\frac{az+b}{d}\right) \quad (n > 0), \\ T(-1) : f(z) \longrightarrow f(-\bar{z}), \quad T(-n) = T(-1)T(n), \end{cases} \tag{1.5}$$

308 i.e.

$$f_j|T(n) = \frac{a_j(n)}{|n|^{\frac{1}{2}}} f_j \qquad (n \in \mathbf{Z},\ n \neq 0) \tag{1.6}$$

(then $a_j(1) = 1$, $a_j(-1) = \pm 1$, and $a_j(n)$ is multiplicative). The functions f_j chosen in this way are called the *Maass eigenforms;* they form an orthogonal (but not orthonormal) basis of $L^2_\circ(\Gamma/\mathsf{H})$, uniquely determined up to order. For each j we define the *Rankin zeta-function* $R_{f_j}(s)$ by

$$R_{f_j}(s) = \frac{\Gamma(\frac{s}{2})^2}{8\pi^s \Gamma(s)} \Gamma\left(\frac{s}{2} + ir_j\right) \Gamma(\frac{s}{2} - ir_j) \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{|a_j(n)|^2}{|n|^s} \quad (\mathrm{Re}(s) > 1). \tag{1.7}$$

We also set

$$R^*_{f_j}(s) = \pi^{-s}\Gamma(s)\zeta(2s)R_{f_j}(s) = \zeta^*(2s)R_{f_j}(s), \tag{1.8}$$

where $\zeta(s)$ denotes the Riemann zeta-function and

$$\zeta^*(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \zeta^*(1-s). \tag{1.9}$$

The Rankin-Selberg method implies that $R^*_{f_j}(s)$ has a meromorphic continuation to all s, is regular except for simple poles at $s = 1$ and $s = 0$ with

$$\mathrm{res}_{s=1} \mathbf{R}^*_{f_j}(s) = \frac{1}{2}(f_j, f_j), \tag{1.10}$$

and satisfies the functional euqation

$$\mathbf{R}^*_{f_j}(s) = \mathbf{R}^*_{f_j}(1-s) \tag{1.11}$$

(the proofs will be recalled in § 2).

We will also need the zeta-functions $\zeta(s, D)$, where D is an integer congruent to 0 or 1 modulo 4. They are defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s, D) = \sum_{Q} \sum_{m,n} \frac{1}{Q(m,n)^s} \qquad (\mathrm{Re}(s) > 1), \tag{1.12}$$

where the first summation runs over all $SL_2(\mathbf{Z})$-equivalence classes of 309
binary quadratic forms Q of discriminant D and the second over all pairs of integers $(m,n) \in \mathbf{Z}^2/\operatorname{Aut}(Q)$ with $Q(m,n) > 0$, where $\operatorname{Aut}(Q)$ is the stabilizer of Q in $SL_2(\mathbf{Z})$. These functions, which were introduced in [10], are related to standard zeta-functions by

$$\zeta(s,D) = \begin{cases} \zeta(s)\zeta(2s-1) & \text{if } D = 0 \\ \zeta(s)^2 \cdot (\text{finite Dirichlet series }) & \text{if } D = \text{square} \neq 0 \ x \\ \zeta_{Q(\sqrt{D})}(s) \cdot (\text{finite Dirichlet series}) & \text{if } D \neq \text{square}, \end{cases} \tag{1.13}$$

where $\zeta_{\mathbf{Q}(\sqrt{D})}(s)$ denotes the Dedekind zeta-function of $\mathbf{Q}(\sqrt{D})$ (for precise formulas see [10], Proposition 3, p. 130). In particular, $\zeta(s,D)$ has a meromorphic continuation in s and $\zeta(s,D)/\zeta(s)$ is holomorphic except for a simple pole at $s = 1$ when D is a square.

Now let $h : \mathbf{R} \longrightarrow \mathbf{C}$ be a function satisfying

$$\begin{cases} h(r) = h(-r); \\ h(r) \quad \text{has a holomorphic continuation to the strip } |\operatorname{Im}(r)| < \dfrac{1}{2}A \\ \qquad \text{for some } A > 1; \\ h(r) \text{ is of rapid decay in this strip} \end{cases} \tag{1.14}$$

("rapid decay" means $O(|r|^{-N})$ for all N). The object of this paper is to compute $\sum_{j=1}^{\infty} \dfrac{h(r_j)}{(f_j, f_j)} R_{f_j}(s)$. In §§2 and 3 we will show that this function equals the function $I(s)$ of §0 and compute it in the strip $1 < \operatorname{Re}(s) < A$ by the Rankin-Selberg method; §§4 and 5 give the analytic continuation in s, computation of the residue at $s = 1$ (Selberg trace formula), and generalization to $\sum_{j=1}^{\infty} a_j(m) \dfrac{h(r_j)}{(f_j, f_j)} \mathbf{R}_{f_j}(s)$, where the $a_j(m)$ are the Fourier coefficients defined by (1.4). We state here the final result for $0 < \operatorname{Re}(s) < 1$ and $m > 0$ in a form which makes the functional equation apparent.

Theorem 1: *Let* $h : \mathbf{R} \longrightarrow \mathbf{C}$ *be a function satisfying the conditions* (1.14) *and* $m \geqslant 1$ *an integer. Then for* $s \in \mathbf{C}$ *with* $0 < \operatorname{Re}(s) < 1$ *we have* *the identity*

$$\sum_{j=1}^{\infty} a_j(m)\frac{h(r_j)}{(f_j,f_j)}R^*_{f_j}(s) = \mathscr{R}(s) + \mathscr{R}(1-s) \tag{1.15}$$

with $\mathscr{R}(s) = \mathscr{R}(s;m,h)$ *given by*

$$\begin{aligned}
\mathscr{R}(s) = &-\frac{1}{8\pi}\zeta^*(s)^2\int_{-\infty}^{\infty}\frac{\zeta^*(s+2ir)\zeta^*(s-2ir)}{\zeta^*(1+2ir)\zeta^*(1-2ir)}\left(\sum_{\substack{a,d\ge 1\\ ad=m}}\left(\frac{a}{d}\right)^{ir}\right)h(r)dr\\
&-\frac{1}{2}\frac{\zeta^*(s)\zeta^*(2s)}{\zeta^*(s+1)}\left(\sum_{\substack{a,d\ge 1\\ ad=m}}\left(\frac{a}{d}\right)^{s/2}\right)h(\frac{is}{2})\\
&+\frac{m^{\frac{s-1}{2}}}{4\pi^2}\frac{\Gamma(s)\Gamma(s-\frac{1}{2})}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{2-s}{2}\right)}\sum_{t=-\infty}^{\infty}\zeta(s,t^2-4m)\times\\
&\times\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma(ir)\Gamma(-ir)}\\
&\times F\left(\frac{1-s}{2}+ir,\frac{1-s}{2}-ir;\frac{3}{2}-s;1-\frac{t^2}{4m}\right)h(r)dr,
\end{aligned} \tag{1.16}$$

where $\zeta^*(s)$ *and* $\zeta(s,t^2-4m)$ *are defined by equations* (1.9) *and* (1.12) *and* $F(a,b;c;z)$ *denotes the hypergeometric function (defined by analytic continuation if* $z<0$*) and can be expressed in terms of Legendre functions for the special values of the parameters* a,b,c *occurring in* (1.16).

For $m<0$ *there is a similar formula with* m *replaced by* $|m|$ *in the first two terms and the function*

$$m^{\frac{s-1}{2}}F\left(\frac{1-s}{2}+ir,\frac{1-s}{2}-ir;\frac{3}{2}-s;1-\frac{t^2}{4m}\right)$$

in the third term replaced by a different hypergeometric function.

Corollary. *The Rankin zeta-function $R^*_{f_j}(s)$ is divisible by $\zeta^*(s)$ for all j.*

Proof of the Corollary: Every term on the right-hand side of equation (1.16) (and of the corresponding formula for $m < 0$) is divisible by $\zeta^*(s)$; since the series converges absolutely, we deduce that $\mathscr{R}(s)$ (and hence, by the functional equation (1.9), also $\mathscr{R}(1-s)$) is divisible by $\zeta^*(s)$. Therefore the expression on the left-hand side of equation (1.15), vanishes (with the appropriate multiplicity) at every zero of the Riemann zeta-function, and the linear independence of the eigenvalues
$a_j(m)h(r_j)(m \in \mathbf{Z} - \{0\}$, h satisfying (1.14)) for different j implies that **311**
the same holds for each $R^*_{f_j}(s)$. A more formal argument is as follows: For $z \in \mathsf{H}$ define

$$\Phi(s,z) = \sum_{j=1}^{\infty} \frac{1}{(f_j, f_j)} f_j(z) R^*_{f_j}(s);$$

then (1.14) and (1.15) imply the identity

$$\Phi(s,z) = \sqrt{y} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} [\mathscr{R}(s; m, h_{my}) + \mathscr{R}(1-s; m, h_{my})] e^{2\pi i m x},$$

where $h_{m,y}(r) = K_{ir}(2\pi|m|y)$. Therefore $\Phi(s,z)$ is divisible by $\zeta^*(s)$ and the corollary follows because $R^*_{f_j}(s)$ equals the scalar product $(\Phi(s,\cdot), f_j)$.

As mentioned in the introduction, the above Corollary, which is the analogue of the result for holomorphic forms proved in [8] and [10], is included in the results of Jacquet-Gelbart [2]. We also observe that, up to gamma factors, the quotient $R^*_{f_j}(s)/\zeta^*(s)$ equals

$$\frac{\zeta(2s)}{\zeta(s)} \sum_{n=1}^{\infty} \frac{|a_j(n)|^2}{n^s}.$$

Using the usual relations among the eigenvalues $a_j(n)$ of a Hecke eigenform, we see that this Dirichlet series has the Euler product

$$\prod_p \frac{1}{(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s})}.$$

where α_p, β_p are defined by

$$\sum_{n=1}^{\infty} \frac{a_j(n)}{n^s} = \prod_p \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}$$

(i.e. $\alpha_p + \beta_p = a_j(p)$, $\alpha_p \beta_p = 1$). Thus the corollary is the case $n = 2$ of the conjecture that the "symmetric power L-functions"

$$L_n(f_j, s) = \prod_p \prod_{m=0}^{n} \frac{1}{(1-\alpha_p^m \beta_p^{n-m} p^{-s})}$$

are entire functions of s for all $n \geqslant 1$.

2 Eisenstein series and the spectral decomposition of $L^2(\Gamma/\mathsf{H})$.

312 In this section we review the definitions and main properties of Eisenstein series, the Rankin-Selberg method, the spectral decomposition formula for $L^2(\Gamma/\mathsf{H})$, the Selberg transform, and the Selberg kernel function. All of this material is standard and may be skipped by the expert reader. We will try to give at least a rough proof of all of the statements; for a more detailed exposition the reader is referred to Kubota's book [4].

Eisenstein Series. For $z \in \mathsf{H}$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$ we set

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty / \Gamma} \mathrm{Im}(\gamma z)^s \quad (\mathrm{Re}(s) > 1), \tag{2.1}$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2(\mathbf{Z}) \right\} / \{\pm 1\} \cong \mathbf{Z}$ is the group of translations in Γ. The series converges absolutely and uniformly and therefore defines a function which is holomorphic in s and real-analytic and Γ-invariant with respect to z. Using the $1 : 1$ correspondence between Γ_∞/Γ and pairs of relatively prime integers (up to sign) given by

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow \pm(c, d),$$

we can rewrite (2.1) as

$$E(z,s) = \frac{1}{2} \sum_{\substack{c,d\in\mathbf{Z} \\ (c,d)=1}} \frac{y^s}{|cz+d|^{2s}} \quad (\mathrm{Re}\, z > 1)$$

and hence

$$\zeta(2s)E(z,s) = \frac{y^s}{2} \sum_{m,n}{}' \frac{1}{|mz+n|^{2s}} \quad (\mathrm{Re}(s) > 1), \tag{2.2}$$

where $\sum'$ denotes a summation over all pairs of integers $(m,n) \neq (0,0)$. This latter function has better analytic properties than $E(z,s)$, namely:

Proposition 1. *The function* (2.2) *can be continued meromorphically to the whole complex s-plane, is holomorphic except for a simple pole at* $s = 1$*, and satisfies the functional equation* **313**

$$E^*(z,s) = E^*(z,1-s), \tag{2.3}$$

where

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = \zeta^*(2s)E(z,s). \tag{2.4}$$

The residue at $s = 1$ *is independent of* z*:*

$$\mathrm{res}_{s=1} E(z,s) = \frac{6}{\pi} \mathrm{res}_{s=1} E^*(z,s) = \frac{3}{4} \quad (z \in \mathsf{H}). \tag{2.5}$$

We will deduce these properties from the Fourier development of $E(z,s)$, which itself will be needed in the sequel. Separating the terms $m = 0$ and $m \neq 0$ in (2.2) gives

$$\zeta(2s)E(z,s) = y^s[\zeta(2s) + \sum_{m=1}^{\infty} \varphi_s(mz)] \qquad (\mathrm{Re}(s) > 1),$$

where

$$\varphi_s(z) = \sum_{n=-\infty}^{\infty} \frac{1}{|z+n|^{2s}} \qquad \left(z \in \mathsf{H}, \mathrm{Re}(s) > \frac{1}{2}\right).$$

The function $\varphi_s(x+iy)$ is periodic in x for fixed y and hence has a Fourier development $\sum\limits_{\pi=-\infty}^{\infty} a(n,s,y)e^{2\pi inx}$ with

$$a(n,s,y) = \int\limits_{-\infty}^{\infty} \frac{e^{-2\pi inx}}{(x^2+y^2)^s}dx$$
$$= \begin{cases} \dfrac{\Gamma(\frac{1}{2}\Gamma(s-\frac{1}{2}))}{\Gamma(s)}y^{1-2s} & (n=0) \\ 2\left(\dfrac{\pi|n|}{y}\right)^{s-\frac{1}{2}} \dfrac{\Gamma(\frac{1}{2})}{\Gamma(s)}K_{s-\frac{1}{2}}(2\pi|n|y)(n \neq 0) \end{cases}$$

[GR 3.251.2 and /8.432.5]. Hence

$$\zeta(2s)E(z,s) = \zeta(2s)y^s + \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)y^{1-s}$$
$$+2\frac{\pi^s y^{\frac{1}{2}}}{\Gamma(s)} \sum_{m=1}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{|n|}{m}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|my)e^{2\pi inmx}$$

314 or, multiplying both sides by $\pi^{-s}\Gamma(s)$,

$$E^*(z,s) = \zeta^*(2s)y^s + \zeta^*(2s-1)y^{1-s} \qquad (2.6)$$
$$+\,2\sqrt{y}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \tau_{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx},$$

where $\zeta^*(s)$ is defined by (1.9) and $\tau_\nu(n)$ by

$$\tau_\nu(n) = |n|^\nu \sum_{d|nd>0} d^{-2\nu} = \sum_{\substack{ad=|n| \\ a,d>0}} \left(\frac{a}{d}\right)^\nu \quad (n \in \mathbf{Z}-\{0\}, \nu \in \mathbf{C}). \qquad (2.7)$$

The infinite sum in (2.6) converges absolutely and uniformly for all s and z, so (2.6) implies that $E^*(z,s)$ can be continued meromorphically to all s, the only poles being simple poles at $s=0$ and $s=1$ with residue $\pm\frac{1}{2}$ (the poles of $\zeta^*(2s)$ and $\zeta^*(2s-1)$ at $s=\frac{1}{2}$ cancel). Also, it is clear from (1.3) and the second formula of (2.7) that $K_\nu(z)$ and $\tau_\nu(n)$ are even functions of ν, so the functional equation of $E^*(z,s)$ follows from (2.6) and (1.9). Another consequence of (2.6) is the estimate

$$E(z,s) = O(y^{\max(\sigma,1-\sigma)}) \quad (y \to \infty), \tag{2.8}$$

where $\sigma = \mathrm{Re}(s)$; this follows because the sum of Bessel functions is exponentially small as $y \to \infty$.

THE RANKIN-SELBERG METHOD. We use this term to designate the general principle that the scalar product of a function $f : \Gamma/\mathsf{H} \to \mathbf{C}$ with an Eisenstein series equals the Mellin transform of the constant term in the Fourier development of f. More precisely, we have:

Proposition 2: *Let $f(z)$ be a Γ-invariant function in the upper half-plane which is of sufficiently rapid decay that the scalar product*

$$(f, E(.,\bar{s})) = \int\limits_{\Gamma/\mathsf{H}} f(z)E(z,s)dz \tag{2.9}$$

converges absolutely for some s with $\mathrm{Re}(s) > 1$. Then for such s

$$(f, E(.,\bar{s})) = \int\limits_0^\infty y^{s-2} A_\circ(f;y)dy \tag{2.10}$$

where $A_\circ(f,y)$ is defined by equation (1.1).

Proof. Substituting (2.1) into (2.9) we find 315

$$\begin{aligned}(f, E(.,\bar{s})) &= \int\limits_{\Gamma/\mathsf{H}} f(z) \sum_{\gamma\in\Gamma_\infty/\Gamma} \mathrm{Im}(\gamma z)^s dz \\ &= \int\limits_{\Gamma_\infty/\mathsf{H}} f(z)\,\mathrm{Im}(z)^s dz \\ &= \int\limits_0^\infty \int\limits_0^1 f(x+iy)y^s \frac{dxdy}{y^2}\end{aligned}$$

which is equivalent to (2.10).

Note that the growth condition on f in the proposition is satisfied if $f(z) = O(y^{-\epsilon})$ as $y \to \infty$ for some $\epsilon > 0$, for then (2.8) implies that the scalar product (2.9) converges absolutely in the strip $-\epsilon < \mathrm{Re}(s) < 1+\epsilon$.

One of the main applications of Proposition (**??**) is the one obtained by choosing $f(z) = |f_j(z)|^2$, where f_j is a Maass eigenform. (This was the original application made by Ranking [5] and Selberg [6], except that they were looking at holomorphic cusp forms.) From (1.4) we find that the constant term of f is given by

$$A_\circ(f;y) = y \sum_{n \neq 0} |a_j(n)|^2 K_{ir_j}(2\pi|n|y)^2$$

(notice that $K_{ir_j}(2\pi|n|y)$ is real by (1.3), since r_j is either real or pure imaginary). Hence (2.10) gives

$$\begin{aligned} \int_{\Gamma/\mathsf{H}} |f_j(z)|^2 E(z,s)dz &= \int_0^\infty y^{s-1} \sum_{n\neq 0} |a_j(n)|^2 K_{ir_j}(2\pi|n|y)^2 dy \\ &= \sum_{n\neq 0} \frac{|a_j(n)|^2}{|n|^s} \int_0^\infty y^{s-1} K_{ir_j}(2\pi y)^2 dy \qquad (2.11) \\ &= R_{f_j}(s) \qquad (\mathrm{Re}(s) > 1) \end{aligned}$$

(the integral is evaluated in [ET 6.8 (45)] and equals the gamma factor in (1.7)). The analytic properties of $R_{f_j}(s)$ given in §1 (meromorphic continuation, position of poles, residue formula (1.10), functional equation (1.11)) follow from (2.11) and the corresponding properties of $E(z,s)$. □

Spectral decomposition. We now give a rough indication, ignoring an-
316 alytic problems, of how the Rankin-Selberg method implies the spec-
tral decomposition formula for $L^2(\Gamma/\mathsf{H})$. This formula states that any

$f \in L^2(\Gamma/\mathsf{H})$ has an expansion

$$f(z) = \sum_{j=0}^{\infty} \frac{(f, f_j)}{(f_j, f_j)} f_j(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(f, E\left(., \frac{1}{2} + ir\right)\right) E\left(z, \frac{1}{2} + ir\right) dr, \tag{2.12}$$

where $\{f_j\}_{j \geq 1}$ is an orthogonal basis for $L^2_\circ(\Gamma/\mathsf{H})$ and $\{f_\circ\}$ for the space of constant functions (we will choose $f_j (j \geq 1)$ to be the normalized Maass eigenforms and $f_\circ(z) \equiv 1$). We prove it under the assumption that f is of sufficiently rapid decay, say $f(z) = O(y^{-\epsilon})$ with $\epsilon > 0$. Let $\Psi(s)$ be the scalar product (2.9). Proposition 2 shows that $\Psi(s)$ is a meromorphic function of s, is regular in $0 < \mathrm{Re}(s) < 1 + \epsilon$ except for a simple pole at $s = 1$ with

$$\mathrm{res}_{s=1} \Psi(s) = \frac{3}{\pi} \int_{\Gamma/\mathsf{H}} f(z) dz = \frac{(f, f_\circ)}{(f_\circ, f_\circ)} f_\circ, \tag{2.13}$$

and satisfies the functional equation

$$\Psi(s) = \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \Psi(1-s). \tag{2.14}$$

On the other hand, (2.10) says that $\Psi(s)$ is the Mellin transform of $\frac{1}{y} A_\circ(f; y)$, so by the Mellin inversion formula

$$A_\circ(f; y) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Psi(s) y^{1-s} ds \quad (1 < C < 1 + \epsilon).$$

Moving the path of integration from $\mathrm{Re}(s) = C$ to $\mathrm{Re}(s) = \frac{1}{2}$ and using (2.13) and (2.14) we find

$$\begin{aligned} A_\circ(f; y) &= \frac{f, f_\circ}{f_\circ, f_\circ} f_\circ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\frac{1}{2} - ir) y^{\frac{1}{2}+ir} \quad dr \\ &= \frac{(f, f_\circ)}{(f_\circ, f_\circ)} f_\circ + \frac{1}{4\pi} \int_{-\infty}^{\infty} \Psi(\frac{1}{2} - ir)(y^{\frac{1}{2}+ir} + \frac{\zeta^*(1-2ir)}{\zeta^*(1+2ir)} y^{\frac{1}{2}-ir}) dr. \end{aligned} \tag{2.15}$$

317 On the other hand, equation (2.6) implies that $y^{\frac{1}{2}+ir} + \dfrac{\zeta^*(1-2ir)}{\zeta^*(1+2ir)} y^{\frac{1}{2}-ir}$ is the constant term of $E(z, \frac{1}{2}+ir)$, so (2.15) tells us that the Γ-invariant function

$$\tilde{f}(z) = f(z) - \frac{(f, f_\circ)}{(f_\circ, f_\circ)} f_\circ(z) - \frac{1}{4\pi} \int_{-\infty}^{\infty} \Psi\left(\frac{1}{2} - ir\right) E\left(z, \frac{1}{2} + ir\right) dr$$

has zero constant term. It is also square integrable, because $f(z)$ is and the non-constant terms in the Fourier expansion of $E(z, \frac{1}{2}+ir)$ are exponentially small. Hence $\tilde{f} \in L^2_\circ(\Gamma/\mathsf{H})$, so $\tilde{f}(z) = \sum_{j=1}^{\infty} \dfrac{(\tilde{f}, f_j)}{(f_j, f_j)} f_j(z)$, and this proves (2.12) since $(\tilde{f}, f_j) = (f, f_j)$ for all $j \geq 1$.

Selberg transform. As in the introduction, let φ be a function on G of sufficiently rapid decay and T_φ the operator given by convolution with φ. Since we are interested only in functions on the upper half-plane $\mathsf{H} = G/K$ (where $K = SO(2)$ and the identification is given by $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} K \leftrightarrow \dfrac{ai+b}{ci+d}\right)$ we can assume that φ is left and right K-invariant. But the map

$$t : K\begin{pmatrix} a & b \\ c & d \end{pmatrix}K \longmapsto a^2 + b^2 + c^2 + d^2 - 2$$

gives an isomorphism between $K\backslash G/K$ and $[0, \infty)$ (Cartan decomposition), so we can think of φ as a map

$$\varphi : [0, \infty) \to \mathbf{C}.$$

An easy calculation shows that

$$t(g^{-1}g') = \frac{|z - z'|^2}{yy'} \qquad (g, g' \in G),$$

where $z, z' \in \mathsf{H}$ are the images of g and g'. Therefore T_φ acts on functions $f : \mathsf{H} \to \mathbf{C}$ by

$$T_\varphi f(z) = \int_H k(z, z') f(z') dz' \quad (z \in \mathsf{H}), \tag{2.16}$$

where 318

$$k(z,z') = \varphi\left(\frac{|z-z'|^2}{yy'}\right) \qquad (z,z' \in \mathsf{H}). \tag{2.17}$$

The growth condition we want to impose on φ is that

$$\varphi(x) = O\left(x^{\frac{1+A}{2}}\right) \qquad (x \to \infty) \tag{2.18}$$

for some $A > 1$; then (2.16) converges for any f in the vector space

$$V = \left\{f; \mathsf{H} \longrightarrow \mathbf{C} | f \text{ is continuous, } f(z) = O\left(y^{-\frac{1+A}{2}}\right)\right\}.$$

Because $k(z,z') = k(gz, gz')$ for any $g \in G$, the operator T_φ commutes with the action of G. A general argument (cf. [7], p. 55 or [4], Theorem 1.3.2) then shows that any eigenfunction of the Laplace operator is also an eigenfunction of T_φ. More precisely,

$$f \in V, \Delta f = -\left(\frac{1}{4} + r^2\right) f \Rightarrow T_\varphi f = h(r) f, \tag{2.19}$$

where $h(r)$, the *Selberg transform* of φ, is an even function of r, depending on φ but not on f. To compute it, we choose $f(z) = y^{\frac{1}{2}+ir}$, which satisfies the conditions is (2.19) if $r \in \mathbf{C}$ with $|\operatorname{Im}(r)| < \dfrac{A}{2}$. Then

$$T_\varphi f(z) = \int_0^\infty y'^{-\frac{3}{2}+ir} \int_{-\infty}^\infty \varphi\left(\frac{(x-x')^2 + (y-y')^2}{yy'}\right) dx'\, dy'.$$

Making the change of variables $x' = x + \sqrt{yy'}v$ in the inner integral gives

$$T_\varphi f(z) = \int_0^\infty y'^{-\frac{3}{2}+ir} \sqrt{yy'}\, Q\left(\frac{(y-y')^2}{yy'}\right) dy',$$

where the function Q is defined by 319

$$Q(w) = \int_{-\infty}^\infty \varphi(w+v^2) dv = \int_w^\infty \frac{\varphi(t)dt}{\sqrt{t-w}} \quad (w \geqslant 0). \tag{2.20}$$

The further change of variables $y' = ye^u$ then gives

$$T_\varphi f(z) = y^{\frac{1}{2}+ir} \int_{-\infty}^{\infty} e^{iru} Q(e^u - 2 + e^{-u}) du.$$

Hence, setting

$$g(u) = Q(e^u - 2 + e^{-u}) \qquad (u \in \mathbf{R}), \tag{2.21}$$

we have

$$h(r) = \int_{-\infty}^{\infty} g(u) e^{iru} du \quad (r \in \mathbf{C}, |\operatorname{Im}(r)| < \frac{A}{2}). \tag{2.22}$$

Formulas (2.20)-(2.22) describe the Selberg transform (the notations Q, g, h, due to Selberg, are by now standard and we have retained them). The inverse transform is easily seen to be

$$\begin{cases} g(u) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} h(r) e^{iru} du, \\ Q(w) = g(2 \sinh^{-1} \frac{\sqrt{w}}{2}), \\ \varphi(x) = \dfrac{-1}{\pi} \displaystyle\int_{-\infty}^{\infty} Q'(x + v^2) dv. \end{cases} \tag{2.23}$$

We can also combine these three integrals, obtaining

$$\begin{aligned} \varphi(x) &= \frac{1}{2\pi^2} \int_{-\infty}^{\infty} r h(r) \int_{\cosh^{-1}(1+\frac{x}{2})}^{\infty} \frac{\sin ru}{\sqrt{e^u + e^{-u-2-x}}} du \, dr \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} P_{-\frac{1}{2}+ir}(1 + \frac{x}{2}) \quad r \tan h\pi r \; h(r) dr, \end{aligned} \tag{2.24}$$

where $P_\nu(z)(\nu \in \mathbf{C}, z \in \mathbf{C} - (-\infty, 1])$ denotes a Legendre function of the **320**
first kind. (For properties of Legendre functions we refer the reader to [EH], Chapter 3; in particular, the integral representation of $P_{-\frac{1}{2}+ir}$ just used follows from formulas 3.7 (4) and 3.3.1 (3) there.) The inversion formula of Mehler and Fock ([EH], p. 175) then gives

$$h(r) = 2\pi \int_0^\infty P_{\frac{1}{2}+ir}\left(1 + \frac{x}{2}\right)\varphi(x)dx \quad \left(|\operatorname{Im}(r)| < \frac{A}{2}\right). \tag{2.25}$$

From (2.20) - (2.23) we see easily that the conditions

$$\begin{aligned}
&\varphi(x) = O\left(x^{-\frac{1+A}{2}}\right),\\
&Q(w) = O(w^{-\frac{A}{2}}),\\
&g(u) = O(e^{-\frac{A}{2}|u|}),\\
&h(r) \text{ holomorphic in } |\operatorname{Im}(r)| < \frac{A}{2}
\end{aligned}$$

are equivalent; this also follows from (2.24) and (2.25) since $P_{-\frac{1}{2}+ir}(x)$ grows like $x^{-\frac{1}{2}+|\operatorname{Im}(r)|}$ as $x \to \infty$ [EH 3.9.2 (19), (20)]. Thus the growth condition (2.18) is equivalent to a holomorphy condition on h, while the condition that φ be smooth is equivalent to the requirement that h be of rapid decay.

Selberg kernel function. Now suppose that the function f in (2.16) is Γ-invariant. Then $T_\varphi f$ is also Γ-invariant and clearly

$$T_\varphi f(z) = \int_{\Gamma/\mathrm{H}} K(z, z')f(z')dz' \tag{2.26}$$

with

$$K(z, z') = \sum_{\gamma\in\Gamma} k(z, \gamma z'), \tag{2.27}$$

i.e. the action of T_φ on Γ-invariant functions is given by the kernel function (2.27). We claim that

$$K(z, z') = O\left(y'^{\frac{1-A}{2}}\right) \quad (z \text{ fixed}, y' \longrightarrow \infty)$$

if φ satisfies (2.18). To see this, write

$$K(z,z') = \sum_{n\in Z} k(z+n,z') + \sum_{\substack{\gamma\in\Gamma_\infty/\Gamma \\ \gamma\notin\Gamma_\infty}} \sum_{n\in Z} k(z+n,\gamma z').$$

321 The first term is easily seen to be $O(y'^{\frac{1-A}{2}})$. In the second term, $\mathrm{Im}(\gamma z')$ is uniformly small as $y'\to\infty$ and from this one easily sees that the inner sum is uniformly $O(\mathrm{Im}(\gamma z')^{\frac{A+1}{2}})$. Therefore the second term is

$$O\left(\sum_{\substack{\gamma\in\Gamma_\infty/\Gamma \\ \gamma\notin\Gamma_\infty}} \mathrm{Im}(\gamma' z)^{\frac{A+1}{2}}\right) = O\left(E\left(z', \frac{A+1}{2}\right) - y'^{\frac{A+1}{2}}\right)$$

which by (2.6) is $O(y'^{\frac{1-A}{2}})$.

From (320) is follows that $K(z,z')$ is in $L^2(\Gamma/\mathsf{H})$ with respect to each variable separately and that the scalar product $(K(\cdot,z'),E(\cdot,s))$ converges for $\dfrac{1-A}{2} < \mathrm{Re}(s) < \dfrac{1+A}{2}$. Using (2.26) and (2.19) we find

$$(K(\cdot,z'),f_j) = h(r_j)f_j\bar{(}z') \quad (j\geq 0), \tag{2.29}$$

where r_j is given by (1.2) for $j\geq 1$ and $r_\circ = \dfrac{i}{2}$, and similarly

$$\left(K(\cdot,z'),E\left(\cdot,\frac{1}{2}+ir\right)\right) = h(r)E\left(z',\frac{1}{2}-ir\right)$$

since $\Delta E(z,\frac{1}{2}+ir) = -\left(\frac{1}{4}+r^2\right)E(z,\frac{1}{2}+ir)$. Therefore the spectral decomposition formula (2.12) applied to $K(\cdot,z')$ gives

$$K(z,z') = \sum_{j=0}^{\infty} \frac{h(r_j)}{(f_j,f_j)} f_j(z)\overline{f_j(z')} + \frac{1}{4\pi}\int_{-\infty}^{\infty} E\left(z,\frac{1}{2}+ir\right)E\left(z',\frac{1}{2}-ir\right)h(r)dr.$$

We restate this formula as

Proposition 3: *Let $h(r)$ be a function satisfying* (1.14) *and set*

$$K_\circ(z,z') = \sum_{j=1}^{\infty} \frac{h(r_j)}{(f_j,f_j)} f_j(z)\overline{f_j(z')} \quad (z,z' \in H), \tag{2.30}$$

where $\{f_j\}$ is an orthogonal basis of $L^2_\circ(\Gamma/\mathsf{H})$ satisfying (1.2)*. Let*

$$k(z,z')(z,z' \in \mathsf{H})$$

be the function defined by (2.17)*, where φ is given by* (2.23) *or* (2.24)*. Then*

$$K_\circ(z,z') = \sum_{\gamma\in\Gamma} k(z,\gamma z') - \frac{3}{\pi} h\left(\frac{i}{2}\right) \tag{2.31}$$

$$- \frac{1}{4\pi} \int_{-\infty}^{\infty} E\left(z, \frac{1}{2} + ir\right) E\left(z', \frac{1}{2} - ir\right) h(r) dr.$$

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We remark that (2.31) can be proved directly, without recourse to the spectral decomposition formula (2.12): Using the formulas for the Selberg transform and Mellin inversion, one can check directly that the expression on the right-hand side of (2.31) has constant term zero with respect to both variables and hence (using the estimate (320)) is a cusp form; equation (2.29) then implies the desired identity. We leave the details as an exercise for the reader.

3 Computation of $I(s)$ for $\Re(s) > 1$.

Let $h(r)$ be a function satisfying (1.14) and define

$$I(s) = \int_{\Gamma\mathsf{H}} K_\circ(z,z) E(z,s) dz, \tag{3.1}$$

where $K_\circ(z,z')$ is defined by (2.30). Since $K_\circ(z,z)$ is of rapid decay, the integral converges for all $s(\neq 1)$, and from (2.11) we have

$$I(s) = \sum_{j=1}^{\infty} \frac{h(r_j)}{(f_j,f_j)} R_{f_j}(s). \tag{3.2}$$

The object of this section is to compute $I(s)$ for $1 < \mathrm{Re}(s) < A$.

By the Rankin-Selberg method (eq. (2.10)) we have

$$I(s) = \int_0^\infty \mathscr{K}(y) y^{s-2} dy \quad (\mathrm{Re}(s) > 1), \tag{3.3}$$

where $\mathscr{K}(y)$ is the constant term of $K_\circ(z, z)$, which we will compute using Proposition (3) above. From (2.6) we find that the constant term of $E(z, \frac{1}{2} + ir)E(z, \frac{1}{2} - ir)$ equals

$$\left[y^{\frac{1}{2}+ir} + \frac{\zeta^*(1-2ir)}{\zeta^*(1+2ir)} y^{\frac{1}{2}-ir}\right] \left[y^{\frac{1}{2}-ir} + \frac{\zeta^*(1+2ir)}{\zeta^*(1-2ir)} y^{\frac{1}{2}+ir}\right]$$
$$+ \frac{8y}{\zeta^*(1+2ir)\zeta^*(1-2ir)} \sum_{n=1}^\infty \tau_{ir}(n)^2 K_{ir}(2\pi n y)^2.$$

323 Of the four terms obtained by multiplying the expressions in square brackets, two are obtained from the other two by replacing r by $-r$ and hence will give the same contribution when integrated against the even function $h(r)$. As to the first term in (2.31), we separate the terms with $\gamma \in \Gamma_\infty$ and $\gamma \notin \Gamma_\infty$; the former are their own constant terms since $k(z, z+n) = \varphi\left(\frac{n^2}{y^2}\right)$ is independent of x. We thus obtain the decomposition

$$\mathscr{K}(y) = \int_0^1 K_\circ(x+iy, x+iy) dx = \sum_{i=1}^4 \mathscr{K}_i(y)$$

with

$$\mathscr{K}_1(y) = \int_0^1 \cdot \sum_{\substack{\gamma \in \Gamma \\ \gamma \notin \Gamma_\infty}}^\infty k(x+iy,\ \gamma(x+iy)) dx,$$

$$\mathscr{K}_2(y) = \sum_{n=-\infty}^\infty \varphi\left(\frac{n^2}{y^2}\right) - \frac{y}{2\pi} \int_{-\infty}^\infty h(r) dr,$$

$$\mathscr{K}_3(y) = -\frac{y}{2\pi}\int_{-\infty}^{\infty} y^{2ir}\frac{\zeta^*(1+2ir)}{\zeta^*(1-2ir)}h(r)dr - \frac{3}{\pi}h(\frac{i}{2}),$$

$$\mathscr{K}_4(y) = -\frac{2y}{\pi}\int_{-\infty}^{\infty}\frac{1}{\zeta^*(1+2ir)\zeta^*(1+2ir)}\times$$

$$\times\left(\sum_{n=1}^{\infty}\tau_{ir}(n)^2K_{ir}(2\pi ny)^2\right)h(r)dr.$$

This gives a corresponding decomposition of $I(s)$ as $\sum_{i=1}^{4} I_i(s)$ with

$$I_i(s) = \int_0^{\infty}\mathscr{K}_i(y)y^{s-2}dy \quad (i = 1, \ldots, 4).$$

Theorem 2. *The integrals $I_i(s)$ converge for $1 < \mathrm{Re}(s) < A$ and are given in that region by the formulas*

$$I_1(s) = \sum_{t=-\infty}^{\infty}\frac{\zeta(s,t^2-4)}{\zeta(2s)}\int_{\mathsf{H}}\varphi\left(\frac{|z^2+1-t^2|4|^2|}{y^2}\right)y^s\,dz,$$

$$I_2(s) = -\frac{i\Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2})}{2^{s+1}\pi^{s+3/2}}\zeta(s)\int_{-\infty}^{\infty}\frac{\Gamma(\frac{s}{2}-ir)}{\Gamma(1-\frac{s}{2}-ir)}rh(r)dr,$$

$$I_3(s) = -\frac{\pi^{\frac{1}{2}}\Gamma(\frac{s}{2})}{2\Gamma(\frac{s+1}{2})}\frac{\zeta(s)}{\zeta(s+1)}h(\frac{is}{2}),$$

$$I_4(s) = -\frac{\pi^{1-s}\Gamma(\frac{s}{2})^2}{4\Gamma(s)}\frac{\zeta(s)^2}{\zeta(2s)}\times$$

$$\times\int_{-\infty}^{\infty}\frac{\Gamma(\frac{s}{2}+ir)\Gamma(\frac{s}{2}-ir)}{\Gamma(\frac{1}{2}+ir)\Gamma(\frac{1}{2}-ir)}\frac{\zeta(s+2ir)\zeta(s-2ir)}{\zeta(1+2ir)\zeta(1-2ir)}h(r)dr.$$

Proof. We begin with $I_4(s)$ since it is, despite appearances, the easiest of the four integrals. The very rapid decay of the Bessel functions allows us to interchange the order of the integrations and summation, obtaining

$$I_4(s) = -\frac{2}{\pi}\left(\sum_{n=1}^{\infty}\frac{\tau_{ir}(n)^2}{n^s}\right)\left(\int_0^{\infty} y^{s-1}K_{ir}(2\pi y)^2 dy\right)\times \\ \times\frac{h(r)}{\zeta^*(1+2ir)\zeta^*(1-2ir)}dr.$$

The first expression in parentheses equals $\dfrac{\zeta(s)^2}{\zeta(2s)}\zeta(s+2ir)\zeta(s-2ir)$ for $\mathrm{Re}(s) > 1$, as one checks by expanding the Dirichlet series as an Euler product. The second expression in parentheses equals

$$\frac{1}{8\pi^s}\frac{\Gamma\left(\frac{s}{2}\right)^2}{\Gamma(s)}\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)$$

(this is the same integral as was used in (2.11)). Putting this together we obtain the formula for $I_4(s)$ given in the theorem; it is valid for $\mathrm{Re}(s) > 1$. (The integral converges for all s with $\mathrm{Re}(s) \neq 0, 1$, as one sees by using Stirling's formula and standard estimates of $\zeta(s)$ and
325 $\zeta(1+it)^{-1}$ as well as the fact that $h(r)$ is of rapid decay.) Since the gamma factors in the formula are exactly those corresponding to the zeta-functions occurring, we can write the result in the nicer form

$$I_4(s) = -\frac{1}{4\pi}\frac{\zeta^*(s)^2}{\zeta^*(2s)}\times \\ \times\int_{-\infty}^{\infty}\frac{\zeta^*(s+2ir)\zeta^*(s-2ir)}{\zeta^*(1+2ir)\zeta^*(1-2ir)}h(r)dr \quad (\mathrm{Re}(s) > 1). \tag{3.4}$$

The integral I_3 is also quite easy to compute. Since $\zeta^*(1-2ir)$ is nonzero for $\mathrm{Im}(r) \geq 0$ and since the poles of $\zeta^*(1+2ir)$ and $\zeta^*(1-2ir)$ at $r = 0$ cancel, the integrand in $\mathscr{K}_3(y)$ is holomorphic in $0 \leq \mathrm{Im}(r) < \dfrac{A}{2}$

except for a simple pole of residue

$$\frac{1}{\zeta^*(2)} y^{-1} h\left(\frac{i}{2}\right) \operatorname{res}_{r=\frac{i}{2}} (\zeta^*(1+2ir)) = \frac{3i}{\pi y} h\left(\frac{i}{2}\right)$$

at $\frac{i}{2}$. Hence we can move the path of integration to $\operatorname{Im}(r) = \frac{C}{2}(1 < C < A)$, obtaining

$$\mathscr{K}_3(y) = \frac{iy}{4\pi} \int_{C-i\infty}^{C+i\infty} y^{-s} \frac{\zeta^*(s)}{\zeta^*(s+1)} h\left(\frac{is}{2}\right) ds \quad (1 < C < A).$$

The Mellin inversion formula then gives

$$I_3(s) = -\frac{1}{2} \frac{\zeta^*(s)}{\zeta^*(s+1)} h\left(\frac{is}{2}\right) \quad (1 < \operatorname{Re}(s) < A), \tag{3.5}$$

in agreement with the formula in Theorem (2).

We now turn to $I_2(s)$, which is somewhat harder. From (2.23) and (2.20) we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) dr = g(0) = Q(0) = \int_{-\infty}^{\infty} \varphi(v^2) dv = \frac{1}{y} \int_{-\infty}^{\infty} \varphi\left(\frac{u^2}{y^2}\right) du,$$

so

$$\mathscr{K}_2(y) = \sum_{n=-\infty}^{\infty} \varphi\left(\frac{n^2}{y^2}\right) - \int_{-\infty}^{\infty} \varphi\left(\frac{u^2}{y^2}\right) du.$$

By the Poisson summation formula this equals **326**

$$\sum_{n \neq 0} \int_{-\infty}^{\infty} \varphi\left(\frac{u^2}{y^2}\right) e^{2\pi i n u} du = 2y \sum_{n=1}^{\infty} \psi(ny),$$

where

$$\psi(y) = \int_{-\infty}^{\infty} \varphi(u^2) e^{2\pi i u y} du. \tag{3.6}$$

Since φ is smooth, ψ is of rapid decay, so we may interchange summation and integration to get

$$I_2(s) = 2\sum_{n=1}^{\infty}\int_0^{\infty}\psi(ny)y^{s-1}dy = 2\zeta(s)\int_0^{\infty}\psi(y)y^{s-1}dy(\mathrm{Re}(s) > 1). \quad (3.7)$$

To calculate the integral we begin by substituting the third equation of (2.23) into (3.6). This gives

$$\psi(y) = -\frac{1}{\pi}\int_{-\infty}^{\infty}e^{2\pi iuy}\int_{-\infty}^{\infty}Q'(u^2+v^2)dv\,du.$$

Changing to polar coordinates $u + iv = re^{i\theta}$ and using the standard integral representation

$$\cdot J_\circ(x) = \frac{1}{2\pi}\int_0^{2\pi}e^{ix\cos\theta}d\theta$$

of the Bessel function of order 0 [GR 3.915.2] we find

$$\psi(y) = -2\int_0^{\infty}J_\circ(2\pi yr)Q'(r^2)r\,dr$$

or, making the substitution $r = 2\sinh\dfrac{u}{2}$ and using (2.21),

$$\psi(y) = -\int_0^{\infty}J_\circ(4\pi y\sinh\frac{u}{2})g'(u)du.$$

Using the formula

$$\int_0^{\infty}J_\circ(2ay)y^{s-1}dy = \frac{\Gamma(\frac{s}{2})}{2a^s\Gamma(1-\frac{s}{2})}\left(0 < \mathrm{Re}(s) < \frac{3}{2}, a > 0\right)$$

327 [ET 6.8 (1)] we find

$$\int_{-\infty}^{\infty} \psi(y) y^{s-1} dy = -\frac{(2\pi)^{-s}\Gamma(\frac{s}{2})}{2\Gamma(1-\frac{s}{2})} \int_{-\infty}^{\infty} \left(\sinh\frac{u}{2}\right)^{-s} g'(u) du \qquad (3.8)$$

$$\left(0 < \mathrm{Re}(s) < \frac{3}{2}\right)$$

(the integral converges at ∞ because $g'(u) = O\left(e^{-\frac{A}{2}|u|}\right)$ and at 0 because $g'(u)$ is an odd function and hence $O(u)$). Substituting

$$g'(u) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} rh(r) \sin ru \, dr$$

and using the Fourier sine transform formula

$$\int_0^{\infty} \frac{\sin ru}{\left(\sin h\frac{u}{2}\right)^s} du = -2^{s-1} i\Gamma(1-s) \left\{ \frac{\Gamma\left(\frac{s}{2} - ir\right)}{\Gamma\left(1 - \frac{s}{2} - ir\right)} - \frac{\Gamma\left(\frac{s}{2} + ir\right)}{\Gamma\left(1 - \frac{s}{2} + ir\right)} \right\}$$

([ET 2.9 (30)]; the conditions for validity are misstated there) gives

$$\int_0^{\infty} \psi(y) y^{s-1} dy = -\frac{i\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{2^{s+2}\pi^{s+3/2}} \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{s}{2} - ir\right)}{\Gamma\left(1 - \frac{s}{2} - ir\right)} rh(r) dr,$$

where we have used the fact that $h(r)$ is an even function, and substituting this into (3.7) we obtain the formula stated in the theorem. Since the integral converges for all s with positive real part, the formula is valid for all s with $\mathrm{Re}(s) > 1$ (not just $1 < \mathrm{Re}(s)\frac{3}{2}$); we can use the elementary identity

$$\frac{r}{2\pi i} \left(\frac{\Gamma\left(\frac{s}{2} - ir\right)}{\Gamma\left(1 - \frac{s}{2} - ir\right)} - \frac{\Gamma\left(\frac{s}{2} + ir\right)}{\Gamma\left(1 - \frac{s}{2} + ir\right)} \right) = \frac{\Gamma\left(\frac{s}{2} + ir\right)\Gamma\left(\frac{s}{2} - ir\right)}{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\Gamma(ir)\Gamma(-ir)}$$

to write it in the more elegant form 328

$$I_2(s) = \frac{\zeta^*(s)}{(4\pi)^{\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)} \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)}{\Gamma(ir)\Gamma(-ir)} h(r)dr \tag{3.9}$$

$(\mathrm{Re}(s) > 1)$.

The proof of (3.9) was rather complicated and required introducing the extraneous function $J_\circ(x)$. We indicate a more natural and somewhat simpler derivation which, however, would require more work to justify since it involves non-absolutely convergent integrals. Interchange the order of integration in

$$\int_0^\infty \psi(y) y^{s-1} dy = \int_0^\infty \int_{-\infty}^\infty \varphi(u^2) \cos 2\pi uy du y^{s-1} dy.$$

Then the inner integral $\int_0^\infty y^{s-1} \cos 2\pi uy dy$ converges (conditionally) for $0 < \mathrm{Re}(s) < 1$ (thus in a region of validity disjoint from that of (3.7)!) and equals $(2\pi|u|)^{-s}\Gamma(s) \cos \dfrac{\pi s}{2}$ there [ET 6.5 (21)]. Using (2.24) we then find

$$\int_0^\infty \psi(y) y^{s-1} dy =$$

$$= \frac{\Gamma(s) \cos \frac{\pi s}{2}}{2(2\pi)^{s+1}} \int_0^\infty x^{-\frac{s+1}{2}} \int_{-\infty}^\infty P_{-\frac{1}{2}+ir}\left(1+\frac{x}{2}\right) h(r) r \tanh \pi r dr\, dx$$

for $0 < \mathrm{Re}(s) < 1$. Interchanging the order of integration again and using the formula

$$\int_0^\infty x^{-\frac{s+1}{2}} P_{-\frac{1}{2}+ir}\left(1+\frac{x}{2}\right) dx =$$

$$= 2^{1-s} \frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1}{2}+ir\right)\Gamma\left(\frac{1}{2}-ir\right)} \quad (0 < \mathrm{Re}(s) < 1)$$

329 [GR 7.134] we find

$$\int_0^\infty \psi(y) y^{s-1} dy =$$

$$\frac{2^{-s-2}\Gamma\left(\frac{s}{2}\right)}{\pi^{s+\frac{3}{2}}\Gamma\left(\frac{1+s}{2}\right)} \int_{-\infty}^{\infty} \Gamma\left(\frac{s}{2} - ir\right)\Gamma\left(\frac{s}{2} + ir\right) h(r) r \sin h\pi r dr,$$

and this now holds whenever $\mathrm{Re}(s) > 0$ (not just $0 < \mathrm{Re}(s) < 1$) since both sides are holomorphic in that range. Substituting into (3.7) again gives (3.9).

To complete the proof of Theorem 2 we must still compute $I_1(s)$ i.e. the contribution from the main term $\sum_{\gamma \notin \Gamma_\infty} k(z, \gamma z)$ of $K_\circ(z, z)$. For each $\gamma \in \Gamma$ denote by $[\gamma]$ the conjugacy class of Γ in Γ. Its elements are of the form $\sigma^{-1}\Gamma\sigma$ where $\sigma \in \Gamma$ is well-defined up to left multiplication with an element of the stabilizer Γ_γ of γ in Γ. Hence

$$\sum_{\substack{\gamma \in \Gamma \\ \gamma \notin \Gamma_\infty}} k(z, \gamma z) = {\sum_{[\gamma]}}' \sum_{\substack{\sigma \in \Gamma_\gamma / \Gamma \\ \sigma^{-1}\Gamma\sigma \notin \Gamma_\infty}} k(z, \sigma^{-1}\gamma\sigma z),$$

where ${\sum_{[\gamma]}}'$ denotes a summation over all non-trivial conjugacy classes (each such class contains at least one element $\notin \Gamma_\infty$) and we have chosen a representative γ for each class. Multiplying σ on the right by an element $\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$ does not affect the condition $\sigma^{-1}\gamma\sigma \notin \Gamma_\infty$ and replaces $k(z, \sigma^{-1}\gamma\sigma z)$ by $k(z + n, \sigma^{-1}\gamma\sigma(z + n))$. Hence

$$\sum_{\substack{\gamma \in \Gamma \\ \gamma \notin \Gamma_\infty}} k(z, \gamma z) = {\sum_{[\gamma]}}' \sum_{\substack{\sigma \in \Gamma_\gamma / \Gamma / \gamma_\infty \\ \sigma^{-1}\gamma\sigma \notin \Gamma_\infty}} \sum_{n=-\infty}^{\infty} k(z + n, \sigma^{-1}\gamma\sigma(z + n))$$

(for this one has to check that $\sigma^{-1}\Gamma_\gamma\sigma \cap \Gamma_\infty = \{1\}$, but this follows easily from $\sigma^{-1}\gamma\sigma \notin \Gamma_\infty$ and the fact that the centralizer of any non-trivial element of Γ_∞ is $Gamma_\infty$). Since the constant term in the Fourier expansion of a sum $\sum\limits_{n=-\infty}^{\infty} f(x+n)$ is the integral $\int\limits_{-\infty}^{\infty} f(x)\,dx$, we find

$$\mathscr{K}_1(y) = \sum_{[\gamma]}{}' \sum_{\substack{\sigma\in\Gamma_\gamma\backslash\Gamma/\Gamma_\infty \\ \sigma^{-1}\gamma\sigma\notin\Gamma_\infty}} \int_{-\infty}^{\infty} k(x+iy, \sigma^{-1}\gamma\sigma(x+iy))dx$$

330 and hence

$$I_1(s) = \sum_{[\gamma]}{}' \sum_{\substack{\sigma\in\Gamma_\gamma\backslash\Gamma/\Gamma_\infty \\ \sigma^{-1}\gamma\sigma\notin\Gamma_\infty}} \int_{\mathsf{H}} k(z, \sigma^{-1}\gamma\sigma z) y^s\, dz.$$

Now for any element $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $\tau \notin \Gamma_\infty$ (i.e. $c \neq 0$) we have

$$\int_{\mathsf{H}} k(z, \tau z) y^s dz = \frac{1}{|c|^s} V(s,t) \qquad (t = tr(\tau))$$

where

$$V(s,t) = \int_{\mathsf{H}} \varphi\left(\frac{|z^2+1-t^2/4|^2}{y^2}\right) y^s dz. \tag{3.10}$$

(To prove this, substitute (2.17) for $k(z,z')$ and make the change of variable $z \to \dfrac{z}{|c|} + \dfrac{a-d}{2c}$.) Hence

$$I_1(s) = \sum_{t=-\infty}^{\infty} \left(\frac{1}{2} \sum_{\substack{[[\gamma]] \\ tr\gamma = t}}{}' \sum_{\substack{\sigma\in\Gamma_\gamma\backslash\Gamma/\Gamma_\infty \\ \sigma^{-1}\gamma\sigma\notin\Gamma_\infty}} \frac{1}{|c(\sigma^{-1}\gamma\sigma)|^s} \right) V(s,t),$$

where $\sum\limits_{[[\gamma]]}{}'$ denotes a sum over conjugacy classes in $SL_2(\mathbf{Z}) - \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $c(\sigma^{-1}\gamma\sigma)$ the element in the lower left-hand corner of $\sigma^{-1}\gamma\sigma$ (we must work in $SL_2(\mathbf{Z})$ rather than Γ in order to have a well-defined trace; notice that $\Gamma_\gamma \subset \Gamma$ and $\sigma^{-1}\gamma\sigma \in SL_2(\mathbf{Z})$ make sense for $\gamma \in SL_2(\mathbf{Z})$, $\sigma \in \Gamma$). Since $V(s,t) = V(s,-t)$, we have

$$I_1(s) = \sum_{t=-\infty}^{\infty} \left(\sideset{}{'}\sum_{\substack{[[\gamma]] \\ tr\gamma = t}} \sum_{\substack{\sigma\in\Gamma_\gamma\backslash\Gamma/\Gamma_\infty \\ c(\sigma^{-1}\gamma\sigma)>0}} \frac{1}{c(\sigma^{-1}\gamma\sigma)^s} \right) V(s,t).$$

There is a (1:1) correspondence between conjugacy classes $[[\gamma]]$ of trace t and $SL_2(\mathbf{Z})$-equivalence classes of binary quadratic forms of dis crim-inant $t^2 - 4$ given by 331

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow Q(m,n) = cm^2 + (d-a)mn - bn^2.$$

There is also a bijection between Γ/Γ_∞ and the set of relatively prime pass of integers $\pm(m,n) \in \mathbf{Z}^2/\{\pm 1\}$ given by mapping an element $\sigma \in \Gamma/\Gamma_\infty$ to its first column, and under this bijection we have $c(\sigma^{-1}\gamma\sigma) = Q(m,n)$ and $\Gamma_\gamma = \mathrm{Aut}(Q)/\{\pm 1\}$. Hence

$$\sum_{\substack{[[\gamma]] \\ tr\gamma = t}} \sum_{\sigma\in\Gamma_\gamma\backslash\Gamma/\Gamma_\infty c(\sigma^{-1}\gamma\sigma)>0} \frac{1}{c(\sigma^{-1}\gamma\sigma)^s} = \frac{\zeta(s,t^2-4)}{\zeta(2s)}$$

where $\zeta(s,t^2-4)$ is defined by (1.12). To complete the proof of the formula

$$I_1(s) = \frac{1}{\zeta(2s)} \sum_{t=-\infty}^{\infty} \zeta(s,t^2-4)V(s,t) \quad (1 < \mathrm{Re}(s) < A) \tag{3.11}$$

given in the theorem, it remains only to verify the convergence and justify the various interchanges of summation and integration made. Since the integrals I_2, I_3 and I_4 have already been shown to be convergent for $1 < \mathrm{Re}(s) < A$ (eqs. (3.4), (3.5), (3.7)) and the function $\mathscr{K}(y)$ is of rapid decay at infinity, the integral $I_1(s)$ is certainly convergent in the same range. By choosing s real and φ positive, we see that this convergence is absolute, and this gives an *a posteriori* proof of the convergence of (3.11) in the range stated and of the validity of the steps leading up to its proof. □

4 Analytic continuation of $I(s)$,

In this section we will give the analytic continuation of $I(s)$ to the critical strip $0 < \mathrm{Re}(s) < 1$ and compute the residue at $s = 1$ (Selberg trace formula). We will also want to study the functional equations of the various terms in the formula for $I(s)$. From the definition (3.1) of $I(s)$ it is clear that $I(s)$ is holomorphic for all $s \neq 1$ and satisfies the functional equation $I^*(s) = I^*(1-s)$ where

$$I^*(s) = \pi^{-s}\Gamma(s)\zeta(2s)I(s) = \int_{\Gamma\backslash \mathsf{H}} K_0(z,z)E^*(z,s)dz.$$

On the other hand, Theorem 2 says that $I^*(s)$ is the sum of the functions

$$\pi^{-s}\Gamma(s)\zeta(s,t^2-4)V(s,t) \quad (t \in Z,\ t \neq \pm 2), \tag{4.1}$$

$$\pi^{-s}\Gamma(s)\zeta(s,0)[V(s,2)+V(s,-2)] + I_2^*(s), \tag{4.2}$$

$$I_3^*(s) + I_4^*(s) \tag{4.3}$$

332 for $1 < \sigma = \mathrm{Re}(s) < A$, where $I_i^*(s) = \zeta^*(2s)I_i(s)$. We will show that each of the functions (4.1)-(4.3) has a meromorphic continuation to the strip $1 - A < \sigma < A$ with poles at most at 0 and 1 and is invariant under $s \to 1 - s$.

We begin by performing one of the integrations in the double integral (3.10) to write $V(s,t)$ as a simple integral, thus obtaining the analytic continuation and functional equation of $V(s,t)$.

Proposition 4. *Let φ be a function satisfying* (2.18)*, $s \in \mathbf{C}$, $t \in bR$, $\Delta = t^2-4$. If $\Delta \neq 0$, the the integral* (3.10) *converges for $-A < \sigma < 1+A$ and is given by*

$$V(s,t) = 2\pi\left|\frac{\Delta}{4}\right|^{s/2}\int_1^\infty \varphi\Big(|\Delta|(u^2-1)\Big)P_{-s}(u)du$$

$$(-A < \sigma < 1+A) \tag{4.4}$$

if $\Delta < 0$ *and by*

$$V(s,t) = \frac{1}{2}\frac{\Gamma\left(\frac{s}{2}\right)^2}{\Gamma(s)}\Delta^{s/2}\times \tag{4.5}$$

$$\times \int_{-\infty}^{\infty} \varphi(\Delta(u^2+1))(1+u^2)^{\frac{1}{2}s} F\left(\frac{s}{2}, \frac{s}{2}; \frac{1}{2}; \frac{u^2}{u^2+1}\right) du (-A < \sigma < 1 + A)$$

if $\Delta > 0$*, where* $F(a, b; c; z)$ *and* $P_v(z)$ *denote hypergeometric and Legendre functions, respectively. In particular,* $V(s, t)$ *satisfies the functional equation*

$$\frac{\pi^{-s}\Gamma(s)}{\gamma(s,\Delta)} V(s,t) = \frac{\pi^{-1+s}\Gamma(1-s)}{\gamma(1-s,\Delta)} V(1-s,t)(\Delta \neq 0), \tag{4.6}$$

where

$$\gamma(s,\Delta) = \begin{cases} (2\pi)^{-s}|\Delta|^{s/2}\Gamma(s) & \text{if } \Delta < 0, \\ \pi^{-s}\Delta^{s/2}\Gamma\left(\dfrac{s}{2}\right)^2 & \text{if } \Delta > 0. \end{cases}$$

For $\Delta = 0$*,* $V(s, t)$ *converges for* $\frac{1}{2} < \sigma < 1 + A$ *and has a meromorphic* **333**
continuation to $0 < \mathrm{Re}(s) < 1 + A$ *given by*

$$V(s, \pm 2) = \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)} \int_0^{\infty} \varphi(u^2) u^{s-1} ds (0 < \sigma < 1 + A). \tag{4.7}$$

We observe that the functions $\zeta(s, \Delta)$ defined by (1.12) satisfy the functional equations

$$\gamma(s,\Delta)\zeta(s,\Delta) = \gamma(1-s,\Delta)\zeta(1-s,\Delta)$$

for $\Delta \neq 0$ ([10], Prop. 3, ii), p. 130), so (4.6) tells us that each of the functions (4.1) is invariant under $s \to 1 - s$.

Proof. We consider first the case $\Delta < 0$. Mapping the upper half-plane to the unit disc bu $z \to \dfrac{z - i\sqrt{|\Delta|\backslash 4}}{z + i\sqrt{|\Delta|\backslash 4}} = re^{i\theta}$, we find

$$V(s,t) = \int\int_{\mathsf{H}} \varphi\left(\frac{|z^2 - \Delta\backslash 4|^2}{y^2}\right) y^s dz$$

$$= 4|\frac{\Delta}{4}|^{s/2} \iint\limits_{\substack{0\leqslant r\leqslant 1\\ 0\leqslant\theta\leqslant 2\pi}} \varphi\left(\frac{4|\Delta|r^2}{(r^2-1)^2}\right)\left(\frac{1-r^2}{1-2r\cos\theta+r^2}\right)^s \frac{rdr\,d\theta}{(1-r^2)^2},$$

and this is equivalent to (4.4) because

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-\cdot r^2}{1-2r\cos\theta+r^2}\right)^s d\theta = P_{-s}\left(\frac{1+r^2}{1-r^2}\right)(0\leqslant r<1,\, s\in\mathbf{C})$$

[EH 3.7(6)]. The functional equation follows since $P_{-s}(z) = P_{s-1}(z)$. If $\Delta > 0$, then we transform the upper half-plane to itself by $z \to \dfrac{z-\sqrt{\Delta/4}}{z+\sqrt{\Delta/4}} = \xi + i\eta$, obtaining

$$V(s,t) = \Delta^{s/2}\int\int_{\mathsf{H}} \varphi\left(\Delta\frac{\xi^2+\eta^2}{\eta^2}\right)\frac{\eta^s}{|1-\xi-i\eta|^{2s}}\frac{d\xi d\eta}{\eta^2}$$

$$= \Delta^{s/2}\int_{-\infty}^{\infty}\frac{\varphi(\Delta(1+u^2))}{(1+u^2)^{s/2}}\int_0^{\infty}\frac{v^{s-1}dv}{\left(1-\frac{2u}{\sqrt{u^2+1}}v+v^2\right)^s}du$$

334 $(u = \xi/\eta,\, v = \sqrt{\xi^2+\eta^2})$. Substituting $v = e^x$ we find

$$\int_0^{\infty}\frac{v^{s-1}dv}{\left(1-\frac{2u}{\sqrt{u^2+1}}v+v^2\right)^s} = 2^{-s}\int_{-\infty}^{\infty}\frac{dx}{\left(\cosh x-\frac{u}{\sqrt{u^2+1}}\right)^s}$$

$$= \frac{\Gamma(s)\Gamma(\frac{1}{2})}{2^{2s-1}\Gamma(s+\frac{1}{2})}F\left(s,s;s+\frac{1}{2};\frac{1}{2}\left(1+\frac{u}{\sqrt{u^2+1}}\right)\right)\ (\mathrm{Re}(s)>0)$$

$$= \frac{\pi}{2^{2s-1}}\frac{\Gamma(s)}{\Gamma\left(\frac{s+1}{2}\right)^2}F\left(\frac{s}{2},\frac{s}{2};\frac{1}{2};\frac{u^2}{u^2+1}\right)$$

$$+ \frac{u}{\sqrt{u^2+1}}\frac{\pi}{2^{2s-2}}\frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)^2}F\left(\frac{s+1}{2},\frac{s+1}{2};\frac{3}{2};\frac{u^2}{u^2+1}\right)$$

[EH 2.12 (10), 2.1.5 (28)]. Since the second term is an odd function of u, we find the formula

$$V(s,t) = \Delta^{\frac{s}{2}} \frac{\pi}{2^{2s-1}} \frac{\Gamma(s)}{\Gamma\left(\frac{s+1}{2}\right)^2} \int_{-\infty}^{\infty} \frac{\varphi(\Delta(1+u^2))}{(1+u^2)^{s/2}} F\left(\frac{s}{2}, \frac{s}{2}; \frac{1}{2}; \frac{u^2}{u^2+1}\right) du,$$

which is equivalent to (4.5); the functional equation follows because

$$(1+u^2)^{-s/2} F\left(\frac{s}{2}, \frac{s}{2}; \frac{1}{2}; \frac{u^2}{u^2+1}\right) = F\left(\frac{s}{2}, \frac{1-s}{2}; \frac{1}{2}; -u^2\right) \quad [EH 2.1.4(22)].$$

Finally, if $\Delta = 0$ then the substitution $z \to -1/z$ gives

$$V(s,\pm 2) = \iint_{\mathsf{H}} \varphi\left(\frac{|z|^4}{y^2}\right) y^s dz = \iint_{\mathsf{H}} \varphi\left(\frac{1}{y^2}\right) \frac{y^2}{|z|^{2s}} dz;$$

making the substitution $u = y^{-1}$, $t = x/y$ and using **335**

$$\int_{-\infty}^{\infty} (1+t^2)^{-s} dt = \frac{\Gamma(\frac{1}{2})\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \quad \left(\mathrm{Re}(s) > \frac{1}{2}\right)$$

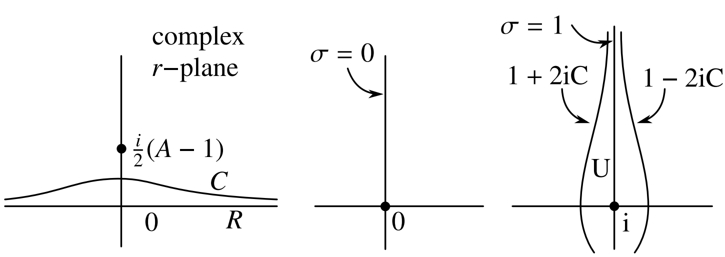
[GR 3.251.2] we obtain (4.7). Notice that, since φ is assumed to be smooth, the integral in (4.7) has a meromorphic continuation to $\sigma < A+1$ with (at most) simple poles at $s = 0, -2, -4, \ldots$; hence $V(s,t)$ can also be meromorphically continued to this range and has (at most) simple poles at $s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots$. This completes the proof of Proposition (4) except for the various assertions about convergence, which can be checked easily using the asymptotic properties of the Legendre and hypergeometric functions.

From (4.5) and (2.18) if follows that $V(s,t)$ grows like $t^{\sigma-1-A}$ as $t \in \infty$ with s fixed, $-A < \sigma < 1+A$. An easy calculation shows that $\zeta(s, t^2-4) = O(t^C)$ for any $C > \max(1-2\sigma, 1-\sigma, 0)$ as $t \to \infty$, and this implies that the sum (3.11) is absolutely convergent for $1-A < \sigma < A$. Thus $I_1(s)$ has a meromorphic continuation to $1-A < \sigma < A$ with (at most) a double pole at $s = 1$ (coming from the double pole of $\zeta(s,0) = \zeta(s)\zeta(2s-1)$) and simple poles at $s = \frac{1}{2}$ and $s = 0$. From (3.5) and our assumptions on $h(r)$ we see that $I_3(s)$ is meromorphic in $-A < \sigma < A$, the only pole in the half-plane $\sigma > 0$ being a simple one at $s = 1$. Thus to obtain a formula for $I(s)$ in the critical strip we must still give the analytic continuations of $I_2(s)$ and $I_4(s)$.

Let $J(s)$ denote the integral in (3.4). As already stated, this integral converges absolutely for all s with $\sigma \neq 0, 1$, because the integrand is of rapid decay as $|r| \to \infty$. However, $J(s)$ is not defined on the lines $\sigma = 0$ and $\sigma = 1$, because the path of integration passes through a pole of the integrand, so the functions defined by the integral in the three regions $\sigma < 0$, $0 < \sigma < 1$ and $\sigma > 1$ need not be (and are not) analytic continuations of one another. To obtain the analytic continuation of $J(s)$ (and hence of $I_4(s)$) to $0 < \sigma < 1$, we set

$$J_C(s) = \int_C \frac{\zeta^*(s+2ir)\zeta^*(s-2ir)}{\zeta^*(1+2ir)\zeta^*(1-2ir)} h(r)dr,$$

336 where C is a deformation of the real axis into the strip $0 < \mathrm{Im}(r) < \frac{1}{2}(A-1)$ which is sufficiently close to the real axis that all zeroes of the Riemann zeta-function lie to the left of $1+2iC$ and $\zeta(1+2ir)^{-1} = O(|r|^{\epsilon})$ for



$r \in C$ (see figure). The integral $J_C(s)$ converges for all $s \in C$ such that $\zeta^*(s+2ir)$ and $\zeta^*(s-2ir)$ remain finite for all $r \in C$, i.e. for $s \notin 1 \pm 2iC$, $\pm 2iC$. In particular, $J_C(s)$ is holomorphic in the region U bounded by $1+2iC$ and $1-2iC$. Clearly $J_C(s) = J(s)$ for s to the right of $1-2iC$, but for s in the right half of U we have

$$J(s) - J_C(s) = \pi \frac{\zeta^*(2s-1)}{\zeta^*(2-s)\zeta^*(s)} h\left(i\frac{s-1}{2}\right) \quad (s \in U, \mathrm{Re}(s) > 1)$$

because the integrand has a simple pole (at $r = \frac{1}{2}(s-1)$) with residue $\frac{1}{2i}\frac{\zeta^*(2s-1)}{\zeta^*(2-s)\zeta^*(s)} h(\frac{i}{2}(s-1))$ in the region enclosed by $\mathbf{R}$ and C. Similarly

$$J(s) - J_C(s) = -\pi \frac{\zeta^*(2s-1)}{\zeta^*(2-s)\zeta^*(s)} h\left(i\frac{s-1}{2}\right) (s \in U, \mathrm{Re}(s) < 1).$$

Therefore the function $J(s)$ in $0 < \sigma < 1$ is $2\pi \frac{\zeta^*(2s-1)}{\zeta^*(2-s)\zeta^*(s)} h\left(i\frac{s-1}{2}\right)$ less than the analytic continuation of the function defined by $J(s)$ for $\sigma > 1$. Together with (3.4) this shows that $I_4^*(s) = \zeta^*(2s)I_4(s)$ has an analytic continuation to $\sigma > 0$ given by

$$I_4^*(s) = \begin{cases} -\frac{1}{4\pi}\zeta^*(s)^2 J(s) \\ -\frac{1}{4\pi}\zeta^*(s)^2 J_C(s) - \frac{1}{4}\frac{\zeta^*(s)\zeta^*(2s-1)}{\zeta^*(s-1)} h\left(i\frac{s-1}{2}\right) & (s \in U), \\ -\frac{1}{4\pi}\zeta^*(s)^2 J(s) - \frac{1}{2}\frac{\zeta^*(s)\zeta^*(2s-1)}{\zeta^*(s-1)} h\left(i\frac{s-1}{2}\right) & (0 < \sigma < 1), \end{cases} \tag{4.8}$$

where we have used the functional equation $\zeta^*(s) = \zeta^*(1-s)$. Of course, **337**
we could use a similar argument to extend past the critical line $\sigma = 0$, but since it is obvious that $J(s) = J(1-s)$, we deduce from (4.8) that $I_4^*(s)$ satisfies the functional equation

$$I_4^*(1-s) = I_4^*(s) - \frac{1}{2}\frac{\zeta^*(s)\zeta^*(2s-1)}{\zeta^*(s-1)} h\left(i\frac{s-1}{2}\right) + \frac{1}{2}\frac{\zeta^*(s)\zeta^*(2s)}{\zeta^*(s+1)} h\left(i\frac{s}{2}\right),$$

and this gives the meromorphic continuation immediately. From (4.8) and (3.5) we find

$$I_3^*(s)+I_4^*(s)=-\frac{1}{4\pi}\zeta^*(s)^2J(s) \tag{4.9}$$
$$-\begin{cases}\dfrac{1}{2}\dfrac{\zeta^*(s)\zeta^*(2s)}{\zeta^*(s+1)}h\left(\dfrac{is}{2}\right) & (1<\sigma<A)\\ \dfrac{1}{2}\dfrac{\zeta^*(s)\zeta^*(2s)}{\zeta^*(s+1)}h\left(\dfrac{is}{2}\right)+\dfrac{1}{2}\dfrac{\zeta^*(s)\zeta^*(2s-1)}{\zeta^*(s-1)}h\left(i\dfrac{s-1}{2}\right) & (0<\sigma<1)\\ \dfrac{1}{2}\dfrac{\zeta^*(s)\zeta^*(2s-1)}{\zeta^*(s-1)}h\left(i\dfrac{s-1}{2}\right) & (1-A<\sigma<0),\end{cases}$$

which proves the invariance of (4.3) under $s\to 1-s$. Notice that the function ${}^{\zeta^*(s)\zeta^*(2s)}/_{\zeta^*(s+1)}h\left(\frac{is}{2}\right)\left(resp.\ \frac{\zeta^*(s)\zeta^*(2s-1)}{\zeta^*(s-1)}h\left(i-\frac{s-1}{2}\right)\right)$ has infini-
338 tely many poles in the half-plane $\sigma<0$ (resp. $\sigma>1$), but drops out of (4.9) before that half-plane is reached. In fact, it is clear from (4.8) and (4.9) that the function $I_3^*(s)+I_4^*(s)$ is holomorphic in $1-A<\sigma<A$ except for double poles at $s=0$ and $s=1$ (the simple poles at $s=\frac{1}{2}$ must cancel since $I_3^*(s)+I_4^*(s)$ is an even function of $s-\frac{1}{2}$).

It remains to treat the function (4.2). Using the formulas (2.23) for the Selberg transform we find

$$\begin{aligned}\int_0^\infty \varphi(u^2)u^{s-1}du &= -\frac{1}{u}\int_0^\infty\int_{-\infty}^\infty Q'(u^2+v^2)u^{s-1}dv\,du\\ &= -\frac{1}{\pi}\int_0^\infty\int_0^\pi Q'(r^2)(r\sin\theta)^{s-1}r\,drd\theta(\mathrm{Re}^{i\theta}=v+iu)\\ &= -\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma(\frac{1}{2})\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty Q'(r^2)r^s dr\\ &= -\frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty\left(2\sinh\frac{u}{2}\right)^{s-1}g'(u)du\end{aligned}$$

and hence, by (4.7),

$$\pi^{-s}\Gamma(s)\zeta(s,0)[V(s,2)+V(s,-2)] =$$
$$= -\frac{(4\pi)^{\frac{1}{2}(s-1)}}{\Gamma\left(\frac{s+1}{2}\right)}\zeta^*(s)\zeta^*(2s-1)\int_0^\infty \left(\sinh\frac{u}{2}\right)^{s-1} g'(u)du;$$

the integral converges for $-1 < \sigma < 1 + A$ and hence gives the analytic continuation of the left-hand side to this strip. On the other hand, formulas (3.7) and (3.8) give

$$I_2^*(s) = -\frac{(4\pi)^{-\frac{1}{2}s}}{\Gamma(1-\frac{s}{2})}\zeta^*(s)\zeta^*(2s)\int_0^\infty \left(\sinh\frac{u}{2}\right)^{-s} g'(u)du,$$

where now the integral converges for $-A < \sigma < 2$. This shows that **339**
the function (4.2) can be continued to the strip $-A < \sigma < 1 + A$ and is invariant under $s \to 1 - s$; equation (3.9) then gives the formula

$$\pi^{-s}\Gamma(s)\zeta(s,0)[V(s,2)+V(s,-2)] = I_2^*(1-s) \tag{4.10}$$
$$= \frac{\zeta^*(s)\zeta^*(2s-1)}{(4\pi)^{\frac{2-s}{2}}\Gamma\left(\frac{2-s}{2}\right)}\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma(ir)\quad\Gamma(-ir)}h(r)dr\ (0<\sigma<1)$$

in the critical strip. A similar discussion to that given for the integral $I_4^*(s)$ now shows that for $\sigma > 1$ we must add $\frac{\pi^{\frac{1}{2}(s-1)}}{\Gamma\left(\frac{s-1}{2}\right)}\zeta^*(s)\zeta^*(2s-1)h\left(i\frac{s-1}{2}\right)$ to the right-hand side of (4.10) and that near the line $\sigma = 1$ we have

$$\pi^{-s}\Gamma(s)\zeta(s,0)[V(s,2)+V(s,-2)]$$
$$= \frac{\zeta^*(s)\zeta^*(2s-1)}{(4\pi)^{\frac{2-s}{2}}\Gamma\left(\frac{2-s}{2}\right)}\int_C \frac{\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma(ir)\Gamma(-ir)}h(r)dr \tag{4.11}$$
$$+\frac{1}{2}\frac{\pi^{\frac{s-1}{2}}}{\Gamma\left(\frac{s-1}{2}\right)}\zeta^*(s)\zeta^*(2s-1)h\left(i\frac{s-1}{2}\right)\quad (s\in U).$$

Again the analytic continuation to $1-A<\sigma\leqslant 0$ follows using the functional equation.

We have thus proved the analytic continuability and functional equation of each of the functions (4.1) - (4.3) in the strip $1-A<\sigma<A$ and
340 given explicit formulas for these functions in each of the five regions $1-A<\sigma<0$, $1-U$, $0<\sigma<1$, U and $1<\sigma<A$ covering this strip. We and this section by using these formulas to compute the residue at $s=1$ of the functions in question.

From the development

$$\zeta^*(s)=\frac{1}{s-1}+\frac{1}{2}(\gamma-\log 4\pi)+O(s-1)\quad(s\to 1)$$

and (4.8) we find

$$\begin{aligned}I_4^*(s)=&-\frac{1}{4\pi}\left[\frac{1}{(s-1)^2}+\frac{\gamma-\log 4\pi}{s-1}+O(1)\right]\\&\times\left[\int\limits_C h(r)dr+(s-1)\int\limits_C z(r)h(r)dr+O(s-1)^2\right]\\&+\frac{1}{8}h(0)\frac{1}{s-1}+O(1)\end{aligned}$$

as $s\to 1$, where $z(r)=\dfrac{zeta^{*'}(1+2ir)}{\zeta^*(1+2ir)}+\dfrac{\zeta^{*'}(1-2ir)}{\zeta^*(1-2ir)}$. Since $z(r)$ is holomorphic for r near the real line (the poles of the two terms at $r=0$ cancel), we can replace C by $\mathbf{R}$ in the two integrals, obtaining

$$\begin{aligned}I_4^*(s)=&-\frac{\kappa}{(s-1)^2}+\Bigg(-\kappa(\gamma-\log 4\pi)+\frac{h(0)}{8}\\&-\frac{1}{4\pi}\int\limits_{-\infty}^{\infty}z(r)h(r)dr\Bigg)(s-1)^{-1}+O(1)\end{aligned}$$

as $s\to 1$, where $\kappa=\dfrac{1}{4\pi}\int\limits_{-\infty}^{\infty}h(r)dr$. From (3.5) we get

$$I_3^*(s)=-\frac{\zeta^*(s)\zeta^*(2s)}{2\zeta^*(s+1)}h\left(\frac{is}{2}\right)=-\frac{1}{2}\frac{h(\frac{i}{2})}{s-1}+O(1)\quad(s\to 1).$$

This takes care of the function (4.3). For (4.2) we use equations (4.11) and (3.9), obtaining (by an argument similar to the one just used for I_4^*) **341**

$$
\begin{aligned}
&\pi^{-s}\Gamma(s)\zeta(s,0)[V(s,2)+V(s-2)]\\
&=\left[\frac{1}{s-1}+\frac{1}{2}(\gamma-\log 4\pi)+O(s-1)\right]\\
&\quad\times\left[\frac{1}{2(s-1)}+\frac{1}{2}(\gamma-\log 4\pi)+O(s-1)\right]\\
&\quad\times\left[\frac{1}{2\pi}+\frac{1}{4\pi}\left(\log 4\pi+\frac{\Gamma'}{\Gamma}(\frac{1}{2})\right)(s-1)+O(s-1)^2\right]\\
&\quad\times\left[\int_C h(r)dr-\frac{s-1}{2}\int_C\left(\frac{\Gamma'}{\Gamma}(ir)+\frac{\Gamma'}{\Gamma}(-ir)\right)h(r)dr\right.\\
&\left.+O(s-1)^2\right]+\frac{h(0)}{8}(s-1)^{-1}+O(1)\\
&=\frac{\kappa}{(s-1)^2}+(\kappa(\gamma-\log 8\pi)\\
&-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+ir)h(r)dr+\frac{1}{8}h(0)\Big)(s-1)^{-1}+O(1)
\end{aligned}
$$

and

$$
I_2^*(s)=\left(\frac{1}{24}\int_{-\infty}^{\infty}h(r)r\tanh\pi r dr\right)(s-1)^{-1}+O(1).
$$

Finally, to compute the residue of (4.1) at $s=1$ we need the values of $V(1,t)$ and $\operatorname{res}_{s=1}\zeta(s,t^2-4)$ for $t\in\mathbf{Z}$, $t\neq\pm 2$. From (4.4) and (4.5) we find

$$
V(1,t)=\begin{cases}\dfrac{\pi}{2}\displaystyle\int_0^{\infty}\varphi(x)\frac{dx}{\sqrt{x+4-t^2}} & (|t|<2)\\[2ex] \dfrac{\pi}{2}\displaystyle\int_{t^2-4}^{\infty}\varphi(x)\frac{dx}{\sqrt{x+4-t^2}} & (|t|>2)\end{cases}
$$

(since $P_0(u) = 1$, $F(0,b;c;x) = 1$). Using the formulas (2.23) for the Selberg transform, we can express this in therms of $h(r)$, obtaining 342

$$V(1,t) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} \frac{e^{-2\alpha r}}{1+e^{-2\pi r}} h(r)dr & \left(|t| = 2\cos\alpha \leqslant 2,\ 0 \leqslant \alpha \leqslant \dfrac{\pi}{2}\right) \\ \dfrac{1}{4}\displaystyle\int_{-\infty}^{\infty} e^{2i\alpha r} h(r)dr & (|t| = 2\cosh\alpha \geqslant 2) \end{cases} \tag{4.12}$$

(we omit the calculation, which is not difficult, since in §5 we will give a general formula for $V(s,t)$ in terms of $h(r)$). As to $\zeta(s,D)$, we have

$$\operatorname{res}_{s=1}\zeta(s,D) = \begin{cases} \dfrac{2\pi}{\sqrt{|D|}} \displaystyle\sum_Q \frac{1}{|\operatorname{Aut}(Q)|} & (D<0) \\ \dfrac{1}{\sqrt{D}} \displaystyle\sum_Q \log \varepsilon_Q & (D>0), \end{cases} \tag{4.13}$$

where $\sum_Q$ and $\operatorname{Aut}(Q)$ have the same meaning as in (1.12) and, in the second formula, ε_Q is the fundamental unit for Q (i.e. the larger eigenvalue of M, where $M \in SL_2(\mathbf{Z})$ is a matrix with positive trace such that $\operatorname{Aut}(Q) = \{\pm M^n, n \in \mathbf{Z}\}$).

We have thus given the principal part of each of the functions (4.1) - (4.3) at the pole $s = 1$. Adding up the expressions obtained, we find that the terms in $(s-1)^{-2}$ cancel and that

$$\begin{aligned} \sum_{j=0}^{\infty} h(r_j) &= \int_{\Gamma/\mathsf{H}} \left[K_0(z,z) + \frac{3}{\pi} h\left(\frac{i}{2}\right)\right] dz = 2\operatorname{res}_{s=1} I^*(s) + h\left(\frac{i}{2}\right) \\ &= \frac{1}{12}\int_{-\infty}^{\infty} h(r) r \tanh \pi r dr \\ &\quad - \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(z(r) + \frac{\Gamma'}{\Gamma}(1+ir) + \log 2\right) h(r)dr \end{aligned} \tag{4.14}$$

$$+\frac{1}{2}h(0)+\frac{2}{\pi}\sum_{\substack{t=-\infty \\ t^2\neq 4}}^{\infty} V(1,t)\mathrm{res}_{s=1}\zeta(s,t^2-4),$$

where 343

$$\begin{aligned} z(r) &= \frac{\zeta^{*'}(1+2ir)}{\zeta^*(1+2ir)}+\frac{\zeta^{*'}(1-2ir)}{\zeta^*(1-2ir)} \\ &= \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1}{2}+ir)-\frac{1}{2}\frac{\Gamma'}{\Gamma}(ir)+\frac{\zeta'}{\zeta}(1+2ir)-\frac{\zeta'}{\zeta}(2ir) \end{aligned}$$

and $V(1,t)$, $\mathrm{res}_{s=1}\zeta(s,t^2-4)$ are given by equations (4.12) and (4.13), respectively. Formula (4.14) is the Selberg trace formula.

§ 5. Complements. In the last section we gave the analytic continuation of $I(s)$ to the strip $1-A<\sigma<A$. To complete the proof of Theorem 1 we must still

1) express $V(s,t)$ in terms of the Selberg transform $h(r)$;

2) generalize the formula obtained for $I(s)$ to the function

$$I^m(s)=\sum_{j=1}^{\infty} a_j(m)\frac{h(r_j)}{(f_j,f_j)}R_{f_j}(s) \qquad (m\in\mathbf{Z}) \tag{5.1}$$

with $m\neq 1$ (notations as in § 1). In this section we will carry out these two calculations and also indicate the generalization to congruence subgroups of $SL_2(\mathbf{Z})$.

The results of § 4 show that $I^*(s)$ equals

$$\begin{aligned} &-\frac{1}{4\pi}\zeta^*(s)^2\int_{-\infty}^{\infty}\frac{\zeta^*(s+2ir)\zeta^*(s-2ir)}{\zeta^*(1+2ir)\zeta^*(1-2ir)}h(r)dr \\ &-\frac{1}{2}\frac{\zeta^*(s)\zeta^*(2s)}{\zeta^*(s+1)}h\left(\frac{is}{2}\right)-\frac{1}{2}\frac{\zeta^*(s)\zeta^*(2s-1)}{\zeta^*(s-1)}h\left(i\frac{1-s}{2}\right) \\ &+\frac{\zeta^*(s)\zeta^*(2s)}{(4\pi)^{\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)}\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)}{\Gamma(ir)\Gamma(-ir)}h(r)dr \end{aligned}$$

$$= \frac{\zeta^*(s)\zeta^*(2s-1)}{(4\pi)^{\frac{2-s}{2}}\Gamma\left(\frac{2-s}{2}\right)} \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma(ir)\Gamma(-ir)} h(r)dr$$

$$+ \pi^{-s}\Gamma(s) \sum_{\substack{t=-\infty \\ t^2\neq 4}}^{\infty} V(s,t)\zeta(s,t^2-4)$$

344 in the critical strip $0 < \sigma < 1$ (cf. equations (4.9) and (4.10). Using the functional equations of $V(s,t)$ and $\zeta(s,D)$ we can write this expression as $\mathscr{R}(s) + \mathscr{R}(l-s)$, where

$$\mathscr{R}(s) = -\frac{1}{8\pi}\zeta^*(s)^2 \int_{-\infty}^{\infty} \frac{\zeta^*(s+2ir)\zeta^*(s-2ir)}{\zeta^*(1+2ir)\zeta^*(1-2ir)} h(r)dr$$

$$- \frac{\zeta^*(s)\zeta^*(2s)}{2\zeta^*(s+1)} h\left(\frac{is}{2}\right)$$

$$+ \frac{\Gamma(s)\Gamma(s-\frac{1}{2})}{2\pi^s\Gamma\left(\frac{2-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}\zeta(s)\zeta(2s-1)\times$$

$$\int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma(ir)\Gamma(-ir)} h(r)dr$$

$$+ \pi^{-s}\Gamma(s) \sum_{\substack{t=-\infty \\ t\neq \pm 2}}^{\infty} v(s,t)\zeta(s,t^2-4),$$

$v(s,t)$ being any function such that

$$V(s,t) = v(s,t) + \frac{\pi^{s-1}\Gamma(1-s)}{\pi^{-s}\Gamma(s)} \frac{\Gamma(s,\Delta)}{\gamma(1-s,\Delta)} v(1-s,t). \tag{5.2}$$

(here $\Delta = t^2 - 4$ as before). Comparing this with (1.16) and observing that $F(a,b;c;0) = 1$, we see that Theorem 1 (for $m = 1$) will follow from □

Proposition 5: For $t \neq \pm 2$ and $0 < \mathrm{Re}(s) < 1$ the function $V(s,t)$
345 defined by (3.10) is given by equation (5.2) with

$$v(s,t) = \frac{\Gamma(s-\frac{1}{2})}{4\Gamma\left(\frac{s+1}{2}\Gamma\left(\frac{2-s}{2}\right)\right)}\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma(ir)\Gamma(-ir)}$$
$$\times F\left(\frac{1-s}{2}+ir, \frac{1-s}{2}-ir; \frac{3}{2}-s; 1-\frac{t^2}{4}\right)h(r)dr.$$

Proof. As in Proposition (4) we must distinguish the cases $\Delta > 0$ and $\Delta < 0$. It will also be useful to introduce symmetrization operators $\mathscr{S}_s^1, \mathscr{S}_r$ with

$$\mathscr{S}_s^1[f(s)] = f(s) + f(1-s), \quad \mathscr{S}_r[f(s)] = f(r) + f(-r)$$

for any function f. Thus the formula we want to prove can be written

$$\frac{\pi^{-s}\Gamma(s)}{\Gamma(s,\Delta)}V(s,t) = \mathscr{S}_s^1\left[\frac{\pi^{-s}\Gamma(s)}{\gamma(s,\Delta)}v(s,t)\right]. \tag{5.3}$$

If $\Delta > 0$, then (4.5) and (2.24) give

$$V(s,t) = \frac{1}{8\pi}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma(s)}\Delta^{\frac{s}{2}}\int_{-\infty}^{\infty} r\tanh\pi r h(r)$$
$$= \times \int_0^1 \mathscr{O}\P_{-\frac{1}{2}+ir}\left(1+\frac{\Delta/2}{1-\xi}\right)(1-\xi)^{\frac{s-3}{2}}F\left(\frac{s}{2},\frac{s}{2};\frac{1}{2};\xi\right)\frac{d\xi}{\sqrt{\xi}}dr,$$

where we have made the change of variables $\xi = \dfrac{u^2}{u^2+1}$. To prove (5.3), we must show that the inner integral equals

$$\mathscr{S}_s^1\left[\frac{2^s}{\Delta^{s/2}}\cosh\pi r\frac{\Gamma(S-\frac{1}{2})\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}\right)-ir}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\right. \tag{5.4}$$
$$\left.\times F\left(\frac{1-s}{2}+ir, \frac{1-s}{2}-ir; \frac{3}{2}-s; 1-\frac{t^2}{4}\right)\right]$$

346 (here and from now one we use standard identities for the gamma function without special mention). Using the identity

$$P_{-\frac{1}{2}+ir}\left(1+\frac{2}{x}\right)=\mathscr{S}_r\left[\frac{\Gamma(2ir)}{\Gamma\left(\frac{1}{2}+ir\right)^2}x^{\frac{1}{2}-ir}F\left(\frac{1}{2}-ir,\frac{1}{2}-ir;1-2ir;-x\right)\right](x>0)$$

[EH 3.2 (19)] and expanding the hypergeometric series, we find that the integral in question equals

$$\mathscr{S}_r\left[\frac{\coth\pi r}{2\pi i}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma\left(n+\frac{1}{2}-ir\right)^2}{\Gamma(n+1-2ir)}\left(\frac{\Delta}{4}\right)^{-n-\frac{1}{2}+ir}\right.$$
$$\left.\times\int_0^1(1-\xi)^{\frac{s}{2}+n-1-ir}F\left(\frac{s}{2},\frac{s}{2};\frac{1}{2};\xi\right)\frac{d\xi}{\sqrt{\xi}}\right].$$

From [EH 2.4(2), 2.8(46)] we have

$$\int_0^1(1-\xi)^{\frac{s}{2}+n-1-ir}\xi^{-\frac{1}{2}}F\left(\frac{s}{2},\frac{s}{2};\frac{1}{2};\xi\right)d\xi$$
$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+n-ir\right)}{\Gamma\left(\frac{1+s}{2}+n-ir\right)}F\left(\frac{s}{2},\frac{s}{2};\frac{1+s}{2}+n-ir;1\right)$$
$$=\frac{\Gamma\left(\frac{s}{2}+n-ir\right)\Gamma\left(\frac{1-s}{2}+n-ir\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}-ir\right)^2},$$

so our integral equals

$$\mathscr{S}_r\left[\frac{\coth\pi r}{2i\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\times\right.$$
$$\left.\times\frac{\Gamma\left(\frac{s}{2}+n-ir\right)\Gamma\left(\frac{1-s}{2}+n-ir\right)}{\Gamma(n+1-2ir)}\left(\frac{\Delta}{4}\right)^{-n-\frac{1}{2}+ir}\right]$$

$$= \mathscr{S}_r\left[\frac{\coth \pi r}{2i\sqrt{\pi}}\frac{\Gamma\left(\frac{s}{2}-ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma(1-2ir)}\left(\frac{\Delta}{4}\right)^{ir-\frac{1}{2}}\right. \tag{5.5}$$

$$\left.\times F\left(\frac{s}{2}-ir, \frac{1-s}{2}-ir; 1-2ir; -\frac{4}{\Delta}\right)\right]$$

$$= \mathscr{S}_r\mathscr{S}_s^1\left[\frac{\coth \pi r}{2i\sqrt{\pi}}\frac{\Gamma(s-\frac{1}{2})\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma\left(\frac{1+s}{2}-ir\right)}\left(\frac{\Delta}{4}\right)^{-\frac{s}{2}}\right.$$

$$\left.\times F\left(\frac{1-s}{2}-ir, \frac{1-s}{2}+ir; \frac{3}{2}-s; -\frac{\Delta}{4}\right)\right]$$

(the last formula is [EH 2.10(2)]), and since **347**

$$\mathscr{S}_r\left[\frac{\coth \pi r}{2i\sqrt{\pi}}\frac{\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma\left(\frac{1+s}{2}-ir\right)}\right] = \frac{\cosh \pi r\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}$$

this agrees with (5.4), completing the proof for $\Delta > 0$.

If $\Delta < 0$, then $\dfrac{\pi^{-s}\Gamma(s)}{\gamma(s,\Delta)} = \delta^{-s/2}$, where $\delta = \dfrac{1}{2}|\Delta| = 1 - \dfrac{t^2}{4}$, so (5.3) is equivalent to

$$\delta^{-s/2}V(s,t) = \mathscr{S}_s^1\left[\delta^{\frac{1}{2}(s-1)}v(1-s,t)\right].$$

On the other hand, from (4.4) and (2.24) we have

$$\pi^{-s/2}V(s,t) = \frac{1}{2}\int_{-\infty}^{\infty} r\tanh \pi r h(r)\int_1^{\infty} P_{-\frac{1}{2}+ir}(1+2\delta(u^2-1))P_{-s}(u)du dr.$$

Denote the inner integral by I. Then (5.3) will be proved if we show that **348**

$$I = \mathscr{S}_s^1\left[\frac{1}{2}\delta^{\frac{1}{2}(s-1)}\frac{\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)}{\Gamma\left(\frac{1}{2}+ir\right)\Gamma\left(\frac{1}{2}-ir\right)\Gamma\left(\frac{2-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}\right.$$

$$\left.F\left(\frac{s}{2}+ir, \frac{s}{2}-ir; \frac{1}{2}+s; \delta\right)\right].$$

By [EH 2.10(1)], this is equivalent to

$$I = \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma\left(\frac{1}{2}+ir\right)\Gamma\left(\frac{1}{2}-ir\right)\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{2-s}{2}\right)}$$
$$\times \delta^{s-\frac{1}{2}} F\left(\frac{s}{2}-ir, \frac{s}{2}+ir; \frac{1}{2}; 1-\delta\right).$$

To prove this this formula, we being by making the substitution $v = u^2 - 1$ in I and substituting for $P_{-\frac{1}{2}+ir}$ by

$$\int_0^\infty e^{-ax} K_{ir}(x) \frac{dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}} \Gamma\left(\frac{1}{2}+ir\right)\Gamma\left(\frac{1}{2}-ir\right) P_{-\frac{1}{2}+ir}(a)$$

[GR 6.628. 7]; after an interchange of integration this gives

$$\sqrt{2\pi}\Gamma\left(\frac{1}{2}+ir\right)\Gamma\left(\frac{1}{2}-ir\right) I$$
$$= \int_0^\infty \left(\int_0^\infty e^{-2\delta x v} P_{-s}\left(\sqrt{1+v}\right) \frac{dv}{\sqrt{1+v}} \right) e^{-x} K_{ir}(x) \frac{dx}{\sqrt{x}}.$$

By [GR 7.146.2] the inner integral equals $(2\delta x)^{-3/4} e^{\delta x} W_{-\frac{1}{4}, \frac{1}{4}-\frac{s}{2}}(2\delta x)$, where $W_{\lambda,\mu}$ is Whittaker's function, and using the Mellin-Barnes integral representation of the latter [GR 9.223] we find that this in turn equals

$$\Gamma\left(\frac{1+s}{2}\right)^{-1} \Gamma\left(1-\frac{s}{2}\right)^{-1} \cdot \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Gamma\left(v+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}-v)\Gamma(\frac{1-s}{2}-v\right) \times$$
$$\times (2\delta x)^{v-\frac{1}{2}} dv,$$

349 where C is chosen such that $-\frac{1}{2} < C < \frac{1}{2}\min(\sigma, 1-\sigma)$. If choose C to satisfy also $C > 0$ then we may interchange the order of integration again, obtaining

$$2\sqrt{\pi}\Gamma\left(\frac{1}{2}+ir\right)\Gamma(\frac{1}{2}-ir)\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1-\frac{s}{2}\right) I$$

$$= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} 2^v \delta^{v-\frac{1}{2}}\Gamma(v+\frac{1}{2})\Gamma\left(\frac{s}{2}-v\right)\Gamma\left(\frac{1-s}{2}-v\right)\times$$

$$\times\int_0^\infty x^{v-1}e^{-x}K_{ir}(x)dx\,dv$$

$$= \frac{\Gamma(\frac{1}{2})}{2\pi i}\int_{C-i\infty}^{C+i\infty}\Gamma\left(\frac{s}{2}-v\right)\Gamma\left(\frac{1-s}{2}-v\right)\Gamma(v+ir)\Gamma(v-ir)\delta^{v-\frac{1}{2}}dv$$

[ET 6.8(28)]. The integral is very rapidly convergent (the integrand is $O(|v|^{-3/2}e^{-2\pi|v|})$), so we may substitute for $\delta^{v-\frac{1}{2}}$ the binomial expansion

$$\delta^{v-\frac{1}{2}} = \delta^{\frac{1}{2}(s-1)}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma\left(\frac{s}{2}-v+n\right)}{\Gamma\left(\frac{s}{2}-v\right)}(1-\delta)^n$$

and integrate term by term. Using "Barnes' Lemma"

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)ds$$
$$= \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}$$

[GR 6.412] we obtain finally

$$2\Gamma\left(\frac{1}{2}+ir\right)\Gamma\left(\frac{1}{2}-ir\right)\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)I = \delta^{\frac{1}{2}(s-1)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{s}{2}+ir+n\right)\Gamma\left(\frac{s}{2}-ir+n\right)\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma\left(\frac{1}{2}+n\right)n!}(1-\delta)^n$$

$$= \Gamma\left(\frac{1}{2}\right)^{-1}\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)\delta^{\frac{s-1}{2}}$$

$$\times F\left(\frac{s}{2}+ir,\frac{s}{2}-ir;\frac{1}{2};1-\delta\right).$$

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This completes the proof of Proposition 5 and hence of Theorem (1) for $m = 1$.

To calculate the function (5.1) for $m > 1$ we set

$$K_0^m(z, z') = \sum_{j=1}^{\infty} a_j(m) \frac{h(r_j)}{(f_j, f_j)} f_j(z)\overline{f_j(z')}.$$

Then $I^m(s) = \int\limits_{\Gamma/\mathsf{H}} K_0^m(z, z)E(z, s)dz$. On the other hand, from (1.6) we see that $K_0^m(z, z') = m^{\frac{1}{2}} K_0(z, z')|T(m)$, where $K_0(z, z')$ is the kernel function (2.27) and $T(m)$ the Hecke operator (1.5), acting (say) on z'. Since the constant function and the Eisenstein series $E(z, s)$ are eigen-functions of $m^{\frac{1}{2}} T(m)$ with eigenvalues $\tau_{\frac{1}{2}}(m)$ and $\tau_{s-\frac{1}{2}}(m)$, respectively ($\tau_\nu(m)$ as in (2.7)), equation (2.31) gives

$$\begin{aligned} K_0^m(z, z') = K^m(z, z') - \frac{3}{\pi}\tau_{\frac{1}{2}}(m) h\left(\frac{i}{2}\right) - \\ - \frac{1}{4\pi} \int_{-\infty}^{\infty} E\left(z, \frac{1}{2} + ir\right) E\left(z', \frac{1}{2} - ir\right) h(r)\tau_{ir}(m)dr, \end{aligned}$$

where

$$K^m(z, z') = \sqrt{m} K(z, z')|T(m) = \frac{1}{2\sqrt{m}} \sum_{\substack{a,b,c,d\in\mathbf{Z} \\ ad-bc=m}} k\left(z, \frac{az' + b}{cz' + d}\right).$$

Hence the constant term $\mathscr{K}^m(y)$ of $K_0^m(z, z)$ equals $\sum\limits_{i=1}^{4} \mathscr{K}_i^m(y)$, where $\mathscr{K}_3^m$ and $\mathscr{K}_4^m$ are defined exactly like $\mathscr{K}_3$ and $\mathscr{K}_4$ but with $h(r)$ replaced
351 by $h(r)\tau_{ir}(m)$ and

$$\mathscr{K}_1^m(y) = \frac{1}{\sqrt{m}} \int_{iy}^{iy+1} \sum_{\substack{ad-bc=m \\ c>0}} k\left(z, \frac{az + b}{cz + d}\right) dz,$$

$$\mathcal{K}_2^m(y) = \frac{1}{\sqrt{m}} \int_{iy}^{iy+1} \sum_{\substack{ad=m\\ a,d>0}} \sum_{b=-\infty}^{\infty} k\left(z, \frac{az+b}{d}\right) dz -$$

$$- \frac{y}{2\pi} \int_{-\infty}^{\infty} h(r)\tau_{ir}(m) dr.$$

As in §3 we then find $I^m(s) = \sum_{i=1}^{4} I_i^m(s)$ for $1 < \sigma < A$, where I_3^m and I_4^m are given by the same formulas as I_3 and I_4 (equations (3.4) and (3.5)) but with $h(r)$ replaced by $\tau_{ir}(m)h(r)$ and

$$\zeta(2s) I_1^m(s) = m^{\frac{s-1}{2}} \sum_{t=-\infty}^{\infty} \zeta(s, t^2 - 4m) V\left(s, \frac{t}{\sqrt{m}}\right).$$

As to I_2^m, from (2.20) and (2.23) we find

$$\mathcal{K}_2^m(y) = \frac{1}{\sqrt{m}} \int_0^1 \sum_{\substack{ad=m\\ a,d>0}} \sum_{b=-\infty}^{\infty} \varphi\left(\frac{(x(d-a)-b)^2 + (a-d)^2 y^2}{my^2}\right) dx$$

$$- \quad \frac{y}{2\pi} \sum_{\substack{ad=m\\ a,d>0}} \int_{-\infty}^{\infty} h(r) \left(\frac{a}{d}\right)^{ir} dr$$

$$= \frac{1}{\sqrt{m}} \sum_{\substack{ad=m\\ a\neq d}} Q\left(\frac{(a-d)^2}{m}\right) - y \sum_{ad=m} g\left(\log\frac{a}{d}\right)$$

$$+ \begin{cases} \dfrac{1}{\sqrt{m}} \displaystyle\sum_{b=-\infty}^{\infty} \varphi\left(\frac{b^2}{my^2}\right) & \text{if } \sqrt{m} \in \mathbf{Z} \\ 0 & \text{if } \sqrt{m} \notin \mathbf{Z} \end{cases}$$

$$= \begin{cases} \dfrac{1}{\sqrt{m}} \mathcal{K}_2(y\sqrt{m}) & \text{if } \sqrt{m} \in \mathbf{Z} \\ 0 & \text{if } \sqrt{m} \notin \mathbf{Z}, \end{cases}$$

352 so

$$I_2^m(s) = \begin{cases} m^{-s/2} I_2(s) & \text{if } \sqrt{m} \in \mathbf{Z}, \\ 0 & \text{if } \sqrt{m} \notin \mathbf{Z}. \end{cases}$$

The analytic continuation to $1 - A < \sigma < 1$ now proceeds as in § 4, the only essential difference being that the terms (4.2) are absent when m is not a square, since I_1^m then has no summands with $t^2 - 4m = 0$ and I_2^m vanishes identically. The final formula is that given in Theorem 1.

If $m < 0$ the proof is similar and in fact somewhat easier (since $t^2 - 4m$ now always has the same sign and the term I_2 is absent), but the calculations with the hypergeometric functions are a little different. Since constant functions and Eisenstein series are invariant under $T(-1)$, the terms $I_3^m(s)$ and $I_4^m(s)$ are equal to $I_3^{|m|}(s)$ and $I_4^{|m|}(s)$, so that first two terms in (1.16) are unchanged except for replacing m by $|m|$. The term I_2^m is always zero since m cannot be a square. Finally, for I_1^m we find

$$I_1^m(s) = |m|^{\frac{s-1}{2}} \sum_{t=-\infty}^{\infty} \frac{\zeta(s, t^2 - 4m)}{\zeta(2s)} V_{s, t|m|^{-\frac{1}{2}}} \quad (m < 0) \tag{5.6}$$

with

$$V_{s,t} = \int_{\mathsf{H}} k\left(z, \frac{1}{\bar{z} + t}\right) y^s dz = \int_{\mathsf{H}} \varphi\left(\frac{\left(|z|^2 - \Delta/4\right)^2}{y^2} + t^2\right) y^s dz,$$

where now $\Delta = t^+4$. This function is easier to compute than $V(s,t)$ since Δ always has the same sign. Making the same substitutions as in the case $\Delta > 0$ of Proposition (4) we find that $V_-(s,t)$ is given by the same integral (4.5) but with $\varphi(\Delta u^2 + t^2)$ instead of $\varphi(\Delta u^2 + \Delta)$, This integral can then be calculated as in the case $\Delta > 0$ of Proposition 5, the only difference being that the function $P_{-\frac{1}{2}+ir}\left(1 + \dfrac{\Delta/2}{1-\xi}\right)$ is replaced by $P_{-\frac{1}{2}+ir}\left(-1 + \dfrac{\Delta/2}{1-\xi}\right)$ and we must use

$$P_{-\frac{1}{2}+ir}\left(-1+\frac{2}{x}\right)=$$

$$=\mathscr{S}_r\left[\frac{\Gamma(2ir)}{\Gamma\left(\frac{1}{2}+ir\right)^2}x^{\frac{1}{2}-ir}F\left(\frac{1}{2}-ir,\frac{1}{2}-ir;1-2ir;x\right)\right]\quad(x>0)$$

[EH 3.2(18)] instead of the corresponding formula for $P_{-\frac{1}{2}+ir}\left(1+\dfrac{2}{x}\right)$. **353**
This has the effect of introducing an extra factor $(-1)^n$ in the infinite sum and hence of replacing the argument $-\dfrac{4}{\Delta}$ in (5.4) by $+\dfrac{4}{\Delta}$. Using the identity

$$\mathscr{S}_r\left[\frac{\coth\pi r}{2i\sqrt{\pi}}\frac{\Gamma\left(\frac{s}{2}-ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)}{\Gamma(1-2ir)}\left(\frac{\Delta}{4}\right)^{ir-\frac{1}{2}}\times\right.$$

$$\left.\times F\left(\frac{s}{2}-ir,\frac{1-s}{2}-ir;1-2ir;\frac{4}{\Delta}\right)\right]$$

$$=\frac{\cosh^2\pi r}{\pi^2}\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-s}{2}-ir\right)\left(\frac{\Delta}{4}\right)^{-\frac{s}{2}}.\times$$

$$\times F\left(\frac{1-s}{2}-ir,\frac{1-s}{2}+ir;\frac{1}{2};1-\frac{\Delta}{4}\right)$$

[EH 2.10(3)] and substituting the expression thus obtained for $V_{-}(s,t)$ into (5.6), we find that the last term in (1.16) must be replaced by

$$\frac{2^{s-4}|m|^{\frac{s-1}{2}}}{\pi^{s+1}}\Gamma\left(\frac{s}{2}\right)^2\sum_{t=-\infty}^{\infty}\zeta(s,t^2-4m)\times$$

$$\times\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)\Gamma\left(\frac{1-s}{2}+ir\right)\Gamma\left(\frac{1-2}{2}-ir\right)}{\Gamma\left(\frac{1}{2}+ir\right)\Gamma\left(\frac{1}{2}-ir\right)\Gamma(ir)\Gamma(-ir)}$$

$$\times F\left(\frac{1-s}{2}+ir,\frac{1-s}{2}-ir;\frac{1}{2};\frac{t^2}{4m}\right)h(r)dr$$

if $m<0$. This completes the proof of Theorem (1).

Finally, we indicate what happens when Γ is replaced by a congruence subgroup Γ_1 in the simplest case $\Gamma_1 = \Gamma_0(q)/\{\pm 1\}$, q prime. There are now two cusps and correspondingly two Eisenstein series E_1 and E_2, given explicitly by **354**

$$E_1(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1} \mathrm{Im}(\gamma z)^s, \quad E_2(z) = \sum_{\gamma \in w^{-1}\Gamma_\infty w \backslash \Gamma_1} \mathrm{Im}(w\gamma z)^s$$

(where $w = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$), and formula (2.31) becomes

$$K_0(z,z') = \sum_{\gamma \in \Gamma_1} k(z,\gamma z') - \frac{1}{\mathrm{vol}(\Gamma_1 \backslash \mathsf{H})} h\left(\frac{i}{2}\right) -$$
$$-\frac{1}{4\pi} \sum_{j=1}^{2} \int_{-\infty}^{\infty} E_j(z, \frac{1}{2} + ir) E_j(z', \frac{1}{2} - ir) h(r) dr,$$

where $K_0(z,z')$ is defined as before but with f_j now running over all Maass cusp forms of weight 0 on Γ_1 (cf. [4]). It is easily checked that

$$E_1(z,s) = \frac{q^s}{q^{2s}-1} E(qz,s) - \frac{1}{q^{2s}-1} E(z,s),$$
$$E_2(z,s) = \frac{q^s}{q^{2s}-1} E(z,s) - \frac{1}{q^{2s}-1} E(qz,s),$$

so that Fourier developments of E_1 and E_2 can be deduced from (2.6). The calculation of $I(s) = \int_{\Gamma_1/\mathsf{H}} K_0(z,z) E_1(z,s) dz$ (which again can be expressed as $\int_0^\infty \mathscr{K}(y) y^{s-2} dy$, $\mathscr{K}(y)$ = constant term of $K_0(z,z)$) now proceeds as in § 3; the final formula is the same except that $I_1(s)$ is replaced by

$$\frac{1}{q^2+1} \zeta(2s)^{-1} \sum_{t=-\infty}^{\infty} \left(1 + \left(\frac{t^2-4}{q}\right)\right) \zeta(s, t^2-4) V(s,t),$$

(where $\left(\frac{\Delta}{q}\right)$ is the Legendre symbol), $I_3(s)$ is multiplied by $\dfrac{q-1}{q^{s+1}-1}$, and the integrand of $I_4(s)$ is multiplied by

$$\frac{1}{1+q^{-s}}\left(\frac{(q+1)(1-q^{-s})(1-q^{1-s})}{(q^{1+2ir}-1)(q^{1-2ir}-1)}+2q^{-s}\right).$$

□

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