Vector Bundles On Algebraic Varieties

VECTOR BUNDLES ON ALGEBRAIC VARIETIES

Papers presented at the Bombay Colloquium 1984, by

ATIYAH AYOUB BARTH BHATWADEKAR
COLLIOT-THÉLÈNE DREZET HIRSCHOWITZ
HORROCKS HULEK KEMPF KNUS LE POTIER
LINDEL MARUYAMA MOORE MUKAI
NARASIMHAN OJANGUREN PARIMALA
RAMANAN RAMANATHAN SESHADRI
SRIDHARAN SRINIVAS TRAUTMANN

Published for the

TATA INSTITUTE OF FUNDAMENTAL RESEARCH,
BOMBAY

OXFORD UNIVERSITY PRESS

BOMBAY DELHI CALCUTTA MADRAS
1987

Oxford University Press, Walton Street, Oxford OX2 6DP

NEW YORK TORONTO
DELHI BOMBAY CALCUTTA MADRAS KARACHI
PETALINGJAYA SINGAPORE HONGKONG TOKYO
NAIROBI DAR ES SALAAM
MELBOURNE AUCKLAND

and associates in

BEIRUT BERLIN IBADAN NICOSIA

Printed in India by K. M. Aarif,
Paper Print India, Worli, Bombay 400018
and published by R. Dayal, Oxford University Press
Oxford House, Apollo Bunder, Bombay 400039

Contents

International Colloquium on Vector Bundles on Algebraic Varieties

Bombay, 9-16 January 1984

REPORT

AN INTERNATIONAL COLLOQUIUM on ‘Vector Bundles on Algebraic Varieties’ was held at the Tata Institute of Fundamental Research, Bombay from January 9 to January 16, 1984. The purpose of the Colloquium was to highlight recent developments in the general area of Vector Bundles as well as principal bundles on both affine and projective varieties. Projective modules and quadratic spaces over general rings were also among the topics covered by the Colloquium. The Colloquium was jointly sponsored by the International Mathematical Union and the Tata Institute of Fundamental Research, and was financially supported by them and the Sir Dorabji Tata Trust.

The Organizing Committee for the Colloquium consisted of Professors Sir M.F. Atiyah, M.P. Murthy, M.S. Narasimhan, M.S. Raghunathan, S. Ramanan and R. Sridharan. The International Mathematical Union was represented by Professors Atiyah and Narasimhan

The following mathematicians gave one-hour addresses at the Colloquium :

M.F. Atiyah, W. Barth, S.M. Bhatwadekar, J.L. Colliot-Thélène, A. Hirschowitz, G. Horrocks, G.R. Kempf, M.A. Knus, J. Le Potier, H. Lindel, M. Maruyama, N. Mohan Kumar, S. Mukai, M. Ojanguren, R. Parimala, S. Ramanan, A. Ramanathan, C.S. Seshadri, V. Srinivas and G. Trautmann.

Besides the members of the School of Mathematics of the Tata Institute, mathematicians from universities and educational institutions in India and France were also invited to attend the Colloquium.

The social programme for the Colloquium included a Tea Party on January 9, a Classical Indian Dance (Bharata Natyam) Performance on January 10, a Concert (Classical Indian Music) on January 12, a Dinner Party at the Institute on January 13, an Excursion to Elephanta Caves on January 14, a Film Show on January 15, and a Farewell Dinner Party on January 16, 1984.

Magnetic Monopoles in Hyperbolic Spaces

By M.F. Atiyah

1 Introduction

1

In recent years, the Penrose twistor transform has been extensively and successfully used to convert certain problems arising in physics into problems of algebraic geometry [1]. More precisely, solutions of the self-dual Yang-Mills equations on $\mathbb{R}^4$ (describing 'instantons') convert into holomorphic bundles on the complex projective 3-space $\mathbb{P}_3$. Similarly, solutions of the Bogomolny equation in $\mathbb{R}^3$ (describing 'magnetic monopoles') convert into holomorphic bundles on $T\mathbb{P}_1$ (the tangent bundle of $\mathbb{P}_1$) [7]. In this talk, I shall consider the analogous problem, for magnetic monopoles, when the Euclidean 3-space $\mathbb{R}^3$ is replaced by the hyperbolic 3-space H^3. Twistor methods still apply and so then 'hyperbolic monopoles' can also be described by holomorphic bundles.

The motivation for studying the hyperbolic case is that, surprisingly, it turns out to be simpler than the Euclidean case, while at the same time preserving all its essential features. Moreover, by varying the curvature of hyperbolic space and letting it tend to zero, the Euclidean case appears as a natural limit of the hyperbolic case. While the details of this limiting procedure are a little delicate, and need much more careful examination than I shall give here, it seems reasonable to conjecture that the moduli (or parameter space) of monopoles remains unaltered by passing to the limit. This conjecture (for $SU(2)$) receives substantial confirmation from the recent result of Donaldson [5] on the moduli space of Euclidean monopoles.

In § 2, I explain how hyperbolic monopoles satisfying suitable decay conditions at infinity, can essentially be viewed as instantons invariant

under a circular rotation. Although this works quite generally for any compact Lie group G, and for any asymptotic value for Higgs field, I
2 will for simplicity concentrate on the $SU(2)$ case. This is the case which (for Euclidean 3-space) has been studied in the greatest detail [7], [8], [9], [10] and it is therefore the most useful for comparative purposes. In particular, I will show how the two splittings (reduction to the triangular group) used by Ward [10] and Hitchin [7] have simple analogues in the hyperbolic case.

The twistor picture for S^1-invariant instantons is developed in § 3 and then the 'mini-twistor' picture is derived in § 4. The analogue of Hitchin's space $T\mathbb{P}_1$ (a quadric cone) is now a non-singular quadric surface Q, and the analogue of Hitchin's spectral curve is now a curve on Q. However, in the hyperbolic space, this spectral curve also has an additional interpretation in terms of the jumping lines of the instanton bundle on the 3-dimensional twistor space.

In § 5, I discuss the limiting process of letting the curvature tend to zero. In the 'mini-twistor' picture, this means that we have a family $Q(t)$ of non-singular quadrics which degenerate to a quadric cone as $t \to 0$. While the general situation is fairly clear, there are many detailed technical questions about this limiting process which I do not enter into and which need thorough investigation.

Finally in § 6, using a recent result of Donaldson [4] on the moduli space of instantons, I will show that the moduli space of S^1-invariant instantons (i.e. of hyperbolic monopoles) can be identified with the space of rational functions (of one complex variable). This result was also derived in [2] in a more general context, but the treatment given here is somewhat more elementary. Moreover, it enables us to identify the rational functions assigned to a monopole as part of the scattering matrix for Hitchin's differential operator. It seems likely that a similar interpretation holds for monopoles in Euclidean space.

3 It will be clear from this introduction that many of the key points rest on results of S.K. Donaldson. I am also indebted to him and to N.J. Hitchin for much valuable discussion on these topics.

2 Instantons and Hyperbolic monopoles

I begin by recalling the self-dual Yang-Mills equations on $\mathbb{R}^4$

$$* F = F \tag{2.1}$$

and the Bogomolny equations on $\mathbb{R}^3$:

$$D\phi = *F \tag{2.2}$$

Here F is the curvature of a connection, ϕ (the Higgs field) is a section of the adjoint bundle. $D\phi$ its corariant derivative and $*$ is the duality operator (relative to the Euclidean metric). With suitable decay conditions at infinity solutions of (1.1) are called *instantons*, and solutions of (2.2) are called *monopoles*. For such solutions the L^2-norm of F is finite in both cases (referred to as the action on $\mathbb{R}^4$ and the energy on $\mathbb{R}^3$).

It is well-known that any solution of (2.1) which is independent of the coordinate x_4 can be reinterpreted as a solution of (2.1) with ϕ being given by the x_4-component of the connection. Thus a monopole can be viewed as a solution of (2.1), but it is a solution with *infinite action*, and so is not an instanton.

Instead of considering *translation*-invariant solutions of (2.1), we can however consider *rotationally*-invariant solutions, relative to angular rotation in say the (x_3, x_4)-plane. It is then convenient to use polar coordinates (r, θ) in this plane and to rewrite the Euclidean metric element ds^2 as

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + dr^2 + r^2 d\theta^2 \\ &= r^2 \left\{ \frac{dx_1^2 + dx_1^2 + dr^2}{r^2} + d\theta^2 \right\} \end{aligned} \tag{2.3}$$

Now the hyperbolic 3-space H^3, of constant curvature -1, can be identified with the upper half-space $z > 0$ in (x, y, z)-space. Thus (2.3) shows that we have a conformal equivalence: 4

$$\mathbb{R}^4 - \mathbb{R}^2 \sim H^3 \times S^1. \tag{2.4}$$

This implies that S^1-invariant solutions of (2.1) over $\mathbb{R}^4 - \mathbb{R}^2$ correspond to solutions of (2.2) on H^3. Since S^1 is compact this correspondence converts finite-action solutions on $\mathbb{R}^4$ into finite-energy solutions on H^3. In particular, *S^1-invariant instantons, defined on the whole of* $\mathbb{R}^4$ (in fact on S^4), *can be interpreted as finite-energy solutions of* (2.2) *on* H^3. We shall see that it is reasonable to *define* a monopole on H^3 as a solution of (2.2) that arises in this way from an S^1-invariant instanton. This bypasses the interesting but technical questions of identifying the precise decay conditions to be required of a monopole.

At this stage, we should look carefully at the notion of S^1-invariance. The unknown function in (2.1) is a connection A on a principal bundle P over $\mathbb{R}^4$, and S^1-invariance means that the S^1-action on $\mathbb{R}^4$ has been lifted to an S^1-action on P preserving A. Notice that the choice of lifting of the S^1-action to P is part of the data.

So far we have not specified our structure group G. From now on, for simplicity, we take $G = SU(2)$ and we shall frequently work with the associated vector bundle E (with fibre $\mathbb{C}^2$). An action of S^1 then gives, on restriction to the $\mathbb{R}^2$-axis (i.e. the (x_1, x_2)-plane), a representation of S^1 on the fibres of E. This representation (up to conjugacy) is independent of the point on the axis and must be of the form:

$$\lambda \to \begin{pmatrix} \lambda^p & 0 \\ 0 & \lambda^{-p} \end{pmatrix} \tag{2.5}$$

5 for some integer $p \geq 0$. *The integer p can be identified with the asymptotic value of the norm $|\phi|$ of the Higgs field.* For this we just have to recall that, intrinsically, the Higgs field is the difference between the Lie derivative and the covariant derivative along the S^1-orbits. At a fixed point of the action, the covariant derivative is zero, while the Lie derivative is just the infinitesimal action on the fibre given by (2.5).

This already shows that our definition of monopoles is some-what restricted since we are requiring $|\phi|$ to tend to an *integer* value at ∞. If we relax this condition (but preserve suitable decay conditions) our monopole would correspond not to an instanton in the strict sense but to a solution of (2.1) with a branch-type singularity along $S^2 \subset S^4$. Such solutions, with non-trivial holonomy around S^2, have been explicitly found in [6]. A systematic treatment of hyperbolic monopoles should certainly include all these but in this talk I shall discuss only the 'integral' case.

Excluding the trivial case $p = 0$, the sub-bundle of E over S^2 corresponding to the positive factor λ^p in (2.5) is well-defined and has an integer first Chern class. This is (up to sign) the *magnetic charge* k of the monopole as usually defined.

To be precise about signs, we must first decide on orientations. We shall use the circle action to orient the $\mathbb{R}^2$ which is rotated, and the orientation of $\mathbb{R}^4$ then induces an orientation on the fixed $\mathbb{R}^2$. Since H^3 has this $\mathbb{R}^2$ as boundary it also inherits an orientation. This turns out to be opposite to the orientation it inherits from (2.4), starting with the orientation of $\mathbb{R}^4$. Hence solutions of the Bogomolny equation on H^3 now correspond to *anti-self-dual* connections on $\mathbb{R}^4$ (independent of θ).

With these sign conventions, the integer k defined above is in fact **6**
positive and can be identified with the magnetic charge of the monopole. Moreover, the anti-instanton number, i.e. the second Chern class c_2 of the bundle over S^4, is related to k and p by the simple formula:

$$c_2 = 2kp. \tag{2.6}$$

This is easily proved by the use of equivariant cohomology as explained in [3]. The equivariant c_2 of E restricted to the fixed $S^2 \subset S^4$ is

$$-(kx - pu)^2$$

where x generates $H^2(S^2)$ and u generates the equivariant H^2 of a point. Since the equivariant c_1 of the normal to S^2 is just u one finds

$$\begin{aligned} c_2 &= \text{ coefficient of } x \text{ in } \; -\frac{(kx - pu)^2}{u} \\ &= 2kp. \end{aligned}$$

Alternatively, (2.6) can be derived by equating the Energy of the monopole (multiplied by 2π) with the Action of the anti instanton, using the formulae

$$pk = \frac{1}{4\pi} \int |F|^2 dx^3$$

$$c_2 = \frac{1}{8\pi^2} \int |F|^2 dx^4$$

The factor 2 arises in (2.6) because $|D\phi|^2$ contributes by (2.2) the same amount to the Energy as $|F|^2$. This integration argument is, of course, an explicit de Rham version of the cohomological argument above (see [3] for further explanation).

We can now proceed to study our hyperbolic monopoles simply be exploiting the available information on instantons.

3 The twistor picture

7 I recall that we have the "twistor" fibration (see [1]

$$\mathbb{P}_3(\mathbb{C}) \to S^4 \tag{3.1}$$

and that $SU(2)$ - anti-instantons on S^4 lift to holomorphic vector bundles F (of rank 2) on $\mathbb{P}_3(\mathbb{C})$. Moreover, the anti-involution σ of $\mathbb{P}_3(\mathbb{C})$ given by

$$\sigma(z_1, z_2, z_3, z_4) = (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3)$$

lifts to F, giving F a "real structure". Finally, since F comes from S^4, it is trivial on all the fibres of (3.1), the "real" lines.

An S^1-invariant anti-instanton on S^4 corresponds in this way to a bundle F on $\mathbb{P}_3(\mathbb{C})$ on which S^1 acts. I now proceed to spell out the details of this action.

First of all, the action of S^1 on S^4 lifts naturally, via (3.1), to an action on $\mathbb{P}_3(\mathbb{C})$ and it then complexifies to give an action of $\mathbb{C}^*$. This action has two fixed lines which we shall denote by $\mathbb{P}_1^+$ and $\mathbb{P}_1^-$. Under the projection (3.1), each gets identified with the fixed $S^2 \subset S^4$ of the rotation. The map $\mathbb{P}_1^+ \to S^2$ is orientation-preserving, while $\mathbb{P}_1^- \to S^2$ orientation-reversing. Moreover, σ interchanges $\mathbb{P}_1^+$ and $\mathbb{P}_1^-$. On the normal bundle to $\mathbb{P}_1^+$ in P_3, the representation of $\mathbb{C}^*$ is just scalar multiplication, while for $\mathbb{P}_1$, it is the dual (or inverse).

When F is restricted to $\mathbb{P}_1^+$, it decomposes as a direct sum

$$F \mid \mathbb{P}_1^+ \cong H^{-k} \otimes L^p \oplus H^k \otimes L^{-p} \tag{3.2}$$

where H is the (positive) Hopf line-bundle (with trivial $\mathbb{C}^*$-action) and L is the standard one-dimensional representation of $\mathbb{C}^*$. This follows from the discussion in § 2, so that k is the magnetic charge and p the limiting value of the norm of the Higgs field. Because σ changes orientation, it follows that

$$F \mid \mathbb{P}_1^- \cong H^{-k} \otimes L^{-p} \oplus H^k \otimes L^p \tag{3.3}$$

The decompositions (3.2) and (3.3) induce splittings of F as described by the following: 8

Proposition 3.4. *On $\mathbb{P}_3 - \mathbb{P}_1^-$ there is a unique holomorphic sub-line-bundle L^+ of F such that*

(i) *F is invariant under the action of $\mathbb{C}^*$,*

(ii) *F, restricted to $\mathbb{P}_1^+$, coincides with the first factor $H^{-k} \otimes L^p$ in* (3.2).

Proof. A sub-line-bundle of F is given by a section of the associated projective bundle $\mathbb{P}(F)$. Consider now the action of $\mathbb{C}^*$ on $\mathbb{P}(F)$. There are four fixed lines α^+, α^-, β^+, β^- corresponding to the four factors of (3.2) and (3.3): α^+, α^- arising (in order) from those of (3.2) and β^+, β^- from (3.3). The weights of the representation of $\mathbb{C}^*$, normal to the lines α^+, α^- are

$$\begin{aligned} \alpha^+ &: (1, 1, -2p) \\ \alpha^- &: (1, 1, 2p). \end{aligned} \tag{3.5}$$

The weights $\pm 2p$ are in the fibre direction while the weights 1 correspond to directions normal to $\mathbb{P}_1^+$ in $\mathbb{P}_3$. Consider now the $\mathbb{C}^*$-orbits in $\mathbb{P}(F)$ lying over $\mathbb{P}_3 - (\mathbb{P}_1^+ \cup \mathbb{P}_1^-)$. As $t \to 0 (t \in \mathbb{C}^*)$, each such orbit acquires a limit point in α_+ or α_-. Since the weights of α_- are all *positive*, most $\mathbb{C}^*$-orbits tend to α_-. Consider the special $\mathbb{C}^*$-orbits which tend to α_+. Since α_+ has just two positive weights, it is not hard to see that these special $\mathbb{C}^*$-orbits define a codimension one complex sub-manifold of $\mathbb{P}(F)$ over $\mathbb{P}_3 - \mathbb{P}_-$, and that this is the graph of the required section. A formal proof can be given on the lines explained in § 6.

Interchanging the roles of $\mathbb{P}_1^+$ and $\mathbb{P}_1^-$ in 3.4, which corresponds to changing t to t^- in $\mathbb{C}^*$, leads similarly to a line-bundle L^- over $\mathbb{P}_3 - \mathbb{P}_1^+$ which extends the first factor $H^{-k} \otimes L^{-p}$ of (3.3). **9**

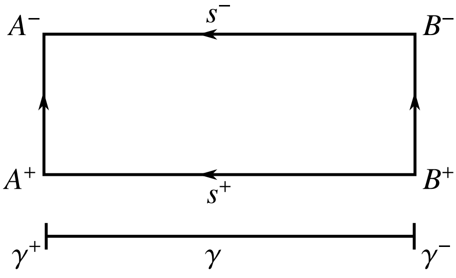
Recall now that a line in $\mathbb{P}_3$ on which F is non-trivial is called a *jumping* line. These are related to our line-bundles L^+, L^- by the following result: □

Proposition 3.5. *The fibres of L^+ and L^- over a point $x \in \mathbb{P}_3 - \mathbb{P}_1^+ \cup \mathbb{P}_1^-$ coincide if and only if the closure in $\mathbb{P}_3$ of the $\mathbb{C}^*$-orbit of x is a jumping line of F.*

Proof. Again, we work with $\mathbb{P}(F)$ so that L^+, L^- determine sections (denoted by the same letters) over the appropriate parts of $\mathbb{P}_3$. Suppose first that the closure of $\mathbb{C}^* x$ (say γ) is *not* a jumping line, so that, over γ, $\mathbb{P}(F)$ is a product. Its sections are therefore just constant and determined by their value at any point. Hence the action of $\mathbb{C}^*$ on these sections is the same as its action on the fibres over the two points

$$\gamma_+ = \gamma \cap \mathbb{P}_1^+, \gamma_- = \gamma \cap \mathbb{P}_1^-.$$

In particular there are just two fixed sections s^+, s^- as indicated in the diagram



Here A^+ is the point of s^+ over γ^+, etc., and the arrows indicate the direction of the $\mathbb{C}^*$-flow as $t \to 0$. As the figure indicates A^+ is the saddle-point over γ^+ so that s^+ must coincide with L^+. Similarly B^- is the saddle-point over γ^- so that s^- coincides with L^-. Since s^+ and s^- are disjoint it follows that $L^+(x)$ and $L^-(x)$ do not coincide.

10 Conversely if $L^+(x) \neq L^-(x)$ then we get a diagram as above and we have to deduce that $\mathbb{P}(F)$ is trivial over γ. For this consider the closure of a generic $\mathbb{C}^*$-orbit in $\mathbb{P}(F)|_\gamma$. From the direction of the arrows in the diagram we see that this gives a section δ passing through A^- and B^+. Moreover from (3.5) it follows that the intersection numbers $\delta.s^+$ and $\delta.s^-$ are both equal to $2p$. This implies that s^+ and s^- are homologous and from this it follows that $\mathbb{P}(F)|_\gamma$ is indeed a product (for a $\mathbb{P}_1$-bundle over $\mathbb{P}_1$ which is not a product, any two non-intersecting sections have opposite self-intersection number).

Note that the real structure on F interchanges L^+ and L^- and has to preserve the set of jumping lines. Moreover since F has no real jumping lines, Proposition 3.5 implies that L^+ and L^- never coincide at points x for which the line γ (closure of $\mathbb{C}^* x$) is real.

The line bundles L^+ and L^- are in fact *algebraic*. This can be proved by noting that, on $\mathbb{P}_3 - (\mathbb{P}_1^+ \cup \mathbb{P}_1^-)$, they descend to the quotient which is the quadric surface. Since algebraic line-bundles correspond to algebraic divisor classes and since divisors on $\mathbb{P}_3 - \mathbb{P}_1$ extend uniquely to $\mathbb{P}_3$ it follows that L^+ and L^- both extend to line-bundles on $\mathbb{P}_3$. Since (ignoring the $\mathbb{C}^*$-action) L^+ restricts to H^{-k} on $\mathbb{P}^{-k}$ on $\mathbb{P}_1^+$ it follows that L^+(and similarly L^-) extend to H^{-k} on $\mathbb{P}_3$. Moreover, since $\mathbb{P}_1^-$ has codimension 2, the homomorphism $L^+ \to F$ extends to a homomorphism $H^{-k} \to F$. In other words $F(k) = F \otimes H^k$ has section σ^+ vanishing on $\mathbb{P}_1^-$, and similarly a section σ^- vanishing on $\mathbb{P}_1^+$. Thus, in the general correspondence between rank 2 bundles on $\mathbb{P}_3$ and (suitable) algebraic curves, F corresponds to a line with a certain multiple structure (i.e. a sheaf with nilpotent elements).

To describe this situation in more detail, it will be necessary to consider the way in which $\mathbb{C}^*$ acts on all this data. First we note that the 11
action of C^* on $\mathbb{P}_3$ comes from the representation

$$\lambda \to \begin{pmatrix} \lambda^{1/2} & & & \\ & \lambda^{1/2} & & \\ & & \lambda^{-1/2} & \\ & & & \lambda^{-1/2} \end{pmatrix}$$

on $\mathbb{C}^4$. The square-roots indicate that this is really a representation of the double cover of $\mathbb{C}^*$: in fact $SO(5) \to SO(4)$ is the spin-representation. Let us now denote by $\mathscr{H}$ the Hopf bundle, regarded as an equivariant bundle for this action of $\mathbb{C}^*$. Then restricting to $\mathbb{P}_1^+$ and $\mathbb{P}_1^-$ we have

$$\mathscr{H}|_{\mathbb{P}_1^+} \cong H \otimes L^{1/2} \quad \mathscr{H}|_{\mathbb{P}_1^-} \cong H \otimes L^{-1/2}. \tag{3.6}$$

Moreover, the line-bundles L^+ and $\mathscr{H}^{-k}$ being isomorphic on $\mathbb{P}_3 - \mathbb{P}_1^-$ have $\mathbb{C}^*$-actions which can differ only by a character of $\mathbb{C}^*$ (since all holomorphic functions on $\mathbb{P}_3 - \mathbb{P}_1^-$ are constant). From (3.6) we can then deduce that, as equivariant bundles,

$$L^+ \cong \mathscr{H}^{-k} \otimes L^{p+k/2}, \quad L^- \cong \mathscr{H}^{-k} \otimes L^{-p-k/2} \tag{3.7}$$

Hence the homomorphism $L^+ \to F$, tensored by $\mathscr{H}^k$, gives a homomorphism $L^{p+k/2} \to F(k)$, showing that the section σ^+ of $F(k)$ which vanishes on $\mathbb{P}_1^-$ is of weight $p + k/2$, i.e.

$$\lambda(\sigma^+) = \lambda^{p+k/2}\sigma^+, \quad \lambda \in \mathbb{C}^* \tag{3.8}$$

To examine the behaviour of σ^+ near $\mathbb{P}_1^-$ we shall use the decomposition (3.3) which, after twisting by $\mathscr{H}^k$, becomes (using (3.6)),

$$F(k)|_{\mathbb{P}_1^-} \cong L^{-p-k/2} \oplus H^{2k} \otimes L^{p-k/2} \tag{3.9}$$

12 Also the canonical bundle N^* of $\mathbb{P}_1^-$ is

$$N^* = H^{-1} \otimes L \otimes \mathbb{C}^2$$

Hence terms of weight $p + k/2$ in the normal Taylor series of σ^+ can arise in just two ways, namely from

$$S^k(N^*) \otimes H^{2k} \otimes L^{p-k/2}$$

and

$$S^{2p+k}(N^*) \otimes L^{-p-k/2}.$$

This shows that, locally, σ^+ is given by a pair of functions (f, g) where

$$\deg g = k, \quad \deg f = 2p + k. \tag{3.10}$$

This implies that $\mathbb{P}_1^-$ is a zero of multiplicity $k(2p+k)$ which proves that

$$c_2(F(k)) = 2pk + k^2$$

so that

$$c_2(F) = 2pk$$

as we have already proved by other methods.

Since (3.9) extends as an exact sequence (rather than as a direct sum) to $\mathbb{P}_3 - \mathbb{P}_1^+$, the function g in (3.10) is globally defined (as a section of H^k), but f is only defined locally. □

4 The mini-twistor picture

In the preceding section, I described the structure of bundles on the twistor space $\mathbb{P}_3$ corresponding to S^1-invariant instantons. Because these 13 bundles are acted on by $\mathbb{C}^*$ it is possible to descend these bundles to the quotient space

$$Q = \frac{\mathbb{P}_3 - (\mathbb{P}_1^+ \cup \mathbb{P}_1^-)}{\mathbb{C}^*} \tag{4.1}$$

Since any point x in $\mathbb{P}_3 - (\mathbb{P}_1^+ \cup \mathbb{P}_1^-)$ lies on a unique transversal to $\mathbb{P}_1^+$ and $\mathbb{P}_1^-$ this transversal is the closure of the $\mathbb{C}^*$-orbit through x. This shows that

$$Q \cong \mathbb{P} \times \mathbb{P}_1^- \tag{4.2}$$

so that Q is (abstractly) a quadric surface. Moreover the real structure on P_3 induces a real structure on Q which interchanges the two factors, and the real points form the "anti-diagonal" Q^σ consisting of pairs $(y, \sigma(y))$. Thus the bundle F on $\mathbb{P}_3$ descends to a bundle, say $\mathscr{F}$, on Q, and $\mathscr{F}$ also has a real structure.

Since $\mathbb{P}_1^+$ and $\mathbb{P}_1^-$ have co-dimension 2 in $\mathbb{P}_3$ two bundles on $\mathbb{P}_3$ which are isomorphic on $\mathbb{P}_3 - (\mathbb{P}_1^+ \cup \mathbb{P}_1^-)$ are automatically isomorphic on $\mathbb{P}_3$. Thus $\mathscr{F}$ uniquely determines F, so that no information has been lost on descending to Q. We proceed now to reinterpret the results on § 3 in terms of bundles on Q.

The sub-line-bundle L^+ of F descends to a sub-line-bundle $\mathscr{L}^+$ of $\mathscr{F}$. If we denote by H_+ and H_- the Hopf bundles on P^+ and $\mathbb{P}_1^-$ pulled back to Q by the factorization (4.2), then $\mathscr{L}^+$ must be of the form $H_+^\alpha \otimes H_-^\beta$ for some integers α, β. To find these we use (3.7) which can also be rewritten

$$L^+ \cong \pi_+^*(H^{-k} \otimes L^p) \cong \pi_1^*(H^{-k} \otimes L^{+k}) \tag{4.2}$$

14 where $\pi_\pm : \mathbb{P}_3 - \mathbb{P}_1^\mp \to \mathbb{P}_1^\pm$ are the natural linear projections. Restricting L^+ to the plane $\pi_+^{-1}(y)$ for $y \in \mathbb{P}_1^+$ and using the first isomorphism in (4.2) then shows that

$$\mathscr{L}^+|_{y \times \mathbb{P}_1^-} \cong H_-^p \tag{4.3}$$

so that $\beta = p$. Similarly the second isomorphism in (4.2) shows that $\alpha = -(p+k)$. Hence

$$\mathscr{L}^+ \cong H_+^{-p-k} \oplus H_-^p; \tag{4.4}$$

similarly, we find

$$\mathscr{L}^- \cong H_+^p \oplus H_-^{-p-k} \tag{4.5}$$

Thus the bundle $\mathscr{F}$ on Q has two different splittings:

$$\begin{gathered} 0 \to \mathscr{L}^+ \to \mathscr{F} \to (\mathscr{L}^+)^* \to 0 \\ 0 \to \mathscr{L}^- \to \mathscr{F} \to (\mathscr{L}^-)^* \to 0 \end{gathered} \tag{4.6}$$

where $\mathscr{L}^+$ and $\mathscr{L}^-$ are given by (4.4) and (4.5). Moreover these splittings are interchanged by the linear structure.

These two splittings coincide where the composite homomorphism

$$\mathscr{L}^+ \to \mathscr{F} \to (\mathscr{L}^-)^*$$

is zero. This is given by the vanishing of a section s of the line bundle

$$(\mathscr{L}^+ \otimes \mathscr{L}^-)^* \cong (H_+ \otimes H_-)^k \tag{4.7}$$

15 The zero set of s is therefore a curve S on Q of bidegree (k,k). The curve S is real and the line-bundles $\mathscr{L}^+$ and $\mathscr{L}^-$. coincide over S so that

$$H_+^{2p+k}|_S \cong H_-^{2p+k|S} \tag{4.8}$$

More geometrically this means that, if D_+, D_- are the divisors cut out on S by the two systems of generators of Q, then

$$(2p+k)(D_1 - D_2) \approx 0 \tag{4.9}$$

where $\approx$ denotes linear equivalence on S.

Following Hitchin [7], we shall call S the *spectral curve* of the hyperbolic monopole. Just as in [7], S determines the monopole uniquely. In fact consider the exact sequence of sheaves on Q

$$0 \to \mathcal{O}(-2p-2k, 2p) \to \mathcal{O}(-2p-k, 2p+k) \to \mathcal{O}_S \to 0$$

where $\mathcal{O}(\alpha,\beta)$ denotes sections of $H_+^{\alpha} \otimes H_-^{\beta}$ and we have used (4.8). Taking

$$\delta(1) \in H^1(Q, \mathcal{O}(-2p-2k, 2p)) = H^1(Q, (\mathscr{L}^+)^2)$$

where δ is the coboundary in the cohomology exact sequence we get the element which defines the first extension in (4.6) and so recover $\mathscr{F}$.

Finally, we note that $\mathscr{F}$, restricted to a generator of Q of either system, has constant type $H^p \oplus H^{-p}$. This follows at once by using the splittings (4.6) together with the isomorphisms (4.4) and (4.5). Thus, for restricted to a $\mathbb{P}_-$ generator, we use the first extension of (4.6) and get an extension

$$0 \to H_-^p \to \mathscr{F}|_{\mathbb{P}_-} \to H_-^{-p} \to 0$$

which splits since $p > 0$. 16

The fact that the restriction of $\mathscr{F}$ to the generators is not generically trivial means that the extension classes in (4.6) are very special. This is borne out by a parameter count which shows that the space of all extensions of the form (4.6) is much larger than the moduli space of monopoles. Moreover the fact that decomposition is of constant type, i.e. that there are no special generators, guarantees that the bundle $\mathscr{F}$ when lifted back to $\mathbb{P}_3 - \mathbb{P}_1^+ \cup \mathbb{P}_1^-)$ *extends* to $\mathbb{P}_3$. Special generators would mean that $\mathscr{F}$ would only extend as a torsion-free sheaf, having a finite number of special points on $\mathbb{P}_1^+$, $\mathbb{P}_1^-$ where it was not locally free.

When $k = 1$, the spectral curve S, being of bidegree $(1, 1)$ is a conic section and so corresponds to a point 0 in H^3. All geodesics through this point are then “spectral lines”. Moreover since the monopole is uniquely determined by S it follows that it is rotationally symmetric about 0. We may therefore refer to 0 as the *centre* of the monopole, and the situation is quite analogous to that for Euclidean 3-space.

5 The limiting process

Comparison of the results of § 4 with those of Hitchin [7] show that the mini-twistor pictures of hyperbolic and Euclidean monopoles are quite similar. The holomorphic bundle on the mini-twistor space has in each case two canonical splittings (conjugates of each other) and a spectral curve where they coincide, which determines the monopole. The main difference is that in the Euclidean case the splittings are defined by asymptotic behaviour, where as in the hyperbolic case we have a compactification which incorporates infinity.

17 To pursue the comparison further, let us, following Hitchin, consider the case of $U(1)$-monopoles on hyperbolic space. For these, the connection is flat and the Higgs field is the constant i. This corresponds to the trivial line bundle on S^4 (or on its twistor space $\mathbb{P}_3$) with the standard S^1-action. As we saw in § 4, this descends from $\mathbb{P}_3$ to the quadric Q to give the line bundle $H_+^{-1} \otimes H_-$. This is therefore the analogue of Hitchin’s line bundle L on $T\mathbb{P}_1$ (the tangent bundle of $\mathbb{P}_1$). Comparing (4.4) and (4.6) with theorem (6.3) of [7] then shows that they are precisely of the same form.

So far, we have only considered the standard hyperbolic space with curvature -1. We now want to vary the curvature and then consider the limiting situation when the curvature tends to zero, giving flat Euclidean space. If, for some positive constant R, we rewrite (2.3) as

$$ds^2 = \frac{r^2}{R^2}\left\{R^2\left\{\frac{dx^2 + dy^2 + dr^2}{r^2}\right\} + R^2 d\theta^2\right\} \tag{5.1}$$

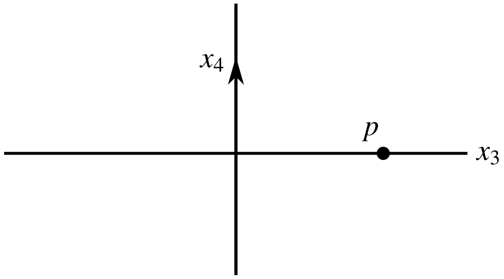
this replaces (2.4) by the conformal equivalence

$$\mathbb{R}^4 - \mathbb{R}^2 \sim H^3(R) \times S^1(R) \tag{5.2}$$

where $S^1(R)$ is the circle of radius R and $H^3(R)$ is hyperbolic space of curvature $-R^{-}$. An S^1-invariant instanton of weight p on R^4 thus defines a monopole on $H^3(R)$ in which, because of the scale change on the circle, the norm of the Higgs field tends to $R^{-1}p$ at infinity. In particular, taking $R = p$ we see that monopoles on H^3 with $|\phi| \to p$ at ∞ are equivalent, by this rescaling, to monopoles on $H^3(p^{-1})$ with $|\phi| \to 1$ at ∞. Note that in Euclidean space, because there is no absolute scale, we can always renormalize the Higgs field to have $|\phi| \to 1$.

18 The idea now is that, as $p \to \infty$, monopoles on the sequence of spaces $H^3(p^{-1})$ should in some appropriate sense converge to monopoles on Euclidean space, the Higgs field having throughout been normalized so that $|\phi| \to 1$ at ∞. More precisely, let us represent $H^3(R)$ as the ball of radius R in Euclidean 3-space. It is easy to show that, as $R \to \infty$, the hyperbolic metrics of these balls converge, on any compact set, to the Euclidean metric. It now makes sense to ask that a sequence of hyperbolic monopoles defined on each $B(p)$, should converge to a Euclidean monopole as $p \to \infty$. If the sequence converges smoothly, it is clear that the limit will indeed be a Euclidean monopole.

If we reinterpret hyperbolic monopoles as S^1-invariant instantons, this limiting procedure amounts to considering (backward) rotations in the (x_3, x_4)-plane having centres at $(0, 0, p, 0)$ and using as S^1-parameter the arc length of the orbit through the origin (divided by 2π). In terms of the generating vector fields, these are normalized to have length one at the origin. As $p \to \infty$, these vector fields converge, on any compact set, to the generator of translation in the x_4-direction.



These vector fields then induce holomorphic vector fields ξ_p on $\mathbb{P}_3$, generating C^*-actions and converging as $p \to \infty$ to a $\mathbb{C}$-action.

For the standard S^1-action considered in § 4 the $\mathbb{C}^*$-fibration over Q defined by (4.1) is easily seen (by calculations similar to those in § 4) to
19 define the line-bundle $H_+ \otimes H_-^{-1}$. Reversing the orientation of S^1 leads to the inverse $H_+^{-1} \otimes H$ while rescaling replaces the bundle by its powers. Hence the vector field ξ_p induces the standard action of $\mathbb{C}^* = \mathbb{C}/2\pi\mathbb{Z}$ on $L_p = H_+^{-p} \otimes H^p$ over the quadric Q_p: the principal bundle of L_p can be identified with the quotient of $\mathbb{P}_3 - (\mathbb{P}_1^+(\mathbb{P}) \cup \mathbb{P}_1^-(p))$ by Z_p, $(P_1(p)$ denoting the zeros of ξ_p).

If we introduce

$$\widetilde{Q}_p = Q_p - Q_p^\sigma \tag{5.3}$$

the complement of the real points of Q_p, then L_p is topologically trivial on $\widetilde{Q}_p$ and arises by exponentiation from an element of $H^1(Q_p, \mathscr{O})$. This element can be viewed geometrically as the -bundle given by the action $t \to \exp 2\pi t \xi_p$ on the universal covering space of $\widetilde{P}_p$, where $\widetilde{P}_p = \pi^{-1}(\mathbb{R}^2 - \mathbb{R}_p^2)$, R_p^2 is the axis of the pth rotation and $\pi : \mathbb{P}_3 \to S^4$ is the twistor fibration. Note that at this stage p could be any positive *real* number, the integrality is only needed if we deal with the whole of Q_p rather than the open set $\widetilde{Q}_p$.

As $p \to \infty$ the quadrics Q_p tend to the quadric cone $T\mathbb{P}_1$, and this degeneration is best viewed by realizing the Q_p as all embeded in a fixed projective 3-space. I shall now describe such a realization.

Recall first that TP_1 parametrizes oriented straight lines in the Euclidean 3-space $V = \mathbb{R}^3$, and that points of $\mathbb{R}^3$ correspond to real sections of $T\mathbb{P}_1$. One way to derive this correspondence is to introduce the affine dual V' of V, i.e. the space parametrizing affine planes in V. If we introduce the projective space $\mathbb{P}$ compactifying V and its projective dual $\mathbb{P}'$ then $V' \subset \mathbb{P}$ is the complement of one point. If x, y, z are coordinates for V, t a fourth homogeneous coordinate and

$$\xi x + \eta y + \zeta z + \tau t = 0 \tag{5.4}$$

the equation of a projective plance, then (ξ, η, ζ, τ) are homogeneous **20**
coordinates for $\mathbb{P}'$ and V' is the complement of the point $(0, 0, 0, 1)$. Now an affine line $L \subset V$ has a dual line $L' \subset V'$ and, if we complexity, the line $L'_{\mathbb{C}}$ meets the complex quadric cone

$$\xi^2 + \eta^2 + \zeta^2 = 0 \tag{5.5}$$

in a pair of complex conjugate points. An orientation of L gives a preferred choice of one of these points and in this way affine lines $L \subset V$ are parametrized by points of the cone (5.5), excluding the vertex $(0, 0, 0, 1)$. Note that (5.5) is the dual of the (purely imaginary) conic at ∞ in $\mathbb{P}_{\mathbb{C}}$ given by

$$x^2 + y^2 + z^2 = 0, \quad t = 0 \tag{5.6}$$

which arises from the Euclidean metric structure of V.

Now consider the hyperbolic space $H^3(R)$ and identify it as before with the ball $B(R) \subset V$. A hyperbolic plane in $B(R)$ is determined by its “circle at ∞”, i.e. on the boundary of $B(R)$. This is cut out by a plane in V, so that the affine dual $B(R)'$ of $B(R)$ can be identified with the subspace of V' representing planes in V whose distance to the origin is less than R. Thus $B(R)$ is given by the inequality

$$\xi^2 + \eta^2 + \zeta^2 - R^{-2}\tau^2 > 0. \tag{5.7}$$

A geodesic in $B(R)$ is therefore represented by a line L' inside the region (5.7). The analogue of (5.5) is the complex quadric in $\mathbb{P}'_{\mathbb{C}}$ given by the equation

$$\xi^2 + \eta^2 + \zeta^2 - R^{-2}\tau^2 = 0 \tag{5.8}$$

which is the dual of the equation **21**

$$x^2 + y^2 + z^2 - R^2t^2 = 0 \tag{5.9}$$

giving the complexification of the boundary of $B(R)$. The complexification $L'_{\mathbb{C}}$ meets (5.8) in a pair of conjugate points and, as before, an orientation of the geodesic gives a preferred choice. Note that as L' lies inside the region (5.7) it does not meet the quadric (5.8), so that our points are never real. In this way, oriented geodesics $B(R)$ are naturally parametrized by the non-real points $\widetilde{Q}(R)$ of the quadric $Q(R)$ given by (5.8).

As $R \to \infty$, the quadric $Q(R)$ tends to the quadric cone given by (5.5) and it is not hard to see that parametrization of (oriented) geodesics is continuous. Thus, if α_R is an oriented geodesic in $B(R)$ which converges (on compact sets) to a straight line as $R \to \infty$, then the corresponding points on $Q(R)$ converge to the required point on the cone.

I shall now describe the family of line-bundles L_R on $Q(R)$ and show that they converge to Hitchin's line bundle L on $Q({}^1\!/_2) = TP_1$.

Let U be the open set $\eta \neq i\xi$ and V the open set $\eta \neq -i\xi$. Their union covers $\widetilde{Q}(R)$ because the excluded line $\xi = \eta = 0$ intersects $Q(R)$ in the *real* points $(0, 0, 1, \pm R)$. The transition function

$$g_{UV} = \frac{\zeta + R^{-1}\tau}{\zeta - R^{-1}\tau}$$

is holomorphic and non-zero on $U \cap V \cap Q_R$ and it defines a line-bundle on Q_R. This is, in fact, the restriction of the line bundle $H_+^{-1} \otimes H_-$ on Q_R.

22 Consider now the line bundle L_R on $\widetilde{Q}(R)$ defined by the transition function $(g_{UV})^R$. As $R \to \infty$, this converges to the limit

$$g_{UV}^{\infty} = \exp(2\tau/\zeta) \tag{5.10}$$

defining a line-bundle L_∞ on $\widetilde{Q}(\infty) = T\mathbb{P}_1$. It remains to check that L_∞ coincides with Hitchin's line-bundle L. Now, in our coordinates, a point $(a, b, c) \in \mathbb{R}^3$ is represented by the plane

$$a\xi + b\eta + c\zeta = \tau. \tag{5.11}$$

If we now parametrize $\mathbb{P}_1$, defined by the homogeneous equation (5.4), by

$$\frac{\xi}{\zeta} = \frac{\rho - \rho^{-1}}{2}, \quad \frac{\eta}{\zeta} = \frac{\rho + \rho^{-1}}{2i}$$

then (5.11) becomes

$$\tau/\zeta = -{}^1\!/\!{}_2 \left\{ (a + ib)\rho^{-1} - 2c - (a - ib)\rho \right\} \tag{5.12}$$

Comparing this with (3.2) of [1] shows that our coordinates are related to Hitchin's by

$$\rho \to \zeta \quad \tau/\zeta \to -\eta/2.$$

Thus our formula (5.10) coincides with Hitchin's formula (5.4) in [7], showing that out L_∞ is his L.

6 The moduli space

In [2] I showed, using a recent result of Donaldson [4], that the moduli space of (based) monopoles of charge k on the hyperbolic space H^3 can naturally be identified with the space of based holomorphic maps 23 $\mathbb{P}_1 \to \mathbb{P}_1$ of degree k, i.e. with the space of rational functions

$$f(z) = \frac{a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k}{z^k + b_1 z^{k-1} + \cdots + b_k}$$

where the numerator and denominator have no common factor. In [2], this was deduced as a special case of a more general result for instantons. I shall now give a more direct and elementary treatment for the case of monopoles alone. Again this rests on [4] and the proof is not essentially different from that in [2], but the presentation is simpler and gives more insight into the way the moduli space arises.

An instanton on $\mathbb{R}^4$ (or S^4) defines a holomorphic bundle on the twistor space $\mathbb{P}_3$ with a real structure (and satisfying further a non-degeneracy condition). If we pick a plane $\mathbb{P}_2$ in $\mathbb{P}_3$ and restrict the bundle, we get a holomorphic bundle on $\mathbb{P}_2$ trivial on the unique real line in $\mathbb{P}_2$. Donaldson's result in [4] asserts that this process identifies the moduli space of based instantons on S^4 with that of holomorphic bundles on $\mathbb{P}_2$ trivialized on $\mathbb{P}_1$. Here a base bundle means a bundle with a preferred choice of trivialization over some base point. The important point in this result of Donaldson is that the bundles on $\mathbb{P}_2$ are purely holomorphic - there is no real or unitary restriction, and $SU(2)$ gets replaced by $SL(2, \mathbb{C})$.

For an S^1-invariant instanton we pick our $\mathbb{P}_2$ to contain the fixed line $\mathbb{P}_1^+$ and we take as base point its intersection A with $\mathbb{P}_1^-$. Then $\mathbb{P}_2$ is also S^1-invariant and the naturality of Donaldson's theorem implies that the moduli space of based monopoles on H^3 can be identified with the moduli space of S^1-invariant based bundles on $\mathbb{P}_2$. Note that the S^1-
24 action does not preserve the trivilization over the base point A. In fact, the representation of S^1 on the fibre over A is determined by the Higgs field.

The action of S^1 complexifies to an action of $\mathbb{C}^*$ and both the bundle on $\mathbb{P}_2$ and the $\mathbb{C}^*$-action are necessarily *algebraic*. The classification of such objects is a fairly elementary matter as we shall now see. As a first step, let us prove:

Lemma 6.1. *Let E be an algebraic vector bundle of rank n over $\mathbb{C}^2$ with an action of $\mathbb{C}^*$ covering the scalar action on $\mathbb{C}^2$. Then E is $\mathbb{C}^*$-isomorphic to the product $\mathbb{C}^2 \times E_O$.*

Proof. $\mathbb{C}^*$-bundles on $\mathbb{C}^2 - 0$ are equivalent to bundles on the quotient space $\mathbb{P}_1$. By Grothendieck's theorem, every vector bundle on $\mathbb{P}_1$ is a direct sum of powers of the Hopf bundle. Hence on $\mathbb{C}^2 - 0$ every $\mathbb{C}^*$-vector bundle is isomorphic to a product bundle of the form $(\mathbb{C}^2 - 0) \times V$, where V is a representation of $\mathbb{C}^*$ (and so a sum of one-dimensional representations). Thus our bundle E and the product $\mathbb{C}^2 \times V$ are $\mathbb{C}^*$-isomorphic on $\mathbb{C}^2 - 0$. The isomorphism then necessarily extends to $\mathbb{C}^2$ and induces in particular an isomorphism of representations of $\mathbb{C}^*$ between E_O and V.

We shall be interested in the case $n = 2$ and representations with weights $(p, -p)$ for $p > 0$. The proof of Lemma 6.1 also shows that the $\mathbb{C}^*$-automorphisms of $E = \mathbb{C}^2 \times E_O$ are all of the form

$$(z_1, z_2; u, v) \to (z_1, z_2; au + c(z_1, z_2)v, bv) \tag{6.2}$$

where a, b are non-zero constants and $c(z_1, z_2)$ is a homogeneous polynomial of degree $2p$. Here $\lambda \in \mathbb{C}^*$ is taken to act on E_O by

$$(u, v) \to \left(\lambda_u^p, \lambda_v^{-p}\right).$$

By lifting a bundle from $\mathbb{C}^1$ to $\mathbb{C}^2$ and applying 6.1 and (6.2), we deduce: □

Corollary 6.3. *Lemma 6.1 holds with $\mathbb{C}^1$ instead of $\mathbb{C}^2$ and automorphisms for $n = 2$ and weights $(p, -p)$ are given by* 25

$$(z; u, v) \to (z;\ au + cz^{2p}, bv) \quad \textit{with} \quad ab \neq 0$$

Remark. The analogue of 6.1 holds for bundles over $\mathbb{C}^m$ for any m, but we only need the cases $m = 1, 2$ and the proof for $m \geq 3$ is slightly more complicated.

Suppose next that E is a $\mathbb{C}^*$-vector bundle over $\mathbb{P}_1 = \mathbb{C} \cup \infty$. Then we have representations E_O and E_∞ which by (6.3) determine E restricted to the two open sets $\mathbb{P}_1 - \infty$ and $\mathbb{P}_1 - 0$. Over the intersection $\mathbb{C}^*$-bundles are equivalent to bundles over a point, so that the data needed to construct E from E_O and E_∞ is just a vector space isomorphism $T : E_O \to E_\infty$ (not necessarily commuting with the action of $\mathbb{C}^*$). For example, T could be fixed by looking at the fibre over the point $z = 1$. Changing the identification

$$E|_{\mathbb{P}_1 - \infty} \cong (\mathbb{P}_1 - \infty) \times E_O$$

by an automorphism of $(\mathbb{P}_1 - \infty) \times E_O$ alters T to TS. In the case we want, when $n = 2$ and the weights of E_O are $(p, -p)$, Corollary (6.3) shows that S is (relative to the standard basis) given by *a triangular matrix*. The coset space of the triangular matrix group in $GL(2, \mathbb{C})$ is just the projective line. More invariantly, let

$$\mathbb{P}(T) : \mathbb{P}(E_O) \to \mathbb{P}(E_\infty)$$

be the projective isomorphism induced by T, and let $p_O \in P(E_O)$ correspond to the positive weight vector. This is the point fixed by the automorphism S. Hence the left cosets of S are determined by the point

$$\mathbb{P}'(T)(p_O) \in \mathbb{P}(E_\infty). \tag{6.4}$$

26

Summing up this discussion, we see that we have proved the following:

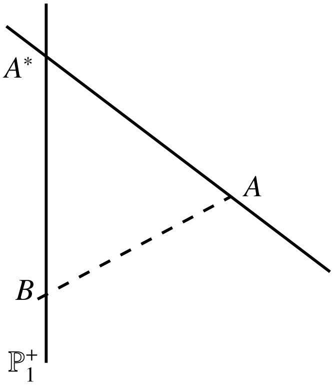
Proposition 6.5. *Let E be a $\mathbb{C}^*$-vector bundle of rank 2 over P_1 with E_O having weights $(p, -p)$, and with a preferred $\mathbb{C}^*$-isomorphism*

$$E|_{\mathbb{P}_1 - 0} \cong (\mathbb{P}_1 - 0) \times E_\infty .$$

Then E corresponds canonically to a point of $\mathbb{P}(E_\infty)$.

Remark. The matrix $T : E_O \to E_\infty$ defined earlier should be viewed as a kind of "scattering matrix" and the point in $\mathbb{P}(E_\infty)$ as the ratio of the "reflection/transmission" coefficients. I shall have more to say about this later.

We are now ready to return to the $\mathbb{C}^*$-bundle E over $\mathbb{P}_2$ corresponding, by Donaldson, to an S^1-invariant instanton and so to a hyperbolic monopole. Recall that $\mathbb{P}_2$ contains a fixed point A *and* a fixed line $\mathbb{P}_1^+$, and a unique real line, joining A to its conjugate point A^* on $\mathbb{P}_1^+$



The bundle E is trivial on the line AA^* and hence on all nearby lines through A^*. Applying the action of $\mathbb{C}^*$ then shows that E is trivial on all lines through A^* except $\mathbb{P}_1^+$. The trivialization of E on AA^* then extends to all these lines. In particular we get a distinguished $\mathbb{C}^*$-isomorphism

$$E|\mathbb{P}_2 - \mathbb{P}_1^+ \cong (\mathbb{P}_2 - \mathbb{P}_1^+) \times E_A .$$

27

Now apply (6.5) to the bundle E restricted to a variable line BA using a third fixed line through A^* to define the unit point (and so identifying BA with the standard $\mathbb{P}_1$). We deduce that E corresponds canonically to a rational map

$$f : \mathbb{P}_1^+ \to \mathbb{P}(E_A). \tag{6.6}$$

Actually, our construction only defines f on $\mathbb{P}_1^+ - A^*$ but, because the trivializations emanate from A^* it is easy to see that f extends by continuity to the whole of $\mathbb{P}_1^+$.

If we compare the definition of (6.6) with proposition (3.5) we see that the poles of f occur precisely at the k points B for which BA is a jumping line of E. To be precise, we parametrize $\mathbb{P}(E_A) \cong \mathbb{P}(E_{A^*})$ so that ∞ corresponds to the minimum weight vector, and 0 to the maximal weight vector. Note that $f(A^*) = 0$.

This completes the proof that the moduli space of based monopoles of charge k on H^3 is the space M_k of based rational maps $\mathbb{P}_1 \to \mathbb{P}_1$ of degree k. Note that the moduli space is independent of the weight p, i.e. of the norm of the Higgs field at ∞. This makes it very plausible that, in the limiting procedure described in § 5 for $p \to \infty$, the moduli spaces should converge to the corresponding moduli space for Euclidean monopoles, which has already been proved by Donaldson [5] to coincide with M_k. Moreover if we allow non-integral values of p we should get a nice continuous one-parameter family of moduli spaces.

So far we have defined the rational map f in (6.6) purely from the holomorphic point of view. In fact it has quite a simple interpretation from the monopole point of view. To see this let us first project from the twistor space $\mathbb{P}_3$ back to S^4. Then our plane $\mathbb{P}_2$ maps to S^4 with the **28** line AA^* collapsing to the point at ∞ and $\mathbb{C}^2 = \mathbb{P}_2 - (AA^*)$ gets identified with $\mathbb{R}^4$. This identification is compatible with the $\mathbb{C}^*$-action so the fixed line $\mathbb{P}_1^+$ (with A^* deleted) maps to the $\mathbb{R}^2$ axis in R^4, while the lines BA (with A deleted) map to the plane orthogonal to this axis. As before we introduce coordinates (x, y, r, θ).

The rational map f determined by a monopole is defined at a point $x_0 + iy_0$ by considering the $\mathbb{C}^*$-bundle over the projective line $\mathbb{P}_1(x_0, y_0)$

(obtained by compactifying the plane $x = x_0$, $y = y_0$) and applying Proposition (6.5). Here the complex coordinate of this $\mathbb{P}_1$ is $w = re^{i\theta}$. The procedure to obtain the value $f(x_0 + iy_0)$ is now as follows.

Over $\mathbb{C} = \mathbb{P}_1 - \infty$ our bundle can be described as a product $\mathbb{C} \times \mathbb{C}^2$ with coordinates $(w; u, v)$ and $\mathbb{C}^*$-action given by

$$\lambda(w; u, v) = (\lambda w; \lambda^p_u, \lambda^{-p}_v).$$

There are two distinguished (invariant) sections, namely

$$\begin{aligned} s_+ &: u = w^p, \quad v = 0, \\ s_- &: u = 0, \quad v = w^{-p}. \end{aligned}$$

Notice that the first is holomorphic on $\mathbb{C}$ while the second has a pole at 0. Similarly over $\mathbb{P}_1 - 0$ we have coordinates $(w'; u', v')$ with similar formulae, and distinguished sections s'_+, s'_- with s'_+ holomorphic at ℓ. In the overlap $\mathbb{P}_1 - (0 \cup \infty)$ we have the "scattering matrix" T expressing $s_\pm$ in terms of $s'_\pm$:

$$\begin{aligned} s_+ &= as'_+ + bs'_- \\ s_- &= cs'_+ + ds'_- \end{aligned}$$

29 with a, b, c, d constants. The value of $f(x_0 + iy_0)$ is then a/b: note that a pole of f, i.e. having $b = 0$, corresponds to $\mathbb{P}_1$ being a jumping line.

We shall now translate this into monopole terminology on the hyperbolic space H^3, given as the upper half space

$$(x, y, r)\ r > 0. \tag{6.7}$$

If we put $\rho = \log r$, so that $\rho + i\theta = \log w$, holomorphic sections s of E over $\mathbb{P}_1$ are defined to be solutions of the equation

$$\nabla_{\rho^s} + i\nabla_{\theta^s} = 0 \tag{6.8}$$

where ∇_ρ and ∇_θ are the components of the connection in the ρ, θ directions. If, as usual, we work in an S^1-invariant gauge and assume s independent of θ (i.e. S^1-invariant), equation (6.8) becomes

$$\nabla_{\rho^s} + i\phi_s = 0 \tag{6.9}$$

where ϕ is the Higgs field. This (up to a sign convention) is the hyperbolic analogue of the equation introduced by Hitchin. The sections $s_\pm$ correspond to solutions of (6.9) which, as $\rho \to -\infty$ (i.e. $r \to 0$), satisfy the asymptotic conditions

$$s_+ \sim \exp(p\rho), \quad s_- \sim \exp(-p\rho)$$

so that s_+ is the decaying solution (unique up to a constant). Similarly $s'_\pm$ correspond to solutions of (6.9) which, as $\rho \to \infty$ (i.e. $r \to \infty$), satisfy the asymptotic conditions

$$s'_+ \sim \exp(-p\rho), \quad s'_- \sim \exp(p\rho)$$

so that s'_+ is again the decaying solution.

30 To sum up, therefore, we have the following procedure to assign a rational function $f(x + iy)$ to a monopole on H^3. For each fixed x, y consider the line in H^3, with r varying and put $\rho = \log r$. Consider the solutions of the linear ordinary differential equation (6.9). Start with the solution which decays exponentially as $\rho \to -\infty$ and express this as a linear combination of the solutions s'_+ and s'_- as $\rho \to \infty$:

$$s_+ = as'_+ + bs'_-$$

Then define

$$f(x + iy) = a/b.$$

Note that, for this to be well defined we need to fix the choice of $s'_\pm$ as $\rho \to \infty$. This is where the gauge fixing at ∞ is used.

Clearly f has poles precisely when $b = 0$, i.e. when our solution decays exponentially at both ends. These are the *spectral lines* defined by Hitchin, and as we have seen, they correspond to the jumping lines.

Thus a monopole is determined by part of its “scattering data” in a fixed direction. Notice, however, that, in usual scattering theory, one used imaginary exponentials and the analogue of (6.9) is a self-adjoint equation.

For a monopole of charge 1, the associated rational function f is just

$$f(z) = a/(z - b) \tag{6.10}$$

with $a \neq 0$. Since $z = b$ gives a spectral line it follows that the centre of the monopole in $H^3 = \mathbb{C} \times \mathbb{R}^+$ is at a point of the form (b, λ), for some real positive number λ. By considering the action of the multiplication
31 group $z \to cz$ one can see that λ is proportional to $|a|$ (the proportionality factor depends on our choice of normalization and is possibly 1), and $\arg(a)$ represents the "phase angle" of the monopole.

More generally, consider a rational function with simple poles.

$$f(z) = \sum_{i=1}^{k} a_i/(z - b_i) \tag{6.11}$$

where the points $b_1, \ldots, b_k$, are 'far apart'. We would like to argue that this corresponds approximately to a superposition of k monopoles of the type (6.10). This can be justified on the following lines. Fix one b_i say b_1 and translate so that this is zero. Now consider the family of rational functions

$$f(z) = (a_1/z) + \sum_{i=1}^{k} a_i/(a - \lambda p_i)$$

and consider the limit as $\lambda \to \infty$. Reverting to their interpretation as holomorphic bundles on $\mathbb{P}_2$ we see that this family converges to a torsion free sheaf, locally free on $\mathbb{P}_2 - (AA^*)$, and having a unique jumping line AB (B the origin). Passing back to the monopole picture shows that the monopoles parametrized by the family f converge (on compact subsets of H^3) to the monopole of charge 1 represented by a_1/z. This shows that (6.11) looks approximately like the first monopole near the origin, provided all $b_i (i > 2)$ are sufficiently large. Repeating the argument for each i then justifies the final claim that the monopole represented by (6.11) does indeed approximate a superposition of k simple monopoles.

Our procedure for assigning a rational function to a hyperbolic monopole appears to extend naturally to the case of Euclidean monopoles. One considers Hitchin's equation (6.9) p now standing for the
32 third Euclidean coordinate, and uses the same scattering prodecure. The fact that the resulting map is indeed holomorphic in $(x + iy)$ is a consequence of the Bogomolny equations. It would be interesting to compare

this with Donaldson's way [5] of constructing a rational function (using Nahm's approach). It seems plausible that the two constructions will agree[1], and that the scattering procedure is continuous as the curvature of hyperbolic space is allowed to tend to zero. This would then show that the rational function (6.11) also represents an approximate superposition of Euclidean monopoles as suggested by Donaldson.

References

[1] M.F. Atiyah, The Geometry of Yang-Mills fields, *Fermi Lectures*, Scuola Normale Superiore, Pisa (1979).

[2] M.F. Atiyah, Instantons in two and four dimensions, *Commun. Math. Phys.* 93 (1984), 437-452.

[3] M.F. Atiyah and R. Bott, The moment map and equivariant cohomology, *Topology*, Vol. 23, No. 1, pp. 1-28 (1984).

[4] S.K. Donaldson, Instantons and geometric invariant theory, *Commun. Math. Phys.* 93 (1984), 453-460.

[5] S.K. Donaldson, Nahm's equations and the classification of monopoles, *Commun. Math. Phys.* 96 (1984), 387-408.

[6] P. Forgacs, Z. Horvath and L. Palla, An exact fractionally charged self-dual solution, Hungarian Academy of Sciences Preprint, KFKI 60 (1980).

[7] N.J. Hitchin, Monopoles and geodesics, *Commun. Math. Phys.* 83 **33**
(1982) 579-602.

[8] N.J. Hitchin, On the construction of monopoles, *Commun. Math. Phys.* 89 (1983), 145-190.

---

[1] This has now been established by J. Hurtubise Commun – *Math. Phys.* 100 (1985), 193-196.

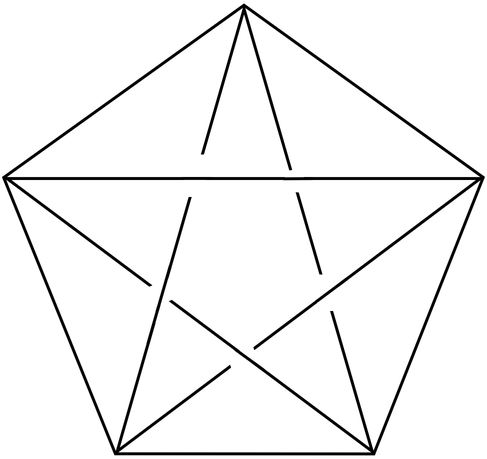
[9] W. Nahm, All multimonopoles for arbitrary gauge groups (preprint) TH 3172 - CERN (1982).

[10] R.S. Ward, A Yang - Mills - Higgs monopole of charge 2. *Commun. Math. Phys.* 79 (1981), 317-325.

Mathematical Institute
St. Giles, Oxford, U.K.

Shioda's Modular Surface $S(5)$ and the Horrocks-Mumford Bundle

By W. Barth, K. Hulek and R. Moore



1 Introduction

38

1.1 Results and motivation

The Horrocks-Mumford bundle $\mathscr{F}$ on $\mathbb{P}_4$ was discovered in 1972 and it is still essentially the only known indecomposable rank-2 bundle on $\mathbb{P}_4$. Since 1972, all efforts to construct more such bundles or to disprove their existence have been in vain. In this situation, we believe it important to understand better the bundle $\mathscr{F}$ itself. Our paper is a step in this direction.

In this paper, we prove the following facts (see section 5.5): The variety of *jumping lines* of $\mathscr{F}$ is an *irreducible rational* 4*-fold* J. The *double* (or triple) *jumping lines* form a *surface* $J_2 \subset J$. Except for singularities along this surface the 4-fold J is *smooth*.

The main tool in our study of the 4-fold J is a birationally equivalent smooth 4-fold $\widetilde{J}$, which in its turn is constructed as the *blow-up of* $\mathbb{P}_4$ with respect to the *ideal sheaf* of a singular surface S_{15} of degree 15 in $\mathbb{P}_4$. This surface S_{15} is a copy of *Shioda's elliptic modular surface* $S(5)$ immersed into $\mathbb{P}_4$ in such a way that it acquires 30 double points, where two branches of the surface meet.

The surface S_{15} is (the closure of) the variety swept out by all *quintic elliptic normal curves*, embedded H_5*-equivariantly* in $\mathbb{P}_4$ for the Schroedinger action of the Heisenberg group H_5 on $\mathbb{P}_4$. Alternatively S_{15} can be described as the *determinantal locus* in $\mathbb{P}_4$ where the matrix of homogeneous polynomials

$$\begin{pmatrix} z_0^2 & z_1^2 & z_2^2 & z_3^2 & z_4^2 \\ z_2z_3 & z_3z_4 & z_4z_0 & z_0z_1 & z_1z_2 \\ z_1z_4 & z_2z_0 & z_3z_1 & z_4z_2 & z_0z_3 \end{pmatrix}$$

39 drops its rank. Or S_{15} can be considered as the image of the surface $S(5)$ under the linear system $|I + 2F|$, where F is the class of an elliptic fibre and I is a divisor on $S(5)$ recentrly considered in another context by Inoue and Livné.

The systematic use of this surface S_{15} is the basic new achievement in our paper. Calculations which have been unaccessible so far become practicable if one uses for appearing parameter points $x \in \mathbb{P}_4$ the distinction, whether they lie outside of S_{15}, on S_{15} away from the 30 double points, or belong to the double points. We study the properties of S_{15} quite carefully. This is motivated mainly by the following three reasons:

1. Elliptic quintic normal curves are mentioned in relation with bundle $\mathscr{F}$ already in [6]. So the appearance of the surface S_{15} is quite natural. Totally unexpected for us is, however, that Shioda's surface $S(5)$ is related to $\mathscr{F}$ in many more ways than the one given in this paper. In sections 1.2, 1.3 and 1.4 we describe what we know about this at present, but *were unable to present with proofs in this paper* for the lack of time.

2. The divisor I is interesting, because Inoue and Livné use it to construct a surface of general type with $c_1^2 = 3c_2$. Little seems to **40** be known about the linear systems $|pI + qF|$, $p, q \in \mathbb{N}$. Now some of them arise naturally in connection with the Horrocks-Mumford bundle.

3. Shioda's modular surface $S(n)$ exists for each $n \in \mathbb{N}$, $n \geq 3$. The surface $S(3)$ is the blow up of $\mathbb{P}_2$ in the 9 base points of the Hesse pencil. This is the pencil of plane cubics invariant under the Heisenberg group H_3. The surface $S(4)$ is swept out by all H_4-invariant elliptic quartic normal curves in $\mathbb{P}_3$. In [9, p. 55] theta-functions are used to identify the image in $\mathbb{P}_3$ as the Fermat quartic. There it is also mentioned that similarly equations can be found defining $S(n)$ for higher n. Our computations are related to properties of theta-functions of level 5 in one variable.

1.2 The surface S_{25} (expository)

The smooth surface $S_{2p} \subset \mathbb{P}_1 \times \mathbb{P}_4$ is introduced in section 3.4. We call it S_{25}, because under the Segre map $\mathbb{P}_1 \times \mathbb{P}_4 \to \mathbb{P}_9$ it goes to a surface of degree 25 in $\mathbb{P}_9$. In fact this surface is the complete intersection of Grass $(\mathbb{P}_1 \subset \mathbb{P}_4) \subset \mathbb{P}_9$ with the (suitably normalized) Segre variety $\mathbb{P}_1 \times \mathbb{P}_4 \subset \mathbb{P}_9$. Notice $\deg(\mathbb{P}_1 \times \mathbb{P}_4) = \deg \text{ Grass } = 5$. Alternatively $S_{25} \subset \mathbb{P}_9$ can be described as the image of Shioda's surface $S(5)$ under the complete linear system $|I + 3F|$.

The surface S_{25} enjoys a strange self-duality. In this way, it parametrizes a 2-dimensional variety of planes in $\mathbb{P}_4$. It turns out that these are the planes Y_v, $v \in S_{15}$ (see 5.1) and that these are precisely the planes $\mathbb{P}_2 \subset \mathbb{P}_4$ on which the restricted bundle $\mathscr{F}|\mathbb{P}_2$ is unstable.

41 We have $c_1(\mathscr{F}(-3)) = -1$. If $\mathscr{F}|\mathbb{P}_2$ is unstable for some plane $\mathbb{P}_2 \subset \mathbb{P}_4$, then $\mathscr{F}(-3)|\mathbb{P}_2$ admits a unique section. This section vanishes in four points. The four pencils of lines in $\mathbb{P}_2$ through these four points are the jumping lines in the plane. The six lines connecting pairs among the four points are double jumping lines.

Upon closer inspection, one finds that each double jumping line lies in this way in three different planes.

1.3 The surface of double jumping lines (expository)

It is easy to see (cf. 5.2) that the only triple jumping lines are the 25 Horrocks-Mumford lines [6, p. 72]

$$z_k = z_{k+2} + \epsilon^j z_{k+3} = z_{k+1} + \epsilon^{3j} z_{k+4} = 0,\ j, k = 0, \ldots, 4,$$
$$\epsilon = e^{2\pi i/5}$$

and that there are no jumping lines of order ≥ 4. In this paper we show that the double jumping lines are parametrized by a surface $J_2 \subset$ Grass $(\mathbb{P}_1 \subset \mathbb{P}_4)$, see section 5.5. There is a birational correspondence between J_2 and the variety of sexti-secants of S_{15}. These sexti-secants are separated in $\widetilde{J}$ after the blow up of S_{15}. The birational map $\widetilde{J} \to J$ essentially consists of blowing down the proper transforms of these sexti-secants.

A "generic" surface in $\mathbb{P}_4$ should admit a finite number of sexti-secants only. But S_{15} comes with a two-dimensional variety. They arise in two ways:

42 (a) Joining a point P on a smooth fibre $E \subset S_{15}$ with $P' \in E$ such that $\mathscr{O}_E(P - P')$ is a 2-torsion class, one obtains a secant of S_{15}. But this secant meets two more smooth fibres in two other pairs

of points differing by a 2-torsion class and thus becomes a sexti-secant. The three fibres are related by the fact that their quotients $E/$(2-torsion class) are isomorphic as elliptic curves with level-5-structure.

(b) Shioda's surface $S(5)$ contains 25 sections $L_{i,j}$ differing by 5-torsion translations on the fibres and 25 3-sections $C_{i,j}$ which consist of all the points differing from $L_{i,j}$ by non-trivial 2-torsion translations on the fibres. It turns out that the images of these curves $C_{i,j}$ on S_{15} are plane sextics (with six nodes). All lines in their 25 planes are sexti-secants.

The sexti-secants of type (a) blown down parametrize the variety of double jumping lines. The sexti-secants of type (b) blown down to 25 points parametrize the 25 triple jumping lines.

The curves $C_{i,j}$ are realizations of Bring's curve C. This is the modular curve $\overline{\mathscr{H}/\Gamma_0(2,5)}$ where

$$H = \{z \in C : \ \mathrm{Im}\, z > \phi\} \quad \text{is the upper half-plane}$$

$$\Gamma_0(2,5) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z) : \begin{array}{l} a \equiv d \equiv 1(5) \\ b \equiv 0(5), c \equiv 0(10) \end{array} \right\}$$

There is a canonical $3:1$ map $C \to \mathbb{P}_1$ inducing a rational $3:1$ map $\rho : S_0(2,5) \to S(5)$, where $S_0(2,5)$ is Shioda's modular surface for the group $\Gamma_0(2,5)$. The fibres of $S(5)$ are elliptic curves with level-5-structure (see 4.2), whereas the fibres of $S_0(2,5)$ are elliptic curves 43
with level-5-structure and a distinguished non-trival 2-torsion element. Forgetting this 2-torsion element is the $3:1$ map $S_0(2,5) \to S(5)$.

So $S_0(2,5)$ birationally parametrizes pairs consisting of sexti-secants of type (a) and of one of their six point of intersection with S_{15}. The variety of sexti-secants themselves is the image of $S_0(2,5)$ under the $6:1$ map

$$S_0(2,5) \xrightarrow{\omega} S_0(2,5) \xrightarrow{\rho} S(5)$$

where ω is the quotient map with respect to the distinguished 2-torsion translation on the fibres. Consequently, J_2 being birationally equivalent to the variety of sexti-secants of type (a), is birational to $S(5)$ again. In fact it is nothing but $S(5)$ with its 25 sections blown down. The corresponding map $S(5) \to \mathrm{Grass}\,(\mathbb{P}_1 \subset \mathbb{P}_4) \subset \mathbb{P}_9$ is given by the linear system $|I + 3F|$. So far this is the only linear system $|pI + qF|$ on $S(5)$ where we know $h^0(\mathcal{O}_{S(5)}(pI + qF)) \neq 0$ and $h^1(\mathcal{O}_{S(5)}(pI + qF)) \neq 0$.

1.4 The surface S_{45}

If a plane varies in the family parametrized by S_{25}, so that the restriction of $\mathcal{F}$ to the plane is unstable, the four points in the plane, where the unique section violating stability vanishes, describe a new surface S_{45}. Its degree turns out to be 45. Again it is an image of Shioda's surface $S(5)$. This time the map is rational, blowing up the 60 double points
44 P_i of singular fibres on $S(5)$ (see section 4.3). It is given by the linear system $|3I + 5F - \Sigma P_i|$.

Associating with a point on S_{45} the (unique) unstable plane containing it we obtain a rational 4 : 1 map $S_{45} \to S_{25}$. This is essentially the quotient map $S(5) \to S(5)$ with respect to the four 2-torsion classes on the fibres.

A double jumping line is contained in three unstable planes and for each one of these contains two points from S_{45}. Strangely enough the double jumping lines being parametrized by sexti-secants of S_{15} are themselves sexti-secants of S_{45}.

1.5 Notations and conventions

The base field will be $\mathbb{C}$ throughout this paper. $\mathbb{P}_n$ will always denote $\mathbb{P}_n(\mathbb{C})$, the n-dimensional complex-projective space. $\mathbb{P}(V)$ is the space of lines in the vector space V. If $v \in V$ is a non-zero vector, we denote by v also the point $\mathbb{C}v \in \mathbb{P}(V)$. We apologize in advance for using this notation. But we believe that the context everywhere explains without ambiguity whether a vector is meant or a point in projective space. More generally, we even apply the following abuse of language: We do not distinguish between a simple p-vector (i.e., an exterior product $x_1 \wedge \ldots \wedge x_p \in \wedge^p V$, $x_i \in V$ linearly independent), the vector space in

V spanned by $x_1, \ldots, x_p$, and the corresponding projective subspace in $\mathbb{P}(V)$.$\mathbb{Z}_n$ denotes the cyclic group of order n.

45 For $X \subset \mathbb{P}_n$, a subvariety with ideal sheaf $\mathscr{I}_x \subset \mathscr{O}_{\mathbb{P}_n}$, we say that homogeneous polynomials $g_1, \ldots, g_k$ describe X schematically if $g_1, \ldots, g_k$ generate $\mathscr{I}_x$ at each point of $\mathbb{P}_n$

The first two authors are indebted to M. Reid and D. Burns for stimulating and helpful conversations which were extremely useful for this work.

2 Some linear algebra

2.1 Contraction and *-operators

Let V be some $\mathbb{C}$-vector space of dimension n. (We are interested only in the case $n = 5$). An element in $\Lambda^p V$ is called *p-vector*, and simple *p-vector* if it decomposes as an exterior product $v_1 \wedge \ldots \wedge v_p$, $v_k \in V$. An element in $\Lambda^p V^*$ is called *p-form*. There is the natural duality $\Lambda^p V^* \otimes \Lambda^p V \to \mathbb{C}$ defined by

$$\langle h_1 \wedge \ldots \wedge h_p, v_1 \wedge \ldots, \wedge v_p \rangle = \det(\langle h_k, v_l \rangle), h_k \in V^*, v_l \in V.$$

It extends to a *contraction map*

$$\Lambda^p V^* \otimes \Lambda^q V \to \begin{cases} \Lambda^{q-p} V & (p \leq q) \\ \Lambda^{p-q} V^* & (p \geq q) \end{cases}$$

$$g \otimes x \to \langle g, x \rangle$$

by

(1) $\langle f, \langle g, x \rangle \rangle = \langle f \wedge g, x \rangle$ for all $f \in \Lambda^r V^* (r = q - p \geq 0)$

(2) $\langle \langle g, x \rangle, y \rangle = \langle g, x \wedge y \rangle$ for all $y \in \Lambda^r V (r = p - q \geq 0)$.

This contraction is the adjoint operator to exterior multiplication. 46
One immediately checks formula (1) and (2) also for all r, $0 \leq r \leq q - p$, resp. $0 \leq r \leq p - q$. For $p = 1$, i.e., $h \in V^*$, and $x_1, \ldots, x_q \in V$ we have

(3) $\langle h, x_1 \wedge \ldots \wedge x_q \rangle = \sum_{k=1}^{q} (-1)^{q+k} \langle h, x_k \rangle x_1 \wedge \ldots \wedge \check{x}_k \wedge \ldots \wedge x_q.$

If $h \in V^*$, $x \in \Lambda^q V$ is arbitrary, then

(4) $\langle h, x \rangle = 0 \Leftrightarrow x \in \Lambda^q U$ with $U = \ker h \subset V$.

Each basis $e_1, \ldots, e_n \in V$ with dual basis $\widehat{e_1}, \ldots, \widehat{e_n} \in V^*$ induces isomorphisms $\Lambda^n V \to \mathbb{C}$, $x \to \langle \widehat{e_1} \wedge \ldots \wedge \widehat{e_n}, x \rangle$, and $\Lambda^n V^* \to \mathbb{C}$, $g \to (g, e_1 \wedge \ldots \wedge e_n \rangle$. More generally we obtain isomorphisms

$$\begin{array}{ll} \Lambda^p V \to \Lambda^{n-p} V^* & \Lambda^q V^* \to \Lambda^{n-q} V \\ x \to x^* := \langle \widehat{e_1} \wedge \ldots \wedge \widehat{e_n}, x \rangle & g \to g* = \langle g, e_1 \wedge \ldots \wedge e_n \rangle \end{array}$$

satisfying as consequence of (1) and (2)

(5) $\langle x^*, y \rangle = (x\, y)^* (x \in \Lambda^p V, y \in \Lambda^q V, p + q \leq n)$

$\langle f, g* \rangle = (f \wedge g) * (f \in \Lambda^p V^*, g \in \Lambda^q V^*, p + q \leq n).$

These isomorphisms are each other's inverses [4, § 8.5, prop. 4], i.e., $(x^*)* = x$ and $(f_*)* = f$.

47 **Lemma 1.** *If $n = 5$, for all $x, y, z \in \Lambda^2 V$ we have*

$$\langle (x \wedge y)^*, z \rangle + \langle (y \wedge z)^*, x \rangle + \langle (z \wedge x)^*, y \rangle = 0 \tag{6}$$

and in particular

$$\langle (x \wedge x)^*, x \rangle = 0 \tag{7}$$

Proof. By linearity formula (6) needs to be verified for simple 2-vectors $x = a_1 \wedge a_2$, $y = a_3 \wedge a_4$, $z = a_5 \wedge a_6$ only. The whole expression (6) is multilinear in $a_1, \ldots, a_6$. From (3) it follows that it is alternating in $a_1, \ldots, a_6$. The assertion then is a consequence of $\Lambda^6 V = 0$. □

2.2 The Horrocks-Mumford maps f^+ and f^-

From now on, V denotes a fixed copy of $\mathbb{C}^5$ with standard basis $e_0, \ldots, e_4$. We use the convention that subscripts are elements of $\mathbb{Z}_5$, i.e., $e_{i+5} = e_i$.

A vector $v \in V$ is always written $v = \sum_0^4 v_i e_i$. In [6] the two maps $f^\pm : V \to \Lambda^2 V$

$$f^+(v) = \sum v_i e_{i+2} \wedge e_{i+3} \qquad f^-(v) = \sum v_i e_{i+1} \wedge e_{i+4}$$

are introduced. In addition, we use the following abbreviations

$$\begin{aligned}
h^+(v) &:= {}^1\!/\!_2(f^+(v) \wedge f^+(v))^* = \sum v_{i+1} v_{i+4} \widehat{e_i} \\
h^-(v) &:= {}^1\!/\!_2(f^-(v) \wedge f^-(v))^* = -\sum v_{i+2} v_{i+3} \widehat{e_i} \\
h^\circ(v) &:= (f^+(v) \wedge f^-(v))^* = \sum v_i^2 \widehat{e_t} \\
v^+ = v^+(v) &:= -\langle h^\circ(v), f^+(v)\rangle = (\sum (v_{i+2} v_{i+4}^2 - v_{i-1}^2 v_{i+3}) e_i \\
v^- = v^-(v) &:= -\langle h^\circ(v), f^-(v)\rangle = \sum (v_{i+1} v_{i+2}^2 - v_{i+3}^2 v_{i+4}) e_i
\end{aligned}$$

48

We observe

$$h^\circ(v) \neq 0 \quad \text{for all} \quad 0 \neq v \in V. \tag{8}$$

As a consequence of (6) and (7), we find

$$\begin{aligned}
\langle h^-(v), f^+(v)\rangle &= \langle {}^1\!/\!_2(f^- \wedge f^-)^*, f^+\rangle = -\langle (f^+ \wedge f^-)^*, f^- >= v \\
\langle h^+(v), f^-(v)\rangle &= \langle {}^1\!/\!_2(f^+ \wedge f^+)^*, f^-\rangle = -\langle (f^+ \wedge f^-)^*, f^+\rangle = v^+ \\
\langle h^+(v), f^+(v)\rangle &= \langle {}^1\!/\!_2(f^+ \wedge f^+)^*, f^+) = 0 \\
\langle h^-(v), f^-(v)\rangle &= \langle {}^1\!/\!_2(f^- \wedge f^-)^*, f^-\rangle = 0
\end{aligned} \tag{9}$$

Using this and (1), we get

$$\begin{aligned}
\langle h^\circ(v), v^\pm\rangle &= -\langle h^\circ, \langle h^\circ, f^\pm\rangle\rangle = -\langle h^\circ \wedge h^\circ, f^\pm\rangle = 0 \\
\langle h^+(v), v^+\rangle &= \langle h^+, \langle h^+, f^-\rangle\rangle = \langle h \wedge h^+, f^-\rangle = 0 \\
\langle h^+(v), v^-\rangle &= -\langle h^+, \langle h^\circ, f^-\rangle\rangle = -\langle h^+ \wedge h^\circ, f^-\rangle \\
&= \langle h^\circ, \langle h^+, f^-\rangle\rangle = \langle h^\circ, v^+\rangle = 0 \\
\langle h^-(v), v^-\rangle &= \langle h^-(v), v^+\rangle = 0 \quad \text{similarly.}
\end{aligned} \tag{10}$$

The exterior products of the forms $h^+(v)$, $h^\circ(v)$, $h^-(v)$ are given by **49**

$$\begin{aligned}(h^\circ(v) \wedge h^\pm(v))_* &= {}^1\!/\!_2\langle h^\circ, f^\pm \wedge f^\pm\rangle && (5)\\ &= \langle h^\circ, f^\pm\rangle \wedge f^\pm && (3)\\ &= -v^\pm \wedge f^\pm(v)\end{aligned}$$

(11)

$$\begin{aligned}(h^+(v) \wedge h^-(v))_* &= {}^1\!/\!_2(h^+, f^- \wedge f^-) && (5)\\ &= \langle h^+, f^-\rangle \wedge f^- && (3)\\ &= v^+ \wedge f^-(v) = -v^- \wedge f^+(v)\\ (h^+(v) \wedge h^-(v) \wedge h^\circ(v))_* &= -((v^- \wedge f^+)^* \wedge h^\circ)_*\\ &= \langle h^\circ, v^- \wedge f^+\rangle && (5)\\ &= \langle h^\circ, v^-\rangle f^+ + v^- \wedge \langle h^\circ, f^+\rangle && (3)\\ &= v^+ \wedge v^-\end{aligned}$$

2.3 The condition $v^+ = v^- = 0$.

The next proposition gives several descriptions of the set $\{v \in V : v^+ = v^- = 0\}$.

50 **Proposition 1.** *For $v \in V$, the following properties are equivalent.*

(i) $v^+ = v^- = 0$

(ii) *$h^+(v)$, $h^\circ(v)$ and $h^-(v)$ are proportional*

(iii) *$\lambda f^+(v) + \mu f^-(v)$ is a simple bivector for two different ratios $(\lambda : \mu) \in \mathbb{P}_1$*

(iv) *v is a multiple of one of the 30 vectors*

$$\begin{aligned}&e_k, k = 0, \ldots, 4\\ &e_{kl} = (1, \epsilon^{k+1}, \epsilon^{2k+3l}, \epsilon^{4k})k, l = 0, \ldots, 4, \epsilon = e^{2\pi i/5}\end{aligned}$$

Proof. **iii) ⇒ ii):** By assumption, there are two different $(\lambda : \mu) \in \mathbb{P}_1$ with $\lambda^2 h^+(v) + \lambda\mu h^\circ(v) + \mu^2 h^-(v) = 0$. This implies that the forms $h^+(v)$, $h^\circ(v)$, $h^-(v)$ span a space of dimension 1.

ii) ⇒ i): By (11), we have $v^+ \wedge f^\pm(v) = v^- \wedge f^\pm(v) = 0$. If $v^+ \neq 0$ or $v^- \neq 0$, this leads to $f^+(v) \wedge f^-(v) = 0$, in conflict with (8).

i) ⇒ iv): The assumption is

$$\begin{array}{ll} v_2 v_4^2 = v_1^2 v_3 & \quad v_1 v_2^2 = v_3^2 v_4 \\ v_3 v_0^2 = v_2^2 v_4 & \quad v_2 v_3^2 = v_4^2 v_0 \\ v_4 v_1^2 = v_3^2 v_0 & \quad v_3 v_4^2 = v_0^2 v_1 \\ v_0 v_2^2 = v_4^2 v_1 & \quad v_4 v_0^2 = v_1^2 v_2 \\ v_1 v_3^2 = v_0^2 v_2 & \quad v_0 v_1^2 = v_2^2 v_3 \end{array}$$

We distinguish between two cases: **51**

Case 1. One of the coordinates v_i vanishes. By $(i \to i+1)$-symmetry we may assume $v_0 = 0$. The equations then are equivalent with

$$v_i v_j = 0 \quad \text{for all} \quad 1 \leq i \neq j \leq 4.$$

So all coordinates but one vanish.

Case 2. $v_i \neq 0$ for all i. We may put

$$v_0 = 1,\ v_1 = \alpha\beta,\ v_2 = \alpha^2\beta^3,$$

and find the equations equivalent with

$$v_3 = \alpha^{-2}\beta^{-4},\ v_4 = \alpha^4\beta^5,\ \alpha^5 = \beta^5 = 1.$$

Putting $\alpha = \epsilon^k, \beta = \epsilon^l$ we obtain $v = e_{kl}$.

iv) ⇒ iii): if $v = e_k$, then $h^+(v) = h^-(v) = 0$ and $f^\pm(v)$ both are simple bivectors. if $v = e_{kl}$, then

$$\begin{aligned} h^\circ(e_{kl}) &= \widehat{e_0} + \epsilon^{2k+2l}\widehat{e_1} + \epsilon^{4k+1}\widehat{e_2} + \epsilon^{4+2l}\widehat{e_3} + \epsilon^{3k}\widehat{e_4} \\ h^+(e_{kl}) &= \epsilon^l h^0(e_{kl}), \quad h^-(e_{kl}) = -\epsilon^{-l} h^\circ(e_{kl}) \end{aligned}$$

show that

$$\lambda^2 h^+(e_{kl}) + \lambda\mu h^\circ(e_{kl}) + \mu^2 h^-(e_{kl})$$

vanishes if and only if 52

$$\epsilon^l \lambda^2 + \lambda\mu - \epsilon^{-l}\mu^2 = 0.$$

This quadratic polynomial has the two different roots

(12) $\lambda/\mu = -\epsilon^{-l}(1 \pm \sqrt{5})/2,$

corresponding to simple bivectors $\lambda f^+(e_{kl}) + \mu f^-(e_{kl})$.

□

We shall need explicitly the polynomial

$$f(\lambda, \mu) = \prod_{l=0}^{4} \left(\epsilon^l \lambda^2 + \lambda\mu - \epsilon^{-l}\mu^2\right)$$

which has the 10 roots (12.) Obviously

$$\begin{aligned} f(\epsilon\lambda, \mu) &= \Pi \epsilon^{l+2}\lambda^2 + \epsilon\lambda\mu - \epsilon^{-l}\mu^2 \\ &= \Pi \epsilon(\epsilon^{l+1}\lambda^2 + \lambda\mu - \epsilon^{-l-1}\mu^2) \\ &= f(\lambda, \mu) \end{aligned}$$

and similarly $f(\lambda, \epsilon\mu) = f(\lambda, \mu)$. This shows

$$f(\lambda, \mu) = a\lambda^{10} + b\lambda^5\mu^5 + c\mu^{10}$$

We have

$$\begin{aligned} a &= 1 \cdot \epsilon \cdot \epsilon^2 \cdot \epsilon^3 \cdot \epsilon^4 = 1 \\ c &= (-1)(-\epsilon)(-\epsilon^2)(-\epsilon^3)(-\epsilon^4) = -1 \\ b &= f(1, 1) \\ &= (\epsilon + 1 - \epsilon^4)(\epsilon^2 + 1 - \epsilon^3)(\epsilon^3 + 1 - \epsilon^2)(\epsilon^4 + 1 - \epsilon) \\ &= 11 \end{aligned}$$

53

(13) $f(\lambda, \mu) = \lambda^{10} + 11\lambda^5\mu^5 - \mu^{10}$

2.4 The condition $v^+ \wedge v^- = 0$.

The next proposition gives several descriptions of the set $\{v \in V : v^+(v) \wedge v^-(v) = 0\}$.

Proposition 2. *The following conditions on* $v \in V$ *are equivalent:*

(i) $v^+ \wedge v^- = 0$

(ii) $h^+(v) \wedge h^\circ(v) \wedge h^-(v) = 0$

(iii) *from some* $(\lambda, \mu) \in \mathbb{C}^2 \backslash 0 : \lambda v^+ - \mu v^- = 0$

(iv) *from some* $(\lambda, \mu) \in \mathbb{C}^2 \backslash 0$

$$\lambda^2 h^+(v) + \lambda\mu h^\circ(v) + \mu^2 h^-(v) = 0$$

(v) *for some* $(\lambda, \mu) \in \mathbb{C}^2 \backslash 0 : \lambda f^+(v) + \mu f^-(v)$ *is a simple bivector.*

Unless $v^+ = v^- = 0$, *the ratio* $(\lambda : \mu) \in \mathbb{P}_1$ *is uniquely determined by* (iii), (iv), *or* (v) *and is the same in all three equations.*

Proof. Properties (i) and (ii) are equivalent by (11). Properties (i) and 54
(iii) are obviously equivalent, and (iv) obviously implies (ii). Properties (iv) and (v) (with the same $(\lambda : \mu)$) are equivalent by definition of h^+, h°, h^-.

To show the direction (ii) ⇒ (iv), we assume

$$c^+ h^+(v) + c^\circ h^\circ(v) + c^- h^-(v) = 0, (c^\pm, c^\circ) \in \mathbb{C}^3 \backslash 0.$$

Contracting this equation with $f^\pm(v)$ and using (9) we find

$$-c^\circ v^+ + c^- v^- = c^+ v^+ - c^\circ v^- = 0.$$

The case $v^+ = v^- = 0$ being settled by proposition 1 we may assume $v^+ \neq 0$ or $v^- \neq 0$. Taking $(\lambda, \mu) \neq (0, 0)$ as in (iii), we find

$$c^\circ \mu - c^- \lambda = c^+ \mu - c^\circ \lambda = 0$$

$$(c^\circ)^2 = c^+ c^-$$

So there is $(\lambda',\mu') \in \mathbb{C}^2\backslash 0$ with $c^+ = (\lambda)2$, $c^\circ = \lambda'\mu'$, $c^- = (\mu')^2$. This proves (iv). Additionally we observe that (λ,μ) and (λ',μ') are proportional. □

If $\lambda f^+(v)+\mu f^-(v)$ is a simple bivector, it describes some 2-dimensional vector space in V. The vector

$$-\langle h^\circ(v), \lambda f^+(v) + \mu f^-(v)\rangle = \lambda v^+ + \mu v^-$$

spans the intersection of this space with the hyperplane $\{u : \langle h^\circ(v), u\rangle = 0\}$, unless $v^+ = v^- = 0$.

55 We observe

(14) $v^\pm \wedge (\lambda f^+(v) + \mu f^-(v)) = 0$

whenever $\lambda f^+(v) + \mu f^-(v)$ is simple.

2.5 The curves $E_{\lambda:\mu}$

We denote by $E_{\lambda:\mu} \subset \mathbb{P}_4(V)$ the set described by vectors v such that $\lambda f^+(v) + \mu f^-(v)$ is simple. It was shown (proposition 2 in 2.4) that for two different ratios $(\lambda : \mu)$ the two sets $E_{\lambda:\mu}$ are disjoint, unless the ratios are 0, ∞ or two from the 10 ones in (12) differing by the sign. By proposition 2 (iv) we have the following quadratic equations for this set

$$\begin{aligned}
Q_0(\lambda,\mu) &= \lambda^2 v_1 v_4 + \lambda\mu v_0^2 - \mu^2 v_2 v_3 = 0\\
Q_1(\lambda,\mu) &= \lambda^2 v_2 v_0 + \lambda\mu v_1^2 - \mu^2 v_3 v_4 = 0\\
Q_2(\lambda,\mu) &= \lambda^2 v_3 v_1 + \lambda\mu v_2^2 - \mu^2 v_4 v_0 = 0\\
Q_3(\lambda,\mu) &= \lambda^2 v_4 v_2 + \lambda\mu v_3^2 - \mu^2 v_0 v_1 = 0\\
Q_4(\lambda,\mu) &= \lambda^2 v_0 v_3 + \lambda\mu v_4^2 - \mu^2 v_1 v_2 = 0
\end{aligned} \tag{15}$$

Proposition 3. *The subscheme $E_{\lambda:\mu} \subset \mathbb{P}_4$ given by equations* (15) *is a smooth curve unless $\lambda\backslash\mu$ takes one of the following 12 values:*

$$0, \infty, -\epsilon^{-l}(1 \pm \sqrt{5})/2 \quad (l = 0, \ldots, 4)$$

56 *Proof.* We show that the matrix of partial derivatives $\partial_i Q_j(\lambda, \mu)$

$$\begin{pmatrix} 2\lambda\mu v_0 & \lambda^2 v_2 & -\mu^2 v_4 & -\mu^2 v_1 & \lambda^2 v_3 \\ \lambda^2 v_4 & 2\lambda\mu v_1 & \lambda^2 v_3 & -\mu^2 v_0 & -\mu^2 v \\ -\mu^2 v_3 & \lambda^2 v_0 & 2\lambda\mu v_2 & \lambda^2 v_4 & -\mu^2 v_1 \\ -\mu^2 v_2 & -\mu^2 v_4 & \lambda^2 v_1 & 2\lambda\mu v_3 & \lambda^2 v_0 \\ \lambda^2 v_1 & -\mu^2 v_3 & -\mu^2 v_0 & \lambda^2 v_2 & 2\lambda\mu v_4 \end{pmatrix}$$

has rank ≥ 3, unless $(\lambda : \mu)$ is one of the 12 special parameters above. We denote by $\det^{ijk}_{lmn}$ the 3×3-subdeterminant in the rows l, m, n and columns i, j, k and compute

$$\begin{aligned} \det{}^{i,i+1,i+4}_{i,i+1,i+4} &= \lambda\{8\lambda^2\mu^3 v_t v_i v_{i+1} v_{i+4} - 2\mu^5 v_i v_{i+2} v_{i+3} \\ &\quad - 2\lambda^4\mu(v_{i+2}v_{i+4}^2 + v_{i+1}^2 v_{i+3}) - \lambda^3\mu^2(v_{i+1}v_{i+2}^2 + v_{i+3}^2 v_{i+4})\} \\ \det{}^{i,i+2,i+3}_{i,i+2,i+3} &= \mu\{-2\lambda^5 v_i v_{i+1} v_{i+4} + 8\lambda^3\mu^2 v_i v_{i+2} v_{i+3}) \\ &\quad + \lambda^2\mu^3(v_{i+2}v_{i+4}^2 + v_{i+1}^2 v_{i+3}) - 2\lambda\mu^4(v_{i+1}v_{i+2}^2 + v_{i+3}^2 v_{i+4})\} \\ \det{}^{i,i+1,i+4}_{i,i+2,i+3} &= \mu\{-2\lambda\mu^4 v_i v_{i+1} v_{i+4} + 2\lambda^5 v_i^3 + 2\lambda^4\mu v_i v_{i+2} v_{i+3} \\ &\quad + \lambda^2\mu^3(v_{i+1}v_{i+2}^2 + v_{i+3}^2 v_{i+4})\} \\ \det{}^{i,i+2,i+3}_{i,i+1,i+4} &= \lambda\{2\lambda\mu^4 v_i v_{i+1} v_{i+4} - 2\mu^5 v_i^3 + 2\lambda^4\mu v_{i+2} v_{i+3} \\ &\quad + \lambda^3\mu^2(v_{i+2}v_{i+4}^2 + v_{i+1}^2 v_{i+3})\} \end{aligned}$$

We may assume $\lambda\mu \neq 0$. Let $v \in E_{\lambda:\mu}$ be some point where the matrix $(\partial_i Q_j(\lambda, \mu))$ has rank ≤ 2. Then 57

$$\begin{pmatrix} 8\lambda^2\mu^3 & 0 & -2\mu^5 & -2\lambda^4\mu & -\lambda^3\mu^2 \\ -2\lambda^5 & 0 & 8\lambda^3\mu^2 & \lambda^2\mu^3 & -2\lambda\mu^4 \\ -2\lambda\mu^4 & 2\lambda^5 & 2\lambda^4\mu & 0 & \lambda^2\mu^3 \\ 2\lambda\mu^4 & -2\mu^5 & 2\lambda^4\mu & \lambda^3\mu^2 & 0 \end{pmatrix} \begin{pmatrix} v_i v_{i+1} v_{i+4} \\ v_i^3 \\ v_i v_{i+2} v_{i+3} \\ v_{i+2}v_{i+4}^2 + v_{i+1}^2 v_{i+3} \\ v_{i+1}v_{i+2}^2 + v_{i+3}^2 v_{i+4} \end{pmatrix}$$

vanishes for $i = 0, \ldots, 4$. The 4×4-minors of the 4×5 matrix are respectively 58

$$8\lambda^3\mu^2(-2\lambda^5 + \mu^5), -8\lambda^2\mu^3(\lambda^5 + 2\mu^5) - 4\lambda^6\mu^4, -12\lambda^5\mu^5, 4\lambda^4\mu^6$$

times the polynomial (13) from 2.3. The matrix has rank 4 unless λ/μ is one of the 12 special parameters. But this would imply

$$\operatorname{rank}\begin{pmatrix} v_0v_1v_4 & v_1v_2v_0 & v_2v_3v_1 & v_3v_4v_2 & v_4v_0v_3 \\ v_0^3 & v_1^3 & v_2^3 & v_3^3 & v_4^3 \\ v_0v_2v_3 & v_1v_3v_4 & v_2v_4v_0 & v_3v_0v_1 & v_4v_1v_2 \\ v_2v_4^2+v_1^2v_3 & v_3v_0^2+v_2^2v_4 & v_4v_1^2+v_3^2v_0 & v_0v_2^2+v_4^2v_1 & v_1v_3^2+v_0^2v_2 \\ v_1v_2^2+v_3^2v_4 & v_2v_3^2+v_4^2v_0 & v_3v_4^2+v_0^2v_1 & v_4v_0^2+v_1^2v_2 & v_0v_1^2+v_2^2v_3 \end{pmatrix} \leq 1$$

Already the first three rows have rank ≥ 2 unless $v = e_k$ or e_{kl} (proposition 1, (ii)) and λ/μ therefore is one of the twelve special values, or some v_i vanishes. In the latter case one easily checks that the rank drops to one only if two more coordinates vanish. Then either the first or the third row vanishes and v belongs to $E_{1:0}$ or $E_{0:1}$. □

Another way of stating the assertion of proposition 3 is this: For each $\lambda : \mu$, not one of the 12 exceptional values, and each $v \in E_{\lambda:\mu}$ the differential

59 (16) $dv \to (\lambda f^+(v) + \mu f^-(v)) \wedge (\lambda f^+(dv) + \mu f^-(dv))$

is a map of rank 3.

3 Elliptic normal curves

3.1 The Heisenberg group of order n^3

Let $n \geq 3$ be some integer. (Again, we are only interested in the case $n = 5$). Denote by V the vector space $\mathbb{C}^n$ with standard basis $e_0, \ldots, e_{n-1}$. The automorphisms

$$\begin{array}{ll} \sigma : e_k \to e_{k-1} & k = 0, \ldots, n-1 \mod n \\ \tau : e_k \to \epsilon^k e_k & \epsilon = e^{\frac{2\pi i}{n}} \end{array}$$

generate a subgroup $H_n \subset GL(n, \mathbb{C})$. For n odd, the group is contained in $SL(n, \mathbb{C})$. The commutator is $\sigma\tau\sigma^{-1}\tau^{-1} = \epsilon.$ id. Therefore we have an exact sequence

$$0 \to \mathbb{Z}_n \to H_n \to \mathbb{Z}_n \times \mathbb{Z}_n \to 0$$

with $\mathbb{Z}_n \subset H_n$ generated by ϵ.id the centre of H_n. The group H_n is called *Heisenberg group* and its representation on *V Schroedinger representation*. This representation is irreducible.

The commutator map induces a nondegenerate skew-symmetric bilinear form

$$(,) : (\mathbb{Z}_n \times \mathbb{Z}_n) \times (\mathbb{Z}_n \times \mathbb{Z}_n) \to \mathbb{Z}_n$$

given by 60

$$\epsilon^{(\alpha,\beta)} = \frac{1}{n}\mathrm{tr}(\alpha\beta\alpha^{-1}\beta^{-1}).$$

If n is odd, then via the map

$$\begin{aligned} H_n &\to \mathbb{Z}_n \times (\mathbb{Z}_n \times \mathbb{Z}_n) \\ \sigma^a\tau^b\epsilon^c &\to \left(c + \frac{1}{2}(n+1)a \cdot b, (a,b)\right) \end{aligned}$$

the group H_n can be described as the set $\mathbb{Z}_n \times (\mathbb{Z}_n \times \mathbb{Z}_n)$ with the group law

$$(a,\alpha) \cdot (b,\beta) = \left(a + b + \frac{1}{2}(n+1)(\alpha,\beta), \alpha + \beta\right)$$

The form $(,)$ is invariant under the action of $SL(2, Z_n)$ on $\mathbb{Z}_n \times \mathbb{Z}_n$. Therefore $SL(2,\mathbb{Z}_n)$ acts on H_n as a group of automorphisms, the action on the central subgroup of H_n being trivial. Write this action as $\gamma(\alpha)$, $\alpha \in H_n$, $\gamma \in SL(2,\mathbb{Z}_n)$. We dentoe by

$$N_n = SL(2,\mathbb{Z}_n) \times H_n$$

the semi-direct product with respect to this action, i.e. the set $N_n = SL(2,\mathbb{Z}_n) \times H_n$ with the group law

$$(\gamma,\alpha)(\gamma',\alpha') = (\gamma\gamma', \alpha \cdot \gamma(\alpha')).$$

For $n = 5$, we have the following result. 61

Lemma 2 (6, §1). *The Schroedinger representation extends uniquely to a representation* $\rho : N_5 \to SL(V)$.

From $SL_2(\mathbb{Z}_5) = N_5/H_5$, we need explicitly the following elements:

$$\bar{\iota} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \bar{\mu} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \bar{\nu} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \bar{\delta} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

They are the images of ι, μ, ν, $\delta \in N_5$ acting on V by

$$\rho(\iota)e_i = e_{-1}, \rho(\mu)e_i = -e_{-3i}, \rho(\nu)e_i = \epsilon^{i^2} e_i$$
$$\rho(\delta)e_i = -\frac{1}{\sqrt{5}} \sum \epsilon^{ij} e_j$$

For ι, ν and μ, see [6, p.66/67]. For $\rho(\delta)$, it follows from the following relations which are easy to verify:

$$\rho(\delta)^2 = \rho(\iota), \rho(\delta^{-1}) = \rho(\delta)\rho(\iota)$$
$$\rho(\delta)\rho(\sigma)\rho(\delta)^{-1} = \rho(\tau)^4, \rho(\delta)\rho(\tau)\rho(\delta)^{-1} = \rho(\sigma).$$

3.2 H_n-invariant elliptic normal curves

Let E be an elliptic curve (over $\mathbb{C}$) and $\mathscr{L}$ some line bundle of degree
62 n. We assumed $n \geq 3$, hence the complete linear system $H^\circ(\mathscr{L})$ is very ample and embeds E as a smooth elliptic curve in $\mathbb{P}_{n-1}(\mathbb{C})$. The image curve is called an *elliptic normal curve of degree n.*

The bundle $\mathscr{L}$ is invariant under all translations of E of order n. We fix one point $O_E \in E$ (from the n^2 ones on E) with the property $\mathscr{L} = \mathscr{O}_E(n.O_E)$. Using O_E as origin, we provide E with a group structure. The n-torsion points form a subgroup $E^{(n)} \subset E$, which is (not canonically!) isomorphic with $\mathbb{Z}_n \times \mathbb{Z}_n$. The group $E^{(n)}$ comes, however, with an intrinsic non-degenerate skew-symmetric $\mathbb{Z}_n$-bilinear form: Write $E = \mathbb{C}/\Omega$ with the lattice $\Omega \subset \mathbb{C}$ canonically isomorphic to $H_1(E, \mathbb{Z})$. This lattice $H_1(E, \mathbb{Z})$ carries the skew-symmetric intersection product $\omega_1 \cdot \omega_2 \in \mathbb{Z}$. Points $\alpha, \beta \in E^{(n)}$ can be represented by complex numbers in $\frac{1}{n}\Omega$ modulo Ω. The form on $E^{(n)}$ then is $(a, \beta) = n\alpha \cdot n\beta$

modulo n. We call a basis $a, \beta \in E^{(n)}$ *symplectic*, if $(\alpha, \beta) = 1$. The group $SL(2, \mathbb{Z}_n)$ permutes the symplectic bases transitively.

The following facts can be found in [8] or [7, chapter 1]:

(a) Thee translation action of $E^{(n)}$ on E lifts to an action of the Heisenberg group on $\mathscr{L}$.

(b) The representation of H_n on $H^\circ(\mathscr{L})$ is isomorphic with the representation of H_n on V^* which is the dual of the Schroedinger representation.

(c) The skew-symmetric form $(,)$ on $H_{n'}/\mathbb{Z}(H_n)$ which is given by the trace of the commutator of the representation (b) coincides with the one given by the intersection product. **63**

To obtain the action of H_n on $H^\circ(\mathscr{L})$, two choices must be made:

1. An epimorphism $H_n \to E^{(n)}$ compatible with the skew-symmetric form. This is equivalent to choosing a symplectic basis $\alpha, \beta \in E^{(n)}$ on which the generators σ, $\tau \in H_n$ are mapped.

2. A lifting of α and β to automorphisms of $\mathscr{L}$ of order n. There are n such liftings for α and for β, each differing by multiplication with ϵ^k, $k = 1, \ldots, n$. These n^2 different liftings are equivalent under the n^2 inner automorphisms of H_n.

Given an action of H_n on $\mathscr{L}$ as above, the isomorphism $H^\circ(\mathscr{L}) \to V^*$ of H_n-modules induces an H_n-equivariant embedding $E \to \mathbb{P}_{n-1}(V)$. The image is an *$H_n$-invariant elliptic normal curve*. Clearly, each elliptic normal curve in $\mathbb{P}_{n-1}(V)$, which is invariant under H_n, is obtained in this way. We have shown

Proposition 4. *There is a bijection between H_n equivariant embedings of elliptic normal curves in $\mathbb{P}_{n-1}(V)$ and pairs (E, α, β) consisting of an elliptic curve E (with origin) and a symplectic basis α, β of $E^{(n)}$. This correspondence associates with an embedding $E \subset \mathbb{P}_{n-1}(V)$ the symplectic basis $\alpha = \sigma(O_E), \beta = \tau(O_E)$ for $E(n)$.*

We are not interested in embeddings, but in the embedded curves. Two symplectic bases α, β and α', β' lead to the same image curve in **64** $\mathbb{P}_{n-1}$, if they differ by an automorphism of the curve. For example α, β and $-\alpha$, $-\beta$ differ by the involution $\iota_E : P \to -P$ on E. An equivalence class under $\mathrm{Aut}(E)$ of symplectic bases for $E^{(n)}$ is called a *level-n-structure* on E. If E is not isomorphic with $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}_i$ or $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}_\omega$, $\omega = e^{2\pi i/3}$, then the level-n-structures are exactly the symplectic bases modulo the action of the involution. We have, therefore, the following consequence of proposition 4:

Proposition 5. *There is a bijection between H_n-invariant elliptic curves in $\mathbb{P}_{n-1}(V)$ and pairs consisting of an elliptic curve and a level-n-structure on it.*

3.3 Quadratic equations

It is known that any elliptic normal curve of degree $n > 3$ in $\mathbb{P}_{n-1}$ is schematically given by quadratic equations. This is a special case of a theorem of Mumford [8, p.349] on abelian varieties. In the case of curves, a proof can be found in [7, lemma IV. 1.1]. In our situation $n = 5$, we need these equations explicitly.

Let $E \subset \mathbb{P}_4$ be an H_5-invariant elliptic normal curve, and let $v_0, \dots, v_4$ be the coordinates on our fixed vector space V, dual to the basis $e_0, \dots, e_4$. The involution ι_E on the curve E can be lifted to I and this defines a unique involution $\iota_E \in SL(V)$ which leaves E invariant. Since, on E, one has

$$\iota_E \sigma \iota_E^{-1} = \sigma^{-1}, \quad \iota_E \tau \iota_E^{-1} = \tau^{-1},$$

65 it follows that

$$\begin{aligned} \iota_E = \rho(\iota) : V &\to V \\ e_m &\to e_{-m}. \end{aligned}$$

Under this involution, the space $V^* = H^\circ(\mathscr{L})$ splits into two eigenspaces, namely

$$V^* = V^+ \oplus V^-$$

where

$$V^+ = \langle v_0, v_1 + v_4, v_2 + v_3\rangle, V^- = \langle v_1 - v_4, v_2 - v_3\rangle.$$

Lemma 3. (i) *The origin O_E lies on the invariant line P*

$$\{v_0 = v_1 + v_4 = v_2 + v_3 = 0\}.$$

(ii) *The non-zero 2-torsion points P_1, P_2, P_3 span the invariant plane* $\{v_1 - v_4 = v_2 - v_3 = 0\}$.

Proof. It is easy to exhibit two ι-invariant subspaces of dimension 1 and 2 in the complete linear system

$$|5O_E| = \mathbb{P}(H^\circ(\mathscr{L})).$$

Let

$$\pi : E \longrightarrow E/\iota = \mathbb{P}_1$$

be the 2 : 1 covering defined by the involution and define

$$\{D; D = O_E + \pi^*(D_2) \quad \text{where} \quad D_2 \in |\mathscr{O}_{\mathbb{P}_1}(2)|\}$$
$$\left\{D', D' = \sum_{i=1}^{3} P_i + \pi^*(D_1) \quad \text{where} \quad D_1 \in |\mathscr{O}_{\mathbb{P}_i}(1)|\right\}$$

These subspaces must coincide with $\mathbb{P}(V^+)$ and $\mathbb{P}(V^-)$ respectively. □ 66

We shall come back to this in section 5.2.

Using the above lemma, it now follows from [3] or [7, prop. IV.2.1] that $E \subset \mathbb{P}_4$ is given by the quadratic equations

$$Q_k(\lambda) = \lambda v_{k+1} v_{k+4} v_k^2 - \frac{1}{\lambda} v_{k+2} v_{k+3}, k = 0, \ldots, 4 \tag{18}$$

The striking point is this: Equations (18) and (15) are identical, up to a change from λ to the projective parameter $(\lambda : \mu)$. This observation is crucial for this whole paper.

The parameter $\lambda = \mathbb{C}$ in (18) depends on the embedded curve, i.e., on the elliptic curve with its level-5-structure. From 2.5, we know that the equations (18) describe a smooth 1-dimensional subscheme in $\mathbb{P}_4$ unless the (projective) parameter λ is

$$\lambda = 0, \infty, \quad \text{or} \quad -\frac{1}{2}\epsilon^{-l}(1 \pm \sqrt{5}), l = 0, \dots, 4 \tag{19}$$

Let us first consider these exceptional values of λ.

For $\lambda = 0$, the equations (18) are

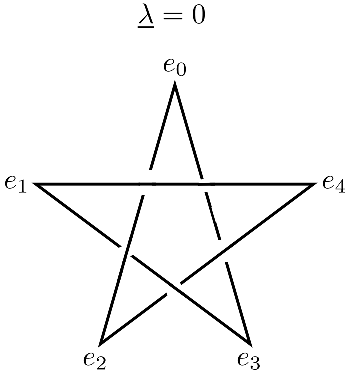
$$v_2 v_3 = v_3 v_4 = v_4 v_0 = v_0 v_1 = v_1 v_2 = 0.$$

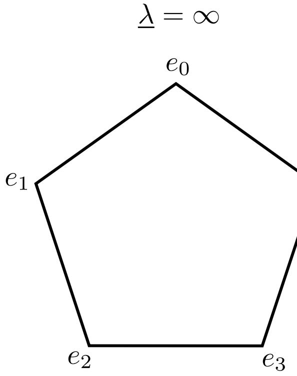
They describe the set consisting of the five lines

$$v_i = v_{i+2} = v_{i+3} = 0 \quad i = 0, \dots, 4$$

67 spanned by the coordinate points e_{i+1}, e_{i+4}. One easily checks that the quadratic equations describe this pentagon of lines even schematically.

Similarly, for $\lambda = \infty$, the equations (18) schematically describe the pentagon consisting of the five lines spanned by the coordinate points $e_{i+2}, e_{i+3}, i = 0, \dots, 4$.





Proposition 6. *For each of the 12 special values* (19) *the equations* (18) *schematically define a pentagon. For* $\lambda = 0, \infty$ *the vertices are the coordinate points for* $\lambda = -\frac{1}{2}\epsilon^{-1}(1 \pm \sqrt{5})$ *the vertices are the points* e_{kl}, $k = 0, \dots, 4$, *from 2.3.*

In one case, say $\lambda = 0$, this can be checked by hand. The other 11 cases are equivalent under the operation of $SL(2, \mathbb{Z}_5)$: Recall μ, ν, $\delta \in SL(2, \mathbb{Z}_5)$ from 3.1. $\rho(\mu)$ changes v_i to $-v_{2i}$ and $Q_k(\lambda)$ into $Q_k(-1/\lambda)$. This transforms $0 \leftrightarrow \infty$, $-\frac{1}{2}\epsilon^{-l}(1+\sqrt{5}) \leftrightarrow -\frac{1}{2}\epsilon^{-l}(1-\sqrt{5}) \cdot \rho(\nu)$ changes v_i to $\epsilon^{-i^2} v_i$ and $Q_k(\lambda)$ into $\epsilon^{-2k^2} Q_k(\lambda\epsilon^3)$. 68

Repeated application permutes cyclically the values $-\frac{1}{2}\epsilon^{-l}(1 \pm \sqrt{5})$ with the same sign.

$\rho(\delta)$ transforms e_k into a multiple of $e_{k,0}$. This transforms the values $0, \infty$ into $-\frac{1}{2}\epsilon^{-l}(1 \pm \sqrt{5})$. □

In face the 6 five-tuplets $\{e_k\}$, $\{e_{kl}, k = 0, \ldots, 4\} l = 0, \ldots, 4$ are the fixed points of the 6 cyclic subgroups in $\mathbb{Z}_5 \times \mathbb{Z}_5$ of order 5 operating on $\mathbb{P}_4$. They are permuted transitively by $SL(2, \mathbb{Z}_5)$.

3.4 The surfaces S_{25}

For the parameters λ different from the 12 special values we know that the curves E_λ are smooth. Among them appear all H_5-equivariantly embedded elliptic curves. Strictly speaking, this does not yet imply that all the smooth E_λ are such elliptic curves. To see this, we put all the curves $E_{\lambda:\mu}$ in a flat family:

Define the surface

$$\begin{aligned} S_{25} = \{(v; \lambda, \mu) \in \mathbb{P}_4 \times \mathbb{P}_1 :& \\ & (\lambda f^+(v) + \mu f^-(v)) \wedge (\lambda f^+(v) + \mu f^-(v)) = 0\} \end{aligned}$$

The projection onto the second factor induces a map $S_{25} \to \mathbb{P}_1$. Its fibre over $(\lambda : \mu)$ projected into $\mathbb{P}_4$ is just the curve $E_{\lambda:\mu}$.

Lemma 4. *S_{25} is a smooth surface.* 69

Proof. We consider the differential map

$$\begin{aligned} & V \times |\mathbb{C}^2 \to \Lambda^4 V \\ & dv, d\lambda, d\mu \to (\lambda f^+(v) + \mu f^-(v)) \wedge (\lambda f^+(dv) + u f^-(dv) + d\lambda f^+(v) + d\mu f^-(v)) \end{aligned}$$

and show that its rank equals 3 for all $(v,\lambda,\mu)\in S_{25}$. The map

$$dv \to (\lambda f^+(v) + \mu f^-(v) \wedge (\lambda f^+(dv) + \mu f^-(dv))$$

has rank 3 for all v belonging to a smooth curve $E_{\lambda:\mu}$ (proposition 3) and for all smooth v on one of the 12 singular curves $E_{\lambda:\mu}$ (proposition 6 above). This proves the assertion for these v. If v is one of the points e_k, e_{kl}, using the symmetries we may assume $v = e_0$, $\lambda = 1$, $\mu = 0$. The differential then is

$$\begin{aligned} &e_2 \wedge e_3 \wedge (\sum dv_i e_{i+2} \wedge e_{i+3} + d\lambda e_2 \wedge e_3 + d\mu e_1 \wedge e_4) = \\ &= d\mu \widehat{e}_0 + dv_2 e_1 + dv_3 \widehat{e}_4 \end{aligned}$$

and obviously has rank 3. ☐

Theorem 1. *The subscheme given by equations* (18) *is a smooth quintic elliptic normal curve* E_λ *embedded* H_5*-equivariantly in* $\mathbb{P}_4$ *is* λ *is not one of the 12 special values* (19) *a pentagon with vertices* e_k, $k = 0,\dots,4, (\lambda = 0,\infty)$ *or* e_{kl}, $k = 0,\dots,4 (\lambda = -\frac{1}{2}\epsilon^{-l}(1\pm\sqrt{5}))$.

70 *Proof.* For λ not one of the 12 special values we know that E_λ is smooth, isomorphic with a smooth fibre of $S_{25} \to \mathbb{P}_1$. All elliptic curves appear as such fibres, so all smooth fibres, hence all smooth E_λ too are (connected) elliptic curves. The equations (18) are H_5-invariant. This implies that each smooth E_λ is an H_5-invariant elliptic quintic. ☐

3.5 Cubic equations

The set $E_{\lambda:\mu}$ is described not only by the quadratic equations (18) alias (15) but also (cf. prop. 2) by the five cubic equations equivalent to $\lambda v^+ - \mu v^- = 0$

$$\lambda(v_{k+2}v_{k+4}^2 - v_{k+3}^2) - \mu(v_{k+1}v_{k+2}^2 - v_{k+3}^2 v_{k+4}) = 0 \qquad (20)$$

At the 30 points e_k, e_{kl} both v^+ and v^- vanish, so these points have to be excluded.

Proposition 7. *Away from the 30 points e_k, e_{kl} the cubic equations* (20) *define the scheme $E_{\lambda:\mu}$.*

Proof. At some $v \in E_{\lambda:\mu}$, $v \neq e_k, e_{kl}$, we compute the rank of the differential

$$d(\lambda v^+ - \mu v^-) : V \to V.$$

For $v = \sum v_i e_i$, $dv = \sum dv_i e_i \in V$, we abbreviate

$$\begin{aligned} h^+(v, dv) &= {}^1\!/\!{}_2(f^+(v) \wedge f^+(dv))^* \\ &= {}^1\!/\!{}_2 \sum (v_{i+4} dv_{i+4} + dv_{i+1} v_{i+4}) \widehat{e_i} \\ h^-(v, dv) &= {}^1\!/\!{}_2(f^-(v) \wedge f^-(dv))^* \\ &= -{}^1\!/\!{}_2 \sum (v_{i+2} dv_{i+3} dv_{i+2} v_{i+3}) \widehat{e_i} \\ h^0(v, dv) &= (f^+(v) \wedge f^-(dv))^* = \sum v_i dv_i \widehat{e_i} \end{aligned}$$

Differentiating $-v^\pm = \langle h^0(v), f^\pm(v) \rangle$ and using (6), we find 71

$$\begin{aligned} -dv^\pm &= \langle 2h^0(v, dv), f^\pm(v) \rangle + \langle h^0(v), f^\pm(dv) \rangle \\ &= \langle h^0(v, dv), f^\pm(v) \rangle - 2\langle h^\pm(v, dv), f^\mp(v) \rangle \\ -d(\lambda v^+ - \mu v^-) &= \langle \lambda h^0(v, dv) + 2\mu h^-(v, dv), f^+(v) \rangle \\ &\quad - \langle \mu h^\circ(v, dv) + 2\lambda h^+(v, dv), f^-(v) \rangle \\ &= \langle ((\lambda f^+(v) + \mu f^-(v)) \wedge f^-(dv)^*, f^+(v) \rangle \\ &\quad - \langle ((\lambda f^+(v) + \mu f^-(v)) \wedge f^+(dv))^*, f^-(v) \rangle \end{aligned}$$

If dv belongs to the kernel of the differential at v, then from

$$\langle ((\lambda f^+(v) + \mu f^-(v)) \wedge f^\pm(dv))^*, \lambda f^+(v) + \mu f^-(v) \rangle = 0 \tag{6}$$

we deduce

$$\langle ((\lambda f^+(v) + \mu f^-(v)) \wedge (\lambda f^+(dv) + \mu f^-(dv)))^*, f^\pm(v) \rangle = 0$$

From (9) we know $\langle h^+(v), f^+(v) \rangle = \langle h^-(v), f^-(v) \rangle = 0$. The space of forms $h \in V^*$ with $\langle h, f^+(v) \rangle = 0$ is one-dimensional unless $f^+(v)$ is a simple bivector and similarly for $f^-(v)$. 72

Applying some transformation from $SL(2,\mathbb{Z}_5)$, we may assume that this is not the case. Since, by assumption, we are away from the 30 points e_k, e_{kl}, from proposition 2 (iv) we conclude $h^+(v) \wedge h^-(v) \neq 0$. Then necessarily

$$(\lambda f^+(v) + \mu f^-(v)) \wedge (\lambda f^+(dv) + \mu f^-(dv)) = 0.$$

So the kernel of $d(\lambda v^+ - \mu v^-)$ coincides with the kernel of $d((\lambda f^+(v) + \mu f^-(v)) \wedge (\lambda f^+(v) + \mu f^-(v))$. The assertion follows from proposition 3 and 6. □

4 Shioda's elliptic modular surface $S(5)$

4.1 The universal elliptic curve with level-n-structure

Let $\mathscr{H}$ denote the *upper half-plane* $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Each $z \in \mathscr{H}$ defines an elliptic curve $E_z = \mathbb{C}/\mathbb{Z} + \mathbb{Z}z$ and in $H_1(E_z,\mathbb{Z})$ a distinguished basis s_z, t_z, namely, the images of 1, z under the canonical isomorphism $\mathbb{Z} + \mathbb{Z}_z \to H_1(E_z,\mathbb{Z})$. Clearly, the intersection number is $s_z \cdot t_z = 1$.
73 The complex numbers $\frac{1}{n}$, $\frac{z}{n}$ have image points σ_z, $\tau_z \in E_z^{(n)}$ forming a symplectic basis for $E_z^{(n)}$.

Lemma 5. *Each elliptic curve E with level-n-structure defined by a symplectic basis $\alpha, \beta \in E^{(n)}$ is isomorphic to $(E_z;\sigma_z,\tau_z)$ for some $z \in \mathscr{H}$.*

Proof. Write $E = \mathbb{C}/\Omega$ and let a, b, $\in \mathbb{C}$ be inverse images of α, β such that na, $nb \in \Omega = H_1(E,\mathbb{Z})$ have intersection number $(na)\cdot(nb) = 1$. This implies $z := nb/na \in \mathscr{H}$. Under multiplication $\mathbb{C} \to \mathbb{C}$, $\zeta \to \zeta/na$, the lattice Ω is mapped onto $\mathbb{Z} + \mathbb{Z}z$ with $na \to 1$, $nb \to z$. This induces an isomorphism $E \to E_z$ with $\alpha \to \sigma_z, \beta \to \tau_z$. □

On $\mathscr{H}$ operates the group $\Gamma = SL(2,\mathbb{Z})$ by

$$\gamma z = \frac{az+b}{cz+d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

The elliptic curves E_z and E_z are isomorphic if and only if $z' = \gamma z$ for some $\gamma \in \Gamma$. Explicitly, the isomorphism $E_z \to E_{\gamma z}$ is given as follows: multiplication $\zeta \to (cz+d)^{-1}\zeta$ induces an isomorphism

$$E_z = \mathbb{C}/\mathbb{Z}(cz+d) + \mathbb{Z}(az+b) \to \mathbb{C}/\mathbb{Z} + \mathbb{Z}\frac{az+b}{cz+d} = E_{\gamma z}.$$

Under this isomorphism, the basis 74

$$\begin{aligned} 1 &= a(cz+d) - c(az+b) \\ z &= -b(cz+d) + d(az+b) \end{aligned}$$

of $H_1(E_z, \mathbb{Z})$ goes to the basis $a - c\gamma z$, $-b + d\gamma z$ of $H_1(E_{\gamma z}, \mathbb{Z})$. The level-$n$-structure given on E_z by σ_z, τ_z goes on $E_{\gamma z}$ to the one determined by $a\sigma_{\gamma z}$, $-c\tau_{\gamma z}$, $-b\sigma_{\gamma z} + d\tau_{\gamma z}$.

Denote by $\Gamma(n) \subset \Gamma$ the *congruence subgroup*

$$\{\gamma \in \Gamma : \gamma \equiv 1 \mod n\}.$$

Lemma 6. *Two curves with level-n-structure,*

$$(E_z, \sigma_z, \tau_z) \quad \textit{and} \quad (E_{z'}, \sigma_{z'}, \tau_z)$$

are isomorphic if and only if $z' = \gamma z$ *with* $\gamma \in \Gamma(n)$.

Proof. If $z' = \gamma z$, $\gamma \in \Gamma(n)$, then, by the above, there is an isomorphism $E_z \to E_{z'}$, mapping $\sigma_z \to \sigma_{z'}$ and $\tau_z \to \tau_{z'}$.

Assume, conversely, that (E_z, σ_z, τ_z) and $(E_{z'}, \sigma_{z'}, \tau_{z'})$ are isomorphic. Then there is some $\gamma \in \Gamma$ with $z' = \gamma z$ and $a\sigma_{z'} - c\tau_{z'}$, $-b\sigma_{z'} + d\tau_{z'}$ differs from the symplectic basis $\sigma_{z'}$, $\tau_{z'} \in E_{z'}^{(n)}$ by an automorphism r of $E_{z'}$. Then there is some $\rho = \left(\begin{smallmatrix} p & q \\ s & t \end{smallmatrix}\right) \in \Gamma$ such that $\rho z' = z'$ and

$$\begin{aligned} a\sigma_{z'} - c\tau_{z'} &= r_*\sigma_{z'} = p\sigma_{z'} - s\tau_{z'}, \\ -b\sigma_{z'} + d\tau_{z'} &= r_*\tau_{z'} = -q\sigma_{z'} + t\tau_{z'}. \end{aligned}$$

Then $\rho^{-1}\gamma \in \Gamma$ maps $z \to z'$, $\sigma_z \to \sigma_{z'}$, $\tau_z \to \tau_{z'}$ and from this, it 75
follows in particular that $\rho^{-1}\gamma \in \Gamma(n)$. ☐

The last lemma means that there is a bijective correspondence between points of $X'(n) := \mathscr{H}/\Gamma(n)$ and isomorphism classes of elliptic curves with level-n-structure. Assume $n \geq 3$, from now on. Then the action of $\Gamma(n)$ on $\mathscr{H}$ has no fixed points.

We denote by N the semi-direct product $\Gamma \times (\mathbb{Z} \times \mathbb{Z})$, i.e. the set $\Gamma \times (\mathbb{Z} \times \mathbb{Z})$ with the group law

$$(\gamma, (m_1, m_2)) \cdot (\gamma', (m_1', m_2')) = (\gamma\gamma', (m_1, m_2) \cdot \gamma' + (m_1, m_2')).$$

This group N acts on $\mathscr{H} \times \mathbb{C}$ by

$$(\gamma, (m_1, m_2)) : (z, \zeta) \to (\gamma z, (\zeta + m_1 z + m_2)(cz + d)^{-1}). \tag{21}$$

Since $n \geq 3$, the induced action of $\Gamma(n) \times (\mathbb{Z} \times \mathbb{Z})$ has no fixed points and

$$S'(n) = \mathscr{H} \times \mathbb{C}/\Gamma(n) \times (\mathbb{Z} \times \mathbb{Z})$$

is a smooth surface. The projection. $\mathscr{H} \times \mathbb{C} \to \mathscr{H}$ induces a map $\varphi : S'(n) \to X'(n)$. The fibre over $(z \mod \Gamma(n)) \in X'(n)$ is isomorphic with the curve E_z. The set $\{(z, 0) \in \mathscr{H} \times \mathbb{C} : z \in \mathscr{H}\}$ is invariant under the
76 group action. Its image in $S'(n)$ is a section $L_{0,0}$ for φ meeting each fibre in its origin. Similarly, the sets $\{(z, \frac{1}{n}) : z \in \mathscr{Z}\}$, $\{(z, \frac{z}{n}) : z \in \mathscr{H}\}$ define sections $L_{1,0}$, $L_{0,1}$ for φ meeting each fibre in the image of σ_z, τ_z. So the two sections $L_{1,0}$, $L_{0,1}$ provide each fibre of φ with a level-n-structure. Lemmas 5 and 6 mean that each elliptic curve with level-n-structure is isomorphic to exactly one fibre. In this sense, $\varphi : S'(n) \to X'(n)$ with zero section $L_{0,0}$ and the two sections $L_{1,0}$, $L_{0,1}$ is the universal elliptic curve with level-n-structure.

4.2 Shioda's elliptic surface $S(n)$ for $n \geq 3$

By adding cusps, the curve $X'(n)$ is completed. The result is the *modular curve* $X(n)$. Shioda [11, p.38] has shown how to complete the surface $S'(n)$ by adding over each cusp a singular fibre of type I_n, i.e., an n-gon of projective lines. The result is the smooth compact *modular surface* $S(n)$, fibred over the modular curve $X(n)$ by $\varphi : S(n) \to X(n)$.

The two sections $L_{1,0}$, $L_{0,1}$ generate a group of n-torsion sections for $\varphi : S'(n) \to X'(n)$. This group is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$. Shioda shows that those sections extend over all of $X(n)$, and that the n^2 sections obtained in this way are the only sections for $\varphi : S(n) \to X(n)$ [11, theorem 5.5]. Of course, over $X'(n)$, the n^2 sections are always disjoint. This is also true over the cusps: Each line in one of the n-gons is met by n sections in n different points.

By Kodaira's theory, cf. [2, V theorem 11.1(b)], the surface $S(n)$ **77**
is uniquely determined by $S'(n)$ and one extended section. In particular, each automorphism of $S'(n)$, mapping the n-torsion sections into themselves, extends uniquely to an automorphism of $S(n)$. We therefore have, acting on $S(n)$,

- the group $\mathbb{Z}_n \times \mathbb{Z}_n$ of n-torsion sections. Denote by $L_{k,l}$ the section corresponding to $(k.l) \in \mathbb{Z}_n \times \mathbb{Z}_n$

- the quotient $SL(2, \mathbb{Z}_n) = \Gamma/\Gamma(n)$. Its action is induced by the action (21) of N.

Now $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in T$ maps

$$\begin{gathered} L_{1,0} \to L_{a,-c} \qquad L_{0,1} \to L_{-b,d} \\ \mathbb{Z}_n \times \mathbb{Z}_n \ni \begin{pmatrix} k \\ l \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \end{gathered}$$

where

$$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \left\{ \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]^t \right\}^{-1} =: \gamma'$$

So up to the automorphism $\gamma \to \gamma'$ of $SL(2, \mathbb{Z}_n)$ this is the action of
$SL(2, \mathbb{Z}_n)$ on $\mathbb{Z}_n \times \mathbb{Z}_n$ from 3.1. The semi-direct product $SL(2, \mathbb{Z}_n) \times (\mathbb{Z}_n \times$ **78**
$\mathbb{Z}_n)$ with respect to this action operates on $S(n)$. Shioda [11] remarks that this is the full automorphism group of $S(n)$. Notice that it is the central quotient $N_n/\mathbb{Z}_n$ with N_n from 3.1.

4.3 Divisors on the surface $S(5)$

The modular curve $X(5)$ is rational, obtained from $X'(5)$ by adding 12 cusps [10, p.23]. The surface $S(5)$ therefore contains 12 fibres of type I_5 (pentagons). Summing over the 12 pentagons, we obtain the Euler number of $S(5)$

$$e = 60.$$

This implies

$$\begin{aligned} \chi(\mathcal{O}_{S(5)}) &= 5 && \text{(Noether's formula)} \\ K_{S(5)} &= \mathcal{O}_{S(5)}(3F), && \text{(Kodaira's formula)} \end{aligned}$$

where F is the class of a fibre,

$$P_g = 4, q = b_1 = 0, b_2 = 58, h^{1,1} = 50.$$

Lemma 7. *The Picard group Pic* $S(5)$ *contains no torsion.*

Proof. [13] Corollary 1.

On $S(5)$, we have the following divisor classes:

F the class of a fibre,

$L_{k,l}$ $k, l = 0, \ldots, 4$, the 25 sections,

$F_1, \ldots, F_{12}$ the twelve singular fibres.

79 For $i = 1, \ldots, 12$, we let $F_{i,0} \subset F_i$ be the component meeting $L_{0,0}$ and label the other components $F_{i,k}$, $k = 1, 2, 3, 4$, in such a way that $F_{i,k} \cdot F_{i,k+1} = 1 (k = 0, \ldots, 4 \bmod 5)$. This determines the notation uniquely up to a change of orientation in the pentagons. We have the following intersection numbers

$$\begin{aligned} &F^2 = 0, F \cdot L_{k,l} = 1, \quad F \cdot F_{i,k} = 0 \\ &L_{k,l}^2 = -5, \quad F_{i,k}^2 = -2 \end{aligned}$$

and each $F_{i,k}$ meets exactly five $L_{k,l}$. □

Lemma 8. *The 50 classes* $L_{0,0}$, F, *and* $F_{i,k}$, $i = 1, \ldots, 12$; $k = 1, 2, 3, 4$, *form a basis of* $\mathrm{Pic}(S(5)) \otimes \mathbb{Q}$.

Proof. Since $h^{1,1}(S(5)) = 50$, it will be sufficient to compute the intersection matrix of these divisor classes. One finds

$$\begin{pmatrix} -5 & 1 & & & \\ 1 & 0 & & & \\ & & A_4 & & \\ & & & \ddots & \\ & & & & A_4 \end{pmatrix} \quad \text{where} \quad A_4 = \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{pmatrix}$$

There are 12 blocks A_4, each with determinant 5. So the intersection matrix has determinant $-5^{12} \neq 0$. □

Next, we define the divisor class

$$I : 5L_{0,0} + 24F - \sum_{i=1}^{12} (2F_{i,1} + 3F_{i,2} + 3F_{i,3} + 2F_{i,4}) \qquad (22)$$

This definition, looking somewhat artificial, is motivated by the following fact. 80

Lemma 9 (Inoue, Livne). *In* Pic$(S(5))$ *one has*

$$5I = \sum_{k,l} L_{k,l}$$

Proof. By lemma 7, it suffices to prove that both divisors are numerically equivalent. By lemma 8, this follows from the equations

$$\begin{aligned} &(L := \textstyle\sum L_{k,l}) \\ &\quad L \cdot L_{0,0} = -5 = 5I \cdot L_{0,0} \\ &\quad L \cdot F = 25 = 5I \cdot F \\ &\quad L \cdot F_{i,1} = L \cdot F_{i,4} = 5 = 5I \cdot F_{i,1} = 5I \cdot F_{i,4} \\ &\quad L \cdot F_{i,2} = L \cdot F_{i,3} = 5 = 5I \cdot F_{i,2} = 5I \cdot F_{i,3}. \end{aligned}$$

□

The divisor classes F and I are invariant under the action of

$$\mathrm{Aut}(S(5)) = SL(2,\mathbb{Z}_5) \times (\mathbb{Z}_5 \times \mathbb{Z}_5).$$

Lemma 10. *Any class $D \in \mathrm{Pic}(S(5))$ invariant under the group $\mathbb{Z}_5 \times \mathbb{Z}_5$ of translations by 5-torsion sections is of the form $D = pI + qF$, $p, q \in \mathbb{Z}$.*

81 *Proof.* By lemma 8, there are a, b, $c_{ik} \in \mathbb{Q}$ such that

$$D = aL_{0,0} + bF + \sum_{i=1}^{12} \sum_{k=1}^{4} c_{ik} F_{ik}$$

If D is invariant, then

$$\begin{aligned} D &= \frac{1}{25} \sum_{\alpha \in \mathbb{Z}_5 \times \mathbb{Z}_5} \alpha * D \\ &= \frac{a}{25} \sum L_{k,l} + bF + 5 \sum_{i=1}^{12} \sum_{k=1} c_{ik} F \\ &= \frac{a}{5} I + (b + 5 \sum c_{ik}) F. \end{aligned}$$

This shows $D = pI + qF$ with $p, q \in \mathbb{Q}$. But

$$p = D \cdot F_{1,0} \in \mathbb{Z}, \quad q - p = D \cdot L_{0,0} \in \mathbb{Z}$$

□

The action of the translation group $\mathbb{Z}_5 \times \mathbb{Z}_5$ lifts trivially to an action on $\mathcal{O}_{S(5)}(F)$. To $\mathcal{O}_{S(5)}(I)$, it lifts as an action of the Heisenberg group: Let σ, $\tau \in \mathbb{Z}_5 \times \mathbb{Z}_5$ be the distinguished generators (cf. 4.1) with $\sigma(L_{0,0}) = L_{1,0}$, $\tau(L_{0,0}) = L_{0,1}$. Denote by O_E the point $L_{0,0} \cap E$ for any fibre E. The explicit formula (22) shows $\mathcal{O}_E(I) = \mathcal{O}_E(5 0_E)$ for any smooth fibre E. We therefore may lift σ, τ to $\mathcal{O}_{S(5)}(I)$ such that over one fixed smooth fibre E they act as in 3.2. This implies $\sigma\tau\sigma^{-1}\tau^{-1} = \epsilon.\ \mathrm{id}$ on all of $\mathcal{O}_{S(5)}(I)$, i.e. the Heisenberg group H_5 acts on $S(5)$ covering the translation group on $S(5)$. By 3.2, the action of H_5 on $\mathcal{O}_E(5 0_E)$ is

82 uniquely determined by the symplectic basi σ, $\tau \in E^{(5)}$ up to an inner automorphism of H_5. So the action of H_5 on all the smooth fibres E is isomorphic to the one described in 3.2.

By tensor products, we define an action of H_5 on all line bundles $\mathscr{O}_{S(5)}(pI + qF)$, $p, q \in Z$.

4.4 The linear system $|I + 2F|$ on $S = S(5)$.

Here we want to describe the map defined by the linear system $|I + 2F|$ on S.

Proposition 8. *Let $D = pI + qF$ be a divisor on S. Then*

(i) *D is ample if and only if $q > p > 0$.*

(ii) *$h^1(\mathscr{O}_S(D))$ vanishes if $p > 0$, $q > p + 3$.*

(iii) *$h^2(\mathscr{O}_S(D))$ vanishes if $p > 0$.*

Proof. (i) If D is ample, then $5p = D \cdot F > 0$ and $q - p = D \cdot L_{0,0} > 0$. To prove the converse, we use the Nakai-Moishezon criterion: If $q > p > 0$, then $D^2 = 5p(2q - p) > 0$. If $C \subset S(5)$ is an irreducible curve, then $C \cdot F > 0$ and $C \cdot I \geq 0$, hence $C \cdot D > 0$, unless one of the following holds:

(a) C is a section. Then $C \cdot D = q - p > 0$

(b) C is a smooth fibre. Then $C \cdot D = 5p > 0$.

(c) C is a component of a singular fibre. Then $C \cdot D = p > 0$. **83**

(ii) follows from Kodaira's vanishing theorem and (i), because $K_S = \mathscr{O}_S(3F)$.

(iii) We have $h^2(\mathscr{O}_S(D)) = h^\circ(\mathscr{O}_S(K - D))$ and $K - D = -pI + (3 - q)F$ cannot be effective if $p > 0$.

□

Lemma 11. *For $D = I + qF$ we have the following table:*

q	$\chi(D) = 5q - 5$	$h^\circ(D)$	$h^1(D)$
0	-5	0	5
1	0	0	0
≥ 2	$5q - 5$	$5q - 5$	0

Proof. Applying Riemann-Roch to D, we find

$$\begin{aligned}\chi(D) &= \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_S) \\ &= \frac{1}{2}(I + qF) \cdot (I + (q - 3)F) + 5 \\ &= 5q - 5\end{aligned}$$

For $q \geq 1$, we show $h^1(D) = 0$ by descending induction on q. If $q \geq 5$, the assertion follows from proposition 8. Assume $q \geq 1$ and consider, for a smooth fibre E, the exact sequence

$$0 \to \mathcal{O}_S(I + qF) \to \mathcal{O}_S(I + (q + 1)F) \to \mathcal{O}_E(5O_E) \to 0$$

84 By induction, $h^1(\mathcal{O}_S(I + (q + 1)F)) = 0$ and we obtain a H_5-equivariant exact sequence

$$H^\circ(\mathcal{O}_S(I + (q + 1)F)) \xrightarrow{\text{restr.}} H^\circ(\mathcal{O}_E(5O_E)) \to H^1(\mathcal{O}_S(I + qF)) \to 0.$$

In view of $h^1(\mathcal{O}_S(I + (q + 1)F)) = 0$, we have $h^\circ(O_S(I + (q + 1)F)) \neq 0$. So the restriction map cannot vanish for all E. But the H_5-space $H^\circ(\mathcal{O}_E(5O_E))$ is irreducible, so restr is surjective for general E and the assertion follows for $q \geq 1$. But, if $\mathcal{O}_S(I + F)$ has no sections, the same is true for $\mathcal{O}_S(I)$. □

Lemma 12. *The linear system $|I + 2F|$ has no base points on S.*

Proof. Lemma 11 implies that restriction

$$H^\circ(\mathcal{O}_S(I + 2F)) \to H^\circ(\mathcal{O}_E(I))$$

is bijective for all fibres E. If E is smooth, then $\mathcal{O}_E(I) = \mathcal{O}_E(5O_E)$ and E cannot contain any base points. So assume that $E = \bigcup_{k=0}^{4} F_{i,k}$ is a pentagon. The restriction $\mathcal{O}_{F_{i,k}}(I)$ has degree 1 on each component $F_{i,k}$. If a smooth point $p \in E$ would be a base point, then by $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry there would be 25 such points on E and all sections in $\mathcal{O}_E(I)$ would vanish, a contradiction. If a singular point $p \in E$ would be a base point, then by $\mathbb{Z}_5 \times \mathbb{Z}_5$-symmetry each component $F_{i,k}$ would contain 85
two base points, which leads to the same contradiction. □

4.5 The surface S_{15}.

In view of lemma 12 above, the linear system $|I + 2F|$ defines a map $S(5) \to \mathbb{P}_4$. We normalize it by identifying $H^\circ(\mathcal{O}_S(I + 2F)) = V^*$ equivariantly with respect to the action of the Heisenberg group H_5.

Proposition 9. *The linear system* $|I + 2F|$ *maps the modular surface onto a surface* $S_{15} \subset \mathbb{P}_4$ *of degree* 15*. The map is injective except for the* 60 *double points of the* 12 *singular fibres. The fibres are paired and the* 5 *double points on both fibres in one pair have the same* 5 *image points.*

Proof. The map $S(5) \to \mathbb{P}_4$ is H_5-equivariant. The fibres on $S(5)$ therefore go the H_5-invariant curves. The smooth fibres go to the smooth quintics from 3.3 Two different fibres are not isomorphic as curves with level-5-structure (see 4.1), so they have different image curves E_λ. By proposition 2, the images are even disjoint. So the map is injective on the open part $S'(5) \subset S(5)$ and the image must be surface S_{15} of degree

$$(I + 2F)^2 = 15.$$

The 12 singular fibres go to non-degenerate H_5-invariant pentagons in 86
$\mathbb{P}_4$. The vertices of such a pentagon are fixed points for some cyclic subgroup $\mathbb{Z}_5 \subset \mathbb{Z}_5 \times \mathbb{Z}_5$ hence belong to the 30 points e_k, $e_{k,l}$. It follows that each singular fibre is mapped isomorphically to one of the 12 singular curves E_λ. Conversely, all curves E_λ are images of fibres on $S(5)$. This follows because the curves E_λ sweep out the irreducible surface which

is the projection of $S_{25} \subset \mathbb{P}_4 \times \mathbb{P}_1$ from 3.4 into $\mathbb{P}_4$. This projection necessarily coincides with S_{15} so each E_λ is contained in the image. By proposition 2, all E_λ are disjoint, except for the pairs of singular ones which have the same vertices. This proves the assertion. □

Next, we show that S_{15} is smooth, except for the 30 double points e_k, $e_{k,l}$, where two smooth branches of S_{15} cross transversally. This follows from

Proposition 10. (i) *The projection $S_{25} \to S_{15}$ is an immersion.*

(ii) *The two branches of S_{15} meeting in one of the 30 points e_k, $e_{k,l}$, are transversal.*

Proof. At $(v; \lambda, \mu) \in S_{25}$ the tangent space to this surface is given by those dv, $d\lambda$ for which

$$(\lambda f^+(v) + \mu f^-(v)) \wedge (\lambda f^+(dv) + \mu f^-(dv) + d\lambda f^+(v) + d\mu f^-(v)) = 0.$$

The projection has maximal rank on this tangent space unless it contains all vectors $(0; d\lambda, d\mu)$. But this would imply

$$2\lambda h^+(v) + \mu h^\circ(v) = \lambda h^\circ(v) + 2\mu h^-(v) = 0.$$

87 Now, recall that

$$\lambda^2 h^+(v) + \lambda\mu h^\circ(v) + \mu^2 h^-(v) = 0$$

is the only relation between h^+, h°, and h^- unless $v = e_k$ or $e_{k,l}$. So, for $v \neq e_k$, e_{kl}, we find $\lambda\mu = 0$, which eventualy leads to the contradiction $h^\circ(v)) = 0$.

If v is one of the 30 singular points on S_{15} we may assume $v = e_0$, without loss of generality. The two points $(e_0; 1, 0)$ and $(e_0; 0, 1)$ lie on S_{25} over e_0. The two tangent planes to S_{25} are

$$\mu = 0 : \; f^+(e_0)(f^+(dv) + d\mu f^-(e_0)) = dv_2\widehat{e_1} + dv_3\widehat{e_4} + d\mu\widehat{e_0} = 0$$
$$\lambda = 0 : \; f^-(e_0)(f^-(dv) + d\lambda f^+(e_0)) = -dv_1\widehat{e_3} - dv_4\widehat{e_2} + d\lambda\widehat{e_0} = 0.$$

They project in $\mathbb{P}_4$ to the two planes $v_2 = v_3 = 0$ and $v_1 = v_4 = 0$. They are tangent to the two branches of S_{15} at e_0 and transversal. □

Finally, we consider the equations defining S_{15}. By definition, S_{15} is the set of points v where $h^+(v) \wedge h^\circ(v) \wedge h^-(v) = 0$. i.e. the determinantal locus where the matrix

$$\begin{pmatrix} v_0^2 & v_1^2 & v_2^2 & v_3^2 & v_4^2 \\ v_2v_3 & v_3v_4 & v_4v_0 & v_0v_1 & v_1v_2 \\ v_1v_4 & v_2v_0 & v_3v_1 & v_4v_2 & v_0v_3 \end{pmatrix}$$

drops its rank. Its ten 3×3 determinants, the 10 coefficients of $v^+ \wedge v^-$ **88**
are ($k = 0, \ldots, 4$).

$$\begin{aligned} &v_kv_{k+2}v_{k+3}(v_{k+2}v_{k+4}^2 - v_{k+1}^2v_{k+3}^2) + v_k^3(v_{k+3}^2v_{k+4} - v_{k+1}v_{k+2}^2) \\ &\qquad + v_{k+1}v_{k+4}(v_{k+1}^3v_{k+2}^3 - v_{k+3}v_{k+4}^3) \\ &v_kv_{k+1}v_{k+4}(v_{k+1}v_{k+2}^2 - v_{k+3}^2v_{k+4}) + v_k^3(v_{k+2}v_{k+4}^2 - v_{k+1}^2v_{k+3}) \\ &\qquad + v_{k+2}v_{k+3}(v_{k+1}v_{k+3}^3 - v_{k+2}^3v_{k+4}) \end{aligned} \tag{23}$$

Proposition 11. *The ten equations* (23) *schematically define S_{15}.*

Proof. As usual, we distinguish between two cases:

- If $v \in S_{15}$ is a smooth point, we check that the differential

  $$d(v^+ \wedge v^-) : V \to \Lambda^2 V$$

  has rank 2. Since $v^- = (\lambda/\mu)v^+$ (if e.g. $\mu \neq 0$) we find

  $$\begin{aligned} d(v^+ \wedge v^-)|_v &= dv^+|_v \wedge v^- + v^+ \wedge dv^-|_v \\ &= ((\lambda/\mu)v \wedge d^+|_v - dv^-|_v) \wedge v^+. \end{aligned}$$

  From 3.5, Proposition 7, we know that the differential $d(\lambda v^+ - \mu v^-)$, λ, μ fixed, has rank 3. This implies that our differential has rank ≥ 2.

- If $v \in S_{15}$ is a singular point, we assume $v = e_0$, without loss of **89**
  generality. The two branches of S_{15} at e_0 are given by functions g_2, g_3 and g_1, $g_4 \in \mathcal{O}_{e_0}$ respectively, where we may assume that $g_i - v_i$ belongs to m^2, m the ideal generated by v_1, v_2, v_3, v_4 at e_0. Then the ideal of S_{15} is generated by g_1g_2, g_1g_3, g_2g_4, and g_3g_4.

□

On the other hand, putting $v_0 = 1$ in equations (23) and reducing modulo m^3 we obtain (only terms containing v_0^4 ramain): v_1v_2, v_1v_3, v_2v_4, v_3v_4. So among the 10 polynomials (23), we may pick four ones, say $h_1, \dots, h_4$ such that

$$g_1g_2 - h_1 \equiv g_1g_3 - h_2 \equiv g_2g_4 - h_3 \equiv g_3g_4 - h_4 \equiv 0 \mod m^3.$$

The polynomials $h_1, \dots, h_4$ belong to the ideal of S_{15}, so we may write them

$$h_i = c_i^{12} g_1g_2 + c_i^{13} g_1g_4 + c_i^{24} c_2g_4 + c_i^{34} g_3g_4, c_i^{kl} \in \mathscr{O}_{e_0}.$$

The congruences above imply

$$c_1^{12} \equiv c_2^{13} \equiv c_3^{24} \equiv c_4^{34} \equiv 1 \mod m$$

$$c_i^{kl} \in m \quad \text{otherwise.}$$

90 The 4×4-matrix

$$\begin{pmatrix} c_1^{12} & c_2^{12} & c_3^{12} & c_4^{12} \\ c_1^{13} & c_2^{13} & c_3^{13} & c_4^{13} \\ c_1^{24} & c_2^{24} & c_3^{24} & c_4^{24} \\ c_1^{34} & c_2^{34} & c_3^{34} & c_4^{34} \end{pmatrix}$$

therefore admits an inverse with coefficients in $\mathscr{O}_{e_0}$.

We therefore may express g_1g_2, g_1g_3, g_2g_4, g_3g_4 in the functions $h_1, \dots, h_4$. This implies that the functions (23) generate the ideal of S_{15} at e_0.

5 Jumping lines of the Horrocks-Mumford bundle

5.1 Jumping lines belonging to $v \in V$.

The Horrocks-Mumford bundle $\mathscr{F}$ is stable with $c_1 = 5$. Therefore, we have $\mathscr{F}|L = \mathscr{O}_{\mathbb{P}_1}(2) \oplus \mathscr{O}_{\mathbb{P}_1}(3)$ for the general line $L \subset \mathbb{P}_4$ (Grauert-Mulich). We define the *jumping order* $k(L)$ for each line $L \subset \mathbb{P}_4$ by

$$\mathscr{F}|L = \mathscr{O}_{\mathbb{P}_1}(2-k) \oplus \mathscr{O}_{\mathbb{P}_1}(3+k).$$

So L is a *jumping line* (resp. *double, triple jumping line*) if $k(L) \geq 1$ (resp. $k(L) \geq 2, 3$). By [1, proposition 1], this jumping order equals the 91
dimension of the kernel of the map

$$V \to 2\Lambda^3 V, v \to f^+(v) \wedge L, f^-(v) \wedge L.$$

We define the variety $\widetilde{J} \subset \mathbb{P}_4 \times \mathrm{Gr}(1,4)$ by

$$\widetilde{J} = \{(v, L) \in \mathbb{P}(V) \times \mathrm{Gr} : f^+(v) \wedge L = f^-(v) \wedge L = 0\}.$$

Then we obtain the following description of $J \subset \mathrm{Gr}$, the *variety of jumping lines for* $\mathscr{F}$: *J is the projection of $\widetilde{J}$ into* Gr. *The fibre of this projection over a jumping line L is a projective subspace in* $\mathbb{P}(V)$ *of dimension* $k(L) - 1$.

We denote by $J_v \subset \mathrm{Gr}$ the projection of the fibre over $v \in \mathbb{P}(V)$ under $J \to \mathrm{Gr}$. This J_v is the variety of *jumping lines determined* by v. First, we describe the variety $Y_v \subset \mathbb{P}(V)$ swept out by the jumping lines determined by v. We use

Lemma 13 ([1, lemma 3])**.** *For v, $x \in \mathbb{P}_4$ the following properties are equivalent:*

(i) *through x there passes a jumping line $x \wedge y$ determined by v,*

(ii) $\langle h^+(v), x\rangle = \langle h^\circ(v), x\rangle = \langle h^-(v), x\rangle = 0$.

It follows that the jumping lines determined by v sweep out the linear subspace

$$Y_v = \{x \in \mathbb{P}_4 : \langle h^+(v), x\rangle = \langle h^0(v), x\rangle = \langle h^-(v), x\rangle = 0\}$$

We have 92

- $\dim Y_v = 1$ if $v \in \mathbb{P}_4 \backslash S_{15}$
- $\dim Y_v = 2$ if $v \in S_{15}$ is a smooth point
- $\dim Y_v = 3$ if $v \in S_{15}$ is singular.

Next we describe the varieties J_v:

Proposition 12. (i) *If* $v \in \mathbb{P}_4 \backslash S_{15}$*, then* J_v *consists of one element, the line* Y_v*.*

(ii) *If* $v \in S_{15}$ *is a smooth point, then* J_v *is a rational curve parametrizing the pencil of lines in the plane* Y_v *with vertex* $v^+ \sim v^-$*.*

(iii) *If* v *is one of the* 30 *double points of* S_{15}*, with* $L_i = \lambda_i f^+(v) + \mu_i f^-(v)$*,* $i = 1, 2$*, the two simple bivectors from proposition 1* (iii)*, then* J_v *is a quadric surface, the* $\mathbb{P}_1 \times \mathbb{P}_1$*'s worth of lines meeting both the lines* L_1 *and* L_2*.*

Proof. (i) follows from lemma 13 above.

(ii) By (11) the four tri-vectors $f^{\pm}(v) \wedge v^{\pm}$ correspond to the plane Y_v (unless some of them vanish, but not all of them do, by proposition 1 (ii). Not all the lines in Y_v can be jumping lines determined by v, for, if $Y_v = x \wedge y \wedge z$, then $f^{\pm}(v) \wedge x \wedge y = 0$ would imply $f^{\pm}(v) = x \wedge x^{\pm} + y \wedge y^{\pm}$. Applying $\wedge x \wedge z$ and $\wedge y \wedge z$, we would find $x^{\pm}, y^{\pm} \in Y_v$ and $f^{\pm}(v)$ decomposable. This would contradict the assumption that $v \in S_{15}$ is smooth.

93 On the other hand, each line $L = v^{\pm} \wedge y$ in Y_v passing through $v^{\pm}$ is a jumping line determined by v:

$$f^{\pm}(v) \wedge L = (f^{\pm}(v) \wedge v^{\pm}) \wedge y = Y_v \wedge y = 0$$

This implies the assertion, because being a jumping line determined by v is a linear condition on the Plucker coordinates of the line.

(iii) The vanishing of v^+ and v^- implies $\langle h^\circ(v), f^{\pm}(v)\rangle = 0$. By (4) the bivectors $f^{\pm}(v)$ belong to $\Lambda^2 Y_v$ and the two lines L_i are lines in Y_v. By proposition 1 (iii), these two lines do not coincide. So we may write the bivectors $f^+(v)$ and $f^-(v)$ as linear combinations of L_1 and L_2. If $L_1 \wedge L_2$ would vanish, then $(\lambda f^+(v) + \mu f^-(v)) \wedge (\lambda f^+(v) + \mu f^-(v))$ would vanish identically a contradiction with (8). So L_1

and L_2 are disjoint. The condition $f^+(v) \wedge L = f^-(v) \wedge L = 0$ is equivalent with $L_1 \wedge L = L_2 \wedge L = 0$. This shows that the jumping lines determined by v are exactly the lines (in Y_v) meeting both L_1 and L_2.

□

5.2 The 25 triple jumping lines.

The 25 sections $L_{i,j} \subset S(5)$ are mapped to lines in $\mathbb{P}_4$ under the linear system $I + 2F$.

Proposition 13. *The* 25 *sections* $L_{i,j}$ *map to the* 25 *Horrocks-Mumford lines [6, p.72].*

$$z_{-i} = z_{-i+2} + \epsilon^j z_{-i+3} = z_{-i+1} + \epsilon^{3j} z_{-i+4} = 0$$

94

Proof. We already saw, in lemma 3, that the origin O_E of a H_5-invariant elliptic curve $E \subset \mathbb{P}_4$ is contained in the line

$$z_0 = z_2 + z_3 = z_1 + z_4 = 0. \tag{24}$$

Hence $L_{0,0}$ must be mapped to this line. The 25 sections $L_{i,j}$ are the translates $\tau^j \sigma^i L_{0,0}$. Their images in $\mathbb{P}_4$ are therefore obtained by applying $\tau^j \sigma^i$. The result is

$$z_{-i} = z_{-i+2} + \epsilon^j z_{-i+3} = z_{-i+1} + \epsilon^{3j} z_{-i+4} = 0.$$

□

Proposition 14. *The* 25 *Horrocks Mumford lines are triple jumping lines for* $\mathscr{F}$*. In fact, they are the only jumping lines L of order* $k(L) \geq 3$.

Proof. To show that these 25 lines are triple jumping lines, by H_5-symmetry it suffices to consider the image (24) of $L_{0,0}$. It is spanned by $e_2 - e_3$ and $e_4 - e_1$. We have

$$(e_2 - e_3) \wedge (e_4 - e_1) = e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_4 - e_3 \wedge e_4$$

$$f^+(v) \wedge (e_2 - e_3) \wedge (e_4 - e_1) = (v_1 - v_4)\widehat{e_0} + (v_2 - v_3)(\widehat{e_2} + \widehat{e_3})$$

$$f^-(v) \wedge (e_2 - e_3) \wedge (e_4 - e_1) = (v_2 - v_3)\widehat{e_0} + (v_4 - v_1)(\widehat{e_1} + \widehat{e_4})$$

95 So this jumping line is determined by all v in the 3-dimensional subspace $v_2 - v_3 = v_1 - v_4 = 0$.

Conversely, if L is a jumping line of some order $3 + k$, $k \geq 0$, then $\mathscr{F}|L = \mathscr{O}_{\mathbb{P}_1}(-1 - k) \oplus \mathscr{O}_{\mathbb{P}_1}(6 + k)$ is not globally generated. But away from the 25 Horrocks-Mumford lines, the bundle is generated by its global sections [6, p. 75]. □

5.3 Smoothness of $\widetilde{J}$.

The aim of this section is to prove

Proposition 15. *The subvariety* $\widetilde{J} \subset \mathbb{P}_4 \times \mathrm{Gr}$ *is smooth of dimension* 4.

Proof. For $(v, L) \in \widetilde{J}$, we again distinguish between the three different cases:

- Over $\mathbb{P}_4 \backslash S_{15}$, the projection $\widetilde{J} \to \mathbb{P}_4$ is bijective. So $\widetilde{J}$ necessarily is smooth there.
- If $v \in S_{15}$ is a smooth point and $L \in J_v$ we show that $(v, L) \in \widetilde{J} \subset \mathbb{P}_4(V) \times \mathbb{P}_9(\Lambda^2 V)$ is a smooth point by differentiating at (v, L) the ten equations $f \pm (v) \wedge L = 0$ defining $\widetilde{J}$. The differential is the map

$$V \times \Lambda^2 V \to 2\Lambda^4 V$$

$$(dv, dL) \to f^{\pm}(dv) \wedge L + f^{\pm}(v) \wedge dL$$

96

A sufficient condition for smoothness is that this map has rank

$$\dim(\mathbb{P}_4 \times \mathbb{P}_9) - 4 = 9.$$

A pair $(t, u) \in 2V$ annihilates the image of this map if and only if

$$(f^+(dv) \wedge t + f^-(dv) \wedge u) \wedge L = 0 \quad \text{for all } dv \in V \tag{25}$$

$$(f^+(v) \wedge t + f^-(v) \wedge u) \wedge dL = 0 \quad \text{for all } \ dL \in \Lambda^2 V.$$

The second condition of course is equivalent with

$$f^+(v) \wedge t + f^-(v) \wedge u = 0. \tag{26}$$

First, we show that $t \wedge u = 0$ in this situation. Indeed, if $t \wedge u \neq 0$, then (26) implies that $t \wedge u$ is a jumping line determined by v.

If e.g. $v^+ \neq 0$, we may write

$$t = t^1 v^+ + t^2 y, u = u^1 v^+ + u^2 y, \quad \text{with} \quad y \in Y_v, t^i, u^i \in \mathbb{C}.$$

Inserting this in (26), we find using (11) that 97

$$t^1 f^+(v) \wedge v^+ + t^2 f^+(v) \wedge y + u^1 f^-(v) \wedge v^+ + u^2 f^-(v) \wedge y$$

$$= -t^1 (h^0(v) \wedge h^+(v)) * -u^1 (h^-(v) \wedge h^+(v)) * +$$

$$(t^2 f^+(v) + u^2 f^-(v)) \wedge y$$

vanishes. Contracting with $h^+(v)$, $h^-(v)$, $h^\circ(v)$ and using (5), (3) and (9), we find

$$u^2 v^+ \wedge y = t^2 v^- \wedge y = (t^2 v^+ + u^2 v^-) \wedge y = 0.$$

Since either $v^+ \wedge y \neq 0$ of $v^- \wedge y \neq 0$, this leads to $t^2 = u^2 = 0$, in conflict with $t \wedge u \neq 0$.

Using $t \wedge u = 0$, we write $t = aw$, $u = bw$ with $w \in V$, a, b, $\in \mathbb{C}$. Then (25) and (26) become

$$(af^+(dv) + bf^-(dv)) \wedge w \wedge L = 0 \quad \text{for all} \quad dv \in V \tag{27}$$

$$(af^+(v) + bf^-(v)) \wedge w = 0. \tag{28}$$

In particular, the bivector $af^+(v) + bf^-(v)$ is simple, so $(a : b) = (\lambda : \mu)$ with λ, μ as in proposition 2. The assertion comes down to showing $w \wedge v^\pm = 0$. Otherwise, we could write (cf. (14))

$$af^+(v) + bf^-(v) = w \wedge v^\pm$$

$$L = y \wedge v^\pm, y \in Y_v.$$

Inserting this in (27), we find 98

$$(\lambda f^+(dv) + \mu f^-(dv)) \wedge (\lambda f^+(v) + \mu f^-(v)) \wedge y = 0 \quad \text{for all } dv \in V.$$

But in (2.5), we showed that the map

$$dv \to (\lambda f^+(dv) + \mu f^-(dv)) \wedge (\lambda f^+(v) + \mu f^-(v))$$

has rank 3 for all smooth points v on the curve $E_{\lambda:\mu}$. The image of this map now is annihilated by w, $v^\pm$, and y. This implies $w \wedge v^\pm \wedge y = 0$ and since $w \wedge v^\pm \neq 0$, the line $\lambda f^+(v) + \mu f^-(v)$ is a jumping line determined by v. Evaluating this we obtain

$$2\lambda h^+(v) + \mu h^\circ(v) = \lambda h^\circ(v) + 2\mu h^-(v) = 0.$$

This contradicts proposition 2 (iv).

– If $v \in S_{15}$ is singular, we assume $v = e_0$ w.l.o.g. Now it does not suffice to differentiate the equations defining $\widetilde{J}$ with respect to $dL \in \Lambda^2 V$, we must consider $dL \in T_{\mathrm{Gr},L}$ and check that

$$\begin{aligned} V \times T_{\mathrm{Gr},L} &\to 2\Lambda^4 V \\ dv, dL &\to f^\pm(dv) \wedge L + f^\pm(v) \wedge dL \end{aligned}$$

99 has rank

$$\dim(\mathbb{P}_4 \times \mathrm{Gr}) - 4 = 6.$$

we write $L = x \wedge y$ with

$$x = x^2 e_2 + x^3 e_3 \quad \text{a point on the line} \quad f^+(e_0) = e_2 \wedge e_3$$

$$y = y^1 e_1 + y^4 e_4 \quad \text{a point on the line} \quad f^-(e_0) = e_1 \wedge e_4$$

Then $dL = x \wedge dy + dx \wedge y$ with $dx, dy \in V$ and we must check that

$$\begin{aligned} &3V \to 2\Lambda^2 V \\ &dv, dx, dy \to f^\pm(dv) \wedge L + f^\pm(e_0) \wedge x \wedge dy + f^\pm(e_0) \wedge dx \wedge y \end{aligned}$$

has rank 6. We compute

$$f^+(e_0) \wedge \times \wedge dy = f^-(e_0) \wedge dx \wedge y = 0$$

$$f^+(e_0) \wedge dx \wedge y = (y^4 dx^1 - y^1 dx^4)\widehat{e_0} - y^4 dx^0 \widehat{e_1} + y^1 dx^0 \widehat{e_4}$$

$$f^-(e_0) \wedge x \wedge dy = (x^2 dy^3 - x^3 dy^2)\widehat{e_0} + x^3 dy^0 \widehat{e_2} - x^2 dy^0 \widehat{e_3}$$

Remembering $(x^2, x^3) \neq (0,0) \neq (y^1, y^4)$, one easily checks that the map in question is given by a 10 × 15 matrix of rank 6.

$\square$

5.4 Blowing-up the ideal of S_{15}

100 First, we describe the blowing-up of a double point, i.e., the blowing-up of $\mathbb{C}^4$ (with coordinates $u_1, \ldots, u_4$) with respect to the ideal sheaf $\mathscr{I} \subset \mathscr{O}_{\mathbb{C}^4}$ generated by the four polynomials

$$u_1 u_2, u_1 u_3, u_2 u_4, u_3 u_4$$

Let $X \to \mathbb{C}^4$ be this blowing-up. By definition [5, p.163], we have $X = \mathrm{Proj}(\bigoplus_{d \geq 0} \mathscr{I}^d)$. There is a surjective map of graded $\mathscr{O}_{\mathbb{C}^4}$-algebras

$$\mathscr{O}_{\mathbb{C}^4}[p_0, p_1] \otimes_{\mathscr{O}_{\mathbb{C}^4}} \mathscr{O}_{\mathbb{C}^4}[q_0, q_1] \to \oplus \mathscr{I}^d$$

$$p_0 \to u_1, p_1 \to u_4, q_0 \to u_2, q_1 \to u_3$$

inducing an embedding $X \to \mathbb{C}^4 \times \mathbb{P}_1 \times \mathbb{P}_1$. The image is the smooth 4-fold with equations

$$p_0 u_4 = p_1 u_1, q_0 u_3 = q_1 u_2 \tag{29}$$

The fibre over some point $u \in \mathbb{C}^4$ is

- a point, if $u \notin \{u_1 = u_4 = 0\} \cup \{u_2 = u_3 = 0\}$
- a copy of $\mathbb{P}_1$, if $u \in \{u_1 = u_4 = 0\} \cup u_2 = u_3 = 0$ but $u \neq 0$

- the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ if $u = 0$.

Equations (29) show that $X \subset \mathbb{C}^4 \times \mathbb{P}_1 \times \mathbb{P}_1$ can be considered as the graph of the rational map $\mathbb{C}^4 \to \mathbb{P}_1 \times \mathbb{P}_1$ sending $(u_1, \dots, u_4)$ to $(u_4 : u_1)$, $(u_3 : u_2)$. This blowing-up is related to blowing up smooth subvarieties as follows: 101

Let $\pi_1 : B_1 \to \mathbb{C}^4$ be the blowing-up in the origin. Introducing an auxiliary $\mathbb{P}_3 = \mathbb{P}(T_{\mathbb{C}^4,0})$ with coordinates $z_1 : \dots : z_4$ we have $B_1 \subset \mathbb{C}^4 \times \mathbb{P}_3$ as the submanifold given by $z_i u_j = z_j u_i$. The two planes $\{u_1 = u_4 = 0\}$, $\{u_2 = u_3 = 0\}$ are separated under π_1. Their proper transforms in B_1 are smooth surfaces meeting the exceptional $\mathbb{P}_3 = \pi_1^{-1}(0)$ in the two lines $L_1 : z_1 = z_4 = 0$, $L_2 : z_2 = z_3 = 0$. Let $\pi_2 : B_2 \to B_1$ be the blowing-up of B_1 in these smooth surfaces. The functions $u_4 : u_1$ and $u_3 : u_2$ are well-defined on B_2, so they induce a map $B_2 \to X \subset \mathbb{C}^4 \times \mathbb{P}_1 \times \mathbb{P}_1$ such that the diagram

$$\begin{tikzcd}
B_2 \arrow[r, "\pi_2"] \arrow[d] & B_1 \arrow[d, "\pi_1"] \\
X \arrow[r] & \mathbb{C}^4
\end{tikzcd}$$

commutes. This map $B_2 \to X$ is an isomorphism over $\mathbb{C}^4 \backslash 0$. But, on $\mathbb{P}_3 = \pi^{-1}(0)$, it contracts to points in $\mathbb{P}_1 \times \mathbb{P}_2$ all the lines meeting both L_1 and L_2. (In B_2, all these lines become disjoint.)

Finally, denote by $\pi : \widehat{\mathbb{P}} \to \mathbb{P}_4$ *the blowing-up of* $\mathbb{P}_4$ *with respect to the ideal sheaf of* S_{15}. After the preceding local discussion, it is clear that $\widehat{\mathbb{P}}$ is a smooth 4-fold and the fibre $\pi^{-1}(v)$, $v \in \mathbb{P}_4$, is

102 - a point if $v \in \mathbb{P}_4 \backslash S_{15}$

- a copy of $\mathbb{P}_1$, if v is a smooth point of S_{15}

- a copy of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$, if v is a singular point of S_{15}.

5.5 The variety of jumping lines

The variety $\widetilde{J} \subset \mathbb{P}_4 \times \mathrm{Gr}$ is a smooth 4-fold. The projection $\widetilde{J} \to \mathbb{P}_4$ is surjective (proposition 12); over $\mathbb{P}_4 \backslash S_{15}$ even an isomorphism. All fibres are irreducible. This proves that $\widetilde{J}$ itself is irreducible.

The map $v^+ \wedge v^- : \mathbb{P}_4 \to \mathrm{Gr}$ is well-defined only on $\mathbb{P}_4 \backslash S_{15}$. There it associates with v the unique jumping line belonging to v. Therefore $\widetilde{J}$ is the graph $\{(v, v^+ \wedge v^-) : v \in \mathbb{P}_4\}$ of the map $v^+ \wedge v^-$ over $\mathbb{P}_4 \backslash S_{15}$.

The 10 components of $v^+ \wedge v^-$ generate the ideal sheaf of S_{15} at each point (proposition 11). This implies that the map $v^+ \wedge v^-$ extends to a regular map $\varphi : \widehat{\mathbb{P}} \to \mathrm{Gr}$ on the blow up of $\mathbb{P}_4$ w.r.t. the ideal sheaf. The map $(\pi, \varphi) : \widehat{\mathbb{P}}_4 \to \mathbb{P}_4 \times \mathrm{Gr}$ has its image in $\widetilde{J}$:

$$\begin{tikzcd}
\widehat{\mathbb{P}}_4 \arrow[rr] \arrow[dr] & & \widetilde{J} \arrow[dl] \\
& \mathbb{P}_4 &
\end{tikzcd}$$

The induced map $\widehat{\mathbb{P}}_4 \to \widetilde{J}$ is birational, over $\mathbb{P}_4 \backslash S_{15}$ an isomorphism. **103**
A point $(v, L) \in \widetilde{J}$ cannot have several pre-images in $\widehat{\mathbb{P}}_4$, because then it could not be a normal point on $\widetilde{J}$, which is smooth. This proves

Proposition 16. *The map* $(\pi, \varphi) : \widehat{\mathbb{P}} \to \mathbb{P}_4 \times \mathrm{Gr}$ *defines an isomorphism* $\widehat{\mathbb{P}} \to \widetilde{J}$.

In particular, $\widehat{\mathbb{P}} = \widetilde{J}$ can be considered as the graph of the rational map $v^+ \wedge v^-$.

The variety J of jumping lines for the Horrocks-Mumford bundle $\mathscr{F}$ is the projection of $\widetilde{J}$ into Gr. The fibres of the projection $\widetilde{J} \to J$ are linear subspaces in $\mathbb{P}_4$. J contains the sub-varieties J_2, resp. J_3 of double, resp. triple jumping lines. They consist of the image points of fibres of dimension ≥ 1, resp. ≥ 2.

Theorem 2. *The variety* $J \subset \mathrm{Gr}$ *is a rational, irreducbile* 4*-fold. It is smooth outside of* J_2*. The variety* $J_2 \subset J$ *has dimension* 2*.*

Proof. For $\mathscr{F}$, as for every rank-2-bundles on $\mathbb{P}_n$ with odd c_1, the jumping lines come in codimension ≤ 2. Therefore, $\dim J \geq 4$ and the general fibre of $\widetilde{J} \to J$ is discrete. This shows that $J_2 \neq J$ and that $\widetilde{J} \to J$ is birational. Since over J_2 all fibres have dimension ≥ 1, necessarily $\dim J_2 \leq 2$. But $J_2 \neq \phi$, because it contains e.g. the 25 points corresponding to the triple jumping lines $L_{k,l}$. Deformation theory shows $\operatorname{codim} J_2 \leq 4$. We therefore find that J_2 is a surface in J.

104 Over $J \backslash J_2$, the map $\widetilde{J} \to J$ is bijective. To show that it is biregular there we must check that at $(v, L) \in \widetilde{J}$ the tangent space $T_{\widetilde{J}}$ is transversal to the fibre $\mathbb{P}_4 \times L$. But this follows immediately from the fact that the equations $f \pm (v) \wedge L = 0$ defining $\widetilde{J}$ are linear in v. As a consequence, J is smooth outside of J_2. □

It is known that through each point $x \in \mathbb{P}_4$ some jumping line has to pass [cf. 12].

Proposition 17. (i) *Through the general point $x \in \mathbb{P}_4$ there passes no double jumping line.*

(ii) *The jumping lines through a general point $x \in \mathbb{P}_4$ are the generators of a cone (with vertex x) over a smooth curve in $\mathbb{P}_3$.*

Proof. (i) Since the double jumping lines are parametrized by a surface, they sweep out at most a threefold in $\mathbb{P}_4$.

(ii) Consider, in $\mathbb{P}_4 \times \mathrm{Gr}$, the variety

$$\{(x, L) : L \in J, x \in L\}$$

It is an irreducible 5-fold which is singular along a 3-fold (lying over J_2). The projection of this 5-fold in $\mathbb{P}_4$ is surjective. The general fibre therefore does not meet the singular locus. By Bertini's theorem, the general fibre must be a smooth curve. This curve is the basis for the cone of jumping lines through the image point of this fibre.

□

References

105

[1] Barth, W. Kummer surfaces associated with the Horrocks-Mumford bundle. In: *Journées de géométrie algébrique d'Angers* 1979, p. 29-48 (Ed. A. Beauville), Sijthoff and Noordhoff, Alphen aan den Rijn 1980.

[2] Barth, W., Peters, C., Van de Ven, A. *Compact complex surfaces,* Springer, Berlin - Heidelberg - New York, 1984.

[3] Bianchi, L. Ueber die Normalformen dritter und fünfter Stufe des elliptischen Integrals erster Gattung, *Math. Annalen* 17, 234-262 (1880).

[4] Bourbaki, N. Eléments de Mathématique VII, Algèbre, Chap. 3, *Algèbre Multilinéaire,* Hermann, Paris 1958.

[5] Hartshorne, R. *Algebraic geometry,* Springer, New York - Heidelberg - Berlin, 1977.

[6] Horrocks, G., Mumford, D. A rank 2 vector bundle on $\mathbb{P}_4$ with 15.000 symmetries, *Topology* 12, 63-81 (1973).

[7] Hulek, K. Projective geometry of elliptic curves, to appear in *l'* Asterisque.

[8] Mumford, D. On the equations defining abelian varieties I, *Invent. Math.* 1, 287-354 (1966).

[9] Mumford, D. *Tata lectures on theta* I, Progress in Math., Vol. 28, 106
Birkhäuser, Boston, 1983.

[10] Shimura, G. *Introduction to the arithmetic theory of automorphic functions,* Iwanami Shoten and Princeton University Press, 1971.

[11] Shioda, T. On elliptic modular surfaces, *J. Math. Soc. Japan* 24, 20-59 (1972).

[12] Van de Ven, A. On uniform vector bundles, *Math. Ann.* 195, 245-248 (1972).

[13] Barth, W., Hulek, K. Projective models of Shioda modular surfaces, *Manuscr. Math.* 50, 73-132 (1985).

*Mathematisches Institut der
Universitat Erlangen-Nurnberg
Bismarckstraβe 1 1/2
D-8520 Erlangen
Bundesrepublik Deutschland

**Department of Mathematics
Australian National University
Institute of Advanced Studies
Canberra, A.C.T. 2600
Australia

Some Results on a Question of Quillen

By S. M. Bhatwadekar

1 Introduction

107 This paper intends to bring to your attention some recent developments on an approach suggested by Quillen to the following well-known conjecture of H. Bass and D. Quillen.

Bass-Quillen Conjecture : Let S *be a regular local ring. Then every finitely generated projective* $S[T]$*- module is free.*

In view of a monic inversion criterion of G. Horrocks (see [5]) it would suffice to show that the extension of the projective module to $S(T)$ (= localization of $S[T]$ at the multiplicatively closed set of monic polynomials) is free. In particular, if all finitely generated projective $S(T)$-modules are free then all finitely generated projective $S[T]$-modules are free.

Let $R = S[T^{-1}]_{(n,T^{-1})}$ where $\mathfrak{n}$ denotes the maximal ideal of S. Then, since S is local, $S(T) = R_{T^{-1}}$. Therefore in [12], Quillen posed the following stronger question.

Question:- Are all finitely generated projective R_f*-modules free,* **108** *when* R *is a regular local ring and* f *is a regular parameter of* R*?*

Since then, the Bass-Quillen Conjecture has been settled in various cases. However in all these cases, the methods employed there throw no light on the corresponding Quillen question. Therefore this question still merits interest. This motivated us (I and R. A. Rao) to investigate the Quillen question at least in those cases where Bass-Quillen Conjecture was known. We find that in some cases the Quillen question is more difficult than the corresponding Bass-Quillen Conjecture and needs subtle arguments. In some other cases, which I shall be describing in this paper, the Quillen question has provided alternative and neater arguments to settle the corresponding Bass-Quillen Conjecture.

When S is a local ring of an affine algebra at a regular prime ideal, the Bass-Quillen Conjecture has been proved by H. Lindel ([6, Theorem]). With Rao, I have settled the corresponding Quillen question. My joint work with Rao had two, perhaps surprising, outcomes. Firstly the special case of the Quillen question - the $S(T)$ case-which implies the Bass-Quillen Conjecture, is actually equivalent to it [3, Theorem 2.4]. The other novelty observed was that the Quillen question in geometric situations (like in Lindel's Theorem) was, up to an etale extension, like $S(T)$-case! This was shown using a refined "Lindel subring" approximation of a geometric local ring.

109 Even prior to Quillen's question, Horrocks has proved that when $\dim R = 3$ and R contains a coefficient field, then every projective R-module of rank 2 is free ([5, Theorem 2]). In this article, I generalize this result by showing that every projective R_f-module of rank $d - 1$ ($d = \dim R$) is free when R contains a coefficient field (see Theorem 3.5). Moreover when $\dim R = 3$, I prove the result of Horrocks under the weaker assumption that R is unramified (see Theorem 3.7). The existence of a coefficient field enables me to approximate (in a broader sense) the ring by a good geometric ring. Using the rank condition, I reduce the problem to one over such an "approxiamte" ring.

During this colloquium, Professor M. Ojanguren brought to my notice a beautiful result of O. Gabber in [4], where he answers the Quillen question affirmatively when $\dim R = 3$.

Finally, let me summarise known results regarding the Bass-Quillen Conjecture and the Quillen question in the following table:

References	**Bass-Quillen Conjecture** $p \in \mathbb{P}(R[T]$; $\dim R = d$	**Quillen question** $P \in \mathbb{P}(R_f)$, $\dim R = d + 1$	**References**
Horrocks [5]	$d = 2$, R contains a coefficient field	$d = 2$, R contains a coefficient field	Horrocks [5]
Murthy [10]	$d = 2$	$d = 2$, R unramified	Bhatwadekar (this paper)
		$d = 2$	Gabber [4]
Lindel-Lutkebohmert [8] Mohan Kumar [9]	R: power series ring over a field		Mohan Kumar [9]
Lindel [6]	R: local ring of an affine algebra at a regular prime ideal		Bhatwadekar-Rao [3]
Lindel [7]	R contains a coefficient field and rank $P = d$		Bhatwadekar (this paper)

This paper has been organized as follows. In § 2, we record some 110
definitions and results. In § 3, we prove our main theorems (Theorem 3.5 and Theorem 3.7). In § 4, we prove a result about projective modules over a Laurent polynomial extension of $S(T)$ when $\dim S = 2$. (Theorem 4.2).

My warmest thanks are due to Ravi Rao, whose intuition and enthusiasm led me to consider Theorem 3.5 of this paper. I also thank Amit Roy for encouragement.

2 Preliminaries

Throughout this paper, all rings will be commutative noetherian and all modules are finitely generated. Further all *projective* modules will be assumed to be of constant rank.

In this section, we collect some definitions and results for later use. A will denote a commutative noetherian ring.

2.1

Given an ideal $\mathfrak{a}$ of A, $ht(\mathfrak{a})$ will denote the height of $\mathfrak{a}$ and $\mu(\mathfrak{a})$ will denote the minimal number of generators of $\mathfrak{a}$.

2.2

Given a projective A-module P and an element p of P we define $O_P(p) = \{\beta(p) | \beta \in \mathrm{Hom}_A(P, A)\}$. We say that p is *uni-modular* if $O_P(p) = A$.

2.3

Given a free A-module A^d we denote an element of A^d by a row vector $[a_1, \ldots, a_d]$. We identify $\mathrm{Aut}_A(A^d)$ with $GL_d(A)$; the group of $d \times d$ invertible matrices over A.

2.4

111 projective A-module P is said to be *A-cancellative* if $P \oplus A^d \simeq Q \oplus A^d$ implies $P \simeq Q$.

2.5

Given a set $\mathfrak{P}$ of prime ideals of A and a function $\delta : \mathfrak{P} \to \mathbb{N} \cup \{0\}$ we define a partial order $\ll$ on $\mathfrak{P}$ by setting $\mathfrak{p} \ll \mathfrak{a}$ if $\mathfrak{p} \subset \mathfrak{a}$ and $\delta(\mathfrak{p}) > \delta(\mathfrak{a})$.

The function δ is called a *generalised dimension function* on $\mathfrak{P}$ if for any ideal $\mathfrak{a}$ of A, $V(\mathfrak{a}) \cap \mathfrak{P}$ has only a finite number of minimal elements with respect to $\ll$.

2.6

Next we quote a theorem from [11]:

Eisenbud-Evans Theorem: Let P be a projective A-module, let $\mathfrak{P}$ be a subset of Spec(A) *and δ a generalized dimension function on $\mathfrak{P}$. Assume rank $P \geq \delta + 1$. Let $(p, a) \in P \oplus A$ be uni-modular at all $\mathfrak{p} \in \mathfrak{P}$. Then there exists an element $q \in P$ such that $p + aq$ is unimodular at all $\mathfrak{p} \in \mathfrak{P}$.*

2.7

Let A be a PID. We say that A is a special PID if $SL_r(A) = E_r(A)$ for all $r \geq 2$.

2.8

Let $A[Y]$ be a polynomial algebra in one variable over A. Then $A(Y)$ will denote the ring obtained from $A[Y]$ by inverting all monic polynomials in Y.

2.9

Quillen-Suslin Theorem ([12], Theorem 3 and and [14], Theorem 1).

Let P be a projective $A[Y]$-module. If $A(Y) \bigotimes_{A[Y]} P$ is free, then P is free. **112**

3 Main Theorem

In this section, we prove the main theorem (Theorem 3.5). For the proof of this theorem we need lemmas and propositions.

Proposition 3.1. *Let A be a ring and P be a projective A-module of rank d. Let T be a multiplicatively closed subset of A and let (p', a') be a unimodular element of $P_T \oplus A_T$. Then there exists $\sigma \in \mathrm{Aut}_{A_T}(P_T \oplus A_T)$ such that*

(1) $\sigma(p', a') = (p, a) \in P \oplus A$

(2) $ht\ O_{P \oplus A}(p, a) \geq d + 1$

Proof. Without loss of generality, we can assume that $T = \{1, t, t^2, \ldots\}$ for some $t \in T$.

Let $I = \{\sigma/\sigma \in \mathrm{Aut}_{A_t}(P_t \oplus A_t); \sigma(p', a') \in P \oplus A\}$. It is obvious that $I \neq \phi$. For $\sigma \in I$, if $\sigma(p', a') = (\widetilde{p}, \widetilde{a}) \in P \oplus A$ then let $N(\sigma)$ denote $ht\ O_{P\oplus A}(\widetilde{p}, \widetilde{a})$. Then it is enough to prove that there exists $\sigma \in I$ such that $N(\sigma) \geq d + 1$. This is proved by showing that for any $\sigma \in I$ with $N(\sigma) \leq d$, there exists $\sigma_1 \in I$ such that $N(\sigma_1) > N(\sigma)$.

Let $\sigma \in I$ be such that $N(\sigma) \leq d$. Let $\sigma(p', a') = (p, a) \in P \oplus A$.
113 Let $\mathfrak{P}$ be a set of prime ideals of A of height $\leq d - 1$ and let $\mathfrak{P}_1 = \mathfrak{P} \cap D(a)$. Then there is a generalized dimension function $\delta : \mathfrak{P}_1 \to \mathbb{N} \cup \{0\}$ such that $\delta \leq d - 1$. Moreover since $\mathfrak{P}_1$ is a subset of $D(a)$, (p, a) is unimodular at all prime ideals belonging to $\mathfrak{P}_1$. Further rank $P = d$. So, by (2.6), there exists a $q \in P$ such that $p + aq$ is unimodular at all prime ideals belonging to $\mathfrak{P}_1$. Since (p, a) can be mapped in to $(p + aq, a)$ by an A-automorphism τ of $P \oplus A$ we have $O_{P\oplus A}(p, a) = O_{P\oplus A}(p + aq, a)$. Therefore $\tau_t\sigma \in I$ and $N(\sigma) = N(\tau_t, \sigma)$. Hence, if necessary, replacing σ by $\tau_t\sigma$ and (p, a) by $(p + aq, a)$, we can assume that p is unimodular at all prime ideals belonging to $\mathfrak{P}_1$. This, in particular, implies that if a prime ideal $\mathfrak{p}$ of A contains $O_P(p)$ but does not contain the element a then $ht\ \mathfrak{p} \geq d$. Therefore, since $d \geq N(\sigma) \geq ht\ O_P(p)$ we have $N(\sigma) = ht\ O_P(p)$.

Let J denote the set of minimal prime ideals of $O_P(p)$. Let J_1 be a subset of J consisting of those members $\mathfrak{p}$ of J which contain the element a and let $J_2 = J - J_1$. Since $N(\sigma) = ht\ O_P(p)$, J_1 is not empty. Moreover since $(p, a)(= \sigma(p', a'))$ is a unimodular element of $P_t \oplus A_t$, we have $t^k \in O_{P\oplus A}(p, a)$ for some positive integer k. Therefore $t \in \mathfrak{p}$ for every $\mathfrak{p} \in J_1$. Since rank $\mathfrak{p} = d$, $\mathfrak{p} \in J$ implies that $ht\ \mathfrak{p} \leq d$ and therefore for every $\mathfrak{p} \in J_2$ we have $ht\ \mathfrak{p} = d$. Hence $\bigcap\limits_{\mathfrak{p}\in J_2} \mathfrak{p} \not\subset \bigcup\limits_{\mathfrak{p}\in J_1} \mathfrak{p}$. Let x be an element of $\bigcap\limits_{\mathfrak{p}\in J_2} \mathfrak{p}$ such that $x \notin \bigcup\limits_{\mathfrak{p}|\in J} \mathfrak{p}$. Since $t \in \mathfrak{p}$ for every $\mathfrak{p}$ in J_1 we have $(tx)^r \in O_P(p)$ for some positive integer r.

Let $\beta : P \to A$ be an A-linear map such that $\beta(p) = (tx)^r$. Let $\widetilde{\theta}$ be
114 an A-automorphism of $P \oplus A$ defined as: $\widetilde{\theta}(q, b) = (q, b + \beta(q))$ and let θ_1 be an A_t-automorphism of $P_t \oplus A_t$ defined as: $\theta_1(q', a') = (q', t'a')$. Let σ_1 denote $\theta_1^{-1}\widetilde{\theta}_t\theta_1\sigma$, then $\sigma_1(p', a') = \theta_1^{-1}\widetilde{\theta}_t\theta_1(p, a) = (p, a + x^r)$.

Therefore, $\sigma_1 \in I$ and

$$N(\sigma_1) = ht\ O_{P \oplus A}(p, a + x^r) > ht\ O_P(p) = N(\sigma).$$

This completes the proof of Proposition 3.1. □

As a consequence of Proposition 3.1, we get the following result.

Corollary 3.2. *Let A be a ring of dimension d and let P be a projective A-module of rank d. Let T be a multiplicatively closed subset of A. If P is A-cancellative then P_T is A_T-cancellative.*

Proof. As before, without loss of generality, we can assume that $T = \{1, t, t^2, \ldots\}$ for some $t \in T$.

Let $(p', a') \in P_t \oplus A_t$ be a unimodular element. We want to show that there exists $\theta \in \text{Aut}_{A_t}(P_t \oplus A_t)$ such that $\theta(p', a') = (0, 1)$.

By Proposition 3.1, there exists $\sigma \in \text{Aut}_{A_t}(P_t \oplus A_t)$ such that

(1) $\sigma(p', a') = (p, a) \in P \oplus A$

(2) $ht\ O_{P \oplus A}(p, a) \geq d + 1$.

Since $\dim A = d$, this implies that (p, a) is a unimodular element of $P \oplus A$. Since P is A-cancellative, there exists $\tau \in \text{Aut}_A(P \oplus A)$ such that $\tau(p, a) = (0, 1)$. Let $\theta = \tau_t \sigma$. Then $\theta \in \text{Aut}_{A_t}(P_t \oplus A_t)$ and $\theta(p', a') = (0, 1)$. 115

As an application of Proposition 3.1 we prove the following result which is a variant of result of Lindel ([7, 2.8 Satz]). □

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and let $(R', \mathfrak{m}')$ be a local subring of $(R, \mathfrak{m})$ such that $\widehat{R'} = \widehat{R}$. Let f be an element of $\mathfrak{m}'$ and let Q be a stably free R_f-module of rank $d - 1$. Then there exists a stably free R'_f-module Q' such that $Q \simeq R_f \otimes_{R'_f} Q'$.*

Proof. Since Q is stably free of rank $d - 1 \geq \dim R_{f'}$ there exists an R_f-isomorphism $\Psi : Q \oplus R_f \to R_f^d (= (R^{d-1})_f \oplus R_f)$. Let $\Psi(0, 1) = [a_1, \ldots, a_d]$. Then, by Proposition 3.1

$$\text{(taking } P = R^{d-1},\ p' = [a_1, \ldots, a_{d-1}],\ a' = a_d),$$

there exists $\sigma \in GL_d(R_f)$ such that

(1) $[a_1, \dots, a_d]\sigma = [b_1, \dots, b_d] \in R^d$

(2) $ht\ O_{R^d}[b_1, \dots, b_d] \geq d.$

If $ht\ O_{R^d}[b_1, \dots, b_d] > d$, then $[b_1, \dots, b_d]$ is a unimodular element of R^d. Since R is local, there exists $\tau \in GL_d(R)$ such that $[b_1, \dots, b_d]\tau = [0, \dots, 1]$. This shows that $Q \simeq R_f^{d-1}$. Then taking $Q' = R'^{d-1}_f$ we are through. So we assume $ht\ O_{R^d}[b_1, \dots, b_d] = d$.

116 Let $\mathfrak{a}$ be an ideal of R generated by $b_1, \dots, b_d$. Then since $\mathfrak{a} = O_R d[b_1, \dots, b_d]$, we have $ht\ \mathfrak{a} = d$. Therefore $\mathfrak{a}$ is $\mathfrak{m}$-primary ideal of R and hence $\mathfrak{m}^\ell \subset \mathfrak{a}$ for some positive integer l. Moreover $d = \mu(\mathfrak{a}) = ht(\mathfrak{a})$.

Since $\widehat{R'} = \widehat{R}$ we have $\mathfrak{m}'^\ell R = \mathfrak{m}^\ell$ and $R'/\mathfrak{m}'^\ell \overset{\approx}{\hookrightarrow} R/\mathfrak{m}^\ell$. So if $\mathfrak{b} = \mathfrak{a} \cap R'$ then (1) $\mathfrak{b}R = \mathfrak{a}$ and (2) $d = \mu(\mathfrak{b}) = ht(\mathfrak{b})$.

Let $c_1, \dots, c_d$ be elements of $\mathfrak{b}$ which generate $\mathfrak{b}$. Then as elements of R, $c_1, \dots, c_d$ generate $\mathfrak{a}$ also. Since R is local and $\mu(\mathfrak{a}) = d$, there exists $\theta \in GL_d(R)$ such that $[b_1, \dots, b_d]\theta = [c_1, \dots, c_d]$.

Consider the following short exact sequence

$$\begin{gathered} 0 \to R'_f \to R'^d_f \to Q' \to 0 \\ 1 \to [c_1, \dots, c_d] \end{gathered}$$

Since $f^\ell \in \mathfrak{a} \cap R' = \mathfrak{b}$, $[c_1, \dots, c_d]$ is a unimodular element of R'^d_f. Therefore Q' is stably free R'_f-module. Obviously $Q \simeq R_F \otimes_{R'_f} Q'$.

Let $(R, \mathfrak{m})$ be a regular local ring of dimension d and let f be a regular parameter of R. If R contains a local ring $(R'\mathfrak{m}')$ such that (1) $f \in R'$ (2) $\widehat{R'} = \widehat{R}$, then Proposition 3.3, shows that to study projective R_f-modules of rank $d-1$ it is enough to consider projective R'_f-modules
117 of rank $d-1$. The following technical lemma gives a sufficient condition for R to contain a local ring R' such that (1) $f \in R'$, (2) $\widehat{R'} = \widehat{R}$ and (3) every projective R'_f-module of rank $d - 1$ is free. □

Lemma 3.4. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d and let f be a regular parameter of R. Suppose R contains a local ring $(S, \mathfrak{n})$ of dimension d_o such that*

(i) *$S \hookrightarrow R$ is a faithfully flat extension*

(ii) *$R/\mathfrak{n}|R$ is a regular local ring with $\overline{f}$ (= image of f in $R/\mathfrak{n}R$) as its one of regular parameters.*

(iii) $S/\mathfrak{n}\overset{\approx}{\hookrightarrow}R/\mathfrak{m}$

(iv) $d_o \leq d - 2$.

Then R contains a regular local ring $(R', \mathfrak{m})$ such that

(1) *f is a regular parameter of R'*

(2) $\widehat{R'} = \widehat{R}$

(3) *Every projective R'_f-module of rank $d - 1$ is free.*

Proof. Since $S \hookrightarrow R$ is a faithfully flat extension and R is regular, so also S. Let $d - d_o = k \geq 2$. Let $f = f_1, f_2, \ldots, f_k$ be elements of R such that $\{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_k\}$ is a regular system of parameters of $R/\mathfrak{n}\, R$(note that $\dim R/\mathfrak{n}R = k$).

Let $B = S[f_1, \ldots, f_k]$ and $\mathfrak{p} = \mathfrak{n}B + (f_1, \ldots, f_k)B$. Then it is easy 118 to see that B is a *polynomial algebra* in k variables over S and $\mathfrak{p}$ is a maximal ideal of B. Let $R' = B_\mathfrak{p}$, $\mathfrak{m}' = \mathfrak{p}B_\mathfrak{p}$. Then obviously R contains R' and $\widehat{R'} = \widehat{R}$. Moreover, $f = f_1$ is a regular parameter of R'.

Let $A = S[f]$, $\mathfrak{p}_o = \mathfrak{n}A + fA$ and $\widetilde{R} = A_{\mathfrak{p}_o}$. Then $\widetilde{R}_f$ is a regular ring of dimension d_o and $\widetilde{R}_f[f_2, \ldots, f_k]$ is a *polynomial algebra* in $k - 1$ variables over $\widetilde{R}_f$. Moreover, R'_f is a localization of $\widetilde{R}_f[f_2, \ldots, f_k]$.

Since $K_0(\widetilde{R}_f[f_2, \ldots, f_k]) = K_0(\widetilde{R}_f) = \mathbb{Z}$ and $\dim \widetilde{R}_f = d_o < d - 1$ (by [15, Theorem 1.1]) every projective $\widetilde{R}_f[f_2, \ldots, f_k]$-module of rank $d - 1$ is free. Now we are through in view of Corollary 3.2, if we note that $K_0(R'_f) = \mathbb{Z}$ and $d - 1 = \dim \widetilde{R}_f[f_2, \ldots, f_k]$. □

Now we prove the main theorem which is a generalization of Theorem 2 of Horrocks ([5]).

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d and let f be a regular parameter of R. If R contains a field L such that $L \hookrightarrow R/\mathfrak{m}$*

is a finite separable extension then every projective R_f-module of rank $\geq d - 1$ is free.

Proof. Let Q be a projective R_f-module of rank $\geq d-1$. Since $K_0(R_f) = K_0(R) = \mathbb{Z}$, Q is stably free. If, rank $Q > d - 1 = \dim R_f$ then by ([1,
119 Corollary 3.5, p. 184]) Q is free. So we assume that rank $Q = d - 1$. Since stably free modules of rank one are always free we assume that $d - 1 \geq 2$ i.e. $d \geq 3$.

Since $L \hookrightarrow R/\mathfrak{m}$ is a finite separable extension, $R/\mathfrak{m}$ is a simple extension of L say $L[\alpha]$. Let $\varphi(Y)$ be a minimal polynomial of α over L. Let $a \in R$ be a lift of α. Then α is separable over L and φ is its minimal polynomial implies $\varphi(a) \in \mathfrak{m}$ but $\dfrac{\partial\varphi}{\partial Y}(a) \notin \mathfrak{m}$.

If $\varphi(a)$ and f are linearly dependent modulo $\mathfrak{m}^2$ then taking $x \in \mathfrak{m}$ such that f and x are linearly independent modulo $\mathfrak{m}^2$, we see that $\varphi(a+x)(= \varphi(a) + \dfrac{\partial\psi}{\partial Y}(a)x + hx^2)$ and f are linearly independent modulo $\mathfrak{m}^2$. Therefore replacing a by $a + x$ (if necessary) we can assume that $\varphi(a)$ and f are linearly independent modulo $\mathfrak{m}^2$.

Let $B = L[a]$ and $\mathfrak{p} = \varphi(a)B$. Then since $\varphi(a)$ is a regular parameter of R, B is a polynomial algebra in one variable over L and $\mathfrak{p}$ is a maximal ideal of B. Let $S = B_{\mathfrak{p}}$, $\mathfrak{n} = \mathfrak{p}B_{\mathfrak{p}}$. Then $(S, \mathfrak{n})$, is a one dimensional regular local ring contained in R. Moreover $S/\mathfrak{n} = L[\alpha] = R/\mathfrak{m}$. It is easy to see that $(S, \mathfrak{n})$ also satisfies the conditions (i) and (ii) of Lemma 3.4.

Since Q is stably free we apply Lemma 3.4 and Proposition 3.3 to conclude that Q is free. □

Corollary 3.6. *Let $(S, \mathfrak{n})$ be a regular local ring of dimension d. Let
120 $S[Y]$ be a polynomial algebra in one variable over S. If S contains a field L such that $L \hookrightarrow S/\mathfrak{n}$ is a finite seperable extension then every projective $S(Y)[X_1, \ldots, X_r]$-module of rank d is free.*

Proof. Let $R = S[Y^{-1}]_{(n,Y^{-1})}$ and $Y^{-1} = f$. Then $R_f = S(Y)$. Therefore by Theorem 3.5 every projective $S(Y)$-module of rank d is free. Hence it is enough to prove that every projective $S(Y)[X_1, \ldots, X_r]$-module of rank d is extended from $S(Y)$. This can be proved using the same arguments as in the proof of 4.1 Satz of Lindel ([7]). $\square$

Now we conclude this section with the following theorem.

Theorem 3.7. *Let R be a regular local ring of dimension 3 and let f be a regular parameter of R. Assume that R is unramified. Then every projective R_f-module is free.*

Proof. Let $\{f = f_1, f_2, f_3\}$ be a regular system of parameters of R. Let $R^0 = R$ and let R^i denote the f_i-adic completion of R^{i-1} for $1 \leq i \leq 3$. Then $R^{i-1} \hookrightarrow R^i$, f_i is a non-zero-divisor of R^i and $R^{i-1}/(f_i) \xrightarrow{\approx} R^i/(f_i)$. Hence by ([13, § 2]) we get the following *cartesian square* (see [1, p. 359])

$$\begin{tikzcd}
\mathbb{P}(R^{i-1}) \arrow[r] \arrow[d] & \mathbb{P}(R^i) \arrow[d] \\
\mathbb{P}(R^{i-1}_{f_i}) \arrow[r] & \mathbb{P}(R^i_{f_i})
\end{tikzcd}$$

where, for a ring A, $\mathbb{P}(A)$ denote the category of all finitely generated projective A-modules.

Therefore, as R^{i-1} is local for $1 \leq i \leq 3$. every projective $R^{i-1}_{f_i}$- **121**
module is free if every projective $R^i_{f_i}$-module is free. Moreover by ([10, Proposition 2′]) every projective $R^i_{f_i}$-module is free if every projective $R^i_{f_{i+1}}$-module is free.

The above discussion shows that to prove the theorem it is enough to prove that every projective $R^3_{f_3}$-module is free. But R^3 is a complete, regular local, unramified ring of dimension 3. Thereofre $R^3 = S[[X, Y]]$ where S is a regular local ring of dimension 1. Again by ([10, Proposition 2′]), we can assume that $f_3 = Y$. Now we are through in view of Lemma 3.4 and Proposition 3.3. $\square$

4 Projective modules over $R_f[X_1, \dots, X_r, Z_1^{\pm 1} \dots, Z_k^{\pm 1}]$

Let R be a regular local ring and let f be a regular parameter of R. We want to study projective modules over $R_f[X_1, \dots, X_r, Z_1^{\pm 1}, \dots, Z_k^{\pm 1}]$. When R is a power series over a field or a geometric local ring over an infinite field then all projective modules over $R_f[X_1, \dots, X_r, Z_1^{\pm 1}, \dots, Z_k^{\pm 1}]$ are free (see [3, Theorem 3.1 and Theorem 2.2]). These results and Corollary 3.6 lead us to ask the following question:
Question. Let R be regular local ring of dimension d satisfying the hypothesis of Theorem 3.5. Let f be a regular parameter of R. Are all projective $R_f[X_1, \dots, X_r, Z_1^{\pm 1}, \dots, Z_k^{\pm 1}]$*-modules of rank* $d - 1$ *free?*

Remark 4.1. Swan has shown that all projective

$$R_f[X_1, \dots, X_r, Z_1^{\pm 1}, \dots, Z_k^{\pm 1}]\text{-modules}$$

122 of rank $> \dim R_f$ are free ([15, Theorem 1.1.]).

In this section, we show that when $d = 3$ and R_f is of special type, then the above question has an affirmative answer. More precisely.

Theorem 4.2. *Let* S *be a regular local ring of dimension* 2*. Let* $S[Y]$ *be a polynomial algebra in one variable over* S*. Let P be a projective* $S(Y)[X_1, \dots, X_r, Z_1^{\pm 1}, \dots, Z_k^{\pm 1}]$*-module. Then P is free.*

For the proof of this theorem we need the following proposition.

Proposition 4.3. *Let A be a UFD and let* π *be an element of A such that* $A/(\pi)$ *is a special PID. Let P be a projective* $A[X_1, \dots, X_r, Z_1^{\pm 1}, \dots, Z_k^{\pm 1}]$*-module such that* P_π *is free. Then P is free.*

Proof. We prove the result by induction on $r + k$, the case $r + k = 0$ being a result of Bass and Murthy ([2, Proposition 9.6]). Let B denote the ring $A[X_1, \dots, X_r, Z_1^{\pm 1}, \dots, Z_k^{\pm 1}]$.

Case (1) : $r > 0$

Let T denote the multiplicatively closed subset of B consisting of
123 monic polynomials in X_1 with coefficients in A. Then

$$B_T = A(X_1)[X_2, \ldots, X_r, Z_1^{\pm 1}, \ldots, Z_k^{\pm 1}].$$

Moreover by ([10, Proposition 1′]) $A(X_1)/(\pi) = A/(\pi)(X_1)$ is a special PID. Obviously P_{T_π} is free. Therefore by induction P_T is free and hence by (2.9) P is free.

Case (2) : $r = 0$.

Let T' denote the multiplicatively closed subset of B consisting of monic polynomials in Z_1 with coefficients in A. Then

$$B_{T'} = A(Z_1)[Z_2^{\pm 1}, \ldots, Z_k^{\pm 1}]$$

and as before we conclude that P_T is free. Therefore by ([15, Lemma 1.3]) there exists a projective $A[Z_1^{-1}, Z_2^{\pm 1}, \ldots, Z_k^{\pm 1}]$-module Q such that $P \simeq B \otimes Q$. Since $P_\pi \simeq B_\pi \otimes Q_\pi$ is free, by (2.9) Q_π is free. Therefore by Case (1) Q is free and hence P is free. □

Proof of Theorem 4.2. Let π be a regular parameter of S. Then $S/(\pi)$ is a discrete valuation ring and therefore a special PID. Hence by ([**?**, Proposition 1′]) $S(Y)/(\pi) = S/(\pi)(Y)$ is a special PID.

Let L denote the quotient field of $S[Y]$. Then $S_\pi[Y] \hookrightarrow S(Y)_\pi \hookrightarrow L$. Moreover, $S(Y)_\pi$ is aUFD of dimension 2 (in fact $S(Y)$ is a UFD of dimension 2). Therefore by ([3, Proposition 2.1]) P_π is free. Hence by virtue of Proposition 4.3, P is free.

References

124

[1] Bass, H. - *Algebraic K-theory*, Benjamin, New York, 1968.

[2] Bass, H. and Murthy, M. P. - Grothendieck groups and Picard groups of abelian groups rings, *Ann. of Math.*, vol. 86 (1967), 16-73.

[3] Bhatwadekar, S. M. Rao, R. A. - On a question of Quillen, Trans. *Amer. Math. Soc.*, vol. 279 (1983), 801-810.

[4] Gabber, O. - Groupe de Brauer, *Séminaire, Les Plans-sur-Bex*, Lecture Notes in Mathematics 844, 129-209, Springer-Verlag (1980).

[5] Horrocks, G. - Projective modules over an extension of a local ring, *Proc. London Math. Soc.,* vol. 14 (1964), 714-718.

[6] Lindel, H. - On a question of Bass - Quillen and Suslin concerning projective modules over polynomial rings, *Invent. Math.,* vol. 65 (1981), 319-323.

[7] Lindel, H. - Erweiterungskriterien fur stabilfrei Moduln uber Polynomringen, *Math. Annalen,* vol. 250 (1980), 99-108.

[8] Lindel, H. and Lütkebohmert, W. - Projective Moduln über Polynomialen Erweiterungen von Potenzreihenalgebren, *Arch. Math.,* vol. 28 (1977), 51-54. **125**

[9] Mohan Kumar, N. - On a question of Bass-Quillen, *J. Indian Math. Soc.,* vol. 43 (1979), 13-18.

[10] Murthy, M. P. - Projective $A[X]$-modules, *J. London Math. Soc.,* vol. 41 (1966), 453-456.

[11] Plumstead, B. R. - The conjectures of Eisenbud and Evans, *Amer. J. of Math.,* vol. 105 (1983), 1417-1433.

[12] Quillen, D. - Projective modules over polynomial rings, *Invent. Math.,* vol. 36 (1976), 167-171.

[13] Roy, A. - Application of patching diagrams to some questions about projective modules *J. Pure and Applied Algebra,* vol. 24 (1982), 313-319.

[14] Suslin, A. A. - Projective modules over a polynomial ring are free, *Soviet Math. Doklady,* vol. 17 (1976), 1160-1164 (English translation).

[15] Swan, R. G. - Projective modules over Laurent polynomial rings, *Trans. Amer. Math. Soc.,* vol. 237 (1978), 111-120.

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005
India

A Problem of Zariski

By J.-L. Colliot-Thelene

IN 1949, ZARISKI raised the following question. Let X and Y be **127**
two algebraic varieties over a field k, let P_k^r be the projective r-space over
k. Assume that $X \times_k P_k^r$ is k-birational to $Y \times_k P_k^r$. Does it follow that
X is k-birational to Y? In particular, if a k-variety X is stably k-rational
(i.e. $X \times_k P_k^r$ is k-birational to P_k^{d+r}), is it k-rational (i.e. is k-birational
to P_k^d)?

Although a positive answer is known in special cases (Segre [13], Nagata [11]), the problem remained open for many years. A positive answer in the particular case would have been of interest for the rationality problem of some moduli spaces of stable vector bundles over curves ([12], [1]), as was pointed out to me by Seshadri at the Colloquium. Unfortunately, the answer to Zariski's problem is negative ([4]):

There exist stably rational surfaces over a suitable non-algebraically closed field k which are not k-rational, and there exist stably rational threefolds over the complex field C which are not rational.

Over a non-algebraically closed field k, with char. $k \neq 2$, which **128**
admits a field extension K/k, Galois with group $\mathfrak{S}_3$ (the symmetric group
on three letters), our examples are surfaces given by an affine equation:

$$y^2 - az^2 = P(x) \tag{1}$$

where $P(x)$ is a separable polynomial with coefficients in k, and $a \in k^*$, and moreover:

(2) P is irreducible of the third degree, and $a = \operatorname{disc}(P) \in k^*$ is not a square.

---

[1] Summary of results appearing in [4].

That such a surface is not k-rational is a special case of a result of Iskovskih ([9]). That it is stably k-rational appeared in the course of arithmetic investigations ([8]) on surfaces as in (1) with $P(x)$ of degree at most 4. As a rule, such surfaces are k-unirational as soon as they have a k-rational point, but they need not be stably k-rational (e.g. if $P(x)$ is a polynomial of the third degree which is split over k, and a is not a square in k; this particular case is the one originally considered by F. Châtelet ([5]).

Iskovskih's result uses the method of linear systems with base points, as was first done over a non-algebraically closed field by B. Segre ([14]). As for the methods of ([8]), they involve a close analysis of principal homogeneous spaces under tori over (smooth compactifications of) surfaces of type (1) – and more generally k-surfaces which become rational after a finite extension of the ground field ([7]).

129 Over the complex field C, examples of stably rational non-rational threefolds are given by affine equations:

$$y^2 - a(t)z^2 = P(x, t) \tag{3}$$

where $P(x, t)$ is an irreducible polynomial, of degree 3 in x, and where $a(t) = \mathrm{disc}_x P$ has no square factor and is of degree at least 5.

That such threefolds are stably rational follows immediately from the stable k-rationality of surfaces of type (1) (2) (take $k = C(t)$). That (3) is not rational uses intermediate jacobians ([6]) and Prym varieties, as was first done by Mumford ([10]). However, the discriminant locus of a conic bundle defined by (3) is a reducible singular curve, and the delicate analysis of [2] and [3] is required to show that (3) is not rational.

References

[1] E. Ballico : Stable rationality for the variety of vector bundles over an algebraic curve, *J. Lond. Math. Soc.*, II. Ser., 30 (1984), 21-26.

130 [2] A. Beauville: Prym varieties and the Schottky problem. *Invent. Math.*, 41 (1977), 149-196.

[3] A. Beauville: Variétés de Prym et jacobiennes intermédiaires. *Ann. Sci. Ecole Norm. Sup.,* 10 (1977), 309-391.

[4] A. Beauville, J.-L. Colliot-Thélène J.-J. Sansuc, Sir P. Swinnerton-Dyer: Variétés stablement rationnelles non rationnnelles, *Ann. of Math.* 121 (1985) 283-318.

[5] F. Châtelet: Points rationnels sur certaines courbes et surfaces cubiques, *Enseign. Math.,* 5 (1959), 153-170.

[6] H. Clemens, P. Griffiths: The intermediate jacobian of the cubic threefold, *Ann. of Math.,* 95 (1972), 281-356.

[7] J.-L. Colliot-Thélène, J.-J. Sansuc: La descente sur les variétés rationnelles, in *Journées de géométrie algébrique* (Angers 1979), A. Beauville ed., Sijthoff & Noordhoff, Alphen aan den Rijn. (1980), 223-237.

[8] J.-L. Colliot-Thélène, J.-J. Sansuc, Sir P. Swinnerton-Dyer: Intersections de deux quadriques et surfaces de Châtelen généralisées (preprint Orsay; 1986 *C. R. Acad. Sc. Paris,* t. 298, Série I, n° 16 (1984), 377-380).

[9] V. A. Iskovskih: Birational properties of a surface of degree 4 in P_k^4, *Mat. Sb.,* 88 (130) (1972), 31-37 (= Math. USSR Sb., 17 (1972), 30-36). **131**

[10] D. Mumford: Prym varieties *I*, in *Contributions to Analysis* (L. Ahlfors et al. ed.), Academic Press, New York (1974), 325-350.

[11] M. Nagata: A theorem on valuation rings and its applications, *Nagoya Math. J,* 29 (1967) 85-91.

[12] P. E. Newstead : Rationality of moduli spaces of stable bundles, *Math. Ann.,* 215 (1975) 251-268; ibid., 249 (1980) 281-282.

[13] B. Segre: Sur un problème de M. Zariski, *Colloque international d'algèbre et de théorie des nombers* (Paris 1949), C.N.R.S. Paris (1950) 135-138.

[14] B. Segre: On the rational solutions of homogeneous cubic equations in four variables, *Math. Notae Univ. Rosario Argentina* 11 (1951) 1-68.

C.N.R.S. Mathematiques,
Bât. 425, Université de Paris-Sud,
F-91405 Orsay

Conditions D'Existence des Fibres Stables de Rang Eleve Sur $\mathbb{P}_2$

By J.M. Drezet and J. Le Potier

EN DEHORS DU paragraphe 9, le travail qui suit est un resume **133** de notre article a paraitre aux Annales de 1'E.N.S [4], auquel le lecteur devra se reporter pour des démonstrations complètes.

Let corps de base est $\mathbb{C}$.

1 Introduction

Soient X use surface projective lisse, $A(X)$ 'anneau de Chow de X. On sait qu'etant donnes un entier $r \geqq 2$, des classes $c_i \in A^i(X)$ il existe des fibres vectoriels algebriques sur X, de rang r, de classes de Chern c_1 et c_2 c'est un resultat qui remonte a Schwarzenberger [13].

Par contre, pour obtenir l'existence de fibrés stables (relativement à **134** une polarisation donnée sur X) de rang r, de classes de Chern c_1 et c_2 on doit imposer des conditions à r, c_1 et c_2: on connaît par exemple le résultat de Bogomolov selon lequel on doit avoir nécessairement

$$c_2 \geqq (r-1)c_1^2/2r.$$

Cette condition n'est pas suffisante pour assurer l'existence de fibres stables. Pour le plan projectif $\mathbb{P}_2$, les conditions necessaires et suffisantes d'existence s'ecrivent en rang $r = 2$:

$$c_2 - c_1^2/4 \geqq \begin{cases} 2 \text{ si } c_1 \text{ est pair} \\ 3/4 \text{ si } c_1 \text{ est impair} \end{cases}$$

---

[1] Presenté par J. Le Potier

Ce résultat est dû lui aussi à Schwarzenberger [4]; on se propose ici de l'étendre en rang quelconque.

2 Formulaire

Sur le plan projectif $\mathbb{P}_2$, on a $A^1(\mathbb{P}_2) = Z$, $A^2(\mathbb{P}_2) = \mathbb{Z}$; si E est un $\mathscr{O}_{\mathbb{P}_2}$ - module coherent, les classes de chern de E seront considerées comme des nombres. Si E est de rang > 0, on definit la pente μ et le discriminant Δ de E par les formules

$$\begin{aligned}\mu &= c_1/r\\ \Delta &= (1/r)(c_2 - (r-1)c_1^2/(2r)).\end{aligned}$$

135 Avec ces conventions, la formule de Riemann-Roch s'écrit

$$\begin{aligned}\chi(E) &= \sum(-1)^i h^i(E)\\ &= r(P(\mu) - \Delta) \qquad (2.1)\end{aligned}$$

où P est le polynôme $P(X) = 1 + 3x/2 + x^2/2$.

Plus généralement, si E et E' sont deux $\mathscr{O}_{\mathbb{P}_2}$-modules cohérents de rangs r et r', de pentes μ, de discriminants Δ et Δ' respectivement, on pose

$$\chi(E, E') = \sum_i (-1)^i \dim \operatorname{Ext}^i(E, E')$$

et la formule de Riemann-Roch s'étend sous la forme suivante:

$$\chi(E, E') = rr'(P(\mu' - \mu) - \Delta - \Delta') \qquad (2.2)$$

En particulier, $\chi(E, E) = r^2(1 - 2\Delta)$.

DUALITE DE SERRE: Soient E et E' des faisceaux algébriques cohérents sur $\mathbb{P}_2$; soit dáutre part $K \cong \mathscr{O}_{\mathbb{P}_2}(-3)$ le fibré canonique sur $\mathbb{P}_2$. Il existe un accouplement

$$\operatorname{Ext}^i(E, E') \times \operatorname{Ext}^{2-i}(E', E \otimes K) \to \mathbb{C}$$

136 qui fait d'un de ces espaces vectoriels le dual de l'autre.

3 Faisceaux semi-stables

La notion de semi-stabilité dont nous aurons besoin est celle de Gieseker [7] et Maruyama [12].

Definition 1. *Un faisceau algebrique coherent E sur $\mathbb{P}_2$ est dit* semi-stable *(resp.* stable*) si E est sans torsion, et si pour tout sous-module $0 \neq E' \subset E$, on a*

$$\mu(E') \leqq \mu(E)$$

et en cas d'égalité $\Delta(E') \geqq \Delta(E)$ (resp. $\Delta(E') > \Delta(E)$).

Soit S une variete algebrique. Considerons le foncteur $\underline{M}(r, \mu, \Delta)$ qui à S associe l'ensemble des classes d'isomorphisme de faisceaux algébriques cohérents E sur $S \times \mathbb{P}_2$, S-plats et tels que pour tout $s \in S$, le faisceau $E(s)$ induit sur $\mathbb{P}_2$ soit semi-stable de rang r, de pente μ, de discriminant Δ. Gieseker [7] et Maruyama [12] ont construit une variété algébrique $M(r, \mu, \Delta)$ et un morphisme fonctoriel

$$\underline{M}(r, \mu, \Delta) \to \mathrm{Mor}(\quad, M(r, \mu, \Delta))$$

qui fait de $M(r, \mu, \Delta)$ un espace de modules grossier, c'est-à-dire qu'il satisfait à la propriété suivante : pour toute variété algébrique M, et tout morphisme fonctoriel

$$M(r, \mu, \Delta) \to \mathrm{Mor}(\quad, M')$$

il existe un et un seul morphisme de variétés $M(r, \mu, \Delta) \to M'$ rendant le diagramme suivant commutatif 137

$$\begin{tikzcd}
 & \mathrm{Mor}(\quad, M(r, \mu, \Delta)) \arrow[dd] \\
M(r, \mu, \Delta) \arrow[ur] \arrow[dr] & \\
 & \mathrm{Mor}(\quad, M')
\end{tikzcd}$$

Ceci caractérise le variété algébrique $M(r, \mu, \Delta)$. Dans $M(r, \mu, \Delta)$ les points qui proviennent de faisceaux stables forment un ouvert lisse M_s;

cet ouvert s'identifie à l'ensemble des classes d'isomorphisme de faisceaux stables; il est de dimension

$$\dim M_s = r^2(2\Delta - 1) + 1$$

FILTRATION DE HARDER-NARASIMHAN: Si E est un faisceau algébrique cohérent sans torsion sur $\mathbb{P}_2$, il a une filtration

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_\ell = E$$

par des sous-modules coherénts tels que

(1) le gradué $gr_i = F_i/F_{i-1}$ soit semi-stable

(2) $\mu(gr_i) \geqq \mu(gr_{i+1})$, et en cas d;égalité $\Delta(gr_i) < \Delta(gr_{i+1})$.

Une telle filtration est déterminée de manière unique par ces conditions; on l'appelle la filtration de Harder-Narasimhan [8].

4 Fibrés exceptionnels

138 **Proposition 1.** *Soit E un faisceau stable de rang r, de pente μ, de discriminant Δ sur $\mathbb{P}_2$. Les assertions suivantes sont equivalentes:*

(1) $\mathrm{Ext}^1(E,E) = 0$

(2) $\Delta < 1/2$

(3) $\chi(E,E) > 0$

En effet, par dualité de Serre, $\mathrm{Ext}^2(E,E)$ est le dual de

$$\mathrm{Hom}(E, E(-3))$$

et par suite est nul par définition de la stabilité. D'autre part, on a $\mathrm{Hom}(E,E) = \mathbb{C}$; compte-tenu de la formule de Riemann-Roch, on obtient

$$\chi(E,E) = r^2(1 - 2\Delta) = 1 - \dim \mathrm{Ext}^1(E,E)$$

ce qui donne l'équivalence voulue.

Definition 2. *Un faisceau algébrique coherent sur* $\mathbb{P}_2$ *est dit exceptionnel (resp.* semi-exceptionnel*) s'il est stable (resp. semi-stable) et si son discriminant C satisfait à la condition*

$$\Delta < 1/2.$$

Examples. Les fibres de rang un sont exceptionnels. Le fibre tangent $T(\mathbb{P}_2)$ est un fibre exceptionnel de pente $3/2$. Le fibre noyau du morphisme canonique d'évaluation

$$\mathrm{ev} : \mathbb{P}_2 \times \Gamma(\mathscr{O}(2)) \to \mathscr{O}(2)$$

est un fibré exceptionnel de rang 5, de pente - $2/5$. Si E est un fibré **139**
exceptionnel, il en est de même du dual, et des fibrés $E(i)$ pour $i \in \mathbb{Z}$.

En un point de l'espace de modules $M(r, \mu, \Delta)$ défini par un faisceau exceptionnel, on a dim $M(r, \mu, \Delta)$ defini par un faisceau exceptionnel, on a dim $M(r, \mu, \Delta) = 0$. Plus précisément, on a le résultat suivant:

Proposition 2. *Pour tout* $\alpha \in Q$*, il existe au plus, à isomorphisme près, un faisceau exceptionnel de pente* α*. Ce faisceau est en fait localement libre; son rang est le plus petit dénominateur* $r_\alpha > 0$ *de* α*, et son discriminant est donné par*

$$\Delta = 1/2(1 - 1/r_\alpha^2)$$

En effet, montrons d'abord que la pente α détermine le rang et le discriminant. On a déjà vu que si E est exceptionnel de rang r, de pente α et de discriminant Δ, on a

$$\chi(E, E) = 1 = r^2(1 - 2\Delta);$$

c'est-à-dire, en revenant aux classes de Chern c_1 *et* c_2

$$1 = r^2 - 2rc_2 + (r - 1)c_1^2.$$

Ceci entraîne que r et c_1 sont premiers entr'eux; ceci signifie que r est
le plus petit denominateur > 0 de α. Le rang r est ainsi déterminé; la **140**

même formule donne pour discriminant $\Delta = 1/2(1 - 1/r^2)$.

Montrons maintenant que l'ouvert $M_S \subset M(r, \mu, \Delta)$ est réduit à un point si $2\Delta < 1$. La méthode ci-dessous, qui évite le recours au résultat d'Ellingsrud [6] nous a été signalée par Mukaï. Soient E et E' deux faisceaux exceptionnels de meme pente μ, de meme rang r et de même discriminant Δ; d'après la formula de Riemann-Roch,

$$\chi(E, E') = r^2(1 - 2\Delta).$$

On a encore $\mathrm{Ext}^2(E, E') = 0$, car cet espace est le dual de

$$\mathrm{Hom}(E', E(-3)),$$

et donc est nul par stabilité. Il en résulte que $\mathrm{Hom}(E, E') \neq 0$; un homomorphisme $f : E \to E'$ est en fait un isomorphisme s'il est non nul, par stabilité. Par suite, E est isomorphe à E'.

Soit E un faisceau exceptionnel de pente α; pour $g \in \mathrm{Aut}(\mathbb{P}_2)$, le faisceau $g^*(E)$ est encore exceptionnel de pente α, et par suite isomorphe à E. Par suite, l'ensemble des points de $\mathbb{P}_2$ au voisinage desquels le faisceau E est localement libre est invariant par le groupe $\mathrm{Aut}(\mathbb{P}_2)$, et ne peut être que $\mathbb{P}_2$ lui-meme.

Proposition 3. *Soit E un faisceau semi-exceptionnel de pente α. Alors E est somme directe de fibrés exceptionnels de pente α.*

141 En effet, les fibrés semi-stables de pente α et de discriminant Δ forment une categorie abélienne, artinienne et noethérienne; par suite, un faisceau semi-stable de pente α, de discriminant Δ a une filtration de Jordan-Hölder

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = E$$

avec $gr_i = F_i/F_{i-1}$ stable de pente α, de discriminant Δ. Si $\Delta < 1/2$, les faisceaux gr_i sont exceptionnels de meme pente α, donc isomorphes; la filtration ci-desses est en fait scindée, d'où la proposition.

5 L'ensemble $\mathfrak{E}$

Soit α un nombre rationnel. On appelle rang de α le plus petit entier $r_\alpha > 0$ tel que $r_\alpha \alpha \in Z$, et discriminant de α le nombre rationnel

$$\Delta_\alpha = 1/2(1 - 1/r_\alpha^2).$$

Soit $\mathfrak{E}$ l'ensemble des nombres rationnels α qui sont pentes de fibrés exceptionnels; si $\alpha \in \mathfrak{E}$, et si E_α est un fibré exceptionnel de pente α, r_α est le rang de E_α et Δ_α son discriminant. La proposition suivante nous permettra de donner une construction de $\mathfrak{E}$.

Proposition 4. *Soit $\mu \in Q$ un rationnel de rang r et de discriminant Δ. Let assertions suivantes sont équivalents:*

(1) $\mu \in \mathfrak{E}$

(2) $r(P(\mu) - \Delta) \in \mathbb{Z}$ *et pour tout* $\alpha \in \mathfrak{E}$ *tel que* $0 < |\alpha - \mu| < 3$*, on a* **142**

$$P(-|\alpha - \mu|) \leq \Delta_\alpha + \Delta$$

(3) $r(P(\mu) - \Delta) \in \mathbb{Z}$ *et pour tout* $\alpha \in \mathfrak{E}$ *tel que* $r_\alpha < r$ *et* $|\alpha - \mu| \leq 1$, *on a*

$$P(-1|\alpha - \mu|) \leqq \Delta_\alpha + \Delta$$

En effet, si $\mu \in \mathfrak{E}$, c'est la pente d'un fibr'e exceptionnel E_μ, et $r(P(\mu) - \Delta)$ doit être la caractéristique d'Euler-Poincaré, donc un entier. L'implication (1) ⇒ (2) repose sur la formule de Riemann-Roch (2.2) : désignons par E_α un fibré exceptionnel de pente α; si $\alpha > \mu$, on a par stabilité

$$\mathrm{Hom}(E_\alpha, E_\mu) = 0$$

D'autre part, $\mathrm{Ext}^2(E_\alpha, E_\mu)$, dual de $\mathrm{Hom}(E_\mu, E_\alpha(-3))$, est nul par stabilité dès que $\alpha - \mu < 3$ par suite $\chi(E_\alpha, E_\mu) \leq 0$. Si $\alpha < \mu$, on voit de même que $\chi(E_\mu, E_\alpha) \leqq 0$ pourvu que $\mu - \alpha < 3$.

L'implication (2) ⇒ (3) est triviale; l'implication (3) ⇒ (1) résulte du théorème d'existence 3 (cf. § 7).

CONSTRUCTION DE $\mathfrak{E}$: Soient α et β deux nombres rationnels Si $3+\alpha-\beta \neq 0$, l'equation en t

$$P(\alpha - t) - \Delta_\alpha = P(t-\beta) - \Delta_\beta$$

a une solution unique, notée $t = \alpha \cdot \beta$, donnée par

$$\alpha \cdot \beta = \frac{\alpha+\beta}{2} + \frac{\Delta_\beta - \Delta_\alpha}{3+\alpha-\beta}$$

143 Soit $\mathfrak{D}$ l'ensemble des nombres rationnels de la forme $p/2^q$ où $p \in \mathbb{Z}$ et où q est un entier ≥ 0. Soit $\epsilon : \mathfrak{D} \to \mathbb{Q}$ l'application construite par récurrence sur q définie par

$$\begin{aligned} &\epsilon(n) = n \quad \text{si} \quad n \in \mathbb{Z} \\ &\epsilon(2p + 1/2^{q+1}) = \epsilon(p/2^q) \cdot \epsilon((p+1)/2^q). \end{aligned}$$

Proposition 5. (1) *L'application $\epsilon : \mathfrak{D} \to \mathbb{Q}$ est bien définie et strictement croissante.*

(2) *Le nombre $\epsilon(p/2^q)$ est de rang $\geqq 2^q$ si p est impair.*

(3) *Pour tout $\alpha \in \epsilon(\mathfrak{D})$, or a $r_\alpha(P(\alpha) \cdot \Delta_\alpha) \in \mathbb{Z}$*

(4) *Pour $\rho \in \mathfrak{D}$, et $n \in \mathbb{Z}$, $\epsilon(\rho + n) = \epsilon(\rho) + n$, et $\epsilon(-\rho) = -\epsilon(\rho)$.*

Il en résulte en particulier que la fonction $\epsilon : \mathfrak{D} \to \mathbb{Q}$ est parfaitement déterminée quand on connaît sa restriction à $\mathfrak{D} \cap [0, 1/2]$. Le calcul effectif donne par exemple:

p	0	1	2	3	4	5	6	7	8
$\epsilon(p/2^4)$	0	$\frac{13}{34}$	$\frac{5}{13}$	$\frac{75}{194}$	$\frac{2}{5}$	$\frac{179}{433}$	$\frac{12}{29}$	$\frac{70}{169}$	$\frac{1}{2}$

Theorem 1. *On a $\epsilon(\mathfrak{D}) = \mathfrak{E}$.*

144 Ceci détermine donc, à isomorphisme près, tous les fibrés exceptionnels. Indiquons le plan de la démonstration: soit $\mathfrak{D}_q$ l'ensemble des nombres de la forme $p/2^q$, où $p \in \mathbb{Z}$, et où q, est un entier tel que $0 \leq q' \leq q$. Pour montrer que $\epsilon(\mathfrak{D}) \subset \mathfrak{E}$, on verifie que $\epsilon(\mathfrak{D}_q) \subset \mathfrak{E}$ par récurrence sur q. Il suffit de demontrer le lemme suivant, qui repose sur la proposition 4:

Lemma 1. *Soient* $\alpha = \epsilon(p/2^q)$, $\beta = \gamma\epsilon((p+1)/2^q)$ *et* $\mu = \alpha \cdot \beta$. *On suppose que* $\alpha \in \mathfrak{E}$, *et* $\beta \in \mathfrak{E}$. *Alors*

(1) *Pour tout* $\alpha' \in \mathfrak{E}$, *tel que* $\alpha < \alpha' < \beta$, *on a* $r_{\alpha'} \geqq r_\mu$

(2) *On a* $\mu \in \mathfrak{E}$.

Inversement, soit $\mu \in \mathfrak{E}$; choisissons q assez grand pour que $2^q \geqq r_\mu$, et p de sorte que

$$\alpha = \epsilon(p/2^q) \leqq \mu < \epsilon(p+1)/2^q$$

Comme $\alpha \cdot \beta$ est de rang $\geqq 2^{q+1}$ d'après la proposition 5, on voit que la partie (1) du lemma ci-dessus impose $\mu = \epsilon(p/2^q)$. Par suite $\mu \in \epsilon(\mathfrak{D})$.

6 Le théorème d'existence

L'étude des faisceaux semi-stables de discriminant $\Delta < 1/2$ ayant deja
été faite, on se limite dans ce paragraphe au cas $\Delta < 1/2$. Remarquons
d'autre part que pour R réel donne, l'ensemble $\mathfrak{E}(R)$ des éléments de $\mathfrak{E}$ **145**
de rang inférieur ou egal à R est localement fini.

Theorem 2. *Soient* r *un entier* $\geqq 2$, c_1 *et* $c_2 \in \mathbb{Z}$; *on pose* $\mu = c_1/r$, $\Delta = (1/r)(c_2 - (r-1)c_1^2/(2r))$, *et on suppose* $\Delta \geqq 1/2$.

On designe par: $\alpha(r,\mu)$ *le plus grand des éléments* $\alpha \in \mathfrak{E}$ *tels que* $r_\alpha \leqq r/2$ *et* $\alpha \leqq \mu$, *et par* $\beta(r,\mu)$ *le plus petit des éléments* $\beta \in \mathfrak{E}$ *tels que* $r_\beta \leqq r/2$ *et* $\mu \leqq \beta$.

Let assertions suivantes sont équivalentes:

(1) *Il existe un fibré stable de rang* r, *de pente* μ, *de discriminant* Δ.

(2) *Il existe un faisceau semi-stable de rang r, de pente μ, de discriminant Δ.*

(3) *Pour tout $\alpha \in \mathfrak{E}$, tel que $\alpha - \mu < 3$, on a*

$$P(-\alpha - \mu|) \leqq \Delta_\alpha + \Delta$$

(4) $\Delta \geqq P(\alpha(r,\mu) - \mu) - \Delta_{\alpha(r,\mu)}$ *et* $\Delta \geqq P(\mu - \beta(r,\mu)) - \Delta_{\beta(r,\mu)}$

En effet les implications (1) ⇒ (2) et (3) ⇒ (4) sont triviales. La démonstration de l'implication (2) ⇒ (3) est tout à fait semblable à celle qui a été vue dans la proposition 4 et repose sur le théorème de dualité de Serre et la formula de Riemann-Roch (2.2). Reste à demontrer que
146 (4) ⇒ (1); pour ceci, on commence par vérifier de manière purement arithmétique que l'assertion (4) entraîne l'assertion suivante:

(5) $\Delta \neq 1/2$, *et pour tout $\alpha \in \mathfrak{E}$ tel que $r_\alpha < r$ et $\alpha - \mu| \leq 1$*

$$P(-1|\alpha - \mu|) \leqq \Delta_\alpha + \Delta$$

L'implication (5) ⇒ (1) résultera du théorème 3 (cf. § 7).

Let théorème 2 permet effectivement de décrire quelles sont exactement les classes de Chern des fibrés stables. Supposons par exemple $r = 20$, $c_1 = 9$ et donc $\mu = 0,45$; le tableau du paragraphe 5 montre que $\alpha(r,\mu) = 0,4$ et $\beta(r,\mu) = 0,5$. Par suite, les conditions (4) s'écrivent

$$\Delta \geqq P(-0,05) - 12/25 \quad \text{et} \quad \Delta \geqq P(-0,05) - 3/5$$

et sont donc équivalentes a $\Delta \geqq 0,55125$, c'est-à-dire $c_2 \geqq 50$.

7 La methode

Pour compléter les démonstrations de la proposition 4 (et donc du théorème 1) et du théorème 2, il suffit de vérifier l'énoncé suivant:

Theorem 3. *Soient r un entier > 1, et deux rationnels tels que $r\ \mu \in \mathbb{Z}$, $r(P(\mu) - \Delta) \in \mathbb{Z}$, $\Delta \neq 1/2$. On* suppose satisfaites les conditions suivantes:

(S) *Pour tout* $\alpha \in \mathfrak{E}$ *tel que* $r_\alpha < r$ *et* $|\alpha - \mu| \leqq 1$, *on a* $P(-|\alpha - \mu|) \leqq \Delta_\alpha + \Delta$

Alors il existe un fibré stable de rang r, de pente μ, *de discriminant* **147**
Δ *sur* $\mathbb{P}_2$.

7.1 Construction d'une grande famille de fibres vectoriels.

Soit $d \subset \mathbb{P}_2$ une droite fixée. On commence par construire une famille de fibrés vectoriels de rang r, de pente μ, de discriminant Δ, paramétrée par une variété algébrique lisse S:

$$E \to S \times \mathbb{P}_2$$

satisfaisant aux conditions suivantes:

(L) Pour tout $s \in S$, $\mathrm{Ext}^2(E(s), E(s)) = 0$

(KS) Le morphisme de déformation infinitésimale de Kodaira et Spencer

$$T_s S \to \mathrm{Ext}^1(E(s), E(s))$$

est suriectif

(R) Pour tout $s \in S$, $E(s)|_d$ est rigide, c'est-à-dire

$$\mathrm{Ext}^1_d(E(s)|_d, E(s)|_d) = 0$$

La construction s'inspire de [11]: considérons le polynôme de Hilbert $H(m) = r(P(\mu + m) - \Delta)$; pour $m + \mu \geqq -3/2$, la suite $m \to H(m)$ est croissante; elle est négative pour $-2 \leqq m + \mu < 0$: on le vérifie pour $1 \leqq m + \mu \leqq 0$ en remarquant que $-m \in \mathfrak{E}$, et donc d'après la condition (S) $P(m+\mu) \leqq \Delta$ et d'autre part, $P(m+\mu-1) = P(m+\mu)-(m+\mu+1) \leqq \Delta$.

Par suite, il existe un entier m_0 tel que $N_o = H(m_o) > 0$, $N_1 =$ **148**
$-H(m_o - 1) \geqq 0$, $N_2 = -H(m_o - 2) \geqq 0$. Soit Q le fibré canonique quotient de rang 2 sur $\mathbb{P}_2$; les morphismes injectifs de fibrés vectoriels

$$(\mathscr{O}(-1))^{N_2} \to (Q^*)^{N_1} \oplus \mathscr{O}^{N_o}$$

forment un ouvert non vide Ω de l'espace vectoriel de tous les morphismes car $2N_1 + N_0 - N_2 = r \geqq 2$. Pour $s \in \Omega$, le conoyau de s définit un fibré vectoriel $F(s)$ de rang r, de pente $\mu + m_o$, de discriminant Δ. La famille de fibrés vectoriels de rang r, de pente μ, de discriminant Δ

$$E(s) = F(s)(-m_o)$$

satisfait aux conditions (L) et (KS). En fait, on a même la condition plus forte que (L)

(L') Pour tout $s \in \Omega$, $\mathrm{Ext}^2(E(s), E(s)(-1)) = 0$

qui permet de vérifier que l'ouvert $S \subset \Omega$ des points $s \in S$ tels que le fibré $E(s)_d$ soit rigide est non vide.

7.2 Stratification de Shatz

149 On considère une famille $E \to S \times \mathbb{P}_2$ de fibrés vectoriels sur $\mathbb{P}_2$ paramétrée par une variété lisse S, et satisfaisant aux conditions (L) et (KS). Soient $(H_1, \ldots, H_\ell)$ des polynômes à coefficients rationnels. On dit que $s \in S$ est de poids $(H_1, \ldots, H_\ell)$ si la filtration de Harder-Narasimhan de $E(s)$

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = E(s)$$

est de longueur ℓ, et si le gradué associé $gr_i(E(s))$ a pour polynôme de Hilbert H_i. Le rang r_i, la pente μ_i, le discriminant Δ_i de $gr_i(E(s))$ sont déterminés à partir de H_i par la formule

$$H_i(m) = r_i(P(\mu_i + m) - \Delta_i).$$

L'énoncé suivant étend à $\mathbb{P}_2$ les résultats obtenus par Atiyah et Bott dans le cadre des surfaces de Riemann [1]:

Proposition 6. *Sour les conditions* (L) *et* (KS), l'ensemble

$$Y(H_1, \ldots, H_\ell)$$

des points $s \in S$ *de poids* $(H_1, \ldots, H_\ell)$ *est une sous-variété localement fermée lisse de codimension*

$$\sum_{i<j} r_i r_j(\Delta_i + \Delta_j - P(\mu_j - \mu_i))$$

Let sous-variétés ainsi definies sont en nombre fini.

7.3 Existence de fibrés semi-stables

L'énoncé ci-dessus s'applique en particulier à la famille construite au paragraphe 7.1. Pour montrer l'existence de fibrés semi-stables de rang **150** r, de pente μ, de discriminant Δ, il suffit de montrer que si $\ell > 1$, les sous-variétés ci-dessus sont de codimension > 0. Compte-tenu de la formule de Riemann-Roch, ceci résulte du lemme suivant.

Lemma 2. *Soit* E *un fibre vectoriel sur* $\mathbb{P}_2$*, de rang* r*, de pente* μ *et de discriminant* Δ *satisfaisant à la condition* (S)*. Soit*

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = E$$

la filtration de Harder-Narasimhan de E*. de gradué* $gr_i(E)$*. On suppose en outre qu'il existe une droite* $d \subset \mathbb{P}_2$ *telle que* $E|_d$ *soit rigide. Alors si* $\ell > 1$

$$\sum_{i<j} \chi(gr_i(E), gr_j(E)) < 0$$

En effet, soient r_i le rang, μ_i la pente, Δ_i le discriminant de $gr_i(E)$ l'existence d'une droite d telle que $E|_d$ soit rigide entraîne

$$0 \leqq \mu_1 - \mu_\ell \leqq 1$$

En particulier, pour $i < j$, on a $0 \leqq \mu_i - \mu_j < 3$ de la semi-stabilité de $gr_i(E)$ il découle

$$\chi(gr_i(E), gr_j(E)) \leqq 0$$

Supposons que pour tout (i, j) tel que $i < j$, $\chi(gr_i(E), gr_j(E)) = 0$, c'est-à-dire, d'apres la formule de Riemann-Roch

$$P(\mu_j - \mu_i) = \Delta_i + \Delta_j.$$

En particulier, si $\mu_1 - \mu_\ell < 0$, ceci entraîne $\Delta_1 + \Delta_\ell < 1$, donc soit $\Delta_1 < 1/2$ soit $\Delta_\ell < 1/2$. Si $\mu_1 = \mu_\ell$, $\Delta_1 < \Delta_\ell$ par définition de la filtration de Harder-Narasimhan; par suite $\Delta_1 < 1/2$. Dans les deux cas, l'un des faisceaux $gr_1(E)$, $gr_\ell(E)$ est semi-exceptionnel. Supposons par exemple $gr_1(E)$ semi-exceptionnel; alors 151

$$\begin{aligned}\chi(gr_1(E),(E) &= \chi(gr_1(E)) + \sum_{i>1} \chi(gr_1(E), gr_i(E)) \\ &= \chi(gr_1(E), gr_1(E))\end{aligned}$$

D'après la proposition 3 qui sonne la structure des faisceaux semi-exceptionnels, cette quantité est positive. On a alors

$$P(\mu - \mu_1) > \Delta + \Delta_1$$

et $|\mu - \mu_1| \leqq 1$. Ceci contradit la condition (S). Dans le cau où c'est $gr_\ell(E)$ qui est semi-exceptionnel, on obtient à nouveau une contradiction en étudiant $\chi(E, gr_\ell(E))$.

7.4 Construction de fibrés stables

Considérons, dans la famille ci-dessus, l'ouvert $\mathfrak{M} \subset S$ correspondant aux fibrés semi-stables. On vient de voir que $\mathfrak{M}$ n'est pas vide.

Lemma 3. *Let points de $\mathfrak{M}$ correspondant aux fibrés stables forment un ouvert $\mathfrak{M}_s$ partout dense dans $\mathfrak{M}$.*

152 En effet, si $\Delta < 1/2$, pour $t \in \mathfrak{M}$, chaque fibré $E(t)$ est semi-exceptionnel; la condition (S) impose en fait que $E(t)$ soit exceptionnel d'après la proposition 3.

Si $\Delta > 1/2$, soit r_i une suite d'entiers tels que $\sum r_i = r$, $r_i > 0$. Let points $t \in \mathfrak{M}$ tels que $E(t)$ ait une filtration (dite de Jordan-Hölder)

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = E(t)$$

dont le gradué gr_i soit stable de rang r_i, de pente μ et de discriminant Δ, est un fermé $Y(r_1, \ldots, r_\ell)$ dont on peut minorer la codimension, de manière tout à fait semblable à ce qui a été vu dans la proposition 6, bien que ces fermés n'aient ici aucune raison d'être lisses:

$$\operatorname{cosim} Y(r_1, \ldots, r_\ell) \geq \sum r_i r_j (2 - 1)$$

Par suite, $Y = \cup Y(r_1, \ldots, r_\ell)$ est un fermé de codimension > 0; le complémentaire de ce fermé est exactement l'ensemble des points $t \in \mathfrak{M}$ tels que $E(t)$ soit stable. Cet ouvert est donc partout dense.

8 Irreductibilité

Un faisceau algébrique cohérent E sur $\mathbb{P}_2$ est dit μ-stable s'il est sans torsion et si pour tout sous-module $F \subset E$ de rang $r(F) < r(E)$, $F \neq 0$ on a

$$\mu(F) < \mu(E).$$

153 Let points de $M(r, \mu, \Delta)$ qui proviennent de faisceaux localement libres μ-stables forment un ouvert $M^o_{\mu s}$; par une méthode semblable a celle qui vient d'être décrite au paragraphe 7, on démontre que cet ouvert est partout dense. Compte-tenu du résultat d'Ellingsrud donnant déjà l'irréductibilité de M^o [6][1], on obtient:

Theorem 4. *L'espace de modules* $M(r, \mu, \Delta)$ *est irréductible.*

9 Groupe de Picard

Soient r un entier > 1, μ et Δ deux rationnels tels que $c_1 = r\mu$ et $\chi = r(P(\mu) - \Delta)$ soient entiers, et $\Delta > 1/2$.

Dans le cas ou r, c_1 et χ sont premiers entr'eux, l'ouvert M_s de l'espace de modules $M(r, \mu, \Delta)$ est egal a $M(r, \mu, \Delta)$: la variété $M(r, \mu, \Delta)$ est alors une variété projective lisse de dimension $r^2(2\Delta - 1) + 1$; c'est un espace de modules fin pour le foncteur quotient

$$S \to \underline{M}(r, \mu, \Delta)(S)/\operatorname{Pic} S$$

---

[1] Pour $r = 2$, l'irréductibilité de $M^o_{\mu s}$ est due à Barth [2] et Hulek [9] Pour $\mu = 0$, elle est due a Hulek [10]. L'irréductibilité de $M(2, \mu', \Delta)$ est connue de Maruyama [12].

où Pic(S) opère sur $\underline{M}(r,\mu,\Delta)(S)$ (cf. § 3) par la formule

$$(L,E) \to E \otimes pr_1{}^*(L)$$

154 ou $L \in \mathrm{Pic}(S)$, et $E \in \underline{M}(r,\mu,\Delta)(S)$.

Considérons l'ensemble $\mathfrak{E}(r/2)$ des éléments $\alpha \in \mathfrak{E}$ tels que $r_\alpha \leq r/2$, et posons

$$\delta(r,\mu) = \mathop{\mathrm{Sup}}_{\substack{\alpha \in \mathfrak{E}(r/2) \\ |\alpha - \mu_1 \leqq 1}} P(-|\alpha-\mu|) - \Delta_\alpha$$

Avec les notations du théorème 2, cette borne superieure est en fait atteinte soit pour $\alpha = \alpha(r,\mu)$, soit pour $\alpha = \beta(r,\mu)$. L'énoncé du théorème 2 peut en fait se lire:

L'espace de modules $M(r,\mu,\Delta)$ n'est pas vide si et seulement si

$$\Delta \geqq \delta(r,\mu)$$

L'énoncé suivant est du au premier auteur; il donne, quand r, c_1 et χ sont premiers entr'eux, le groupe de Picard de la variété $M(r,\mu,\Delta)$:

Theorem 5. *[3] On suppose que $\Delta > 1/2$, et que r, $c_1 = r\mu$ et $\chi = r(P(\mu)-\Delta)$ sont des entiers premiers entr'eux. Alors le groupe de Picard de la variété $M(r,\mu,\Delta)$ est donné par*

$$\mathrm{Pic}\, M(r,\mu,\Delta) = \begin{cases} \mathbb{Z} & si \quad \Delta = \delta(r,\mu) \\ \mathbb{Z}^2 & si \quad \Delta > \delta(r,\mu) \end{cases}$$

155 De plus, en utilisant le fibré universel, on peut donner une base pour le groupe de Picard.

Considérons par exemple le cas $r = 4$, $\mu = 1/2$, $\Delta = 5/8$: alors $c_1 = 2, \chi = 5$, et par suite, l'espace de modules $M(r,\mu,\Delta)$ correspondant est une variété projective lisse de dimension 5; comme $\delta(4.1/2) = 5/8$, son groupe de Picard est $\mathbb{Z}$. On peut en fait vérifier que cet espace de modules est isomorphe à $\mathbb{P}_5$.

Signalons que le cas $r = 1$ a éte étudié par G. Elencwajg et P. Le Barz [5]; le cas $r = 2$ a également été traité récemment par S.A. Strømme [16].

Si r, c_1 et χ ne sont plus premiers entréux la variete $M(r, \mu, \Delta)$ est encore projective et normale, mais il apparaît en général des singularités dans $M(r, \mu, \Delta)$. Il est encore possible de décrire le groupe de Picard de l'ouvert de lissité $M_{\mathrm{reg}}(r, \mu, \Delta)$ de $M(r, \mu, \Delta)$ et le résultat est semblable au précédent [3]:

$$\operatorname{Pic} M_{\mathrm{reg}}(r, \mu, \Delta) = \begin{cases} \mathbb{Z} & \text{si} \quad \Delta = \delta(r, \mu) \\ \mathbb{Z}^2 & \text{si} \quad \Delta > \delta(r, \mu). \end{cases}$$

Sauf pour quelques exceptions qu'il est possible d'énumérer, cet ouvert de lissité coïncide en fait avec l'ouvert M_s correspondant aux faisceaux stables.

References

156

[1] M.F. Atiyahet R. Bott. The Yang-Mills equations over Riemann Surfaces, *Phil. Trans. Roy. Soc. London,* A 308 (1982)

[2] W. Barth. Moduli of vector bundles on the projective plane, *Invent. Math.,* 42 (1977), p. 63-91.

[3] J.M. Drezet. Groupes de Picard des varietes de modules de faisceaux stables sur $\mathbb{P}_2$, en preparation.

[4] J.M. Drezet et J. Le potier. Fibres stables et fibres exceptionnels sur $\mathbb{P}_2$, a paraitre aux Ann. Sc. Ec. Norm. Sup.

[5] G. Elencwajg et P. Le Barz. Picard groups of polygons, Preprint, Nice (1983).

[6] G. Ellingsrud. Sur l'irréductibilité du module des fibrés stables sur $\mathbb{P}_2$, Preprint, Nice (1982).

[7] D. Gieseker. On the moduli of vector bundles on an algebraic surface, *Ann. of Math.*, 106 (1977), p. 45-60.

157 [8] G. Harder et M.S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves, *Math. Ann.* 212 (1975), p. 215-248.

[9] K. Hulek. Stable rank-2 vector bundles on $\mathbb{P}_2$ with c_1 odd, *Math. Ann.* 242 (1979), p. 241-266.

[10] K. Hulek. On the classification of stable rank-r vector bundles over the projective plane, *Vector Bundles and Differential Equations*, Proceedings, Nice 1979, Progress in Math., 7 (1980), p. 113-144.

[11] J. Le Potier. Stabilité et amplitude sur $\mathbb{P}_2(\mathbb{C})$, *Vector Bundles and Differential Equations*, Proceedings, Nice 1979, Progress in Math., 7 (1980), p. 146-181.

[12] M. Maruyama. Moduli of stable sheaves II, Journal *of Mathematics of Kyoto University* 18(3) (1978), p. 557-614.

[13] R.L.E. Schwarzenberger. Vector bundles on algebraic surfaces *Proc. London Math. Soc.*, (3) 11(1961), p. 601-622.

[14] R.L.E. Schwarzenberger. Vector bundles on the projective plane, *Proc. London Math. Soc.*, (3) (1961), p. 623-640.

158 [15] S. Shatz. The decomposition and specialization of algebraic families of vector bundles, *Compos. Math.*, 35 (2) (1977), p. 163-187.

[16] S.A. Strømme. *Ample divisors on fine moduli spaces on the projective plane*, Preprint, Bergen (Norvege) 1984.

Université Paris 7
U.E.R. de Mathématiques et L.A. 212
Aile 45-55
2, Place Jussieu
75251 Paris Cedex 05

Rank Techniques and Jump Stratifications

By A. Hirschowitz

IT IS A fundamental duty and tool for geometers to study natural stratifications. The theory of special divisors on projective curves is devoted to such a matter. Much progress has been made in this field during the last few years (see some references in 7.2.1). In the present lecture, I want to emphasize the fact that some of the tools, which I may call rank techniques and which were developed mainly for application to questions on special divisors, have much wider applicability. **159**

Rank techniques are reviewed in sections 1-4. In particular the rank-Kodaira-Spencer map is introduced in section 3. It turns out that many subvarieties in projective spaces are constructed via rank stratifications (or via similar stratifications). This is explained in sections 5, 6. Jump stratifications are introduced in section 7 where it is shown how far the rank techniques are expected to be useful. In particular, Petri's comorphism is defined, generalizing the Petri's map of the theory of special divisors.

In section 8, I give a sample of applications. The first one should concern special divisors and projective curves. Instead the reader is invited to see [47]. The second one computes the number of minimal sections in a general geometrically ruled surface. The third one is a "simplification" in the proof of the theorem of natural cohomology for rank two vector bundles over $\mathbb{P}^3$ (cf. [56]). And the last one is an echo from Barth's lecture in the Colloquium: any rank two vector bundle on $\mathbb{P}^4$ with $c_1 = 1$, $c_2 = 4$ has a two dimensional family of unstable planes. **160**

From the general definition of jumping loci, the jumping points emerge. Surprisingly, although considered in several special cases, the jumping points have not been studied systematically, not even in the case of curves. Section 9 gives a first account of this matter.

The results exposed below developed slowly in my mind during my collaboration with Brun, Hartshorne, Marlin and Narasimhan, for which I am indebted. This knowledge grew much faster during the two months after the conference, as I wrote the present paper. I thank the organizers who gave me the opportunity to think about this in an accurate way. I also thank them for the kind invitation, and with them, all those people in the Tata Institute who made my stay so pleasant.

1 Rank Stratifications

Many natural stratifications may be viewed as rank stratifications in the following sense; moreover, the strata of many further natural stratifications may be viewed as strata of such rank stratifications.

Definition 1.1. *Let X be a quasiprojective smooth connected variety, E and F locally free sheaves on X and $u : E \longrightarrow F$ a morphism. We define the rank stratification X_u associated with u as follows. First denote by p the smaller among rank E and rank F; then X_u^i is defined to be the zero-scheme associated with $\Lambda^{p-i+1}u$, and the stratification X_u is the* 161 *collection $(X^i)_{i\in\mathbb{N}}$.Observe that X_u and X_{u^*} are identical so that we may assume either rank $E = p+\delta$ or as well rank $F = p+\delta$ (with $\delta \geq 0$). We set $\mathring{X}_u^i := X_u^i - X_u^{i+1}$. Also we consider the relative grassmannian variety G_E over X of linear subspaces in the fibers of E, and the subscheme G_u of subspaces where u vanishes identically. Of course G_E (and in general G_u) have several connected components corresponding to various dimensions.*

1.2 Program.

The study of such a rank stratification splits: from the global study, we expect answers to the questions: "What are the non-void strata X_u^i? What are their degrees, rational equivalence classes, irreducible components?" and to similar questions concerning G_u. From the local study, we expect answers to be questions: "What is the maximum rank of u?

What are the dimensions of the strata, are they reduced? Are the X^i_u smooth?" and to similar questions concerning G_u.

1.3 Tools.

As for the global study : Porteous' formula (Porteous [91], Kempf-Laksov [68]) gives the rational equivalence class of each stratum in terms of Chern classes of E and F, provided the stratum has the expected codimension (namely $i(\delta+i)$ for X^i_u). This is also a powerful tool to prove existence results (cf. [71]); and a theorem of Fulton-Lazarsfeld [35] says that in case X is projective, and $E^\vee \otimes F$ is ample, then each stratum with positive expected dimension (i.e. $i(\delta + i) < \dim X$) is connected.

Now locally, rank stratifications are obtained by pulling-back some universal rank-stratification. Thus the local study splits into the study of universal rank-stratifications and the study of base-change. This approach appears in Arbarello-Cornalba [1], and both parts are developed in the next two sections.

2 Universal Rank Stratifications [1], [67], [67], [48]

162

2.1 The Universal Objects.

We choose an integer $\delta \geq 0$ and another integer $p > 0$, and we consider the vector space $R_{\delta,p}$ of matrices with p columns and $p + \delta$ rows, and coefficients in the base field k. On $R_{\delta,p}$ viewed as an affine variety, we have two tautological morphisms:

$$\tau : \mathscr{O}^p \to \mathscr{O}^{p+\delta} \quad \text{and} \quad \tau^\vee : \mathscr{O}^{p+\delta} \to \mathscr{O}^p$$

As observed above, the stratifications associated with τ and $\tau^\vee$ are identical. We also denote them with $R_{\delta,p}$ so that $R^0_{\delta,p}$ is $R_{\delta,p}$ and the other strata are the $R_{\delta,p}$'s for $i > 0$.

We denote by $G_{\delta,p}$ the connected component $\quad_{\tau}$ (cf. 1.1) consisting of subspaces of dimension i and by $G^i_{\delta,p^\vee}$ the connected component of $G_{\tau^\vee}$ consisting of subspaces of dimension $\delta + i$. If G (resp. $G_\vee$)is the grassmannian variety of subspaces of k^p (resp. $k^{p+\delta}$), then G_τ (resp. $G_{\tau^\vee}$) is a closed subscheme in $G \times R_{\delta,p}$ (resp. $G_\vee \times R_{\delta,p}$).

2.2 Some Properties of $R_{\delta,p}$.

The following properties of $R_{\delta,p}$ are well known : $\mathring{R}^i_{\delta,p}$ is smooth of codimension $i(\delta + i)$. For $i > 0$, $R^{i+1}_{\delta,p}$ is the singular locus of $R^i_{\delta,p}$. Each $R^i_{\delta,p}$ is reduced, determinantal and Cohen-Macaulay. On the other hand $G^i_{\delta,p}$ and $G^i_{\delta,p^\vee}$ are smooth and both $G^i_{\delta,p} \to R^i_{\delta,p}$ and $G^1_{\delta,p^\vee} \to R^i_{\delta,p}$ are (rational) desingularizations of $R^i_{\delta,p}$. In case $\delta = 0$, $p = 2$, $i = 1$, this is the simple example, explained so long ago by Hironaka, of a threefold with an isolated singularity, with two minimal desingularizations, the fiber product of which is the blowing-up of the singular point. Another way to desingularize $R^i_{\delta,p}$ in the general case consists in blowing up
163 successively (the strict transform of) $R^p_{\delta,p}$, $R^{p-1}_{\delta,p}, \ldots, R^{i+1}_{\delta,p}$.

Now we review the normal spaces : at a point u in $\mathring{R}^1_{\delta,p}$, the normal space is the space of linear maps $L(\operatorname{Ker} u, \operatorname{Coker} u)$ and the projection of the tangent space to the normal space is the natural map : $R_{\delta,p} = L(K^p, k^{p+\delta}) \to L(\operatorname{Ker} u, \operatorname{Coker} u)$. Similarly at a point (W, u) in $G^j_{\delta,p}$, the normal space of $G^j_{\delta,p}$ in $G \times R_{\delta,p}$ is $L(W, k^{p+\delta})$ and the projection from the total tangent space is the natural map

$$L(W, k^p/W) \oplus L(k^p, k^{p+\delta}) \to L(W, k^{p+\delta})$$
$$(\alpha, \beta) \to \overline{u} \circ \alpha + \beta \circ w,$$

where w is the injection $W \to k^p$ and u induces u on k^p. The same holds, mutatis mutandis for $G^j_{\delta,p^\vee}$.

2.3 From $R_{\delta,p}$ to $R_{\delta,p}$.

Let us see that the shape of $R_{\delta,p}$ does not depend too heavily upon p.

Proposition 2.3.1. *Let u be a point in $\mathring{R}^i_{\delta,p}$. There exists an open neighbourhood U of u in $R_{\delta,p}$ and a morphism $\varphi : U \to L$ (*Ker *u,* Coker *u) such that:*

(i) *The pull-back by φ of the (universal) rank stratification on*

$$L(\operatorname{Ker} u, \operatorname{Coker} u)$$

is the universal rank stratification $R_{\delta,p}$ restricted to U,

(ii) *the derivative $d_u\varphi$ of φ at u is the natural projection* (cf. 2.2).

Proof. We choose decompositions $k^p = \operatorname{Ker} u \oplus A$, $k^{\delta+p} = \operatorname{Im} u \oplus$
$\operatorname{Coker} u$, so that a point v in $R_{\delta,p}$ reads $v = \left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right)$. We define U to **164**
be the open set where v_{12} is an isomorphism. It clearly contains u. And
we define φ by $\varphi(v) = v_{21} - v_{22}v_{12}^{-1}v_{11}$. One checks easily (i) and (ii). □

3 Base Change

The general hope concerning a bundle morphism is that it has maximal rank (at the generic point) and that the associated rank-stratification "looks like" some universal rank-stratification. We make precise definitions and then introduce the rank-Kodaira-Spencer map, in order to give differential criteria for the above hopes to be fulfilled.

Definition 3.1. *By a stratification indexed by $\mathbb{N}$, I mean a family $(X^i)_{i\in\mathbb{N}}$ of (quasi-projective) schemes together with compatible closed embeddings:*

$$(i < j)X^j \hookrightarrow X^i,$$

so that the X^t's are subschemes of X^0. The differences $X^i - X^{i+1}$ are locally closed subschemes in X^0, which we denote with $\mathring{X}^i$. Let us say that such a stratification $(X^i)_{i\in\mathbb{N}}$ is R_δ – like at a point $x \in \mathring{X}^j$ *if there exists an etale neighborhood X' of xX^0 and a smooth morphism $\varphi : X \to R_{\delta,j}$ such that $(X^i)_{i\in\mathbb{N}}$ and $R_{\delta,j}$ induce the same stratification on X'. Let us say further that $(X^i)_{i\in\mathbb{N}}$ is R_δ-like if it is R_δ-like at every point.*

It follows from 2.3 that for any δ, p, the stratification $R_{\delta,p}$ is R_δ-like, and that whenever a stratification is R_δ-like at a point, it is R_δ-like in a neighbourhood.

3.2 The RKS-MAP.

165 Let $u : E \to F$ be a morphism between locally free sheaves on the quasi-projective smooth variety X. We denote by $L(E, F)$ the geometric bundle associated with $\mathrm{Hom}(E, F)$, also usually denoted by $\mathbb{V}(E \otimes F^\vee)$. Let π be the projection $L(E, F) \to X$. Then over $L(E, F)$, we have a tautological morphism

$$\tau_{EF} : \pi^* E \to \pi^* F$$

Furthermore, u may be viewed as section of π characterized by $u = u^* \tau_{EF}$.

One checks easily that the rank stratification $(L(E, F)^i)_{i \geq 0}$ associated with τ_{EF} is R_δ-like where $\delta = |\operatorname{rank} E - \operatorname{rank} F|$. Also the stratification X_u is the pull-back by (the section) u of the stratification $(L(E, F)^i)_{i \geq 0}$.

Let O be a point in X, E_O and F_O the fibers of E and F at O, and u_O the image of O by u in $L(E, F)$. Let i be such that u_O is in $L(E, F)^i - L(E, F)^{i+1}$. First $L(E, F)^i$ is smooth in u_O, second it is transversal to the fibre $\pi^{-1}(O) = L(E_O, F_O)$ in $L(E, F)$. Thus the normal space of $L(E, F)^i$ in $L(E, F)$ at u_O is canonically identified with the normal space of $L(E_O, F_O)^t$ in $L(E_O, F_O)$ at u_O, which is $L(\operatorname{Ker} u_O, \operatorname{Coker} u_O)$ (cf. 2.2) so that the differential of u induces a linear map from the tangent space $T_0 X$ of X at O into $L(\operatorname{Ker} u_0, \operatorname{Coker} u_0)$. This map we call the rank *Kodaira-Spencer map* (or briefly *RKS*-map) of u at O and we denote it by $RKS(u, O)$.

3.3 Criterion for Maximal Rank.

Let A, B, C be vector spaces and $f : A \to L(B, C)$ a linear map. We shall say that f *reaches the maximal rank* if, for a sufficiently general, $f(a)$ is injective or surjective.

166 **Proposition.** *Let $u : E \to F$ be a vector bundle morphism over a smooth connected variety X. Suppose there exists a point O in X where $RKS(u, O)$ reaches the maximal rank. Then u has maximal rank (at the generic point).*

Proof. The statement being local, we may suppose that E and F are trivial. Also we may suppose that X is a smooth curve. Then u corresponds to an immersion $u : X \to L(k^p, K^{p+\delta})$. Let us define i by $u(O) \in \mathring{R}^i_{\delta,p}$. And let us consider the map φ given by 2.3. According to 2.3 ii, the derivative of $\varphi \circ u$ at O is $RKS\,(u, O)$ and hence reaches the maximal rank. This means that $d_O(\varphi \circ u)$ takes some values outside of $R^1_{\delta,i}$ which is its own tangent cone. Hence the same is true of $\varphi \circ u$ itself which, according to 2.3.i, implies that u takes some values outside $R^1_{\delta,p}$. □

Remark. We could prove easily, when the field has characteristic zero, that $RKS\,(u, x)$ vanishes for x sufficiently general. In particular, if for any x in X^1_u, $RKS\,(u, x)$ does not vanish, then u has maximal rank.

3.4 Criterion for R_δ-Likehood

Proposition. *Let X, E, F, δ, u be as above. Then X_u is R_δ-like at a point x if and only if $RKS\,(u, x)$ is surjective.*

Proof. We define i and φ as in the previous proof. Then X_u is pulled-back by $\varphi \circ u$ of $R_{\delta,i}$; and the derivative of $\varphi \circ u$ is $RKS\,(u, x)$. If the latter is surjective, then X_u is R_δ-like at x by definition. On the other hand, for $(\varphi \circ u)^*(O)$ to be smooth of codimension $i(\delta + i)$ it is necessary that the kernel of $d_X(\varphi \circ u)$ has codimension $i(\delta + i)$, hence that its image has dimension $i(\delta + i)$. □

3.5 Criterion for Smoothness of G_u (cf. [1] 3.3).

167

Proposition. *Let X, E, F, δ, u and x be as above and let W be a linear subspace in* $\operatorname{Ker} u(x)$ *of dimension i if* $\operatorname{rank} E \leq \operatorname{rank} F$, *and of dimension $\delta + i$ otherwise. Then G_u is smooth of codimension $i(\delta + i)$ at W in G_E (cf. 1.1) if and only if $w \circ RKS\,(u, x)$ is surjective where w is the restiction map from* $L(\operatorname{Ker} u(x), \operatorname{Coker} u(x))$ *to* $L(W, \operatorname{Coker} u(x))$.

Proof. We suppose, for instance, that $\operatorname{rank} E > \operatorname{rank} F$. Again we consider (locally) u as a morphism from X to $R_{\delta,p}$. Now W corresponds to a point in $G_\vee$ (cf. 2.1). We extend trivially u as a map $\widetilde{u} : G_\vee \times X \to$

$G_\vee \times R_{\delta,p}$, so that $G_\vee$ is just the pull-back by $\widetilde{u}$ of $G_{\tau^\vee}$. Let $\delta + i$ be the dimension of W. Then, as above, the necessary and sufficient condition for G_u to be smooth of codimension $i(\delta+i)$ is that the derivative $d\widetilde{u}(W,x)$ ranges over the whole normal space of $G^i_{\delta,p^\vee}$ in $G_\vee \times R_{\delta,p}$. According to 2.2, this means that the map

$$\begin{gathered} L(W, k^{p+\delta}/W) \oplus T_x X \to L(W, k^p) \\ (\alpha, \epsilon) \mapsto \overline{u}^\vee \circ \alpha + du^\vee(x,\epsilon) \circ w \end{gathered}$$

is surjective where w is the injection $W \to k^{p+\delta}$ (here we identify k^p and $k^{p+\delta}$ with their duals). Now the image of $\alpha \mapsto \overline{u}^\vee \circ \alpha$ is precisely $L(W, \operatorname{Im} u^\vee)$ so that the condition means that the map

$$\begin{gathered} T_x X \to L(W, \operatorname{Coker} u^\vee) \\ \epsilon \to \gamma \circ du^\vee(x,\epsilon) \circ w \quad \text{is surjective} \end{gathered}$$

where γ is the projection $k^p \to \operatorname{Coker} u^\vee$. To conclude, we just have to observe that $RKS\,(u,x)(\epsilon)$ is the composition

$$\operatorname{Ker} u^\vee \to k^{p+\delta} \xrightarrow{du^\vee(x,\epsilon)} k^p \to \operatorname{Coker} u^\vee .$$

168 □

4 Stable Properties [21]

In [21], Brun and I emphasized stable properties of stratifications. In this section, after recalling the definitions, we review R_δ-likehood from this point of view of stability.

4.1 Definitions.

Although everybody understands what is meant by a property being satisfied by stratifications indexed by $\mathbb{N}$, let us give a careful definition: let us say that it is a subset of the set of isomorphism classes of stratifications indexed by $\mathbb{N}$. Now we say that a property P (satisfied by $(X^t)_{i\in\mathbb{N}}$

as soon as there exists an étale covering $(X_\alpha)_{\alpha\in A}$ of X° such that P is enjoyed by each induced stratification $(X^i\alpha)_{i\in\mathbb{N}}$. Finally we say that the property P is *stable* if, whenever it is satisfied by a stratification $(T^i)_{i\in\mathbb{N}}$ with T^0 smooth, it also holds:

(a) for any submersion $R \to T^0$, by the induced stratification $(R^i)_{i\in\mathbb{N}}$,

(b) for any submersion $T^0 \to R$, with R connected, by the induced stratification on the general fiber $T^0(r)$.

Examples 4.2. Here are the most usual examples:

(a) X^i is s-codimensional in X^0.

(b) $\mathring{X}^i$ is smooth.

(c) X^i is locally complete intersection.

(d) X^i is determinantal.

(e) X^{i+1} is the singular locus of X^i.

(f) locally, in the étale topology, X^i is trivial along X^{i+1}.

(g) X^{p+1} is empty, and X^i may be desingularized by blowing-up suc- **169**
cessively (the strict transforms of) X^p, $X^{p-1}, \dots, X^{i+1}$ (cf. 2.2).

4.3 Stability of R_δ - Likehood.

Proposition. *R_δ-likehood is a stable property.*

Proof. The property is clearly stable under submersions. Hence we may consider a submersion $s : X \to Y$ where X is endowed with the stratification induced by another submersion $f : X \to L(k^j, k^{\delta+j})$. By Bertini's theorem, the general fiber $Z = s^{-1}(y)$ intersects transversally the strata $\mathring{X}^i$. Let x be a point in $Z \cap \mathring{X}^i$. We know that around x, the stratification on X is induced by a submersion $f' : X \to L(k^i, k^{\delta+i})$. Since Z is transversal to the special fiber X^i of f', the restriction of f' to Z is again a submersion, which proves that the stratification induced on Z is R_δ-like at x. □

Remark 4.4. Another point of view is as follows. Call any local stable property satisfied by every $R_{\delta,p}$ a stable property satisfied by R_δ. Then one proves easily that a stratification indexed by $\mathbb{N}$ is R_δ-like if and only if it satisfies all the stable properties of R_δ. Of course R_δ is not a stratification itself. However, as pointed out by M.F. Atiyah, it may be thought of as the stratification by the corank of the open set of Fredholm operators with index δ in the Hilbert space (of course only in the complex case.) Stratification on infinite dimensional manifolds is a main tool in Atiyah-Bott [3] (see also [69]).

5 Other Stratifications

There are several other standard ways to construct stratifications with corresponding universal stratifications and KS-maps. We briefly review two of them.

5.1 Symmetric Rank-Stratifications ([24], [12])

170

Instead of general morphisms $u : E \to F$, we consider symmetric morphisms $\mu : E \to E^\vee$. The universal stratifications are induced by $R_{0,n}$ on the vector subspaces of symmetric maps in $L(k^n, k^n)$. The range of the KS-map is the vector space $L_s(\mathrm{Ker}\, u_O\ \mathrm{Coker}\, u_O)$ of symmetric linear maps from $\mathrm{Ker}\, u_O$ to $\mathrm{Coker}\, u_O$ (since u_O is symmetric, $\mathrm{Ker}\, u_O$ and $\mathrm{Coker}\, u_O$ are dual to each other).

5.2 Σ-Stratificatons ([84], [102]).

Let $Y \to X$ be a locally (in the étale topology) trivial bundle over X with typical fiber $\mathbb{P}^1$. Let E be a rank-two vector bundle over Y. The splitting type of the restriction $E|_{Y(x)}$ of E to the fiber defines a (scheme) stratification of X. The corresponding universal stratifications live on semi-universal deformations of rank two vector bundles over $\mathbb{P}^1$. These split into two families Σ^0 and Σ^1 according to the parity of the degrees. As in the rank case, one can define Σ^0-like and Σ^1-like stratifications.

In the present case, the KS-map is the classical one with values in $H^1(Y(O), \operatorname{End} E(O))$.

According to Gruson-Peskine ([44, p.1]), the corresponding universal stratifications may also be seen as rank stratificatons on vector spaces of so-called per-symmetric matrices.

5.3 S-Likehood.

In each of the case we have encountered so far, instead of a single universal stratification, we have seen a sequence of universal stratifications (which we could glue over a Hilbert manifold, see 4.4). So the following definition seems relevant.

Definition. *Let S be a set of stratifications, each indexed by $\mathbb{N}$ and let $(X^i)_{i\in\mathbb{N}}$ be a further stratification. We say $(X^i)_{i\in\mathbb{N}}$ is S-like in case it* **171** *enjoyes any local stable property enjoyed by any member of S.*

For this definition to fit with that of R_δ-likehood, we have to consider R_δ as the set $\{R_{\delta,p}\}_{p>0}$.

6 Constructing Subvarieties

After having worked some time on vector bundles (or any other topic) one asks himself why. The answer, though not immediate, arises, in general, from connections with other fields whose interest is presumably more clear. I am satisfied with the connection between vector bundles and subvarieties say of projective spaces. First vector bundles are a tool in the study of subvarieties. Second, vector bundles provide new natural varieties through their moduli. Finally, many subvarieties are constructed starting from vector bundles. The original idea was to consider zero-sets of sections of vector bundles. It turns out that several generalizations of this idea deal with rank stratifications. That is what is explained in the present section.

6.1 Zero-Sets.

It is a standard tool, called the Serre-Horrocks construction and reactived by Barth - van de Ven [14], [15] in the seventies. Given a rank two vector bundle E and a section s, we get an exact sequence:

$$0 \to \mathscr{O}_f \xrightarrow{s} E \to I_Y(c_1) \to 0.$$

Provided s just vanishes in codimension two, I_Y is the ideal sheaf of $s^{-1}(0)$ and c_1 is the determinant of E. This is, on the one hand, a starting point for studying vector bundles and their moduli [51], [73], [57], [19], [26] through subvarieties and their Hilbert scheme and, on the other
172 hand, this is a tentative technique to find out subvarieties [62]. Here is a picture of corresponding properties for vector bundles and from subvarieties point of view (over projective space):

Subvarieties side	connection	vector bundles side
Y smooth	$(\exists) \Longleftarrow$	E globally generated
Y connected	$\Longleftrightarrow$	$h^1(E(-c_1)) = 0$
postulation $h^\circ(I_Y(s))$	$\leftrightsquigarrow$	cohomology $h^\circ(E(s-c_1))$
liaison	$\Longleftarrow$	twist, cf. [93], [94]
cohomology of the normal bundle	$\leftsquigarrow$	cohomology of $E(s)$, $E \otimes E(s)$, cf. [5]
compactification: Hilbert scheme		compactification: Maruyama's moduli

6.2 First Generalization.

Suppose we have a linear algebraic group G acting (linearly) on k^N and a stratification $S = (S^i)_{i\in I}$ of $\mathbb{A}_k^N$ (i.e. $S^\circ = \mathbb{A}_k^N$), which is invariant under the action of G. Suppose we have a rank N vector bundle on X with structure group G. Then the total space $\mathbb{V}(E^\vee)$ inherits a stratification $(E^i)_{i\in I}$, locally induced by $(S^i)_{i\in I}$ through trivializations. Furthermore, if σ is a section of E over X, viewed as section of $\mathbb{V}(E^\vee) \to X$ it induces a stratification X_σ defined by $X_\sigma^i = \sigma * E^i$. Now this is used through the following general nonsense:

Proposition. (cf. [70], [107]) *In case E is globally generated (i.e.* 173

$$H^0(E) \otimes \mathscr{O} \to E$$

is surjective) and σ is general enough, then X_σ is S-like (cf. 5.3). *In particular, if S is a universal rank stratification $R_{\delta,p}$ then X_σ is R_δ-like.*

Proof. Let $e : H^0(E) \times X \to \mathbb{V}(E^\vee)$ be the evaluation map. Since E is globally generated, e is a submersion. For σ in $H^0(E)$, X_σ is the stratification induced by $e^*(E^i)_{i \in I}$ on $\{\sigma\} \times X$. The projection $H^0(E) \times X \to H^0(E)$ is again a submersion. Thus the statement follows from the definition of stability. □

Program. In order to invoke this proposition, one has to

(a) choose the group G, the representation and the stratification (supposed to be well known).

(b) find a globally generated vector bundle with structure group G acting through the chosen representation. One way, for instance, if $G = GL(n, k)$, is to start with a globally generated rank n vector bundle and consider the rank N vector bundle associated with the representation. If the representation is suitably “positive”, this vector bundle will be globally generated again. Also remember ([85, p. 99]) that for E to be globally generated, a sufficient condition (on $\mathbb{P}^n$) is

$$0 = H^1(E(-1)) = H^2(E(-2)) = \ldots = H^n(E(-n)).$$

(c) study global properties of the obtained strata. Here methods of (1.2), or analogous methods should be helpful.

6.3 Examples

6.3.1

We first see how zero sections fit into this frame. Here $G = GL(n, k)$, the representation is the identity, and the stratification is:

$$S^0 = \mathbb{A}^n, \quad S^1 = 0 \in \mathbb{A}^n.$$

174

6.3.2

Here $G = GL(V)$ acts diagonally on V^r, and $N = nr$. The stratification is by the rank (observe $V^r = L(k^r, V)$). In case we have $r \leq n$ and $d < 2(n - r + 2)$ where d is the dimension of X, this leads to smooth subvarieties of codimension $n - r + 1$ ([107], [94], [98], [58]).

6.3.3

More generally, $G = GL(V) \times GL(W)$ acts on $L(V, W)$ and the universal rank stratification is invariant. We just come back to rank stratifications as introduced in 1.1 to which 6.2 applies whenever $\mathscr{H}om(E, F)$ is generated by global sections. Examples over $\mathbb{P}^3$ with E and F sums of line bundles arise in [32]. Let us observe that the main theorem in [56], giving many vector bundles globally generated over $\mathbb{P}^3$ leads to much more curves than suggested in ([56, p. 365]). More generally, this technique would yield many smooth subvarieties of dimension d in $\mathbb{P}^n$, for roughly $2d \leq n$, if we could prove that there are many globally generated vector bundles over $\mathbb{P}^n$ at least in higher rank.

6.3.4

Here $G = GL(V)$ acts on $S^2V, N = \dfrac{n(n+1)}{2}$. The stratification is again by the rank (cf. 5.1). This leads to hypersurfaces with a 3-codimensional signular locus of double points, provided the base dimension is at most 5 (cf. [24], [12]). Furthermore there is an extension of Porteous' formula available for this case [66], [104], [48], [49].

6.4 Second Generalization

Previous considerations extend in some sense to sheaves with tame singularities. In the rank two case over a 3-fold, suitable sheaves are reflexive sheaves of which the number of singular points is precisely the third Chern class. Starting from such a globally generated sheaf, the **175**
method of 6.3.1 leads to nonsingular curves [54], while the method of 6.3.4 leads to surfaces with nodes [60].

7 Jump Stratifications (cf. [21, § 1])

Many stratifications occurring in the theory of vector bundles or in the theory of subvarieties of projective spaces are what we define below as jump stratifications. In the present section, we explain how far rank stratification techiniques help in their study. In particular, we generalize Petri's map which, in good cases, describes the first order local behaviour of the jump stratification for semi-universal families.

Definition 7.1. *Let $\pi : X \to S$ be a projective morphism and $(E_\alpha)_{\alpha\in A}$ be a finite set of S-flat coherent sheaves over X. According to the semi-continuity theorem, the dimensions $h^i(E_{\alpha,X(s)})$ define upper semi-continuous integer-valued functions on S. All together, these functions define on S the jump stratification associated with $(E_\alpha)_{\alpha\in A}$. For the moment, the jump stratification is set-theoretic. An additional scheme structure may be defined, at least in the cases we are interested in. Also, observe that the jump stratification is not indexed for the moment; or, it is indexed by the huge set of integer valued functions on $I \times A$ where $I = \{0, \dots, n\}$ and n is the (maximal) dimension of the fibers of π. Indeed, for such a function g, we may set*

$$S^g = \left\{ s \in S \mid \forall i \quad \forall \alpha, h^i(E_{\alpha\ X(s)}) \geq g(i,\alpha) \right\}.$$

7.2 Examples

7.2.1 Special Divisors

This is the fundamental example: S is the Jacobian variety of isomorphism classes of line bundles of degree d over a smooth projective curve **176** C and X is $C \times S$. Finally, E is some Poincaré bundle over X (cf. e.g. [40], [37] [35], [27], [28], [103] and for an introduction [47]).

7.2.2 Postulation

S is a subvariety of the Hilbert scheme of $\mathbb{P}^n$, X is $S \times \mathbb{P}^n$; we denote by U the universal subscheme and by I_U its ideal sheaf. The considered sheaves are $I_U(l)$, $\mathscr{O}_U(l)$, l running in a big subset of $\mathbb{Z}$ (outside of which nothing happens) (cf. e.g. [32], [42], [43], [45], [52], [55], [59], [4], [6], [7], [23]).

7.2.3 Jump Loci

Let Y be a projective variety and $(\mathscr{F}_\alpha)_{\alpha \in A}$ a finite set of vector bundles over Y (for instance a set of twists $E(1)$ of a single bundle E). Let S be a subvariety in the Hilbert scheme of Y, with universal subscheme U and corresponding ideal sheaf I_U. We consider the jump stratification associated with the sheaves $\mathscr{F}_\alpha \boxtimes I_U$, $\mathscr{F}_\alpha \boxtimes \mathscr{O}_U$ (cf. [20], [21], [22], [25], [13]). This stratification is a fundamental tool in the classification of vector bundles over projective spaces [99], [105], [14], [15], [10], [11], [63].

7.2.4 Line Subbundles Over Curves

Let S, C, X, E be as in 7.2.1 and let F be a rank two vector bundle over C. Then the jump stratification associated with $E^\vee \boxtimes F$ distinguishes line subbundles of F (cf. [86], [79], [101], [46], [75], [95]). One of the obtained strata is the starting point for the classification of (semi-) stable rank two vector bundles over C (cf. [87], [88], [89]).

7.2.5 Normal Bundles

Let S be a subvariety of the Hilbert Scheme of smooth space curves and $X \subset S \times \mathbb{P}^3$ be the tautological curve. The normal bundle N is a rank two vector bundle over X. Among the twisted bundle $N(l) := N \boxtimes \mathcal{O}_{\mathbb{P}^3(\ell)}$, $N(-1)$ stands out as the only one with zero Euler-Poincare characteristic. Hence it will be the most sensitive to jump phenomena. **177**
So we are interested in the jump stratification associated with $N(-2)$ (cf. [36], [96], [29], [30], [34], [65], [31], [5], [33], [64]). One has also studied the restricted tangent bundle and other restricted standard bundles [106], [18], [69].

7.2.6 Stratification of Quot-Schemes

Let $(Y, Y, (1))$ be a projective variety, $\mathcal{F}$ a coherent sheaf over Y and H a (Hilbert) polynomial. This defines a Grot-scheme S of quotients of $\mathcal{F}$ with Hilbert polynomial H. Over $X = S \times Y$ we have the universal quotient, the universal subsheaf and their twists. This situation generalizes 7.2.2.

7.3 Generalization

We want to consider some among the above situations with parameters. For instance, in 7.2.1, we want to consider C as a point in the moduli of curves. The hope here is to understand in a better way the original stratification as the trace of a (well-understood) stratification on something bigger. Some trouble arises because in general there is no Poincare bundle over moduli see [87], [92], [76], [61], [81], [82]. Fortunately, in order to define jump stratifications, we only need sheaves up to line bundles over S. So we could define a pseudo-S-sheaf over X to be a section over S of the sheaf in the etale topology associated with the presheaf P for which $P(T)$ is the set of isomorphism classes of sheaves over X_T, modulo $\operatorname{Pic} T$. The obstruction for a pseudo-S-sheaf over X to come from a sheaf is in $H^2_{\text{ét}}(S, \mathcal{O}_S *)$. Now, on the one hand, jump-stratifications are clearly defined for pseudo S-sheaves and on the other, there exists a (unique) Poincaré (pseudo-S-sheaf on $S \times Y$

whenever S is a moduli of stable sheaves on Y. So the new picture is $E_\alpha \to X \to S \to M$ where E_α is a pseudo-S-sheaf and where M stands to remind us that we are interested not only in the stratification over S
178 but also in its restriction to the fibers of $S \to M$. Observe that we get a natural stratification on M by flattening the jump stratification on S.

7.4 Examples

7.4.1

Example 7.2.1 extends by letting C move in the moduli of curves (rigidified in such a way that the relative Jacobian exists). The corresponding stratification on the moduli M selects the hyperelliptic locus, the trigonal locus, and so on.

7.4.2

Example 7.2.3 extends by letting the sheaves $\mathscr{F}_\alpha$ move in their moduli. In this way, we get a stratification (by jumping loci) on moduli say of stable vector bundles (see an example of result concerning this stratification in [77, Prop. 5.2]).

7.4.3

Examples 7.2.4 extends by letting the vector bundle F move in some moduli. The corresponding stratification of the moduli is finer than by the degree of stability (cf. [75]).

7.4.4

Notations are those of 7.2.5. Let $J \to S$ be a relative Jacobian associated with $X \to S$ and P a Poincaré (pseudo-J-) bundle over $J \underset{S}{\times} X$. We consider the (pseudo-J-) sheaf $\mathscr{H}om(P, N)$ on $J \underset{S}{\times} X$. The corresponding stratification on $M := S$ is finer than by the degree of stability of N (cf. [97], [90]).

7.4.5

S is a moduli of stable vector bundles over a projective variety Y, X is $S \times Y$, and E is a Poincaré (pseudo-J-) sheaf on $S \times Y$. This is another generalization of 7.2.1 (cf. [11], [51], [53], [19], [73], [56], [22]).

7.5 Programme

In order to describe a jump stratification, the first task consists in discovering the generic value of the h^i's. This is the corresponding maximal rank problem. Here the general hope is that the generic value should be minimal among values suitable for the Riemann-Roch formula (cf. for instance [22]). The next step is to find a simpler way of indexing. The hope here is that very few among the h^i's should be relevant, preferably only two, related by Riemann-Roch. Observe here that among a family of twisted sheaves $E(l)$, the most sensitive to jump phenomena should be the ones for which the Hilbert polynomial $\chi(E(l))$ assumes its minimal absolute value. In any case, one should discover all the possible values of the h^i's. The final hope is to identify strata as strata of some rank stratification for which the techiniques reviewed above could help, the goal being, of course, to find degrees, rational equivalence classes, irreducible (or connected) components of the strata, and to describe the local shape. **179**

7.6 Global Tools

For the rest of the present section, we suppose that we are concerned with a single torsion-free sheaf E such that the restrictions $E|_{X(S)}$-are torsion-free too and that we know that, for any $s \in S$, and any $i \geq 2$, $h^i(E|_{X(s)})$ vanishes. Then $h^0(E|_{X(s)}) - h^1(E|_{X(s)})$ is a constant χ. Moreover the jump stratification is a rank stratification. Indeed, we can embed E in a vector bundle F with no higher direct images (i.e. $R^i\pi * F = 0$, $i > 0$) in such a way that the restricted morphisms $E|_{X(s)} \to F|_{X(s)}$ are injections too. Then the cokernel G of $E \to F$ is flat and has no higher direct images as well because, for $i > 1$, $R^i\pi_*E$ vanishes. By the Riemann-Roch formula, $h^0(F|_{X(s)})$ and $h^0(G|_{X(s)})$ are locally constant.

Thus, by the base change theorem, π_*F and π_*G are locally free and
the jump stratification fits with the rank stratification associated with $u :$
$\pi_*F \to \pi_*G$. Incidentally, our stratification acquires a scheme structure
180 (which according to the theory of Fitting ideals does not depend upon
the choice of $E \to F$). So the global tools of 1.3 apply. More precisely,
in case X and S are smooth, the rational equivalence classes of the strata
S^i, supposed to have the right dimension $i(|\chi| + i)$ where χ^1 is the Euler-characteristic of $E(s)$, are, modulo torsion given by an expression of the
form Δ_i $(\log(\pi_*(\mathrm{ch}\, E.\ \mathrm{td}\, X/S)))$ where *td* is the Todd class, ch is the
Chern character, log is a universal polynomial map from $A^*(S) \otimes \mathbb{Q}$ into
itself and Δ; another one, depending on the index of the stratum (cf.
[44], [16], [22]).

7.7 Petri's Morphisms

For the study of the local behavior, we still require the assumptions of 7.6 and suppose further that E is locally free and that $\pi : X \to S$ is trivial (say for simplicity). Then at any point $s \in S$, we have:

- the map $RKS(u, s) : T_sS \to L(H^0(E(s)), H^1(E(s)))$
- the deformation KS map:

$$DKS(E, s) : T_sS \to H^1(X(s), \mathrm{End}\, E(s))$$

Petri's comorphism:

$$PC(E(s)) : H^1(X(s), \mathrm{End}\, E(s)) \to L(H^0(E(s)), H^1E(s))) :$$

this is the natural map associated with $E(s) \otimes \mathrm{End}\, E(s) \to E(s)$. I call it comorphism because, in the theory of special divisors, Petri's map is the dual of the present map. As one would expect (up to sign!), we have:

Proposition. *$RKS(u, s) = -PC(E(s)) \circ DKS(E, s)$.*

Proof. Here we set $X = Y \times S$ and we may suppose that the bundle F introduced in 7.6 is of the form $\mathrm{pr}_1 * F'$. We may suppose $S =$

$\mathrm{Spec}(k[\epsilon]/\epsilon^2)$. So we have exact sequences:

$$0 \to E \to \mathrm{pr}_1^* F' \xrightarrow{q} G \to 0$$
$$0 \to H^0(E(O) \to H^0(F') \to H^0(G(O)) \xrightarrow{\alpha} H^1(E(O)) \to 0.$$

Over the generic point y of Y, we can choose a splitting: 181

$$F'(y) = E(O)(y) \oplus G(O)(y).$$

We know that the tangent space to $\mathrm{Quot}\, F$ corresponding to $G(O)$ is $\mathrm{Hom}(E(O), G(O))$ and that the image of the canonical tangent vector of S in this tangent space is a map $\varphi : E(O) \to G(O)$ such that over $\{y\} \times S$, E is the graph, in $(S \times E(O)(y)) \times G(O)(y) \subset S \times F')$, of $\epsilon\varphi$. Now let γ be a section of $E(O)$, and let σ be the corresponding constant section of $\pi_* F$. We want to compute $q(\sigma)$. Over the generic point y, $\sigma(y) + \epsilon\varphi(\sigma(y))$ is a section of $E(y)$, so that $q(\sigma(y) + \epsilon\varphi(\sigma(y))) = 0$. This implies that $q(\sigma) = -\epsilon\varphi(\sigma)$. Now $DKS(E, s)$ is known to be the image of φ under the natural map

$$\beta : \mathrm{Hom}(E(O), G(O)) \to \mathrm{Ext}^1(E(O), E(O))$$

and we conclude because for any section δ of $E(O)$, the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}(E(O), G(O)) \arrow[r, "\beta"] \arrow[d, "\circ\gamma"] & \mathrm{Ext}^1(E(O), E(O)) \arrow[d, "\gamma^*"] \\
\mathrm{Hom}(\mathscr{O}, G(O)) = H^0(G(O)) \arrow[r, "\alpha"] & H^1(E(O)) = \mathrm{Ext}^1(\mathscr{O}, E(O)).
\end{tikzcd}$$

□

Remarks. (a) Hence the surjectivity of the RKS-map follows if we know the surjectivity of the DKS-map *and* of Petri's comorphism.

(b) Similar considerations hold in case, instead of h^0, h^1, only the highest group h^n, h^{n-1} are not identically zero. Also, by duality, one can pass from one case to the other. 182

8 Applications

One of the main applications of the techniques reviewed above is to the study of space curves. In the first version of the present text, I devoted a section to it. Then my attention was drawn to [47] where this study is performed much more completely. That is why there is no section 8.1 below.

8.2 Ruled Surfaces

Let C be a curve of genus g and E be a stable rank two vector bundle over C, of degree d. Let J denote the Jacobian of C and P be a Poincaré bundle over $J \times C \rightarrow J$ of relative degree d'. Consider over $J \times C$ the bundle $\mathrm{Hom}(P, p_2^* E)$. The associated jump stratification on J selects classes of line bundles which inject (as sheaves) in E. Now a theorem of Nagata ([86], see also [101] [46] [75] [74]) says that whenever $d' \leq \dfrac{d-g+1}{2}$, such line bundles exist. This result could be reproved in the vein of [71]. Instead, we give the following sample.

Theorem. *If C and E are general enough, and $d-g$ is odd, then E as exactly $2g$ subbundles of degree $\dfrac{d-g+1}{2}$.*

Proof. We may suppose $d = g-1$ and set $\alpha = \left[\dfrac{g}{2}\right]$, $\alpha' = g - \alpha$ so that $\alpha \leq \alpha' \leq \alpha + 1$. We may suppose that Petri's condition holds for C. This implies that for any line bundle L over C of degree α or α', we have $h^0(C, L) \leq 1$. Now we choose two line bundles L_α and $L_{\alpha'}$ of degree α and α' and set $E_0 = L_\alpha \oplus L_{\alpha'}$. We consider the semi universal deformation of E_0: this is a vector bundle E over $S \times C$ where S is a smooth variety with base point O, such that the fiber $E(O)$ is E_O and the tangent space $T_O S$ is identified with $H^1(C, \mathrm{End}\, E_0)$ (cf. [72] or for a sketch of the algebraic proof [17]). Now, we denote by J the Jacobian of C, and by P a Poincaré bundle over $J \times C$ of relative degree 0, and we consider the bundle $\mathrm{Hom}(P, E)$ over $J \times S \times C$ as a family of bundles over C parametrized by $J \times S$. I claim that the associated jump stratification **183** is R_{g-1}-like along $J \times \{O\}$. Indeed, its $R.K.S$ map at a point $L \times \{O\}$, restricted to the subspace $\{O\} \times T_oS$ of the tangent space $T_LJ \times T_oS$ is the natural map

$$H^1(\operatorname{End} E_O) \to L(H^o(L^\vee \otimes E_O), H^1(L^\vee \otimes E_O)).$$

In order it prove that it is surjective, we prove that its four parts are surjective namely:

$$u_1 : H^1(\operatorname{End} L_\alpha) \to L(H^0(L^\vee \otimes L^q), H^1(L^\vee \otimes L_\alpha))$$
$$u_2 : H^1(\operatorname{End} L_{\alpha'}) \to L(H^0(L^\vee \otimes L_{\alpha'}), H^1(L^\vee \otimes L_{\alpha'}))$$

are surjective by Petri's assumption on C and

$$u_3 : H^1(\operatorname{Hom}(L_\alpha, L_{\alpha'})) \to L(H^0(L^\vee \otimes L_\alpha), H^1(L^\vee \otimes L_{\alpha'}))$$
$$u_4 : H^1(\operatorname{Hom}(L_{\alpha'}, L_\alpha)) \to L(H^0(L^\vee \otimes L_{\alpha'}), H^1(L^\vee \otimes L_\alpha))$$

are transposed in

$$u_3^* : H^0(L^\vee \otimes L_\alpha) \otimes H^0(\Omega \otimes L_{\alpha'}^\vee \otimes L) \to H^0(\Omega \otimes L_\alpha \otimes L_{\alpha'}^\vee)$$
$$u_4^* : H^0(L^\vee \otimes L_{\alpha'}) \otimes H^0(\Omega \otimes L_\alpha^\vee \otimes L) \to H^0(\Omega \otimes L_{\alpha'} \otimes L_\alpha^\vee)$$

which are injective because $h^0(L^\vee \otimes L_\alpha) \leq 1$, $h^0(L^\vee \otimes L_{\alpha'}) \leq 1$. Hence by 4.3, for s general in S, the jump stratification associated with $\operatorname{Hom}(P, E(s))$ is R_{g-1}-like. This means that $E(s)$ has a finite number n of subbundles of degree 0 and this number is given by Porteous' for- **184**
mula (cf. 7.3). Now we prove $n = 2^g$. It is proved in [71], although not explicitly stated, that if P' is a Poincaré bundle over $C \times J$ of relative degree $d > 2g - 2$ on the fibers of $C \times J \to C$, and of relative first Chern class 0 on the fibers of $C \times J \to C$, then the Chern class of $\mathfrak{p}_*P'$ is $\exp(-\theta) = 1 - \theta + \dfrac{\theta^2}{2} + \cdots$ modulo numerical equivalence, where θ is the polarization class. In order to compute the Chern class of $p_{2!}(E_O \otimes P'^\vee)$, we choose a very negative line bundle L_β so that the line bundle $L'_\beta := (\det E_O) \otimes L_\beta^\vee$ is very positive. One checks easily that E_O

is a generalization of $E_1 := L_\beta \oplus L_{\beta'}$, so that we compute Chern class of $p_{2!}(E_1 \otimes P'^\vee)$, which is

$$C(p_{2*}(L'_\beta \otimes P'^\vee))/C^\vee(p_{2*}(\Omega \otimes L_\beta \otimes P'))$$

and by the quoted result, the inverse Chern class is $\exp(2\theta)$ whose top component 2^g is the number we look for. □

Remark. This was known to Segre (cf. [108]). For a more general formulation of the problem solved above, see [109].

8.3 Cohomology of Instanton Bundles

The maximal rank problem for the moduli of rank two stable vector bundles over $\mathbb{P}^3$ is solved in [56], at least in one irreducible component of each moduli $M(c_1, c_2)$. The proof consists in looking at a non-locally free specialization of some simple bundles in order to reduce the problem to a suitable general position statement. We show how this reduction can be achieved using 3.3 instead of specialization (this was known to Hartshorne long ago):

Proposition. *Let* $a = 2$ *(resp.* $a = 3$*). Suppose that there exists a disjoint union* $Y \subset \mathbb{P}^3$ *of* r *lines (resp. conics) and a morphism* $\beta : \mathscr{O}(-a) \to \mathscr{O}_Y$ *such that for all* $n \in \mathbb{Z}$*, taking* $\alpha = (\beta, \rho)$*, the induced maps*

$$H^0(\alpha(n)) : H^0(\mathbb{P}^3, \mathscr{O}(n-a) \oplus \mathscr{O}(n)) \to H^0(Y, \mathscr{O}_Y(n)) \quad \textit{and}$$
$$H^0(\rho(n)) : H^0(\mathbb{P}^3, \mathscr{O}(n)) \to H^0(Y, \mathscr{O}_Y(n))$$

185 *are of maximal rank. Then there exists a rank two vector bundle* E *over* $\mathbb{P}^3$ *with natural cohomology and* $c_1(E) = 0$ *(resp* -1*),* $c_2(E) = r - 1$ *(resp.* $2r - 2$*).*

Proof. We choose a vector bundle E_O sitting in an extension

$$0 \to \mathscr{O}(1-a) \to E_O \to I_Y(1) \to 0.$$

Such an E_O exists and it corresponds to a smooth point in the moduli (cf. [51]). Hence, according to 3.3 and 7.7, it is enough to prove that $PC(E_O(l-1))$ reaches the maximal rank, for each l. We see that $H^1(E_O(l-1)) = H^1(I_Y(l-1))$. If this group vanishes there is nothing to prove; so, we may suppose that $H^0(I_Y(l-1))$ vanishes. Then we see that $H^0(E_O(l-1) = H^0(\mathscr{O}(l-a))$. We also see that
$\mathrm{Ext}^1(E_O, E_O) \to \mathrm{Ext}^1(E_O, I_Y(1))$ is surjective as well as
$\mathrm{Ext}^1(E_O, I_Y(1)) \to \mathrm{Ext}^1(\mathscr{O}(1-a), I_Y(1))$ and
$\mathrm{Hom}(\mathscr{O}(1-a), \mathscr{O}_Y(1)) \to \mathrm{Ext}^1(\mathscr{O}(1-a), I_Y(1))$. We choose β in $\mathrm{Ext}^1(E_O, E_O)$ having the same image in $\mathrm{Ext}^1(\mathscr{O}(1-a), I_Y(1))$ as β, and we check that the assumption on β implies that $PC(E_O(l-1))(\overline{\beta})$ has maximal rank. □

Remark. The general position statement in [56] is weaker since only the first series of maps is required to have maximal rank. However, the second series of maps is known to have maximal rank for a general 186
union of lines [55], and should not be too difficult to handle for a general union of $r \geq 3$. Moreover, this approach fits in a better way with [52] p. 109.

8.4 Unstable Planes

Let E be a vector bundle over $\mathbb{P}^n$, $n \geq 3$ with rank $E = 2$, $-1 \leq c_1(E) \leq 0$ and let $\chi = 2 + c_1 - c_2$. Then E is expected to have at least a $(|\chi| + 1)$-codimensional family of unstable planes (i.e. planes H for which $H^0(E_H) \neq 0$) provided a certain polynomial in c_2 (with coefficients depending on n, c_1, χ) does not vanish. This polynomial can be computed in any case, given time. We give an example corresponding to Barth's lecture.

Proposition. *Any rank two vector bundle over $\mathbb{P}^4$ with $c_1 = -1$, $c_2 = 4$ has at least a two-dimensional family of unstable planes.*

Proof. In case there exists a two-dimensional family of planes H with $H^2(E_H) \neq 0$, we are done because, by Serre-duality, such planes are

unstable. So we suppose that the family of such planes is at most one-dimensional. Let G' be the complement of this family in the grassmannian variety of planes in $\mathbb{P}^4$. Over G', the family of restrictions of E satisfies the conditions of 7.6. Hence it is enough to prove that the expected rational equivalence class is not zero in G'. Although straightforward, the computation seems quite tedious. Fortunately, we can avoid it thanks to [13]. Indeed, they prove that the family of unstable planes of the Horrocks-Mumford bundle F is non-empty and two-dimensional. In order to conclude, it is sufficient to prove that for any plane H, $H^2(F_H)$ vanishes. And this follows [64], because $F(3)$ is globally generated
187 outside 25 lines, and has a section with exactly 6 zeroes on each of these lines, so that, for any H, $F_H(3)$ has a section vanishing in codimension two. From the eact sequence

$$0 \to \mathcal{O}_H \to F_H(3) \to I_Y(5) \to 0$$

we deduce $H^0(F_H(-2)) = 0$ which, by Serre-duality, implies the desired result. □

Remark. It was announced during the Colloquium that someone (in USSR) has constructed a rank two vector bundle on ^{4} with $c_1 = -1$, $c_2 = 4$, which is not isomorphic to a Horrocks-Mumford bundle.

9 Jumping Points

Let Y be a projective non-singular variety, and E a vector bundle over Y (or $(E(l))_{l\in A}$ a set of twists of E). We may consider Y as a subvariety in its own Hilbert scheme either in the trivial way (simple points) or as a variety of non-reduced points (big points, see 9.1). According to 7.2.3, we get corresponding jump stratifications on Y itself. Jumping points in Y are points in the non-dense strata. While jumping lines appeared as a fundamental tool from the very beginning of the theory of vector bundles over projective spaces (see [99], [105], [14], [15]), jumping points were considered in a much more discrete way (see 9.2). The general hope here around is to describe (stable) vector bundles (and consequently their moduli) in terms of some of their jumping loci together

with possible additional data living there. The prototype is Barth's result [11, § 2.3.4] (See also [63, § 7.2, 7.3]). From this point of view, the current knowledge about $M_{\mathbb{P}^3}(-1,2)$ [57] and $M_{\mathbb{P}^3}(0,2)$ [52] is quite satisfactory. In case of more involved moduli, one has to study each **188**
stratum separately (see e.g. [8]). In this section, after some generalities, we look at some examples of loci of jumping simple points (9.2), then we solve the maximal rank problem for big points jumping with respect to rank two stable vector bundles on $\mathbb{P}^2$ with even first Chern class (9.3).

9.1 Generalities

Let Y, E be as above, and let d be the dimension of Y, and r be the rank of E. Choose an integer $s \geq 0$ and define an s-big-point in Y to be the subscheme associated with the $(s+1)$-th power of a maximal ideal sheaf. So, for $s = 0$, an s-big-point is just a (simple) point. For each s, if we set $N(s) = \binom{s+d}{s}$, we have an embedding $Y \to \mathrm{Hilb}^{N(s)}\, Y$, corresponding to s-big-points. The corresponding universal sub-scheme U_s in $Y \times Y$ is the s-th infinitesimal neighbourhood of the diagonal. We see that the only sensible groups are $H^0(E \boxtimes I_{U_s}(y))$ and $H^1(E \boxtimes I_{U_s}(y))$ and that the jump stratification is the rank stratification associated with

$$\mathrm{ev}_s : H^0(E) \otimes \mathscr{O}_Y \to J^s E$$

where $J^s E$ is the jet-bundle: $J^s E = \mathrm{pr}_{-1*}(E \otimes \mathscr{O} U_s)$; and ev_s is the corresponding evaluation.

So we are interested in the Chern polynomial $c_t(J^s E)$. From the exact sequences.

$$0 \to S^s\Omega \otimes E \to J^s E \to J^{s-1} E \to 0,$$

we get $c_t(J^s E) = \prod_{l=0}^{s} c_t(S^{-1}\Omega \otimes E)$, where $S^l\Omega$ is the l-th symmetric power of the cotangent bundle. Also we observe that sensitive s will achieve small values of $|h^\circ(E) - rN(s)|$.

9.2 Simple Points

189 The jump locus of simple points first occurred in the study of unstable rank two vector bundles over $\mathbb{P}^2$ (Grauert-Mulich [38]). For stable bundles, it appeared in the study of special moduli over $\mathbb{P}^3$, rather than over $\mathbb{P}^2$.

Example 9.2.1. $M_{\mathbb{P}^3}(0,2)$, cf. Hartshorne [52], see also [61]. Rank two stable bundles over $\mathbb{P}^3$ with $c_1 = 2$, $c_2 = 3$ have two sections whose dependency locus is a smooth quadric Q. The bundle and its other jumping loci (lines, planes) are described in terms of a certain pencil over this jumping locus Q.

Example 9.2.2. $M_{\mathbb{P}^3}(-1,2)$, cf. Hartshorne-Sols [57], see also [78], [83]. Rank two stable bundles over $\mathbb{P}^3$ with $c_1 = 1$, $c_2 = 2$ have (up to scalars) only one non-zero section whose zero scheme is a double line. This jumping locus plays the central role in the classification. For the similar study of $M_{\mathbb{P}^3}(-1,4)$, see [8].

Example 9.2.3. $M_{\mathbb{P}^2}(1,2)$. A rank two stable bundle E over $\mathbb{P}^2$ with $c_1 = 1$, $c_2 = 2$ has two sections whose dependency locus is a line L (the unique jumping line). The image of $H^0(E)$ in $E|_L$ is a line bundle of degree two over L so that $H^0(E)$ defines a pencil of degree two. The two double points in this pencil characterize E. The second kind of jumping lines are the lines through one among these two points (see [76, p. 236] and [63, p. 256]).

Example 9.2.4. $M_{\mathbb{P}^2}(0,4)$. A general rank two stable bundle E over $\mathbb{P}^2$ with $c_1 = 2$, $c_2 = 5$ has two sections whose dependency locus is a smooth conic, where they define a pencil of degree 5. Jumping lines are lines cutting the conic in two points of a divisor in the pencil.

Example 9.2.5. $M_{\mathbb{P}^2}(0,5)$. A general rank two stable bundle E over $\mathbb{P}^2$ with $c_1 = 2$, $c_2 = 6$ has (up to scalars) only one non-zero section, vanishing at six points (see [11, p. 84]).

190 **Remark.** The jumping loci considered in 9.2.4, 9.2.5 lead to a nice description of an open dense subset in the moduli which generalizes to cases where the Euler-Poincaré characteristic is two or one.

9.3 Jumping Big-Points

9.3.1 Line Bundles over Curves

Let L be a general line bundle of degree $d \geq g$ over a curve C of genus g. We have a morphism from C into its Jacobian of degree $g-1$ through $x \mapsto [L(-(d+1-g)x)]$. The intersection with the θ-divisor gives $g(d+1-g)^2$ points. This is the jump locus for $(d+1-g)$-big points.

9.3.2 Rank Two Vector Bundles Over $\mathbb{P}^2$, With Even c_1

What we saw in 9.2.4, 9.2.5 generalizes to the case of big points as soon as the Euler-Poincaré characteristic is of the form s^2+s-1, s^2+s or s^2+s+1. To see that the corresponding jumping loci are actually non-trivial, we have to solve a maximal rank problem, namely

Theorem. *Let c_1, c_2 be integers with c_1 even and $c^2 - 4c_2 \leq -8$. Then, for the general rank two stable vector bundle E over $\mathbb{P}^2$ with Chern classes c_1, c_2, the evaluations*

$$\mathrm{ev}_s : H^0(E) \otimes \mathscr{O} \to J^s E$$

have maximal rank.

Proof. First, remember [99], [80] that the required condition on c_2 is necessary and sufficient for the moduli $M_{\mathbb{P}^2}(c_1, c_2)$ of rank two stable **191** vector bundles with these Chern classes to be non-empty, and that this moduli is irreducible. Next remember that an open dense subset in the moduli consists of classes of bundles with "natural" cohomology (cf. eg. [19]). Now, observe that if ev_s is injective, so is ev_t for any $t > s$. Also, observe that the property for ev_s^- to have maximal rank is open in flat families of torsion free sheaves $\mathscr{F}_t$ with constant $h^0(\mathscr{F}_t)$. So it is sufficient to prove that for any s, there exists a bundle E in $M_{\mathbb{P}^2}(c_1, c_2)$ with natural cohomology and ev_s of maximal rank. In fact, it is sufficient to produce a torsion-free sheaf $\mathscr{F}$ satisfying:

(i) $\mathrm{ev}_s : H^0(\mathscr{F}) \otimes \mathscr{O} \to J^s \mathscr{F}$ has maximal rank

(ii) $H^1(\mathscr{F}) = 0$ (or $H^0(\mathscr{F}) = 0$).

(iii) $\mathscr{F}$ deforms to a rank two stable bundle with Chern classes c_1, c_2.

The Euler-Poincaré characteristic for our bundles is

$$\chi = 2^{\frac{1}{2}}c_1(c_1 + 3) + 2 - c_2.$$

So we may suppose $c_1 > 0$, $c_2 < \frac{1}{2}c_1(c_1 + 3) + 2$, otherwise, for the general E, $H^0(E)$ vanishes.

First, we let $c_1 = 2t$, and set $c_2 = t^2 + 2d + \epsilon$ with $0 \leq \epsilon \leq 1$. We know $d \geq 0$. We choose $\mathscr{F} = I_X(t) \oplus I_Y(t)$ where X is a general set of d points in $\mathbb{P}^2$ and Y a general set of $d + \epsilon$ points in $\mathbb{P}^2$. So we have $0 \leq \chi(I_X(t)) - \chi(I_Y(t)) \leq 1$ and $\chi(I_X(t)) + \chi(I_Y(t)) = \chi$. So for X and Y general enough, $H^1(\mathscr{F})$ vanishes. Now (i) and (ii) follow from the lemmas: ☐

Lemma 1. *For a sufficiently general subset Z in* $\mathrm{Hilb}^d\,\mathbb{P}^2$*,*

$$\mathrm{ev}_s : H^0(I_Z(t)) \otimes \mathscr{O} - J^s(I_Z(t))$$

has maximal rank.

192 **Lemma 2.** *Let t, d_1, d_2 be integers satisfying*

$$\frac{(t-2)(t-1)}{2} < d_1 \leq d_2 \leq \frac{t(t+1)}{2}$$

and for $i = 1, 2$ let Y_i be in $\mathrm{Hilb}^{d\ i}\mathbb{P}^2$*. Then $I_{Y_1} \oplus I_{Y_2}$ deforms to a stable vector bundle.*

Proof of Lemma 1. It is enough to prove that the general union T of one s-big point and d points has maximal rank (i.e. the restrictions

$$H^0(\mathbb{P}^2, \mathscr{O}(l)) \to H^0(\mathbb{P}^2, \mathscr{O}_T(l))$$

have maximal rank). Adding or deleting points, it is enough to treat the case where $d = 0$ (evident) or

$$\frac{(l+1)(l+2)}{2} = \frac{s(s+1)}{2} + d.$$

Observe that $s+d \geq l+1$. Thus we may choose $l+1-s$ points on a line meeting the s-big point and we proceed by induction in the usual way (cf. [55], [59]).

Proof of Lemma 2. We may suppose Y_1 and Y_2 smooth and general so that for any point p in Y_i, $H^0(I_{Y_i-\{\mathfrak{p}\}}(t-3)) = 0$. Thus, there exist exact sequences (cf. [39], [19]):

$$0 \to \mathscr{O} \to E_i \to I_{Y_i}(t) \to 0$$

where E_1, E_2 are locally free, Furthermore, we have

$$H^1(E_i(-1)) = H^1(I_{Y_i}(t-1)) = 0$$
$$\text{and} \quad H^2(E_i(-2)) = H^2(I_{Y_i}(t-2)) = 0.$$

Hence, by Castelnuovo's criterion ([85, p. 99]), E_i is generated by global sections. Now we have an exact sequence 193

$$0 \to \mathscr{O} \oplus \mathscr{O} \xrightarrow{u_0} E_1 \oplus E_2 \to I_{Y_1}(t) \oplus I_{Y_2}(t) \to 0.$$

Moving u_0 in the vector space $L(\mathscr{O}\oplus\mathscr{O}, E_1\oplus E_2)$ we get a deformation of $I_{Y_1}(t)\oplus I_{Y_2}(t)$ which is flat, because the Hilbert polynomial is constant (cf. [50] III. 9.9 and its proof). Now by 6.2, for general u, Coker u is locally free. So $I_{Y_1} \oplus I_{Y_2}$ deforms to a vector bundle as well and this vector bundle, by semicontinuity has no non-zero section hence is stable.

Remarks. A similar statement and proof should hold in case c_1 is odd and also over $\mathbb{P}^3$ at least for the so-called general instanton bundle.

References

[1] E. Arbarello - M. Cornalba : Su una congettura di Petri. *Comment. Math. Helvetici* 56 (1981), 1-38.

[2] E. Arbarello – M. Cornalba : A few remarks about variety of irreducible plane curves of given degree and genus, *Ann. Scient. Ec. Norm. Sup.,* 16 (1983) 467-488.

[3] M.F. Atiyah - R. Bott : The Yang-Mills equations over Riemann surfaces, *Phil. Trans. R. Soc. Lond.* A 308, (1982) 523-615.

194 [4] E. Ballico - Ph. Ellia : Generic curves of small genus in $\mathbb{P}^3$ are of maximal rank, *Math. Ann.,* 264 (1983) 211-225.

[5] E. Ballico - Ph. Ellia : Some more examples of curves in $\mathbb{P}^3$ with stable normal bundle, J. Reine Angew. Math., 350 (1984), 87-93.

[6] E. Ballico - Ph. Ellia : Sur la postulation des courbes de $\mathbb{P}^n$ et de leurs projections, *C.R.A.S.* Paris, 299 (1984), 237-240.

[7] E. Ballico - Ph. Ellia : The Maximal rank conjecture for non-special curves in P^3, *Invent. Math.,* 79 (1985) 541-555.

[8] C. Banica - N. Manolache : Rank 2 stable vector bundles on $\mathbb{P}^3(C)$ with Chern classes $c_1 = 1$, $c_2 = 4$, Preprint, Bucarest (1983).

[9] W. Barth : Submanifolds of low codimension in projective space, *Proceedings of the International Congress of Mathematicians,* Vancouver, 1974, 409-413.

[10] W. Barth : Some properties of stable rank-2 vector bundles on $\mathbb{P}_n$, *Math. Ann.,* 226, 125-150 (1977).

[11] W. Barth : Moduli of vector Bundles on the projective plane. *Invent. Math.,* 42, 63-91 (1977).

[12] W. Barth : Counting singularities of quadratic forms on vector bundles, in *Vector Bundles and Differential Equations,* Prog. in Math., 7 Birkhauser, Boston 1980, 1-19.

195 [13] W. Barth - K. Hulek - R. Moore : Shioda's modular surface $S(5)$ and the Horrocks-Mumford bundle. These Proceedings, 1984.

[14] W. Barth - A. Van de Ven : A decomposability criterion for algebraic 2-bundles in projective spaces, *Invent. Math.,* 25 (1974), 91-106.

[15] W. Barth - A. Van de Van : On the geometry in codimension 2 of Grassmann manifolds, *Springer L. N. in Maths.* 412, Springer-Verlag (1974), 1-35.

[16] J. Bertin - I. Sols : Quelques formules énumératives concernant les fibres vectoriels sur $\mathbb{P}^r$, *C.R.A.S.,* Paris, 294 (1982), 197-201.

[17] J. Bingener : Appendix in [21].

[18] A. Bruguieres : Le schéma des morphismes d'une courbe elliptique dans une grassmannienne, These 3e cycle, Paris VII (1984).

[19] J. Brun : Les fibrés de rang dues sur $\mathbb{P}_2$ et leurs sections, *Bull. Soc. Math. France,* 108 (1980) 457-473.

[20] J. Brun - A. Hirschowitz : Droites de seut des fibrés stable de rang élevé sur $\mathbb{P}_2$, *Math. Zeit,* 181 (1982), 171-178.

[21] Variétés des droites sauteuses du fibré instanton général, *Compos. Math.,* 53 (1984), 325-336.

[22] J. Brun - A. Hirschowitz : Restrictions planes du fibre instanton general I, II, In preparation. **196**

[23] J. Brun - A. Hirschowitz : Sur la stratification par la postulation du schema de Hilbert de $\mathbb{P}^2$, Preprint, Nice (1984).

[24] F. Catanese : Babbage's conjecture, contact of surfaces, symmetrical determinant varieties and applications, *Invent. Math.,* 63 (1981), 433-465.

[25] M.C. Chang : A bound on the order of jumping lines, *Math. Ann.,* 262 (1983), 511-516.

[26] L. Ein : Some stable vector bundles on $\mathbb{P}^4$ and $\mathbb{P}^5$, *J. Reine Angew. Math.,* 337 (1982), 142-153.

[27] D. Eisenbud - J. Harris : A simpler proof of the Gieseker-Petri theorem on special divisors, *Invent. Math.,* 74 (1983), 269-280.

[28] D. Eisenbud - J. Harris : Divisors on general curves and cuspidal rational curves, *Invent. Math.,* 74 (1983), 371-418.

[29] D. Eisenbud - A. Van de Ven : On the normal bundle of smooth rational space curves, *Math. Ann.,* 256 (1981), 453-463.

[30] D. Eisenbud - A. Van de Ven : The irreducibility of the space of rational space curves with given degree and normal bundle, *Invent. Math.,* 67 (1982), 89-100.

[31] Ph. Ellia : Examples de courbes de $\mathbb{P}^3$ a fibre normal semi-stable, stable, *Math. Ann.,* 264 (1983) 389-396.

197 [32] G. Ellingsrud : Sur le schéma de Hilbert des variétiés de codimension 2 dans $\mathbb{P}^e$ á cône de Cohen-Macaulay, *Ann. Sci. Ecole Norm. Sup. Paris,* 8 (1975), 423-432.

[33] G. Ellingsrud - A. Hirschowitz : Sur le fibré normal des courbes gauches, *C.R.A.S.,* Paris, 299 (1984), 245-248.

[34] G. Ellingsrud - D. Laksov : The normal bundle of elliptic space curves of degree five, in *18th Scand. Congress of Math. Proc.* 1980, Prog. in Math., *11* Birkhäuser, Boston, 1981, 258-287.

[35] W. Fulton - R. Lazarsfeld : On the connectedness of degeneracy locus and special divisors, *Acta Math.,* 146 (1981), 271-283.

[36] F. Ghione - G. Sachiero : Normal bundles of rational curves, *Manuscr. Math.,* 33 (1980), 111-128.

[37] D. Gieseker : Special divisors and stable curves, *Invent. Math.,* 66 (1982), 251-275.

[38] H. Grauert - G. Mülich : Vektorbundel vom Rang 2 über dem n-dimensionalen komplex-projektiven Raum, *Manuscr. Math.,* 16 (1975), 75-100 and 18 (1976), 213-214.

[39] P.A. Griffiths - J. Harris : Residues and zero-cycles on algebraic varieties, *Ann. of Math.,* 108 (1978), 461-505.

[40] P.A. Griffiths - J. Harris : On the variety of special linear systems **198**
on a general algebraic curve, *Duke Math. Journal,* 47 (1980), 233-270.

[41] A. Grothendieck : Techniques de construction en géométrie algébrique: Les schémas de Hilbert, *Sém. Bourbaki,* Exposé 221 (1960-61).

[42] L. Gruson - Ch. Peskine : Genre de courbes de l'espace projectif I, in *Algebraic Geometry,* Tromsϕ 1977, Springer Lecture Notes in Math. 687 (1978), 31-59.

[43] L. Gruson - Ch. Peskine : Genre des courbes de l'espace projectif II. *Ann. Sci. Ecole Norm. Sup. Paris,* 15 (1982), 401-418.

[44] L. Gruson - Ch. Peskine : Courbes de l'espace projectif : varietes de secantes, in *Enumerative Geometry and Classical Algebraic Geometry.* Progress in Math., 24, Birkhauser, Boston (1982), 1-31.

[45] L. Gruson - R. Lazarsfeld - Ch. Peskine : On a theorem of Castelnuovo and the equations defining space curves, *Invent. Math.,* 72 (1983), 491-506.

[46] R.C. Gunning : On the divisor order of vector bundles of rank 2 on a Riemann surface, *Bull. Inst. Math. Acad. Sinica* 6, (1978) 295-302.

[47] J. Harris : Curves in projective space, *Sém. Math. Sup.* (1982), Presses Univ. Montréal.

[48] J. Harris - L. Tu : Chern numbers of kernel and cokernel bundles, *Invent. Math.,* 75 (1984), 467-475.

[49] J. Harris - L. Tu : On symmetric and skew-symmetric determinan- **199**
tal varieties, To appear in Topology.

[50] R. Hartshorne : *Algebraic Geometry,* Graduate Texts in Math. 52, Springer-Verlag, New York (1977), XVI+496 pp.

[51] R. Hartshorne : Stable vector bundles of rank 2 on $\mathbb{P}^3$, *Math. Ann.*, 238 (1978), 229-280.

[52] R. Hartshorne : On the classification of algebraic space curves, in *Vector bundles and Differential equations,* Prog. in Math. 7, Birkhauser, Boston, 1980, 83-112.

[53] R. Hartshorne : Stable reflexive sheaves, *Math. Ann.*, 254 (1980), 121-176.

[54] R. Hartshorne - A. Hirschowitz : Nouvelles courbes de bon genre via la cohomologie des faisceaux réflexifs (In preparation).

[55] R. Hartshorne - A. Hirschowitz : Droites en position générale dans l'espace projectif, in *Algebraic Geometry,* Proc. La Rabida, 1981, Springer L. N. in Math. 961 (1982), 169-189.

[56] R. Hartshorne - A. Hirschowitz : Cohomology of a general instanton bundle, *Ann. Scient. Ec. Norm Sup.,* 4e série, t. 15 (1982), 365-390.

[57] R. Hartshorne - I. Sols : Stable rank 2 vector bundles on $\mathbb{P}^3$ with $c_1 = -1, c_2 = 2$, *J. Reine Angew. Math.,* 325 (1981), 145-152.

200 [58] R. Hernandez - I Sols : On a family of rank 3 bundles on $\mathrm{Gr}(1,3)$, Preprint, Madrid (1983).

[59] A. Hirschowitz : Sur la postulation générique des courbes rationnelles, *Acta Math.,* 146 (1981) 209-230.

[60] A. Hirschowitz - R. Marlin : Nouvelles surfaces a noeuds dans $\mathbb{P}_3$, *Math. Ann.,* 267 (1984), 83-89.

[61] A. Hirschowitz - M.S. Narasimhan : Fibrés de 't Hooft spéciaux et applications, in *Enumerative Geometry and Classical Algebraic Geometry,* Prog. in Math., 24, Birkhauser, Boston (1982), 143-164.

[62] G. Horrocks - D. Mumford : A rank two vector bundle on $\mathbb{P}^4$ with 15,000 symmetries, *Topology,* 12 (1973), 63-81.

[63] K. Hulek : Stable rank 2 vector bundles on $\mathbb{P}_2$ with c_1 odd, *Math. Ann.,* 242 (1979), 241-266.

[64] K. Hulek : Projective geometry of elliptic curves, Preprint, Erlangen (1983)

[65] K. Hulek - G. Sacchiero : On the normal bundle of elliptic space curves, *Archiv der Math.,* 40 (1983), 61-68.

[66] T. Jozefiak - A. Lascoux - P. Pragacz : Classes of determinantal varieties associated with symmetric and antisymmetric matrices, Quoted in [12].

[67] G. Kempf : On the collapsing of homogeneous bundles, *Invent. Math.,* 37 (1967), 229-239. **201**

[68] G. Kempf - D. Laksov : The determinantal formula of Schubert calculus, *Acta Math.,* 132 (1973), 153-162.

[69] F.C. Kirwan : On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles, Preprint, Harvard (1984).

[70] S.L. Kleiman : Geometry on Grassmannians and applications to splitting bundles and smoothing cycles, *Publ. Math. IHES,* 36 (1969), 281-298.

[71] S.L. Kleiman - D. Laksov : Another proof of the existence of special divisors, *Acta Math.,* 132 (1974), 163-176.

[72] K. Kodaira - D.C. Spencer : On deformations of complex analytic structures I, II, *Ann. of Math.,* 67 (1957), 328-401, 403-465.

[73] H. Lange : Invertierbare Untergarben maximalen Grades von Rang-2 Vektor-bündeln auf der projektiven Ebene, *Archiv der Math.,* 34 (1980), 313-321.

[74] H. Lange : Higher secant varieties of curves and the theorem of Nagata on ruled surfaces, *Manuscr. Math.,* 47 (1984), 263-269.

202 [75] H. Lange - M.S. Narasimhan : Maximal subbundles of rank two vector bundles on curves, *Math. Ann.,* 266 (1983), 55-72.

[76] J. Le Potier : Fibrés stables de rang 2 sur $\mathbb{P}_2(\mathbb{C})$, *Math. Ann.* 241 (1979), 217-256.

[77] J. Le Potier : Stabilite et amplitude sur $\mathbb{P}^2(\mathbb{C})$, in *Vector Bundles and Differential Equations,* Prog. in Math. 7, Birkhauser, Boston, 1980, 145-182.

[78] N. Manolache : Rank 2 stable vector bundles on $\mathbb{P}^3$ with Chern classes $c_1 = -1$, $c_2 = 2$, Rev. Roum. Math. Pures et Appl., 26 $n°$ 9 (1981), 1203-1209.

[79] M. Maruyama : On classification of ruled surfaces, *Lectures in Math., Kyoto Univ.,* 3 Tokyo (1970).

[80] M. Maruyama : On a family of algebraic vector bundles, *Conference in honour of Y. Akizuki,* Kinokuniya, Tokyo, 1973.

[81] N. Mestrano : Poincare bundles for projective surfaces, *Annales Inst. Fourier* (To appear).

[82] N. Mestrano - S. Ramanan : Poincaré bundles for families of curves, *J. Reine Angew. Math* (To appear).

[83] R. Moore - R. Wardelmann : PGL(4) acts transitively on $M(-1, 2)$, J. Reine Angew. Math., 364 (1984), 48-53.

203 [84] G. Mülich - Familien holomorpher Vektorbündel über der projektiven Geraden und unzerlegbare holomorphe 2-Bündel über der projektiven Ebene, Dissertation, Göttingen (1974).

[85] D. Mumford : Lectures on curves on an algebraic surface, *Ann. of Math. Studies* 59, Princeton (1966).

[86] M. Nagata : On self-intersection number of vector bundles of rank 2 on a Riemann surface, *Nagoya Math. J.,* 37 (1970), 191-196.

[87] M.S. Narasimhan - S. Ramanan : Vector bundles on curves, Proc. of the International Bombay Colloq. on *Algebraic Geometry,* Oxford (1969) 335-346.

[88] M.S. Narasimhan - S. Ramanan : Moduli of vector bundles on a compact Riemann surface, *Ann. of Math.,* 89 (1969), 14-51.

[89] M.S. Narasimhan - S. Ramanan : 2θ-linear systems on Abelian varieties, These Proceedings, 1984.

[90] P.E. Newstead : A space curve whose normal bundle is stable, *J. London Math. Soc.,* (2) 28 (1983), 428-434.

[91] I.R. Porteous : Simple singularities of maps, *Springer Lecture Notes in Math.,* 192 (1971), 286-307.

[92] S. Ramanan : The moduli spaces of vector bundles over an algebraic curve, *Math. Ann.,* 200 (1973), 69-84.

[93] P. Rao : Liaison among curves in $\mathbb{P}^3$, *Invent. Math.,* 50 (1979), 205-217. **204**

[94] P. Rao : Liaison equivalence classes, *Math. Ann.,* 258 (1981), 169-173.

[95] M. Raynaud : Sections des fibrés vectoriels sur les courbes, Bull. Soc. Math. France, 110 (1982).

[96] G. Sacchiero : On the varieties parametrizing rational space curves with fixed normal bundle, Preprint Series, Oslo, 1981.

[97] : Exemple de courbes de $\mathbb{P}^3$ de fibré normal stable, *Comm. Algebra,* 11 (1983), 2115-2121.

[98] M. Schneider : Einschränkung stabiler Vektorraum-bündel vom Rang 3 auf Hyperebenen des projektiven Raumes, *J. Reine Angew. Math.,* 323 (1981), 177-192.

[99] R.L.E. Schwarzenberger : Vector bundles on the projective plane, *Proc. London Math. Soc.,* (3) 11 (1961), 623-640.

[100] G. Sernesi : On the existence of certain families of curves, *Invent. Math.*, 75 (1984), 25-57.

[101] U. Stuhler : Unterbündel maximalen Grades von Vektorbündeln auf algebraischen Kurven, *Manuscr. Math.*, 27 (1979), 313-321.

205 [102] T. Suwa : Stratification of local moduli spaces of Hirzebruch manifolds, In *Rice Univ. Studies*, 59 (1973), 129-146.

[103] M. Teixidor : For which Jacobi varieties is Sing θ reducible? Preprint, Barcelona, 1983.

[104] L. Tu. : Variation of Hodge structure and the local Torelli Problem, Thesis, Harvard (1979).

[105] A. Van de Ven : On uniform vector bundles, *Math. Ann.*, 195 (1972), 245-248.

[106] J.L. Verdier : Two dimensional σ-Models and Harmonic Maps from S^2 to S^{2n}, in Group Theoretical Methods in Physics, *Lecture Notes in Physics*, n° 180, Springer-Verlag.

[107] H. Vogelaar : Constructing vector bundles from codimension two subvarieties, Ph. D. Thesis, Leiden, 1979.

[108] F. Ghione : Quelques resultats de Corrado Segre sur les surfaces reglées, *Math. Ann.*, 255 (1981), 77-95.

[109] F. Ghione : Un probleme du type Brill Noether pour les fibres vectoriels, in *Algebraic Geometry, – Open problems, Proceedings Ravello* 1982, Springer Lecture Notes in Math., 997, 1983, 197-209.

Department of Mathematics
University of Nice
Parc Valrose
F–06034 NICE CEDEX

Vector Bundles on the Punctured Spectrum of a Local Ring II

By G. Horrocks

GIVEN A SCHEME and a divisor what are the obstructions to extending a bundle supported by the divisor to a bundle on the ambient scheme? Schwarzenberger's integrality conditions [5] give congruences for the Chern classes when the divisor is a hyperplane in $\mathbb{P}^{d+1}$, and in [3] Atiyah and Rees obtain an independent mod 2 condition on the dimensions of the holomorphic cohomology spaces when the divisor has odd dimension and the bundle is self-dual in a suitable sense. **207**

In the present article, the ambient variety Y is the punctured spectrum of a local ring and the divisor X belongs to a regular element. There are no additive obstructions to extending a bundle from X to Y. So consider self-dual bundles on X and whether they have self-dual extensions. When X has odd Krull dimension, there are again no additive obstructions but for even dimension, the length mod 2 of the middle cohomology group of the bundle is the unique additive obstruction to extendibility. For the complex field this obstruction is also a topological invariant and may be identified with an element of $K\widetilde{S}$ or $K\widetilde{O}$ depending on the mod 4 residue class of the dimension. For arbitrary residue fields of the local ring, the obstruction is invariant for self-dual algebraic equivalence. More generally, it is invariant for confluence of bundles.

Finally there are non-additive obstructions to extendibility for bundles which need not be self-dual, for example, the k-th exterior power of a bundle with rank $2k$ is self-dual and for some of these the obstruction to self-dual extendibility is non-zero.

1 No additive obstructions

208 Let B be a regular local ring of dimension $d + 2 \geq 4$, $\mathfrak{n}$ be its maximal ideal, $Y = \operatorname{Spec} B - \{\mathfrak{n}\}$, x be an element of $\mathfrak{n} - \mathfrak{n}^2$, $A = B/xB$, $\mathfrak{m} = \mathfrak{n}/xB$, and X the divisor of x in Y. So $\dim X = d$.

Complexes are cochain complexes indexed by $\mathbb{Z}$ and their differentials are indexed by the degree of their source.

Identify vector bundles on Y with reflexive B-modules locally free on Y, and call them Y-bundles. An X-bundle E extends to a Y-bundle F if E is isomorphic to the A-bidual $(F/xF)^{**}$. There are no additive obstructions to extendibility with values in an abelian group because of:

Theorem 1.1. *Let E be an X-bundle. Then there exists an extendible X-bundle E' such that $E \oplus E'$ is extendible.*

Proof. In the equicharacteristic case, we may assume A complete ([6, § 8]) and $B = A[[x]]$. Let V be the complex with non-zero components $V^{-1} = B$, $V^{\circ} = B$ and differential ∂ multiplication by x. Take the dual P of an A-projective resolution of E^* as in [6] and form the complex $(B \underset{A}{\otimes} P) \otimes V$. Then the module of cycles of degree zero extends $E \oplus \operatorname{Ker}(P^1 \to P^2)$, and the theorem follows by induction on the projective dimension of E^*.

In the general case, take $P^i (i \geq 0)$ as before and in lower degrees take it to be an A-projective resolution of E. Let $Q \to P$ be a homomorphism of a B-projective complex onto P inducing isomorphisms of cohomology groups. Take the tensor product $\otimes_B A$ in the derived category of B-complexes to obtain a morphism $Q \underset{B}{\otimes} A \to P \underset{B}{\otimes} V$ inducing isomorphisms of cohomology. So $E \oplus \operatorname{Ker}(P^1 \to P^2)$ extends after adding a free summand and the theorem follows by induction on projective dimension. □

2 Self-dual bundles

209 Self-dual bundles are assumed to have a given pairing. This is described by an isomorphism between the bundle and its dual. A self-dual X-

bundle E is said to be extendible if it has a Y-bundle extension F and its pairing $E \cong E^*$ extends to a pairing $F \cong F^*$.

Theorem 2.1. *Suppose that d is odd and let E be a self-dual X-bundle. Then there exists an extendible self-dual X-bundle $\widehat{E}$ such that $E \oplus \widehat{E}$ extends as a self-dual bundle.*

Proof. First construct a self-dual A-projective complex J such that:

$$H^\circ(J) \cong E; \;\; H^i(J) = 0 (i < 0); \;\; J^i = 0 \text{ if } |i| > d/2.$$

The construction starts from the complex P in the proof of 1.1. Kill the cohomology of P in dimensions less than $d/2$ to obtain a complex Q. Lift the isomorphisms in dimensions greater than $d/2$ to a homomorphism of P into Q and add free modules to P so that the homomorphism is surjective. Then J is its kernel.

Since J is self-dual, identify $J^{-i} = (J^i)^* (i < 0)$ and $(\partial^{-i-1} = \partial^i)^* (i > 0)$. The pairing determines an isomorphism $q : (J^\circ)^* \cong J^\circ$. The differential ∂^{-1} is $q(\partial^\circ)^*$.

In the equicharacteristic case put $K = J \underset{A}{\otimes} B$, and denote the induced differential and pairing by ∂, q also. Define a self-dual complex L with differential d by

$$L^\circ = K^{-1} \oplus K^\circ \oplus K^1, L^i = K^i \oplus K^{i+1} (i > 0), L^{-i} = (L^i)^* (i > 0),$$

$$d^\circ(a, b, c) = (\partial^\circ b + xc, \partial^1 c), d^i(a, b) = (\partial^i a + xb, \partial^{i+1} b)(i > 0),$$

$$d^{-i-1} = (d^i)^* (i > 0), r(a, b, c) = (c, qb, a) \text{ and } d^{-1} = r(d^\circ)^*.$$

Then $H^\circ(L)$ is a self-dual Y-bundle extending $E \oplus E' \oplus (E')^*$ where $E = \operatorname{Ker}(J^1 \to J^2)$.

In the general case, modify this procedure as in the proof of 1.1. **210**
First let J^+ be the complex obtained from J by replacing J^i by zero if $i < 0$. Then construct a complex of free B-modules Q^+ and an epimorphism $Q^+ \to j^+$ inducing isomorphisms of cohomology in degrees greater than zero. Let $s : (Q^\circ)^* \cong Q^\circ$ left $q : (J^\circ)^* \cong J^\circ$. Let δ be the differential of Q^+. Define a self-dual complex L by

$$L^i = Q^i, L^{-i} = (L^i)^* (i > 0), \partial^i = \delta^i \partial^{-i-1} = (\partial^i)^* i > 0,$$

$$L^\circ = Q^\circ \oplus Q^\circ, \partial^\circ(a,b) = \delta^\circ a - \delta^\circ b, \partial^{-1}(a) = (s(\delta^\circ)^* a, s(\delta^\circ)^* a).$$

Then $H^\circ(L)$ extends $E \oplus E' \oplus (E')^* \oplus$ free. So the proof follows from 1.1.

Now suppose that $d = 2r$ is even. There is an additive obstruction to extendibility for self-dual bundles. For suppose F extends E. The exact cohomology sequence

$$H^r(F) \to H^r(F) \to H^r(E) \to H^{r+1}(F) \to H^{r+1}(F)$$

and Serre-Grothendieck duality show that $H^r(E)$ has even length as an artinian A-module. For any self-dual X-bundle E, define the obstruction $\mu(E)$ to be the length of $H^r(E)$mod 2. □

Theorem 2.2. *Suppose that d is even and the residue field k is algebraically closed. Let E be a self-dual bundle with $\mu(E) = 0$. Then there exists an extendible self-dual bundle $\widehat{E}$ such that $E \oplus \widehat{E}$ extends as a self-dual bundle.*

Proof. Since k is algebraically closed, $H^r(E)$ has a submodule M whose quotient is isomorphic to M^* via the isomorphism induced by the paining. Construct a self-dual A-projective complex J such that: $H^\circ(J) \cong E$; $H^i(J) = 0\,(i < 0)$; $H^r(J) \cong M$. The construction is similar to the first stage of the proof of 2.1. The change is to construct the complex Q by killing the cohomology of P in dimensions less than r and only the part
211 M in dimension r. The proof can now be completed as for 2.1.

To generalize this result to arbitrary fields, it is necessary to use the Witt group for bilinear forms. □

3 Confluence and algebraic equivalence

Suppose that F is a reflexive B-module locally free except on a 1-dimensional closed subset of Spec B intersecting X in the empty set. Its ideal is an intersection of distinct 1-dimensional primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ of B which do not contain x. Localizing B, F at $\mathfrak{p}_i$ gives a regular local ring A_i and a vector bundle E_i on the punctured spectrum of A_i. The bundle $E = (F/xF)^*$ is called a confluence of $E_1, \ldots, E_s$. The following result has been proved in [4]:

Theorem 3.1. *Suppose that d is even and F is self-dual; then*

$$\mu(E) = \sum_{i=1}^{s} \mu(E_i).$$

Proof. The biduality spectral sequence gives exact sequences

$$0 \to \operatorname{Ext}_B^{d+2}(H^{r+1}(F), B) \to H^r(F) \to \operatorname{Ext}^{d+1}(H^r(F), B) \to 0$$
$$0 \to \operatorname{Ext}_B^{d+2}(H^r(F), B) \to H^{r+1}(F) \to \operatorname{Ext}^{d+1}(H^{r-1}(F), B) \to 0.$$

In both cases the kernels have support $\mathfrak{n}$ and the cokernels are x-torsion free. So the two kernels are dual to each other. The result now follows from the exact cohomology sequence for F/xF.

There is a simple application to the extension of bundles from $\mathbb{P}^d$ to $\mathbb{P}^{d+1}$ (d even). Let $\mathscr{E}$ be a self-dual bundle on $\mathbb{P}^d$. It always has a self-dual extension as a sheaf to $\mathbb{P}^{d+1}$, for, example by extending the pull-back of $\mathscr{E}$ to the punctured cone on $\mathbb{P}^d$ over the vertex.

Let $\mathscr{F}$ be any self-dual extension to $\mathbb{P}^{d+1}$ which is locally free except
for singularities at points $a_1, \ldots, a_s$ of $\mathbb{P}^{d+1}$. The sheaf $\mathscr{F}$ determines 212
reflextive sheaves $\mathscr{F}_1, \ldots, \mathscr{F}_s$ at each of these points. Apply 3.1 to the
punctured cone over $\mathscr{P}^{d+1}$ in the neighbourhood of the vertex. We find
that

$$\sum_{i=1}^{s} \mu(\mathscr{F}) = \mu(\mathscr{E})$$

where $\mu(\quad)$ is defined by lifting $\mathscr{E}$ to the punctured cone on $\mathbb{P}^d$.

A second application is to algebraic equivalence. Define two X-bundles to be algebraically equivalent if they can be joined by a sequence of confluences. Let E_1, E_2 be two X-bundles, not necessarily self-dual, with even rank $2t$ and assume d is even. The t-th exterior power of a rank $2t$ bundle has a natural pairing and $\gamma(E) = (\mu(\Lambda^t E))$ is an obstruction to extending E. Applying 3.1 shows that if E_1, E_2 are algebraically equivalent then $\gamma(E_1) = \gamma(E_2)$. In particular let E be the X-bundle of rank d which is the second syzygy of an ideal of A generated by a system of parameters for A. Then $\gamma(E)$ is the multiplicity of the ideal mod 2 and E is not algebraically equivalent to a trivial bundle if the multiplicity is odd.

When E is a rank $2X$-bundle coming by pull-back from $\mathbb{P}^d$ and $d \geq 4$, M. Cohen has shown by means of the Riemann-Roch Theorem that $\mu(E) = 0$. □

4 Formalism for μ

Assume $d = 2r$ is even. Form the exterior algebra $\Lambda(d+1)A)$ and let ξ be an element of $(d+1)A$ whose coordinates generate m. Put

$$\Theta = \operatorname{Ker}({}_{\wedge}\xi : \Lambda^{r+1} \to \Lambda^{r+2}).$$

It is a self-dual X-bundle with a pairing induced by the exterior algebra structure and its cohomology is given by

$$H^i(\Theta) = 0 (1 \leq i \leq d-1, i \neq r), H^r(\Theta) = k.$$

213 So $\mu(\Theta) = 1$. Now let E be any self-dual X-bundle and put $\rho(E)$ equal to the rank of E modulo two, and put

$$\widehat{\mu}(E) = \rho(E) + \mu(E)t \in \mathbb{Z}_2[t], t^2 = 0$$

Theorem 4.1. (i) $\rho(E) = \mu(E \otimes \Theta)$.

(ii) $\mu(\Theta^p) = 0$, $p \geq 2$.

(iii) $\widehat{\mu}$ *is a homomorphism of the ring of self-dual vector bundles onto* $\mathbb{Z}_2[t]$.

(iv) $\mu(\Lambda^p\Theta) = 0$, $p \geq 2$.

Proof. First prove (ii) for $p = 2$. Since Θ extends to a vector bundle on $\operatorname{Spec} A$ when $\mathfrak{m}$ is blown up, it is sufficient to consider the graded case. The dualizing line bundle for Θ^2 regarded as a sheaf on $\mathbb{P}^d$ has even degree and the dualizing line bundle for $\mathbb{P}^d$ has odd degree. Serre duality implies that $\mu(\Theta^2)$ vanishes.

Now we prove (i). By 2.2, there is an extendible self-dual X-bundle E' such that $E \oplus \mu(E)\Theta \oplus E'$ is self-dually extendible. So tensoring with Θ and using (ii) with $p = 2$ shows that it is sufficient to note that $\rho(\Theta) = 0$ and to prove (i) for an extendible self-dual X-bundle E. Let $\mathfrak{q}$ be the ideal generated by a base of $\mathfrak{m}$ lifted to B. Construct a B-module Φ from the exterior algebra on $(d+1)B$ using this lifted base in the same way as Θ was constructed from $(d+1)A$ and a base for $\mathfrak{m}$. The module Φ is reflexive, free outside the variety of $\mathfrak{q}$, and extends Θ. Choose a self dual Y-bundle F extending E' and apply 3.1 to $F \otimes \Phi$. Since F is locally free except at $\mathfrak{n}$, the localization $F_{\mathfrak{q}}$ is free with the same rank as E. So

$$\mu(E \otimes \Theta) = \mu(F_{\mathfrak{q}} \otimes \Phi_{\mathfrak{q}}) = \rho(E)\mu(\Phi_{\mathfrak{q}}) = \rho(E),$$

because $\mu(\Phi_{\mathfrak{q}}) = 1$.

Since $\rho(\Theta) = 0$, (ii) now follows for $p > 2$.

To prove (iii) it is sufficient to show that $\widehat{\mu} = 0$ defines an ideal. This is easily reduced to showing that if E is extendible and of even rank then $\widehat{\mu}(E \otimes \Theta) = 0$. But this follows (i). 214

Finally (iv) follows by reduction to the graded case as in the proof of (ii). Serre duality and consideration of the total weights of the indecomposable representations contained in $H^r(\Lambda^p\Theta)$ show that the representations occur in dual pairs. So $\mu(\Lambda^p\Theta) = 0$. □

5 Topological invariance

Take the residue field of A to be $\mathbb{C}$. The category of X-bundles is determined up to a canonical equivalence by the completion of A and it is easily verified that an X-bundle E determines up to bundle isomorphism a topological bundle on a punctured neighbourhood of the origin of $\mathbb{C}^{d+1}$. So E determines a topological bundle $|E|$ on S^{2d+1}. If E extends then $|E|$ is trivial.

Suppose that $d = 2r$. The K-group for self-dual bundles has been defined in [1, p. 636]. For S^{4r+1} it is isomorphic to $\mathbb{Z}_2$. So the topological invariance of μ is equivalent to:

Theorem 5.1. *Θ maps to the non-trivial element of the K-group of self-dual bundles on S^{4r+1}.*

Proof. At the conference I gave a proof via [3, Theorem 4.2] which depends on the Atiyah-Singer Index Theorem. Briefly let W be the non-Kähler manifold obtained by factoring $\mathbb{C}^{d+1} - \{0\}$ by an infinite cyclic subgroup of $\mathbb{C}^*$. It has a holomorphic fibration $f : W \to \mathbb{P}^d$ with a topological factorization through S^{4r+1}. Now Θ is the pull back to X of θ the r-th exterior power of the tangent bundle of $\mathbb{P}^d$. The stable homotopy invariant β of [3] (the holomorphic semicharacteristic) is now easily computed for $f * \theta$ and seen to be non-zero. So, Θ is non-trivial
215 in the sense of the stable homotopy theory of bundles with a pairing.

In the course of the conference, M.F. Atiyah gave me a direct proof of this result which I outline here. First the K-group of self-dual bundles on S^{4r+1} can be identified with $K\widetilde{O}(S^{8m+1})$ if $r = 2m$ and with $K\widetilde{S}(S^{8m+5})$ if $r = 2m+1$. In each case S^{4r+1} is a homogeneous space for the appropriate group (Spin or Symplectic). Consider the case $r = 2m$. Then $S^{8m+1} = \text{Spin}(8m+2)/\text{Spin}(8m+1)$ and the representations of $\text{Spin}(8m+1)$ determine real vector bundles on S^{8m+1}. The generator of $K\widetilde{C}$ corresponds to the basic spin representation of dimension 2^{4m} [[2], [7, p.270]]. Now $\mathbb{P}^{4m} = U(4m+1)/U(4m)\times U(1)$ and the exterior power representation Λ^{2m} of $U(4m)$ determines the bundle θ. To compare θ with the generator of KO express S^{8m+1} as $SU(4m+1)/SU(4m)$. An easy character computation shows that the basic spin representation and Λ^{2m} give equivalent representations of $SU(4m)$ modulo sums of pairs of dual representations. The case $r = 2m+1$ can in a similar way be reduced to a character computation. □

References

[1] D.W. Anderson, The real K-theory of classifying spaces, *Proc. N. A. S.* Vol. 51 (1964), 634-6.

[2] M.F. Atiyah, R Bott and A. Shapiro, Clifford modules, *Topology*, Vol. 3, Suppl. 1, 1964, 3-38.

[3] M.F. Atiyah and E. Rees, Vector bundles on projective 3-space, *Invent. Math.* 35, 131-53 (1976).

[4] M. Cohen, Algebraic families of vector bundles on projective space and the punctured spectrum of a local ring, Ph.D. thesis, Newcastle upon Tyne, 1978. **216**

[5] F. Hirzebruch, *Topological Mehtods in Algebraic Geometry,* 3rd Ed., Springer-Verlag, Berlin 1966.

[6] G. Horrocks, Vector bundles on the punctured spectrum of a local ring, *Proc. London Math. Soc.* 1964, 689-713.

[7] H. Weyl, *The Classical Groups*, Princeton Math. Series, Princeton 1946.

School of Mathematics,
The University,
Newcastle upon Tyne,
NE1 7RU, England.

Some Metrics on Picard Bundles

By G.R. Kempf

THE PICARD BUNDLES on the Jacobian J of a smooth complete 217
curve C of genus g have been studied by various methods [1], [2], but their nature still remains mysterious. In this paper, I will present some naturally occurring Hermitian metrics on certain Picard bundles in the complex case. These metrics are special among the myriad of such metrics. Hopefully, they will be useful in the application of the methods of metric geometry to the study of Picard bundles.

These metrics are induced by metrics on the analog of Picard bundles on the Jacobian. The rich geometry of abelian varieties provides metrics on these analogs. The connection between these analogs and the Picard bundles themselves has been developed by R.C. Gunning [1]. The first two sections of this paper are algebraic and apply to the geometry over an arbitrary field k. In them, I will develop the abstract version of Gunning's connection. In the third section, we will be working over the complex field to define the matrics.

1 Some linear systems on C and J

Let V denote the set of effective divisors D on C of degree g such that the complete linear system $|D|$ consists of one point. We may regard V as an open dense subset of the g-th symmetric product $C^{(g)}$. Let $\int_g : C^{(g)} \to \mathrm{Pic}_g$ be the universal abelian integral onto the g-th Picard variety of C, which classifies isomorphism classes of invertible sheaves on C of degree g. Then $\int_g$ induces an isomorphism between V and an open dense subset U of Pic_g.

---

[1] Partly supported by NSF grant #MPS 75-05578.

Let E be a divisor on C of degree ng for some integer $n \geq 2$. Then we have

218 **Lemma.** *The divisors $D_1 + \cdots + D_n$ where the D_i's in V are dense in the complete linear system $|E|$.*

Proof. Let F be a member of $|E|$. We may write $F = G_1 + \cdots + G_n$ where the G_i's are effective divisors of degree g. Let $\mathscr{L}_1, \ldots, \mathscr{L}_n$ denote invertible sheaves on C of degree zero such that $\bigotimes_{1 \leq i \leq n} \mathscr{L}_i \approx \mathscr{O}_C$. As $n \geq 2$ for a general choice of the $\mathscr{L}_i$'s, each sheaf $\mathscr{L}_i(G_i)$ is contained in U. In this case $\mathscr{L}_i(G_i) \approx \mathscr{O}_C(D_i)$ where D_i is in V. Hence $D_i + \cdots + D_n \in |E|$ and F is contained in the closure of such divisors as V is dense in $C(g)$. *Q.E.D.* ☐

Recall that the Jacobian J possesses a principal polarization; i.e., there is a divisor θ on J which is determined upto translation with $\dim \Gamma(J, \mathscr{O}_J(\theta)) = 1$. For any integer $n > 0$, we have the multiple polarization. Any divisor in this class is linearly equivalent to $n\theta$ for some θ. Choose an element α of Pic_1. Then we have the embedding $i : C \hookrightarrow J$ given by $i(C) = \int(C) - \alpha$. With this embedding, we can study the relation between the linear systems on J and those on C.

Proposition. *If $n \geq 2$, the linear system $|n\theta|$ cuts out a complete linear system on C of degree ng. Equivalently, the restriction $\Gamma(J, \mathscr{O}_j, (n\theta)) \to \Gamma(C, \mathscr{O}_J(n\theta)|_C)$ is surjective.*

Proof. First recall [4, for instance] that the linear equivalence class $i^*\theta$ is of degree g. Furthermore, if the inverse image $i^1(\theta + j)$ is defined for a point j of J, it is a divisor in V and all divisors in V occur this way. Let E denote a member of the class $i^*(n\theta)$. Take a divisor $D_1 + \cdots + D_n$ in $|E|$ with the D_i's in V. Thus we may find points j_i of J such that $i^{-1}(\theta + j_i)$ is defined and equals D_i. Therefore,

$$D_1 + \cdots + D_n = i^{-1}((\theta + j_1) + \cdots + (\theta + j_n)).$$

As $(\theta + j_1) + \cdots + (\theta + j_n)$ is algebraically equivalent to $n\theta$ and its intersection with C is linearly equivalent to E, $(\theta + j_i) + \cdots + \cdots + (\theta + j_n)$

219 is contained in $|n\theta|$ by the autoduality of the Jacobian. As $|n\theta|$ cuts out a linear system, this linear system is complete by the Lemma. Q.E.D. □

2 The connection between the Picard sheaves

Let $\mathscr{L}$ be an invertible sheaf on $C \times J$ which is a universal family of invertible sheaves on C of degree d. If $d > 2g - 2$, the direct image $\mathscr{W}_d \equiv \pi_{J_*}\mathscr{L}$ is a locally free sheaf on J of rank $d + 1 - g$. This sheaf $\mathscr{W}_d$ is called a Picard bundle of degree d. Furthermore, the formation of the direct image commutes with base extension. In particular, for any point j of J, we have a natural isomorphism

$$\mathscr{W}_d \otimes k(j) \xrightarrow{\approx} \Gamma(C, \mathscr{L}|_{C\times\{j\}}).$$

Thus $\mathscr{W}_d$ arises naturally when one is studying variational problems involving complete linear systems of degree d on C. Let X be an abelian variety with dual $X\hat{}$. Similarly, let $\mathscr{M}$ be an invertible sheaf on $X \times X$ which gives a universal family of invertible sheaves on X which are algebraically equivalent to $\mathscr{N} \equiv \mathscr{M}|_{X\times 0}$. We will assume that $\mathscr{M}$ is isomorphic to $\pi_X^*\mathscr{N} \otimes \mathscr{P}$ where $\mathscr{P}$ is a Poincaré sheaf on $X \times X\hat{}$. If $\mathscr{N}$ is ample on X, the direct image $\mathscr{V}\{\mathscr{N} \equiv \pi_{X\hat{}}* \mathscr{M}$ is a locally free sheaf on $X\hat{}$of rank $\dim\Gamma(J, \mathscr{N})$ [3]. Furthermore, the formation of this direct image also commutes with base extension. In particular, for each point y of x, we have a natural isomorphism

$$\mathscr{V}\{\mathscr{N}\} \otimes k(y) \to \Gamma(X, \mathscr{M}|_{X\times y})$$

In the special case $X = J$, we have $X\hat{}\cong J$ as J is principally polarized. If $\mathscr{N} \approx \mathscr{O}_J(n\theta)$ for some integer $n > 0$, the sheaf $V\{\mathscr{N}\}$ will be denoted by $\mathscr{V}_n$. As $\dim\Gamma(J, \mathscr{O}_J(n\theta)) = n^g$, $\mathscr{V}_n$ is *a* locally free sheaf on J of rank n^g. This sheaf contains variational information about the various linear systems $|n\theta|$.

Now consider the embedding $C \hookrightarrow J$, the sheaf $C_J(n\theta) \otimes \mathscr{P}|_{C\times J}$ is a universal family of invertible sheaves on C of degree ng paramerized by J because of the autoduality of J. Furthermore, we have the natural restriction of direct images.

$$\alpha : \mathscr{V}_n = \pi_{2^*}(\pi_1^* \mathscr{O}_J(n\theta) \otimes \mathscr{P}) \to$$
$$\to \pi_{J^*}(\pi_1^* \mathscr{O}_J(n\theta) \otimes \mathscr{P}|_{C\times J}) = \mathscr{W}_{ng}.$$

220 The main connection between these two kinds of Picard sheaves is the following.

Proposition 2. *If* $n \geq 2$, *the above homomorphism* $\alpha : \mathscr{V}_n \to \mathscr{W}_{ng}$ *is surjective.*

Proof. It will suffice to show $\alpha \otimes k(j)$ is surjective for each point j of J. As the formation of both sheaves commutes with base extension, this is equivalent to the surjectivity of $\Gamma(J, \mathscr{H}) \to \Gamma(C, \mathscr{H}|_C)$ where $\mathscr{H} \approx \mathscr{O}_J(n\theta)$ for some choice of θ. Thus this proposition follows from the last one. Q.E.D. □

The sheaf $\mathscr{V}\{\mathscr{M}\}$ has a clear description which we will use in the next section. The last proposition may be used to give a description of the Picard sheaf $\mathscr{W}_4$ when C is a curve of genus 2. In this case $\mathscr{W}_4$ has rank 8 and $\mathscr{V}_2$ has rank 4.

Claim. (a) We have an exact sequence

$$0 \to \mathscr{O}_J(-\theta) \xrightarrow{\beta} \mathscr{V}_2 \xrightarrow{\alpha} \mathscr{W}_4 \to 0.$$

(b) Any homomorphism $\mathscr{O}_J(-\theta)$ to $\mathscr{V}_2$ is a multiple of β.

Proof. As $g = 2$, we can assume that $C = \theta$.

On $J \times J$ we have an exact sequence

$$0 \to \pi_1^* \mathscr{O}_J(\theta) \otimes \mathscr{P} \to \pi_1^* \mathscr{O}_J(2\theta) \otimes \mathscr{P} \to \pi_1^* \mathscr{O}_J(2\theta) \otimes \mathscr{P}|_C \to 0.$$

The sequence in (a) is just the direct image sequence. To see the claim (a) it will suffice to compute $\pi_{2*|}(\pi_1^* \mathscr{O}_J(\theta) \otimes \mathscr{P})$. Here we may take

$$\mathscr{P} = (\pi_1 + \pi_2)^* \mathscr{O}_J(\theta) \otimes \pi_1^* \mathscr{O}_J(-\theta) \otimes \pi_2^* \mathscr{O}_J(-\theta).$$

Thus we need to compute $\pi_{2*}((\pi_1+\pi_2)^*\mathcal{O}_J(\theta)\otimes\pi_2^*\mathcal{O}_J(-\theta))$ which equals $\pi_{2*}(\pi_1+\pi_2)^*\mathcal{O}_J(\theta))\otimes\mathcal{O}_J(-\theta)$ by the projection formula. Lastly, we need **221** to show that $\pi_2^*((\pi_1+\pi_2)^*\mathcal{O}_J(\theta))$ is trivial. To see this, note that we have the π_2 - isomorphism $(\pi_1+\pi_2,\pi_2)$ between $\pi_1+\pi_2$ and π_1. Hence

$$\pi_{2*}((\pi_1+\pi_2)^*\mathcal{O}_J(\theta)) \approx \pi_{2*}(\pi_1^*\mathcal{O}_J(\theta)) = \mathcal{O}_J\otimes_k\Gamma(J,\mathcal{O}_J(\theta))$$

The desired result follows because $\Gamma(J,\mathcal{O}_J(\theta)) = k$.

For (b), note that $\mathrm{Hom}(\mathcal{O}_J(-\theta),\mathcal{V}_2) =$

$$\begin{aligned}&\Gamma(J,\mathrm{Hom}\,\mathcal{O}_J(\mathcal{O}_J(-\theta),\pi_{2*}\mathcal{O}_J(2\theta)\otimes\mathcal{P})))\\ =&\Gamma(J\times J,\pi_2^*\mathcal{O}_J(\theta)\otimes\pi_1^*\mathcal{O}_J(2\theta)\otimes\mathcal{P}).\end{aligned}$$

Using our formula for $\mathcal{P}$, we need to show that the space of global sections of $(\pi_1+\pi_2)^*\mathcal{O}_J((\theta)\otimes\pi_1^*\mathcal{O}_J(\theta)$ is one dimensional. Using the isomorphism $(\pi_1,\pi_1+\pi_2)$ of $J\times J$ the last space is isomorphic to

$$\Gamma(J\times J,\pi_1^*\mathcal{O}_J(\theta)\otimes\pi_2^*\mathcal{O}_J(\theta)) = \Gamma(J,\mathcal{O}_J(\theta)) = \otimes(J\Gamma(J,\mathcal{O}_J(\theta)) = k$$

by the Kunnëth formula. This proves (b). □

3 The natural Hermitian metrics on the Picard bundles

We return to the situation of the last section where $\mathcal{N}$ is an ample invertible sheaf on an abelian variety X with dual $X\hat{}$. Recalling from [5] that we have an isogeny $\phi_{\mathcal{N}} : X \to X$ which sends a point x in X to the isomorphism class $T_x\mathcal{N}\otimes\mathcal{N}^{\otimes -1}$ where T_x denotes translation by x. The kernel of $\phi_{\mathcal{N}}$ is the finite group scheme H. Furthermore, we have Mumford's theta group G which is given by a central extension

$$1 \hookrightarrow \mathbb{G}_m \to G \to H \to 0.$$

An element of G is a specific isomorphism $\alpha : T_k\mathcal{N} \cong \mathcal{N}$ for some point k of H. In fact, if G acts on X via translation by H, we have a G-linearization of the sheaf $\mathcal{N}$ on X where the centre $\mathbb{G}_m$ of G acts by multipliacation. Consequently, $\Gamma(X, \mathscr{N})$ is a representation of G where G_m acts by multiplication. This representation is the unique irreducible representation of G with this condition on G_m and order $(H) = [\alpha\Gamma, ???\Gamma(X, \mathscr{N})]^2$.

222 Consider the sheaf $\Gamma(X, \mathscr{N}) \otimes_k \mathscr{N}^{\otimes -1}$ on J. This sheaf possesses a natural action of H. Just let G act naturally on $\Gamma(X\mathscr{N})$ and contragradiently on $\mathscr{N}^{\otimes -1}$; then the tensor product has an induced G-linearization given by the tensor product of the two actions. As the center of G acts trivially on the tensor product, we have an action of H on the sheaf. A central result about Picard sheaves on abelian varieties is

Proposition 3 (See [3]). *We have an H-isomorphism*

$$\phi^*_{\mathscr{N}}(\mathscr{V}\{\mathscr{N}\}) \overset{\approx}{\to} F(X\mathscr{N}) \otimes_k \mathscr{N}^{\otimes -1}$$

where the H-action on $\phi^*\mathscr{N}(\mathscr{V}\{\mathscr{N}\})$ *is the tautological one which determines* $\mathscr{V}\{\mathscr{N}\}$ *by descent theory.*

From now on, we will assume that the ground field k is the complex numbers $\mathbb{C}$. In this case, all the group schemes are reduced. The invertible sheaf $\mathscr{N}$ (or any invertible sheaf, for that matter) on our abelian variety X possesses an almost canonical Hermitian metric by a result of Mumford's [6]. This metric is determined upto a positive real multiple by the condition that its curvature is an invariant differential form on the abelian group X. The group G possesses a maximal compact subgroup K which is an extension of H by $\{c \in G| \, |c| = 1\}$. In fact, K consists of elements of G where the isomorphism α is an isometry. Also this metric globally induces a Hermitian inner product on the vector space $\Gamma(J\mathscr{N})$ which is invariant under K. Here one uses an invariant normalized Haar measure μ on X and defines

$$\langle\alpha,\beta\rangle = \int_X \langle\alpha_X,\beta_X\rangle_X\mu.$$

Returning to our sheaf $\Gamma(J\mathscr{N}) \otimes_{\mathbb{C}} \mathscr{N}^{\otimes -1}$ we may give this sheaf a metric by taking the tensor product of the one on $\Gamma(J, \mathscr{N})$ with the
223 dual metric on $\mathscr{N}^{\otimes -1}$ for a given choice of a Mumford type metric on

$\mathscr{N}$. The metric is immediately seen to be independent of the last choice and to be invariant under the action of H on $\Gamma(J, \mathscr{N}) \otimes_{\mathbb{C}} \mathscr{N}^{\otimes -1}$. Using Proposition 3 and descent theory, we may descent this metric to get one on the quotient $\mathscr{V}\{\mathscr{N}\}$. Thus we have

Proposition 4. *The Picard sheaf* $\mathscr{V}\{\mathscr{N}\}$ *possesses a canonical Hermitian metric which pulls back via* $\phi_{\mathscr{N}}$ *to the one given above.*

Returning to the curve case, we have

Corollary. *If* $n \geq 2$, *the Picard sheaf* $\mathscr{W}_{ng}$ *possesses a canonical Hermitian metric.*

Proof. By Proposition 2, we may use the restriction $\mathscr{N}_n \to\!\!\to \mathscr{W}_{ng}$ to give a quotient metric from the canonical metric on $\mathscr{V}_n$. Q.E.D. □

Remark. One may check that the Chern forms (elementary invariants of the curvature) of the Hermitian sheaf $\mathscr{V}\{\mathscr{N}\}$ are invariant differential forms. This follows immediately after pulling back via $\phi_{\mathscr{N}}$. One may ask for a computation of the Chern forms of the Picard shaves $\mathscr{W}_{ng}$. Their cohomology class is well-known.

References

[1] R.C.Gunning, On generalized theta functions, *Amer. J. Math.,* 104 (1982) pp. 183-208.

[2] G. Kempf, Inversion of Abelian integrals, *Bulletin of A.M.S.,* 9(1982) pp. 25-32.

[3] G. Kempf, Toward the inversion of Abelian integrals II, *Amer. J. Math.,* 101(1979) pp. 184-202.

[4] G. Kempf, Inverse images of theta divisors, *Ill. J. Math.,* to appear. **224**

[5] D. Mumford, *Abelian Varieties,* Oxford University Press, Oxford, 1970.

[6] D. Mumford, On the equations defining abelian varieties III, *Invent. Math.*, 3(1967) pp. 215-244.

Johns Hopkins University
Baltimore, Maryland, U.S.A.

Bundles on $\mathbb{P}^2$ with a Quaternionic Structure

By M.-A. Knus

Introduction

225

In this paper, we give a survey of some recent results on 2-bundles over $\mathbb{P}^2_{\mathbb{C}}$, which have a quaternionic structure. These bundles were studied in [7] in connection with projective ideals of the polynomial ring in two variables over the real quaternions $\mathbb{H}$.

We present some of the main results of [7] in a slightly different form and give a few generalizations. In particular we compute the Chern classes of bundles arising from an explicit family of ideals in $\mathbb{H}[x, y]$ and we indicate how to calculate the curves of jump lines for some of them. Finally, we mention connections with bundles over quadrics and over $\mathbb{P}^3_{\mathbb{C}}$.

We do not mention any related results for bundles of higher rank (see [6], [10], [12] and report of Parimala at this Conference).

Most of the results described here were obtained by M. Ojanguren, 226 R. Parimala, R. Sridharan and the author, mostly as joint work (in many different combinations), but the author is responsible for possible errors in the presentation of the results given here.

We thank the Tata Institute for its hospitality and we are thankful to many participants in the Colloquium for the useful discussions we had with them in connection with this work.

1 Quaternion algebras and quadratic bundles

1.1

Let X be a scheme over $\mathbb{R}$ and let $\mathscr{H}$ be a sheaf of algebras over X which is a locally free $\mathscr{O}_X$-module of rank 4 and such that the fiber at each real closed point is isomorphic to the real quaternion division algebra $\mathbb{H}$. We call such a sheaf of algebras a *quaternion algebra* over X. Any quaternion algebra $\mathscr{H}$ has a quadratic structure given by the *reduced norm* n. We recall that a *quadratic space* (or a *quadratic bundle*) over a K-scheme X, (K any field of characteristic $\neq 2$) is a vector bundle $\mathscr{E}$ over X with a symmetric bilinear form $b : \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E} \to \mathscr{O}_X$ which is nonsingular, i.e. which induces an isomorphism $\mathscr{E} \xrightarrow{\sim} \mathscr{E}^*$. Composing b with the diagonal $\mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E}$, we get the quadratic form q associated with b. We denote a quadratic bundle by (E, q) or simply q.

1.2

Let $\mathscr{E}$ be a locally free sheaf of rank 4 over X which is a right $\mathscr{H}$-module, $\mathscr{H}$ a quaternion algebra over X. The restriction of $\mathscr{H}$ to any affine open set $U = \operatorname{Spec} A$ is a separable A-algebra H. Hence any projective A-module, which is also an H-module, projective as H-module.
227 Thus $\mathscr{E}$ is locally free of rank one as an $\mathscr{H}$-module (for the Zariski topology of X). We denote the category of such $\mathscr{H}$-modules $\mathscr{E}$ by $P(\mathscr{H})$ and call them simply $\mathscr{H}$-modules of rank one.

1.3

Let $\mathscr{H}$ be a quaternion algebra over X. We say that a quadratic space (E, q) over X is of *type* $\mathscr{H}$ if

(a) $\mathscr{E} \in P(\mathscr{H})$

(b) $q \circ m = q \otimes \mathscr{O}_X n$, where $m : \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{H} \to \mathscr{E}$ defines the operation of $\mathscr{H}$ on $\mathscr{E}$ and n is the norm of $\mathscr{H}$.

Let $X = \operatorname{Spec} A$. Then a quadratic space over X is a pair (P, q), where P is a finitely generated projective A-module and $q : P \to A$ is a nonsingular quadratic form. Condition (a) means that P is a projective right H-module of rank one for a quaternion algebra H over A and condition (b) means that $q(xh) = q(x)n(h)$, $x \in P$, $h \in H$.

1.4

Let X be an $\mathbb{R}$-scheme, let $\mathscr{H}$ be a quaternion algebra over X and let (E, q) be a quadratic bundle over X of type $\mathscr{H}$. Then q is locally similar to the norm n of $\mathscr{H}$. Hence we see that at the fiber of any real closed point of X, the quadratic form q is definite. We say that a quadratic bundle is *positive definite* if the fiber at any real closed point is positive definite. In the following, we always assume that a quadratic bundle of type $\mathscr{H}$ is positive definite. This is not a serious restriction since q or $-q$ is positive definite if X is connected.

Proposition 1.5. *Let $\mathscr{H}$ be a quaternion algebra over a real scheme X and let $\mathscr{E} \in P(\mathscr{H})$. Then there exists a universal quadratic map $q : \mathscr{E} \to \mathscr{N}_{\mathscr{E}}$ (in the category of coherent sheaves over X) satisfying property (b) of (1.3). The sheaf $\mathscr{N}_{\mathscr{E}}$ is invertible and the pair $(q, \mathscr{N}_{\mathscr{E}})$ is uniquely determined up to isomorphisms by (1.3)* and *the condition im $b = \mathscr{N}_{\mathscr{E}}$, where b is the symmetric bilinear map associated with q.* 228

Proof. If X is affine, we construct $\mathscr{N}_{\mathscr{E}}$ in the standard way by generators and relations. In order to prove $\mathscr{N}_{\mathscr{E}}$ is an invertible sheaf, we may assume by localization that $\mathscr{E} \simeq \mathscr{H}$. In this case the pair $(n, \mathscr{O}_X)$, where n is the norm of $\mathscr{H}$, is universal. Thus $\mathscr{N}_{\mathscr{H}} \simeq \mathscr{O}_X$ is invertible. The construction clearly globalizes for any real scheme. □

We call the quadratic map $q : \mathscr{E} \to \mathscr{N}_{\mathscr{E}}$ the *norm* of $\mathscr{E}$. The following results are consequences of the universal property of $\mathscr{N}_{\mathscr{E}}$.

Corollary 1.6. *Let $(\mathscr{E}, q)$ be a quadratic bundle of type $\mathscr{H}$ over X. Then we have $\mathscr{N}_{\mathscr{E}} \simeq \mathscr{O}_X$. Conversely, if $\mathscr{E} \in P(\mathscr{H})$ is such that $\mathscr{N}_{\mathscr{E}} \simeq \mathscr{O}_X$, then $\mathscr{E}$ is of type $\mathscr{H}$ for some quadratic structure q. The form*

q is uniquely determined up to a multiple $\lambda \in H^\circ(X, G_m)$ (and up to isometries).

229 **Corollary 1.7.** *Assume that $H^\circ(X, G_m) = \mathbb{R}^X$. Let $(\mathscr{E}, q)$ and $(\mathscr{E}', q')$ be two positive definite quadratic bundles of type $\mathscr{H}$. If $\mathscr{E}$ and $\mathscr{E}'$ are isomorphic as $\mathscr{H}$-modules, they are isometric as quadratic spaces.*

Proposition 1.8. *Let X be a real scheme such that $H^\circ(X, \mathscr{O}_X) = \mathbb{R}$ and let $(\mathscr{E}, q)$ be a positive definite quadratic bundle of type $\mathscr{H}$ over X. If $\mathscr{E} \neq \mathscr{H}$, then $H^\circ(X, \mathscr{E}) = 0$.*

Proof. Let $s \in H^\circ(X, \mathscr{E})$, then $q(s) \in H^\circ(X, \mathscr{O}_X)$ is a constant $\lambda \in \mathbb{R}$. Since q is positive definite, λ is positive if $s \neq 0$. The section s defines a homomorphism $\mathscr{H} \to \mathscr{E}$ which is an isometry if $\lambda \neq 0$. Thus $\mathscr{E} \simeq \mathscr{H}$ if $\mathscr{E}$ has a nontrivial global section. □

The next result is again a consequence of the universal property of $\mathscr{N}_{\mathscr{E}}$.

Lemma 1.9. *Let X be a real scheme, let $\mathscr{I}$ be an invertible sheaf on X and let $\mathscr{E} \in P(\mathscr{H})$ for some quaternion algebra $\mathscr{H}$ over X. Then we have $\mathscr{N}_{\mathscr{E}} \otimes_{\mathscr{O}_X} \mathscr{I} \cong \mathscr{N}_{\mathscr{E}} \otimes_{\mathscr{O}_X} \mathscr{I}^2$.*

2 Bundles over projective spaces

2.1

Let $X = \mathbb{P}^n_{\mathbb{R}}$ be the real projective n-space, let $\mathscr{H}$ be a quaternion algebra over X and let $\mathscr{E} \in P(\mathscr{H})$. In view of (1.5), we have $\mathscr{N}_{\mathscr{E}} \simeq \mathscr{O}(n)$ for some $n \in \mathbb{Z}$. We claim that n is even. By (1.9), we are reduced to the cases $n = 0$ and $n = 1$. For any real affine n-space $U \subset X$, the restriction $\mathscr{N}_{\mathscr{E}U}$ is free, hence $\mathscr{E}|_U$ carries a quadratic form of type $\mathscr{H}|_U$. As noticed in (1.4) we may assume that the form is positive definite. But
230 if $n = 1$, the form would change sings for a real closed point of some real affine n-space U.

Proposition 2.2. *Let $\mathscr{H}$ be a quaternion algebra over $\mathbb{P}^n_{\mathbb{R}}$ and let $\mathscr{E} \in P(\mathscr{H})$. Then, for some $m \in \mathbb{Z}$, $\mathscr{E}(m)$ carries a quadratic form q of type $\mathscr{H}$ which is positive definite. The form q is unique up to isometry.*

Proof. The existence of q follows from (2.1). Let q and q' be two quadratic structures on $\mathscr{E}$ of type $\mathscr{H}$ and which are positive definite. By (1.6) we have $\lambda \in \mathbb{R}^X$ such that $q' = \lambda q$. Since both forms are positive definite, λ is positive and the forms are isometric. □

Remark 2.3. The uniqueness part of (2.2) follows more generally from the fact that a vector bunder over a real projective scheme carries at most one positive-definite quadratic structure (see [6]).

2.4

We call the bundle $\mathscr{E}(m)$ of (2.2) the *normalization* of $\mathscr{E}$ and we say that $\mathscr{E} \in P(\mathscr{H})$ is *normalized* if it carries a quadratic structure of type $\mathscr{H}$.

2.5

Let $\mathscr{H}_0$ be the constant sheaf $H \otimes_{\mathbb{R}} \mathscr{O}_X$ of quaternion algebras over a real scheme X. Let $\mathscr{E}$ be a locally free $\mathscr{H}_0$-module. The embedding $\mathbb{C} \to \mathbb{H}$ induces a complex structure on $\mathscr{E}$. Hence we can associate with $\mathscr{E}$ a complex bundle over $X_{\mathbb{C}} = X \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$. We denote this bundle by $\mathscr{E}_c$.

Proposition 2.6. *Let $X = \mathbb{P}^n_{\mathbb{R}}$ and let $\mathscr{E} \in P(\mathscr{H}_0)$ such that $\mathscr{E}(m) \not\xrightarrow{\sim} \mathscr{H}_0$ for all $m \in \mathbb{Z}$. Then $\mathscr{E}_c$ is a stable 2-bundle over $\mathbb{P}^n_{\mathbb{C}}$.*

Proof. We may assume by (2.2) that $\mathscr{E}$ is a quadratic bundle of type $\mathscr{H}_0$. **231** In view of (1.8), we have $H^0(\mathbb{P}^n_{\mathbb{C}}, \mathscr{E}_c) = 0$. Further, $c_1(\mathscr{E}_c) = 0$, since $\mathscr{E}$ is a quadratic bundle. Thus $\mathscr{E}_c$ is stable. □

Proposition 2.7. *Let $\mathscr{E}$ be a quadratic bundle of type $\mathscr{H}_0$. Then the restriction of $\mathscr{E}$ to any real line L is trivial.*

Proof. The restriction of $\mathscr{E}$ to L gives an anisotropic bundle over $\mathbb{P}^1_{\mathbb{R}}$ and the claim follows by a result of Scharlau [14]. □

2.8

Let $\sigma : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ be the usual real structure of $\mathbb{P}^n_{\mathbb{C}}$, given by complex conjugation of the variables. For any $\mathscr{E} \in P(\mathscr{H}_0)$, *the* multiplication by $j \in \mathbb{H}$ induces an isomorphism $\mathscr{E}_c \to \sigma * \mathscr{E}_c$, also denoted by j, such that $\sigma * j \circ j = -1$. Conversely, any isomorphism $j \cdot \mathscr{F} \to \sigma * \mathscr{F}$, $\mathscr{F}$ a 2-bundle over $\mathbb{P}^n_{\mathbb{C}}$, such taht $\sigma * j \circ j = -1$ defines the structure of an $\mathscr{H}_0$-module of rank one on $\mathscr{F}$ considered as a 4-bundle over $\mathbb{P}^n_{\mathbb{R}}$. If n is odd, then $\mathbb{P}^n_{\mathbb{C}}$ has another real structure, without real points. A stable 2-bundle $\mathscr{E}$ over $\mathbb{P}^3_{\mathbb{C}}$ with a quaternionic structure j with respect to the nonstandard real structure of $\mathbb{P}^3_{\mathbb{C}}$, such that $c_1(\mathscr{E}) = 0$ and $\mathscr{E}$ is trivial over real lins is an instanton bundle. Over $\mathbb{P}^2_{\mathbb{C}}$, there is only one real structure, the usual one. Thus, quadratic bundles of type $\mathscr{H}_0$ are in some sense instantons over $\mathbb{P}^2_{\mathbb{C}}$.

2.9

Let $M(0, k)$ be the moduli space of stable 2-bundles over $\mathbb{P}^2_{\mathbb{C}}$ with $c_1 = 0$ and $c_2 = k$. By results of Barth [1], $M(0, k)$ is an irreducible complex variety of dimension $4k - 3$. Instantons correspond to real points of $M(0, k)$ for the real structure of $M(0, k)$ induced functorially by the real structure of $\mathbb{P}^2_{\mathbb{C}}$. Thus, this set is a real open smooth manifold of real dimension $4k - 3$. Thus, the moduli space of quadratic bundles of type
232 $\mathscr{H}_0$ over $\mathbb{P}^2_{\mathbb{R}}$ is a manifold of dimension $4k - 3$. An explicit description of this manifold is given in section 9 for $k = 2$.

3 Extensions of isometries

3.1

Let K be a field of characteristic $\neq 2$ and let X be a K-scheme. We say that a quadratic bundle $(\mathscr{E}, q)$ is *anisotropic* if its restriction to any affine open set is anisotropic. If $X = \operatorname{Spec} A$, then a quadratic space over X is called anisotropic if $q(x) = 0$ implies $x = 0$, $x \in P$. The property of anisotropicity is birational. If $K = \mathbb{R}$, $X_{\mathbb{R}}$ is connected and $X(\mathbb{R})$ not empty, then positive definite quadratic bundles are anisotropic.

Proposition 3.2. *Let X be an irreducible noetherian scheme over K and let Y be the closed subscheme defined by a sheaf of locally principal ideals. Let $(\mathscr{E}, q)$, $(\mathscr{E}', q')$ be quadratic spaces over X with an isometry $\varphi : (\mathscr{E}, q) \to (\mathscr{E}', q')$ defined over $X - Y$. If $(\mathscr{E}, q)$ and $(\mathscr{E}', q')$ are anisotropic over Y, then φ can be extended to a unique isometry $(\mathscr{E}, q) \to (\mathscr{E}', q')$ over X.*

Proof. In view of the asserted uniqueness, the question is local on X. Thus, we may assume that X is affine and the claim follows by proposition (1.1) of [7]. □

Example 3.3. Let $X = \mathbb{P}^n_{\mathbb{R}}$ and let $U \subset X$ be a real affine subspace of dimension n, i.e., $U = \mathbb{P}^n_{\mathbb{R}} - V(L)$, where $V(L)$ is a real hyperplane in $\mathbb{P}^n_{\mathbb{R}}$. Let $(\mathscr{E}, q)$, $(\mathscr{E}', q')$ be positive definite quadratic bundles over $\mathbb{P}^n_{\mathbb{R}}$. They are positive definite over $V(L)$, hence anisotropic. Thus, any isometry $\varphi : (\mathscr{E}, q) \to (\mathscr{E}', q')$ over U extends to a unique isometry over $\mathbb{P}^n_{\mathbb{R}}$.

3.4

233 Let $\mathscr{E}$ and $\mathscr{E}'$ be $\mathscr{H}$ modules of rank one over $\mathbb{P}^n_{\mathbb{R}}$, for $\mathscr{H}$ some quaternion algebra over $\mathbb{P}^n_{\mathbb{R}}$. Let U as in (3.3) and let $\varphi : \mathscr{E} \to \mathscr{E}'$ be an isomorphism of $\mathscr{H}$ modules over U. By (1.6) $\mathscr{E}'|U$ and $\mathscr{E}'|_U$ are quadratic spaces of type $\mathscr{H}|_U$ and by (1.7) φ is an isometry $\mathscr{E}|_U \xrightarrow{\sim} \mathscr{E}'|_U$. In view of (3.3) φ extends to an isometry of the normalizations $\mathscr{E}(m)$, resp. $\mathscr{E}'(m')$ of $\mathscr{E}$ and $\mathscr{E}'$. This isometry is an isomorphism of $\mathscr{H}$-modules. Thus, we see that if $\mathscr{E}|_U \xrightarrow{\sim} \mathscr{E}'|_{U'}$ then $\mathscr{E}(k) \xrightarrow{\sim} \mathscr{E}'$ for some $k \in \mathbb{Z}$.

4 Bundles over the affine and the projective plane

4.1

Let $\mathbb{P}^2_{\mathbb{R}}$ be the real projective plane with coordinates X_1, X_2, X_3 and let $\mathbb{A}^2_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{R}}$ be the affine plane $\mathbb{P}^2_{\mathbb{R}} - V(X_3)$, with affine coordinates $x = X_1/X_3, y = X_2/X_3$. We use the notation $A = \mathbb{R}[x, y]$ and $H = H \otimes_{\mathbb{R}} A$. Let

P be a finitely generated projective (right) H-module of rank n. Since the ring $\mathbb{H}(x)[y] = \mathbb{H}\otimes_{\mathbb{R}} R(x)[y]$. is a P.I.D., the module P becomes a free module over $\mathbb{H}(x)[y]$. Thus, there is a monic polynomial $f \in \mathbb{R}[x]$ such that $P\otimes_A A[1/f] = H^n\otimes_A A[1/f]$. Let f_{X_3} be the X_3-homogenization of f preserving the degree of f. We extend P to $U = D(X_3)\cup(D(X_1)\cap D(f_{X_3}))$ by taking the free H-module H^n over $D(X_1) \cap D(f_{X_3})$ and glueing with an isomorphism $P \otimes_A A[1/f] \simeq H^n \otimes_A A[1/f]$ over $D(X_3) \cap D(X_1) \cap D(f_{X_3}) = \operatorname{Spec}\mathbb{R}[x, y, x^{-1}, f^{-1}]$. We thus have an extension $\widetilde{P}$ of P to $U = \mathbb{P}^2_{\mathbb{R}} - (0, 1, 0)$. The direct image $i * \widetilde{P}$ of $\widetilde{P}$ over $\mathbb{P}^2_{\mathbb{R}}$ is a coherent sheaf and by [2, p. 110] is reflextive. Since a finitely generated reflexive module over a regular local ring of dimension two is free, the sheaf $i * \widetilde{P}$ is locally free. Thus we have a bundle $\mathscr{E} = i * \widetilde{P}$ which extends P to $\mathbb{P}^2_{\mathbb{R}}$.

234 **Theorem 4.2.** *Let $\mathscr{H}_0$ be the constant sheaf of quaternion algebras over $\mathbb{P}^2_{\mathbb{R}}$. Let $\mathbb{A}^2_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{R}}$ be any real affine plane contained in $\mathbb{P}^2_{\mathbb{R}}$. Then the embedding $i : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ induces a one-to-one correspondence between isomorphism classes of normalized $\mathscr{H}_0$-bundles of rank one over $\mathbb{P}^2_{\mathbb{R}}$ and isomorphism classes of projective H-modules of rank one, where $H = \mathscr{H}_0(U)$.*

Proof. By a linear change of coordinates, we may assume that $\mathbb{A}^2_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}} - V(X_3)$. Then the construction described in (4.3) shows that any projective H-module is the restriction of some $\mathscr{H}_0$-module. The claim now follows from (3.4). □

Remark 4.3. A projective $\mathbb{H}[x, y]$-module P gives a module over an affine plane $\mathbb{A}^2_{\mathbb{R}}$. Its extension to $\mathbb{P}^2_{\mathbb{R}}$ will in general depend on the particular choice of the embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. Fixing the embedding as $\mathbb{A}^2 = \mathbb{P}^2 - V(X_3)$, we denote by $\mathscr{E}(P)$ the normalized bundle which extends the projective $\mathbb{H}[x, y]$-module P of rank one.

4.4 The examples of Ojanguren-Sridharan

A construction of nonfree projective $D[x, y]$-modules of rank one, D any division ring, was described in [9]. For $D = \mathbb{H}$, the construction runs as follows: Let $f, g \in \mathbb{R}[x, y]$ and let $\varphi : \mathbb{H}[x, y]^2 \to \mathbb{H}[x, y]$ be the $\mathbb{H}[x, y]$-linear map given by $\varphi((1, 0)) = f + i$ and $\varphi((0, 1)) = g + j$. Then the sequence

$$0 \to P_{f,g} \to H^2 \xrightarrow{\varphi} H \to 0$$

with $P_{f,g} = \ker \varphi$ and $H = \mathbb{H}[x, y]$ is exact and splits. Thus, $P_{f,g}$ is a projective H-module of rank one. It is shown in [13] by explicit computations that the modules P_{x,y^m} are mutually nonisomorphic. In contrast, **235**
projective $\mathbb{H}[x, y]$-modules of rank > 1 are free (see [5] or for more general results [15]). Thus, we can identify projective $\mathbb{H}[x, y]$-modules or rank one with projective ideals of $\mathbb{H}[x, y]$.

4.5

The correspondence given by (4.2) has application in both directions. In one direction, we deduce results about projective $\mathbb{H}[x, y]$-modules of rank one from results about stable bundles over $\mathbb{P}^2_{\mathbb{C}}$. In the other, known examples of projective $\mathbb{H}[x, y]$-modules give examples of bundles over $\mathbb{P}^2_{\mathbb{C}}$. A first application is to define the second Chern class of a projective $\mathbb{H}[x, y]$-module P of rank one as the second Chern class of the complex 2-bundle $\mathscr{E}(P)_c$. With this definition, we deduce from (2.9) that the moduli space of projective $\mathbb{H}[x, y]$-modules of rank one with fixed second Chern class k is a real manifold of dimension $4k - 3$. Another application is to define the curve $C(P)$ of jump lines of P as the curve of jump lines of $\mathscr{E}(P)_c$.

Since the restriction of $\mathscr{E}(P)_c$ to any real line is trivial, the curve $C(P)$ does not have any real closed points. Thus its degree must be even. Since, by a result of Barth [11], the degree of the curve of jump lines is the second Chern class, we see that the second Chern class of a projective $\mathbb{H}[x, y]$-module of rank one is even. This integer is computed for some examples in section 7 of this paper.

5 Galois cohomology

5.1

Let $\varphi : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \xrightarrow{\sim} M_2(\mathbb{C})$ be the fixed isomorphism of $\mathbb{C}$-algebras

$$s \otimes (u + vj) \to s \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}, \quad u,\ v \in \mathbb{C}.$$

236 Let $R = \mathbb{R}[x, y]$, $C = \mathbb{C} \otimes_{\mathbb{R}} R$ and $H = H \otimes_{\mathbb{R}} R$. For any (right) projective ideal P of H, $\varphi(\mathbb{C} \otimes_{\mathbb{R}} P)$ is a projective ideal of $M_2(C)$ of rank one, hence free. We choose $\gamma \in M_2(C)$ such that

$$\varphi(\mathbb{C} \otimes_{\mathbb{R}} P) = \gamma \cdot M_2(C). \tag{5.2}$$

Let $\tau \otimes 1 : \mathbb{C} \otimes_{\mathbb{R}} H \to \mathbb{C} \otimes_{\mathbb{R}} H$ the conjugation map on $\mathbb{C}$ and let $\sigma = \varphi \circ \tau \otimes 1 \circ \varphi^{-1}$ its transport on $M_2(C)$ through φ. The element $\sigma(\gamma)$ is also a generator of $\varphi(\mathbb{C} \otimes_{\mathbb{R}} P)$ and we have $\sigma(\gamma)\alpha = \gamma$ for some $\alpha \in GL_2(C)$. The element α satisfies the idenity $\sigma(\alpha) = \alpha^{-1}$, i.e., α is a *1-cocycle.* Two 1-cocycles σ and β are cohomologous if $\sigma(\nu)\alpha = \beta\nu$ for some $\nu \in GL_2(C)$. They correspond to isomorphic ideals.

5.2

It follows from $\sigma(\alpha) = \alpha^{-1}$ that $\det \alpha = \overline{\det \alpha^{-1}}$. Thus, by Hilbert 90, there is $\rho \in \mathbb{C}$ such that $\det \alpha = \rho\overline{\rho}^{-1}$. Replacing α by the cohomologous cocycle

$$\begin{pmatrix} \overline{\rho} & 0 \\ 0 & 1 \end{pmatrix} \alpha \begin{pmatrix} 1 & 0 \\ 0 & \rho^{-1} \end{pmatrix},$$

we may assume that $\det \alpha = 1$. We call such a cocycle *normalized*. An explicit computation shows that

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix}, \quad a, b, c, d, \in C.$$

Thus, if $\det \alpha = 1$, the cocycle condition $\sigma(\alpha) = \alpha^{-1}$ reduces to the condition $\overline{\alpha}^t = \alpha$, that is α is a 2×2-hermitian matrix.

237 **Example 5.3.** Let $P_{f,g}$, $f, g \in \mathbb{R}[x,y]$, be a projective ideal of $\mathbb{H}[x,y]$ as constructed in (4.3). A generator $\gamma_{f,g}$ for $\varphi(\mathbb{C} \otimes_{\mathbb{R}} P_{f,g})$ was computed by Parimala in [11] and [13] (see also the computation in [4]):

$$\gamma_{f,g} = \begin{pmatrix} 1+g^2 & g(f-i) \\ g(1+g^2) & g^2(f-i)-2i \end{pmatrix}$$

The corresponding hermitian 2 × 2-matrix is

$$\alpha_{f,g} = \begin{pmatrix} 1+f^2g^4 & -fg(1+g^2)+ig(1+f^2g^2) \\ -fg(1+g^2)-ig(1+f^2g^2) & 4+g^2(1+f^2) \end{pmatrix}$$

We remark that $\gamma_{f_1,g}$ with $f_1 \equiv f \bmod (g^2+1)$ in $\mathbb{R}[x,y]$, is also a generator of $\varphi\mathbb{C} \otimes_{\mathbb{R}} P_{f,g})$. This allows us sometimes to simplify computations.

5.4

Let $\alpha \in GL_2(\mathbb{C}[x,y])$ be a hermitian matrix. We identify $\mathbb{C}[x,y]$ with $\mathbb{R}[x,y]^2$ as $\mathbb{R}[x,y]$-module and define a quadratic form q on $\mathbb{R}[x,y]^4 = \mathbb{C}[x,y]^2$ by

$$q(\xi) = \bar{\xi}^t \alpha \xi, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{C}[x,y]^2.$$

If $\alpha = a + ib$ is the decomposition of a into real and imaginary parts, the 238
form q is given by the real symmetric 4 × 4-matrix $S = \left(\begin{smallmatrix} a & b \\ b^t & a \end{smallmatrix}\right)$. Thus, the matrix S defines a quadratic bundle of rank 4 over $\mathbb{A}^2_{\mathbb{R}}$.

5.5

Let P be a projective ideal of $\mathbb{H}[x,y]$ and let α be a normalized cocycle corresponding to P. By Galois descent we have $\varphi(P) \simeq \{\gamma\eta | \sigma(\gamma\eta) = \gamma\eta, \eta \in M_2(C)\}$. γ is a generator of $\varphi(\mathbb{C} \otimes_{\mathbb{R}} P)$ such that $\sigma(\sigma)\alpha = \gamma$. Thus $\varphi(P) \simeq \{\gamma\eta \mid \sigma(\eta) = \alpha\eta\}$ and P is isomorphic to $Q = \{\eta \in M_2(C) \mid \sigma(n) = \alpha\eta\}$, where $\mathbb{H}[x,y]$ acts on Q (as a right module) through φ. Let

$$f(\eta) = \bar{\eta}^t \alpha\eta, \quad \eta \in Q.$$

We have $f(\eta) = \overline{\eta}^t\sigma(\eta) = \det(\overline{\eta}) = \det(\eta)$ since $\det(\eta) = \det(\sigma(\eta)) = \det(\alpha\eta) = \det(\overline{\eta})$. From this, it is easy to check that f is a quadratic form of type $\mathbb{H}[x,y]$ on Q. Thus, we know by (1.6) that the quadratic form on $\mathbb{R}[x,y]^4$ given by a normalized cocycle α of P is isometric to the norm of P.

6 Transition maps

6.1

Let X_1, X_2, X_3 be the variables of $\mathbb{P}^2_{\mathbb{R}}$ and let $\cup U_i = \mathbb{P}^2$ be the covering given by $U_1 = D(X_i) = \operatorname{Spec} R_i$, where $R_i = \mathbb{R}[X_1, X_i, X_2/X_i, X_3/X_i]$ is a polynomial ring in two variables over $\mathbb{R}$. Further, we introduce the notations $H_i = \mathbb{H} \otimes_{\mathbb{R}} R_i$ and $C_i = \mathbb{C} \otimes_{\mathbb{R}} R_i$. For any module $\mathscr{E}$ of rank one over the constant sheaf $\mathscr{H}_0$ of quaternion algebras over $\mathbb{P}^2_{\mathbb{R}}$, the restriction of $\mathscr{E}$ to U_i defines a projective H_i-module of rank one which **239** is isomorphic to an ideal P_i of H_i. Let, as in (5.2), γ_i be a generator for $\varphi(C \otimes_{\mathbb{R}} P_i)$ in $M_2(C_i)$. Restricting to $U_i \cap U_j$, we can choose the image of γ_i or γ_j as generator. Thus we have

$$\gamma_j = \gamma_i t_{ij}, \quad t_{ij} \in GL_2(C_{ij}) \tag{6.2}$$

where $U_i \cap U_j = \operatorname{Spec} R_{ij}$ and $C_{ij} = \mathbb{C} \otimes_{\mathbb{R}} R_{ij}$. The family $\{t_{ij}\}$ is a family of transition maps for a complex 2-bundle $\mathscr{F}_c$ over $\mathbb{P}^2_{\mathbb{C}}$. We claim that $\mathscr{F} \simeq \mathscr{E}_c(\ell)$ for some $\ell \in \mathbb{Z}$. For this we may first assume that the γ_i are such that the corresponding 1-cocyles α_i are hermitian 2×2-matrices with determinant one. Further, we may assume that the quadratic forms q_i associated with the a_i as in (5.4) are positive definite. We deduce from (6.2) and the relation $\sigma(\gamma_i)a_i = \gamma_i$ that

$$\det \overline{(t_{ij})} \alpha_j = \overline{t_{ij}}^t \alpha_i t_{ij}$$

Since the α_i are positive definite, we have $\det(\overline{t_{ij}}) = \lambda^2_{ij} \in R^X_{ij}$ as in (2.1) and

$$\alpha_j = \overline{u_{ij}}^t \alpha_i u_{ij} \tag{6.3}$$

with $u_{ij} = t_{ij} \cdot \lambda_{ij}^{-1}$. The u_{ij} are transition maps of a bundle $\mathscr{F}(n)$. The relations (6.3) show that this bundle is a quadratic bundle with quadratic form q_i on U_i. Since q_i is of type $\mathscr{H}_i$ on U_i, we see that $F(n)$ is of type $\mathscr{H}_0$ on $\mathbb{P}^2_{\mathbb{R}}$. The bundle $\mathscr{E}(m)$ also is of type $\mathscr{H}_0$ for some $m \in \mathbb{Z}$ (see (2.2)). Since the restrictions of the quadratic bundles $\mathscr{F}(n)$ and $\mathscr{E}(m)$ to U_i are isometric, we conclude by (3.4) that $\mathscr{F} \simeq \mathscr{E}_c(\ell)$ for some ℓ as claimed.

Remark 6.4. Assume that the bundle $\mathscr{E}$ in (6.1) is normalised. Then $\mathscr{E}$ is isomorphic to the bundle $\mathscr{F}(n)$ with the transition maps u_{ij}. The relations (6.3) show that the quadratic structure of $\mathscr{E}$ (or $\mathscr{F}(n)$) is induced by a *hermitian structure* on $\mathscr{E}_c$ with respect to the real structure of $\mathbb{P}^2_{\mathbb{C}}$. Such a structure (which is called *σ-hermitian* in [6], [7] or [10]) is given on a complex bundle $\mathscr{G}$ by an isomorphism $\varphi : \mathscr{G} \to \sigma^*\mathscr{G}^*$ such that $(\sigma^*\varphi)^t = \varphi$. It can be shown, that for any real scheme X, the quadratic structure of a bundle $\mathscr{E}$ of type $\mathscr{H}_0$ is induced by a hermitian structure on $\mathscr{E}_c$. **240**

Example 6.5. Let $P_{x,y}$ be the projective ideal of $\mathbb{H}[x, y]$ given by the Ojanguren-Sridharan construction for $f = x$ and $g = y$ and let $\mathscr{E}_{x,y} = \mathscr{E}(P_{x,y})$ be the normalized extension of $P_{x,y}$ to $\mathbb{P}^2_{\mathbb{R}}$ given by (4.2) (for the identification $\mathbb{A}^2_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}} - V(X_3)$). The restriction of $\mathscr{E}_{x,y}$ to U_3 has $\gamma_3 = \gamma_{x,y}$ (see 5.3) as generator, where $x = X_1/X_3$ and $y = X_2/X_3$. Generators for the restrictions $\mathscr{E}_1$ and $\mathscr{E}_2$ on U_1 and U_2 were computed in [4]: we have

$$\gamma_1 = \begin{pmatrix} 2y^2 - iz(y^2+1) & -3yz - iy(2-z^2) \\ -iy(1-y^2) & y^2 + iz(2-y^2) \end{pmatrix}$$

$$\text{where} \quad y = \frac{X_2}{X_1} \quad \text{and} \quad z = \frac{X_3}{X_1}$$

$$\gamma_2 = \begin{pmatrix} 1 - ixz & -x + iz \\ z(1+ixz) & -i(z^2+2) + zx \end{pmatrix}$$

$$\text{where} \quad x = \frac{X_1}{X_2} \quad \text{and} \quad z = \frac{X_3}{X_2}.$$

7 Chern Classes

241 **Proposition 7.1.** *Let $\mathscr{E}_{x,y}$ be as in (6.4). We have $c_1(\mathscr{E}_{x,y}) = 0$ and $c_2(\mathscr{E}_{x,y}) = 2$.*

Proof. Let $E_i = \mathscr{E}_{x,y}|_{U_i}$ and let γ_i be the generator of E_i given in (6.5). Let $\mathscr{F}$ be the complex 2-bundle given by the transition maps t_{ij} satisfying $\gamma_j \simeq \gamma_i t_{ij}$. We have $c_1(\mathscr{F}) = -2$, thus $\mathscr{E}_{x,y} \simeq \mathscr{F}(1)$. Writing the 2×2-matrix γ_i as $\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$, we check that

$$t_{ij}\begin{pmatrix} d_j \\ c_j \end{pmatrix} = \begin{pmatrix} d_i \\ -c_i \end{pmatrix}\left(\frac{X_i}{X_j}\right)^2$$

Thus $\left(\begin{smallmatrix} d_i \\ -c_i \end{smallmatrix}\right)$ is a global section of $\mathscr{G} = \mathscr{F}(2)$. The zeros of this section are $(1,0,0)$ and $(1,\pm 1, i)$ with multiplicity one. Thus we have $c_2(\mathscr{G}) = 3$. It follows from the formula $c_2(\mathscr{G}(m)) = c_2(\mathscr{G}) + mc_1(\mathscr{G}) + m^2$, that $c_2(\mathscr{E}_{x,y}) = c_2(\mathscr{G}(-1)) = 3 - 2 + 1 = 2$ as claimed. □

Theorem 7.2. *Let $f, g \in R[x,y]$ be algebraically independent. Then we have $c_2(\mathscr{E}_{f,g'}) = 2 \cdot [\mathbb{C}(x,y) : \mathbb{C}(f,g)]$.*

Proof. Let $f'(x,y) = f(x+y^r, y)$, $g'(x,y) = g(x+y^r, y)$. It can be shown that $c_2(\mathscr{E}_{f',g'}) = c_2(\mathscr{E}_{f,g})$. Thus, by choosing r big enough, we may assume that f and g have as terms of higher degree monomials y^n and y^m. Assume that $n \geq m$.

Let $x = X_1/X_3$, $y = X_2/X_3$ and let F, resp. G be the X_3-homogenization
242 of f, resp. g. We define a rational map $\Psi : \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ which extends the map $\varphi : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^2_{\mathbb{R}}$ given by $x \to f$, $y \to g$ by $\Psi(X_1) = F$, $\Psi(X_2) = GX_3^{n-m}$, $\Psi(X_3) = X_3^n$. The map Ψ is not defined at $(1,0,0)$. By a sequence of blow-ups at real closed points we construct a resolution of Ψ:

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & \\
\mathbb{P}^2_{\mathbb{R}} \arrow[rr, "\Psi"'] & & \mathbb{P}^2_{\mathbb{R}}
\end{tikzcd}$$

Since $P_{f,g} = \varphi^* P_{x,y}$, we have $\pi^* \mathscr{E}_{f,g} \simeq \phi^* \mathscr{E}_{x,y}$ outside of

$$Z = \pi^{-1}(\{(1,0,0)\}) \subset Y.$$

The set Z is a set of real lines. Therefore $\pi^* \mathscr{E}_{f,g}$ and $\phi^* \mathscr{E}_{x,y}$ are anisotropic over Z. In view of (3.2), the two bundles $\pi^* \mathscr{E}_{f,g}$ and $\phi^* \mathscr{E}_{x,y}$ are isomorphic over Y. Thus we have $c_2(\mathscr{E}_{f,g}) = c_2(\pi^* \mathscr{E}_{f,g}) = c_2(\phi^* \mathscr{E}_{x,y})$. $\deg \phi = 2$. $[\mathbb{C}(x,y) : \mathbb{C}(f,g)]$. □

8 Curves of jump lines

8.1

Let $\mathscr{E}$ be a stable 2-bundle over $\mathbb{P}^2_{\mathbb{C}}$ with $c_1(\mathscr{E}) = 0$ and let $C(\mathscr{E})$ be the curve of jump lines of $\mathscr{E}$. By [1] we know that the degree of $C(\mathscr{E})$ is equal to $c_2(\mathscr{E})$. We compute the equation of the curve of jump lines for the following classes of bundles $\mathscr{E}_{f,g}$:

(1) $g = y$

(2) $f \in \mathbb{R}[x,y]$ is monic and has degree n as a polynomial in x. By the last remark of (5.3) we may assume that f is linear in y.

By (7.2), we know that the degree of $C_{f,g} = C(\mathscr{E}_{f,g})$ is $2n$.

8.2

243

Let E_3 be the restriction of $\mathscr{E}_{f,g}$ to $U_3 = D(X_3)$. A generator γ_3 for $\mathbb{C} \otimes_{\mathbb{R}} E_3$ is given in (5.3) using affine coordinates $x = X_1/X_3$ and $y = X_2/X_3$. We have $\det \gamma_3 = (X_2^2 + X_3^2)/X_3^2$. Let $V_2 = D(X_2) \cap D(X_2^2 + X_3^2)$ and let H_2 be the restriction of the constant sheaf $\mathscr{H}_0$ to V_2. Since γ_3 invertible on $U_3 \cap V_2$, γ_3 induces an isomorphism $E_3|_{U_3 \cap V_2} \xrightarrow{\sim} H_2|_{U_3 \cap V_2}$. Thus we can extend E_3 to a bundle $\mathscr{F}$ over $U = U_3 \cap V_2$ by taking H_2 on V_2 and glueing with φ_3 over $U_3 \cap V_2$. Since $\mathbb{P}^2 - U = \{(1,0,0)\}$, the bundle $\mathscr{F}$ on U has a unique extension to $\mathbb{P}^2$, also denoted by $\mathscr{F}$. Since $\mathscr{F}|_{U_3} \simeq \mathscr{E}|_{U_3}$, the bundles $\mathscr{F}$ and $\mathscr{E}$ only differ by a twist on $\mathbb{P}^2$. Hence they have the same curves of jump lines.

8.3

The equation $X_1 = uX_2 + vX_3$, u, $v \in \mathbb{R}$ gives a real line L contained in $U = U_3 \cup V_2$ (see (8.2)). We compute the lines L which are jump lines for $\mathscr{F}$. Using affine coordinates $x = X_1/X_3$ and $y = X_2/X_3$ and substituting $x = uy + v$ in γ_3, we obtain a transition map

$$\Gamma = \begin{pmatrix} 1+y^2 & y(f_1 - i) \\ y(1+y^2) & y^2(f_1 - i) - 2i \end{pmatrix}$$

for $\mathscr{F}|_L$ on $U_3 \cap L \cup U_2 \cap L$, where $f_1(y) = f(uy+v, y)$. Since $U_3 \cap L = \operatorname{Spec}\mathbb{R}[y]$ and $V_2 \cap L = \operatorname{Spec}\mathbb{R}[y^{-1}, (1+y^{-2})^{-1}]$, we see that we have to diagonalize Γ, operating on the right over $\mathbb{R}[y]$ and on the lest over $\mathbb{R}[y^{-1}, (1+y^{-2})^{-2}]$, to determine the type of $\mathscr{F}|_L$. In particular we can reduce f_1 modulo $1 + y^2$ by column operations. Thus we may assume
244 that $f_1 = A(u,v)y + B(u,v)$ is linear in y. By further row and column operations we check that a pair (u,v) gives a jump line if and only if $A(u,v)^2 + B(u,v)^2 + 1 = 0$. Under the given hypothesis for f, this is an equation of degree $2n$, which by (7.2) is the degree of the curve of jump lines. Thus we have the affine part of the curve. Homogenizing, we obtain the full curve of jump lines.

Example 8.4. Let $f = x$, then $f_1 = uy + v$ and the equation of the curve of jump lines of $\mathscr{E}_{x,y}$ is

$$U^2 + V^2 + W^2 = 0.$$

9 Bundles with $c_2 = 2$ and $c_2 = 4$

9.1

Let $\mathscr{H}_0$ be the constant sheaf of quaternion algebras over $\mathbb{P}^2_{\mathbb{R}}$ and let $\mathscr{E}$ be a normalised $\mathscr{H}_0$-module of rank one with $c_2(\mathscr{E}) = 2$. We know that the curve $C(\mathscr{E})$ of jump lines of $\mathscr{E}$ is a real conic without real points. Such a conic is induced by the conic (8.4) through a real projective transformation $\rho \in PGL_3(\mathbb{R})$. By a result of Barth [1], 2-bundles over $\mathbb{P}^2_{\mathbb{C}}$ with $c_1 = 0$ and $c_2 = 2$ are isomorphic if and only if they have the same curve of jump lines. Thus $\rho^*\mathscr{E}_{x,y}$ and $\mathscr{E}$ are isomorphic as $\mathscr{H}_0$-modules. Conversely, for any $\rho \in PGL_3(\mathbb{R})$, $\rho^*\mathscr{E}_{x,y}$ is a normalized $\mathscr{H}_0$-module. Thus the set of isomorphism classes of $\mathscr{H}_0$-modules of rank one with $c_1 = 0$ and $c_2 = 2$ is in bijection with the orbit of the conic $V(U^2 + V^2 + W^2)$ under the action of group $PGL_3(\mathbb{R})$. This gives an explicit description of the 5-dimensional moduli space of these bundles (see (2.9)).

9.2

Let $\mathscr{E}$ be as in (9.1) and let $\rho \in PGL_3(\mathbb{R})$ such that $\rho^*\mathscr{E}_{x,y} \simeq \mathscr{E}$. By **245**
modifying ρ by an element in the isotropy group of $V(U^2 + V^2 + W^2)$, we may assume that ρ fixes the line at infinity $V(X_3)$. Then we have $\rho^*\mathscr{E}_{x,y} = \mathscr{E}_{f,g}$ with $f = ax + by + c$, $g = a'x + b'y + c'$, both linear. The bundles $\mathscr{E}_{f,g}$ and $\mathscr{E}_{x,y}$ are isomorphic if and only if $c = c' = 0$ and $\left(\begin{smallmatrix} a & b \\ a' & b' \end{smallmatrix}\right)$ belongs to the orthogonal group of the quadratic form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. By a similar argument, we see that the one-to-one correspondence between normalized $\mathscr{H}_0$-modules of rank one over $\mathbb{A}^2_{\mathbb{R}}$ and $\mathbb{P}^2_{\mathbb{R}}$ does not depend on the embedding $\mathbb{A} \to \mathbb{P}^2$ for bundles with $c_2 = 2$.

Example 9.3. As noticed by parimala, we can replace xy by x in $\gamma_{x,y}$ (see (5.3)) and obtain a new matrix

$$\gamma' = \begin{pmatrix} 1 + y^2 & x - iy \\ y(1 + y^2) & y(x - iy) - 2i \end{pmatrix}$$

with corresponding 1-cocycle

$$a' = \begin{pmatrix} 1 + x^2y^2 & -x(1 + y^2) + iy(1 + x^2) \\ -x(1 + y^2) - iy(1 + x^2) & 4 + x^2 + y^2 \end{pmatrix}$$

This cocycle belongs to some projective ideal P of $\mathbb{H}[x, y]$ with reduced norm given by the symmetric 4×4-matrix.

$$S = \begin{pmatrix} 1 + x^2y^2 & -x(1 + y^2) & 0 & y(1 + x^2) \\ -x(1 + y^2) & 4 + x^2 + y^2 & -y(1 + x^2) & 0 \\ 0 & -y(1 + x^2) & 1 + x^2y^2 & -x(1 + y^2) \\ y(1 + x^2) & 0 & -x(1 + y^2) & 4 + x^2 + y^2 \end{pmatrix}$$

The matrix S gives the simplest known example of a nontrivial quadratic bundle over $\mathbb{A}^2_{\mathbb{R}}$. We claim that S is isometric to the norm on $P_{x,y}$ or that $P \simeq P_{x,y}$. Let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be the quadratic transformation given by $x \to xy$, $y \to y$. We have $P_{x,y} = \varphi^* P$. Since φ is a birational map, $c_2(P) = c_2(P_{x,y}) = 2$ and the curve of jump lines of P is a conic. By computations as in (8.3), we obtain that $C(P) = V(U^2 + V^2 + W^2)$. Thus $P \simeq P_{x,y}$ as claimed. 246

9.4

Let $f \in \mathbb{R}[x, y]$ be monic and of degree 2 as polynomial in x and let $g = y$. As already noticed we may assume that f is linear in y. In view of (7.2) the curve of jump lines of $\mathscr{E}_{f,g}$ has degree 4.

We obtain by (8.3) an affine equation for the curve of jump lines of the form

$$(u^2 + v^2)^2 + p(u, v) = 0$$

where p is of degree 3. For example, $f = x^2 + a$ gives the equation

$$(u^2 + v^2)^2 + 2a(u^2 - v^2) + (a^2 + 1) = 0.$$

247 These curves which have the two cyclic points. $(1, \pm i, 0)$ as double points, are called *bicircular quartics*. Since a quartic with two double points depends on 12 parameters, we see that it is impossible to obtain all bundles $\mathscr{E}$ with $c_1 = 0$ and $c_2 = 4$ which are $\mathscr{H}_0$-modules of rank one, as translates of some $\mathscr{E}_{f,y}$, f quadratic in x, for the action of the group $PGL_3(\mathbb{R})$.

10 Bundles over a quadric and extensions to $\mathbb{P}^3$

10.1

Another way to extend the projective $\mathbb{H}[x, y]$-module $P_{x,y}$ to a bundle over $\mathbb{P}^2_{\mathbb{R}}$ is to homogenize the sequence 4.3: We put $H = \mathbb{H}[X_1, X_2, X_3]$ and define a map $\varphi : H^2 \to H$ by $\varphi((1, 0)) = X_1 + iX_3$ and $\varphi((0, 1)) =$

$X_2 + jX_3$. The kernal of the map φ is a graded module over $\mathbb{R}[X_1, X_2, X_3]$ and the associated sheaf over $\mathbb{P}^2_{\mathbb{R}}$ is a bundle which extends $P_{x,y}$.

This suggests the following construction of a bundle over $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$: We put $H = \mathbb{H}[X_1, X_2, U_1, U_2]$ and define $\varphi : H^2 \to H$ by $\varphi((1,0)) = X_1 + iX_2$, $\varphi((0,1)) = U_1 + jU_2$. The kernel defines a bundle $\mathscr{E}$ over $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$. Assuming that $\mathscr{E}$ is normalized, we have $c_2(\mathscr{E}) = 2$, and the jump lines of $\mathscr{E}$ are the four lines $X_1^2 + X_2^2 = 0$, $U_1^2 + U_2^2 = 0$. Let $\mathbb{Q} \subset \mathbb{P}^3$ be the image of $\mathbb{P}' \times \mathbb{P}^1$ by the Segre embedding. It follows by results of Le Potier [8] that $\mathscr{E}$ as a bundle over $\mathbb{Q}$ is the restriction of two instanton bundles over $\mathbb{P}^3$. We can embed $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ in such a way that the nonstandard real structure on $\mathbb{P}^3_C$ induces the usual real structure on $\mathbb{P}^1 \times \mathbb{P}^1$. Thus the restriction of an instanton bundle over $\mathbb{P}^3_{\mathbb{C}}$ to $\mathbb{P}^1 \times \mathbb{P}^1$ is a normalised $\mathscr{H}_0$-bundle of rank one over $\mathbb{P}^1 \times \mathbb{P}^1$. By a result of Le potier mentioned above, any normalized $\mathscr{H}_0$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ with $c_2 = 2$ is such a restriction. It would be interesting to know if this is true **248** for higher values of c_2.

10.2

Similarly, one can consider the bundles over $\mathbb{P}^2$ which are restrictions of instanton bundles over $\mathbb{P}^3$. By results of Donaldson [3], these are stable bundles which are trivial on the line at infinity. Hence $\mathscr{H}_0$-modules of rank one over $\mathbb{P}^2$ are such restrictions, since they are trivial over any real line. But there is, *a priori*, no relation between the quaternionic structures on the bundles over $\mathbb{P}^2$ and over $\mathbb{P}^3$. Let $\pi : \mathbb{A}^3_{\mathbb{R}} \to \mathbb{A}^2_{\mathbb{R}}$ be the projection given by $(x, y, z) \to (x, y)$. As communicated to us by Parimala, nontrivial $\mathscr{H}_0$-modules of rank one over $\mathbb{A}^3_{\mathbb{R}}$ of the form $\pi^* P$, P an $\mathscr{H}_0$-module over $\mathbb{A}^2_{\mathbb{R}}$, cannot be extended as $\mathscr{H}_0$-modules to $\mathbb{P}^3_{\mathbb{R}}$.

References

[1] Barth, W. : Moduli of vector bundles on the projective plane, *Invent. Math.* 42 (1977), 63-91.

[2] Colliot-Thélène, J:-L.; J.-J. Sansuc : Fibrés quadratiques et composantes connexes réelles, *Math. Ann.* 244 (1979), 105-134.

[3] Donaldson, S. K. : Instantons and Geometric Invariant Theory, Preprint.

[4] Knus, M. -A. : Quaternionic modules over $\mathbb{P}^2_{\mathbb{R}}$, *Springer Lecture Notes* 917, 245-259, 1982,

249 [5] Knus, M. -A., M. Ojanguren: Modules and quadratic forms over polynomial algebras, *Proc. Amer. Math. Soc.* 66 (1977), 223-226.

[6] Knus, M.-A., M. Ojanguren, R. Parimala: Positive definite quadratic bundles over the plane, *Comment. Math. Helvetici* 57 (1982), 400-411.

[7] Knus, M.-A., R. Parimala, R. Sridharan: Nonfree projective modules over $\mathbb{H}[x,y]$ and stable bundles over $\mathbb{P}^2(C)$, *Invent. Math.* 65 (1981), 13-27.

[8] Le Potier, J.: Sur l'espace des modules des fibres de Yang et Mills, *Progress in Math.* 37; Mathematique et Physique, Ed.: L. Boutet de Monvel, A. Douady, J.-L. Verdier, Birkhauser, 65-137, 1983.

[9] Ojanguren, M., R. Sridharan: Cancellation of Azumaya algebras, *J. Algebra* 18 (1971), 501-505.

[10] Ojanguren, M., R. Parimala, R. Sridharan: Indecomposable quadratic bundles of rank $4n$ over the real affine plane, *Invent. Math.* 71 (1983), 643-652.

[11] Parimala, R.: Failure of a quadratic analogue of Serre's conjecture, *Amer. J. Math.* 100 (1978), 913-924.

[12] Parimala, R.: Indecomposable quadratic spaces over the affine plane, to appear in *Advances in Mathematics.*

250 [13] Parimala, R., R. Sridharan: Projective modules over polynomial rings over division rings, *J. Math. Kyoto Univ.* 15 (1975), 129-148.

[14] Scharlau, W.: Remark on symmetric bilinear forms over euclidean domains, Preprint.

[15] Stafford, J. T.: Projective modules of polynomial extensions of division rings, *Invent. Math.* 59 (1980), 105-117.

Mathematik, ETH-Zentrum.
CH-8092, Zurich.

On Projective Modules Over Positively Graded Rings

By H. Lindel

Introduction

251

Let A be a noetherian ring of finite Krull dimension $\dim(A) = d$. A fundamental result of Serre says that every projective A-module P of $r = \mathrm{rank}(P) > d$ has a unimodular element p, i.e., an element p such that P splits into a direct sum $P = Q \oplus Ap$. This reduces the study of the structure of projective modules over finite dimensional noetherian rings to the case $\overset{\vee}{=} d$. The maximal $r \in |N$ such that there exists a projective A-module without unimodular elements is called the *Serre dimension* Serre-dim(A) of A. In 1979, Plumstead showed that, for a polynomial ring $R = A[T]$ in one indeterminate T, that Seree-dim ($A[T]$) $\leq \dim(A)$, thus settling a question of Eisenbud and Evans [[7]). In 1981, Bhatwadekar and Roy generalized Plumstead's result by proving that Serre-dim ($A[T_1, \ldots, T_n]$) $\leq \dim(A)$ ([4, Theorem 3.1], and Theorem 2.4 below). Recently, this result was extended to Laurent polynomial rings $R = A[T_1, \ldots, T_n, U_1^{\pm 1}, \ldots, U_m^{\pm 1}]$ establishing a positive answer to a question of Bass and Murthy ([3, Theorem 4.1]], and [2, § 9]). To look for further cases of ring extensions R of A with Serre-dim (R) $\leq \dim(A)$ it seems to be sensible to consider rings R that **252** are birationally equivalent to a polynomial ring $A[T_1, \ldots, T_n]$. If, in this case $n = 1$, we know by a result of Rao ([9, Theorem 1.1]) that indeed Serre-dim (R) $\leq \dim(A)$.

In studying the structure of projective modules, it can be useful to consider projective modules over positively graded rings that we write in the usual form $R = \bigoplus_{i \geq 0} R_i$ (See, for example, the use of Criterion *I* in [3]). The purpose of this paper is to give an elementary approach to some basic theorems on projective modules over such rings. we give some applications, but the approach itself is the main point.

In § 1, we generalise Quillen's patching theorem ([8, Theorem 1]) from polynomial rings to positively graded rings (cf. Theorem 1.3):

Let M be a finitely presented module over a positively graded ring $R = \bigoplus_{i \geq 0} R_i$, $A = R_0$. Then the set $J(A, M)$ of all $u \in A$, for which M_u is extended from A_u, is an ideal in A.

In the case of a projective M or a flat extension R/A, Murthy has observed ([6, Theorem 3.6]) that Quillen's technique applies. Our proof is based on the crucial Matrix-Lemma 1.1 and does not depend on Quillen's techniques. In particular, we need no flatness assumption. The above result was obtained independently by Artamonow ([1]).

As a first application, we derive a lovely result of Vorst ([10, Theorem 3.2], and Theorem 1.5 below).

253 *Let A be a noetherian ring such that all projective $A[T_1, \ldots, T_n]$-modules are extended from A. Then every projective module over discrete Hodge A-algebras is extended from A.*

In § 2, we apply our matrix lemma to obtain in correspondence with Criterion I in [3] (cf. (2.1)).

Let P be a projective module over a positively graded ring $R = \bigoplus_{i \geq 0} R_i$. Assume there exists an element $q \in P$, whose canonical images in the localizations P_{1+R} and $P_{1+J}(A, R)$, $A = R_0$, are unimodular. Then P has a unimodular element p with $p - q \in R^+ P$, $R^+ = \bigoplus_{i \geq 1} R_i$.

As a first application, we deduce the results of Plumstead and Bhatwadekar/Roy on the Serre dimension of polynomial rings (cf. (2.4)).

If $\dim(R) \geq \dim(A) + 1$ and R is an affine algebra over a field, then in the situation of Theorem 2.1 it suffices to know that the canonical image q_{1+JR^+} of q in P_{1+JR^+}, $J = J(A, P)$, is unimodular (cf. (2.5)).

In § 2, we apply our results to projective modules over "Segre extensions" of a field k, i.e., to a k-algebra $S_{mn} = k[x_{ij}]$, $1 \leq i \leq m$, $1 \leq j \leq n$ with $x_{ij}x_{ls} - x_{is}x_{lj} = 0$, $1 \leq i \leq m$, $1 \leq j \leq n$. We obtain the following results (Theorem 2.7 and Theorem 2.8)

(1) *Every stably free S_{mm}-module P of rank* $(P) \geq 1 + \min\{m, n\}$, *is free.* 254

(2) *If k is infinite, then every projective S_{mn}-module P of* $\operatorname{rank}(P) \geq 1 + \min\{m, n\}$ *contains a unimodular element.*

We have not made serious attempts to remove the additional assumption on k in the assertion 2. It remains an open question, whether all projective S_{mn}-modules are free or at least stably free.

The Segre extensions can also be written as $S_{mn} = A[\mathfrak{m}T_2, \ldots, \mathfrak{m}T_n]$ where $A = k[X_1, \ldots, X_m]$ and $\mathfrak{m}$ is the canonical maximal ideal $\sum\limits_{i=1}^{m} AX_i$ of A, $T_2, \ldots, T_n$ indeterminates. If, in particular, $n = 2$, S_{mn} appears as the blowing up of A in $\mathfrak{m}$, i.e. as the Rees ring $S_{m2} = \bigoplus\limits_{i\geq 1} \mathfrak{m}^n = A[\mathfrak{m}T]$. Assume, more generally that A is a regular ring and let $\mathfrak{p}$ be a prime ideal in A such that $A/\mathfrak{p}$ is regular. We are interested in an example where projective $A[T]$-modules and projective $A/\mathfrak{p}[T]$-modules are extended from A but projective $A[\mathfrak{p}T]$-modules are not extended from A.

Acknowledgements

We would like to thank the colleagues and officials from the Tata Institute of Fundamental Research for their hospitality during the International Colloquium on Algebraic Vector Bundles in January 1984. In particular, we are grateful to S.M. Bhatwadekar and R.A. Rao for many 255 fruitful discussions.

1 The basic matrix lemma and the graded patching theorem

Let M be a module over a ring R and let A be a subring of R. M is called *extended from A*, if there exists an A-module N with $M = R \otimes_A N$. If $R = \bigoplus\limits_{i\geq 0} R_i$ is a positively graded ring with $A \cong R_0$ then $N = M/R^+M$.

In terms of relation modules, extendability can be expressed by saying that $M = R^n/RK$, where K is a submodule of A^n and RK is the R-module generated by the vectors in K considered as vectors in R^n. This is a consequence of right exactness of the tensor product. Hence M is extended from A if and only if it can be presented by a matrix with coefficients in A.

Now assume that there exist comaximal elements $u_1, u_2 \in A$ such that $M_i = M_{u_i}$ is extended from $A_i = A_{u_i}$, $i = 1, 2$. It is easy to see that there exist two submodules N_1, N_2 of M with the following properties: (1) $M = N_1 + N_2$, (2) $M_i = (N_i)_i = (N_i)_{u_i}$, $i = 1, 2$ (3) N_i *has a relation module* $L_i \subset R$ *such that* R_iL_i, $R_i = R_{u_i}$, *is generated by vectors with components in the canonical image of* A *in* A_i, $i = 1, 2$. The epimorphisms $R^{s_i} \to N_i$ with kernel L_i give rise to a presentation of M of the form $R^m \xrightarrow{j} R^{s_1} \oplus R^{s_2} \to M \to 0$, where $j(R^m)$ is the fiber product of R^{s_1} and R^{s_2} over M with respect to the compositions of $R^{s_i} \to N_i$ and the **256** natural inclusion $N_i \to M$, $i = 1$, 2. It follows from (2) that $u_i^l M \subset N_i$ for a suitable $l \in \mathbb{N}$, $i = 1, 2$, and (1) allows to assume that j is given by a matrix $\underline{D}$ of the form

$$\underline{D} = \left(\begin{array}{c|c} \underline{V}_2 & \underline{0} \\ \hline u_1^l\underline{E} & \underline{B}_1 \\ \hline \underline{B}_2 & u_2^l\underline{E}' \\ \hline \underline{0} & \underline{V}_1 \end{array}\right), \quad \underline{E} \text{ and } \underline{E}' \text{ unit matrics.}$$

The submatrices $\underline{V}_i$ present N_i, $i = 1, 2$. They can be chosen in the form $\underline{V}_i = \left(\begin{array}{c} \underline{V}_i \\ \hline \underline{C}_i \end{array}\right)$, where the canonical image of $\underline{V}'_i$ in $\mathrm{Mat}(A_i)$ presents $(N_i)_i$, $i = 1, 2$. This implies that, over R_1, the row vectors of $\underline{D}$ are generated by the row vectors of

$$\underline{D}_1 = \left(\begin{array}{c|c} \underline{V}'_2 & \underline{0} \\ \hline u_1^l\underline{E} & \underline{B}_1 \end{array}\right),$$

and that, over R_2, they are generated by those of

$$\underline{D}_2 = \left(\begin{array}{c|c} \underline{B}_2 & u_2^l\underline{E}' \\ \hline \underline{0} & \underline{V}_1 \end{array}\right).$$

Since u_1 and u_2 are comaximal, we see that in $\underline{D}$ the matrices $\underline{V}_i$ can be replaced by $\underline{V}_i'$, $i = 1, 2$. Therefore, we may assume that the matrices $\underline{V}_i$ themselves have coefficients in A. Notice that there exists a $t \in \mathbb{N}$ such that $u^t(u_1^l u_2^l \underline{E}' - \underline{B}_2\underline{B}_1)$ is a right multiple of $\underline{V}_1$ and $u^t(u_1^l u_2^l \underline{E} - \underline{B}_1\underline{B}_2)$ **257**
is a right multiple of $\underline{V}_2$. To fix terminology, let us call a matrix $\underline{D}$ of the described form *fibred* (*over A with respect to u_1, u_2*). We are going to show that every fibred matrix over a positively graded ring $R = \bigoplus_{i\geq 0} R_i$ is equivalent to its "constant term", where "equivalence" means that we need only elementary transformations.

Let us first fix some notation concerning positively graded rings $R = \bigoplus_{i\geq 0} R_i$. As usual, we set $R^+ = \bigoplus_{i\geq 0} R_i$ and call a $r \in R_i$ homogenous of degree i. If $r = \sum_{i\geq 0} r_i$, then r_0 is the constant term of r. In correspondence with the decomposition of R into a direct sum of homogenous components R_i, we have a decomposition of matrices $\underline{B} \in \mathrm{Mat}(R)$ into a sum $B = \sum_{i\geq 0} B_i$, $\underline{B}_i \in \mathrm{Mat}(R_i)$, which is canonically induced by the decomposition of the coefficients. Every $a \in R_0$ induces a R_0-algebra homomorphism $h_a : R \to R$ with $h_a\left(\sum_{i\geq 0} r_i\right) = \sum_{i\geq 0} a^i r_i$. If $\underline{D}$ is a matrix over R, then $h_a \cdot \underline{D}(0)$ denotes the constant term of $\underline{D}$.

The next lemma is the crucial observation in our approach.

Lemma 1.1. *Let $R = \bigoplus_{i\geq 0} R_i$ be a positively graded ring and let D be a matrix with coefficients in R of the form*

$$\underline{D} = \left(\begin{array}{c|c} u^l\underline{E} & \underline{B}_1 \\ \hline \underline{B}_2 & \underline{B}_3 \\ \hline \underline{0} & \underline{V} \end{array}\right),$$

where $\underline{E}$ *is a unit matrix,* $\underline{V}$ *has coefficients in* R_0*, and* $u \in R_0$*. Assume* 258 *that* $u^t(u^l B_3 - B_2 B_1) = GV$*,* G *a matrix,* t *a natural number. Then, for all* $n > t + l$ *and all* $a, c \in R_0$*, the matrices* $h_a(\underline{D})$ *and* $h_{a+cu^n}(\underline{D})$ *are equivalent.*

Proof. We extend R to a polynomial ring $R' = R[T]$, T an indeterminate R' can be considered as a positively graded ring with homogeneous components $R'_i = R_i[T]$, $i \geq 0$. The R_0-algebra homomorphisms h_a, $a \in R_0$, extend canonically to R'_0-algebra homomorphisms of R'. Set $h = h_{a+cu^n}$. The matrix $h_a(\underline{D})$ is equivalent to a matrix

$$\underline{D}_0 = \left(\begin{array}{c|c} u^l\underline{E} & h(\underline{B}_1) \\ \hline h(B_2) & \underline{C} \\ \hline \underline{O} & \underline{V} \end{array}\right)$$

where C is a submatrix, whose relations with $h(\underline{B}_3)$ and $\underline{V}$ will now be described. There exist matrices $\underline{L}$, $\underline{L}'$ with

$$\left(\begin{array}{c|c} \underline{E} & \underline{O} \\ \hline \underline{L} & \underline{E}' \end{array}\right) \left(\begin{array}{c|c} u^l\underline{E} & h_a(\underline{B}_1) \\ \hline h_a(\underline{B}_2) & h_a(\underline{B}_3) \end{array}\right) \left(\begin{array}{c|c} \underline{E} & \underline{L} \\ \hline \underline{O} & \underline{E} \end{array}\right) = \left(\begin{array}{c|c} u^1\underline{E} & h(\underline{B}_1) \\ \hline h(\underline{B}_2) & \underline{C} \end{array}\right)$$

$\underline{E}'$ a unit matrix. A simple calculation leads to

$$u^l\underline{C} - h(\underline{B}_2)h(\underline{B}_1) - u^l h_a(\underline{B}_3) - h_a(\underline{B}_2)h_a(\underline{B}_1)$$

and hence to

$$u^l(\underline{C} - h_a(\underline{B}_3)) = h(\underline{B}_2)h(\underline{B}_1) - h_a(\underline{B}_2)h_a(\underline{B}_1).$$

Since $h(r) - h_a(r) \in R'T$ for all $r \in R^1$, we obtain that $\underline{C} - h_a(\underline{B}_3)$ and $h(\underline{B}_3) - h_a(B_3)$ are divisible by T. Hence $\underline{C} - h(\underline{B}_3)$ is divisible by T and we have $\underline{C} - h(\underline{B}_3) = \underline{H}T$, $\underline{H}$ a matrix. We have

$$u^t(u^l\underline{C} - h(\underline{B}_2)h(\underline{B}_1)) = u^t(u^l h_a(\underline{B}_3) - h_a(\underline{B}_2)h_a(\underline{B}_1)) = h_a(G)V.$$

Applying h to

$$u^t(u^l\underline{B}_3 - \underline{B}_2\underline{B}_1) = \underline{GV}$$

we obtain

$$u^t(u^l h(\underline{B}_3) - h(\underline{B}_2)h(\underline{B}_1) = h(\underline{G})\underline{V},$$

and hence 259

$$u^{t+1}(\underline{C} - h(\underline{B}_3)) = \underline{W}_0\underline{V}_0, \underline{W}_0 = h_a(\underline{G}) - h(\underline{G}).$$

Because $\underline{C} - h(\underline{B}_3) = \underline{H}T$ and $\underline{V}$ has coefficients in R_0, T divides $\underline{W}_0$ and we have $\underline{W}_0 = \underline{W}T$ for a matrix $\underline{W}$. This implies $u^{t+l}\underline{H} = \underline{W}\underline{V}$. Now we return from R' to R by the substitution $T \to cu^n$, $n \geq t + l$, $c \in R_0$, the result of which we denote by a star "$*$". We have $\underline{C}^* - h^*(\underline{B}_3) = cu^n\underline{H}^* = \underline{S}\,\underline{V}$, $\underline{S} = cu^{n-1-t}\underline{W}^* h^* = h_{a+cu^{n+1}}$. We obtain

$$\underline{D}^* = \left(\begin{array}{c|c} u^l\underline{E} & h^*(\underline{B}_1) \\ \hline h^*(\underline{B}_2) & h^*(\underline{B}_3) + \underline{S}\,\underline{V} \\ \hline \underline{0} & \underline{V} \end{array}\right).$$

Hence $\underline{D}^*$ it is equivalent to $h^*(\underline{D})$. Because $h_a(\underline{D})$ is equivalent to $\underline{D}_0$ with coefficients in R, it is equivalent to $\underline{D}_0^*$, whence to $h^*(\underline{D})$. This proves the assertion. ☐

In terms of modules, Lemma 1.1 implies the following

Theorem 1.2. *Let M be a finitely presented module over a positively graded ring $R = \bigoplus_{i\geq 0} R_i$. Assume that $M = R \otimes_{R'} M'$, $R' = h_a(R)$ for an $a \in R_0$ and M' a finitely presented R'-module. If M'_u is extended from $(R_0)_u$ for a $u \in R_0$, then M is extended from $h_{a+cu^n}(R)$ for all $c \in A$ and sufficiently high $n \in \mathbb{N}$.*

Proof. Since M'_u is extended from $(R_0)_u$' it can be presented by a matrix of the form $h_a(\underline{D})$ where $\underline{D}$ has the form as in Lemma 1.1. By Lemma 260
1.1, M can be presented by every matrix $h_{a+cu^n}(\underline{D})$, $c \in R_0$, $n \in \mathbb{N}$ sufficiently high. ☐

Now it is easy to generalize Quillen's patching theorem ([8, Theorem 1]) from polynomial rings to positively graded rings.

Theorem 1.3. *Let M be a finitely presented module over a positively graded ring $R = \bigoplus_{i\geq 0} R_i$ and let $J(A, M)$ be the set of all $u \in A$, $A = R_0$ for which M_u is extended from A_u. Then $J(A, M)$ is an ideal in A. If, for all maximal ideals $\mathfrak{m}$ of A, the localizations $M_{\mathfrak{m}} = M_{A\setminus\mathfrak{m}}$ are extended from $A_{\mathfrak{m}}$, then M is extended from A.*

Proof. It suffices to show that M is extended from A, if there exist two comaximal elements $u, v \in J(A, M)$. If $a = 1$ it follows from (1.2) that M is extended from $h_a(R)$, $a = 1 + dv^n$, for all $d \in A$ and sufficiently high $n \in \mathbb{N}$. Since u and v are comaximal, M can be presented by a fibred matrix D with respect to v and u. Then $h_a(\underline{D})$ is also fibred with respect to v and u and it presents M by Lemma 1.1. Hence M is not only extended from $h_a(R)$ but also an extension of a $h_a(R)$-module M' such that M'_u is extended from A_u. Hence M is extended from $h_{a+cu^n}(R)$ for all $c \in A$ and sufficiently high $n \in \mathbb{N}$. Since u, v are comaximal, it follows that M is extended from $A = h_0(R)$. □

1.4

The importance of Quillen's patching theorem in case of a polynomial
261 ring $R = A[T]$ comes from the possibility to generalize the "local Horrocks theorem" to the affine Horrocks theorem. Avoiding the use of monic polynomials the local Horrocks theorem can be formulated as follows: *Let P be projective module over a polynomial ring $A[T]$, where A is local. Assume there exists a free submodule F of P such that P/F is a finite A-module. Then P is free.* Quillen's patching theorem shows that this theorem remains valid for an arbitrary ring A. In the case of positively graded rings, one can show the following generalized version of Horrocks theorem: *Let P be a projective module over a positively graded ring $R = \bigoplus_{i\geq 0} R_i$, $A = R_0$ a noetherian ring of finite Krull dimension d and $R = A[x_1, \ldots, x_n]$, x_i homogeneous of positive degree. Assume that* $\dim R = d + 1$ *and that* $(Rx_1 : R^+) \cap R^+ = Rx_1$. *If there exists a projective submodule F of F such that F is extended from A and P/F is a finite A-module, then P is extended from A.* This result does not look very satisfactory. We have not found really interesting applications

and omit the proof.

Let us add the remark that Roitman's converse of Quillen's patching theorem also extends to the graded case: *Let R be a positively graded ring as above with A as its zero component and let S be any multiplicative subset of A. If all projective R-modules are extended from A then all projective R_s-modules are extended from A then all projective R_s-modules are extended from A_s.*

We finish this section with an interesting result of Vorst ([10, Theorem 1.1]).

Theorem 1.5 (Vorst). *Let A be a noetherian ring such that all projective $A[T_1, \ldots, T_n]$-modules are extended from A. Then every projective module P over a discrete Hodge A-algebra is extended from A.* 262

Proof. Recall that R is a discrete A-Hodge algebra, if R is isomorphic to a residue class ring $A[T_1, \ldots, T_n]/\mathfrak{a}$, where $\mathfrak{a}$ is generated by monomials. It is easy to see that it suffices to treat the case that R is reduced. Then $\mathfrak{a}$ is generated by square free polynomials. We induct on n. If $n \leq 1$, then $R = A$ or $R = A[T]$ if allow $\mathfrak{a} = 0$. Suppose $n \geq 2$. The ideal $\mathfrak{a}$ can be written as $\mathfrak{b}_0 + \mathfrak{b}_1 T$, $T = T_n$, where b_0 and b_1 are generated by square free monomials in $B = A[T_1, \ldots, T_{n-1}]$; $C = B/\mathfrak{b}_0$ is a discrete Hodge A-algebra and by the induction hypothesis all projective C-modules are extended from A. We have $R = C[T]/\mathfrak{b}_1 T$, where the bar denotes residue class formation modulo $\mathfrak{b}_0 \cdot D = C/\mathfrak{b}_1 \cong B/(\mathfrak{b}_0, \mathfrak{b}_1)$, is a discrete Hodege A-algebra and hence $D[T]$ is a discrete Hodge $A[T]$-algebra. By the induction hypothesis, all projective D-modules are extended from A. Now, we have the following result: □

Lemma 1.6. *Let $C[T]$ be a polynomial ring over a ring C and let $\mathfrak{b}$ be an ideal in C and $R = C[T]/\mathfrak{b}T$. If every projective $C/\mathfrak{b}[T]$-module is extended from $C/\mathfrak{b}$ then every projective R-module is extended from C.*

Proof. Let t denote the residue class of T modulo $\mathfrak{b}T$. One has $\mathfrak{b}t = 0$ in $R = C[t]$ and hence $R\mathfrak{b} = \mathfrak{b}$. Since $R/R\mathfrak{b} = C/\mathfrak{b}[T]$, for every projective R-module P, the factor module $P/\mathfrak{b}P$ is extended from $C/\mathfrak{b}$. Notice that R is a graded C-algebra, $R = \bigoplus_{i \geq 0} R_i$ with $R_i = Ct_t^i$. By Theorem 1.3, 263 it suffices to handle the case that C is local. Then $P/\mathfrak{b}P$ is free. Set $r = \operatorname{rank} P/\mathfrak{b}P$. There exist a submodule $F = \sum_{i=1}^{r} Rf_i$ such that the residue classes f_i of f_i modulo, $\mathfrak{b}P$, $1 \leq i \leq r$, form a basis of $P/\mathfrak{b}P$. We have $P = F + \mathfrak{b}P$. Since $R\mathfrak{b} = \mathfrak{b}$, there exists a $b \in \mathfrak{b}$ such that $(1 + b)P \subset F$. But C is local and hence $1 + b$ a unit. So $P = F$. If $\sum r_i f_i = 0$, then $r_i \in \mathfrak{b}$ for all $1 \leq i \leq n$, because $F/\mathfrak{b}F$ is free and $R\mathfrak{b} = \mathfrak{b}$. This implies that P is extended from C. □

Due to this lemma applied to $D = C/\mathfrak{b}_1$, we obtain that projective R-modules are extended from C, whence, from A, by the induction hypothesis. This proves Vorst's result.

2 Unimodular elements in projective modules over graded rings

Let P be a projective module over a commutative ring R, and let $P^* = \operatorname{Hom}_R(P, R)$ be the dual of P. An element $P \in P$ is called *unimodular* if the ideal $0_P(p) = \{\varphi(p), \varphi \in aP^*\}$ equals the whole ring R. This is equivalent to the property that Rp is a direct summand of P. If P is generated by elements $p_1, \ldots, p_m$ and P^* by elements $q_1^*, \ldots, q_n^*$, the module P is isomorphic to the module that is generated by the rows $(q_1^*(p_i), \ldots, q_n^*(p_i))$, $1 \leq i \leq m$, of the matrix $(q_i^*(p_i))$, $1 \leq i \leq m$, $1 \leq j \leq n$. We call such a matrix a *presenting matrix* of P. In this "matrix picture" an element p is unimodular if and only if the coefficients of the corresponding row generate the whole ring.

264 Assume that there exists an element s in a subring A of R such that P_s is free. Set $t = \operatorname{rank} P_s$. It is easy to see that then there exist a submodule F of P with a system $f_1, \ldots, f_t$ of generators and homomorphisms $f_1^*, \ldots, f_t^* \in P^*$ and $l \in \mathbb{N}$ such that $s^l P \subset F$ and $f_i^*(f_j) = s^l \delta_{ii}$ where δ_{ij} is the Kronecker symbol. If we include the f_i, $1 \leq i \leq t$, in a system of generators of P, we obtain a presenting matrix D of P with a submatrix $s^l\underline{E}$, E *a* $t \times t$-unit matrix, and with rank $\underline{D}_s = t$ where $\underline{D}_s$ denotes the canonical image of $\underline{D}$ in $\operatorname{Mat}(R_s)$. Let us call a matrix D with this porperty *s-distinguished.*

The following theorem corresponds to Criterion *I* in ([3]). It is the main result of this section.

Theorem 2.1. *Let P be a projective module over a positively graded ring $R = \bigoplus\limits_{i\geq 0} R_i$. Assume that there exists an element $q \in P$ such that $q_{1+R^+} \in Um(P_{1+R^+})$ and $q_{1+J} \in Um(P_{1+J})$, $J = J(R_0, P)$. Then P contains a unimodular element p with $p - q \in JR^+P$.*

Proof. The assumptions on q_{1+R^+} and q_{1+J} mean that $(0_P(q), R^+) = R$ and $(0_P(q) \cap A, J) = A$, $A = R_0$. Hence it follows that there exist a finitely generated ideal $I = (u_1, \dots, u_t) \subset J$ with $(0_P(q) \cap A, I) = A$. Because P is projective, we may assume, without loss of generality, that P_{u_i} is free. $1 \leq i \leq t$. By the remark preceding the theorem, we see that P is presentable by a matrix $\underline{D}$ that is u_i-distinguished for all i, $1 \leq i \leq t$. Let $I_j = (u_1, \dots, u_j) \subset I$, $1 \leq j \leq t$. We show by induction that to every **265** j there exists a $n \in \mathbb{N}$ such that $\underline{D}$ is equivalent to all matrices $h_a(\underline{D})$, $a \in 1+I_j^n$. Since $\underline{D}$ is u_1-distinguised it has the form described in Lemma 1.1 with $u = u_1$ (up to permutations of rows and columns) including the additional assumption stated in the lemma. So the assertion in case of $j = 1$ follows from Lemma 1.1 with $u = u_{j+1}$, $j < t$, and hence we obtain that $h_a(\underline{D})$ and so $\underline{D}$ itself is equivalent to all matrices $h_{a+cu_{j+1}^n}(\underline{D})$, $c \in A$ and n suitably high. This finishes the induction. In case $j = t$, we have that for a suitably high $n \in \mathbb{N}$ the matrix $\underline{D}$ is equialent to all $h_a(\underline{D})$, $a \in 1 + I^n$. It follows from $A = (0_P(q) \cap A, I)$ that there exists a $w \in I^n$ with $1 + w \in 0_P(q)$, and hence $\underline{D}$ is equivalent to $\underline{D}' = h_{1+w}(\underline{D})$. If q corresponds to a row of the form $(q_1^*(q), \dots, q_l^*(q))$, $P^* = \sum Rq_i^*$, then the corresponding row of D' has the form $(v_1^*(p), \dots, v_1^*(p))$ for a $p \in P$ and $P^* = \sum Rv_i^*$ such that $v_i^*(p) = h_{1+w}(q_i^*(q))$, $1 \leq i \leq l$. Since $h_{1+w}(a) = a$ for $a \in A$ we obtain $0_P(q) \cap A \subset 0_P(p) \cap A$, whence $1 + w \in 0_P(p)$. Therefore $0_P(q)$ contains the constant terms of the $v_i^*(p)$, $1 \leq i \leq l$, which are equal to the constant terms of $q_i^*(q)$ for each i. But these constant terms generate A. This shows that p is unimodular and that $p - q \in wR^+P \subset JR^+P$. □

The following result of Eisenbud and Evans ([5]) is crucial for our applications of Theorem 2.1.

266 **Theorem 2.2.** *Let P be a projective module of rank r over a noetherian ring A and let $p = (p_1, a)$ be a unimodular element of $P \oplus A$. Then there exists a $p_2 \in P$ such that $ht(0_p(p_1 + ap_2)) \geq \min\{r, ht(0_p \oplus_A (p)\}$.*

For example, this theorem implies a well known theorem of Serre:

Theorem 2.3. *Let A be a noetherian ring of finite Krull dimension d. Every projective A-module P of rank $(P) \geq d + 1$ has a unimodular element.*

In 1979, Plumstead [*P*] settled a question of Eisenbud and Evans, which asked whether (2.3) remains valid for a polynomial rign $A[T]$. In 1981, Bhatwadekar and Roy generalized Plumstead's result making available Plumstead's patching technique for many indeterminates. As a kind of justification of our matrix pictures, we now give a short deduction of their result from Theorem 2.1.

Theorem 2.4 (Bhatwadekar/Roy)**.** *Let P be a projective module over a polynomial ring $R = A[T_1, \dots, T_n]$, A a noetherian ring of* $\dim A = d$*. If rank $(P) \geq d + 1$, then P has a unimodular element.*

Proof. We use induction on n. It suffices to handle the case that R is reduced. If $n = 0$, the assertion follows from (2.3). Suppose $n \geq 1$. If $d = 0$, then A is a direct product of fields and P is even free, by the Theorem of Quillen and Suslin ([8]). Let $d \geq 1$. The localization P_S
267 of P at the set S of non zero divisors of R is free because $\dim R_S = 0$. Therefore, there exists $s \in S$ such that P_s is free, and hence the Quillen ideal $J = J(A, P)$ has height ≥ 1. Following the procedure of Bhatwadekar and Roy, we consider the factor module $\overline{P} = P/JTP$ over $\overline{R} = R/(JT) = \overline{A}[T_1, \dots, T_{n-1}]$, where $\overline{A} = A[T]/(JT)$, $T = T_n$. By the induction hypothesis, P contains a unimodular element. This means that P contains an element q with $(0_P(q), JT) = R$. Theorem 2.2 allows to assume that $ht(0_P(q)) = d + 1$. There exists a Nagata transformation of indeterminate of the form $T_i' = T_i + T^{r_i}$, $1 \leq i \leq n - 1$, $T' = T$, such that $0_P(q)$ contains a polynomial $f(T)$ that is monic in T over the ring $R' = A[T_1, \dots, T_{n-1}']$. Furthermore, $0_P(q)$ contains an element g of the form $g = 1 + hT$ with $h \in RJ$. The resultant res (f, g) of f and g in R'

has the form $1 + h'$, $h' \in R'J$. Hence $(0_P(q) \cap R', J(R', P)) = R'$. Now Theorem 2.1 implies that P contains a unimodular element. □

The following theorem generalizes Plumstead's result ((2.4) with $n = 1$):

Theorem 2.5. *Let* $R = \bigoplus_{i \geq 0} R_i$ *be a positively graded ring,* $A = R_0$ *noetherian of finite dimension* d. *Assume that* R *is finitely generated,* $R = A[t_1, \dots, t_n]$, t_i *homogeneous in* R^+, $1 \leq i \leq n$ *with* $\dim R = d + 1$. *Further assume that the kernel of an* A*-epimorphism* φ *from* $A[T_1, \dots, T_n]$ *to* R *with* $\varphi(T_1) = t_i$ *has* $ht(\ker \varphi) \geq n - 1$. *Let* P *be a projective* R*=module of rank* $P \geq \dim R$. *If* P_{1+JR^+}, $J = J(A, P)$, *has a unimodular element, then* P *has a unimodular element.*

Proof. There exist a $q \in P$ such that $(0_P(q), JR^+) = R$. By the result of 268
Eisenbud and Evans (see 2.2), we may assume that $ht(0_p(q)) \geq d + 1$. Since $ht(\ker \varphi) \geq n - 1$, the inverse image $\mathfrak{b} = \varphi^{-1}(0_P(q))$ of $0_P(q)$ has $ht(\mathfrak{b}) \geq n + d = \dim B$. After suitable Nagata transformations of indeterminates (cf. the proof of (2.4)) we obtain indeterminates $T_1', \dots, T_n'$ with $B = A[T_1', \dots, T_n']$ and a sequence of polynomials $f_1(T_1'), \dots, f_n(T_n)$ such that $f_1(T_i)$ is monic in T_i with coefficients in $A[T_1, \dots, T_{i-1}]$, $1 \leq i \leq n$. It follows that $\overline{B} = B/\mathfrak{b}$ is a finite $A/\mathfrak{b} \cap A$-module, and hence we conclude from $\overline{B}J = \overline{B}$ that $(0_P(q) \cap A, J) = A$. Now the assertion follows from Theorem 2.1. □

Remark 2.6. We do not know, if Theorem 2.5. remains valid without the special assumption on $ht(\ker \varphi)$. But in case of affine algebras over a field, this assumption is fulfilled. To have a simple example, let us handle a well-known case: Let $R = k[x, y, z]$, $z^n - xy = 0$, k a field. Murthy has shown that projective R-modules are free. Since R is graded and normal, we have $\mathrm{Pic}(R) = 0$. So it remains to show that a given projective R-module P of rank $P \leq 2$ has a unimodular element. R is a graded A-algebra with $A = k[x]$ and $\dim R = 2 = 1 + \dim A$. Moreover, $R_x = A_x[z]$, z algebraically independent over A_x. So P_x is free and $x \in J$. This implies $\dim R/JR^+ = 1$, $R^+ = (y, z)$, $\deg y = n$, $\deg z = 1$. By (2.3)

P/JR^+P has a unimodular element. Now we conclude from (2.5) that P has a unimodular element.

Now we shall consider projective modules over Segre extensions of a field k. Let us first fix some notation.

269 Let k be a field and let $\underline{X} = (X_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a matrix of indeterminats X_{ij} over k. Let $\mathfrak{d}_{mn}$ be the ideal in the polynomial ring $k[X]$ which is generated by the 2×2-minors of $\underline{X}$. We call the residue class ring $S_{mn} = K[\underline{X}]/\mathfrak{d}_{mn}$ a *Segre extension* of k. The residue classes of X_{ij} will be denoted by x_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$. So S_{mn} is defined by the relations $x_{ij}x_{lt} - x_{it}x_{lj} = 0$, $1 \leq i, l \leq m$, $1 \leq j, t \leq n$. The maximal ideal in S_{mn} which is generated by the x_{ij} is denoted by $\mathfrak{n}(S_{mn})$. It is easy to see that $ht(\mathfrak{d}_{mn}) = (m-1)(n-1)$, $\dim S_{mn} = m + n - 1$ and that S_{mn} is normal. One has an ascending chain $S_{m0} \subset S_{m1} \subset \ldots \subset S_{mn}$ with $\dim S_{m,i+1} = \dim S_{mi} + 1$ and S_{mj} is in an obvious way, a positively graded S_{mi}-algebra and birationally equivalent to a polynomial ring in one variable over $S_{m,j-1}$, $1 \leq i \leq j \leq n$. Notice that $(S_{mn})_{x_{ij}}$ is isomorphic to a Laurent polynomial ring $k[Y_1, \ldots, Y_{m+n-1}, Y_1^{-1}]$. Hence the localizations $P_{x_{ij}}$ of a projective S_{mn}-module P at x_{ij} are free for $1 \leq i \leq m$, $1 \leq j \leq n$. This implies that the Quillen ideal J of P in $S_{m,n-1}$ contains the maximal ideal $\mathfrak{n}(S_{m,n-1})$. Therefore *P is extended from $S_{m,n-1}$, if P_z is extended from $(S_{m,n-1})_z$ for a $z \in \mathfrak{n}(S_{m,n-1})$.* By Rao's theorem ([9, Theorem 1.1]) we know that *every projective S_{mn}-module P of rank $(P) \geq \dim(S_{mn})$ has unimodular element and that stably free S_{mn}-modules of rank equal to $\dim(S_{mn})$ are free.*

In this special case, we can prove stronger results. At first, we consider stably free modules.

270 **Theorem 2.7.** *Let $R = S_{mn}$ be a Segre extension of a field k with $m \leq n$. Every stably free R-module P of rank $(P) \geq m + 1$ is free.*

Proof. Let us first assume that k is infinite. We proceed by induction on n. If $n = 1$, P is even free, because S_{m0} is a polynomial ring in m indeterminate over k. Suppose $n \geq 2$. We show that P is extended from $S_{m,n-1}$. It suffices to treat the case $P \oplus R = R^{r+1}$, $r = \text{rank}(P)$. Then P can be presented by a unimodular vector $\underline{v} \in Um_{r+1}(R)$. By known results of ideal theory, we can assume that $ht(\mathfrak{a}) = r$ and $(\mathfrak{a}, \mathfrak{n}(R)) = R$ where q is the ideal generated by the first r components of v. The inverse image $\acute{\mathfrak{a}}$ of $\mathfrak{a}$ in the polynomial ring $R' = k[X]$ has height $ht(\mathfrak{a}\text{'}) = r+(m-1)(n-1) \geq (m-1)n+2$. This implies that the contractions, $\mathfrak{b}_i = \grave{\mathfrak{a}} \cap B_i$ of $\acute{\mathfrak{a}}\text{'}$ to the subring $B_i = k[X_{il}, \dots, X_{in}]$, have $ht(\mathfrak{b}_i) \geq 2$, because, by this contraction, $n(m-1)$ indeterminates are eliminated, $1 \leq i \leq m$. Since $ht(\mathfrak{b}_i) \geq 2$, the $\mathfrak{b}_i$ contain a *homogenous* polynomial f_i, $1 \leq i \leq m$. Because k is infinite, there exist $c_j \in k$ such that after the homogenous linear transformation $X_{in} = X_{in}$, $X_{ij} = X_{ij} + c_j X_{in}$, $1 \leq j \leq n-1$, the f_i are monic in X_{in}, $1 \leq i \leq m$. Since $X_{ij}X_{lt} - X_{it}X_{lj} = X_{ij}X_{lt} - X_{it}X_{lj}$, $1 \leq i$, $1 \leq m$, $1 \leq j, t \leq n$, we may assume, without loss of generality, that the f_i are monic in X_{in}, $1 \leq i \leq m$. It follows that $(\mathfrak{a}, R\mathfrak{n}(B)) = R$, $B = S_{m,n-1}$. Moreover, $R/\mathfrak{a}$ is a finite $B/\mathfrak{a} \cap B$-module. Hence we obtain $(\mathfrak{a} \cap B, \eta(B)) = B$. Thus, we have shown that there exists a $z \in B \backslash \mathfrak{n}(B)$ such that P_z is free and hence $J(B, P) \neq \mathfrak{n}(B)$. As remarked above, we have $\mathfrak{n}(B) \subset J(B, P)$, whence $B = J(B, P)$. This shows that P is extended from B. If k is finite, adjoin an indeterminate U to R. Since 271 $k(U)$ is infinite, $k(U) \otimes_k P$ is free, and hence $R[U] \otimes_R P$ is free by the affine Horrocks theorem. Therefore P is free. □

Theorem 2.8. *Let P be a projective module over a Segre extension $R = S_{mn}$ of an infinite field k, $m \leq n$. Every projective module P of rank* $(P) \geq m+1$ *has a unimodular element.*

Proof. We proceed by induction on n. If $n = 1$, P is even free. Let $n = 2$ and let $\mathfrak{p}$ denote the (prime) ideal in R which is generated by the coefficients of the n-th column of the matrix $(x_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$. $R = R/\mathfrak{p}$ is a Segre extension isomorphic to $S_{m,n-1}$. By the induction hypothesis $P = P/\mathfrak{p}P$ has a unimodular element, and hence P contains an element q with $(0_P(q), \mathfrak{p}) = R$. By the theorem of Eisenbud and Evans (see (2.2)), we may assume that $ht(0_P(q)) \geq m+1$. As shown in the proof of Theorem 2.6, we can assume without loss of generality that $R/0_P(q)$ is a finite $B/B \cap 0_P(q)$-module, $B = S_{m,n-1}$. Furthermore we may assume that $(0_P(q), R\mathfrak{n}(B)) = R$. This implies that $B = (0_P(q) \cap B, \mathfrak{n}(B))$. Since R is a positively graded B-algebra with $R^+ = \mathfrak{p}$ and since $J(B, P) \supset \mathfrak{n}(B)$, we conclude from Theorem 2.1 that P has a unimodular element. □

Remarks 2.9. Theorem 2.8 should be valid for arbitrary fields, but we have no convincing argument to show this. We must leave it as an open question, if all projective S_{mn}-modules are free. Because we do not believe that this question has an affirmative answer, we are interested in calculating $K_0(S_{mn})$.

References

272 [1] Artamonow, W. A., Quillen's theorem for graded algebras, Westnik MGU *Math., Nr.* 3, 59-61 (1983) (In Russian).

[2] Bass, H., Murthy, M. P., Grothendieck groups and Picard groups of abelian group rings, *Ann. of Math.* 86, 16-73 (1967).

[3] Bhatwadekar, S. M., Lindel, H., Rao, R. A., The Bass-Murthy Question: Serre Dimension of Laurent Polynomial Extensions, *Invent. Math.* 81, 189-203 (1985).

[4] Bhatwadekar, S. M., Roy, A., Some theorems about projective modules over polynomial rings, J. of Alg.86, (1983).

[5] Eisenbud, D, Evans, G., Generating modules efficiently – Theorems from Algebraic *K*-theory, *J. of Alg.* 27, 278-305, (1973).

[6] Murthy, M. P., *Lectures on Projective Modules,* Notes by J. Yanik (1977).

[7] Plumstead, B. R., The Conjectures of Eisenbud and Evans, *Amer. J. of Math.* 105, 1417-1433 (1983).

[8] Quillen, D. Projective modules over polynomial rings, *Invent. Math.* 36, 167-171 (1976).

273 [9] Rao, R. A., Stability theorems for overrings of polynomial rings II, *J. of Alg.* 78, 437-444 (1982).

[10] Vorst, A. C. F., The Serre Problem for Discrete Hodge Algebras *Math. Zeit.*, 184, 425-433 (1983).

Mathematisches Institut
Universität Munster
Einsteinstr. 62
44 Münster
Federal Republic of Germany

Vector Bundles On $\mathbb{P}^2$ And Torsion Sheaves On The Dual Plane

By M. Maruyama

Introduction

275 W. Barth found a beautiful relationship between rank-2 nets of quadrics and stable vector bundles of rank-2 on the projective plane $\mathbb{P}^2$ with the first Chern class zero ([2]). Then, in [7], K. Hulek defined a pre-stable Kronecker module and succeeded in describing s-stable vector bundles on $\mathbb{P}^2$ in terms of it. The notion of Kronecker modules is a generalization of nets of quadrics. In fact, a net of quadrics is nothing but a symmetric Kronecker module.

For a non-degenerate Kronecker module α, we can define the discriminant curve $\Delta(\alpha)$ in the dual plane and an $\mathscr{O}_{\Delta(\alpha)}$-module $\mathscr{L}(\alpha)$. Moreover, the couple $(\Delta(\alpha), \mathscr{L}(\alpha))$ determines α ([7, p. 124, 125]). When α is a net of quadrics $\mathscr{L}(\alpha)$ becomes a θ-characteristic on $\Delta(\alpha)$, inheriting the symmetry of α. Thus we come to the known result that the classification of non-degenerate nets of quadrics reduces to that of couples of a plane curve and a generalized, ineffective θ-characteristic (see, for example, [4] and Corollary 2.12.4 of this article).

If α is obtained from a vector bundle F on $\mathbb{P}^2$, the non degeneracy 276
means that

(1.10.2) for general lines ℓ in $\mathbb{P}^2$, $F\,|_{\ell} \simeq \mathscr{O}_{\ell}^{\oplus r(F)}$.

In this way, an s-stable vector bundle on $\mathbb{P}^2$ with the property (1.10.2) gives rise to a couple of a curve in the dual plane and a coherent sheaf on it. The main purpose of this work is to study the inverse of this process.

A remarkable property of $\mathscr{L}(\alpha)$ is

$$H^0(\Delta(\alpha), \mathscr{L}(\alpha)) = H^1(\Delta(\alpha), \mathscr{L}(\alpha)) = 0.$$

Taking note of this, we shall start with, instead of a Kronecker module, a coherent sheaf L on the dual plane P^* with the following property:

(2.3.1) L is a torsion sheaf such that $H^0(P^*, L) = H^1(P^*, L) = 0$.

Then we have the following resolution (Proposition 2.5):

$$(2.5.1) \quad 0 \to \mathscr{O}_{P^*}(-1)^{\oplus n} \xrightarrow{\alpha} \mathscr{O}_{P^*}^{\oplus n} \to L(1) \to 0.$$

α is represented by an $n \times n$-matrix $\alpha(L)$ whose entries are linear forms on P^*. The $\alpha(L)$ is a Kronecker module and the curve in P^* defined by $\det \alpha(L) = 0$ is the discriminant curve $\Delta(\alpha(L))$ The given L carries an $\mathscr{O}_{\Delta(\alpha(L))}$-module structure which is the $\mathscr{L}(\alpha(L))$ stated in the above.

277 Therefore, we shall take, on one hand, the full subcategory $\mathscr{C}$ of the category of coherent $\mathscr{O}_{P^*}$-modules whose objects have the properties (2.3.1) and (2.3.2). The property (2.3.2) corresponds to a half of 1.2.2 of [7]; $\dim\left(\alpha^T(\varphi \otimes V^*)\right) \geqq 2$ (see Corollary 2.13.2). On the other hand, let $\mathscr{V}$ be the full subcategory of the category of coherent sheaves on $\mathbb{P}^2$ such that an F is contained in $\mathscr{V}$ if F is a vector bundle with the properties (1.10.2), $H^0\left(\mathbb{P}^2, F\right) = 0$ and $c_2(F) = r(F)$. Our main result (Main Theorem 2.16) is then stated as follows.

Main Theorem. *There is an equivalence between two categories* $\mathscr{V}$ *and* $\mathscr{C}$.

Barth [2] and Hulek [7] used the monads to get vector bundles from nets of quadrics and Kronecker modules. We shall construct a vector bundle in $\mathscr{V}$ from a member of $\mathscr{C}$ by exploiting a flag manifold and direct image functors. An observant reader will find the idea of elementary transformations behind our construction of vector bundles.

We know two types of good monads for s-stable vector bundles E on $\mathbb{P}^2$ with $c_1(E) = 0$([3], §6).

$$M_1 : \mathscr{O}_P(-1)^{\oplus n} \to \mathscr{O}_P^{\oplus(2n+r)} \to \mathscr{O}_P(1)^{\oplus n}$$

$$M_2 : \mathscr{O}_P(-1)^{\oplus n} \xrightarrow{a} \Omega_P(1)^{\oplus n} \xrightarrow{b} \mathscr{O}_P^{\oplus(n-r)}$$

where $P = \mathbb{P}^2$, $c_2(E) = n$ and $r(E) = r$. The former was used by K. Hulek and W. Barth employed the latter in the case of $r = 2$. If we describe a vector bundle F in $\mathscr{V}$ by M_2, then $b = 0$ and hence F is **278** isomorphic to coker(α) because $n = r$. By using the resolution (2.5.1) and Corollary 3.6.1, we also get to the monad.

For general s-stable vector bundles, the connection between our results and the monadology will be revealed by Theorem 4.8. Thus our results must be interpreted in terms of the monad of type M_2 if one follows the work of W. Barth in [2]. The author hopes, nevertheless, that his results will serve deeper understanding of the relation between vector bundles on $\mathbb{P}^2$ and plane curves.

We have a nice application of our viewpoint to the study of moduli spaces of stable vector bundles on $\mathbb{P}^2$ with the first Chern class zero. It is easy to see that the second Chern class c_2 is greater than or equal to the rank r. If one fixes c_2, then the extreme case $c_2 = r$ is, in some sense, most important. In fact, we shall show that a good part of the other moduli space is a subscheme of that of the extreme case (Proposition 1.7 and Theorem 5.6). The structure of the moduli space of the extreme case will be studied in a forthcoming paper.

This work was completed while the author was staying at Tata Institute of Fundamental Research. He wishes to express his hearty thanks to the mathematicians at the Institute for their hospitality and their stimulation given to him.

Notation and convention.

A variety in this article is geometrically integral algebraic scheme over a field. For a coherent sheaf F on a variety X, $h^i(X, F)$ denotes **279** $\dim H^i(X, F)$ and $r(F)$ does the rank of F, that is, $F\ |_U \simeq\ \mathscr{O}_u^{\oplus r(F)}$ on a non-empty open set U. F^* is the dual $\mathrm{Hom}_{\mathscr{O}_X}(F, \mathscr{O}_X)$ of F. If E is a coherent sheaf on $\mathbb{P}^2$, we can define the first and the second Chern classes $c_1(E)$, $c_2(E)$. Since all the cycles on $\mathbb{P}^2$ are determined by their degrees up to the rational equivalence, $c_1(E)$ and $c_2(E)$ can be regarded as integers.

§1 Vector bundles on $\mathbb{P}^2$ with $c_1 = 0$ and $c_2 = r$.

First of all, let us recall the definition of stable sheaves.

Definition 1.1. Let $(X, \mathscr{O}_X(1))$ be a couple of a non-singular projective variety X over a field k and an ample line bundle $\mathscr{O}_X(1)$ on X. For a coherent sheaf G on X with $r(G) > 0$, $P_G(m)$ is the polynomial $\chi(G(m))/r(G)$, where $\chi(G(m)) = \sum(-1)^i h^i(X, G(m))$. $\mu(G)$ is defined to be the rational number $d(G, \mathscr{O}_X(1))/r(G)$, where $d(G, \mathscr{O}_X(1))$ is the degree of $c_1(G)$ with respect to $\mathscr{O}_X(1)$. For an overfield K of k and for a coherent sheaf G on $X_K = X \otimes_k K$, $P_G(m)$ and $\mu(G)$ denote those with respect to $(X_K, \mathscr{O}_{X_K}(1))$, respectively. A coherent sheaf E on X is stable (semi-stable, μ-stable or μ-semi-stable) with respect to $\mathscr{O}_X(1)$ is (a) E is torsion free and (b) for every coherent subsheaf F of $E \otimes_k \bar{k}$ with $0 < r(F) < r(E)$, we have $P_F(m) < P_E(m)$ for all large $m(P_F(m) \leqq (P_E(m)$ for all large m, $\mu(F) < \mu(E)$ or $\mu(F) \leqq \mu(E)$, resp.). When E is locally free, a stable (semi-stable, μ-stable or μ-semi-stable) sheaf is called a stable (semi-stable, μ-stable or μ-semi-stable, resp.) vector bundle.

For a coherent sheaf on $\mathbb{P}^2$ which we are mainly concerned with in
280 this article, the above four notions are independent of the choice of the
ample line bundle.

Notation 1.2. We shall fix a base field k which is not necessarily algebraically closed. P denotes the projective plane $\mathbb{P}^2_k$.

The following seems to be known.

Lemma 1.3. *Let E be a torsion free coherent sheaf on P. If $c_1(E) = 0$, $c_2(E) \leqq 0$ and if E is μ-semi-stable, then E is isomorphic to $\mathscr{O}_P^{\oplus r}$.*

Proof. Let us prove the lemma by induction on $r = r(E)$. If $r = 1$, then $E' = (E^*)^*$ is $\mathscr{O}_P$. Since $c_2(E) = h^0(P, E'/E)$ is positive or zero according as $E \neq E'$ or $E = E'$, we see that $E = E' \simeq \mathscr{O}_P$ by our assumption that $c_2(E) \leqq 0$. Assume that $r(E) = r > 1$ and if $r(E) < r$, our assertion is true. By our assumption, we see that for $E' = (E^*)^*$, $c_1(E') = 0$ and $c_2(E') \leqq c_2(E)$ whose equality holds if and only if

$E = E'$. Thus, replacing E by E', we may assume that E is locally free. The μ-semi-stability of E implies that $h^2(P, E) = h^0(P, E^*(-3)) = 0$ and hence $h^1(P, E) = -r + c_2(E)$ if $h^0(P, E) = 0$. This is not the case because $c_2(E) \leqq 0$. Therefore, by using a non-zero section of E, we have an exact sequence

$$0 \to \mathscr{O}_P \to E \to F \to 0.$$

By the assumption that E is μ-semi-stable we see that F is torsion free. Since $c_1(F) = c_1(E)$ and $c_2(F) = c_2(E)$, our induction hypothesis implies that F is isomorphic to $\mathscr{O}_P^{\oplus(r-1)}$ and then $E \simeq \mathscr{O}_P^{\oplus r}$. Q.E.D

The above is said in other words: **281**

Corollary 1.3.1. *If E is a μ-semi-stable sheaf on P with $c_1(E) = 0$, then $c_2(E) > 0$ or $E \simeq \mathscr{O}_P^{\oplus r}$.*

The results of this section are based on the following.

Lemma 1.4. *Let E and F be coherent sheaves on P with $c_1(E) = c_1(F) = 0$. Assume there exists an exact sequence*

$$0 \to E \to F \xrightarrow{\Psi} \mathscr{O}_P^{\oplus r}.$$

(1) If F is stable, $c_2(F) = r(F)$ and if Ψ is generically surjective, then Ψ is surjective.

(2) If Ψ is surjective, E is μ-stable, F is locally free and $H^0(P.F) = 0$, then F is stable.

Proof. (1) Put $\operatorname{im}(\Psi) = E'$ and $\mathscr{O}_P^{\oplus r}/E' = T$. Since Ψ is generically surjective, T is a torsion sheaf. If $\dim \operatorname{Supp}(T) = 1$, then $d(E'\mathscr{O}_P(1)) < 0$ and hence F is not μ-semi-stable, a fortiori, not stable. Thus $\dim \operatorname{Supp}(T) \leqq 0$ and then it is easy to see that $c_2(E') = h^0(P, T) = s$ and $c_1(E') = c_1(E) = 0$. Suppose that $s < r \cdot t = h^0(P, E') \geqq h^0\left(P, \mathscr{O}_P^{\oplus r}\right) - h^0(P, T) = r - s > 0$. Now Ψ is surjective if and only if $t = r$. Assume that $t < r$. Since $\mathscr{O}_P^{\oplus t} = H^0(P, E') \otimes_k \mathscr{O}_P$ is a subbundle of $\mathscr{O}_P^{\oplus r} = H^0\left(P, \mathscr{O}_P^{\oplus r}\right) \otimes_k \mathscr{O}_P$, the $\mathscr{O}_P^{\oplus t}$ is a subsheaf of E'. By replacing E

by $\Psi^{-1}\left(\mathscr{O}_P^{\oplus t}\right)$, we may assume that $s \geqq r$ because $c_2\left(E'/\mathscr{O}_P^{\oplus t}\right) = c_2(E') = s$ and $r\left(E'/\mathscr{O}_P^{\oplus t}\right) = r - t$. By virtue of our assumption $c_2(E) = r(F) - s$. Then Riemann-Roch Theorem shows that $P_E(m) = m^2/2 + 3m/2 - (r(F) - s)/r(E) + 1$. On the other hand, $P_F(m) = m^2/2 + 3m/2 - r(F)/r(F) + 1$. These together with the inequality
282 $-(r(F) - s)/r(E) \geqq (-r(F) + r)/r(E) = -r(E)/r(E) = -1$
contradict the stability of F.

(2) Let F' be a coherent subsheaf of F such that $0 < r(F') < r(F)$ and F/F' is torsion free. Since F is μ-semi stable, $d(F', \mathscr{O}_P(1)) \leqq 0$. If $d(F', \mathscr{O}_P(1)) < 0$, it is obvious that $P_{F'}(M) < P_F(m)$ for all large m. We may assume, therefore, that $c_1(F') = 0$. It is well-known that F' is locally free. For $E' = F' \cap E$, F'/E' can be regarded as a subsheaf of $\mathscr{O}_P^{\oplus r}$. If $E' = 0$ then F' itself is a subsheaf of $\mathscr{O}_P^{\oplus r}$. Then F' must be $\mathscr{O}_P^{\oplus s}$ because it is locally free and $c_1(F') = 0$. This violates the assumption that $H^0(P, F) = 0$. Hence we have that $r(E') > 0$. Since E is μ-stable, $d(E', \mathscr{O}_P(1)) < 0$ or $r(E) = r(E')$. The former is not the case because $d(F', \mathscr{O}_P(1)) = d(F'/E', \mathscr{O}_P(1)) + d(E', \mathscr{O}_P(1)) < 0$ if that holds. In the latter case, the difference between $P_{F'}(m)$ and $P_F(m)$ is at most on constant terms which are $-c_2(F')/r(F') + 1$ and $-c_2(F)/r(F) + 1$, respectively. On the other hand, E/E' is a subsheaf of F/F' which is torsion free. This and $r(E') = r(E)$ imply that $E = E'$. Then, we see that $P_{F'}(m) < P_F(m)$ for all large m because

$$\frac{-c_2(F')}{r(F')} = \frac{\left\{c_2(E) + c_2\left(\frac{F'}{E}\right)\right\}}{r(F') \leqq \frac{-c_2(E)}{r(F')}} < \frac{-c_2(E)}{r(E)} = \frac{-c_2(F)}{r(F)} = \frac{-c_2(F)}{r(F)}$$

by virtue of Corollary 1.3.1

Q.E.D

For a vector bundle F on P with $c_1(F) = 0$ and $c_2(F) = r(F)$, Lemma 1.4 provides us with a relation between the stability and the s-stability ([7]).

Corollary 1.4.1. *If F is a stable vector bundle on P with $c_1(F) = 0$ and $c_2(F) = r(F)$ and if $h^0(P, F^*) = s$, then F is an extension of $\mathscr{O}_P^{\oplus s}$ by a vector bundle E with $H^0(P, E^*) = 0$. Moreover, E is uniquely determined by F.*

Proof. Since F is stable, F^* is μ-semi-stable. Then it is easily seen that $\mathscr{O}_P^{\oplus s} = H^0(P, F^*) \otimes_k \mathscr{O}_P$ is subsheaf of F^*. The sheaf $E = \left(F^*/\mathscr{O}_P^{\oplus s}\right)^*$ is locally free and we have the following exact sequence 283

$$0 \to E \to F \xrightarrow{\Psi} \mathscr{O}_P^{\oplus s}$$

Ψ is generically surjective and we can apply Lemma 1.4, (1) to the above sequence. F is, therefore, an extension of $\mathscr{O}_P^{\oplus s}$ by E. The rest of our assertion is obvious.

Let E be a μ-semi-stable vector bundle on P such that $H^0(P, E) = 0$ and $c_1(E) = 0$. Since E^* is μ-semi-stable, $h^2(P, E) = h^0(P, E^*(-3)) = 0$. By Riemann-Roch Theorem, we have $h^1(P, E) = r(E) - c_2(E) = t$. Pick a basis $\{\eta_1, \dots, \eta_t\}$ of $H^1(P, E)$. Then $\eta = (\eta_1, \dots, \eta_t)$ can be regarded as an element of $\mathrm{Ext}^1_{\mathscr{O}_P}\left(\mathscr{O}_P^{\oplus t}, E\right) \simeq H^1(P, E)^{\oplus t} \cdot \eta$ defines an extension

$$0 \to E \to F \to \mathscr{O}_P^{\oplus t} \to 0,$$

where F is locally free and $r(F) = c_2(E) = c_2(F)$. F is uniquely determined by E up to isomorphisms so long as $\{\eta_1, \dots, \eta_t\}$ is a basis of $H^1(P, E)$. Moreover, by the construction of F we have $H^0(P, F) = 0$. Q.E.D

Definition 1.5. $V(r, n)_k$ is the set of isomorphism classes of vector bundles E on P with the following properties:

$$E \text{ is } \mu - \text{semi-stable.} \tag{1.5.1}$$

$$H^0(P, E) = 0. \tag{1.5.2}$$

$$c_1(E) = 0,\ c_2(E) = n \text{ and } r(E) = r. \tag{1.5.3}$$

The above discussion shows that if $r > n$, then $V(r,n)_k = \phi$ and that **284**
by mapping E to the extension F, we obtain a map $\sigma(r,n)_k$ of $V(r,n)_k$ to $V(n,n)_k$.

Definition 1.6. $V(r,n)^s_k$ $\left(\text{or } V(r,n)^\mu_k\right)$ is the subset of consisting of stable (or, μ-stable, resp.) vector bundles.

Lemma1.4. (2) means that $\sigma(r,n)_k = \left(V(r,n)^\mu_k\right) \subset V(r,n)^s_k$. Let $M(r,n)_0$ $\left(\text{or } M(r,n)^\mu_0\right)$ be the moduli space of stable vector bundles (or, μ-stable vector bundles, resp.) E of rank r on P such that $c_1(E) = 0$ and $c_2(E) = n$. When k is algebraically closed, $V(r,n)^s_k = M(r,n)_0(k)$ and $V(r,n)^\mu_k = M(r,n)^\mu_0(k)$ as sets.

Proposition 1.7. *There exists an immersion* $\Psi(r,n)$ *of* $M(r,n)^\mu$ *to* $M(n,n)_0$ *such that for all algebraically closed fields* K *containing* k, $\sigma(r,n)_k$ *is induced by* $\Psi(r,n)(K)$.

Proof. Set $M(r,n)^\mu_0 = M_1$. There exist a principal $PGL(N)$-bundle $p : Q \to M_1$ and a vector bundle $\widetilde{E}$ on $P\times Q$ with $GL(N)$-linearization ([10, Proposition 6.4], and [9], §4) such that for all x in $Q(K)$, $\widetilde{E}(x) = \widetilde{E}\otimes k(x)$ is the vector bundle corresponding to the point $p(x)$. For the projection $\pi : P \times Q \to Q$, $G = R^1\pi_*(\widetilde{E})$ is a vector bundle of rank $n - r$ on Q. Since $\pi_*\left(\pi^*(G^*\otimes \widetilde{E}\right) = G^*\otimes \pi * \left(\widetilde{E}\right) = 0$ and $R^1\pi_*\left(\pi^*(G^*)\otimes\widetilde{E}\right) = G^*\otimes R^1\pi_*\left(\widetilde{E}\right) = G\otimes G^*$, $\text{Ext}^1\ \mathscr{O}_{P\times Q}\left(\pi^*(G),\widetilde{E}\right) \overset{\sim}{\longleftarrow} H^1\left(P\times Q, \pi^*(G^*)\otimes\widetilde{E}\right) \to H^0(Q, G^*\otimes G) \simeq \text{Hom}_{\mathscr{O}_Q}(G,G)$. Thus $\text{Ext}^1_{\mathscr{O}_{P\times Q}}\left(\pi^*(G),\widetilde{E}\right)$ contains a special element ζ which corresponds to id_G of $\text{Hom}_{\mathscr{O}_Q}(G,G)$. Let us consider the extension defined by ζ:

$$0 \to \widetilde{E} \to \widetilde{F} \to \pi^*(G) \to 0.$$

285

It is easy to see that for all points x of $Q(K)$, $F(x)$ is $\sigma(r,n)_K(E(x))$. Moreover, this construction of F is compatible with base changes. By the universality of $M(n,n)0$, we have a morphism f' of Q to $M(n,n)_0$. The $GL(N)$- linearization of E and the compatibility with base changes imply that there is a morphism f of M_1 to $M(n,n)_0$ such that $fp = f'$.

For $M(n,n)_0$, we have a principal $PGL(N')$-bundle $q' : R \to M(n,n)_0$ and a vector bundle H' on $P \times R'$ with the same properties as in the case of M_1. Set $R = \{y \in R' | (h^0(P_{k(y)}, \widetilde{H}',\ (y)^*) = n - r$ and

$$\widetilde{H}'(y)\widetilde{H}^0(P_{k(y)}, H'(y)^*) \otimes \mathscr{O}_{P_{k(y)}} \text{ is } \mu\text{-stable}\}.$$

Then R is locally closed in R' and $PGL(N')$-invariant. Hence $M_2 = q'(R)$ is locally close in $M(n,n)_0$. If we endow R and M_2 with the reduced structure, then we have a principal $PGL(N')$-bundle: $q : R \to M_2$. Since M_1 is smooth, f is a morphism of M_1 to M_2. Put $\widetilde{H'}\ |_{P\times R} = \widetilde{H}$. Corollary 1.4.1 shows that $\left(\widetilde{H}^*/\tau^*\tau_*(\widetilde{H}^*)\right)^* = \widetilde{D}$ is a vector bundle of rank r such that for all y of $R(K)$, $D(y)$ is a member of $V(r,n)^{\mu}_{K}$, where τ is the projection of $P \times R$ to R. Thus we have a morphism g' of R to M_1. By an argument similar to the case of f', we obtain a morphism of M_2 to M_1 such that $gq = g'$. Take a point x of Q. If one looks into the construction of f' ([9], § 5), then he will find a morphism f'' of an open neighbourhood U of x to R which covers f. We also get an open neighbourhood V of $f''(x)$ and a morphism g'' to U covering g. Furthermore, $(1 \times f'')^*(D)$ is isomorphic to $\widetilde{E}\ |_U$ and $(1 \times g'')^*(E\ |_U) \simeq D\ |_V$. Thus $1 \times g'' f'')^* \left(\widetilde{E}\ |_U\right)$ is isomorphic to E in a neighbourhood U' of x. Since Q is an open set of a Quot-scheme and E is the universal quotient sheaf, the universality of the $\left(Q, \widetilde{E}\right)$ shows that $g'' f'' = \mathrm{id}$ on U' after replacing F'' by its composition with the action of an element of $PGL(N)$. We **286** have, therefore, that $\pi = \pi g'' f'' = g' F'' = g\tau f'' = g f' = g f \pi$. Since π is faithfully flat, this means that $gf = \mathrm{id}$. Thus f is birational. Since $M(r,n)^{\mu}_{0}$ is smooth and f is bijective, we deduce from this and ZMT that f is isomorphic. Q.E.D

Each member of $V(n,n)$ has beautiful properties.

Lemma 1.8. *If F is a member of $V(n,n)$, then F is* 1*-regular* (for the definition of regularity, see [8]).

Proof. Since F is μ-semi-stable, so is F^*. This and the assumption that $c_1(F) = 0$ imply that $h^2(P, F(-1)) = h^0(P, F^*(-2)) = 0$. Then, as we have seen in the above, we get that $h^1(P, F) = c_2(F) - r(F)$ which is equal to zero by virtue of the assumption (1.5.3) for $V(n,n)$. Q.E.D

Corollary 1.8.1. *If we write* $F\mid_{\ell}=\bigoplus_{i=1}^{n}\mathscr{O}_{\ell}(a_i)$ *for an* F *in* $V(n,n)$ *and a line* ℓ *in* P*, then we have* $a_i\geqq -1$ *for all* i*.*

Proof. Since F is 1-regular, $F(1)$ is generated by its global sections and then so is $F(1)\mid_{\ell}$. Thus $a_i+1\geqq 0$ for all i. Q.E.D

Another remarkable property of $V(n,n)$ is

Lemma 1.9. *If* F *is a member of* $V(n,n)$*, then* F *is semi-stable. Moreover, if* E *is a locally free subsheaf of* F *with* $P_E(m)=P_F(m)$*, then* E *is contained* $V(t,t)$*.*

Proof. Since F is μ-semi-stable, the subsheaf F' which may disturb the
287 semi-stability of F is of degree zero. We may assume that F is locally free. Since F is μ-semi-stable and since $H^0(P,F')\subset H^0(P,F)=0$, we see that $c_2(F')-r(F')=h^1(P,F')\geqq 0$. On the other hand, $P_{F'}(m)=m^2/2+3m/2-c_2(F')/r(F')+1$ and $P_{F'}(m)=m^2/2+3m/2$. Thus $P_{F'}(m)\leqq P_F(m)$ for all large m and $P_{F'}(M)=P_F(m)$ if and only if $c_2(F')=r(F')$, that is, F is a member of $V(r(F'),r(F'))$. Q.E.D

Corollary 1.9.1. *A vector bundle* E *on* P *is contained in* $V(n,n)$ *if and only if it has the properties* (1.5.3) *and (1.9.2)* E *is semi-stable.*

Proof. Assume that E has the properties (1.5.3) and (1.9.2). Then, obviously it has the property (1.5.1). If $H^0(P,E)\neq 0$, then $\mathscr{O}_P$ is a subsheaf of E and it violates the semi-stability of E. Thus we obtain the property (1.5.2). The converse was proved in Lemma 1.9. Q.E.D

The main aim of this article is to study the following category.

Definition 1.10. $\mathscr{V}$ is the full subcategory of the category of coherent sheaves on P whose objects are vector bundles E on P with the properties (1.5.2) and

$$c_2(E)=r(E), \tag{1.10.1}$$

for general lines ℓ in $P_{\overline{k}}$, $E\otimes_k \overline{k}\mid_{\ell}\simeq \mathscr{O}_{\ell}^{\oplus r(E)}$ (1.10.2)
$ob(\mathscr{V})$ is a disjoint union of $\mathscr{V}(n)=\{E\in\mathscr{V}\mid c_2(E)=n\}$.

§2 A class of torsion sheaves on $\mathbb{P}^2$.

Let V be a 3-dimensional vector space over the field k which was fixed **288**
in Notation 1.2. Then P is isomorphic to $\mathrm{Proj}(S(V))$. For the dual space $V^* = \mathrm{Hom}_k(V, k)$ of V, *we set* $P^* = \mathrm{Proj}(S(V^*))$. P^* is the dual plane of P. Let $\mathbb{F}$ be the flag manifold which defines the incidence correspondence between P and P^*. Then we have the following diagram which is the most fundamental in the following:

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ P & & P^* \end{tikzcd} \tag{2.1}$$

where p (or q) is isomorphic to the projective bundle $\mathbb{P}(T_P(-1))$ (or, $\mathbb{P}(T_{P^*}(-1))$, resp.) associated with the tangent bundle T_P of P (or, T_{P^*} of P^*, resp.). $p^*(\mathscr{O}_P(1))$ (or, $q^*(\mathscr{O}_{P^*}(1))$) is the tautological line bundle of $T_{P^*}(-1)$ (or $T_P(-1)$, resp.).

Notation 2.2. For a coherent sheaf G on $\mathbb{F}$, we denote

$$G \otimes p^*(\mathscr{O}_P(a)) \otimes q^*(\mathscr{O}^*_P(b)) \text{ by } G(a, b).$$

For a coherent sheaf L on P^*, we shall consider the following properties:

$$\mathrm{Supp}(L) \neq P^* \text{ and } H^0(P^*, L) = H^1(P^*, L) = 0. \tag{2.3.1}$$

$$q^*\left(\mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}(-3))\right)(1, 0) \text{ is generated by its global sections.} \tag{2.3.2}$$

In the first place, let us study (2.3.1). Assume that L has the property (2.3.1). The property that $\mathrm{Supp}(L) \neq P^*$ implies that $H^2(P^*, L(a)) = 0$ **289**
for all integers a. Then, by the property that $H^1(P^*, L) = 0$, we see that L is 1-regular.

Lemma 2.4. *If a coherent sheaf L on P^* has the property* (2.3.1)*, then L is* 1*-regular.*

Let $c_1(L) = n$. Since $\mathrm{Supp}(L) \neq P^*$ and $H^0(P^*, L) = 0$, $\mathrm{Ass}(L)$ consists of codimension one points and hence $n > 0$. By Riemann-Roch Theorem, we have

$$\chi(L) = n(n+3)/2 - c_2(L)$$

because $r(L) = 0$. Our assumption (2.3.1) shows that $c_2(L) = n(n+3)/2$. Using this, it is easy to see that $c_1(L(1)) = n$ and $c_2(L(1)) = \left(n^2 + n\right)/2$. Then, by Riemann-Roch Theorem again and by the 1-regularity of L, we have

$$h^0\left(P^*, L(1)\right) = \chi(L(1)) = \frac{n(n+3)}{2} - \frac{(n^2+n)}{2} = n.$$

Proposition 2.5. *Let L be a coherent sheaf on P^*. L has the property* (2.3.1) *and $c_1(L) = n$ if and only if there is an exact sequence*

$$0 \to \mathscr{O}_{P^*}(-1)^{\oplus n} \xrightarrow{\lambda} \mathscr{O}_{P^*}^{\oplus n} \to L(1) \to 0. \tag{2.5.1}$$

Proof. "If" part is obvious because $\mathrm{Supp}(L) = \{\det(\lambda) = 0\}$ and $\det(\lambda) \not\equiv 0$. Conversely, by Lemma 2.4, $L(1)$ is generated by its global sections. Then the above computation gives rise to a surjection $\mathscr{O}_{P^*}^{\oplus n} \xrightarrow{\nu} L(1)$ which
290 induces an isomorphism of $H^0\left(P^*, \mathscr{O}_{P^*}^{\oplus n}\right)$ to $H^0\left(P^*, L(1)\right)$ Let K be the kernal of ν. Since $\mathrm{depth}_{\mathscr{O}_{P^*,x}} L(1)_x = 1$ at every closed point x in $\mathrm{Supp}(L)$, K is locally free. Now let us consider the following exact sequence

$$0 \to K(a) \to \mathscr{O}_{P^*}(a)^{\oplus n} \xrightarrow{\nu(a)} L(a+1) \to 0$$

If $a \leqq -1$, then $H^0\left(P^*, L(a+1)\right) = 0$ and hence $H^1\left(P^*, K(a)\right) = 0$. When $a = 0$, $H^0(\nu(a))$ is surjective by the construction of ν. Thus $H^1\left(P^*, K\right) = 0$. Since $H^2\left(P^*, K(-1)\right) H^1\left(P^*, L\right) = 0$, we see that K is 1-regular and hence $H^1(P^*, K(a)) = 0$ for all $a \geqq 0$. By a well-known result on vector bundles on $\mathbb{P}^2$, K is a direct sum of line bundles. Moreover, $H^0(P^*, K) = 0$, $r(K) = n$ and $c_1(K) = -n$. This K is isomorphic to $\mathscr{O}_P * (-1)^{\oplus n}$. Q.E.D

The λ in (2.5.1) can be represented by an $n \times n$-matrix $\alpha(L)$ whose entries are all linear forms on P^*, that is, members of $H^0(P^*, \mathscr{O}_{P^*}(1)) \cdot$ $\alpha(L)$ is determined up to the choice of bases of $\mathscr{O}_{P^*}^{\oplus n}$ and $\mathscr{O}_{P^*}(-1)^{\oplus n}$, in other words, $\alpha(L)$ and $\alpha'(L)$ represent the same λ if and only if there are two elements β and γ of $GL(n, k)$ such that $\alpha(L) = \beta\alpha'(L)\gamma$. Thus the curve defined by $\det(\lambda) = \det \alpha(L) = 0$ is independent of the choice of the bases of $\mathscr{O}_{P^*}^{\oplus n}$ and $\mathscr{O}_{P^*}(-1)^{\oplus n}$.

Definition 2.6. The discriminant of L is the curve (effective Cartier divisor) in P^* defined by $\det \alpha(L) = 0$ and it is denoted by $S(L)$.

The definition is justified by the following which is proved by a simple argument in linear algebra.

Lemma 2.7. *L is an $\mathscr{O}_{S(L)}$-module and* Supp(L) *is equal to the support $|S(L)|$ of the divisor $S(L)$.* 291

The proof of the next lemma is also obvious and hence we omit it.

Lemma 2.8. *For an exact sequence of coherent sheaves on P^**

$$0 \to L' \to L \to L'' \to 0,$$

if two of L', L and L'' have the property (2.3.1)*, then so does the third. In that case, we have that $S(L) = S(L') + S(L'')$ as Cartier divisors on P^*.*

Let us list some of basic results on coherent sheaves on P^* with the property (2.3.1)

Proposition 2.9. *Let L be a coherent sheaf on P^* with the property* (2.3.1).

1. *If $S(L)$ is smooth at x, then L is an invertible $\mathscr{O}_{S(L)}$ module at x.*
2. *If $S(L)$ is a non-singular curve, then L is a line bundle on $S(L)$ with* $\deg L = g - 1$*, where g is the genus of $S(L)$.*
3. *$\mathscr{H}om_{\mathscr{O}_{s(L)}}(L, \omega_{S(L)}) = L'$ has the property* (2.3.1)*, where $\omega_{S(L)}$ is the canonical sheaf of $S(L)$. Moreover, $\alpha(L') = t_\alpha(L)$ and hence $S(L) = S(L')$.*

(4)

$$\mathscr{E}xt^i_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}) \simeq \begin{cases} \mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \mathscr{O}_{S(L)}(n)) & \text{if} \quad i = 1 \\ 0 \quad \text{if} \quad i \neq 1 \end{cases}$$

292 *where* $n = c_1(L)$. *Hence* $\mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}(-3)) \simeq L'$.

(5) $\mathscr{E}xt^i_{\mathscr{O}_{S(L)}}(L, \mathscr{O}_{S(L)}) = 0$ *for all* $i > 0$.

(6) Let $\delta : \mathscr{O}_{S(L)} \oplus n \to L(1)$ *be the restriction of the map* $\nu : \mathscr{O}_{P^*}^{\oplus n} \to L(1)$ *to* $S(L)$ *and* $M = \ker(\delta)$. *Then* $\mathscr{E}xt^i_{\mathscr{O}S(L)}(M, \mathscr{O}_{S(L)}) = \mathscr{E}xt^i_{\mathscr{O}_{S(L)}}(M^*, \mathscr{O}_{S(L)}) = 0$ *for all* $i > 0$ *and* $\mathscr{E}xt^1_{\mathscr{O}_{P^*}}(M, \mathscr{O}_{P^*}) \simeq M^*(n)$, *where* $M^* = \mathscr{H}om_{\mathscr{O}_{S(L)}}(M, \mathscr{O}_{S(L)})$.

(7) The canonical homomorphisms

$$L \to (L^*)^* = \mathscr{H}om_{\mathscr{O}_{S(L)}}(\mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \mathscr{O}_{S(L)}), \mathscr{O}_{S(L)})$$

and $M \to (M^*)^*$ *are isomorphism.*

Proof. (1) is due to Barth. In fact, by [1] Lemma 7 and (2.5.1), $1 = \mathrm{int}_x(S(L), \ell) = h^0(\ell, L_{|\ell})$ for a general line ℓ passing through x. Thus the minimal number of generators of L at x is 1. This holds at every smooth point of $S(L)$. Then, as is well-known, L is invertible on the open set of smooth points of $S(L)$. If $S(L)$ is non-singular, L is invertible on $S(L)$ 293 by virtue of (1). Riemann-Roch Theorem on $S(L)$ and (2.3.1) provide us with $\deg L - g + 1 = 0$ which proves (2). For the proof of (4), look at the sequence (2.5.1). It supplies us with a locally free resolution of $L(1)$. Thus

$$\mathscr{E}xt^i_{\mathscr{O}_{p^*}}(L, \mathscr{O}_{p^*}) = \mathscr{E}xt^i_{\mathscr{O}_{P^*}}(L(1), \mathscr{O}_{P^*})(1) = 0 \quad \text{if} \quad i \geqq 2.$$

Since L is a torsion sheaf on P^*, $\mathscr{H}om_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}) = 0$. By the exact sequence

$$0 \to \mathscr{O}_{P^*} \to \mathscr{O}_{P^*}(n) \to \mathscr{O}_{S(L)}(n) \to 0,$$

we have the exact sequence

$$0 \to \mathscr{H}om_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{S(L)}(n)) \to \mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}) \xrightarrow{\alpha} \mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}(n))$$

Since $\mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*})$ is an $\mathscr{O}_{S(L)}$-module and α is nothing but the multiplication by an equation of $S(L)$, α is zero. Therefore, we have that $\mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \mathscr{O}_{S(L)}(n)) \simeq \mathscr{H}om_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{S(L)}(n)) \simeq \mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}.$ By using the spectral sequence $E_2^{p,q} = H^p(P^*, \mathscr{E}xt^q_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}(-3))) \Rightarrow E^{p+q} = \mathscr{E}xt^{p+q}_{\mathscr{O}P^*}(L, \mathscr{O}_{P^*}(-3)))$ and (4), we get isomorphisms

$$H^i(P^* \mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \mathscr{O}_{S(L)}(n-3))) \simeq \mathscr{E}xt^{i+1}(L, \mathscr{O}_{P^*}(-3)),\ i = 0, 1.$$

On the one hand, $O_{S(L)}(n-3)$ is isomorphic to $\omega_{S(L)}$ and on the other hand, $\mathrm{Ext}^{i+1}_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}(-3))$ is a dual space of $H^{1-i}(P^*, L)$ by virtue of Serre duality. We infer from these and (2.3.1) for L that $H^1(P^*, L') = 0$ for $i = 0, 1$. It is obvious that L' is a torsion sheaf. Thus L has the **294** property (2.3.1). To prove the latter half of (3), dualizing the sequence (2.5.1) and tensoring $\mathscr{O}_{P^*}(-1)$ to it, we have the exact sequence

$$\begin{aligned} 0 \to \mathscr{O}_{P^*}(-1)^{\oplus(n)} \xrightarrow{t\alpha(L)} \mathscr{O}_{P^*}^{\oplus n} \to \mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L(1), \mathscr{O}_{P^*}(-1)) \\ \to \mathscr{E}xt^1_{\mathscr{O}_{P^*}}(\mathscr{O}_{P^*}, \mathscr{O}_{P^*}(-1)) = 0. \end{aligned}$$

On the other hand, we derive the following isomorphisms from (4)

$$\begin{aligned} &\mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L(1), \mathscr{O}_{P^*}(-1)) \simeq \mathscr{H}om(L, \mathscr{O}_{S(L)}(n-3))(1) \simeq \\ &\mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \mathscr{O}_{S(L)})(1) = L'(1). \end{aligned}$$

Hence we see that $\alpha(L') = t\alpha(L)$. As for (5), since the problem is local, it is enough to show the following if (4) is taken into account. Q.E.D

Lemma 2.10. *Let a be a non-zero divisor of a commutative ring B, $A = B/aB$ and N an A-module. Then*

$$\mathscr{E}xt^i_A(N, A) \simeq \mathscr{E}xt^{i+1}(N, B).$$

Proof. The exact sequence $0 \to B \xrightarrow{\times a} B \to A \to 0$ supplies us with the complex

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_B(B, B) \arrow[r, "\times a"] \arrow[d, equal] & \mathrm{Hom}_B(B, B) \arrow[r] \arrow[d, equal] & 0 \\ & B & B & \end{tikzcd}$$

whose cohomology is $\mathrm{Ext}^i(A, B)$. Thus $\mathrm{Ext}^1_B(A, B) = A$ and $\mathrm{Ext}^i_B(A, B) =$ **295**
0 if $i \neq 1$. Then, making use of the spectral sequence ([5, p. 349])

$$E_2^{p,q} = \mathrm{Ext}^p_A(N, \mathrm{Ext}^q_B(A, B)) \Rightarrow E^{p+q} = \mathrm{Ext}^{p+q}_B(N, B),$$

our proof is completed.

Now let us come back to the proof of Proposition 2.9. It remains to prove (6) and (7). From the exact sequence (2.5.1) we have the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathscr{O}_{P^*}(-n)^{\oplus n} \arrow[r, equal] \arrow[d] & \mathscr{O}_{P^*}(-n)^{\oplus n} \arrow[d] & & \\
0 \arrow[r] & \mathscr{O}_{P^*}(-1)^{\oplus n} \arrow[r] \arrow[d] & \mathscr{O}_{P^*}^{\oplus n} \arrow[r] \arrow[d] & L(1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathscr{O}_{S(L)}^{\oplus n} \arrow[r] \arrow[d] & L(1) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By the same argument as in the case of L, we see that

$$\mathscr{E}xt^1_{\mathscr{O}_{P^*}}(M, \mathscr{O}_{P^*}(-n)) \simeq M^*.$$

Then the left column of the above diagram gives rise to an exact sequence:

$$0 \to \mathscr{O}_{P^*}(-n+1)^n \to \mathscr{O}_{P^*}^{\oplus n} \to M^* \to 0$$

296 Thus $\mathscr{E}xt^i_{\mathscr{O}_{P^*}}(M, \mathscr{O}_{P^*}) = \mathscr{E}xt^i_{\mathscr{O}_{P^*}}(M^*, \mathscr{O}_{P^*}) = 0$ for all $i > 1$. Applying Lemma 2.10 to the above, (6) is proved. By (3), (5) and (6), we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathscr{O}_{S(L)}^{\oplus n} \arrow[r] \arrow[d, "\wr"] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (M^*)^* \arrow[r] & \mathscr{O}_{S(L)}^{\oplus n} \arrow[r] & (L^*)^* \arrow[r] & 0
\end{tikzcd}$$

Thus the canonical homomorphism $\rho : L \to (L^*)^*$ is surjective. On the other hand, the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_{P^*}(-1)^{\oplus n} \arrow[r, "\alpha(L)"] \arrow[d, equal] & \mathscr{O}_{P^*}^{\oplus n} \arrow[r] \arrow[d, equal] & L(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_{P^*}(-1)^{\oplus n} \arrow[r, "{}^t({}^t\alpha(L))"] & \mathscr{O}_{P^*}^{\oplus n} \arrow[r] & \mathscr{H}om\mathscr{O}_{S(L)}(L', \omega_{S(L)})(1) \arrow[r] & 0
\end{tikzcd}$$

provides us with an isomorphism θ of L to

$$\mathscr{H}om_{\mathscr{O}_{S(L)}}(L', \omega_{S(L)}) \simeq \mathscr{H}om_{\mathscr{O}_{S(L)}}(\mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \omega_{S(L)}), \omega_{S(L)} \simeq (L^*)^*.$$

Since L is a coherent sheaf on a noetherian scheme, ρ must be isomorphism. Then, the natural homomorphism of M to $(M^*)^*$ is isomorphic, too. Q.E.D

Definition 2.11. Assume that L has the property (2.3.1). L is said to be quadratic, if there is an isomorphism $\gamma : L \to L' = \mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \omega_{S(L)})$. A quadratic sheaf (L, γ) is called symmetric (or, symplectic) if ${}^t\gamma$: $(L')' = \mathscr{H}om_{\mathscr{O}_{S(L)}}(\mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \omega_{S(L)}), \omega_{S(L)}) \simeq L \to L'$ is equal to γ (or, $-\gamma$, resp.).

The following is an interpretation of the definition.

Proposition 2.12. *An L is symmetric (or, symplectic) if and only if $\alpha(L)$* **297**
is symmetric (or, skew-symmetric, resp.) with respect to suitable bases of $\mathscr{O}_{P^*}^{\oplus n}$ and $\mathscr{O}_{P^*}(-1)^{\oplus n}$.

Proof. Let H be the vector space $H^0(P^*, L(1))$. Then the exact sequence (2.5.1) can be written in the form

$$0 \to \mathscr{O}_{P^*}(-1)^{\oplus n} \to \mathscr{O}_{P^*} \otimes H \to L(1) \to 0. \tag{2.12.1}$$

Dualizing this sequence and tensoring $\mathscr{O}_{P^*}(-1)$ to it, we have

$$0 \to \mathscr{O}_{P^*}(-1)\otimes H^* \to \mathscr{O}_{P^*}^{\oplus n} \to \mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \omega_{S(L)})(1) = L'(1) \to 0. \tag{2.12.2}$$

Identifying $L'(1)$ with $L(1)$ through $\gamma(1)$, the middle of the above can be regarded as $\mathscr{O}_{P^*}\otimes H$. For these bases, (2.12.1) and (2.12.2) turn out

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_{P^*}(-1)\otimes H^* \arrow[r] \arrow[d, "\beta"'] & \mathscr{O}_{P^*}\otimes H \arrow[r] \arrow[d, "\delta"'] & L(1) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathscr{O}_{P^*}(-1)\otimes H^* \arrow[r] & \mathscr{O}_{P^*}\otimes H \arrow[r] & L'(1) \arrow[r] & 0
\end{tikzcd} \tag{2.12.3}$$

By the choice of the basis of H, we see that $\delta = \mathrm{id}$. To see β, let us make the dual diagram of (2.12.3):

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_{P^*}(-1)\otimes H^* \arrow[r] & \mathscr{O}_{P^*}\otimes H \arrow[r] & \mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L(1), \mathscr{O}_{P^*})(-1) \arrow[r, "\sim"] & \mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \omega_{S(L)})(1) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O}_{P^*}(-1)\otimes H^* \arrow[r] \arrow[u, equal] & \mathscr{O}_{P^*}\otimes H \arrow[r] \arrow[u] & \mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L'(1), \mathscr{O}_{P^*})(-1) \arrow[r, "\sim"] \arrow[u, "\epsilon"'] & \mathscr{H}om_{\mathscr{O}_{S(L)}}(L', \omega_{SL})(1) \arrow[r] \arrow[u, "{}^t\gamma"] & 0
\end{tikzcd}$$

298 where $\epsilon = \mathscr{E}xt^1_{\mathscr{O}_{P^*}}(\gamma, \mathscr{O}_{P^*})(-1)$. Though the isomorphisms ν and ν' depend on the choice of the equation of $S(L)$, the square of ν, ν', ${}^t\gamma$ and ϵ is commutative once we fix the equation of $S(L)$. Therefore, ${}^t\beta = \mathrm{id}$ or $-\mathrm{id}$ according as γ is symmetric or symplectic. The converse is obvious. Q.E.D

This proposition and Proposition 2.5 show the following (cf. [4, Proposition 6.23])

Corollary 2.12.4. *Giving a non-degenerate net of quadrics is equivalent to giving a coherent sheaf which has the property* (2.3.1) *and is symmetric.*

Now we shall study the property (2.3.2). For a coherent sheaf L on P^*, L has the properties (2.3.1) and (2.3.2) if and only if $L \otimes_k \overline{k}$ has them. Thus we assume, up to Remark 2.14, that the ground field k is algebraically closed. The following shows that the condition (2.3.2) is quite mild.

Proposition 2.13. *Assume that a coherent sheaf L on P^* has the property* (2.3.1). *Then L has the property* (2.3.2) *if and only if for each line M contained in $S(L)$ and for each point x on M, there exists a point y on M different from x such that*

$$Z(y) = \left\{ s \in H^0(P^*, L'(1)) \mid s(y) = 0 \right\}$$

generates $L'(x) = L \otimes k(x)$.

Proof. Let D be the closed set $q^{-1}(|S(L)|)$ of F (see the diagram (2.1)). Pick a point z on D and put $x = q(z)$. There exists a unique line M
passing through x such that z is on the minimal section Γ of $q^{-1}(M) \simeq$ 299
F_1. For a point y of M different from x, $H = p^{-1}pq^{-1}(y)$ is isomorphic to F_1 and contains the Γ as a fibre. We have the exact sequence on $\mathbb{F}$:

$$0 \to \mathcal{O}_{\mathbb{F}} \to \mathcal{O}_{\mathbb{F}}(1,0) \to \mathcal{O}_H(0,1) \otimes \mathcal{O}_H(-E) \to 0,$$

where E is the exceptional divisor on $H \simeq F_1$, that is, $q^{-1}(y)$. Tensoring the above sequence with $q^*(L')$, we have

$$0 \to q^*(L') \to q^*(L')(1,0) \to q^*(L') \otimes \mathcal{O}_H(0,1) \otimes \mathcal{O}_H(-E) \to 0.$$

Note that the equation of H is a non-zero divisor of $q^*(L')$ and hence $q^*(L') \to q^*(L')(1,0)$ is injective. Now we need

Lemma 2.13.1. *Let S be a locally noetherian scheme over a field k and $\pi : X \to S$ a $\mathbb{P}^n$-bundle in the category of k-schemes. Then, for every coherent $\mathcal{O}_S$-module F and every line bundle L on X, we have a natural isomorphism of $F \otimes_{\mathcal{O}_S} R^i\pi_*(L)$ to $R^i\pi_*(\pi^*(F) \otimes_{\mathcal{O}_X} L)$.*

Proof. Since the problem is local with respect to S, we may assume that $X = S \times_k \mathbb{P}^n$ and S is a noetherian affine scheme. Let π' be the second

projection. Then, by the base change theorem, $L \otimes_{\mathscr{O}_X} \pi'^*(\mathscr{O}_{\mathbb{P}^n}(-m)) \simeq \pi^*(N)$ for some integer m and a line bundle N on S. For these N and m, we have that $\pi^*(F) \otimes_{\mathscr{O}_X} L \cong \pi^*(F \otimes_{\mathscr{O}_S} N) \otimes_{\mathscr{O}_X} \pi'^*(\mathscr{O}_{\mathbb{P}^n}(m))$. Since $H^j(S, F \otimes_{\mathscr{O}_S} N) = 0$ for all $j > 0$, Künneth's formula provides us with an
300 isomorphism of $H^0(S, F \otimes_{\mathscr{O}_s} N) \otimes_k H^i(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(m))$ to $H^i(X, \pi^*(F) \otimes_{\mathscr{O}_X} L)$. This means that

$$(F \otimes_{\mathscr{O}_S} N) \otimes_{\mathscr{O}_S} R^i\pi_*(\pi'^*(\mathscr{O}_{\mathbb{P}^n}(m))) \xrightarrow{\sim} R^i\pi_*(\pi^*(F) \otimes_{\mathscr{O}_X} L)$$

On the other hand, by the projection formula, we have that

$$N \otimes_{\mathscr{O}_S} R^i\pi_*(\pi'^*(\mathscr{O}_{\mathbb{P}^n}(m))) \simeq R^i\pi_*(\pi^*(N) \otimes_{\mathscr{O}_X} \pi'^*(\mathscr{O}_{\mathbb{P}^n}(m))) \simeq R^i\pi_*(L).$$

Q.E.D

Let us come back to the exact sequence before the lemma. Thanks to (2.3.1) for L' and the above lemma, we obtain that $H^0(\mathbb{F}, q^*(L')(1,0)) \simeq H^0(H \cap D, q^*(L') \otimes \mathscr{O}_H(0,1) \otimes \mathscr{O}(-E))$.

Case 1. Assume that $M \not\subset S(L)$, If y is chosen so that it is not a point on $S(L)$, then $q^*(L') \otimes \mathscr{O}_H(0,1) \otimes \mathscr{O}(-E)$ can be identified with $L'(1)$. Since $L'(1)$ is generated by its global sections, $(q^*(L') \otimes \mathscr{O}_H(0,1) \otimes \mathscr{O}(-E))(z) = q^*(L')(0,1)(z)$ is generated by global sections of $q^*(L') \otimes \mathscr{O}_H(0,1) \otimes \mathscr{O}(-E)$ which are the same as those of $q^*(L')(1,0)$.

Case 2. Assume that $M \subset S(L)$. By applying Lemma 2.13.1 to the sequence before the lemma, we have the following exact sequence

$$0 \to L' \xrightarrow{\sigma} L' \otimes T_{P^*}(-1) \to q_*(q^*(L') \otimes \mathscr{O}_H(0,1) \otimes \mathscr{O}(-E)) \to 0$$

The map σ is indused by $\mathscr{O}_{P^*} \xrightarrow{\times s} T_{P^*}(-1)$ with s the section defining H. Therefore, the last term of the sequence is $L'(1) \otimes m_y$, where m_y is the ideal of the point y in O_{P^*}. Since $H^0(P^*, L'(1) \otimes m_y) = Z(y)$, we see that $Z(y)$ generates $L'(x)$ if and only if $q^*(L')(1,0)$ is generated by its global sections at z.

301 Now suppose that L has the properties (2.3.1) and (2.3.2) and that $S(L)$ contains a line M. Note that L' is isomorphic to $\mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}(-3))$

by Proposition 2.9, (3) and (4). Pick a point x on M and let z be the point on the minimal section of $q^{-1}(M) \cong F_1$ and on the fibre $q^{-1}(x)$. Applying the case II above to this situation we see that L must have the property of our proposition for this M. Conversely, assume that the property of the proposition holds for an L with the property (2.3.1). Then, combining the cases I and case II, it is easily seen that L has the property (2.3.2).

An interpretation of the above proposition through $\alpha(L)$ is the following.

Corollary 2.13.2. *Let L be a coherent sheaf on P^* with the property* (2.3.1) *and $H' = H^0(P^*, L'(1))$. Let us consider the sequence in Proposition 2.5 for L'*

$$0 \to \mathscr{O}_{P^*}(-1) \otimes_k H \to \mathscr{O}_{P^*} \otimes_k H' \to L'(1) \to 0$$

where H is a vector space of dimension n. Then L has the property (2.3.2) *if and only if the following holds:*

(2.13.3) for each line M in $S(L)$, there is a couple of points x and y of M such that $H' = \alpha(x)(H) + \alpha(y)(H)$, where for a point z of M, $\alpha(z) = \alpha(L') \otimes k(z) : H \to H'$.

Proof. In the first place, let us note the condition (2.13.3) is independent 302
of the choice of the couple x and y. In fact, if we take another couple $(\lambda x + \mu y, \lambda' x + \mu' y)$, then $\alpha(\lambda x + \mu y) = \lambda\alpha(x) + \mu\alpha(y)$ and $\alpha(\lambda' x + \mu' y) = \lambda'\alpha(x) + \mu'\alpha(y)$. Let h be an element of H'. By (2.13.3), there are u and v in H such that $h = \alpha(x)(u) + \alpha(y)(v)$. For the matrix $\begin{bmatrix} t & s \\ t' & s' \end{bmatrix} = \begin{bmatrix} \lambda & \lambda' \\ \mu & \mu' \end{bmatrix}^{-1}$, take $u' = tu + sv$ and $v' = t'u + s'v$. Then $\alpha(\lambda x + \mu y)(u') + \alpha(\lambda' x + \mu' y)(v') = h$ as required. Now assume that the condition in Proposition 2.13 holds. The exact sequence

$$H \xrightarrow{\alpha(y)} H' \to L'(1) \otimes k(y) \to 0$$

shows that $\alpha(y)(H) = Z(y)$. Thus every element of H' is contained in $Z(y)$ module $\alpha(x)(H)$ by the condition of Proposition 2.13. This is

(2.13.3). Conversely, it is obvious that (2.13.3) implies that the condition of Proposition 2.13 for the given x in (2.13.3) is satisfied. On the other hand, as we have seen in the above, (2.13.3) is independent of the choice of (x, y).

Thus our proof is completed. Q.E.D

Corollary 2.13.4. *If the condition of Proposition 2.13 is satisfied by a point an a line M in $S(L)$, then it is satisfied by all the points on M.*

Remark 2.14. It is easy to see that (2.13.3) corresponds to the $(\alpha 2)$ in [2].

Now let us introduce a subcategory of the category of coherent sheaves on P^*.

303 **Definition 2.15.** Let $\mathscr{C}$ be the full subcategory of the category of coherent sheaves on P^* whose objects are coherent sheaves with properties (2.3.1) and (2.3.2). ob($\mathscr{C}$) is a disjoint union of $\mathscr{C}(n)$, where $\mathscr{C}(n) = \{L \in \mathscr{C} \mid c_1(L) = n\}$.

Our main result of this article is the following.

Main Theorem 2.16. *For a member F of $\mathscr{V}(n)$, $R^1q_*(p^*(F)(-1,-1))$ is contained in $\mathscr{C}(n)$.*

(2) The functor $\Phi : \mathscr{V} \ni F \to R^1q_*(p^*(F)(-1,-1)) \in \mathscr{C}$ gives rise to an equivalence of categories of $\mathscr{V}$ to $\mathscr{C}$.

Example 2.17. Let L be a coherent sheaf on P^* with the property (2.3.1) and with $c_1(L) = 1$. Then $S(L)$ is a line M in P^* and $L \simeq \mathscr{O}_M(-1)$. Since $L'(1) \simeq \mathscr{O}_M$, L does not have the property (2.3.2). Therefore, $\mathscr{C}(1) = \phi$. On the other hand, $\mathscr{V}(1)$ is empty, too.

§3 From $\mathscr{V}$ to $\mathscr{C}$.

In this section, we shall study $R^1q_*(p^*(F)(-1,-1))$ for members F of $\mathscr{V}$ and prove that (1) of our Main Theorem holds and that Φ is fully faithful. Let F be a vector bundle on $P = \mathbb{P}^2_k$ with the properties (1.5.2), (1.10.2) and

$$(1.10.1)_n \quad c_2(F) = r(F) = n,$$

that is, F is a member of $\mathscr{V}(n)$. Take a general line ℓ in P so that $F\mid_\ell \cong$ **304**
$\mathscr{O}_\ell^{\oplus n}$. Then we have the following exact sequence:

$$0 \to F(-1) \to F \to \mathscr{O}_\ell^{\oplus n} \to 0$$

This supplies us with another exact sequence:

$$\begin{gathered} 0 \to q_* p^*(F(-1)) \to q_* p^*(F) \to \mathscr{O}_{P^*}^{\oplus n} \\ \to R^1 q_*(p^*(F(-1))) \to R^1 q_*(p^*(F)), \end{gathered}$$

where p and q are the same as in the diagram (2.1). The leftmost term is, on one hand, torsion free because $F(-1)$ is so. On the other hand, it is torsion, thanks to the property (1.10.2). Hence it vanishes. Since F is locally free, P^* is smooth and since $\dim P^* = 2$, $G = q_* p^*(F)$ is locally free. By (1.10.2) again, $R^1 q_*(p^*(F(-1)))$ is torsion and hence $r(G) = n$. By virtue of Corollary 1.8.1 and the base change theorem, we see that $R^1 q_*(p^*(F)) = 0$. Putting $L = \Phi(F)$, we have the following exact sequence:

$$0 \to G \to \mathscr{O}_{P^*}^{\oplus n} \to L(1) \to 0.$$

Lemma 3.1. *(1) L has the property* (2.3.1).

(2) $G \simeq \mathscr{O}_{P^*}(-1)^{\oplus n}$.

Proof. Since $H^0(P^*, G) \simeq H^0(\mathbb{F}, p^*(F)) \simeq H^0(P, F)$, we see that $H^0(P^*, G) = 0$, by the property (1.5.2) for F. $H^1(P^*, G)$ is a subspace of $H^1(\mathbb{F}, p^*(F))$ by a spectral sequence of Leray. On the other hand, we have an exact sequence

$$H^1(P, F) \to H^1(\mathbb{F}, p^*(F)) \to H^0\left(P, R^1 \mid p_*(p^*(F))\right).$$

Since p is a $\mathbb{P}^1$-bundle, we have that $R^1 p_*(p^*(F)) = 0$. By virtue **305**
of Lemma 1.8, $H^1(P, F)$ must vanish. These show that $H^1(P^*, G) = 0$. Thus the map $H^0\left(P^*, \mathscr{O}_{P^*}^{\oplus n}\right) \to H^0(P^*, L(1))$ is bijective. Since $H^0(P^*, R^1 q_*(p^*(F)(0, -1)) = 0$ by Corollary 1.8.1, $H^2(P^*, G(-1))$ is a subspace of $H^2(\mathbb{F}, p^*(F)(0, -1))$ by a spectral sequence of Leray. As is easily seen, $R^1 p^*(p^*(F)(0, -1)) = 0$ for all i. Therefore,

$$H^2(\mathbb{F}, p^*(F)(0, -1)) = 0$$

and hence $H^2(P^*, G(-1)) = 0$. Similarly we have that $H^1(P^*G(-1)) = 0$. Since L is a torsion sheaf, the proof of (1) is completed. The proof of (2) is completely the same as that of Proposition 2.5, because the homomorphism of $H^0(P^*, \mathscr{O}_{P^*}^{*n})$ to $H^0(P^*, L(1))$ is isomorphic. Q.E.D

The natural homomorphism of $q^*(G)$ to $p^*(F)$ is generically isomorphic because of the property (1.10.2) of F and the base change theorem. Since G is locally free, the map is injective;

$$0 \to q^*(G) \to p^*(F) \to A \to 0. \tag{3.2}$$

Let us determine $A = \operatorname{coker}(q^*(G) \to p^*(F))$. If ℓ is a sufficiently general line in P, then for $p^{-1}(\ell) = H \simeq F_1$, H is isomorphic to P^* on $\operatorname{Supp}(L)$ and $H \cap \operatorname{Ass}(A) = \phi$. We have the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & q^*(G)(-1,0) \arrow[r] \arrow[d] & p^*(F(-1)) \arrow[r] \arrow[d] & A(-1,0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & q^*(G) \arrow[r] \arrow[d] & p^*(F) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & q^*(G)\mid_H \arrow[r] \arrow[d] & p^*(F)\mid_H \arrow[r] \arrow[d, equal] & A\mid_H \arrow[r] \arrow[d] & 0 \\
 & 0 & p^*(F\mid_\ell) \arrow[d] & 0 & \\
 & & 0 & &
\end{tikzcd}$$

306 Taking the direct image of the above by q, another exact commutative

diagram is obtained:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G \arrow[r, "\sim"] \arrow[d] & G = q_*p^*(F) \arrow[r] \arrow[d] & q_*(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[d] & \mathscr{O}_{P^*}^{\oplus n} \arrow[r, "\sim"] \arrow[d] & q_*\,(A\,|_H) \simeq A\,|_H \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & L(1) \arrow[r] \arrow[d] & R^1q_*(A(-1,0)) \arrow[r] \arrow[d] & 0 \\
& & 0 = R^1q_*p^*(fF) \arrow[r] & R^1q_*(A) \arrow[r] & R^1q_*q^*(G) = 0
\end{tikzcd}$$

From the top row, we deduce that $q_*(A) = 0$. We infer from the bottom row that $R^1q_*(A) = 0$. Thus we obtain an isomorphism

$$L(1) \cong A\,|_H\ . \tag{3.3}$$

Note that we can regard $A\,|_H$ as a coherent sheaf on P^* because 307

$$\mathrm{Supp}(A) \subseteq q^{-1}(\mathrm{Supp}(L))$$

and H is isomorphic to P^* on $\mathrm{Supp}(L)$. Abusing the notation as in (3.3), the following sequence is exact;

$$0 \to A \to A(1,0) \to L(1) \otimes p^*(\mathscr{O}_P(1)) \to 0$$

on the support of $L(1) = A\,|_H$, $p^*(\mathscr{O}_P(1)) = \mathscr{O}_{P^*}(1)$. Taking this into account, let us make the direct image by q of the above sequence;

$$0 \to q_*(A) \to q_*(A(1,0)) \to L(2) \to R^1q_*(A).$$

Since $q_*(A) = R'q_*(A) = 0$, we see

$$q_*(A(1,0)) \simeq L(2). \tag{3.4}$$

From another exact sequence

$$0 \to q^*(G)(1,0) \to P^*(F)(1,0) \to A(1,0) \to 0$$

we have

$$0 \to G \otimes T_{P^*}(-1) \to q_*p^*(F(1)) \to q_*(A(1,0)) \to R'q_*(q^*(G)(1,0))$$

308 Obviously the last term $R^1q_*(q^*(G)(1,0))$ is zero. Putting the above together, we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & q^*(G \otimes T_{P^*}(-1)) \arrow[r] \arrow[d, "u"'] & q^*q_*p^*(F(1)) \arrow[r] \arrow[d, "v"'] & q^*q_*(A(1,0) \arrow[r] \arrow[d, "\omega"'] & 0 \\
0 \arrow[r] & q^*(G)(1,0) \arrow[r] & p^*(F(1)) \arrow[r] & A(1,0) \arrow[r] & 0
\end{tikzcd} \tag{3.5}$$

By the base change theorem u is surjective. Thanks to Corollary 1.8.1 and the base change theorem again we see that v is surjective, too. Thus ω is surjective. Setting $K' = \ker(u)$, $K = \ker(v)$ and $K'' = \ker(\omega)$, we get the following exact sequence by the snake lemma:

$$0 \to K' \xrightarrow{\zeta} K \to K'' \to 0.$$

Since $r(q^*(G \otimes T_{P^*}(-1))) = 2n = r(q^*q_*p^*(F(1)))$ and since both K' and K are locally free, we know that $\mathrm{Supp}(K'')$ is of pure codimension 1, in fact, $\mathrm{Supp}(K'')$ is the divisor defined by $\det(\zeta) = 0$. The first Chern class of $q_*p^*(F(1))$ is equal to zero by the top row of (3.5) because $c_1(G \otimes T_{P^*}(-1)) = -n$ and $c_1(q_*(A(1,0))) = c_1(L(2)) = n$ by (3.4). Then we have

$$c_1(K') = c_1(\mathscr{O}_{\mathbb{F}}(0,-n)) - c_1(\mathscr{O}_{\mathbb{F}}(n,-n)) = c_1(\mathscr{O}_{\mathbb{F}}(n,0)) \quad \text{and}$$
$$c_1(K) = 0 - c_1(\mathscr{O}_{\mathbb{F}}(n,0)) = c_1(\mathscr{O}_{\mathbb{F}}(n,0)).$$

309 These imply that $c_1(K'') = 0$, which means that $K'' = 0$. We have therefore

Proposition 3.6. *For a vector bundle F in $\mathscr{V}(n)$, there exists an exact sequence*

$$0 \to \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n} \to p^*(F) \xrightarrow{\pi} q^*(L)(-1,2) \to 0,$$

where $L = R^1q_*(p^*(F)(-1))$.

An obvious corollary to the above is

Corollary 3.6.1. *$p_*(\pi)$ is an isomorphism of F to $p_*(q^*(L)(-1,2))$.*

For the above L, let us put $C = S(L)$ and $D = q^{-1}(C)$. C and D are effective Cartier divisors on P^* and $\mathbb{F}$, respectively. Then the exact sequence in Proposition 3.6 is displayed in the following exact commutative diagram:

$$\begin{tikzcd}
& 0 & 0 & & \\
0 \arrow[r] & N \arrow[u] \arrow[r] & p^*(F)\,|_D \arrow[u] \arrow[r] & q^*(L)(-1,2) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n} \arrow[u, "\nu"] \arrow[r] & p^*(F) \arrow[u] \arrow[r] & q^*(L)(-1,2) \arrow[u, equal] \arrow[r] & 0 \\
& p^*(F)(0,-n) \arrow[u] \arrow[r, equal] & p^*(F)(0,-n) \arrow[u] & & \\
& 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{3.7}$$

The canonical homomorphism 310

$$q_C^*(\mathscr{E}xt^i_{\mathscr{O}_C}(L, \mathscr{O}_C)) \to \mathscr{E}xt^i_{\mathscr{O}_D}(q^*(L), \mathscr{O}_D)$$

is an isomorphism, thanks to the flatness of $q_C : D \to C$, and hence $\mathscr{E}xt^i_{\mathscr{O}_D}(q^*(L)(-1,2), \mathscr{O}_D) = 0$ for all $i > 0$. Indeed,

$$\mathscr{E}xt^i_{\mathscr{O}_D}(q^*(L)(-1,2), \mathscr{O}_D) \simeq \mathscr{E}xt^i_{\mathscr{O}_D}(q^*(L), \mathscr{O}_D)(1,-2)$$

$$\simeq q_C^*(\mathscr{E}xt^i_C(L, \mathscr{O}_c))(1, -2) = 0$$

by Proposition 2.9, (5) because, as we have seen in Lemma 3.1, L has the property (2.3.1). We have therefore

$$\mathscr{E}xt^i_{\mathscr{O}_D}(N, \mathscr{O}_D) = 0 \text{ for all } i > 0. \tag{3.8}$$

We shall determine $\ker(\nu_D) = T$, where ν_D is the restriction of to D. The restriction of 3.7 to D gives rise to the following exact commutative diagram;

$$\begin{tikzcd}
 & 0 \arrow[r] & N \arrow[r] & p^*(F)\mid_D \\
{} \arrow[r] & \mathscr{T}or_1\, \mathscr{O}_{\mathbb{F}}(q^*(L)(-1,2), \mathscr{O}_D) \arrow[r] & \mathscr{O}_D(0,-1)^{\oplus n} \arrow[r, "\theta"] \arrow[u, "\nu_D"'] & p^*(F)\mid_D \arrow[u, equal]
\end{tikzcd}$$

From the diagram, we can deduce clearly that $T = \ker(\nu_D) = r(\theta) = \mathscr{T}or_1\, \mathscr{O}_{\mathbb{F}}(q^*(L)(-1,2), \mathscr{O}_D)$. On the other hand, the solution of $\mathscr{O}_D$ by locally free sheaves

$$0 \to \mathscr{O}_{\mathbb{F}}(-D) \simeq \mathscr{O}_{\mathbb{F}}(0, -n) \overset{\delta}{\to} \mathscr{O}_{\mathbb{F}} \to \mathscr{O}_D \to 0$$

provides us with an isomorphism $\mathscr{T}or_1\, \mathscr{O}_{\mathbb{F}}(q^*(L)(-1,2), \mathscr{O}_D) \simeq (1 \otimes \delta : q^*(L)(-1, 2-n) \to q^*(L)(-1,2))$. Since δ is the multiplication by the local equation of D which annihilates L), we know that $1 \otimes \delta = 0$, whence we have the following exact sequence:

$$0 \to q^*(L)(-1, 2-n) \to \mathscr{O}_D(0,-1)^{\oplus n} \to N \to 0. \tag{3.9}$$

311 Taking the dual of the above sequence and tensoring with $\mathscr{O}_D(0,-1)$, we get

$$\begin{gathered}0 \to \mathscr{H}om_{\mathscr{O}_D}(N, \mathscr{O}_D(0,-1)) = N^*(0,-1) \to \mathscr{O}_D^{*n} \to \\ \mathscr{H}om_{\mathscr{O}_D}(q^*(L), \mathscr{O}_D)(1, n-3) \to \mathscr{E}xt^1_{\mathscr{O}_D}(N, \mathscr{O}_D)(0,-1).\end{gathered}$$

By (3.8) and the fact that $\mathscr{O}_D(0, n-3) \simeq q^*(\omega_C)$, we have the exact sequence

$$0 \to N^*(0,-1) \to \mathscr{O}_D^{\oplus n} \to q^*(L)(1,0) \to 0. \tag{3.10}$$

In fact, the natural homomorphism

$$q^*(L) \simeq q^*(\mathscr{H}om_{\mathscr{O}_C}(L, \omega_C)) \to \mathscr{H}om_{\mathscr{O}_D}(q^*(L), q^*(\omega_C))$$

is isomorphic because of the flatness of q_C. Now we have

Proposition 3.11. $L = R' q_*(p^*(F)(-1,-1))$ *enjoys the property 2.3.2.*

Proof. Since $\mathscr{E}xt^1_{\mathscr{O}_{P^*}}(L, \mathscr{O}_{P^*}(-3)) \simeq L'$ by Proposition 2.9, (4), our assertion is obvious if one looks at the exact sequence (3.10).

Combining Proposition 3.6 with Proposition 3.11, we have a part of our Main Theorem. Q.E.D

312

Corollary 3.11.1. *(1) For a member F of $\mathscr{V}(n)$, $R^1 q_*(p^*(F)(-1,-1)$ is contained in $\mathscr{C}(n)$.*

(2) The functor Φ in Main Theorem 2.16 is fully faithful.

Proof. (1) is done Lemma 3.1 and Proposition 3.11. Let F_1 and F_2 be two objects in $\mathscr{V}$ and set $L_i = \Phi(F_i)$. For a given homomorphism f of F_1 to F_2, the exact sequences for F_1 and F_2 in Proposition 3.6 provide us with the following commutative diagram:

$$\begin{tikzcd}
p^*(F_1) \arrow[r, "\pi_1"] \arrow[d, "p^*(f)"] & q^*(L_1)(-1,2) \arrow[r] \arrow[d, "\psi_f(-1{,}2)"] & 0 \\
p^*(F_2) \arrow[r, "\pi_2"'] & q^*(L_2)(-1,2) \arrow[r] & 0
\end{tikzcd} \qquad (3.11.2)_f$$

because the kernel of π_i is $q_* p^*(F_i)$, where ψ_f is a homomorphism of $q^*(L_1)$ to $q^*(L_2)$. Tensoring the above diagram with $\mathscr{O}_F(-1,-1)$ and applying $R^1 q_*$ to it, we get

$$\begin{tikzcd}
\Phi(F_1) \arrow[r] \arrow[d, "\Phi(f)"] & L_1 \otimes R^1 q_*(\mathscr{O}_{\mathbb{F}}(-2,1)) \arrow[d, "\xi_f"] \\
\Phi(F_2) \arrow[r] & L_2 \otimes R^1 q_*(\mathscr{O}_{\mathbb{F}}(-2,1))
\end{tikzcd} \qquad (3.11.3)$$

by Lemma 2.13.1. Since $R^1 q_*(\mathscr{O}_{\mathbb{F}}(-2,1)) \simeq \mathscr{O}_{P^*}$, $\xi_f = q_*(\psi_f)$ and $\psi_f = q^*(\xi_f)$. Now assume that for two elements f and g of

$\mathrm{Hom}_{\mathscr{V}}(F_1, F_2)$, we have $\Phi(f) = \Phi(g)$. Since (3.11.3) is canonical, we get $\xi_f = \xi_g$. If one takes the direct images of $(3.11.2)_f$ and $(3.11.2)_g$ by p, he obtains that $f = p^*(\pi_2)^{-1} p_*(\psi_f(-1,2)) \cdot p_*(\pi_1) = p_*(\pi_2)^{-1} \cdot p_*(\psi_g(-1,2)) \cdot p_*(\pi_1) = g$ because $\psi_f = q^*(\xi_f) = q^*(\xi_g) = \psi_g$. Thus 313

$$\mathrm{Hom}_{\mathscr{V}}(F_1, F_2) \to \mathrm{Hom}_{\mathscr{C}}(\Phi(F_1), \Phi(F_2))$$

is injective. To prove the surjectivity of the map, let us pick a member ξ of $\mathrm{Hom}_{\mathscr{C}}(\Phi(F_1), \Phi(F_2))$ and set $\psi = q^*(\xi)(-1,2)$. By Corollary 3.6.1, $p^*(\pi_i)$ is an isomorphism of

$$p_* p^*(F_i) \quad \text{to} \quad p_*(q^*(L_i)(-1,2)).$$

Identifying F_i with $p_* p^*(F_i)$, set $f_\xi p_*(\pi_2)^{-1} \cdot p_*(\psi) \cdot p_*(\pi_1)$. Then f_ξ is an element of $\mathrm{Hom}(F_1, F_2)$ and we have the following commutative diagram:

$$\begin{tikzcd}
 & & p^* p_*(q^*(L_1)(-1,2)) \arrow[dl, "k_1"] \arrow[ddd, "p^* p_*(\psi)"] \\
p^*(F_1) \arrow[r, "\pi_1"'] \arrow[urr, "p^* p_*(\pi_1)", "\sim"'] & q^*(L_1)(-1,2) \arrow[d, "\psi"] & \\
p^*(F_2) \arrow[r, "\pi_2"] \arrow[drr, "\sim", "p^* p_*(\pi_2)"'] & q^*(L_2)(-1,2) & \\
 & & p^* p_*(q^*(L_2)(-1,2)) \arrow[ul, "k_2"']
\end{tikzcd}$$

Since $p^*(f_\xi) = p^* p_*(\pi_2)^{-1} \cdot p^* p_*(\psi) \cdot p^* p_*(\pi_1)$, we see that $\pi_2 \cdot p^*(f_\xi) = k_2 \cdot p^* p_*(\psi) \cdot p^* p_*(\pi_1) = \psi \cdot k_1 \cdot p^* p_*(\pi_1) = \psi \cdot \pi_1$. Then, as before, we have that $\Phi(f_\xi) = \xi$.

Q.E.D

§4 From $\mathscr{C}$ to $\mathscr{V}$

The remaining part of our proof of the Main Theorem is that if L is an 314 object of $\mathscr{C}$, then there is an F in $\mathscr{V}$ such that $\Phi(F)$ is isomorphic to

L. By the property (2.3.2) for L and Proposition 2.9, (4), $q^*(L)(1,0)$ is generated by its global sections, where $L' = \mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \omega_{S(L)})$. Set $S(L) = C$ and $q^{-1}(C) = D$ as in the preceding section. Assume that L is a member of $\mathscr{C}(n)$. Since $q^*(q_*(L')(1,0)) \simeq L' \otimes T_{P^*}(-1)$ by Lemma 2.13.1, we have that $h^0(\mathbb{F}, q^*(L')(1,0)) = h^0(P^*, L' \otimes T_{P^*}(-1))$. By the exact sequence

$$0 \to L' \to L'^{\oplus 3} \to L' \otimes T_{P^*}(1) \to 0$$

we see that $h^0(P^*, L' \otimes T_{P^*}(-1)) = h^1(P^*, L'(-1))$ because L' has the property (2.3.1) by Proposition 2.9, (3). On the other hand, the exact sequence

$$0 \to \mathscr{O}_{P^*}(-3)^{\oplus n} \to \mathscr{O}_{P^*}(-2)^{\oplus n} \to L'(-1) \to 0$$

(Proposition 2.9, (3) and Proposition 2.5) implies that $h^1(P^*, L'(-1)) = h^2(P^*, \mathscr{O}_{P^3}(-3)^{\oplus n}) = n$. Thus we obtain that $h^0(\mathbb{F}, q^*(L')(1,0)) = n$.

Let N be the kernal of a homomorphism $\mathscr{O}_D^{\oplus n} \xrightarrow{\tau} q^*(L')(1,0)$ such that $H^0(\tau)$ is isomorphic:

$$0 \to N \to \mathscr{O}_D^{\oplus n} \xrightarrow{\tau} q^*(L')(1,0) \to 0. \tag{4.1}$$

Note here that $n > 1$, by Example 2.17. Since $\mathscr{E}xt^i_{\mathscr{O}_C}(L', \mathscr{O}_C) = 0$ for all $i > 0$, we see, as in the proof of (3.8), that

$$\mathscr{E}xt^i_{\mathscr{O}_D}(N, \mathscr{O}_D) = \mathscr{E}xt^i_{\mathscr{O}_D}(q^*(L'), \mathscr{O}_D) = 0 \ \text{ for all } \ i > 0. \tag{4.2}$$

And also, by the flatness of q_C and Proposition 2.9, (3) and (7), we see 315
that

$$\begin{aligned} &(q^*(L'))^* \simeq q^*(L'^*) \text{ and the canonical homomorphism} \\ &((q^*(L'))^*)^* \to q^*(L') \text{ is an isomorphism, where} \end{aligned} \tag{4.3}$$

$$(q^*(R))^* = \mathscr{H}om_{\mathscr{O}_D}(q^*(R), \mathscr{O}_D).$$

Dualizing (4.1) and tensoring with $\mathscr{O}_{\mathbb{F}}(0, n-1)$, we have, by (4.2), (4.3) and Proposition 2.9, (3),

$$0 \to q^*(L)(-1,2) \to \mathscr{O}_D(0, n-1)^{\oplus n} \xrightarrow{\sigma} N^*(0, n-1) \to 0. \tag{4.4}$$

Let $\widetilde{F}$ be the kernel of the homomorphism $\mathscr{O}_{\mathbb{F}}(0,n-1)^{\oplus n} \to \mathscr{O}_D(0,n-1)^{\oplus n} \xrightarrow{\sigma} N^*(0,n-1)$. Then the following exact commutative diagram is obtained

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & q^*(L)(-1,2) \arrow[r] \arrow[u] & \mathscr{O}_D(0,n-1)^{\oplus n} \arrow[r] \arrow[u] & N^*(0,n-1) \arrow[r] & 0 \\
0 \arrow[r] & \widetilde{F} \arrow[r] \arrow[u] & \mathscr{O}_{\mathbb{F}}(0,n-1)^{\otimes n} \arrow[r] \arrow[u] & N^*(0,n-1) \arrow[r] \arrow[u, equal] & 0 \\
 & \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n} \arrow[r, equal] \arrow[u] & \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n} \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{4.5}$$

Proposition 4.6. *(1) $\widetilde{F}$ is locally free.*

(2) There exists a vector bundle F on P such that $\widetilde{F} \simeq p^*(F)$.

316 **Proof.** Let x be a closed point of D and $y = q(x)$. Since $q_c : D \to C$ is a $\mathbb{P}^1$-bundle and since $\mathrm{depth}_{\mathscr{O}_{P^*,y}}(L_y) = 1$ (see Proposition 2.5) $\mathrm{depth}_{\mathscr{O}_{\mathbb{F},x}}(q^*(L')_x) = 2$. On the other hand, $\mathrm{depth}_{\mathscr{O}_{\mathbb{F},x}}(\mathscr{O}_{D,x}) = 2$. Then (4.1) shows that $\mathrm{depth}_{\mathscr{O}_{\mathbb{F},x}}(N_x) = 2$. Since F is a non-singular, projective three-fold, this implies that there is a locally free resolution of N of length 1; $0 \to E_1 \to E_0 \to N \to 0$. This gives rise to an exact sequence

$$0 \to E_0^* \to E_1^* \to \mathscr{E}xt^1_{\mathscr{O}_{\mathbb{F}}}(N, \mathbb{O}_{\mathbb{F}}) \to 0$$

On the other hand, as in the proof of Proposition 2.9. (4), $N^*(0,n) \simeq \mathscr{E}xt^1_{\mathscr{O}_{\mathbb{F}}}(N, \mathscr{O}_{\mathbb{F}})$ is proved. This and the above exact sequence show that $\mathrm{depth}_{\mathscr{O}_{\mathbb{F},x}}(N^*)_x = 2$ for all closed points x of D. Then our assertion (1) follows from the definition of $\widetilde{F}$ or the middle row of (4.5).

Let U be the open set $\left\{z \in P \mid p^{-1}(z) \not\subset D\right\}$ in P. U is the set of points z such that there exists a line ℓ passing through z but not contained in $S(L) = C$. It is clear that $P - U$ is a finite set of points. For $az \in U$, pick such a line ℓ and set $H = p^{-1}(\ell)$. For the M in Proposition 2.9, we have the following exact sequence

$$0 \to \mathscr{O}_{P^*}(-n+1)^{\oplus n} \to \mathscr{O}_{P^*}^{\oplus n} \to M^* \to 0$$

(see the proof of Proposition 2.9 (6)). Since $n \geqq 2$, $h^0(P^*, M^*) = n$. Obviously, we see that

(4.6.1) If $\psi : \mathscr{O}_{P^*}^{\oplus n} \to M^*$ is a homomorphism such that $H^*(\psi)$ is an **317**
isomorphism, then $\ker(\psi)$ is isomorphic to $\mathscr{O}_{P^*}(-n+1)^{\oplus n}$.

No member of $\mathrm{Ass}(\mathscr{O}_D)$ does not contain the equation of H and $\mathrm{Ass}(N) = \mathrm{Ass}(q^*(L')) = \mathrm{Ass}(\mathscr{O}_D)$. This and (4.1) provides us with the following exact sequence

$$0 \to N \mid_H \to \mathscr{O}_D^{\oplus n} \mid_H \to q^*(L')(1,0) \mid_H \to 0.$$

From the choice of l, it turns out that H is isomorphic to P^* in a neighborhood of $S(L) = C$ and hence we can identify $D \cap H$ with C. Moreover, by this identification we have that $L'(1) \simeq q^*(L')(1,0) \mid_H$. Therefore, the above exact sequence can be regarded as that on P^*.

$$0 \to N \mid_H \to \mathscr{O}_C^{\oplus n} \xrightarrow{\delta} L'(1) \to 0. \tag{4.6.2}$$

Since $H^0(\mathbb{F}, q^*(L')) = H^0(P^*, L') = 0$ and $H^1(\mathbb{F}, q^*(L')) = H^1(P^*, L') = 0$, the exact sequence $0 \to q^*(L') \to q^*(L')(1,0) \to L'(1) \to 0$ gives us an isomorphism $H^0(\mathbb{F}, q^*(L')(1,0)) \xrightarrow{\sim} H^0(C, L'(1))$. The map

$$H^0(\mathbb{F}, \mathscr{O}_D^{\oplus n}) \to H^*(C, \mathscr{O}_C^{\oplus n})$$

is obviously isomorphic:

$$\begin{tikzcd}
H^0(\mathbb{F}, \mathscr{O}_D^{\oplus n}) \arrow[r, "\sim"] \arrow[d, "\wr"] & H^0(\mathbb{F}', q^*(L')(1,0)) \arrow[d, "\wr"] \\
H^0(C, \mathscr{O}_C^{\oplus n}) \arrow[r, "H^o(\delta)"] & H^0(C, L'(1))
\end{tikzcd}$$

$H^0(\delta)$ is, therefore, bijective. Then, by the definition of M, we see **318**

$$N \mid_H \simeq M. \tag{4.6.3}$$

Our next claim is

$$N^* \mid_H \simeq (N \mid_H)^* = \mathscr{H}om_{\mathscr{O}_C}(N \mid_{H'} \mathscr{O}_C). \tag{4.6.4}$$

Indeed, from the exact sequence

$$0 \to \mathscr{O}_D(-1,0) \to \mathscr{O}_D \to \mathscr{O}_C \to 0,$$

we obtain another

$$0 \to N^*(-1,0) \to N^* \to \mathscr{H}om \mathscr{O}_D(N, \mathscr{O}_C) \to \mathscr{E}xt^1 \mathscr{O}_D(q^*(L'), \mathscr{O}_D).$$

(4.2) and the exact sequence

$$\begin{tikzcd}
\mathscr{E}xt^1 \mathscr{O}_D(\mathscr{O}_D^{\oplus n}, \mathscr{O}_D) \arrow[r] \arrow[d, equal] & \mathscr{E}xt^1 \mathscr{O}_D(N, \mathscr{O}_D) \arrow[r] & \mathscr{E}xt^2 \mathscr{O}_D(q^*(L'), \mathscr{O}_D(-1,0)) \\
0 & &
\end{tikzcd}$$

which is obtained from (4.1) show that $\mathscr{E}xt^1_{\mathscr{O}_D}(N, \mathscr{O}_D(-1,0)) = 0$. Then our claim is clear, because $\mathscr{H}om_{\mathscr{O}_D}(N, \mathscr{O}_C) \simeq \mathscr{H}om_{\mathscr{O}_C}(N \mid_H, \mathscr{O}_C) = (N \mid_H)^*$.

(4.6.3) and (4.6.4) yield an isomorphism

$$N^* \mid_H \simeq M^* \tag{4.6.5}$$

319

From the middle row of (4.5) we also have an exact sequence

$$0 \to F \mid_H (0, 1-n) \to \mathscr{O}_H^{\oplus n} \xrightarrow{\eta} N^* \mid_H \to 0. \tag{4.6.6}$$

For the η, we claim the following:

$$H^0(\eta) \text{ is isomorphic.} \tag{4.6.7}$$

Since $q^*(L)(-1,2)\ |_H \simeq L(1)$, we get the following exact commutative diagram by restricting (4.5) to H and tensoring with $\mathscr{O}_F(0,1-n)$:

$$\begin{tikzcd}
0 \arrow[r] & L(2-n) \arrow[r] & \mathscr{O}_C^{\oplus n} \arrow[r, "\epsilon"] & N^*\ |_H \arrow[r] \arrow[d, equal] & 0 \\
 & & \mathscr{O}_H^{\oplus n} \arrow[r, "\eta"] \arrow[u, "\zeta"] & N^*\ |_H \arrow[r] & 0
\end{tikzcd}$$

By (4.6.5) $h^0(C, N^*\ |_H) = h^0(C, M^*)$, which is equal to n, as we have seen in the first part of this proof. Since $n \geqq 2$, $h^0(C, L(2-n)) = 0$. Hence $H^0(\epsilon)$ is isomorphic. $H^0(\zeta)$ is clearly an isomorphism. Thus our claim is proved.

Since $\mathscr{O}_H^{\oplus n} \simeq (q\ |_H)^* \left(\mathscr{O}_{P^*}^{\oplus n}\right)$ and since $N^*\ |_H$ can be regarded as a sheaf on P^*, there is a vector bundle E on P^* which is fitted in an exact sequence

$$0 \to E \to \mathscr{O}_{P^*}^{\oplus n} \xrightarrow{\zeta} N^*\ |_H \to 0$$

whose pull-back to H is (4.6.6). Since $H^0(\xi) = H^0(\eta)$ is isomorphic, 320 (4.6.1) and (4.6.5) imply that $E \simeq \mathscr{O}_{P^*}(n-1)^{\oplus n}$ and hence $F\ |_H \simeq (q\ |_H)^*\ (\mathscr{O}_{P^*}(1-n)^{\oplus n})(0, n-1) \simeq \mathscr{O}_H^{\oplus n}$. What we have proved so far is that for all points z of U, $F(z)$ is a trivial bundle. Therefore, $p_*\left(\widetilde{F}\right) = F$ is a vector bundle of rank n and the natural homomorphism $\lambda : p^*p_*\left(\widetilde{F}\right) \to \widetilde{F}$ isomorphic on $p^{-1}(U)$. The set where λ is not isomorphic is $\det(\lambda) = 0$ which is pure codimension 1 in $\mathbb{F}$. On the other hand, $\mathbb{F} - p^{-1}(U)$ is at least codimension 2. Thus λ is an isomorphism. Q.E.D

The above proposition and the following lemma complete the proof of Main Theorem 2.16.

Lemma 4.7. *The vector bundle F in Proposition 4.6 is contained in $\mathscr{V}$ and $\Phi(F) \simeq L$.*

Proof. Since $R^iq_*(\mathscr{O}_{\mathbb{F}}(0.-1)^{\oplus n}) = 0$ for all $i > 0$ and $q_*(\mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n}) \simeq \mathscr{O}_{P^*}(-1)^{\oplus n}$, we see that $H^i(\mathbb{F}, \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n}) = H^i(P^*, \mathscr{O}_{P^*}(-1)^{\oplus n}) = 0$ for all i. All the cohomology groups of $q^*(L)(-1,2)$ vanish, too, because $R^iq_*(q^*(L)(-1,2)) \simeq L \otimes R^iq_*(\mathscr{O}_{\mathbb{F}}(-1,2)) = 0$ for all i. By the leftmost column of (4.5), these show that $H^i\left(\mathbb{F}, \widetilde{F}\right) = 0$ for all i, Since $H^i\left(\mathbb{F}, \widetilde{F}\right) = H^i(P, F)$ for all i, F has the property (1.5.2) and moreover $0 = -c_2(F) + n$, by Riemann-Roch Theorem for F (Note that $c_1(F) = 0$ and $r(F) = n$). Thus $r(F) = c_2(F)$. Pick a point outside $S(L)$. Then, by the construction of $\widetilde{F}$, $\widetilde{F} \mid_q -1_{(x)}$ is trivial. Since $F \mid_{pq-1(x)} \simeq \widetilde{F} \mid_q -1_{(x)}$, we see that F enjoys the property (1.10.2). The second assertion is easily proved by using the leftmost column of (4.5) again. Q.E.D

321 For an L in $\mathscr{C}$, $L' = \mathscr{H}om_{\mathscr{O}_{S(L)}}(L, \omega_{S(L)})$ is not necessarily contained in $\mathscr{C}$. If both L and L' are members of $\mathscr{C}$, the structure of corresponding bundles is clearer.

Theorem 4.8. *Let F be a member of $\mathscr{V}(n)$ and set $\Phi(F) = L$. Then L' contained in $\mathscr{C}$ (i.e. L' has the property* (2.3.2)*) if and only if there is an exact sequence of vector bundles*

$$0 \to E \to F \to \mathscr{O}_P^{\oplus r} \to 0 \tag{4.8.1}$$

with $H^0(P, E^*) = 0$ Moreover L' corresponds to the vector bundle $\sigma(n - r, n)_k(E^*)$ by Φ.

Proof. Assume that we have the exact sequence (4.8.1). Since $H^0(P, E^*)$ and $E^* \mid_\ell \simeq \mathscr{O}_\ell^{\oplus(n-r)}$ for general lines ℓ, the isomorphism class E^* is contained in $V(n - r, n)_k$. Let us consider a vector bundle F' whose isomorphism class is $\sigma(n - r, n)_k(E^*)$. Then F' is fitted in an exact sequence

$$0 \to E^* \to F' \to \mathscr{O}_P^{\oplus r} \to 0.$$

Now it is obvious that

$$\Phi(F) \simeq R^1q_*(P^*(E)(-1, -1))$$

and

$$\Phi(F') \simeq R^1q_*(P^*(E^*)(-1, -1)).$$

The dual sequence of (4.8.1) shows that $R^1q_*(p^*(E^*)(-1, -1))$ is isomorphic to $R^1q_*(p^*(F^*)(-1, -1))$. On the other hand, dualizing the exact sequence of Proposition 3.6, we have

$$0 \to p^*(F^*) \to \mathscr{O}_{\mathbb{F}}(0, 1)^{\oplus n} \to \mathscr{E}xt^1_{\mathscr{O}_{\mathbb{F}}}(q^*(L)(-1, 2), \mathscr{O}_{\mathbb{F}}) \to 0. \tag{4.8.2}$$

322 Since

$$\mathscr{E}xt^1_{\mathscr{O}_{\mathbb{F}}}(q^*(L)(-1,2),\mathscr{O}_{\mathbb{F}}) \simeq q^*(\mathscr{E}xt^1_{O_{P^*}}(L,\mathscr{O}_{P^*}(-3)))(1,1) \simeq q^{\mathbb{F}_*}(L')(1,1),$$

we get an isomorphism $R^1q_*(p^*(F^*)(-1,-1)) \simeq L'$, by tensoring the above sequence with $\mathscr{O}_{\mathbb{F}}(-1,-1)$ and then applying the direct image functor of q to it. Combining the above results, we see that $\Phi(F') \simeq L'$ and hence L' is contained in $\mathscr{C}$.

The proof of the converse consists of several steps.

Step I. Let F' be the bundle which corresponds to L'. We shall construct a homomorphism of $\widetilde{F^*} = p^*(F^*)$ to $\widetilde{F'} = p^*(F')$. Setting $H = p^{-1}(\ell)$ for a general line ℓ in P, we have the following exact sequence:

$$0 \to \widetilde{F^*}(-1,0) \to \widetilde{F^*} \to \mathscr{O}_H^{\oplus n} \to 0.$$

Since $R^1q_*\left(\widetilde{F^*}(-1,0)\right) \simeq L'(1)$ as we have seen in the above, the above sequence provides us with another exact sequence

$$0 \to G' = q_*\left(\widetilde{F^*}\right) \to \mathscr{O}_{P^*}^{\oplus n} \to L'(1) \to R^1q_*\left(\widetilde{F^*}\right) \to 0.$$

Since $\widetilde{F^*}$ is locally free, so is G'. If one denotes the cokernel of the natural map of $q^*(G')$ to $\widetilde{F^*}$ by T, one gets the following exact commutative diagram (see 4.8.2)

$$\begin{tikzcd}
 & 0 & & & \\
 & T \arrow[u] & & 0 & \\
0 \arrow[r] & \widetilde{F^*} \arrow[u] \arrow[r] & \mathscr{O}_{\mathbb{F}}(0,1)^{\oplus n} \arrow[r] & q^*(L')(1,1) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & q^*(G') \arrow[u] \arrow[r] & q^*q_*(\mathscr{O}_{\mathbb{F}}(0,1))^{\oplus n} \arrow[u, equal] \arrow[r] & q^*(B) \arrow[u] \arrow[r] & 0 \\
 & & & R \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd} \tag{4.8.3}$$

where B is $\operatorname{im}(\beta)$ in the following exact sequence **323**

$$\begin{aligned} 0 \to G' \to \mathscr{O}_{P^*}(1)^{\oplus n} \xrightarrow{\beta} q^*(q^*(L')(1,1)) \simeq L' \otimes T_{P^*} \\ \to R^1 q_*\left(\widetilde{F}^*\right) \to R^1 q_*(\mathscr{O}_{\mathbb{F}}(0,1)^{\oplus n}) = 0. \end{aligned}$$

Let us consider another exact commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & & \\
 & q^*(L')(1,1) \arrow[u] \arrow[r, equal] & q^*(L')(1,1) \arrow[u] & & \\
0 \arrow[r] & q^*(B) \arrow[u] \arrow[r] & q^*(L' \otimes T_{P^*}) \arrow[u] \arrow[r] & q^*\left(R^1 q_*\left(\widetilde{F}^*\right)\right) \arrow[r] & 0 \\
0 \arrow[r] & R \arrow[u] \arrow[r] & q^*(L')(-1,2) \arrow[u] \arrow[r] & Q \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

The first diagram gives rise to an isomorphism of T to R. Composing this with the injection of the bottom row of the second diagram, we obtain an injection $\delta_0 : T \to q^*(L')(-1,2)$. Note that we have the following isomorphism

$$q^*(L')(-1,2)/\delta_0(T) \simeq q^*\left(R^1 q_*\left(\widetilde{F}^*\right)\right). \tag{4.8.4}$$

324 From the exact sequence in Proposition 3.6 for F', we obtain the exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathscr{O}_{\mathbb{F}}}\left(\widetilde{F}^*, \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n}\right) \to \operatorname{Hom}_{\mathscr{O}_{\mathbb{F}}}\left(\widetilde{F}^*, \widetilde{F}'\right) \to \\ \operatorname{Hom}_{\mathscr{O}_{\mathbb{F}}}\left(\widetilde{F}^*, q^*(L')(-1,2)\right) \to \mathscr{E}xt^1_{\mathscr{O}_{\mathbb{F}}}\left(\widetilde{F}^*, \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n}\right). \end{aligned}$$

As easily seen, $\operatorname{Hom}_{\mathscr{O}_{\mathbb{F}}}\left(\widetilde{F}^*, \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n}\right) \simeq H^0\left(\mathbb{F}, \widetilde{F}(0,-1)^{\oplus n}\right) = 0$ and $\operatorname{Ext}^1_{\mathscr{O}_{\mathbb{F}}}\left(\widetilde{F}^*, \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n}\right) \simeq H^1\left(\mathbb{F}, \widetilde{F}(0,-1)^{\oplus n}\right) = 0$. Thus there exists a unique homomorphism $\widetilde{\delta} : F^* \longrightarrow F'$ which covers δ_0. Obviously, for $\delta = p_*\left(\widetilde{\delta}\right) : F^* \to F'$, $p^*(\delta) = \widetilde{\delta}$.

Step II. Set $K = \ker(\delta)$. We shall prove that $K \simeq \mathscr{O}_P^{\oplus r}$, where $r = h^0(P, F^*)$. Since both F^* and F' are μ-semi-stable and since $c_1(F^*) = c_1(F') = 0$, for $E' = \mathrm{im}(\delta)$, $c_1(E') = 0$ and hence $c_1(K) = 0$. Let S be the torsion part of F'/E'. Since $0 \leqq c_1((F'/E')/S) = c_1(F'/E') - c_1(S) = -c_1(S)$, we see that S is supported by, at most, a finite set of points. This implies that for a general z of P^*,

$$0 \to \widetilde{K} \mid_q -1_{(z)} \to \widetilde{F}^* \mid_q -1_{(z)} \to \widetilde{F}' \mid_q -1_{(z)}$$

is exact, where $\widetilde{K} = p^*(K)$. From this, we deduce that

(4.8.5) $\widetilde{K} \mid_{q^{-1}(z)}$ is a trivial bundle of rank r for all general points z 325
in P^* with $r = r(K)$ because both $\widetilde{F}^* \mid_{q^{-1}(z)}$ and $\widetilde{F}' \mid_{q^{-1}(z)}$ are trivial for all points z in $P^* - S(L)$. If ℓ is a sufficiently general line in P, then for $H = p^{-1}(\ell)$, we have the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{K}(-1,0) \arrow[r] \arrow[d] & \widetilde{K} \arrow[r] \arrow[d] & \widetilde{K} \mid_H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{F}^*(-1,0) \arrow[r] \arrow[d] & \widetilde{F}^* \arrow[r] \arrow[d] & \widetilde{F}^* \mid_H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{F}'(-1,0) \arrow[r] & \widetilde{F}' \arrow[r] & \widetilde{F}' \mid_H \arrow[r] & 0
\end{tikzcd} \tag{4.8.6}$$

On the other hand, the leftmost colum of (4.8.3) yields the exact sequence

$$0 = R^1q_*(q^*(G')(-1,0)) \to R^1q_*\left(\widetilde{F}^*(-1,0)\right) \to R^1q_*(T(-1,0)) \to 0,$$

which implies that $L'(1) = R^1q_*\left(\widetilde{F}^*(-1,0)\right) \simeq R^1q_*(T(-1,0))$. Since $R^1q_*(q^*R^1q_*\left(\widetilde{F}^*(-1,0))\right) \simeq R^1q_*\left(\widetilde{F}^*(-1,0)\right) \otimes R^1q_*\left(\mathscr{O}_{\mathbb{F}}\right) = 0$, (4.8.4) shows that

$$R^1q_*(\delta_0(-1,0)) : R^1q_*(T(-1,0)) \to R^1q_*(q^*(L')(-2,2)) \simeq L'(1)$$

is surjective and then it is an isomorphism because $R^1q_*(T(-1,0))$ is isomorphic to $L'(1)$ as we have seen in the above. In the commutative diagram

$$\begin{tikzcd}
R^1q_*\left(\widetilde{F}^*(-1,0)\right) \arrow[r] \arrow[d, "u"] & R^1q_*(T(-1,0)) \arrow[r] \arrow[d, "R^1q_*(\delta_0(-1,0))"] & 0 \\
R^1q_*\left(\widetilde{F}'(-1,0)\right) \arrow[r] & R^1q_*(q^*(L')(-2,2)) \arrow[r] & 0
\end{tikzcd}$$

326 we have showed that the maps except for u were in bijective correspondence, whence u is bijective, too. Now, (4.8.6) supplies us with the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & q_*\left(\widetilde{K}\right) \arrow[r] \arrow[d] & q_*\left(\widetilde{K}\mid_H\right) \arrow[r] \arrow[d] & R^1q_*\left(\widetilde{K}(-1,0)\right) \arrow[d] & \\
0 \arrow[r] & G' \arrow[r] \arrow[d] & \mathscr{O}_{P^*}^{\oplus n} \arrow[r] \arrow[d] & L'(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_{P^*}(-1)^{\oplus n} \arrow[r] & \mathscr{O}_{P^*}^{\oplus n} \arrow[r] & L'(1) \arrow[r] & 0
\end{tikzcd}$$

Look at the two exact sequences

$$0 \longrightarrow \widetilde{K}(-1,0) \longrightarrow \widetilde{F}^*(-1,0) \longrightarrow \widetilde{E}'(-1,0) \longrightarrow 0$$

$$0 \longrightarrow \widetilde{E}'(-1,0) \longrightarrow \widetilde{F}'(-1,0)$$

where $\widetilde{E}'$ is the torsion free $p^*(E')$. From these, we have

$$\begin{tikzcd}
R^1q_*\left(\widetilde{F}^*(-1,0)\right) \arrow[r, "\nu"] \arrow[dr, "u"'] & R^1q_*\left(\widetilde{E}'(-1,0)\right) \arrow[r] \arrow[d] & 0 \\
 & R^1q_*\left(\widetilde{F}'(-1,0)\right) &
\end{tikzcd}$$

327 Since u is bijective, v is injective, whence this is isomorphic. By (4.8.5), for general points z of P^*, $\widetilde{E}' \mid_{q^{-1}(z)}$ is a trivial bundle of rank $n - r$, which implies that $q_*\left(\widetilde{E}'(-1,0)\right)$ vanishes. We see, therefore, that $R^1q_*\left(\widetilde{K}(-1,0)\right) = 0$. The above diagram then shows that $q_*\left(\widetilde{K}\right)$ is isomorphic to $q_*\left(\widetilde{K} \mid_H\right)$ which is a trivial bundle thanks to the middle column of the diagram. The leftmost column gives us that $h^0\left(P^*, q_*\left(\widetilde{K}\right)\right) = h^0\left(P^*.G'\right) = h^0\left(\mathbb{F}, \widetilde{F}^*\right) = h^0(P, F^*)$. Therefore, $q_*\left(\widetilde{K}\right) \simeq \mathscr{O}_{P^*}^{\oplus r}$ with $r = h^0(P, F^*)$. By (4.8.5), the natural homomorphism ω of $\mathscr{O}_{\mathbb{F}}^{\oplus r} \simeq q^*q_*\left(\widetilde{K}\right)$ to $\widetilde{K}$ is injective and $r = r\left(\widetilde{K}\right)$. Thus coker$(\omega)$ is supported by the divisor det$(\omega) = 0$. On the other hand, $c_1(\text{coker}(\omega)) = 0$ because $c_1\left(\widetilde{K}\right) = c_1\left(\mathscr{O}_{\mathbb{F}}^{\oplus r}\right) = 0$. These mean that ω is an isomorphism, that is, $K \simeq \mathscr{O}_{\mathbb{F}}^{\oplus r}$. Hence we obtain that $K \simeq \mathscr{O}_P^{\oplus r}$ with $r = h^0(P, F^*)$ as required.

Step III. Let us prove that $E' = \text{im}(\delta)$ is locally free. Set $I = F'/E'$. Since $c_1(E') = 0$, the torsion part U of I is supported by a finite set of points. On the other hand, setting $J = \text{coker}(q_*(\delta) : G' \to \mathscr{O}_{P^*}(-1)^{\oplus n})$ and taking (4.8.4) into account, we have the following exact commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & q^*(G') \arrow[r] \arrow[d] & \widetilde{F}^* \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_{\mathbb{F}}(0,-1)^{\oplus n} \arrow[r] \arrow[d] & \widetilde{F}' \arrow[r] \arrow[d] & q^*(L')(-1,2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & q^*(J) \arrow[r] \arrow[d] & \widetilde{I} \arrow[r] \arrow[d] & q^*R^1q_*(F^*) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The bottom row Lemma 2.13.1 assert that the natural map of $q^*q_*\left(\widetilde{I}\right)$ to **328** $\widetilde{I}$ is bijective. Since p and q are flat, the torsion part of $\widetilde{I}$ is, on one hand, $p^*(U)$ and, on the other hand, $q^*(V)$, where V is the torsion part of $q^*\left(\widetilde{I}\right)$. Thus $p^{-1}(\text{Supp}(U)) = q^{-1}(\text{Supp}(V))$, which implies that (Supp$U$) is empty or equivalently $U = 0$, that is, I is torsion free. This implies that E' is locally free.

Step IV. We shall complete the proof of the theorem. In the first place, $\chi(F') - \chi(I) = \chi(E') = \chi(F^*) - \chi(\mathscr{O}_P^{\oplus r})$ and $\chi(F') = \chi(F^*) = 0$. Thus $\chi(I) = r$. Since $c_1(I) = 0$ and $r(I) = r$, we see, by Riemann-Roch Theorem, that $c_2(I) = 0$. Moreover, I is μ-semi-stable because for a general line ℓ in P, $I\,|_\ell \simeq \mathscr{O}_\ell^{\oplus r}$. Then applying Lemma 1.3 to I, we know that I is isomorphic to $\mathscr{O}_P^{\oplus r}$. Thus we have two exact sequences

$$0 \to \mathscr{O}_P^{\oplus r} \to F^* \to E' \to 0$$
$$0 \to E' \to F' \to \mathscr{O}_P^{\oplus r} \to 0.$$

Setting $E = E'^*$, we get our assertion. In fact, since $H^0(P, F^*) = r$, $H^0(P, E^*)$ vanishes. Q.E.D

Corollary 4.8.7. *If a member L of $\mathscr{C}$ is quadratic, then we have an exact sequence*

$$0 \to E \to F \to \mathscr{O}_P^{\oplus r} \to 0$$

where $H^0(P, E) = 0$, $E \simeq E^*$ and $r = h^0(P, F^*)$.

329 **Proof.** We have only to prove that E is isomorphic to E^*. Since $L \simeq L'$, F' in the proof of Theorem 4.8 is isomorphic to F. Thus we have two exact sequences:

$$0 \to \mathscr{O}_P^{\oplus r} \to F^* \to E^* \to 0$$
$$0 \to \mathscr{O}_P^{\oplus r} \to F^* \to E \to 0$$

Since $h^0(P, F^*) = r$, $E^* = F^*/H^0(P, F^*) \otimes \mathscr{O}_P$ by the first sequence and it is isomorphic to E by the second. Q.E.D

Let us show by an example that for an L in $\mathscr{C}$, L' is not necessarily contained in $\mathscr{C}$.

Example 4.9. Let E be a member of $\mathscr{V}(2)$ and I an ideal of a rational point x of P. By a well-known spectral sequence, we have the following exact sequence:

$$0 \to H^1(P, \mathscr{H}om_{\mathscr{O}_P}(I, E)) \to \mathrm{Ext}^1_{\mathscr{O}_P}(I, E) \to$$

$$H^0(P, \mathscr{E}xt^1_{\mathscr{O}_P}(I, E)) \to H^2(P, \mathscr{H}om_{\mathscr{O}_P}(I, E)).$$

Another exact sequence

$$0 \to I \to \mathscr{O}_P \to k(x) \to 0$$

shows that $\mathscr{H}om_{\mathscr{O}_P}(I, E) \simeq E$ and $\mathscr{E}xt^1_{\mathscr{O}_P}(I, E) \simeq \mathscr{E}xt^2_{\mathscr{O}_P}(k(x), E) \simeq k(x)^{\oplus 2}$. Therefore, $\mathrm{Ext}^1_{\mathscr{O}_P}(I, E) \simeq H^0(P, \mathscr{E}xt^1_{\mathscr{O}_P}(I, E)) \simeq k(x)^{\oplus 2}$ because E is a **330** member of $\mathscr{V}(2)$. In a neighborhood U of x, $E \mid_U \simeq \mathscr{O}_U e_1 \oplus \mathscr{O}_U e_2$. We may assume that by the map $\mathscr{E}xt^1 \mathscr{O}_P(I, E) \to \mathscr{E}xt^1 \mathscr{O}_U(I \mid_{U'} \mathscr{O}_U e_1 \oplus \mathscr{O}_U e_2) \to k(x)^{\oplus 2}$, e_1 is sent to the $(1, 0) = \xi \cdot \xi$. defines an extension

$$0 \to E \to F \to I \to 0$$

$F \mid_U$ is isomorphic to $\mathscr{O}_U e_2 \oplus G$, where G is the extension of $I \mid_U$ by $\mathscr{O}_U e_1$ defined by a generator of $\mathscr{E}xt^1 \mathscr{O}_U(I \mid_U, \mathscr{O}_U e_1) \simeq k(x)$. By virtue of a result of Serre ([12]), G is locally free and hence so is F. Now it is clear that $H^0(P, F) = 0$, $c_2(F) = r(F) = 3$ and for a general line ℓ in P, $F \mid_\ell$ is trivial. Thus F is a member of $\mathscr{V}(3)$. However, $E' = F^*/H^0(P, F^*) \otimes \mathscr{O}_P$ is a subsheaf of E^* but not equal to E^* at x. Thus E' is not locally free, this and Theorem 4.8 together show that for $L = \Phi(F)$, L' is not contained in $\mathscr{C}$. Note that the discriminant curve $S(L)$ of L is a union of non-singular conic and a line. This F also supplies us with an example to show that Lemma 1.4, (1) does not hold without assuming the stability of F. In fact, this F is semi-stable but not stable.

§5 Applications and remarks

Let us give some applications of our Main Theorem and Theorem 4.8 and some remarks. The first remark is

Lemma 5.1. *(1) A member F of $\mathscr{V}$ is stable if and only if $\Phi(F)$ is a simple object in $\mathscr{C}$.*

(2) Let L be a member of $\mathscr{C}$. If $S(L)$ is irreducible and reduced, then a vector bundle corresponding to L is stable. **331**

Proof. (1) If $\Phi(F)$ contains a proper subsheaf L which is a member of $\mathscr{C}$, then F contains a subsheaf E corresponding to L. Since E is a member of $\mathscr{V}(t)$ for some t, we see that $P_E(m) = m^2/2 + 3m/2 = P_F(m)$, which implies that F is not stable. Conversely, if F is not stable, then it is semi-stable and contains an E of a $\mathscr{V}(t)$ by Lemma 1.9 $\Phi(E)$ is then a proper, non-zero subsheaf of $\Phi(F)$.

(2) Take an F in $\mathscr{V}(n)$ which corresponds to L. Assume that F is not stable. By virtue of Lemma 1.9, we see that F is semistable and contains an E in $\mathscr{V}(t)(t < n)$ as a subsheaf. Then $M = \Phi(E)$ is a proper subsheaf of $L = \Phi(F)$ and then $S(M)$ is a subsheme of $S(L)$ (Lemma 2.8) which violates our assumption.

Q.E.D

Which curve can be a discriminant curve of L in $\mathscr{C}$ or the curve of jumping lines of a vector bundle in $\mathscr{V}$? The following answers partly this question.

Lemma 5.2. *Let D be an effective divisor on P^* whose support does not contain any line. Then there exists an L in $\mathscr{C}$ such that $S(L) = D$.*

Proof. Since the support of D contains no lines, we do not need to care about the property (2.3.2), by virtue of Proposition 2.13. Let D be $\sum n_i D_i$, where n_i's are positive integers and D_i's are mutually distinct irreducible, reduced divisors. If we have L_i for each D_i such that 332 $S(L_i) = D_i$, then $L = \oplus L_i^{\oplus n_i}$ has the property (2.3.1) and $S(L) = D$ (see Lemma 2.8). Thus we may assume that D is irreducible and reduced. Let $g : \widetilde{D} \to D$ be the normalization of D. Then, as is easily seen, there is a line bundle $\widetilde{L}$ on $\widetilde{D}$ such that $H^0\left(\widetilde{D}, \widetilde{L}\right) = H^1\left(\widetilde{D}, \widetilde{L}\right) = 0$. Set $L = g_*\left(\widetilde{L}\right)$. Since g is a finite morphism, $H^0(P^*, L) = H^0\left(\widetilde{D}, \widetilde{L}\right) = 0$ and $H^1(P^*, L) = H^1\left(\widetilde{D}, \widetilde{L}\right) = 0$. Thus L has the property (2.3.1). At smooth points of D, g is isomorphic. This means that $c_1(L) = D$. We see, therefore, that $S(L) = D$. Q.E.D

The next question is on the existence of μ-stable vector bundles F with $S(F)$ smooth.

Proposition 5.3. *Let r and c_2 be integers with $c_2 \geqq r \geqq 2$. Then there exists a μ-stable vector bundle F on $P = \mathbb{P}^2$ of rank r with $c_1(F) = 0$ and $c_2(F) = c_2$ such that the curve of jumping lines $S(F)$ of F is smooth.*

Proof. We shall fix a c_2 and prove our assertion by induction on r. In the case of $r = 2$, the proposition is well known (see [2] or [11]). Suppose that Proposition 5.3 is true up to $r-1$. Pick a μ-stable vector bundle E on P of rank $r-1$ with $c_1(E) = 0$, $c_2(E) = c_2$ and $S(E)$ smooth. Since $H^0(PE) = H^2(P, E) = 0$, we have $h^1(P, E) = c_2 - r + 1 > 0$. Then we have a non-trivial extension

$$0 \to E \to F' \to \mathscr{O}_P \to 0$$

which is stable by Lemma 1.4, (2). Since

$$R^1 q_*(p^*(E(-1))) \simeq R^1 q_*(p^*(F'(-1)),$$

$S(F') = S(E)$ which is a smooth curve in P^*. Let M be the irreducible 333
component of the moduli space of stable vector bundles on P which contains the F' in the above. Since $S(F')$ is smooth, for general points F of M and general line ℓ in P, $F\,|_\ell \simeq \mathscr{O}_\ell^{\oplus r}$ and $S(F)$ is smooth. Assume that F is not μ-stable. Then F is μ-semi-stable. Let E' be a maximal proper subsheaf of F with $c_1(E') = 0$. Then E' is locally free, $E'' = F/F'$ is torsion free and $c_1(E') = 0$. Moreover, both E and E' are μ-semi-stable. On the one hand, $S(F)$ is smooth and, on the other hand, $S(F) = S(E') + S(E')$. This implies that one of $S(E')$ and $S(E')$ is zero; in other words, either $c_2(E')$ or $c_2(E')$ is zero. By Lemma 1.3, we see that $E' \simeq \mathscr{O}_P^{\oplus s}$ or $E' \simeq \mathscr{O}_P^{\oplus(r-s)}$. If E' is trivial, $h^0(P, F) \geqq h^0(P, E') \neq 0$ which contradicts the stability of F. Therefore F is fitted in the extension

$$0 \to E' \to F \to \mathscr{O}_P^{\oplus t} \to 0$$

Now we shall assume that our assertion is false for r and show that it leads us to a contradiction. Since $h^0(P, F^*)$ is upper semi-continuous on M and $h^0(P, F'^*) = 1$, t must be 1 for general F. Furthermore, by the openness of μ-stability and the fact that E^* is μ-stable, we see that $E'^* \simeq F^*/\mathscr{O}_P$ is μ-stable for general F. Thus there exists a non-empty open set U of M whose points correspond to stable vector bundles F such that F is an extension of $\mathcal{O}_P$ by a μ-stable E:

$$0 \to E \to F \to \mathcal{O}_P \to 0. \tag{5.3.1}$$

334 The dimension of the moduli space of μ-stable vector bundles E with $c_1(E) = 0$, $c_2(E) = c_2$ and $r(E) = r - 1$ is

$$2(r-1)c_2 - (r-1)^2 + 1([10, \text{Proposition 6.9}]).$$

Since $h^1(P, E) = c_2 - r + 1$, the dimension of the space of the extensions (5.3.1) is $c_2 - r$. By Proposition 6.9 of [10] again, we have that $\dim U = 2rc_2 - r^2 + 1$. Then we get $\dim U - \{2(r-1)c_2 - (r-1)^2 + 1 + c_2 - r\} = c_2 - r + 1 > 0$. This means that general members F of U cannot be fitted in the extension (5.3.1). This is a contradiction. Q.E.D

Remark 5.4. The existence of μ-stable vector bundles E on P such that $c_1(E) = 0$, $c_2(E) = c_2$ and $r(E) = r$ for the c_2 and r given in the above proposition is a very special case of the result by J.M. Drezet and J. Le Potier [6].

In view of Theorem 4.8, we can say, on stable vector bundles in $V(n, n)$, a little more than Corollary 1.4.1 under the additional assumption (1.10.2).

Proposition 5.5. *Let F be a stable vector bundle in $\mathcal{V}$.*

(1) *Both $\Phi(F)$ and $\Phi(F)$ are contained in $\mathcal{C}$.*

(2) *We have an exact sequence of vector bundles*

$$0 \to E \to F \to \mathcal{O}_P^{\oplus r} \to 0$$

335 *with E μ-stable, where $r = h^0(P, F^*)$. In particular, if $H^0(P, F^*) = 0$, then F is μ-stable.*

Proof. (1) is corollary to Corollary 1.4.1 and Theorem 4.8. We proved (2) in Corollary 1.4.1 except for the μ-stability of E. Assume that E is not μ-stable. Since E is μ-semi-stable, there is a coherent subsheaf E_1

of E such that $c_1(E_1) = 0$, E_1 is locally free and $r(E_1) < r(E)$. Set $E_2 = E/E_1$. Then E_2 is torsion free and for general lines ℓ in P, we see that $E_l \mid_\ell$, $E \mid_\ell$ and $E_2 \mid_\ell$ are all trivial, which particularly implies that $q_*(p^*(E_2))(-1,-1))$ is torsion. On the other hand, since $p^*(E_2)$ is torsion free, thanks to the flatness of p, so is $q_*(p^*(E_2)(-1,-1))$. Hence $q_*(p^*(E_2)(-1,-1)) = 0$. Therefore, putting $L_i = R^1q_*(p^*(E_i)(-1,-1))$ and $L = R^1q_*(p^*(E)(-1,-1)) = \Phi(F)$, we obtain the following exact sequence:

$$0 \to L_1 \to L \to L_2 \to 0.$$

Since $R^iP_*(p^*(E_j)(-1,-1)) = 0$ for all i and j (see Lemma 2.13.1), $H^i(\mathbb{F}, p^*(E_j)(-1,-1)) = 0$ for all i and j. Combining this and the fact that $R^iq_*(p^*(E_j)(-1,-1)) = 0$ unless $i = 1$, we know that L_1 and L_2 have the property (2.3.1). By virtue of Proposition 2.9, (4), we get the exact sequence

$$0 \to L_2' \to L' \to L_1' \to 0.$$

Then it is obvious that L_1 has the property (2.3.2) because so does L. By Lemma 5.1, L is a simple object in $\mathscr{C}$ and hence either L_1 or L_2 **336** vanishes. On the other hand, it is not difficult to see that $\deg S(L_i) = c_2(E_i)$ (cf. [11] Proposition 1.7). Thus $c_2(E_1) = 0$ or $c_2(E_2) = 0$. Applying Lemma 1.3 to E_i, we get that E_1 or E_2 is trivial. The former violates the condition that $H^0(P, F) = 0$ and the latter contradicts the fact that $H^0(P, E^*) = 0$. This completes the proof.

Let $M'(r,n)_0^\mu$ be the moduli space of μ-stable vector bundles E of rank r and P with $c_1(E) = 0$ and $c_2(E) = n$ which have the property (1.10.2). Then $M'(r,n)_0^\mu$ is an open subscheme of $M(r,n)_0^\mu$. If the base field k is of characteristic zero, then $M'(2,n)_0^\mu = M(2,n)_0^\mu$, by the theorem of GrauertMülich. If we define $M'(n,n)_0$ to be the moduli space of stable vector bundles E with $c_1(E) = 0$ and $c_2(E) = r(E) = n$ which have the property (1.10.2), then $M'(n,n)_0$ is also an open subscheme of $M(n,n)_0$ and $\psi(r,n)(M'(r,n)_0^\mu) = \psi(r,n)(M(r,n)_0^\mu) \cap M'(n,n)_0$ (see Proposition 1.7). Let $\psi'(r,n)$ be the morphism of $M'(r,n)_0^\mu$ to $M'(n,n)_0$ induced by $\psi(r,n)$ of Proposition 1.7. Relations among $M'(r,n)_0^\mu$ are clearer than those among $M(r,n)_0^\mu$ which we have seen in Proposition 1.7.

Theorem 5.6. *There are subschemes* $Z(r,n)$ *in* $M'(n,n)_0 (r = 2,3,\dots,n)$ *such that* (1) $\psi'(r,n)$ *induces an isomorphism of* $M'(r,n)_0^{\mu}$ *to* $Z(r,n)$, (2) $M'(n,n)_0 = \coprod_{r=2}^{n} Z(r,n)$ *and* (3) $\overline{Z(r,n)} = \coprod_{s \leqq r} Z(s,n)$, *where* $\overline{}$ *means the closure in* $M'(n,n)_0$.

337 **Proof.** By Proposition 1.7 and Proposition 5.5, (2), our assertions are obvious except for (3). (3) is also easy if one takes the irreducibility of the moduli space of stable bundles on P ([6, see, for example,]) into account.

Let us close this article by the following question.

Question 5.7. Let $\overline{M}(n,n)$ be the moduli space of semi-stable sheaves E of rank n on P with $c_1(E) = 0$ and $c_2(E) = n$ ([10]). It is known that $\overline{M}(n,n)$ is a projective, normal variety. What is the closure of $Z(r,n)$ in $\overline{M}(n,n)$?

References

[1] W. Barth, Some properties of stable rank-2 vector bundles on $\mathbb{P}^n$, *Math. Ann.*, 226,1977, 125–150.

[2] W. Barth, Moduli of vector bundles on the projective plane, *Invent. Math.*, 42, 1977, 63–91.

[3] W. Barth and K. Hulek, Monads and moduli of vector bundles, *Manuscr. Math.*, 25, 1978, 323–347.

[4] A. Beauville, Varietés de Prym et jacobiennes intermédiaires, *Ann. Sci. École Norm. Sup.*, 10, 1977, 304–391.

338 [5] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, Princeton, New Jersey, 1956.

[6] J. M. Drezet and J. Le Potier, Fibrés stables et fibrés exceptionnels sur $\mathscr{P}_2$, *Annales Scient. Éc. Norm. Sup.*, 18, 1985, 193–244.

[7] K. Hulek, On the classification of stable rank-r vector bundles over the projective plane, Proceedings, *Vector bundles and differential equations,* Nice, 1979 (ed. A. Hirschowitz), progress in Math. 7, Birkhauser, Boston-Basel-Stuttgart.

[8] S. Kleiman, Les théorémes de finitude pour le foncteur de Picard, Séminaire de Geométrie Algebrique du Bois Marie 1966/67, Lect. Notes in Math., 225, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1971.

[9] M. Maruyama, Moduli of stable sheaves, I, *J. Math. Kyoto Univ.*, 17, 1977, 91–126.

[10] M. Maruyama. Moduli of stable sheaves, II, *J. Math. Kyoto Univ.*, 18, 1978, 557–614.

[11] M. Maruyama, Singularities of the curves of jumping lines of a vector bundle of rank 2 on $\mathbb{P}^2$, *Algebraic Geometry*, Proc. of Japan-France Conf., Tokyo and Kyoto, 1982, Lect. Notes in Math., 1016, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[12] J.-P Serre, Sur les modules projectifts, *Sém. Dubreil Pisot* 1960/61. **339**

Department of Mathematics
Faculty of Science
Kyoto University
Kyoto 606, Japan

On The Moduli Space Of Bundles On $K3$ Surfaces, I

By S. Mukai

341 IN [12], WE have shown that the moduli space M_S of stable sheaves on a $K3$ or abelian surface S is smooth and has a natural symplectic structure. In this article, we shall study M_S more precisely in the case S is of type $K3$. We shall show that every compact 2 dimensional component of M_S is a $K3$ surface isogenous to S (Definition 1.7 and 1.8) and describe its period explicitly (Theorem 1.4). As an application of this result, we shall show that certain Hodge cycles on a product of two $K3$ surfaces are algebraic (Theorem 1.9). As a corollary, we have that two $K3$ surfaces with Picard number $\geqq 11$ are isogeneous in our sense if and only if their transcendental Hodge structures T_S and $T_{S'}$ are isogenous, i.e., isomorphic over $\mathbb{Q}$ (Corollary 1.10).

This work was done during the author's stay at the Institute for Advanced Study in 1981-2, at the Max Planck Institut fur Mathematik first in 1982 and later in 1983 and at the Mathematics Institute in University of Warwick in 1982-3. The stay at MPI was partially supported by SFB 40 and others by Educational Projects for Japanese mathematical Scientists.

1 Introduction

342 Let S be an algebraic $K3$ surface over the complex number field τ. The cohomology group $H^2(S,\mathbb{Z})$ with the cup product pairing is an even unimodular lattice and isomorphic to $\wedge = U^{\perp 3} \perp E_8^{\perp 2}$ which we call a $K3$ lattice, where U is the hyperbolic lattice $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and E_8 is an even

unimodular negative definite lattice of rank 8. We define a bilinear form and a Hodge structure of weight 2 on the cohomology ring $H^*(S,\mathbb{Z})$. The integral bilinear form $(\cdot)$ on $H^*(S,\mathbb{Z})$ is defined by

$$(\alpha\cdot\beta) = -\alpha^{0\cup}\beta^4 + \alpha^{2\cup}\beta^2 - \alpha^{4\cup}\beta^0 \in H^4(S,\mathbb{Z}) \cong \mathbb{Z} \tag{1.1}$$

for every $\alpha = \left(\alpha^0,\alpha^2,\alpha^4\right)$ and $\beta = \left(\beta^0,\beta^2,\beta^4\right)$ in $H^*(S,\mathbb{Z})$, where we identify $H^4(S,\mathbb{Z})$ with $\mathbb{Z}$ by the fundamental cocycle $\omega \in H^4(S,\mathbb{Z})$. The Hodge decomposition of $H^*(S,\mathbb{C}) = H^*(S,\mathbb{Z}) \oplus \mathbb{C}$ is defined by

$$\begin{aligned}
&H^{*,2,0}(S,\mathbb{C}) = H^{2,0}(S,\mathbb{C}),\\
&H^{*,0,2}(S,\mathbb{C}) = H^{0,2}(S,\mathbb{C})\\
\text{and}&\\
&H^{*,1,1}(S,\mathbb{C}) = H^0(S,\mathbb{C}) \oplus H^{1,1}(S,\mathbb{C}) \oplus H^4(S,\mathbb{C}).
\end{aligned} \tag{1.2}$$

$H^*(S,\mathbb{Z})$ with the bilinear form (1.1) and the Hodge structure (1.2) is denoted by $\widetilde{H}(S,\mathbb{Z}) \cdot H^2(S,\mathbb{Z})$ is a sublattice and a Hodge substructure of $\widetilde{H}(S,\mathbb{Z})$.

343 Let E be a sheaf on S. Since $H^2(S,\mathbb{Z})$ is an even lattice, the Chern character $ch(E)$ of E belongs to $H^*(S,\mathbb{Z})$. We denote $ch(E)\cdot\sqrt{td_S} \in H^*(S,\mathbb{Z}) = \widetilde{H}(S,\mathbb{Z})$ by $v(E)$ (Definition 2.1) The $H^0(S)$ -component of $v(E)$ is the rank $r(E)$ of E (at the generic point) and $H^2(S)$-component is the 1st Chern class $c_1(E)$. The $H^4(S)$-component of $v(E)$ is denoted by $s(E)$. By the Riemann-Roch theorem, we have $s(E) = r(E) + ch^2(E) = \mathcal{X}(E) - r(E)\cdot v(E)$ is of type (1,1) with respect to the Hodge structure defined in (1.2). For sheaves E and F on S, $\mathcal{X}(E,F)$ denotes the alternating sum $\sum_i(-1)^i \dim \operatorname{Ext}^i \mathcal{O}_S(E,F)$. By the Riemann-Roch theorem, we have (see Proposition 2.2)

$$\mathcal{X}(E,F) = -(v(E)\cdot v(F)).$$

Let v be a vector of $\widetilde{H}(S,\mathbb{Z})$ of Hodge type (1,1), and let $M_A(v)$ be the moduli space of stable sheaves E on S with $v(E) = v$ which are stable with respect to A in the sense of [2]. Then $M_A(v)$ is smooth and each component has dimension $(v^2) + 2$. Assume that v is isotropic, i.e., $(v^2) = 0$ and that v is primitive, i.e., not divisible by any integer $\geqq 2$. Then $M_A(v)$ is 2-dimensional. The orthogonal complement $v^\perp$ of v in $\widetilde{H}(S,\mathbb{Z})$ contains v and the quotient $\dfrac{v^\perp}{\mathbb{Z}v}$ is a free $\mathbb{Z}$-module of rank 22. The quadratic form on $\widetilde{H}(S,\mathbb{Z})$ defined in (1.1) induces a quadratic form on $\dfrac{v^\perp}{\mathbb{Z}v}$ with signature (3, 19). Since v is of type (1.1), the Hodge decomposition of $\widetilde{H}(S,\mathbb{C})$ induces that of $\left(\dfrac{v^\perp}{\mathbb{Z}v}\right) \otimes \mathbb{C}$. Hence $\dfrac{v^\perp}{\mathbb{Z}v}$ carries the polarized Hodge structure of the same kind as $H^2(S,\mathbb{Z})$.

Theorem 1.4. *Let S be an algebraic $K3$ surface and v a primitive isotropic vector of $\widetilde{H}(S,\mathbb{Z})$. Assume that the moduli space $M_A(v)$ is* 344 *nonempty and compact. Then $M_A(v)$ is irreducible and is a (minimal) $K3$ surface. Moreover, there is an isomorphism of Hodge structures between $H^2(M_A(v),\mathbb{Z})$ and $\dfrac{v^\perp}{\mathbb{Z}v}$ which is compatible with the cup product pairing on $H^2(M_A(v))$ and the bilinear form $\dfrac{v^\perp}{\mathbb{Z}v}$ induced by that on $\widetilde{H}(S,\mathbb{Z})$.*

The above theorem and the Torelli theorem for $K3$ surfaces ([7], [20]) determine the isomorphism class of $M_A(v)$ uniquely There are many pairs of v and A for which the moduli spaces $M_A(v)$ are compact (Proposition 4.1 and Proposition 4.3).

Remark. Even if $M_A(v)$ is not compact, every component of $M_A(v)$ is birationally equivalent to a $K3$ surface M and the period of M is isomorphic to $\dfrac{v^\perp}{\mathbb{Z}v}$.

Now we show how the isomorphism between $H^2(M_A(v),\mathbb{Z})$ and $\dfrac{v^\perp}{\mathbb{Z}v}$ is obtained. The isomorphism is induced by a natural algebraic cycle on $S \times M_A(v)$. There exists a sheaf $\mathscr{E}$ on $S \times M_A(v)$ which we call a quasi-universal sheaf (Definition A.4 and Theorem A.5). $\mathscr{E}$ is flat over $M_A(v)$ and the restriction to $S \times m$ is isomorphic to $E_m^{\oplus\sigma}$ for every point $m \in M_A(v)$, where E_m is a stable sheaf in $M_A(v)$ corresponding to m. The integer $\sigma = \sigma(\mathscr{E})$ does not depend on m and is called the similitude of E. Let $ch(\mathscr{E}) \in H^*(S \times M_A(v), \mathbb{Q})$ be the Chern character of $\mathscr{E}$. Put $Z_{\mathscr{E}} = \left(\pi_S^* \sqrt{td_S}\right) \cdot ch(\mathscr{E}) \cdot \left(\frac{\pi_M^* \sqrt{td_M}}{\sigma(\mathscr{E})}\right)$, where td_S is the Todd class of S and $M = M_A(v) \cdot Z_{\mathscr{E}}$ is an algebraic cycle on $S \times M_A(v)$ (with $\mathbb{Q}$ coefficient) and induces the homomorphism

$$\begin{tikzcd}
f_{Z_{\mathscr{E}}} : \widetilde{H}(S, \mathbb{Q}) \arrow[r] \arrow[d, equal] & \widetilde{H}(M_A(v), \mathbb{Q}) \arrow[d, equal] \\
H^*(S, \mathbb{Q}) \arrow[r] & H^*(M_A(v), \mathbb{Q}). \\
t \arrow[r] \arrow[u, phantom, sloped, "\in"] & \pi_{M,*}(Z_{\mathscr{E}} \cdot \pi_S^* t) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

345 $f_{Z_{\mathscr{E}}}$ is a homomorphism of Hodge structures. $f_{Z_{\mathscr{E}}}$ sends v to the fundamental cocycle $w \in H^4(M_A(v), \mathbb{Z})$ (Lemma 4.11) and maps $v^\perp$ into $H^0(M_A(v), \mathbb{Q}) \oplus H^2(M_A(v), \mathbb{Q})$. Hence $f_{Z_{\mathscr{E}}}$ induces the homomorphism $\varphi_{\mathbb{Q}} = \frac{(v^\perp \otimes \mathbb{Q})}{\mathbb{Q}v} \to H^2(M_A(v), \mathbb{Q})$.

Theorem 1.5. *Assume that v is an isotropic vector and that $M_A(v)$ is nonempty and compact. Then we have*

(1) $\varphi_{\mathbb{Q}}$ does not depend on the choice of a quasi-universal family $\mathscr{E}$ on $S \times M_A(v)$,

(2) $\varphi_{\mathbb{Q}}$ is an isomorphism of Hodge structures and compatible with the bilinear forms on $\frac{(v^\perp \otimes \mathbb{Q})}{\mathbb{Q}v}$ and $H^2(M_A(v), \mathbb{Q})$, and

(3) $\varphi_{\mathbb{Q}}$ is defined over $\mathbb{Z}$, i.e., $\varphi_{\mathbb{Q}}\left(\frac{v^\perp}{\mathbb{Z}v}\right) = H^2(M_A(v), \mathbb{Z})$.

If $\mathscr{E}$ is a universal family (i.e., $\sigma(\mathscr{E}) = 1$), then $Z_{\mathscr{E}}$ is integral and $f_{Z\mathscr{E}}$ gives an Hodge isometry of between $\widetilde{H}(S, \mathbb{Z})$ and $\widetilde{H}(M, \mathbb{Z})$ (Theorem 4.9).

Remark 1.6. The relation between the periods of a variety X and the 346
moduli space of bundles on X was studied in the case X is a curve in [16]: Let M be the moduli space of stable rank 2 bundles with a fixed determinant ξ. If $\deg \xi$ is odd, then M is compact and the two polarized Hodge structures $H^1(C, \mathbb{Z})$ and $H^3(M, \mathbb{Z})$ are isomorphic and the isomorphism is given by using the Chern class of a universal family on $C \times M$. (Since the weights are odd, in this case, the polarization is not symmetric but skew symmetric).

The following is a natural analogue of the notion of isogeny of abelian surfaces.

Definition 1.7. An algebraic cycle $Z \in H^4(S \times S', \mathbb{Q})$ on a product of two $K3$ surfaces S and S' is an isogeny, if the homomorphism $f_Z : H^2(S, \mathbb{Q}) \to H^2(S', \mathbb{Q}), t \to \pi_{S}, \cdot_*(Z.\pi_S^* t)$, is an isometry, i.e. an isomorphism compatible with cup product pairings.

f_Z is an isometry if and only if so is the homomorphism $f'_Z : H^2 (S', \mathbb{Q}) \to H^2(S, \mathbb{Q})$, $t' \to \pi_S, *(Z, \pi_S^* t')$ because f_Z and f'_Z are adjoint to each other with respect to the cup product pairings. In fact, we have $(t' \cdot f_Z(t)) = (\pi_S^*, t' \cdot Z \cdot \pi_S^* t) = (f'_Z(t) \cdot t)$ for every $t \in H^2(S, \mathbb{Q})$ and $t' \in H^2(S', \mathbb{Q})$.

Definition 1.8. Two $K3$ surfaces S and S' are isogenous if there exists an isogeny $Z \in H^4(S \times S', \mathbb{Q})$ on $S \times S'$.

Let N_S be the Néron-Severi group of S.N_S is canonically isomorphic 347
to $H^{1,1}(S, \mathbb{Z})$ and is a primitive sublattice of $H^2(S, \mathbb{Z})$. The orthogonal complement T_S of N_S is called the *transcendental lattice* of S. Every cohomology class in N_S is of type (1,1) and any cohomology class in T_S is not so. $H^2(S, \mathbb{Z})$ contains $N_S \perp T_S$ as a sublattice of a finite index and $H^2(S, \mathbb{Q})$ is isomorphic to $(N_S \times \mathbb{Q}) \perp (T_S \times \mathbb{Q})$. Hence the cohomology group $H^4(S \times S', \mathbb{Q})$ is the direct sum of 4 vector spaces $N_S \times N_{S'} \otimes \mathbb{Q}, N_S \otimes T_{S'} \otimes \mathbb{Q}$, $T_S \otimes N_{S'} \otimes \mathbb{Q}$ and $T_S \otimes T_{S'} \otimes \mathbb{Q}$. Neither $N_S \otimes T_{S'} \otimes \mathbb{Q}$ nor $T_S \otimes N_{S'} \otimes \mathbb{Q}$ contains a cohomology class of type (2,2). Hence if $Z \in H^4(S \times S', \mathbb{Q})$ is a Hodge cycle, then Z is the sum of $Z_v \in N_S \otimes N_{S'} \otimes \mathbb{Q}$ and $Z_\tau \in T_S \otimes T_{S'} \otimes \mathbb{Q}$. Z_v is always an algebraic cycle. Hence a Hodge cycle Z is algebraic if and only if so is $Z_\tau \cdot Z_\tau$ induces the homomorphism $f_Z^\tau : T_S \otimes \mathbb{Q} \to T_{S'} \otimes \mathbb{Q}$. In particular, S

and S' are isogeneous if and only if there exists an algebraic cycle Z on $S \times S'$ such that $f_Z^\tau : T_S \otimes \mathbb{Q} \to T_{S'} \otimes \mathbb{Q}$ is an isometry. By Theorem 1.5, $Z_{\mathscr{E}}$ is an isometry and S and $M_A(v)$ are isogeneous. As an application of this fact, we have

Theorem 1.9. *Let S and S' be algebraic $K3$ surfaces and $Z \in H^4(S \times S', \mathbb{Q})$ a Hodge cycle on $S \times S'$. Assume that $f_Z^\tau : T_S \otimes \mathbb{Q} \to T_{S'} \otimes \mathbb{Q}$ is an isometry and that the lattice $T = T_S \cap (f_Z^\tau)^{-1} T_S'$ can be primitively embedded into a $K3$ lattice $\wedge$. Then Z is an algebraic cycle.*

If $\rho(S) \geqq 11$, then rank $T \leqq 11$ and T can be primitively embedded into $\wedge$ by Corollary 1.12.3 in [17]. Hence we have

348 **Corollary 1.10.** *If $\rho(S) \geqq 11$ and if $f_Z^\tau : T_S \otimes \mathbb{Q} \to T_S \otimes \mathbb{Q}$ is an isometry, then the Hodge cycle Z is algebraic.*

Remark 1.11. By the corollary, two $K3$ surfaces S and S' with $\rho \geq 11$ are isogenous if and only if the Hodge structures T_S and $T_{S'}$ are so. This partially answers to the question posed in [21]. For $K3$ surfaces with $\rho = 20$, this has been proved by Shioda-Inose [22]. Moreover, Inose [4] has proved that if T_S and $T_{S'}$ are isogenous for such two $K3$ surfaces S and S', then there exist rational maps of finite degree from S to S' and from S' to S.

In [10], Morrison has proved that if T_S has a primitive embedding $T_S \hookrightarrow U^{\perp 3}$, then there exist an abelian surface A and a certain algebraic correspondence on $S \times A$ which induces $T_S \cong T_A$. By this result and the above corollary, we have

Theorem 1.12. *Let S be an algebraic $K3$ surface. If $T_S \otimes \mathbb{Q}$ can be embedded into $(U \otimes \mathbb{Q})^{\perp 3}$ as a lattice, then there exists an algebraic cycle on $S \times A$ which induces an isometry between $T_S \otimes \mathbb{Q}$ and $T_A \otimes \mathbb{Q}$*

This was conjectured in [10] by modifying Oda's conjecture in [19].

Notation. A $K3$ surface always means a minimal algebraic $K3$ surface over $\mathbb{C}$, throughout this article. For a complex manifold X over $\mathbb{C}$,
349 $H^*(X, \mathbb{Z})$ is the cohomology ring of X. The even (resp. odd) part of

$H^*(X, \mathbb{Z})$ is denoted by $H^{ev}(X, \mathbb{Z})$ (resp. $H^{odd}(X, \mathbb{Z})$. $*$ is the involution of $H^{ev}(X, \mathbb{Z})$ which is $+1$ on $\bigoplus_n H^{4n}(X)$ and -1 on $\bigoplus_n H^{4n+2}(X)$.

A sheaf on X is a choerent $\mathscr{O}_X$-module. $h^i(E)$ is the dimension of the cohomology group $H^i(X, E)$ and $\mathcal{X}(E)$ is the alternating sum $\sum(-1)^i$ $h^i(E)$. For an ample line bundle A and a nontorsion sheaf E, the rational number $\dfrac{(c_1(E) \cdot A^{\dim X-1})}{r(E)}$ is called the slope of E with respect to A and denoted by $\mu_A(E)$. A torsion free sheaf E is μ-stable (resp. μ-semi-stable) with respect ot A, if $\mu_A(F) > \mu_A(E)$ (resp. $\mu_A(F) \geqq \mu_A(E)$) for every proper nontorsion quotient sheaf F of E. The set of isomorphism classes of all μ-stable (resp.μ-semi-stable) sheaves on X is denoted by $M_X^\mu \left(\text{resp.} S\, M_X^\mu\right)$. M_X^μ is an open subset of the moduli space M_X of stable (in Gieseker's sense) sheaves on X. For a sheaf E on $X, E^\vee$ denotes the dual sheaf $\mathrm{Hom}_{\mathscr{O}X}(E, \mathscr{O}_X) \cdot ch(E) \in H^{ev}(X, \mathbb{Q})$ is the Chern character of E. If E is locally free, then we have $ch(E^v) = ch(E)^*$.

A *lattice* over a ring R is a free R-module L with a symmetric bilinear form $(\cdot) : L \times L \to R$ and a lattice means a lattice over $\mathbb{Z}$. A sublattice L_0 of L is primitive if $\dfrac{L}{L_0}$ has no torsion and a vector v of L is primitive if $\mathbb{Z}v$ is a primitive sublattice. An isomorphism $f : L \overset{\sim}{\longrightarrow} L'$ between two lattices L and L' is an isometry if f is compatible with the bilinear forms on L and L.

For an algebraic variety X, the Néron-Severi group N_X is the Picard **350** group Pic (X) modulo algebraic equivalence. The Picard number $\rho(X)$ is the rank of N_X. If S is a $K3$ surface, then the natural map $\mathrm{Pic}(S) \to N_S$ is a bijection. For $\ell \in N_S$, we denote by $\mathscr{O}_S(\ell)$ the line bundle corresponding to ℓ.

2 Generalities

In this section, we assume that S is an abelian or $K3$ surface. The Todd class td_S of S is equal to $1 + 2\epsilon w$, where $1 \in H^0(S, \mathbb{Z})$ is the unit element of the cohomology ring $H^*(S, \mathbb{Z})$, $w \in H^4(S, \mathbb{Z})$ is the fundamental cocycle of S and ϵ is equal to 0 or 1 according as S in abelian or of type $K3$. The positive square root $\sqrt{td_S} = 1 + \epsilon w$ lies in the even part $H^{ev}(S,\mathbb{Z})$ of $H^*(S,\mathbb{Z})$. Let E be a sheaf on S. Then the Chern character $ch(E)$ belongs to $H^{ev}(S,Z)$.

Definition 2.1. For a sheaf E, we put $v(E) = ch(E).\ \sqrt{td_S} \in H^{ev}(S,\mathbb{Z})$ and call it the vector associated to E.

We define a symmetric integral bilinear form $(\,\cdot\,)$ on $H^{ev}(S,\mathbb{Z})$ by

$$(u \cdot u') = \alpha^{\cup}\alpha' - r^{\cup}s' - s^{\cup}r' \in H^4(S,\mathbb{Z}) \cong \mathbb{Z}$$

for every $u = (r,\alpha,s)$ and $u' = (r',\alpha',s') \in H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$. We denote $H^{ev}(S,\mathbb{Z})$ with this inner product $(\,\cdot\,)$ by $\widetilde{H}(S,\mathbb{Z})$. 351 $\widetilde{H}(S,\mathbb{Z})$ is an even lattice of rank $8(1+2\epsilon)$ and isomorphic to $U^{\perp 4} \perp E_8^{\perp 2}\epsilon$ as an abstract lattice. The inner product $(u \cdot u')$ is equal to the $H^4(S,\mathbb{Z})$-component of $-u^* \cdot u \in H^{ev}(S,\mathbb{Z})$. Hence, for sheaves E and F on S, $(v(E) \cdot v(F))$ is equal to the $H^4(S)$-component of $-ch(E)^* \cdot ch(F) \cdot td_S$. Therefore, by the Riemann-Roch theorem, we have

Proposition 2.2. *Let E and F be sheaves on S and put* $\mathcal{X}(E,F) = \sum_i (-1)^i \dim \operatorname{Ext}^i_{\mathcal{O}_S}(E,F)$. *Then we have* $\mathcal{X}(E,F) = -(v(E) \cdot v(F))$.

Proof. If E is locally free, then $\operatorname{Ext}^i_{\mathcal{O}_S}(E,F)$ is canonically isomorphic to $H^i(S, E^\vee \otimes F)$ for every i and $-ch(E)^* \cdot ch(F) \cdot td_S$ is equal to $-ch(E \otimes F) \cdot td_S$. Hence our assertion follows from the usual Riemann-Roch theorem. If $0 \to E' \to E \to E'' \to 0$ is an exact sequence, then $\mathcal{X}(E,F)$ and $(v(E) \cdot v(F))$ are equal to $\mathcal{X}(E',F) + \mathcal{X}(E'',F)$ and $(v(E') \cdot v(F)) + (v(E'') \cdot v(F))$, respectively. Since E has a resolution by locally free, sheaves, we have our assertion for every sheaves E and F. q.e.d

The dualizing sheaf ω_S of S is trivial. Hence the Serre duality is simple in form and is a very effective tool of our study.

Proposition 2.3. *Let E and F be sheaves on S. Then the pairing* $\operatorname{Ext}^i_{\mathcal{O}_S}$ 352 $(E,F) \times \operatorname{Ext}^{2-i}_{\mathcal{O}_S}(F,E) \to H^2(\mathcal{O}_S), (\alpha,\beta) \to tr^2(\alpha \circ \beta)$ *is nondegenerate for every i, where* $tr^2 : \operatorname{Ext}^2_{\mathcal{O}_S}(F,F) \to H^2(\mathcal{O}_S)$ *is the trace homomorphism of* $\operatorname{Ext}^2_{\mathcal{O}_S}(F,F)$. *In particular we have* $\dim \operatorname{Ext}^2_{\mathcal{O}_S}(E,F) = \dim \operatorname{Hom}_{\mathcal{O}_S}(F,E)$ *and* $\dim \operatorname{Ext}^1(E,F) = \dim \operatorname{Ext}^1_{\mathcal{O}_S}(F,E)$.

Proof. The usual Serre duality says that the natural pairing $H^i(S,G)\times \mathrm{Ext}^{2-i}_{\mathscr{O}_S}(G,\omega_S)\to H^2(S,\omega_S)$ is nondegenerate for every sheaf G on S. In the case where E is locally free, applying this Serre duality for $G = E^\vee\otimes F$, we have our proposition. In the general case, take locally free resolutions $0\to E^m\to E^{m-1}\to\cdots\to E^0\to E\to 0$ and $0\to F^m\to F^{n-1}\to\cdots F^0\to F\to 0$ of E and F, and apply the Serre duality for $\mathrm{Hom}^{\bullet}_{\mathscr{O}_S}(E^\bullet,F^\bullet)$ in the derived category $D(S)$ of S ([3]), where $E^\bullet = [0\to E^m\to E^{m-1}\to\cdots\to E^0\to 0]$ and $F^\bullet = [0\to F^n\to F^{n-1}\to\cdots\to F^0\to 0]$. Then we have our proposition. q.e.d

In the special case where $E = F$, the Serre pairing is a non degenerate bilinear form on $\mathrm{Ext}^1_{\mathscr{O}_S}(E,E)$ which we call the Serre bilinear form. This form is skew symmetric.

By Proposition 2.2 and 2.3, we have

Proposition 2.4. $(v(E)\cdot v(F)) = \dim\mathrm{Ext}^1_{\mathscr{O}_S}(E,F)\dim\mathrm{Hom}_{\mathscr{O}_S}(E,F) - \dim\mathrm{Hom}_{\mathscr{O}_S}(F,E)$.

Corollary 2.5. $\dim\mathrm{Ext}^1_{\mathscr{O}_S}(E,E) = (v(E)^2) + 2\dim\mathrm{End}_{\mathscr{O}_S}(E)$ *for every sheaf E on S. In particular,* $\dim\mathrm{Ext}^1_{\mathscr{O}_S}(E,E)$ *is always an even integer. If E is simple, then* $\dim\mathrm{Ext}^1_{\mathscr{O}_S}(E,E) = (v(E)^2+2)$ *and hence* $(v(E)^2)\geqq -2$. 353

The tangent space of Spl_S (or M_A) at the point $[E]\in\mathrm{Spl}_S$ is canonically isomorphic to $\mathrm{Ext}^1_{\mathscr{O}_S}(E,E)$. Since Spl_S is smooth ([12]), we have

Corollary 2.6. *Let v be a vector of $\widetilde{H}(S,\mathbb{Z})$. Then every component of $Spl_S(v)$ is smooth and has dimension* $\left(v^2\right)+2$.

Next we prove some inequalities for $\left(v(E)^2\right)$ and $\dim\mathrm{Ext}^1_{\mathscr{O}_S}(E,E)$ which play an important role for our study of sheaves on S.

Proposition 2.7. *Let* $X :\to F\xrightarrow{f} E\xrightarrow{g} G\to 0$ *be an exact sequence of sheaves on S such that* $\mathrm{Hom}_{\mathscr{O}_S}(F,G) = 0$. *Define* $i : \mathrm{Ext}^1_{\mathscr{O}_S}(G,F)\to\mathrm{Ext}^1_{\mathscr{O}_S}(E,E)$ *and* $j : \mathrm{Ext}^1_{\mathscr{O}_S}(E,E)\to\mathrm{Ext}^1_{\mathscr{O}_S}(F,G)$ *by* $i(\alpha) = f\circ\alpha\circ g$ *and* $j(\beta) = g\circ\beta\circ f$. *Let I be the image of i and J the kernel of j. Then we have*

(1) *$I \subset J$ and the quotient $\dfrac{J}{I}$ is isomorphic to*

$$\mathrm{Ext}^1_{\mathscr{O}_S}(F, E) \oplus \mathrm{Ext}^1_{\mathscr{O}_S}(G, G),$$

(2) *Let $e \in \mathrm{Ext}^1_{\mathscr{O}_S}(G, F)$ be the extension class of X and define the homomorphism $h : \mathrm{End}_{\mathscr{O}_S}(F) \oplus \mathrm{End}_{\mathscr{O}_S}(G) \to \mathrm{End}_{\mathscr{O}_S}(G, F)$ by $h(e_F, e_G) = e_F \circ e - e \circ e_G$. Then the sequence (2.7.1) $0 \to \mathrm{End}_{\mathscr{O}_S}(E) \to \mathrm{End}_{\mathscr{O}_S}(F) \oplus \mathrm{End}_{\mathscr{O}_S}(G) \xrightarrow{h} \mathrm{Ext}_{\mathscr{O}^i_S}(G, F) \xrightarrow{i} \mathrm{Ext}_{\mathscr{O}_S}$*
354 *$(E.E)$ is exact (since $\mathrm{Hom}_{\mathscr{O}_S}(F, G) = 0$, every endomorphism of E preserves X and induces endomorphisms of F and G), and*

(3) *J is the orthogonal complement $I^{\perp}$ of I with respect to the Serre bilinear form on $\mathrm{Ext}^1_{\mathscr{O}_S}(E, E)$ and I is totally isotropic.*

Proof. (1) Since $g \circ f = 0$, $j \circ i = 0$ and J contains I. We show that $\dfrac{J}{I}$ is isomorphic to $\mathrm{Ext}^1_{\mathscr{O}_S}(F, F) \oplus \mathrm{Ext}^1_{\mathscr{O}_S}(G, G)$. If $\alpha \in \mathrm{Ext}^1_{\mathscr{O}_S}(E, E)$ belongs to J, then $(g \circ \alpha) \circ f = 0$. Hence there exists $\alpha_G \in \mathrm{Ext}^1_{\mathscr{O}_S}(G, G)$ such that $g \circ \alpha = \alpha_G \circ g$. Since $\mathrm{Hom}_{\mathscr{O}_S}(F, G) = 0$, such an α_G is unique. In a similar way, there exists a unique $\alpha_F \in \mathrm{Ext}^1_{\mathscr{O}_S}(F, F)$ such that $\alpha \circ f = f \circ \alpha_F$. It is easy to see that the map $\varphi : J \to \mathrm{Ext}^1_{\mathscr{O}_S}(F, F) \oplus \mathrm{Ext}^1_{\mathscr{O}_S}(G, G), \alpha \mapsto (\alpha_F, \alpha_G)$ is a homomorphism.

Claim. Ker $\varphi = I$.

If $\alpha \in I$, then $g \circ \alpha = \alpha \circ f = 0$. Hence $\alpha_F = \alpha_G = 0$ and I is contained in Kerφ. Assume that α belongs to Kerφ. Then we have $\alpha \circ f = g \circ \alpha = 0$. Hence there exists $\beta \in \mathrm{Ext}^1_{\mathscr{O}_S}(E, F)$ such that $\alpha = f \circ \beta$. Since $f \circ (\beta \circ f) = \alpha \circ f = 0$ and since $\mathrm{Ext}^1_{\mathscr{O}_S}(F, F)$ $f^{\circ} \xrightarrow{*} \mathrm{Ext}^1_{\mathscr{O}_S}(F, E)$ is injective, we have $\beta \circ f = 0$. Hence $\beta = \gamma \circ g$ for some $g \in \mathrm{Ext}^1_{\mathscr{O}_S}(F, G)$. Therefore, α is equal to $f \circ \gamma \circ g$ and belongs to I.

Claim. φ is surjective.

355 By the Serre duality and by our assumption, we have $\operatorname{Ext}^2_{\mathscr{O}_S}$ $(G,F)=0$. Hence the homomorphism $\operatorname{Ext}^1_{0_S}(E,F)\xrightarrow{*\circ f}$ $\operatorname{Ext}^1_{\mathscr{O}_S}(F,F)$ is surjective. Therefore, for every $\alpha_F\in\operatorname{Ext}^1_{\mathscr{O}_S}(F,F)$, there exists $\beta\in\operatorname{Ext}^1_{\mathscr{O}_S}(E,F)$ such that $\alpha_F=\beta\circ f$. Put $\alpha=f\circ\beta\in$ $\operatorname{Ext}^1_{\mathscr{O}_S}(E,E)$. Then it is easy to see that $\varphi(\alpha)=(\alpha_F,0)$. In a similar way, for every $\alpha_G\in\operatorname{Ext}^1_{\mathscr{O}_S}(G,G)$, we obtain $\alpha\in\operatorname{Ext}^1_{\mathscr{O}_S}(E,E)$ such that $\varphi(\alpha)=(0,\alpha_G)$. Hence φ is surjective.

(2) If $h(e_F,e_G)=0$, then $e_F\circ e=e\circ e_G$ which means that two endomorphisms e_F and e_G of F and G are compatible with respect to the extension class of X. Hence there exists an endomorphism of E which induces e_F and e_G. Therefore, the sequence (2.7.1) is exact at $\operatorname{End}_{\mathscr{O}_S}(F)\oplus\operatorname{End}_{\mathscr{O}_S}(G)$. Since $f\circ e=e\circ g=0$, we have $h\circ i=0$. Assume that $\alpha\in\operatorname{Ext}^1_{\mathscr{O}_S}(G,F)$ and $i(\alpha)=0$, i.e., $f\circ(\alpha\circ g)=0$. Then there exists $\beta\in\operatorname{Hom}_{\mathscr{O}_S}(E,G)$ such that $\alpha\circ g=$ $e\circ\beta$. Since $\operatorname{Hom}_{\mathscr{O}_S}(F,G)=0$, there exists an endomorphism γ_G of G such that $\beta=\gamma_G\circ g$. Since $(\alpha-e\circ\gamma_G)\circ g=0$, there exists an endomorphism γ_F of F such that $\alpha-e^\circ\gamma_G=\gamma_F\circ e$. Therefore, α lies in the image of h and the sequence (2.7.1) is exact at $\operatorname{Ext}^1_{\mathscr{O}_S}(G,F)$.

(3) Since ω_S is trivial, the homomorphisms i and j are dual to each other by the Serre duality. Hence I and $\operatorname{Ext}^1_{\mathscr{O}_S}\dfrac{(E,E)}{J}$ are dual to each other. If $\alpha\in I$ and $\beta\in J$, then $\alpha\circ\beta\in\operatorname{Ext}^2_{\mathscr{O}_S}(E,E)$ is zero. Hence I and F are perpendicular with respect to the Serre bilinear form on $\operatorname{Ext}^1_{\mathscr{O}_S}(E,E)$. Since the Serre bilinear form is nondegenerate, J coincides with I^ℓ.

q.e.d

Corollary 2.8. *([11]) Let X be same as above. Then we have* 356

$$\dim\operatorname{Ext}^1_{\mathscr{O}_S}(F,F)+\dim\operatorname{Ext}^1_{\mathscr{O}_S}(G,G)\leqq\dim\operatorname{Ext}^1_{\mathscr{O}_S}(E,E).$$

Remark 2.9. If S is a surface and $|\ K_S\ |\neq\phi$, then $\operatorname{Hom}_{\mathscr{O}_S}(F,G)=0$ implies $\operatorname{Ext}^2_{\mathscr{O}_S}(G,F)=0$. Hence (1) and (2) of the proposition and the corollary are true for such surface (1) of the proposition says that every infinitesimal deformation of F and G can be lifter to an infinitesimal deformation of E.

The following proposition and its proof are quite similar to above ones. In fact, these propositions are equivalent if one consider them in the derived category $D(S)$ of S.

Proposition 2.10. *Let* $X : 0 \to E \xrightarrow{g} G \xrightarrow{e} F \to 0$ *be an exact sequence of sheaves on* S *such that* $\mathrm{Ext}^1_{\mathscr{O}_S}(F, G) = 0$. *Let* $f \in \mathrm{Ext}^1_{\mathscr{O}_S}(F.E)$ *be the extension class of* X. *Define* $i = \mathrm{Hom}_{\mathscr{O}_S}(G, F) \to \mathrm{Ext}^1_{\mathscr{O}_S}(E, E)$ *and* $j : \mathrm{Ext}^1_{\mathscr{O}_S}(E, E) \to \mathrm{Ext}^2_{\mathscr{O}_S}(F, G)$ *by* $i(\alpha) = f \circ \alpha \circ g$ *and* $j(\beta) = g \circ \beta \circ f$. *Let* I *be the image of* i *and* J *the kernel of* j. *Then we have (1) and (3) in Proposition 2.7 and (2) define the homomorphism* $h : \mathrm{End}_{\mathscr{O}_S}(F) \oplus \mathrm{End}_{\mathscr{O}_S}(G) \to \mathrm{Hom}_{\mathscr{O}_S}(G, F)$ *by* $h(e_F, e_G) = e_F \circ e - e \circ e_G$ *for* $e_F \in \mathrm{End}_{\mathscr{O}_S}(F)$ *and* $e_G \in \mathrm{End}_{\mathscr{O}_S}(G)$. *Every endomorphism of* E *is induced by that of* G *and the sequence* $0 \to \mathrm{End}_{\mathscr{O}_S}(E) \to \mathrm{End}_{\mathscr{O}_S}(F) \oplus \mathrm{End}_{\mathscr{O}_S}(G) \xrightarrow{h}$
357 $\mathrm{Ext}^1_{\mathscr{O}_S}(G, F)^i \to \mathrm{Ext}^1_{\mathscr{O}_S}(E, E)$ *is exact. In particular, if* $I = 0$, *then* h *is surjective.*

Proof. By the Serre duality, we have $\mathrm{Ext}^1_{\mathscr{O}_S}(G, F) = 0$. (1) and (3) can be proved in a similar way to Proposition 2.7. Since $\mathrm{Ext}^1_{\mathscr{O}_S}(F, G) = 0$, the map $\mathrm{End}_{\mathscr{O}_S}(G) \to \mathrm{Hom}_{\mathscr{O}_S}(E, G)$ is surjective. Hence every endomorphism of E is a restriction of an endomorphism of G. Hence the homomorphism $\mathrm{End}_{\mathscr{O}_S}(E) \to \mathrm{End}_{\mathscr{O}_S}(F) \oplus \mathrm{End}_{\mathscr{O}_S}(G)$ is well defined. The exactness of the sequence can be proved in a similar way to Proposition 2.7. q.e.d

Corollary 2.11. *Let* X *be same as above. Then we have* $\dim \mathrm{Ext}^1_{\mathscr{O}_S}(F, F) + \dim \mathrm{Ext}^1_{\mathscr{O}_S}(G, G) \leqq \dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E)$.

Let E be a torsion free sheaf and $\widetilde{E}$ the double dual of E. Then the natural homomorphism $E \to \widetilde{E}$ is injective and the cokernel M is of finite length. We have the exact sequence

$$0 \to E \to \widetilde{E} \xrightarrow{e} M \to 0.$$

Since $\widetilde{E}$ is locally free, we have $\mathrm{Ext}^1_{\mathscr{O}_S}\left(M, \widetilde{E}\right) \cong \mathrm{Ext}^1_{\mathscr{O}_S}\left(\widetilde{E}, M\right)^{\vee} = 0$. Since $\left(v(M)^2\right) = 0$. $\dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E)$ is equal to $2 \dim \mathrm{End}_{\mathscr{O}_S}(E)$ by Corollary 2.5. Hence we have 358

Corollary 2.12. *Let E be a torsion free sheaf on S and $\widetilde{E}$ and M be as above. Then we have*

$$\dim \mathrm{Ext}^1_{\mathscr{O}_S}\left(\widetilde{E}, \widetilde{E}\right) + 2 \dim \mathrm{End}_{\mathscr{O}_S}(M) \leqq \dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E).$$

If equality holds in the above relation, then the natural homomorphism $\mathrm{End}_{\mathscr{O}_S}\left(\widetilde{E}\right) \oplus \mathrm{End}_{\mathscr{O}_S}(M) \to \mathrm{Hom}_{\mathscr{O}_S}\left(\widetilde{E}, M\right)$, $(\alpha, \beta) \mapsto e^{\circ}\alpha - \beta \circ e$, *is surjective.*

Lemma 2.13. *Let $(R, \mathfrak{m})$ be a local ring and M an artinian R-module. Then we have length* $(\mathrm{End}_R(M)) \geqq$ *length* (M). *If equality holds, then M is isomorphic to $\frac{R}{I}$ for an ideal I of R.*

Proof. We prove by induction on length (M). Let M_0 be the submodule $\{x \in M; \mathfrak{m}x = 0\}$ of M. Every endomorphism of M maps M_0 into itself. Hence we have the exact sequence

$$0 \to \mathrm{Hom}_R(M, M_0) \to \mathrm{End}_R(M) \to \mathrm{End}_R\left(\frac{M}{M_0}\right) \to 0.$$

Since M is artinian, M_0 is nonzero. Hence by induction hypothesis, we have length $\left(\mathrm{End}_R\left(\frac{M}{M_0}\right)\right) \geqq$ length $\left(\frac{M}{M_0}\right)$. Since $\mathfrak{m}M_0 = 0$, every homomorphism from M to M_0 factors through $\frac{M}{\mathfrak{m}M}$. Hence $\mathrm{Hom}_R(M, M_0)$ is isomorphic to the vector space $\frac{\mathrm{Hom}_R}{\mathfrak{m}(M/\mathfrak{m}M, M_0)}$. Therefore, we have

$$\begin{aligned}
&\text{length } (\mathrm{End}_R(M)) = \text{ length } \left(\mathrm{End}_R\left(\frac{M}{M_0}\right)\right) + \text{ length } (\mathrm{Hom}_R(M, M_0)) \\
&\geqq \text{ length } \left(\frac{M}{M_0}\right) + \text{ length } \left(\frac{M}{\mathfrak{m}M}\right) \text{ length } (M_0) \\
&\geqq \text{ length } (M)
\end{aligned}$$

which shows the first half of the lemma. If equalities hold in the above 359
relations, then we have length $\left(\mathrm{End}_R\left(\frac{M}{M_0}\right)\right) =$ length $\left(\frac{M}{M_0}\right)$ and length $\left(\frac{M}{\mathfrak{m}M}\right) = 1$. By the latter equality and the Nakayama's lemma, M is generated by one element. Hence M is isomorphic to $\frac{R}{I}$ for an ideal I. q.e.d

By Corollary 2.12 and the above lemma, we have

Proposition 2.14. *Let E be a torsion free sheaf on S, $\widetilde{E}$ the double dual of E and $M = \frac{\widetilde{E}}{E}$. Then we have*

$$\dim \mathrm{Ext}^1_{\mathscr{O}_S}\left(\widetilde{E}, \widetilde{E}\right) + 2\ \textit{length}\ (M) \leqq \dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E).$$

If equality holds, then the natural map $\mathrm{End}_{\mathscr{O}_S}\left(\widetilde{E}\right) \oplus \mathrm{End}_{\mathscr{O}_S}(M) \to \mathrm{Hom}_{\mathscr{O}_S}\left(\widetilde{E}, M\right)$ *is surjective and M is isomorphic to $\frac{\mathscr{O}_S}{\mathscr{I}}$ for an ideal $\mathscr{I}$ of $\mathscr{O}_S$.*

Remark 2.15. Since $\widetilde{E}$ is locally free, $\mathrm{Ext}^1_{\mathscr{O}_S}\left(\widetilde{E}, M\right) = \mathrm{Ext}^1_{\mathscr{O}_S}\left(M, \widetilde{E}\right) = 0$ for any surface S. Hence Corollary 2.12 and the above proposition are true for any (smooth) surface.

Let $0 \to F \to E \to G \to 0$ be an exact sequence of nontorsion sheaves on S. Since $\upsilon(E) = \upsilon(F) + \upsilon(G)$ and $r(E) = r(F) + r(G)$, we have

$$\frac{\left(\upsilon(F)^2\right)}{r(F)} + \frac{\left(\upsilon(E)^2\right)}{r(G)} - \frac{\left(\upsilon(E)^2\right)}{r(E)} = \frac{r(F)r(G)}{r(E)}\left(\frac{\upsilon(F)}{r(F)}\frac{\upsilon(G)}{r(G)}\right)^2$$

$$\text{Since } \frac{\upsilon(F)}{r(F)} - \frac{\upsilon(G)}{r(G)} = \left(0, \frac{c_1(F)}{r(F)} - \frac{c_1(G)}{r(G)}, \frac{s(F)}{r(F)} - \frac{s(G)}{r(G)}\right),$$

360 the right hand side of the above equality is equal to $\frac{r(F)r(G)}{r(E)}$
$\left(\frac{c_1(F)}{r(F)} - \frac{c_1(G)}{r(G)}\right)^2$. Hence we have

Proposition 2.16. *Let* $0 \to F \to E \to G \to 0$ *be an exact sequence of nontorsion sheaves. Then we have*

$$\frac{\left(\upsilon(F)^2\right)}{r(F)} + \frac{\left(\upsilon(G)^2\right)}{r(G)} - \frac{\left(\upsilon(E)^2\right)}{r(E)} = \frac{r(F)r(G)}{r(E)}\left(\frac{c_1(F)}{r(F)} - \frac{c_1(G)}{r(G)}\right)^2$$

If $\rho(S) = 1$, then the right hand side is always nonnegative because we are assuming that S is algebraic. Hence we have

Corollary 2.17. *If (S is algebraic and)* $\rho(S) = 1$, *then* $\frac{\left(\upsilon(F)^2\right)}{r(F)} + \frac{\left(\upsilon(G)^2\right)}{r(G)} \geqq \frac{\left(\upsilon(E)^2\right)}{r(E)}$. *Here equality holds if and only if* $\frac{c_1(F)}{r(F)} = \frac{c_1(G)}{r(G)}$.

If F and G have the same slope with respect to an ample line bundle A i.e., $\mu_A(F) = \mu_A(G)$, then we have $\left(A.\frac{c_1(F)}{r(F)} - \frac{c_1(G)}{r(G)}\right) = 0$. Hence, by the Hodge index theorem $\left(\frac{c_1(F)}{r(F)} - \frac{c_1(G)}{r(G)}\right)^2$ is always nonpositive and is equal to zero if and only if $\frac{c_1(F)}{r(F)} = \frac{c_1(G)}{r(G)}$. Hence we have

Corollary 2.18. *Assume that F and G have the same slope with respect to an ample line bundle. Then we have*

$$\frac{\left(\upsilon(F)^2\right)}{r(F)} + \frac{\left(\upsilon(G)^2\right)}{r(G)} \leqq \frac{\left(\upsilon(E)^2\right)}{r(E)}.$$

and equality holds if and only if $\frac{c_1(F)}{r(F)} = \frac{c_1(G)}{r(G)}$.

Let E be a μ-semi-stable sheaf. Then there is a filtration

$$E_* : 0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

such that every successive quotient $F_i = \frac{E_i}{E_{i-1}}$ is μ-stable and has the same slope as E. Such a filtration E_* is called a $\mu - JHS$ filtration of E. Applying the above corollary repeatedly for this filtration, we have the following:

Proposition 2.19. *Let E be a μ-semi-stable sheaf and $F_i (1 \leqq i \leqq n)$ the successive quotients of a μ-JHS filtration of E. Then we have*

$$\sum_{i=1}^{n} \frac{\left(\upsilon(F_i)^2\right)}{r(F_i)} \leqq \frac{\left(\upsilon(E)^2\right)}{r(E)}$$

Equality holds if and only if $\dfrac{c_1(F_i)}{r(F_i)}$ *is equal to* $\dfrac{c_1(E)}{r(E)}$ *for every* $1 \leqq i \leqq n$.

Remark 2.20. If E is a semi-stable sheaf. Then there is a filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

362 such that F_i is stable, has the same slope as E and $\dfrac{s(F_i)}{r(F_i)} = \dfrac{s(E)}{r(E)}$ for every $i = 1, \ldots, n$. Such a filtration is called a *JHS* filtration of E. The above proposition is also true for a semi-stable sheaf E and its *JHS* filtration.

Now we assume that S is a $K3$ surface and prove a result which we shall need in § 5. Let F be a sheaf on S which satisfies

$$\begin{array}{l} \text{the canonical homomorphism } f : H^0(S, F) \otimes \mathcal{O}_S \to F \\ \text{is injective and } H^2(S, F) = 0. \end{array} \tag{2.21}$$

We construct a sheaf E on S from F, which we call the *reflection* of E (from the left), such that $r(E) = -s(F)$, $c_1(E) = c_1(F)$ and $s(E) = -r(F)$. We show that E is simple if and only if F is so. This result is a very special case of the theory of the reflection functor of S, which we will discuss systematically in [14].

Let $\overline{F}$ be the cokernel of the canonical homomorphism

$$f : H^0(S, F) \otimes \mathcal{O}_S \to F.$$

We have the exact sequence

$$0 \to H^0(S, F) \otimes \mathcal{O}_S \xrightarrow{f} F \to \overline{F} \to 0. \tag{2.22}$$

Since $H^1(S,\mathscr{O}_S) = H^2(S,F) = 0$, the above sequence induces the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^1(S,F) \arrow[r, "\alpha"] & H^1\left(S,\overline{F}\right) \arrow[r] & H^0(S,F)\otimes H^2(S,\mathscr{O}_S) \arrow[r] \arrow[d, equal, "\wr"] & 0. \\ & & & H^0(S,F) & \end{tikzcd} \tag{2.23}$$

Construct an exact sequence 363

$$0 \to \overline{F} \to E \to H^1(S,F) \to H^1(S,F)\otimes \mathscr{O}_S \to 0 \tag{2.24}$$

so that the coboundary map $\delta : H^1(S,F)\otimes H^0(S,\mathscr{O}_S) \to H^1(S,\overline{F})$ is equal to α. We call this extension E of $H^1(S,F)\otimes\mathscr{O}_S$ be $\overline{F}$ the reflection of F (from the left). Since $H^2(S,F) = 0$ by our assumption, $\mathcal{X}(F)$ is equal to $h^0(F) - h^1(F)$. Hence we have.

$$\begin{aligned} v(E) &= v(F) + h^1(F)v(\mathscr{O}_S) \\ &= v(F) - h^0(F)v(\mathscr{O}_S) + h^1(F)v(\mathscr{O}_S) \\ &= v(F) - \mathcal{X}(F)v(\mathscr{O}_S). \end{aligned}$$

Since $\mathcal{X}(F) = r(F) + s(F)$ and $v(\mathscr{O}_S) = (1,0,1)$, we have $r(E) = -s(F), c_1(E) = c_1(F)$ and $s(E) = -r(F)$. (By our assumption, $\mathcal{X}(F) \leqq h^0(F) \leqq r(F)$. Hence $s(F)$ is nonpositive.)

Proposition 2.25. *Assume that F satisfies (2.21) and let E be the reflection of F. Then we have* $\mathrm{End}_{\mathscr{O}_S}(E) \cong \mathrm{End}_{\mathscr{O}_S}(F)$.

Proof. We have constructed E canonically from F. It is almost clear that every endomorphism of F induces an endomorphism of E. Let φ be an endomorphism of E. We show that φ is induced by an endomorphism of F. Since $\mathrm{Hom}_{\mathscr{O}_S}(F,\mathscr{O}_S) = 0$ by our assumption and the Serre duality, we have $\mathrm{Hom}_{\mathscr{O}_S}(F,\mathscr{O}_S) = 0$. Hence φ preserves the exact sequence (2.24) and induces an endomorphism $\overline{\psi}$ of $\overline{F}$ and f_1 of $H^1(S,F)$. Since

$\overline{\psi}$ and f_1 are induced by φ, the following diagram

$$\begin{tikzcd}
H^1(S,F)\otimes H^0(S,\mathscr{O}_S) \arrow[r, "\delta=\alpha"] \arrow[d, "f_1"'] & H^1\left(S,\overline{F}\right) \arrow[d, "H^1(\overline{\psi})"] \\
H^1(S,F)\otimes H^0(S,\mathscr{O}_S) \arrow[r, "\delta=\alpha"'] & H^1\left(S,\overline{F}\right)
\end{tikzcd}$$

364 is commutative. Hence f_1 preserves the exact sequence (2.23) and induces an endomorphism f_0 of $H^0(S,F)$. From the long exact sequence $\mathrm{Ext}^*_{\mathscr{O}_S}$ ((2.22), $\mathscr{O}_S$), we obtain the exact sequence

$$0 \to H^0(S,F)^\vee \xrightarrow{\delta'} \mathrm{Ext}^1_{\mathscr{O}_S}\left(\overline{F},\mathscr{O}_S\right) \to \mathrm{Ext}^1_{\mathscr{O}_S}(F,\mathscr{O}_S) \to 0.$$

This sequence is the dual of the exact sequence (2.22) via the Serre duality. Hence we have the following commutative diagram:

$$\begin{tikzcd}
H^0(S,F)^\vee \arrow[r, "\delta"] \arrow[d, "f_0^\vee"] & \mathrm{Ext}^1_{\mathscr{O}_S}\left(\overline{F},\mathscr{O}_S\right) \arrow[d, "\mathrm{Ext}^1_{\mathscr{O}_S}\left(\overline{\psi},\mathscr{O}_S\right)"] \\
H^0(S,F)^\vee \arrow[r, "\delta'"] & \mathrm{Ext}^1_{\mathscr{O}_S}\left(\overline{F},\mathscr{O}_S\right)
\end{tikzcd}$$

Therefore, there exists an endomorphism ψ of F which Preserves the exact sequence (2.22) and induces $\overline{\psi}$ on F and f_0 on $H^0(S,F)$. By our construction, this ψ induces φ.

For our requirements in §4, we show a vanishing of higher direct image sheaf $R^i f_* F$, which was essentially proved in [15].

365 **Proposition 2.26.** *Let $f : X \to Y$ be a proper morphism of noetherian schemes and F a Y-flat coherent $\mathscr{O}_X$ -module. Let Z be a closed subscheme which is locally complete intersection in Y. For $y \in Y$, let F_Y be the restriction of F to the fibre $X_y = f^{-1}(y)$. Assume that $H^i(X_y,F_y)$ vanishes for every $i <$ codim Z and $y \in Y - Z$. Then $R^i f_* F = 0$ for every $i <$ codim Z.*

Proof. We may assume that $Y = \mathrm{Spec} A$ is affine and Z is defined by a regular sequence $x_1, \dots x_n \in A$, $n = \mathrm{codim}\, Z$. By the theorem in § 5 [15], there exists a finite complex $K^{\bullet}$ of finitely generated projective A-modules such that $H^i(K^{\bullet}) \cong R^i f_* F$. By the base change theorem and by our assumption $R^i f_* F$ has a support on Z for every $i < n$. Hence there exists an integer N such that $\mathfrak{a}^N H^i(K^{\bullet}) = 0$ for every $i < N$, where $\mathfrak{a} = (x_1, \dots, x_n)A$. Our proposition follows from the following:

Lemma. *Let $K.$ be a finite complex of finitely generated projective A-module and $\mathfrak{a}$ an ideal of A generated by a regular sequence $x_1 \dots x_n$ of A. If $\mathfrak{a}^N H^i(K^{\bullet}) = 0$ for every $i < n$, then $H^i(K^{\bullet}) = 0$ for every $i < n$.*

This can be proved in the same way as the lemma in ([15, p.127]) by using induction on n. q.e.d

3 Semi-rigid sheaf

In this section, we shall study sheaves E on a $K3$ surface S with small 366
$\mathrm{Ext}^1_{\mathscr{O}_S}(E, E)$.

Definition 3.1. A sheaf E on S is *rigid* if $\mathrm{Ext}^1_{\mathscr{O}_S}(E, E) = 0$. By Proposition 2.5, we have

Proposition 3.2. *If E is simple, then the following are equivalent:*

(1) *E is rigid,*

(2) $\left(\upsilon(E)^2\right) = -2$, *and*

(3) $\left(\upsilon(E)^2\right) < 0$.

By Proposition 2.14, we have

Proposition 3.3. *If E is rigid and torsion free, then E is locally free.*

If E is a rigid sheaf and if $\upsilon(F) = a\upsilon(E)$ for a rational number a, then $\mathcal{X}(E, F)$ is equal to $a\mathcal{X}(E, E)$ and is positive. Hence we have

Proposition 3.4. *Let E be a rigid sheaf and F a sheaf with $v(F) = av(E), a \in \mathbb{Q}$. Then either* $\mathrm{Hom}_{\mathscr{O}_S}(E, F) \neq 0$ *or* $\mathrm{Hom}_{\mathscr{O}_S}(F, E) \neq 0$.

If E is stable and F is semi-stable and if $v(E) = v(F)$, then every nonzero homomorphism between E and F is an isomorphism. Hence we have **367**

Corollary 3.5. *Let E be a stable rigid bundle. If F is semi-stable and $v(F) = v(E)$, then F is isomorphic to E.*

Corollary 3.6. *Let v be a vector of $\widetilde{H}^{1,1}(S, \mathbb{Z})$ with $\left(v^2\right) = -2$. Then the moduli space $M_A(v)$ is empty or a reduced one point.*

Proof. By Corollary 3.5, if $M_A(v)$ is nonempty, then $M_A(v)$ is one point. The tangent space of $M_A(v)$ at the point $[E] \in M_A(v)$ is canonically isomorphic to $\mathrm{Ext}^1_{\mathscr{O}_S}(E, E) = 0$. Hence $M_A(v)$ is reduced. q.e.d

$\dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E)$ is always an even integer (Corollary 2.5). Hence if $\mathrm{Ext}^1_{\mathscr{O}_S}(E, E) \neq 0$, then $\dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E) \geqq 2$.

Definition 3.7. A simple sheaf E on S is semi-rigid if E satisfies the following equivalent conditions:

(1) $\dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E) = 2$, and

(2) $v(E) \in \widetilde{H}^{1,1}(S, \mathbb{Z})$ is isotropic, i.e. $\left(v(E)^2\right) = 0$.

368 Proposition 3.3 is not true for semi-rigid sheaf. In fact, there is a semi-rigid torsion free sheaf which is not locally free. The simplest example is a maximal ideal $\mathfrak{m}$ of $\mathscr{O}_S$. We can construct many such semi-rigid sheaves from a rigid bundle. Let F be a simple rigid vector bundle of rank r. Take a point $s \in S$ and put $V = F \otimes k(s)$ and $\widetilde{F} = F \otimes_k V^\vee \cdot \widetilde{F}$ is a rigid bundle of rank r^2 and $\widetilde{F} \otimes k(s)$ is isomorphic to $\mathrm{End}(V)$. Let E be the kernel of the homomorphism $f : \widetilde{F} \to \widetilde{F} \otimes k(s) \cong \mathrm{End}(V) \xrightarrow{tr} k(s)$, where tr is the trace map of $\mathrm{End}(V)$. Every endomorphism of E is induced by an endomorphism α of $\widetilde{F}$. Since α preserves f, α is a constant multiplication and hence E is simple. It is easy to check that $v(E)$ is isotropic. We call this E the *semi-rigid sheaf associated to F*. We have proved the following:

Proposition 3.8. *Let F be a simple rigid bundle of rank r. Then, for every point $s \in S$, there exists a semi-rigid sheaf E of rank r^2 and an exact sequence*

$$0 \to E \to F^{\oplus r} \to k(s) \to 0.$$

The above examples of semi-rigid torsion free sheaves are locally free except at one point. This is true in general. In fact, by Proposition 2.14, we have

Proposition 3.9. *Let E be a torsion free sheaf with* $\dim \operatorname{Ext}^1_{\mathscr{O}_S}(E, E) = 2$. *Let $\widetilde{E}$ be the double dual of E and assume that E is not locally free. Then* 369
the quotient $\dfrac{\widetilde{E}}{E}$ is isomorphic to $k(s)$ for a point $s \in S$. Moreover, E is a rigid vector bundle and the natural homomorphism α : $\operatorname{End}_{\mathscr{O}_S}\left(\widetilde{E}\right) \to \operatorname{Hom}_{\mathscr{O}_S}\left(\widetilde{E}, k(s)\right)$ *induced by the exact sequence* $0 \to E \to \widetilde{E} \to k(s) \to 0$ *is surjective.*

Corollary 3.10. *Let E be a μ-stable semi-rigid sheaf. If E is not locally free, then* $r(E) = 1$ *and E is isomorphic to $L \otimes \mathfrak{m}$ for a line bundle L and a maximal ideal $\mathfrak{m}$ of $\mathscr{O}_S$.*

Proof. Since E is μ-stable, so is E. Hence E is simple. Since α is surjective and $\dim \operatorname{End}_{\mathscr{O}_S}\left(\widetilde{E}\right) = 1$, we have $\dim \operatorname{Hom}_{\mathscr{O}_S}\left(\widetilde{E}, k(s) \leqq 1\right)$. Therefore, $\widetilde{E}$ is a line bundle. q.e.d

Remark 3.11. If F is a stable rigid bundle, then the semi-rigid sheaf E associated to F is stable. Hence the above corollary is not true for stable semi-rigid sheaves.

If E is semi-rigid and $\upsilon(F) = \upsilon(E)$, then $\mathcal{X}(E, F) = -(\upsilon(E).\upsilon(F)) = 0$. Hence, if $\operatorname{Hom}_{\mathscr{O}_S}(E, F) = \operatorname{Hom}_{\mathscr{O}_S}(F, E) = 0$, then we have $\operatorname{Ext}^1_{\mathscr{O}_S}(E, F) = 0$.

Proposition 3.12. *Let E be a stable semi-rigid sheaf and F a semi-stable sheaf with $\upsilon(F) = (\upsilon(E))$. If E is not isomorphic to F, then* $\operatorname{Ext}^i_{\mathscr{O}_S}(E, F)$ *and* $\operatorname{Ext}^i_{\mathscr{O}_S}(F, E)$ *vanish for every i.*

Proof. By the assumption of (semi-) stablility of E and F, every homo- 370
morphism between E and F is either zero or an isomorphism. Hence, if $E \ncong F$, then $\mathrm{Hom}_{\mathcal{O}_S}(E,F) = \mathrm{Hom}_{\mathcal{O}_S}(F,E) = 0$. Since $\mathcal{X}(E,F) = \mathcal{X}(F,E) = 0$, we have our assertion by Proposition 2.4 q.e.d

If $M_A(v) \neq \phi$, then $M_A(av)$ is empty for every $a \neq 1$, In fact, we have

Proposition 3.13. *Let E be a stable semi-rigid sheaf and F a simple semi-stable with* $v(F) = av(E), \in \mathbb{Q}$. *Then every nonzero homomorphism between E and F is an isomorphism.*

Proof. Let $f : E \to F$ be a nonzero homomorphism. Then f is injective and the cokernel of F is semi-stable by our assumption on (semi-) stability of E and F.

Claim. F is E-potent, i.e., has a filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_n = F$ such that $\dfrac{F_i}{F_{i-1}} \cong E$ for every $i = 1, \ldots, n$.

We define $F_1 = \mathrm{Im}(f)$ and F_i inductively for $i \geqq 2$. Assume that F_i has been defined and $F_i \neq F$. Let G_i be the quotient $\dfrac{F}{F_i}$, Since E is simple, $\mathrm{Hom}_{\mathcal{O}_S}(G_i, E) = 0$. Since $G_i \neq 0$ and F is simple, the exact sequence $0 \to F_i \to F \to G_i \to 0$ does not split. Hence
371 $\mathrm{Ext}^1_{\mathcal{O}_S}(G_i, F_i) \neq 0$. Since F_i is E-potent and $\mathrm{Ext}^1_{\mathcal{O}_S}$ is an additive functor, we have $\mathrm{Ext}^1_{\mathcal{O}_S}(G_i, E) \neq 0$. Since $\mathcal{X}(G_i, E) = -(v(G_i).v(E)) = (i-a)\left(v(E)^2\right) = 0$, we have $\dim \mathrm{Hom}_{\mathcal{O}_S}(E, G_i) = \dim \mathrm{Ext}^1_{\mathcal{O}_S}(G_i, E) - \dim \mathrm{Hom}_{\mathcal{O}_S}(G_i, E) > 0$. Hence there exists a nonzero homomorphism $f_i : E \to G_i$. Let F_{i+1} be the pull-back of $\mathrm{Im}(f_i)$ by $F \twoheadrightarrow G_i$. Since G_i is semi-stable, f_i is injective and $\dfrac{F_{i+1}}{F_i}$ is isomorphic to E. So F_{i+1} is well defined.

If $g : F \to E$ is a nonzero homomorphism, then g is surjective. By the same argument, we have our claim in this case. Since F is simple, F is isomorphic to E by the above and f and g are isomorphisms. q.e.d

Next we investigate the stability of semi-rigid sheaves.

Proposition 3.14. *Let S be an algebraic $K3$ surface with Picard number 1 and E a simple torsion free sheaf on S. Assume that E is rigid or semi-rigid and that $\upsilon(E)$ is primitive in $\widetilde{H}^{1,1}(S,\mathbb{Z})$. Then E is stable.*

Proof. Since $\rho(S) = 1$ and $\upsilon = \upsilon(E)$ is primitive, every semi-stable sheaf E' with $V(E') = \upsilon$ is stable. Hence it suffices to show that E is semi-stable. Assume that E is not so. Let F_1 be the β-subsheaf of E, i.e., F_1 maximizes the polynomial $\dfrac{X(F_1(n))}{r(F_1)}$ among all subsheaves of E and 372
then maximizes $r(F_1)$ among such subsheaves. The quotient $F_2 = \dfrac{E}{F_1}$ is torsion free and $\mathrm{Hom}_{\mathscr{O}_S}(F_1, F_2) = 0$ by our choice of F_1. Hence, by Corollary 2.8, we have $(*)$ $\dim \mathrm{Ext}^1_{\mathscr{O}_S}(F_1, F_1) + \dim \mathrm{Ext}^1_{\mathscr{O}_S}(F_2, F_2) \leqq \dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E)$. Since $\dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E) = \left(\upsilon(E)^2\right) + 2 \leqq 2$, we have $\dim \mathrm{Ext}^1_{\mathscr{O}_S}(F_i, F_i) \leqq 2$ for both $i = 1$ and 2. Hence

$$\left(\upsilon\,(F_i)^2\right) = \dim \mathrm{Ext}^1_{\mathscr{O}_S}(F_i, F_i) - 2 \dim \mathrm{End}_{\mathscr{O}_S}(F_i) \leqq 0$$

for both $i = 1$ and 2. Since $r(F_i) < r(E)$, we have, by Corollary 2.17,

$$\left(\upsilon(F_1)^2\right) + \left(\upsilon(F_2)^2\right) \geqq \left(\upsilon(E)^2\right).$$

Hence we have

$$\begin{aligned} &\dim \mathrm{Ext}^1_{\mathscr{O}_S}(F_1, F_1) + \dim \mathrm{Ext}^1_{\mathscr{O}_S}(F_2, F_2) \\ &\geqq \dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E) + 2 \dim \mathrm{End}_{\mathscr{O}_S}(F_1) \\ &+ 2 \dim \mathrm{End}_{\mathscr{O}_S}(F_2) - 2 \\ &> \dim \mathrm{Ext}^1_{\mathscr{O}_S}(E, E), \end{aligned}$$

Which contradicts $(*)$. q.e.d

Remark 3.15. If F is a rigid bundle of rank $\geqq 2$, then the semi-rigid sheaf E associated to F is not μ-stable. Hence, even if $\rho(S) = 1$, it is not 373
always true that simple semi-rigid torsion free sheaf is μ-stable.

In the following two propositions, we consider the case where $c_1(E)$ is ample and study the stablility of E with respect to $c_1(E)$.

Proposition 3.16. *Let E be a semi-rigid sheaf with $v(E) = (r, \ell, s)$. Assume that ℓ is ample and E is stable with respect to ℓ. If s is divisible by r and $v(E)$ is primitive, then E is μ-stable with respect to ℓ.*

Proof. Assume that E is not μ-stable. Then E has a proper quotient sheaf E_1 with $\mu(E_1) = \mu(E)$. We choose E_1 so that $r(E_1)$ is minimum among such quotients. Put $v(E_1) = (r_1, \ell_1, s_1)$. Since $\mu(E_1) = \mu(E)$, we have $(\ell \cdot \ell_1 - r_1\ell/r) = 0$. Since E is semi-rigid, we have $\ell^2 = 2rs$. Therefore, we have

$$\begin{aligned}\left(v(E_1)^2\right) &= \left(\left(\ell_1 - r_1\frac{\ell}{r}\right) + r_1\frac{\ell}{r}\right)^2 - 2r_1s_1\\ &= \left(\ell_1 - r_1\frac{\ell}{r}\right)^2 + \left(r_1\frac{\ell}{r}\right)^2 - 2r_1s_1\\ &= \left(\ell_1 - r_1\frac{\ell}{r}\right)^2 + 2r_1\left(\frac{r_1s}{r - s_1}\right).\end{aligned}$$

Since $v(E)$ is primitive, r and ℓ are coprime. Hence $\ell_1 - r_1\dfrac{\ell}{r}$ is not
374 zero. Since $\left(\ell_1 - r_1\dfrac{\ell}{r.\ell}\right) = 0$ and ℓ is ample, $\left(\ell_1 - r_1\dfrac{\ell}{r}\right)^2$ is negative by the Hodge index theorem. On the other hand, since E is stable, the integer $\dfrac{r_1s}{r - s_1}$ is negative. Therefore, we have $(v(E_1)^2) < -2r_1 \leqq -2$, which contradicts Corollary 2.5 because E_1 is μ-stable and simple by our choice. q.e.d

Proposition 3.17. *Let $v = (r, \ell, s)$ be a primitive isotropic vector of $\widetilde{H}^{1,1}(S, \mathbb{Z})$ and E a sheaf with $v(E) = v$. Assume that ℓ is ample and E is semi-stable but not stable with respect to ℓ. Let*

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E, n \geqq 2$$

be a JHS-filtration of E. Then the successive quotients $F_i = \dfrac{E_i}{E_{i-1}}$ are rigid for every $i = 1, \ldots, n$.

Proof. By Proposition 2.19 and Remark 2.20, we have $\left(v(F_i)^2\right) \leqq 0$ for every i. Since v is primitive, equality is not attained for any i. Hence F_i is rigid by Proposition 3.2 q.e.d

Corollary 3.18. *Let v be as above. Then the complement of $M_\ell(v)$ in the moduli space $\overline{M}_\ell(v)$ of semi-stable sheaves E with $v(E) = v$ is a* 0*-dimensional set.*

4 Surface components of the moduli space

375 Let $v = (r, \ell, s)$ be an isotropic vector of $\widetilde{H}^{1,1}(S, \mathbb{Z})$ and A an ample line bundle. Then each component of $M_A(v)$ has dimension 2. In this section, we study $M_A(v)$ in the case it is compact and we prove Theorem 1.4 and Theorem 1.5. By Langton's result [6] (see also [9, § 5]), the moduli space of semi-stable sheaves on S is compact. Hence we have

Proposition 4.1. *$M_A(v)$ is compact if and only if every semi-stable sheaf E with $v(E) = v$ is stable. This is the case, e.g., if the greatest common divisor of r, $(\ell.A)$ and s is equal to* 1.

The above is true for every vector v. Using this proposition, we give some further sufficient conditions for $M_A(v)$ to be compact for given primitive isotropic vector v. Let c be the greatest common divisor of r, (ℓ, m) and s, where m runs over all divisor class of S. Then there exists an ample line bundle A such that the greatest common divisor of r, (A, ℓ) and s is equal to c. Hence if $c = 1$, then $M_A(v)$ is compact for such an ample line bundle A. For an application in § 6, we consider the case $c \geqq 2$. We show that $M_A(v)$ is compact for an ample line bundle A in this case, too. Let N_S be the Neron-Severi group of S. N_S is a sublattice of $H^2(S, \mathbb{Z})$. Let N be the sub module generated by N_S and $\dfrac{\ell}{c}$ in $N_S \otimes \mathbb{Q}$. 376 Since $\left(\ell^2\right) = 2rs, \left(\ell^2\right)$ is divisible by $2c^2$. By the definition of c, the bilinear form on $N_S \otimes \mathbb{Q}$ is integral and even on N. Hence N is an even lattice which contains N_S as a sublattice of index c.

Proposition 4.2. *Let A be an ample line bundle on S such that G.C.D. $(r, (A.\ell), s) = c$. If there are no (-2) vectors α in N with $(A.\alpha) = 0$, then $M_A(V)$ is compact.*

Proof. Let E be a semi-stable sheaf with $v(E) = v$ and

$$E_* : 0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

be a JHS-filtration of E We show that $n = 1$. Put $F_i = \dfrac{E_i}{E_{i-1}}$ and $v(F_i) = (r_i, \ell_i, s_i)$ for every $i = 1, \ldots, n$. Since E_* is a JHS-filtration, we have $r_i : (A.\ell_i) : s_i = r : (A.\ell) : s$ for every i. There exists an integer a_i such that $r_i = a_i\dfrac{r}{c}, \dfrac{(A.\ell)}{c}$ and $s_i = \dfrac{a_i s}{c}$. Put $m_i = \ell_i - \dfrac{a_i\ell}{c} \in N$. Then we have $(A.m_i) = 0$ and $\left(v(F_i)^2\right) = \left(m_i + a_i\dfrac{\ell}{c}\right)^2 - r_i s_i = \left(m_i^2\right) + \dfrac{2a_i(m_i.\ell)}{c}$. Since $\sum\limits_{i=1}^{m} m_i = 0$, there exists an i such that $(m_i.\ell) \leqq 0$. For this i, we have $\left(m_i^2\right) \geqq \left(v(F_i)^2\right) \geqq -2$. Since $(A.\mathrm{m}_i) = 0$, $\left(m_i^2\right)$ is non-positive by the Hodge index theorem. Hence by our assumption, we have $\left(m_i^2\right) = 0$ and
377 $m_i = 0$ by the Hodge index theorem. Therefore, we have $v(F_i) = a_i\dfrac{v}{c}$. Since v is primitive, $v(F_i)$ is equal to v. Hence E is stable. q.e.d

As an application of the above, we have the following proposition:

Proposition 4.3. *Assume that there exists a semi-rigid sheaf E with $v(E) = v$ which is μ-stable with respect to an ample line bundle A'. Then there exists an ample line bundle A such that*

(1) E is μ-stable with respect to A, and

(2) $M_A(v)$ is nonempty and compact.

Proof. There exists a neighbourhood U of A' in $\mathbb{P}(N_S \otimes \mathbb{R})$ such that E is μ-stable with respect to A for every ample line bundle $A \in U$. Let $\alpha_1, \ldots, \alpha_n$ be all the (-2) vectors in N which are perpendicular to A'. If A_1 is an ample line bundle in $U - \bigcup\limits_{i=1}^{n} \alpha_i^{\perp}$ and if A_1 is sufficiently near A', then $(A_1.\alpha) \neq 0$ for any (-2) vector α in N. Take such A_1 from $U - \bigcup_{i=1}^{n} \alpha_i^{\perp}$, and take an ample line bundle A_2 such that G.C.D $(r, (A_2.\ell), s) = c$. If n is sufficiently large, then $A = ncA_1 + A_2$ belongs to U and satisfies the last assumption of the preceding proposition. There are infinitely many n's such that G.C.D. $(r, (A.\ell), s) = c$. Hence there exists an integer n **378** such that $M_A(v)$ is compact and nonempty. q.e.d

If $M_A(v)$ is compact, then $M_A(v)$ is irreducible. In fact, we have

Proposition 4.4. *Assume that $M_A(v)$ contains a connected component M which is compact and every member of M is locally free. Then we have*

(1) $M_A(v)$ is irreducible, and

(2) every semi-stable sheaf E with $v(E) = v$ is stable.

Proof. Since $M_A(v)$ is smooth, M is irreducible. We show that every semi-stable sheaf F with $v(F) = v$ belongs to M. Let $\mathscr{E}$ be the restriction to $S \times M$ of a quasi-universal family on $S \times M_A(v)$ (see Appendix 2). We consider the functor $\Phi^i(F) = R^i\pi_{M,*}\left(\mathscr{E}^\vee \otimes \pi_S^* F\right), i = 0, 1$ and 2, of $\mathscr{O}_S$-module F into the category of $\mathscr{O}_M$ -modules. If F is semi-stable, then, for every stable sheaf E with $v(E) = v(F)$, $H^l(S, E^\vee \otimes F) \neq 0$ is equivalent to $F \cong E$. Hence if F is semi-stable and $v(F) = v$, then $\Phi^i(F)$ is supported at most one point. Therefore, by Proposition 2.26, we have $\Phi^0(F) = \Phi^1(F) = 0$. Since $\dim S = 2$, $\Phi^2(F)$ is canonically isomorphic to $H^2(S, E^\vee \otimes F)$ at the point $[E]$ of M, that is, $\Phi^2(F) \otimes k([E]) \cong H^2(S, E^\vee \otimes F)$. Hence $\Phi^2(F)$ is nonzero if and only if F is stable and belongs to M. On the other hand, the cohomology class $\alpha(F) = ch\left(\Phi^0(F)\right) - ch\left(\Phi^1(F)\right) + ch\left(\Phi^2(F)\right) \in H^*(M, \mathbb{Q})$ does not **379** depend on F but depends only on $v(F)$ by the Grothendieck-Riemann-Roch theorem. If F belongs to M, the $\alpha(F)$ is nonzero. Hence $\alpha(F)$ is nonzero for every sheaf F with $v(F) = v$. Therefore every semi-stable sheaf f with $v(F) = v$ is stable and belongs to M, which proves (1) and (2). q.e.d

Remark 4.5. In the above proposition, the assumption that every member of M is locally free is superfluous. The proof works without this assumption, if one defines that functor Φ^i by $\Phi^i(F) = \pi_M - \operatorname{Ext}\left(\mathscr{E}, \pi_S^* F\right)$, where $\pi_M \operatorname{Ext}(*, *)$ is the sheaf associated to the presheaf assigning

$$\operatorname{Ext}_{\mathscr{O}_{S\times U'}}(* \,|_{S\times U'} \; * \,|_{S\times U})$$

for every open subset U of M.

Corollary 4.6. *If every semi-stable sheaf E with $v(E) = v$ is stable, the $M_A(v)$ is compact and irreducible.*

We assume that the moduli space $M = M_A(v)$ is compact. Since the canonical bundle of M is trivial, ([12, Corollary 0.2]), $M_A(v)$ is abelian or of type $K3$. We first consider the case where a universal family exists on $S \times M$.

Lemma 4.7. *For every sheaf $\mathscr{E}$ on $S \times M$, the Chern character $ch(\mathscr{E})$ of E is integral, i.e., belongs to $H^*(S \times M, \mathbb{Z})$.*

380 **Proof.** Put $ch(\mathscr{E}) = \sum_{i=0}^{4} ch^i(\mathscr{E}) \in \bigoplus_{i=0}^{4} H^{2i}(S \times M, \mathbb{Q}).ch^1(\mathscr{E})$ is the first Chern class $c_1(\mathscr{E})$ of $\mathscr{E}$ and is integral. Since $H^1(S) = 0$, $H^2(S \times M)$ is the direct sum of $H^2(S)$ and $H^2(M)$. Hence $c_1(\mathscr{E})$ is equal to $c_1, s(\mathscr{E}) + c_1, M(\mathscr{E}) \in H^2(S, \mathbb{Z}) \oplus H^2(M, \mathbb{Z})$. Since both S and M have trivial canonical bundles, both $c_1, s\,(\mathscr{E})^2$ and $c_{1,M}\,(\mathscr{E})^2$ are even. Hence $ch^2(\mathscr{E}) = \frac{1}{2} c_1\,(\mathscr{E})^2 - c_2(\mathscr{E})$ is integral. By the Grothendieck-Riemann-Roch theorem, the $H^*(S) \otimes H^4(M)$ -component of $ch(\mathscr{E}) \cdot td_M$ is equal to $\left(\sum_j (-1)^j ch(R^j \pi_S, *\mathscr{E}\right) \otimes w$, where $w \in H^4(M)$ is the fundamental cocycle of M. Hence $ch^2(\mathscr{E})$ and the $H^2(S) \otimes H^4(M)$-component of $ch^3(\mathscr{E})$ are integral. Interchanging S and M, we have that the $H^4(S) \otimes H^2(M)$ component of $ch^3(\mathscr{E})$ is also integral. Since $H^6(S \times M)$ is the direct sum of $H^2(S) \oplus H^4(M)$, $ch^3(\mathscr{E})$ is integral. q.e.d

Let $\mathscr{E}$ be a universal family on $S \times M$. Put $\mathbb{Z} = \pi_S * \sqrt{td_S} ch(\mathscr{E})^* \cdot \pi_M * \sqrt{td}_M$. By the lemma, $\mathbb{Z}$ belongs to $H^*(S \times M, \mathbb{Z}).Z$ defines a homomorphism

$$\begin{tikzcd} f : H^*(S,\mathbb{Z}) \arrow[r] & H^*(M,\mathbb{Z}). \\ \alpha \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \pi_M,*(\mathbb{Z}.\pi_S * \alpha) \arrow[u, phantom, sloped, "\in"] \end{tikzcd} \tag{4.8}$$

Theorem 4.9. *Under the above situation, we have*

1. *M is K3 surface,*
2. *f is an isometry form $\widetilde{H}(S,\mathbb{Z})$ onto $\widetilde{H}(M,\mathbb{Z})$ with respect to the quadratic forms defined in* (1.1)*, and* 381
3. *the inverse of f is equal to the homomorphism*

$$\begin{tikzcd} f' : H^*(M,\mathbb{Z}) \arrow[r] & H^*(S,\mathbb{Z}) \\ \beta \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \pi_S,*\left(Z' \cdot \pi_M^*\beta\right) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

defined by $Z' = \pi_S^* \sqrt{td_S} \cdot ch(\mathscr{E}) \cdot \pi_M^* \sqrt{td}_M$.

For the proof, the following is essential.

Proposition 4.10. *Let $\mathscr{E}$ be a universal family on $S \times M$. Let π_{12} and π_{13} be the two projections of $S \times M \times M$ onto $S \times M$. Then $\pi_{M\times M} - \mathrm{Ext}^i \left(\pi_{1,2}^*\mathscr{E}, \pi_{13}^*\mathscr{E}\right)$ is zero if $i \neq 2$ and $\pi_{M\times M} - \mathrm{Ext}^2 \left(\pi_{12}^*\mathscr{E}, \pi_{13}^*\mathscr{E}\right)$ is supported on the diagonal sub scheme Δ of $M\times M$ and is a line bundle on Δ.*

Proof. If $E, F \in M_A(v)$ and $E \not\cong F$, then $\mathrm{Ext}^i_{\mathcal{O}_S}(E, F) = 0$ for every i by Proposition 3.8. Hence the relative Ext-sheaf $\pi_{M\times M} - \mathrm{Ext}^i\left(\pi_{12}^*\mathscr{E}, \pi_{13}^*\mathscr{E}\right)$ has a support on Δ. Since Δ is locally complete intersection, the relative Ext-sheaf is zero for both $i = 1$ and 2, by Proposition 2.26. By the base change theorem, $\pi_{M\times M} - \mathrm{Ext}^2\left(\pi_{12}^*\mathscr{E}, \pi_{13}^*\mathscr{E}\right)$ is canonically isomorphic to the 1-dimensional vector space $\mathrm{Ext}^2_{\mathcal{O}_S}(E, E) \cong \mathrm{End}_{\mathcal{O}_S}(E)^\vee$ at the point $([E], [E]) \in \Delta$. Since M is a moduli space and $\mathscr{E}$ is a universal family, the sheaf $\pi_{M\times M} - \mathrm{Ext}^2\left(\pi_{12}^*\mathscr{E}, \pi_{13}^*\mathscr{E}\right)$ is annihilated by the ideal $\mathscr{I}_\Delta$ of Δ.
382 Therefore, $\pi_{M\times M} - \mathrm{Ext}^2\left(\pi_{12}^*E, \pi_{13}^*E\right)$ is a line bundle on Δ. q.e.d

Proof of theorem 4.9. The following is the key to our proof.

Claim. The endomorphism $f \circ f'$ of $H^*(M, \mathbb{Z})$ is the identity.

The homomorphisms f and f' are given by cycles Z and Z' on $S \times M$. Using the projection formula, it can be easily shown that $f \circ f'$ is given by the cycle $\widetilde{Z} = \pi_{M\times M}, *\left(\pi_{12}^*Z \cdot \pi_{13}^*Z'\right)$, where π_{12} and π_{13} are same as in the above proposition. Precisely speaking, $(f \circ f')(\beta) = \pi_{1,*}\left(\widetilde{Z} \cdot \pi_2^*\beta\right)$ for every $\beta \in H^*(M, \mathbb{Z})$, where π_1 and π_2 are two projections of $M \times M$ onto M. By the definition of Z and Z', we have $\widetilde{Z} = \left(\pi_1^*\sqrt{td_M}\right).\left(\pi_2^*\sqrt{td}_M\right)\pi_{M\times M}, *(U)$, where $U = \dfrac{\left(\pi_{12}^*ch(\mathscr{E})^*\right)}{\pi_S^*td_S.\left(\pi_{13}^*ch\left(\mathscr{E}\right)\right)}$. By the Grothendieck-Riemann-Roch theorem, the cycle $\pi_{M\times M}*(U)$ is rationally equivalent to $\sum_i(-1)^i ch\left(\pi_{M\times M}\mathrm{Ext}^i\left(\pi_{12}^*\mathscr{E}, \pi_{13} * \mathscr{E}\right)\right)$. By the above proposition, $\widetilde{Z}$ is rationally equivalent to $\pi_1^*\sqrt{td}_M.ch(\delta_*L).\pi_2 * \sqrt{td}_M$, where L is a line bundle M and $\delta : M \to M \times M$ is the diagonal embedding. Therefore, $f \circ f'$ is the multiplication by $ch(L) \in H^*(M)$, i.e., $(f \circ f')(\beta) = \beta \cdot ch(L)$ for every $\beta \in H^*(M, \mathbb{Z})$. Let ρ be the factor change of $M \times M$. Then $(1 \times \rho)^*U$ is equal to U^*. Hence, we have $\rho^*(\pi_{M\times M}, *U) = (\pi_{M\times M}, *U)^*$. On the other hand, since $\pi_{M\times M}, *U$ has a support on Δ, we have $\rho^*(\pi_{M\times M}, *U) = \pi_{M\times M} * U$. Hence we have $ch(\delta_*L)^* = ch(\delta_*L)$. Since S is a $K3$ surface, the line bundle L is trivial. Therefore, $f \circ f'$ is the identity.

383 By the claim, $H^*(M, \mathbb{Z})$ is a direct summand of $H^*(S, \mathbb{Z})$. Since Z

and Z' belong to $H^{ev}(S \times M, \mathbb{Z})$, f and f' preserve the decompositions $H^* = H^{ev} \oplus H^{odd}$ of the cohomology groups $H^*(M, \mathbb{Z})$ and $H^*(S, \mathbb{Z})$. Hence $H^{odd}(M, \mathbb{Z})$ is a direct summand of $H^{odd}(S, \mathbb{Z})$ which is zero, since S is a $K3$ surface. Since M has a trivial canonical bundle, we have, (1). By (1), $H^*(M, \mathbb{Z})$ and $H^*(S, \mathbb{Z})$ have the same rank (= 24). Therefore, f is an isomorphism, which shows (3). Let $\gamma = \gamma_S : S \to \mathrm{Spec}\mathbb{C}$ be the structure morphism of S. Then our inner product $(\alpha.\alpha')$ on $\widetilde{H}(S, \mathbb{Z}) = \widetilde{H}^*(S, \mathbb{Z})$ is equal to $\gamma_*(\alpha^*.\alpha')$ Hence, by the projection formula, we have

$$\begin{aligned}(\alpha.f'(\beta)) &= \gamma_{S,*}\left(\alpha^* \cdot \pi_{S,*}\left(\pi_S^* \sqrt{tdS} \cdot ch(\mathscr{E}) \cdot \pi_M^* \sqrt{td_M} \cdot \pi_M^* \beta\right)\right) \\ &= \gamma_S, *\pi_S, *\left(\pi_S^* \alpha^* \cdot \pi_S^* \sqrt{tdS} \cdot ch(\mathscr{E}) \cdot \pi_M^* \sqrt{td}_m \cdot \pi_M^* \beta\right) \\ &= \gamma_{S \times M,*}\left(\pi_S^* \alpha^* \cdot \pi_M^* \beta \cdot ch(\mathscr{E}). \sqrt{td_{s \times m}}\right).\end{aligned}$$

for every $\alpha \in H^*(S, Z)$ and $\beta \in H^*(M, Z)$. In a similar way, we have

$$(\beta \cdot f(\alpha)) = \gamma_{S \times M,*}\left(\pi_M^* \beta^* \cdot \pi_S^* \alpha \cdot ch(\xi)^* \cdot \sqrt{td_{S \times M}}\right).$$

Therefore $(\alpha \cdot f'(\beta)) = (f(\alpha).\beta)$ for every $\alpha \in H^*(S \cdot \mathbb{Z})$ and $H^*(M, \mathbb{Z})$, that is, f and f' are adjoint to each other with respect to the inner products $(\cdot)$ on $H^*(S, \mathbb{Z})$ and $H^*(M, \mathbb{Z})$. By (3), $f' \circ f$ is the identity. Hence we have $(f(\alpha) \cdot f(\alpha')) = (\alpha \cdot f'(f(\alpha))) = (\alpha \cdot \alpha')$ for every $\alpha, \alpha' \in H^*(S, \mathbb{Z})$, which proves (2). q.e.d

Now we assume only that $M = M_A(v)$ is compact and that $\mathscr{E}$ is a **384**
quasi-universal family on $S \times M$ and prove Theorem 1.4 and 1.5. Let $\sigma(\mathscr{E})$ be the similitude of $\mathscr{E}$ and put $Z = \pi_S^* \sqrt{td}_S ch(\mathscr{E}).\pi_M^* \dfrac{\sqrt{td_M}}{\sigma(\mathscr{E})} \in H^{ev}(S \times M, \mathbb{Q}).Z$ induces a homomorphism

$$\begin{tikzcd} f : H^*(S, \mathbb{Q}) \arrow[r] & H^{ev}(M, \mathbb{Q}) \\ \Cup & \Cup \\ \alpha \arrow[r, mapsto] & \pi_M, *(Z.\pi_S^* \alpha) \end{tikzcd}$$

The $H^0(M,\mathbb{Q})$ -component of $f(\alpha)$ is equal to $(\upsilon.\alpha)$. Hence the orthogonal complement $\upsilon^{\perp}$ of υ in $H^*(S,\mathbb{Q})$ is sent into $H^2(M,\mathbb{Q})\oplus H^4(M,\mathbb{Q})$ by f.

Lemma 4.11. *$f(v)$ is equal to the fundamental cocycle $\omega \in H^4(M,\mathbb{Z})$.*

Proof. Let F be a member of $M = M_A(\upsilon)$ and let $\Phi^2(F)$ be same as in the proof of Proposition 4.4 and Remark 4.5. By the Grothendieck-Riemann-Roch theorem, we have $ch(\Phi^2(F)) = \pi_M, *(ch(\mathscr{E})^* \cdot \pi_S^*(ch(F)\cdot td_S)) = \sigma(\mathscr{E})\sqrt{td_M}^{-1}\cdot f(ch(F)\cdot\sqrt{td_S}) = \sigma(\mathscr{E})\sqrt{td_M}^{-1} f(\upsilon)$. Now $\Phi^2(F)$ has a support at the point $x \in M$ corresponding to F and $\Phi^2(F)\otimes k(x)$ is canonically isomorphic to $\mathrm{Ext}^2_{\mathscr{O}_S}(\mathscr{E}_{S\times x}, F)$. Since $\mathscr{E}$ is a quasi-universal family, $\mathscr{E}\,|_{S\times x}$ is isomorphic to $F^{\oplus\sigma(\mathscr{E})}$. Hence $\Phi^2(F)\otimes k(x)$ is a $\sigma(\mathscr{E})$ dimensional vector space. On the other hand, since M is the moduli space and $\mathscr{E}$ is a quasi-universal family, $\Phi^2(F)$ is annihilated by
385 the maximal ideal at x. Hence $\Phi^2(F)$ is isomorphic to $k(x)^{\oplus}\sigma(\mathscr{E})$ and $ch(\Phi^2(F)) = \sigma(\mathscr{E})\omega$, which proves our lemma. q.e.d

By this lemma, we see that f induces a homomorphism

$$\varphi_{\mathbb{Q}} : \frac{(\upsilon^{\perp} \text{ in } H^*(S,\mathbb{Q}))}{\mathbb{Q}\upsilon} \to H^2(M,\mathbb{Q}).$$

Proof of Theorem 1.4 and Theorem 1.5 : If $\mathscr{E}$ is a quasi-universal family on $S\times M$. then so is $\mathscr{E}\otimes\pi_M^*V$ for every vector bundle V on M. We first show that the two homomorphisms $\varphi_{\mathbb{Q}}$ and $\varphi_{\mathbb{Q},V}$ for $\mathscr{E}$ and $\mathscr{E}\otimes\pi_M^*V$ are same. The similitude $\sigma(\mathscr{E}\,|_{\otimes}\,\pi_M^*V)$ is equal to $\sigma(\mathscr{E})r(V)$. Hence $\dfrac{ch(\mathscr{E}\otimes\pi_M^*V)}{\sigma(\mathscr{E}\otimes\pi_M^*V)}$ is equal to $\dfrac{ch(\mathscr{E})}{\sigma(\mathscr{E})}.\pi_M^*\left(\dfrac{Ch(V)}{r(V)}\right)$. Therefore, we have $f_{\mathbb{Q},v}(\alpha) = f_{\mathbb{Q}}(\alpha)\left(\dfrac{ch(v)}{r(V)}\right)$ for every $\alpha \in H^*(S,\mathbb{Q})$. If $(\upsilon.\alpha) = 0$, then $H^0(M)$-component of $f_{\mathbb{Q}}(\alpha)$ is zero. Hence the $H^2(M)$ component of $f_{\mathbb{Q},V}(\alpha)$ is same as that of $f_{\mathbb{Q}}(\alpha)$. Therefore, $\varphi_{\mathbb{Q},V}$ and $\varphi_{\mathbb{Q}}$ are same. If $\mathscr{E}$ and $\mathscr{F}$ are quasi-universal families on $S\times M$, then there exist vector bundles U and V on M such that $\mathscr{E}\otimes\pi_M^*U = \mathscr{F}\otimes\pi_M^*V$ (Definition A.4). Hence, by what we have shown, the two homomorphisms $\varphi_{\mathbb{Q}}s$ for $\mathscr{E}$ and $\mathscr{F}$ are same, which shows (1) of Theorem 1.5

We prove (2) and (3) of Theorem 1.5 by a deformation argument. Both are reduced to the case where a universal family exists on $S \times M$. Let T be the moduli space of $K3$ surface S' with isometric markings $i' : H^2(S',\mathbb{Z}) \to H^2(S',\mathbb{Z})$. Let T_0 be the subspace of T consisting of (S',i')'s for which $i'(c_1(A))$ and $\ell = i'(\ell)$ lie in $H^{1,1}(S')$ and $i'(c_1(A))$ is positive. T_0 contains (S, id) and has dimension 18 or 19 according as **386** $c_1(A)$ and ℓ are linearly independent or not. Let A' be an ample divisor on S' such that $c_1(A') = i'(c_1(A))$ and put $v' = (r, \ell', s)$. The family of moduli spaces $M_{A'}(v')$ is smooth over an etale covering of T_0 ([12] Theorem 1.17). There exists a family of quasi-universal families $\mathscr{F}_t$ on $S_t \times M_{A_t}(v_t), t \in T_0$, which is flat over an etale covering of T_0. By Proposition 4.1, the compactness of $M_{A'}(V')$ is an open condition: There exists an open neighbourhood U of (S, id) such that $M_{A'}(v')$ is compact for every $(S', i') \in U$. On the other hand the set of $(S', i^{'})$ which satisfy

$(*)$ there exists a divisor class $m \in H^{1,1}(S',\mathbb{Z})$ such that G.C.D. $(r, (\ell.m), s) = 1$.

is dense in T_0. By Theorem A.6 and Remark A.7, for such S', there exists a universal family on $S \times M_{A'}(v')$. Hence there exists a pair (S', i') for which $M' = M_{A'}(v')$ is compact and a universal family $\mathscr{E}'$ exists on $S \times M'$. By Theorem 4.9, M' is a $K3$ surface and (2) and (3) of Theorem 1.5 are true for this S' and $\mathscr{E}$. Hence M is a $K3$ surface and (2) and (3) of Theorem 1.5 are true for this S' and for every quasi-universal family $\mathscr{F}'$ on $S \times M'$. Since (S, id) and $\mathscr{F}$ is a flat deformation of (S', i') and $\mathscr{F}'$, (2) and (3) are also true for S. The second half of Theorem 1.4 follows from (2) and (3) of Theorem 1.5 q.e.d.

5 Existence of simple μ-semi-stable semi-rigid sheaves

In this section, we show the existence of simple μ-semi-stables sheaves E with $v(E) = v$ for primitive isotropic vectors v of $\widetilde{H}^{1,1}(S,\mathbb{Z})$.

Theorem 5.1. *Let $v = (r, \ell, s)$ be a primitive isotropic vector of* **387** *$\widetilde{H}^{1,1}(S,\mathbb{Z})$ of rank $r \geqq 1$ and A an arbitrary ample divisor. Then there* *exists a simple μ-semi-stable sheaf E with $v(E) = v$, i.e. $SM_A(v)$ is nonempty.*

By virtue of Theorem A. 1, this theorem is equivalent to the following stronger version:

Theorem 5.2. *Let m be a divisor class of S. Then the simple μ-semi-stable sheaf E can be chosen so that E satisfies the following condition:* $(*)$ $\dfrac{(c_1(F) \cdot m)}{r(F)} \geqq \dfrac{(c_1(E) \cdot m)}{r(E)}$ *holds for every non-torsion quotient sheaf F of E with $\mu(F) = \mu(E)$.*

In fact, if $n \gg 0$, then $nA + m$ is ample. By Theorem 5.1, there exists a simple sheaf E_n with $v(E_n) = v$ and which is μ-semi-stable with respect to $A + \dfrac{1}{n}m$. By Theorem A.1, there exists a simple sheaf E which is μ-semi-stable with respect to infinitely many $A + \dfrac{1}{n}m$. It is easy to see that this E satisfies $(*)$ in Theorem 5.2 We prove these theorems by induction on r. In the case $r = 1$, $E = \mathcal{O}_S(\ell) \otimes \mathfrak{m}$ satisfies our requirement for a maximal ideal $\mathfrak{m}$ of $\mathcal{O}_S$. In fact, $v(E) = n$ and v is μ-stable with respect to any ample line bundle. Assume that Theorem 5.2 is true in the case of rank $< r$. Under this assumption, we shall show that Theorem 5.1 is true fore every v of rank r.

Step I. Assume that $-r < s < 0$ and $(\ell.A) = 0$. Then there exists a simple μ-semi-stable sheaf E with $v(E) = v$.

388

Proof. By the induction hypothesis, there exists a simple μ-semi-stable sheaf F with $v(F) = (-s, \ell, -r)$. Since $\mu(F) = 0$, the canonical homomorphism $f : H^0(S, F) \otimes \mathcal{O}_S \to F$ is injective and for every nonzero homomorphism $g : F \to \mathcal{O}_S$, the cokernel of g is of finite length. Here we apply Theorem 5.2, putting $m = -\ell$. Then we can take F so that

$$\frac{-(c_1(G) \cdot \ell)}{r(G)} \geqq \frac{-(\ell^2)}{r(F)}$$

holds for every nontorsion quotient G of F with $\mu(G) = \mu(F)$. Since $(\ell^2) = 2rs < 0, (c_1(G) \cdot \ell)$ is negative. Hence, for this F, we have

$\mathrm{Hom}_{\mathscr{O}_S}(F, \mathscr{O}_S) = 0$. Therefore, by the Serre duality, $H^2(S, F) = 0$ and F satisfies (2.21). Let E be the reflection of F (see § 2). Then $v(E) = v$ and there is an exact sequence

$$0 \to H^0(S, F) \otimes \mathscr{O}_S \xrightarrow{f} F \to E \to H^1(S, F) \otimes \mathscr{O}_S \to 0.$$

Since F is μ-semi-stable and $\mu(F) = \mu(\mathscr{O}_S)$, the cokernel of f is torsion free and μ-semi-stable. Hence E is torsion free and μ-semi-stable. By Proposition 2.25, E is simple. q.e.d

We do not use the full strength of the above step but only the existence of simple torsion free sheaves on monogonal $K3$ surfaces. A quasi-polarized $K3$ surface (S, A) is called *monogonal* if there exists a smooth elliptic curve C on S with $(A.C) = 1$. put $g = \frac{1}{2}\left(A^2\right) + 1$. Then $(A - gC)^2 = -2$ and $(C.A. - gC) = 1$. Hence there exists an effective divisor D such that $D \sim A - gC$.

If $\rho(S) = 2$, then Pic S is generated by C and D and D is a smooth rational curve. S is a double cover of the $\mathbb{P}^1$-bundle $\mathbb{F}_2 = \mathbb{P}(\mathscr{O} \oplus \mathscr{O}(2))$ over $\mathbb{P}^1$. A divisor $aC + b(C + D)$ on S is ample if and only if $a > b > 0$. 389

Step.II. *Assume that S is monogonal and $\rho(S) = 2$. Then there exists a simple torsion free sheaf E on S with $v(E) = v$.*

Proof. ℓ is equal to $aC + b(C + D)$ for some integers a and b. Take an integer b' so that $b' \equiv b \mod r$ and $|b'| \leqq r/2$. Then take an integer a' congruent to a modulo r so that $r/2 < |a'| \leqq 3r/2$ and $a'b' < 0$ if $b' \neq 0$ and so that $-r < a' \leqq 0$ if $b' = 0$. Put $\ell' = a'C + b'(C + D) \cdot \ell'$ is congruent to ℓ modulo r and $s' = \left(\ell'^2\right)/2r$ is an integer. We show the existence of a simple torsion free sheaf E' on S with $v(E') = (r, \ell', s')$. Then $E = E' \otimes \mathscr{O}_S\left(\frac{(\ell - \ell')}{r}\right)$ is a simple torsion free sheaf and satisfies $v(E) = v$. If $b' \neq 0$, then $\frac{-3r^2}{4} \leqq a'b' < 0$ by our choice of a' and b'. Since $\left(\ell'^2\right) = 2a'b'$, we have $\frac{-3r}{4} \leqq s' < 0$. Put $H = a'C - b'(C + D)$. If $b' \neq 0$, then H or $-H$ is ample. Since $(H \cdot \ell') = 0$, there exists a simple torsion free sheaf E' with $v(E') = (r, \ell', s')$ by Step I. If $b' = 0$, then $s' = 0$. Since v' is primitive, r and a' are coprime. Hence there exists a simple vector bundle ξ on the elliptic curve C of rank $-a'$ and degree r by [1] (see also § 2 [18]). ξ is generated by global sections and $H^1(C,\xi) = 0$ (see Lemma 5.3 below). We regard ξ as a sheaf on S supported by C. Let E' be the kernel of the natural homomorphism $\varphi : H^0(S,\xi) \otimes \mathscr{O}_S \to \xi$. Then φ is surjective and E' is a vector bundle. Since $\dim H^0(S,\xi) = \dim H^0(C.\xi) = r$. the rank of E' is equal to r. 390 Since ξ is a simple sheaf and since $H^1(S,\xi) = 0, E'$ is simple. (Every endomorphism of E comes from that of ξ.) q.e.d

Lemma 5.3. *Let E be an indecomposable vector bundle of rank r and degree d on an elliptic curve C. If $d > r$. then E is generated by global sections and $H^1(C,E) = 0$.*

Proof. Let h be the greatest common divisor of r and d. Then E has a filtration

$$0 = E_o \subset E_1 \subset \ldots \subset E_b = E$$

such that $\dfrac{E_i}{E_{i-1}}$ is indecomposable and has rank $\dfrac{r}{h}$ and degree $\dfrac{d}{h}$ for every $i = 1, 2, \ldots, h$. Hence we may assume that r and d are coprime. Then, by Lemma 2.2 [1], E is simple. Let $\dfrac{d'}{r'}$ be the greatest irreducible fraction with $\dfrac{d'}{r'} < \dfrac{d}{r}$ and $0 < r' < r$. There exists a simple vector bundle E' on C with rank r' and degree d'. Since $r'd - rd' = 1$, we have $\mathcal{X}(E',E) = 1$ by the Riemann-Roch theorem. Applying Part II [1] for $E'^{\vee} \otimes E$, we have $\mathrm{Ext}^1_{\mathscr{O}_C}(E',E) = 0$ and $\dim \mathrm{Hom}_{\mathscr{O}_C}(E'E) = 1$.

Since E' and E are stable, the canonical homomorphism $\varphi : E' \otimes \mathrm{Hom}_{\mathscr{O}_C}(E',E) \to E$ is injective and the cokernel E'' has no torsion. Since $\mathrm{Hom}_{\mathscr{O}_C}(E'',E') = 0$, we have $\mathrm{Ext}^1_{\mathscr{O}_C}(E',E'') = 0$ by the Serre duality. Hence every endomorphism of E'' is induced by that of E. Therefore, E'' is simple. So we have obtained an exact sequence of simple vector bundles

$$0 \to E' \to E \to E'' \to 0$$

391 Case for which $\dfrac{d'}{r'} > 1$: By the induction hypothesis, our assertion is

true for E' and E''. Hence so is for E.

Case for which $\dfrac{d'}{r'} = 1$: By our choice of $\dfrac{d'}{r'}$, we have $r' = d' = 1$ and $d = r+1$. E' is a line bundle of degree 1 and isomorphic to $\mathscr{O}_C(p)$ for a point p on C. By the induction hypothesis, E'' is generated by global sections. Hence E is generated by global sections except at p. Let L be the kernel of the canonical homomorphism $\psi : H^0(C, E) \otimes \mathscr{O}_C \to E$. Since ψ is generically surjective and since $h^0(C, E) = r(E) + 1$, L is a line bundle. If $\dim \operatorname{Hom}_{\mathscr{O}_C}(L, \mathscr{O}_C) < h^0(C, E)$, then E would be decomposable. Hence we have $h^0(C, L^{-1}) \geqq d = r+1$. By the Riemann-Roch theorem we have $\deg L \leqq -d$ and deg (Image ψ)$\geqq d$. Hence ψ is surjective. q.e.d

Next we study the case where A is primitive and ℓ is a multiple of A, say $\ell = kA$ for an integer k. In this case, the moduli space $M_{S,A}(v)$ is defined for every polarized $K3$ surface (S, A) of a fixed degree, say $d = (A^2)$. Let F_d (resp. $\overline{F}_d$) be the moduli space of polarized (resp. quasi-polarized) $K3$ surfaces (S, A) of degree d. By the Torelli theorem ([7], [20]), F_d and F_d are irreducible.

Step III. There is a nonempty open subset U of F_d such that $M_{S,A}$ is nonempty for every polarized $K3$ surface $(S, A) \in U$.

Proof. If (S, A) is monogonal and $\rho(S) = 2$, then there exists simple 392
torsion free sheaf E with $v(E) = v$. Since $\{Spl_S(v)\}_{(S,A)\in F_d}$ is a smooth family over an etale covering of F_d ([12, Theorem1.17]), there exists a simple torsion free sheaf E' on S' with $v(E') = (r, kA', s)$ for every small deformation (S', A') of (S, A). The polarized $K3$ surfaces (S', A') with $\rho(S') = 1$ form a dense subset in F_d. Hence there exists a polarized $K3$ surface (S', A') with $\rho(S') = 1$ and a simple torsion free sheaf E' on S' with $v(E') = (r, kA', s)$. Since $(\nu(E')^2) = 0$ and $\rho(S') = 1$, E' is stable, by virtue of Proposition 3.14 Since $\{M_{S,A}(v)\}_{(S,A)\in F_d}$ is a smooth family over an etale covering of $F_{d'}$ there exists an open neighbourhood U of (S', A') which satisfies our requirement. q.e.d

Step IV. *If ℓ is a multiple of A, then there exists a sheaf E with $v(E) = v$ and which is stable with respect to A, i.e., $M_{S,A}(v)$ is nonempty for every (S, A).*

Proof. By Langton's theorem ([6] see also [9] § 5), the family

$$\{\overline{M}_{S,A}(v)\}_{(S,A)\in F_d}$$

of the moduli spaces of semi-stable sheaves is proper over F_d. By Step III, $\overline{M}_{S,A}(v)$ is nonempty over a dense open subset of F_d. Therefore $\overline{M}_{S.A}(v)$ is nonempty for every $(S,A) \in F_d$. Let $\pi : \mathscr{S} \to F$ be a family of polarized $K3$ surfaces. Then, by Maruyama [9]§ 4, the (coarse) moduli space $\prod : \overline{M}_{\mathscr{S}/F} \longrightarrow F$ to semi-stable sheaves on $\dfrac{\mathscr{S}}{F}$ exists and each fibre of $\prod$ is canonically isomorphic to the moduli space of semi-stable sheaves on the corresponding fibre of π. In particular, the function $F_d \ni (S,A) \mapsto \dim \overline{M}_{S,A}(v$ is upper semi-continuous. Since
393 $\dim \overline{M}_{S,A}(v) \geqq \dim \overline{M}_{S,A}(v) = 2$ for every member (S,A) of U in Step II, we have $\dim \overline{M}_{S,A}(v) \geqq 2$ for every polarized $K3$ surface (S,A). By Proposition 3.14, the complement of $M_{S,A}(v)$ in $\overline{M}_{S,A}(v)$ discrete. Hence $M_{S,A}(v)$ is nonempty for every $(S,A) \in F_d$. q.e.d

Now we return to the general case.

Step V. There exists a simple sheaf E with $v(E) = v$ and which is μ-semi-stable with respect to A.

Proof. If a sheaf E is stable with respect to A, then $E \otimes L$ is simple and μ-stable with respect to A for every line bundle L. Hence, by Step IV, our assertion is true of $\ell \equiv kA \mod r$ for an integer k. In particular, $SM^{\mu}_{rnA+\ell}(r,\ell,s)$ is nonempty for every $n \gg 0$. Since the sequence $\{A + \ell/rn\}$ $\mathbb{Q}$-divisors converges to A, we have, by Theorem A. 1, $SM^{\mu}_{A}(r,\ell,s)$ is nonempty. q.e.d

We have completed the proof of Theorem 5.1 and Theorem 5.2 By Step IV, we have also proved the following.

Theorem 5.4. *Let $v = (R,\ell,s)$ be a primitive isotropic vector of $\widetilde{H}^{1,1}(S,\mathbb{Z})$ and assume that ℓ is ample. Then there exists a sheaf E with $v(E) = v$ and stable with respect to ℓ, i.e., $M_{\ell}(r,\ell,s) \neq \phi$.*

6 Application to the Hodge conjecture

394 In this section, we apply the results in § 4 and 5 to show that certain Hodge cycles Z on a product $S \times S'$ of two algebraic $K3$ surfaces S and S' are algebraic (Theorem 1.9). We first consider the special case for which $T_S \cong \varphi(T_S)$, where $\varphi = f_Z^{\tau}$ as in Theorem 1.9.

Step I. *Let* $\varphi : T_S \overset{\sim}{\longrightarrow} T_{S'}$ *be a Hodge isometry between the transcendental lattices of* S *and* S'. *Then there exists an algebraic cycle* $\omega \in H^4(S \times S', \mathbb{Q})$ *on* $S \times S'$ *such that* $\varphi(\alpha) = \pi'_{S'}, * \, (W \cdot \pi_S^* \alpha)$ *for every* $\alpha \in T_S$.

We remark that there exists an isomorphism $f : S' \to S$ such that $f^* = \varphi$ on T_S if $\rho(S) > 11$ (Proposition 6.2). But this is not true in general if $\rho(S) \leqq 11$. In fact, there is a pair of $K3$ surfaces S and S' such that $T_S \cong T_{S'}$ but $N_S \ncong N_{S'}$ as lattices. We note that two lattices $\widetilde{H}^{1,1}(S, \mathbb{Z})$ and $\widetilde{H}^{1,1}(S', \mathbb{Z})$ are isomorphic to each other, which is the key of our proof of Step I. More strongly, by Theorem 1.14.2 and 1.14.4 in [17], we have

Proposition 6.1. *Let* $\varphi_1, \varphi_2 : T \to H$ *be two primitive embeddings of a lattice* T *into an even unimodular lattice.* H. *Assume that the orthogonal complement* N *of* $\varphi_1(T)$ *in* H *satisfies one of the following:*

(1) N *contains the hyperbolic lattice* $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *as a sublattice or*

(2) N *is indefinite and rank* $N \geqq$ *rank* $T + 2$.

Then φ_1 *and* φ_2 *are equivalent, i.e., there exists an isometry* γ *of* H *such* 395
that $\varphi_1 = \gamma \circ \varphi_2$.

We give a proof of the fact remarked above, which is a prototype of our proof of Step I.

Proposition 6.2. *Let* S *and* S' *be algebraic* $K3$ *surfaces and* $\varphi : T_S \to T_{S'}$ *be a Hodge isometry. If* $\rho(S) > 11$, *then there exists an isomorphism* $f : S \to S$ *such that* $f^* = \varphi$ *on* T_S.

For the proof, we need a version of Torelli theorem of $K3$ surfaces:

Proposition 6.3. *Let S and S' be $K3$ surfaces and $\psi : H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ be a Hodge isometry. Then there exists an isomorphism $f : S' \to S$ such that $f^* = \psi$ on T_S.*

Proof. By the strong Torelli theorem ([7]), there exists an isomorphism $f : S'' \to S$ and reflections $r_i (i = 1, \dots, n)$ by (-2) curves $C_i \cong \mathbb{P}^1$ on S such that $\psi = f^* \circ r_1 \circ \cdots \circ r_n$. Since $[C_i]$ is perpendicular to T_S, r_i is identity on T_S for every $i = 1, \dots, n$. Hence we have our proposition. q.e.d

Proof of theorem 6.2. Apply (2) of Proposition 6.1 to two primitive embeddings $T_S \hookrightarrow H^2(S,\mathbb{Z})$ and $T_{S'} \hookrightarrow H^2(S,\mathbb{Z})$. Since $H^2(S,\mathbb{Z})$ and $H^2(S',\mathbb{Z})$ are isomorphic to each other as lattices, we obtain isometry $\widetilde{\varphi}$:
396 $H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ such that $\widetilde{\varphi} \mid T_S = \varphi$. By the above proposition, there exists an isomorphism $f : S' \to S$ such that $f^* = \widetilde{\varphi}$ which proves our proposition. q.e.d

Proof of Step I. The orthogonal complement of T_S in the extended $K3$ lattice $\widetilde{H}(S,\mathbb{Z})$ is isomorphic to $N_S \perp U$. Applying (1) of Proposition 6.1 to the embedding of T_S and $T_{S'}$ into $\widetilde{H}(S,\mathbb{Z})$ and $\widetilde{H}(S',\mathbb{Z})$, we see that there exists an isometry $\Phi : \widetilde{H}(S,\mathbb{Z}) \to \widetilde{H}(S,\mathbb{Z})$ such that $\Phi \mid T_S = \varphi$. Put $v = \Phi(0,0,1) = (r,\ell,s)$ and $u = \Phi(1,0,0) = (p,k,q)$. Φ maps $\widetilde{H}^{1,1} \cdot (S')$ onto $\widetilde{H}^{1,1}(S)$. Hence both ℓ and k are divisor classes on S. Let m be a divisor class on S. The Chern character e^m of the line bundle $\mathcal{O}_S(m)$ is a unit of the cohomology ring $H^*(S,\mathbb{Z})$. Hence the multiplication by e^m induces a Hodges isometry Φ_m, $\Phi_m(r,\ell,s) = \left(r, \ell + rm, s + (m.\ell) + \frac{r}{2}\left(m^2\right)\right)$ of the extended $K3$ lattice $\widetilde{H}(S,\mathbb{Z}) \cong H^*(S,\mathbb{Z})$. Replacing Φ by $\Phi_m \circ \Phi$ for a sufficiently ample divisor m, we can choose Φ so that s is positive. Since the change of r and s is an Hodge isometry, we choose Φ so that r is positive. Since $(u.v) = -1$, the greatest common divisor r, (ℓ, k) and s is equal to 1.

Claim. There exists an integer n such that r and $s + n(\ell.k)$ are coprime. Let d be the greatest common divisor of s and $(\ell.k)$. Since $\frac{s}{d}$ and $\frac{(\ell.k)}{d}$ are coprime, there exists an integer n such that r and $\frac{s}{d} + \frac{n(l.k)}{d}$ are coprime. Since r and d are coprime, so are r and $s + n(\ell.k)$.

397 Take n as in the claim and replace Φ by $\Phi_{nk} \circ \Phi$. Then by the claim, r and s are coprime. Replace Φ by $\Phi_{rA} \circ \Phi$ again for a sufficiently ample divisor A. Then r and s are still coprime and ℓ become ample. Let M be the moduli space $M_\ell(v)$ of sheaves E with $v(E) = v$ which is stable with respect to ℓ. By Theorem 5.4 M is nonempty. Since r and s are coprime, every semi-stable sheaf is stable. Hence M is compact and hence irreducible by Corollary 4.6 By Theorem A.6 and Remark A.7, there exists a universal family $\mathscr{E}$ on $S \times M$. By Theorem 4.9, the cycle $Z = \pi_S * \sqrt{td_S} \cdot ch(\mathscr{E})\pi_M^* \sqrt{td_M}$ induces a Hodge isometry $\psi : \widetilde{H}(M,\mathbb{Z}) \to \widetilde{H}(S,\mathbb{Z})$, with $\Psi(\delta) = v$, where $\delta = (0,0,1)$.$\Phi^{-1} \circ \Psi$ is an isometry and sends δ to δ. Hence $\Phi^{-1} \circ \Psi$ induces a Hodge isometry from $H^2(M,\mathbb{Z}) = \Psi^{-1}\left(\frac{v^\perp}{\mathbb{Z}v}\right)$ onto $H^2(S',\mathbb{Z}) = \Phi^{-1}\left(\frac{v^\perp}{\mathbb{Z}v}\right)$.

By Proposition 6.3, there exists an isomorphism $f : S' \to M$ such that $f^* : H^2(M,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ coincides with $\Phi^{-1} \circ \Psi$ on T_M. Then the Chern character $ch((i \times f) * \mathscr{E}) \in H^*(S \times S',\mathbb{Z})$ of $(1 \times f)^* E$ induces a Hodge isometry $\Psi' : \widetilde{H}(S',\mathbb{Z}) \to \widetilde{H}(S,\mathbb{Z}$ which coincides with Φ (or equivalently φ) on T'_S. The $H^4(S \times S')$ component W of Z induces a homomorphism τ of the Hodge structure $H^2(S',\mathbb{Z})$ to $H^2(S',\mathbb{Z})$. τ maps T'_S onto T_S and coincides with ϕ on T'_S. q.e.d

Let $v = (r,\ell,s)$ be a primitive isotropic vector of $\widetilde{H}^{1,1}(S',\mathbb{Z})$ and assume that the moduli space $M = M_A(v)$ of stable sheaves E with $v(E) = v$ is nonempty and compact. Then, by Theorem 1.5, there exists an algebraic cycle Z on $S \times M$ defined by using the Chern character of a quasi- 398
universal family and Z induces a Hodge isometry $\phi : \frac{v^\perp}{\mathbb{Z}v} \to H^2(M,\mathbb{Z})$. The transcendental lattice T_S regarded as a sublattice of $\widetilde{H}(S,\mathbb{Z})$ is perpendicular to ν and $T_S \cap \mathbb{Z}\nu = 0$. Hence $\frac{v^\perp}{\mathbb{Z}v}$ contains a sublattice isomorphic to T_S and φ induces a Hodge isometry $\varphi : T_S \to T_M \cdot \varphi$ is injective but not surjective in general.

Proposition 6.4. *Let $v = (r,\ell,s)$ m and φ be as above. Let $n = n(v)$ be the minimum of $|(u.v)|$, where u runs over all vectors of $\widetilde{H}^{1,1}(S,\mathbb{Z})$ with*

$(u.v) \neq 0$. *Then we have*

(1) *the cokernel of φ is a cyclic group of order n,*

(2) *there exists a transcendental cycle $\lambda \in T_S$ such that $\ell + \lambda \in H^2(S,\mathbb{Z})$ is divisible by n, and*

(3) *if λ satisfies (2), then $\varphi(\lambda) \in T_M$ is divisible by n and $\frac{\varphi(\lambda)}{n}$ generates the cokernel of φ.*

Proof. For every $\upsilon \in H^{1,1}(S,\mathbb{Z})$, $\frac{(u.v)}{n}$ is an integer. Since $\widetilde{H}(S,\mathbb{Z})$ is unimodular and $\widetilde{H}^{1,1}(S,\mathbb{Z})$ is a primitive sublattice, there exists $w \in \widetilde{H}(S,\mathbb{Z})$ such that $\frac{(u.\upsilon)}{n} = (w.v)$ for every $\upsilon \in \widetilde{H}^{1,1}(S,\mathbb{Z}).\lambda = nw - u \in \widetilde{H}(S,\mathbb{Z})$ is perpendicular to $\widetilde{H}^{1,1}(S,\mathbb{Z})$ and hence lies in T_S. It is clear that λ satisfies (2). Assume that λ satisfies (2). Then $w = \frac{(\lambda + \upsilon)}{n}$ lies in $\upsilon^\perp$ and nw is congruent to λ modulo $\mathbb{Z}\upsilon$. Hence $\frac{\varphi(\lambda)}{n}$ lies in T_M. We show that $\frac{\varphi(\lambda)}{n}$ generates the cokernel of φ. The transcendental lattice T_M is isomorphic to $\frac{\left(\upsilon^\perp \cap \widetilde{H}^{1,1}(S,\mathbb{Z})\right)^\perp}{\mathbb{Z}\upsilon} \cong (\mathbb{Q}\upsilon \oplus T_S \otimes \mathbb{Q}) \in \frac{\widetilde{H}(S,\mathbb{Z})}{\mathbb{Z}\upsilon}$. Let α be a vector of $(\mathbb{Q}\upsilon \oplus T_S \otimes \mathbb{Q}) \cap \widetilde{H}(S,\mathbb{Z})$. Then $\alpha = a\upsilon + v$ for
399 $a \in \mathbb{Q}$ and $v \in T_S \otimes \mathbb{Q}$. Take a vector $u \in \widetilde{H}^{1,1}(S,\mathbb{Z})$ such that $(u.\upsilon) = n$. Then we have an $= a(u.\upsilon) = (\alpha.\upsilon) \in \mathbb{Z}$. Since $\upsilon = nw - \lambda$, we have $\alpha = (an)w + (\upsilon - a\lambda)$. Since an is an integer, $\upsilon - a\lambda$ lies in T_S and α is congruent to $(an)w$ modulo T_S. Hence $\frac{\varphi(\lambda)}{n}$ generates the cokernel of φ, which shows (3). If $\frac{m\varphi(\lambda)}{n}$ lies in T_S, then mw lies in $T_S + \mathbb{Z}\upsilon$ and is equal to $\lambda' + b\upsilon$ for $\lambda' \in T_S$ and $b \in \mathbb{Z}$. We have $m(\lambda + \upsilon) = n)(\lambda' + bv)$. Since $T_S \cap Zv = 0$, m is equal to nb and divisible by n. Hence $\frac{\varphi(\lambda)}{n}$ has order n is Coke φ, which shows (1) q.e.d

We have thus proved the following

Corollary 6.5. *Let M be a compact surface component of the moduli space of stable sheaves on S. Then there exists an algebraic cycle on $S \times M$ which induces a homomorphism $\varphi : T_S \to T_M$ such that $\varphi \times \mathbb{Q}$ is an isometry and the cokernel of φ is a finite cyclic group.*

Conversely, we have

Proposition 6.6. *Let S be an algebraic $K3$ surface and $\Psi : T_S \to T$ be an embedding of the transcendental lattice T_S of S into an even lattice T. Assume that the cokernel of Ψ is a cyclic group of order $r < \infty$. Then there exists a compact component M of the moduli space of stable sheaves of rank r on S which satisfies the following:*

(1) there is an isometry $i : T \xrightarrow{\sim} T_M$ and

(2) there is an algebraic cycle on $S \times M$ which induces $i \circ \psi$. 400

Proof. Take a transcendental cycle $\tau \in T_S \otimes \mathbb{Q}$ so that $(\psi \otimes \mathbb{Q})(\tau)$ belongs to T and generates T modulo $\psi(T_S)$. By our assumption, $\lambda = r\tau$ belongs to T_S. Since $\psi \otimes \mathbb{Q}$ is an isometry, $(\tau.\beta)$ is equal to $((\psi \otimes \mathbb{Q})(\tau) \cdot \psi(\beta))$ and is an integer for every $\beta \in T_S$. Since $H^2(S, \mathbb{Z})$ is a unimodular lattice and since T_S is a primitive sublattice of $H^2(S, \mathbb{Z})$, there exists a cycle $\alpha \in H^2(S, \mathbb{Z})$ such that $(\alpha \cdot \beta) = (\tau \cdot \beta)$ for every transcendental cycle $\beta \in T_S$. Then, the cycle $\ell = r(\alpha - \tau)$ belongs to $H^2(S, \mathbb{Z})$ and perpendicular to T_S. Hence ℓ is a divisor class of S. Moreover, $\ell + \lambda$ is equal to $r\alpha$ and divisible by r in $H^2(S, \mathbb{Z})$. Replacing α by α+(a sufficiently ample divisor), we can choose α so that ℓ becomes an ample divisor class. We put $s = \dfrac{(\ell^2)}{2r} = \dfrac{r(\alpha - \tau)^2}{2}$ and $v = (r, \ell, s) \in \widetilde{H}^{1,1}(S, \mathbb{Z})$ and consider the moduli space $M = M_A(v)$ of stable sheaves E with $v(E) = v$. Since (τ^2) is an even integer, so is $(\alpha - \tau)^2$. Hence s is divisible by r. Since τ is transcendental, $(\ell \cdot m)$ is equal to $r(\alpha \cdot m)$ and hence divisible by r for every divisor class m of S. Hence the number $n(v)$ (see Proposition 6.4) is equal to r. $M_\ell(v)$ is nonempty, by Theorem 5.4 and $M_\ell^\mu(v)$ is nonempty, by Proposition 3.16. Hence by Proposition 4.3, there exists an ample line bundle A such that $M = M_A(v)$ is nonempty and compact and irreducible. By Proposition 6.4, there exists an isometry $i : T \to T_M$ such that $\varphi = i \cdot \Psi$ and φ is induced by an algebraic cycle on $S \times M$. q.e.d

401 Step II. Let $\varphi : T_S \to T_S$ be a homomorphism of Hodge structures and assume that $\varphi \otimes \mathbb{Q}$ is an isometry. Then there exists an algebraic cycle $W \in H^4(S \times S', \mathbb{Q} |)$ on $S \times S'$ which induces φ.

Proof. We prove our assertion by induction on the length ℓ of the cokernel of φ. In the case $\ell = 1$, our assertion was proved in Step I. Hence we assume that $\ell > 1$. Take a sublattice T of $T_{S'}$ such that $\varphi(T_S) \underset{\neq}{\subset} T$ and $\dfrac{T}{\varphi(T_S)}$ is a cyclic group. Then, by Proposition 6.6, there exists a $K3$ surface M which is a compact component of the moduli space of stable sheaves such that $T_M \cong T$ and there exists an algebraic cycle W_1 on $S \times M$ which induces $T_S \to T \cong T_M$. By induction hypothesis, there exists an algebraic cycle W_2 on $M \times S'$ which induces $T_M \cong T \to T_{S'}$. Then, the cycle $Z = \pi_{S \times S'}, *(\pi^*_{S \times M} W_1 \cdot \pi_{M \times S'} * W_s)$ on $S \times S'$ is algebraic and induces φ on T_S. q.e.d

Proof of Theorem 1.9. By our assumption, there exists a primitive embedding $T \hookrightarrow \wedge$ of T into a $K3$ lattice $\wedge$. Since $T \otimes \mathbb{Q} \cong T_S \otimes \mathbb{Q}$ the Hodge decomposition of $T_S \otimes \mathbb{C}$ induces that of $T \otimes \mathbb{C}$ We regard T as a polarized Hodge structure by this Hodge decomposition. The orthogonal complement of T in $\wedge$ is a hyperbolic lattice, i.e., has signature $(1, *)$. By virtue of the surjectivity theorem of the period map for $K3$ surfaces [23], there exists a $K3$ surface S'' and an isometry $i : \wedge \to H^2(S'', \mathbb{Z})$ such that $i(T) = T_{S''}$ and $i\,|_T$ is a homomorphism of Hodge structures. Both T_S and $T_{S'}$ contain $T_{S''}$ as a sublattice of finite index. By Step II, there exist algebraic cycles on $S'' \times S$ and on $S'' \times S'$ which induce 402 the isometries $T_{S''} \hookrightarrow T_S$ and $T_{S''} \hookrightarrow T_{S'}$, respectively. Therefore, the composition of the two algebraic cycles induces the Hodge isometry between $T_S \otimes \mathbb{Q}$ and $T_{S'} \otimes \mathbb{Q}$. q.e.d

Appendix 1. Boundedness and existence of μ-semi-stable sheaves.

In this section, S is an arbitrary complete algebraic surface over $\mathbb{C}$. We study the behaviour of moduli spaces of μ-semi-stable sheaves with respect to $A_n, n = 1, 2, 3, \ldots$, when ample $\mathbb{Q}$ -divisors A_n converge to an ample divisor A.

Theorem A. 1. *Let $\{A_n\}$ be a sequence of ample $\mathbb{Q}$-divisors which converges to an ample divisor A. Let c_1 be a numerical equivalence class of divisors and c_2 an integer. Assume that, for every n, there exists a sheaf E_n on S with Chern classes c_1 and c_2 (modulo numerical equivalence) and which is μ-semi-stable with respect to A_n. Then there exists a sheaf E on S which satisfies the following:*

(1) there exists an infinite subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that E is μ-semi-stable with respect to every A_{n_k}, and

(2) E is μ semi-stable with respect to A.

Let P be a Zariski-open condition for sheaves on S which is independent of A_n, e.g., simpleness or local freeness. If the open condition P holds for every E_n, then E can be chosen so that E satisfies P.

For the proof of the above theorem, a certain boundedness of μ- **403**
semi-stable sheaves is essential. Let $\mathscr{A}$ be the ample cone in $H^{1,1}(S,\mathbb{R})$ and $\mathscr{A}$ its closure.

Theorem A. 2. *Let H be an ample divisor and B a bounded subset of $\overline{\mathscr{A}} \cap H^2(S,\mathbb{Q})$. Let $S^r_A(c_1,c_2)$ denote the set of isomorphic classes of rank r sheaves with Chern classes c_1 and c_2 modulo numerical equivalence and which are μ-stable with respect to an ample $\mathbb{Q}$-divisor A. Then the union $\bigcup\limits_{b\in B} S^r_{H+b}(c_1,c_2)$ is bounded.*

In the case $B = \{0\}$, this was proved by Maruyama in [8] and our proof of Theorem A.2 is quite parallel to his proof in § 2 [8]. Let $\alpha_1,\dots,\alpha_{r-1}$ be a sequence of $r-1$ rational numbers and let

$$S^r_B(\alpha_1,\dots,\alpha_{r-1} : c_1,c_2)$$

be the set of isomorphism classes of rank r torsion free sheaves of type $\alpha_1,\dots,\alpha_{r-1}$ with respect to $H+b$ for some $b\in B$ ([8, see p.28]) and with Chern classes c_1 and c_2 modulo numerical equivalence. Our Theorem A. 2 is a special case of the boundedness of $S^r_B(\alpha_1,\dots,\alpha_{r-1} : c_1,c_2)$ which follows from Theorem A. 3 below and Theorem 1.14 in [8].

Theorem A. 3. *Let a be an integer and let* $S^r_{B,a}(\alpha_1, \dots, \alpha_{r-1} : c_1)$ *be the union of* $S^r_B(\alpha_1, \dots, \alpha_{r-1} : c_1, c_2)$ *for all* $c_2 \leqq a$*. Then there are two constants* b_0 *and* b_1 *(independent of each* c_2*) such that for any member E of* $S^r_{B,a|}(\alpha_1, \dots, \alpha_{r-1} : c_1)$, $\dim H^0(S, E) \leqq b_0$ *and* $\dim H^0(C, E \otimes \mathscr{O}_C) \leqq b_1$ *for any curve C in an open set* $U(E)$ *of* $|H|$*, where* $U(E)$ *may depend on E.*

404 **Proof.** Our proof is quite similar to that of Theorem 2.5 in [8]. We only indicate the parts to be modified. It suffices to show the theorem for the subset $VS^r_{B,a}(\alpha_1, \dots, \alpha_{r-1} : c_1)$ of $S^r_{B,a}(\alpha_1, \dots \alpha_{r-1} : c_1)$ consisting of vector bundles in $S_{B,a}(\alpha_1, \dots, \alpha_{r-1} : c_1)$. We prove our theorem by induction on r. Assume that the theorem is true in the case rank $r - 1$. Under this assumption, we shall show that our theorem holds for $VS^r_{B,a}(\alpha_1, \dots, \alpha_{r-1} : c_1)$. Since B is bounded, there exists an integer n such that $H^0(S, E(n)) \neq 0$ for every member E of $VS^r_{B,a}(\alpha_1, \dots, \alpha_{r-1} : c_1)$ (cf. Lemma 2.1 in[8]), where $E(n)$ is the abbreviation of $E \otimes H^{\otimes n}$. Hence, for every member E of $VS^r_{B,a}(\alpha_1, \dots, \alpha_{r-1} : c_1)$, there exists an exact sequence

$$0 \to \mathscr{O}_S(D) \otimes H^{\otimes(-n)} \to E \to F \to 0$$

where D is an effective divisor and F is a torsion free sheaf of rank $r - 1$. Let L be the set of effective divisors D such that $\mathscr{O}_S(D) \otimes H^{\otimes(-n)}$ is contained in some member E of $VS^r_{B,a}(\alpha_1, \dots, \alpha_{r-1} : c_1)$.

Claim. L is bounded.

$\mathscr{O}_S(D)$ is a subsheaf of $E(n)$ and $E(n)$ is of type $\alpha_1, \dots, \alpha_{r-1}$ with respect to $H + b$ for some $b \in B$. Hence we have

$$\begin{aligned}(D \cdot H + b) \leqq \mu_{H+b}(E(n)) + \frac{\alpha_{r-1}}{(r-1)} &= \frac{(c_1 \cdot H + b)}{r + n(H \cdot H + b)} + \frac{\alpha_{r-1}}{(r-1)} \\ &= \frac{(c_1 \cdot H)}{r + n(H^2)} + \left(\frac{c_1}{r} + nH.b\right) + \frac{\alpha_{r-1}}{(r-1)}.\end{aligned}$$

405 Since B is bounded, $R = \underset{b \in B}{sup}(c_1/r + nH \cdot b) < \infty$. Since b belongs to $\mathscr{A}$

and D is effective, $(b\cdot D)$ is nonnegative. Hence we have

$$(D\cdot H) \leqq (D\cdot H + b) \leqq \frac{(c_1\cdot H)}{r+n(H^2)} + R + \frac{\alpha_{r-1}}{(r-1)}.$$

Therefore, L is bounded.

Let G be a rank s quotient sheaf of F. Since G is a quotient of E and since E is of type $\alpha_1,\ldots\alpha_{r-1}$ with respect to $H+b$, we have

$$\mu_{H+b}(E) - \alpha_s \leqq \mu_{H+b}(G).$$

Put $\alpha_{s,D,b} = \alpha_S + \{n(H\cdot H+b) + (c_1/r - D\cdot H + b)\}/(r-1)$. Then we have $\mu_{H+b}(E) - \alpha_s = \mu_{H+b}(F) - \alpha_{s,D,b}$. Put $\alpha'_s = \sup\limits_{D\in L, b\in B} \alpha_{s,D,b.}$ Then we obtain $\mu_{H+b}(F) - \alpha_s \leqq \mu_{H+b}(G)$. Hence F is of type $\alpha_1,\ldots,\alpha_{r-2}$ with respect to $H+b$. Let Q be the set of isomorphic classes of $F's$ which are obtained from some E in $VS^r_{B,a}(\alpha_1,\ldots,\alpha_{r-1} : c_1)$ as above. Then, by the above result, Q is a subset of

$$\coprod_{\lambda\in\wedge}\ \coprod_{c_2\leqq\alpha+\beta} S^{r-1}_B\left(\alpha'_1,\ldots,\alpha'_{r-2} : c_1 - \lambda + nc_1(H), c_2\right)$$

where $\wedge = L/($ (numerical equivalence) and $\beta = \max\limits_{D\in L}\{-(c_1 - D + nH\cdot D - nH)\}$. 406 By induction hypothesis, our theorem is true for any member F of Q and our proof can be completed in the same way as Theorem 2.5 in [8]. q.e.d

Proof of Theorem A.1. Take an integer N so that $NA_n - A$ is ample for every n. Applying Theorem A.2 for $H = A$ and $B = \{NA_n - A\}$, we see that the set $\mathscr{E}$ of isomorphic classes of sheaves on S which are μ-semi-stable with respect to A_n for some n is bounded. All $E_n s$ belong to $\mathscr{E}$ and hence there exists a subfamily $\{F_t : t\in V\}$ of $\mathscr{E}$ parametrized by a variety V which contains E_n for infinitely many n, say, for $n = n_1, n_2, \ldots$ Since μ-semi-stability is an open condition, for each n_k, there exists a Zariski open set U_k of V such that F_u is μ-semi stable with respect to A_{n_k} (and satisfies the property P) for every $u\in U_k$. V is a variety over $\mathscr{C}$ and is a Baire space. Hence the intersection of all U_k s is nonempty. Therefore, we have (1). (2) follows immediately from (1), because $\mu_A(F) = \lim\limits_{k\to\infty}\mu_{A_{n_k}}(F)$ for every sheaf F on S q.e.d

Appendix 2. Existence of a (quasi-) universal family

Let X be a scheme and $\mathscr{M}$ a connected component of the moduli functor $\mathscr{S}pl_X$ of simple sheaves on S.

Definition A. 4. (1) Let T be a scheme. A sheaf $\mathscr{E}$ on $X \times T$ is a quassi-family of sheaves in $\mathscr{M}$ if $\mathscr{E}$ is T-flat and if, for every $t \in T$ there exists an integer σ and a member E of $\mathscr{M}$ such that
407 $\mathscr{E}\,|_{X\times T} \cong E^{\oplus\sigma}$. If T is connected, then the positive integer σ does not depend on $t \in T$ and called the similitude of $\mathscr{E}$.

(2) Two quasi-families $\mathscr{E}, \mathscr{E}'$ of sheaves in $\mathscr{M}$ on $X \times T$ are equivalent if there exist vector bundles V and V' on T such that $\mathscr{E} \otimes \pi_T^* V \cong \mathscr{E} \otimes \pi_T^* V'$.

(3) A sheaf $\mathscr{E}$ on $X \times M$ is a quasi-universal family of sheaves in $\mathscr{M}$ if $\mathscr{E}$ is a quasi-family and, for every scheme T and quasi family $\mathscr{F}$ on $X \times T$, there exists a unique morphism $f : T \to M$ such that $f^*\mathscr{E}$ and $\mathscr{F}$ are equivalent.

By definition, if $\mathscr{E}$ on $X \times M$ and $\mathscr{E}'$ on $X \times M'$ are quasi-universal families, the M and M' are isomorphic to each other and $\mathscr{E}$ and $\mathscr{E}'$ are equivalent.

Theorem A. 5. *Assume that $\mathscr{M}$ is representable by a scheme M of finite type in the usual topology (if $k = \mathbb{C}$) or in the etale topology. Then there exists a quasi-universal family on $X \times M$.*

Proof. For simplicity, we assume that $k = \mathbb{C}$ and M is representable in the usual topology. There exists an open covering $M = \bigcup_i U_i$ (in the usual topology) and a universal family $\mathscr{E}_i$ on $U_i \times X$ for every i. Take a sufficiently ample line bundle L such that all higher cohomology groups $H^i(X, E \otimes L)$ vanish for every member E of M. By the base change theorem, the direct image $V_i = \pi_i * (\mathscr{E}_i \otimes L)$ is a vector bundle on U_i,
408 where π_i is the projection of $X \times U_i$ onto U_i Shrink the covering $\bigcup_i U_i$ so that $\mathrm{Pic}(U_i \cap U_j) = 0$ for every $i \neq j$. Then there exists an isomorphism $f_{ij} : \mathscr{E}_i\,|_{X\times(U_i\cap U_j)} \xrightarrow{\sim} \mathscr{E}_j\,|_{X\times(U_i\cap U_j)}$. f_{ij} induces an isomorphism $\overline{f}_{ij} =$

$\pi_{ij} * (f_{ij} \otimes L) : V_i \xrightarrow{\sim} V_j$, on $U_i \cap U_j$ where π_{ij} is the projection of $X \times (U_i \cap U_j)$ onto $U_i \cap U_j$. We put $\Phi(f_{ij}) = f_{ij} \otimes \pi_{ij}^*\left(\overline{f_{ij}^{-1}}\right)^\vee : \mathscr{E}_i \otimes \pi_i^* V_i^\vee |_{X\times(U_i\cap U_j)} \xrightarrow{\sim} \mathscr{E}_j \otimes \pi_j^* V_j^\vee |_{X\times(U_i\cap U_j)}$.

Claim. $\Phi(f_{ij}) \circ \Phi(f_{jk}) \circ \Phi(f_{ki})$ is identity over $X \times (U_i \cap U_j \cap U_k)$ for every i, j and k.

By the functoriality of Φ, $\Phi(f_{ij}) \circ \Phi(f_{ik}) \circ \Phi(f_{ki})$ is equal to $\Phi(g_{ijk})$, where $g_{ijk} = f_{ij} \circ f_{jk} \circ f_{ki} \cdot g_{ijk}$ is an automorphism of $\mathscr{E}_i |_{X\times(U_i\cap U_j\cap U_k)}$ over $U_i \cap U_j \cap U_k$. Since $\mathscr{E}_i$ is simple over U_i, the automorphism g_{ijk} of $\mathscr{E}_i$ over $U_i \cap U_j \cap U_k$ is multiplication by an invertible element of $\mathscr{O}_{U_i} \cap U_j \cap U_k$. Hence $\Phi(g_{ijk})$ is identity.

By the claim, $\mathscr{E}_i \otimes \pi_i^* V_i^\vee$ can be glued together by $\Phi(f_{ij})$'s. We obtain a sheaf $\mathscr{E}$ on $X \times M$ whose restriction to $U_i \times X$ is isomorphic to $\mathscr{E}_i \otimes \pi_i^* V_i^\vee$ for every i. We show that $\mathscr{E}$ is a quasi-universal family. Let $\mathscr{F}$ be a quasi-family of sheaves in M on $X \times T$. Since $\mathscr{E}_i$ are universal families, there exist a unique morphism $f : T \to M$, a vector bundle F_i on $f^{-1}(U_i)$ and an isomorphism $h_i : \mathscr{F} |_{X\times f^{-1}(U_i)} \xrightarrow{\sim} ((1 \times f)^* \mathscr{E}_i) \otimes \mathscr{O}_T F_i$ for every i. We show that two quasi-families $\mathscr{F}$ and $\mathscr{G} = (1 \times f)^* \mathscr{E}$ on $X \times T$ are equivalent. Define the homomorphism $\varphi : \mathscr{G} \otimes \pi^* \pi_* \mathrm{Hom}\mathscr{O}_{X\times T}(\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}\mathscr{O}_{X\times T}(\pi^*\pi_*\mathrm{End}_{\mathscr{O}_{X\times T}}(\mathscr{G}), \mathscr{F})$ by $\varphi(g \otimes f)(e) = f(e(g))$ for every $g \in \mathscr{G}$, $f \in \pi^*\pi_*\mathrm{Hom}\mathscr{O}_{X\times T}(\mathscr{G}, \mathscr{F})$ and 409 $e \in \pi^*\pi_*\mathrm{End}\mathscr{O}_{X\times T}(\mathscr{G})$, where π is the projection of $X \times T$ onto T. By using the isomorphisms h_i, it can be easily checked that this φ is an isomorphism. Since $\pi_*\mathrm{Hom}_{\mathscr{O}_{X\times T}}(\mathscr{G}, \mathscr{F})$ and $\pi_*\mathrm{End}_{\mathscr{O}_{X\times T}}(\mathscr{G})$ are vector bundles on T, two quasi-families F and G are equivalent. q.e.d

A quasi-universal family of similitudes 1 is nothing but a universal family. On the existence of a universal family, we have the following by an argument similar to the above and by an idea in [16] (and its improvement Theorem 6.11 in [9]).

Theorem A. 6. *Let the assumption be same as in above theorem. Let μ be the greatest common divisor of $\mathscr{X}(E \otimes N)$, where E is a member of $\mathscr{M}$ and N runs over all vector bundles on X. If $\mu = 1$, then there exists a universal on $X \times M$.*

Proof. Let μ_0 be the greatest common divisor of $\mathcal{X}(E \otimes N)$, where N runs over all vector bundles on X which satisfy $(*)$ all higher cohomology groups $H^i(X, E \otimes N)$ vanish for every member E of M.

We show that $\mu_0 = 1$. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. Then there exists an integer m_0 such that $N(m)$ satisfies $(*)$ for every $m \geqq m_0$. $\mathcal{X}(E \otimes N(m))$ is divisible by μ_0 for every $m \leqq m_0$ by definition. Since $Q\mathcal{X}(E \otimes N(m))$ is a numerical polynomial on m, $\mathcal{X}(E \otimes N)$ is divisible by μ_0. Since N is an arbitrary vector bundle, μ_0 divides μ
410 and hence $\mu_0 = 1$ by our assumption. Therefore, there exist vector bundles N_j with the property $(*)$ and integers $a_\upsilon (1 \leqq \upsilon \leqq n)$ such that $\sum_{\upsilon=1} a_\upsilon \mathcal{X}(E \otimes N_\upsilon) = -1$. Let $M = \cup_i U_i, \mathscr{E}_i$ and $f_{ij} : \mathscr{E}_{i\ X\times(U_i\cap U_j)} \xrightarrow{\sim} \mathscr{E}_{j\ X\times(U_i\cap U_j)}$ be same as in the proof of Theorem A.5. By the Property $(*), \pi_{i,*}(\mathscr{E}_i \otimes \pi_X * N_\upsilon)$ is a vector bundle of rank $\mathcal{X}(E \otimes N_\upsilon)$ on U_i for every i and υ. Put $L_i = \bigotimes_{\upsilon=1}^{n} \det(\pi_{i,*}(\mathscr{E}_i \otimes \pi_X^* N_\upsilon))^{\otimes a_\upsilon}$, where det denotes the highest nonzero exterior power of a vector bundle. The isomorphism f_{ij} induces the isomorphism $p_{ij} : L_i \mid_{U_i\cap U_j} \xrightarrow{\sim} L_j \mid_{U_i\cap U_j}$ for every i, j. By the same argument as in Theorem A.5, we can show that $\mathscr{E}_i \otimes \pi_i^* L_i$ on $X \times U_i$ can be glued together by the isomorphisms $f_{ij} \otimes p_{ij}$ and we obtain a sheaf $\mathscr{E}$ on $X \times M$ whose restriction to $X \times U_i$ is isomorphic to $\mathscr{E}_i \otimes \pi_i^* L_i$ for every i. Since $\mathscr{E}_i$ are universal families, $\mathscr{E}$ is a universal family. q.e.d

Remark A. 7. If X is smooth, then every sheaf on X has a resolution by a locally free sheaves. Hence μ in the theorem is equal to the greatest common divisor of $\mathcal{X}(E \otimes N')$ where $E \in \mathscr{M}$ and N' runs over all sheaves on X. If X is smooth and $\dim X = 2$, then μ is equal to the greatest common divisor of $r(E), (c_1(E).D)$ and $\mathcal{X}(E)$, where D runs over all divisors of X.

References

[1] Atiyah, M. F.: Vector bundles over an elliptic curve, *Proc.London Math. Soc.*, 7(1957)., 414–452.

[2] Gieseker, D.: On the moduli space of vector bundles on an algebraic surface, *Ann. of Math.*, 106(1977), 45–60.

[3] Hartshorne, R.: *Residues and duality*, Lecture Notes in Mathematics, Vol. 20. Berlin-Heidelberg-Now York: Springer 1966. **411**

[4] Inose, H.: Defining equations of singular $K3$ surfaces and a notion of isogeny, *Intl. Symp. on Algebraic Geometry* Kyoto 1977, 495–502: Kinokuniya 1978.

[5] Kleiman, S.: Les théorémes des finitude pour le foncteur de Picard, *Séminaire de Geometrie Algébrique du Bois Marie*, 1966/67 (S.G.A.6), Exposé XIII, Lecture Notes in Math., Vol. 225, Berlin-Heidelberg-New York: Springer 1971.

[6] Langton, S. G.: Valuative criteria for families of vector bundles on algebraic varieties, *Ann. of math.*, 101, (1975) 88–110.

[7] Looijenga, E., Peters. C.: Torelli theorems for Kähaler $K3$ surfaces, *Compos. Math.*, 42(1981), 145–186.

[8] Maruyama, M.: Stable vector bundles on an algebraic surface, *Nagoya Math. J.*, 58(1975), 25–68.

[9] Maruyama, M.: Moduli of stable sheaves, II, *J. Math. Kyoto Univ.*, 18(1978), 557–614.

[10] Morrison. D. R.: On $K3$ surfaces with large Picard number *Invent. Math.*, 75(1984), 105–121.

[11] Mukai, S.: On the classification of vector bundles over an abelian surface (in Japanese), *Recent Topics in Algebraic Geometry*, RIMS Kokyuroku 409(1989), 103–127. **412**

[12] Mukai, S.: Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface, *Invent, math.*, 77 (1984), 101–116.

[13] Mukai, S.: On the moduli space of bundles on $K3$ surfaces, II (in preparation)

[14] Mukai, S.: On reflection functors. (in preparation)

[15] Mumford, D.: *Abelian varieties*, Oxford University Press 1974.

[16] Mumford, D., Newstead, P. E.: Periods of a moduli space of bundles on curves, *Amer. J. Math.*, 90 (1968), 1200–1208.

[17] Nikulin, V. V.: Integral symmetric bilinear forms and some of their applications, English translation, *Math. USSR Izvestija*, 14(1980), 103–167.

[18] Oda, Tadao: Vector bundles on an elliptic curve, *Nagoya Math. J.*, 43(1971), 41–171.

[19] Oda, Takayuki: A note on the Tate conjecture for $K3$ surfaces, *Proc. Japan Acad.*, Ser. A 56(1980), 296–300.

413 [20] Piateckii-Shapiro, I., Shafarevitch, I. R.: A Torelli theorem for algebraic surfaces of type $K3$, English translation, *math. USSR Izvestija* 5(1971), 547–587.

[21] Shafarevitch, I. R.: Le théorème de Torelli pour les surfaces algebraique de type $K3$, *Acte Congrés Intl. Math.*, 1(1970), 413–417.

[22] Shioda, T., Inose, H.: On singular $K3$ surfaces, *Complex analysis and Algebraic Geometry*, Iwanami-Shoten, Tokyo (1977), 113–136.

[23] Todorov, A. N.: Applications of the Kähler-Einstein Calabi-Yau metric to moduli of $K3$ surfaces, *Invent. math.*, 61(1980), 251–265.

Department of Mathematics
Nagoya University
Furo-cho, Chikusa-ku
Nagoya, 464 Japan

2θ-Linear Systems On Abelian Varieties

By M. S. Narasimhan and S. Ramanan

1 Introduction and Statement of Main Theorem

415 We would like to consider the 2θ-linear system on an abelian variety with a principal polarisation θ. In the case when the abelian variety is the Jacobian of a projective non-singular curve X, there is a close relationship between semistable vector bundles of rank 2 and trivial determinant on X and the 2θ-linear system. On the one hand, one can describe the moduli $SU_X(2)$ of such bundles on X in terms of the 2θ-linear system. On the other, classical questions regarding Kummer varieties or the Schottky relation may be better understood in terms of this 'new' variety $SU_X(2)$.

We first considered this relationship some fifteen years ago in the case of genus 2 and proved

Theorem 1 ([5]). *The variety $SU_X(2)$ is canonically isomorphic to the projective space of divisors on the Jacobian, linearly equivalent to 2θ.*

This result was somewhat of a surprise for the following reason. For every line bundle j on any curve X, consider the semistable bundle $j \oplus j^{-1}$. This imbeds in $SU_X(2)$, the Kummer variety $\mathscr{K} = j/i$, where $J = \mathrm{Pic}^{\circ}(X)$ is the Jacobin of X and i is the involution $x \to x^{-1}$ of J. It 416 is easy to see that $SU_X(2) - \mathscr{K}$ is smooth. That $SU_X(2)$ is itself smooth in the case of genus 2 is surprising n view of the following

Theorem 2 ([5]). *The kummer variety $\mathscr{K}$ is precisely the singular locus of $SU_X(2)$, is $g \geq 3$.*

Some of the ideas relating to Theorem 1 have been generalised [2] to hyperelliptic curves of arbitrary genus $g \geq 2$. In particular, we have

Theorem 3 ([2]). *If X is hyperelliptic of genus 3 and i the involution of $SU_X(2)$ induced by the hyperelliptic involution on X, then $\dfrac{SU_X(2)}{i}$ is a quadric in $\mathbb{P}^7$.*

The aim of this paper is the following generalisation of Theorem 1.

Main Theorem. *If X is non-hyperlliptic of genus 3, then $SU_X(2)$ isomorphic to a quartic hypersurface in $\mathbb{P}^7$*

In particular, of course, the Kummer variety is imbedded in $\mathbb{P}^7$ and is the singular locus of a quartic hypersurface. Suppose f is the quartic polynomial in the homogenous coordinates $(Z_1, \ldots, Z_8)$ defining $SU_X(2)$. Then Theorem 2 implies that $\mathscr{K}$ is defined by $\dfrac{\partial f}{\partial z_i} = 0, i = 1, \ldots 8$. Thus we have, as an application, the

Corollary. *$\mathscr{K}$ can be defined by cubic polynomials.*

417 Wirtinger [7] had shown that $\mathscr{K}$ can be defined by quartics and it was an open problem if cubics would suffice. (See Coble ([1, p. 106])).

2 The relationship between vector bundles and 2θ-linear systems

Let us now make explicit the map of $SU_X(2)$ into the projective linear system P of 2θ, which exists for any g. If we denote by J^d the space of line bundles on X of degree d, notice that the natural divisor θ lives only in J^{g-1} Hence the linear system of 2θ is a system of divisors in J^{g-1}. The map that we have in mind associates to a vector bundle E of rank 2 with trivial determinant, the subset $D_E = \left\{\xi \epsilon J^{g-1} : \Gamma(\xi \otimes E) \neq 0\right\}$. If E is not semistable, then it has a line subbundle L of positive degree. For every $\xi \epsilon J^{g-1}$, we have $\Gamma(\xi \otimes L) \neq 0$ so that $D_E = J^{g-1}$ in this case. On the other hand, Raynaud [6] showed that if E is semistable, the D_E is a proper subset of J^{g-1} Then it is easy to see that D_E is the support of a divisor linearly equivalent to 2θ.

Actually, one can associate a divisor D_E to E (and not merely its support) as follows. Take $x_1, \dots x_N \epsilon X$, with N sufficiently large. If Z is the divisor $x_1 + \cdots + x_N$, tensorise the exact sequence

$$0 \to \mathscr{O}(-Z) \to \mathscr{O}_X \to \mathscr{O}_Z \to 0$$

with $\xi \otimes E$. If N is large enough so that $\Gamma(\xi \otimes E \otimes \mathscr{O}(-Z)) = 0$, then clearly $\Gamma(\xi \otimes E)$ is the kernel of the connecting homomorphism

$$\sum_{i=1}^{N} (\xi \otimes E)_{x_i} \to H^1(\xi \otimes E \otimes \mathscr{O}(-Z))$$

We will now allow ξ to vary over J^{g-1}. In others words, taking ξ to be the universal line bundle of degree g^{-1} parametrised by J^{g-1} we get two vector bundles on J^{g-1}, both of rank $2N$ and a homomorphism. This defines a section of a line bundle, namely the difference between their determinants. It is easy to compute this line bundle and show that it is isomorphic to 2θ. **418**

3 Generalities on polarised abelian varieties

Let A be an abelian variety and τ a principal polarisation on 4. Consider $B = \mathrm{Pic}^{\tau} A$, namely the space of isomorphism classes of line bundles on A with Chern class τ. Then A acts on B by $a.\xi$=translation by a of ξ. All line bundles $\xi \epsilon B$ are ample and hence $H^i(\xi) = 0$ for $i \geq 1$ ([3, p. 150]). On the other hand, $\Gamma(\xi)$ is 1-dimensional, since the polarisation is principal. Hence the line bundles $\xi \epsilon B$ may be identified with their divisors in A. Let $\theta \subset B$ be defined by

$$\theta\{\xi \epsilon B : \text{ divisor of } \xi \text{ passes through } 0\}$$

This belongs to the principal polarisation τ in the sense, that for any choice of a point in B, the natural identification $A \to B$ leads to a divisor in A whose class is τ. Moreover, $A \coprod B$ can be made into a group, using the involution $i : B \to B$ given by $\xi \to i^*\xi$ where i is the morphism $x \to -x$ of A. We have then a natural exact sequence

$$0 \to A \to A \coprod B \to \mathbb{Z}/2 \to 0.$$

419 Of course, this sequence splits, but the point is that there is no canonical splitting. Elements of $A_2 = \{a \epsilon A : 2a = 0\}$ are called *period characteristics* and elements of $B_2 = \{b \epsilon B : 2b = 0\}$ are called *theta characteristics*. As a group A_2 is isomorphic to $\mathbb{F}_2^2 g$ and the principal polarisation on A gives a nondegenerate alternating form τ on A_2. Moreover, B_2 can be identified with quadratic forms on A_2 whose associated alternating form is τ. The Arf invariant then distinguishes between odd and even theta characteristics. We have the well-known [4]

Theorem. *The parity of the multiplicity of the divisor θ at $b \epsilon B_2$ determines the parity of the theta characteristic.*

On $A \times B$, we may consider the divisor obtained by pulling back θ by the natural group action $A \times B \to B$. This bundle serves as a Poincaré bundle for line bundles on A with Chern class τ, and also as one for line bundles on B with Chern class τ. Indeed, the space B and the divisor θ are characterized by this property. It then follows that if A is the Jacobian of a curve X of genus g, then B may be identified with J^{g-1}.

Although there is no canonical θ-divisor on A, there is a divisor class Φ with Chern class 2τ. For every $b \epsilon B$, $T_b^*\theta$, is the divisor in A, corresponding to b. The linear equivalence class of $T_b^*\theta + T_{-b}^*\theta$ is independent of $b \epsilon B$ and this defines Φ. More elegantly, this may be constructed as follows. Consider the map $\mu : A \times B \to B \times B$
420 given by $(a, b) \mapsto (a + b, a - b)$. The pull back of tensor product of $P_1^*\mathscr{O}(\theta) \otimes P_2^*\mathscr{O}(\theta)$ on $B \times B$ to $A \times B$ may be looked upon as a family of line bundles on B parametrised by A. But then for any a, this gives the isomorphism class of $\mathscr{O}(2\theta)$. Hence there is a line bundle Φ on A such that

$$P_1^*\phi \otimes P_2^*\mathscr{O}(2\theta) \approx \mu^*\left(P_1^*\mathscr{O}(\theta) \otimes P_2^*\mathscr{O}(\theta)\right).$$

Moreover, the situation comes with a section of the line $P_1^*\Phi \otimes P_2^*\mathscr{O}(2\theta)$, so that we have a canonical element of $\Gamma(A, \Phi) \otimes \Gamma(B, \mathscr{O}(2\theta))$. This can be proved to give a perfect pairing so that we have

Theorem 3.1. *There is a canonical duality between* $\Gamma(A, \Phi)$ *and* $\Gamma(B, \mathscr{O}(2\theta))$.

We have a natural morphism $A \to P\Gamma(A, \Phi)^*$ given by linear system of Φ. On the other hand, for $a \epsilon A$, we have the divisor $T_a\theta + T_a\theta$ linearly equivalent to 2θ. In other words, we have a morphism $A \to P\Gamma(B, \mathscr{O}(2\theta))$. Under the duality of Theorem 3.1, these two morphism are compatible.

When A is the Jacobian and $B = J^{g-1}$ of the curve X, we may interpret the morphism $A \to P\Gamma(B, \mathscr{O}(2\theta))$ defined above as follows. For any $a \epsilon A$, consider the semistable vector bundle $E = a \oplus (a)^*$. Then E is of rank 2 and trivial determinant so that there is a divisor D_E linearly equivalent to 2θ defined in § 2. It is clear that this is the same as the map given above. Thus one may say that the morphism $SU_X(2) \to P\Gamma(J^{g-1}, \mathscr{O}(2\theta))$ given by $E \mapsto D_E$ restricts to the Kummer variety $\mathscr{K}$ as the morphism given by the linear system of Φ. It has been shown by Andreotti and Meyer over $\mathbb{C}$ that for θ irreducible, Φ imbeds the Kummer **421** variety. This does not seem to have been proved in the literature for characteristic $P > 0$. It is easy enough to show that if θ is irreducible, then the Kummer variety is mapped injectively. We can prove (following ideas of Wirtinger[7]) over any field of characteristic $\neq 2$, the following

Theorem. *Let A be any abelian variety,* τ *a principal polarisation on it and* $B = \mathrm{Pic}^\tau A$. *If no even theta characteristic lies on the canonical theta divisor in B,then* $\mathscr{K}$ *is imbedded by the canonical divisor class* Φ *in A with Chern class* 2τ.

4 Proof of the Main Theorem

We will sketch here the alinea of proof of our main theorem. The detailed proofs will appear elsewhere.

Lemma 4.1. *Let ξ be a line bundle of degree 1 on X. Then X may be imbedded in J by $x \to \xi^{-1} \otimes \mathscr{O}(x)$. The induced map $\Gamma(J, \Phi) \to \Gamma(X, K \otimes \xi^2)$ is onto.*

Proof. Any divisor linearly equivalent to $K \otimes \xi^2$ may be written as a sum of two divisors of degree g each. Let these belong to classes $K \otimes \xi \otimes \alpha^{-1}$ and $\xi \otimes \alpha$ for $\alpha \epsilon J^{g-1}$. It is easy to see that 'most' sections in $\Gamma(K \otimes \xi^2)$ can be thus split up in which $\Gamma\left(K \otimes \xi \otimes \alpha^{-1}\right)$ and $\Gamma(\xi \otimes \alpha)$ are of dimension one each. It is clear then that these divisors are images of elements in $\Gamma(\Phi)$ given by $\alpha\theta + \alpha^{-1}\theta \sim \Phi$.

Consider the injective map

$$H^1\left(X, \xi^{-2}\right) \xrightarrow{\sim} \Gamma\left(X, K \otimes \xi^2\right)^* \to \Gamma(J, \Phi)^* \xrightarrow{\sim} \Gamma(J^{g-1}, \mathscr{O}(2\theta)).$$

422 We wish now to interpret the map $PH^1\left(X, \xi^{-2}\right) \to P\Gamma(J^{g-1}, \mathscr{O}(2\theta))$.

Lemma 4.2. *Let ξ be a line bundle on X, of degree 1. Any point of $PH^1\left(X, \xi^{-2}\right)$ gives a nontrivial extension $0 \to \xi^{-1} \to E \to \xi \to 0$, where E is a semi-stable vector bundle of rank 2 and trivial determinant.*

Proof. See ([5, Lemma 5.1]).

By allowing ξ to vary over J^1, one can thus construct a bundle V on J^1 whose fibre over ξ is identifiable with $H^1\left(X, \xi^{-2}\right)$, and an exact sequence

$$0 \to V \to \Gamma(J^{g-1}, \mathscr{O}(2\theta))_J 1 \to W \to 0$$

on J^1 It can be shown that the fibre V_ξ is the subspace of $\Gamma(J^{g-1}, \mathscr{O}(2\theta))$ consisting of sections which vanish on ξX.

Lemma 4.3. *The map $PH^1\left(X, \xi^{-2}\right) \to P\Gamma(J^{g-1}, (2\theta))$ can be interpreted as follows. If E is the vector bundle associated to $\upsilon \epsilon PH^1\left(X, \xi^{-2}\right)$, then the image of υ is the divisor D_E.*

Lemma 4.4. *Let X be of genus 3. If $E \epsilon SU_X(2)$ is stable and $\xi X \subset D_E$ for some $\xi \epsilon J^1$, then $\Gamma(\xi \otimes E) \neq 0$.*

These two lemmas ensure that the morphism $E \mapsto D_E$ is injective. Firstly, it is easy to show (see 5.4 below for a slightly stronger statement)

that there exists $\xi \epsilon J^1$ with $\Gamma(\xi \otimes E) \neq 0$. This implies of course that $\xi X \subset D_E$. If then $D_E = D_E$ it follows from 4.4 that $\Gamma(\xi \otimes E') \neq 0$ as well, which implies that both E and E' occur as extensions of ξ by ξ^{-1}. 423
Now 4.3 implies that $E \mapsto D_E$ is injective.

We have also to show that the morphism $E \rightarrow D_E$ is an imbedding. From the injectivity, we see that the image is a hypersurface in $\mathbb{P}^7 = \mathbb{P}\Gamma\left(J^{g-1}\mathscr{O}(2\theta)\right)$. We will compute the differential of the morphism $P(V) \rightarrow \mathbb{P}^7$. Let $E \epsilon SU_X(2)$ be a stable bundle. Then we will show that in the diagram

$$\begin{tikzcd}
P(V) \arrow[rr] \arrow[dr] & & \mathbb{P}^7 \\
& SU_X(2) \arrow[ur] &
\end{tikzcd}$$

there exists a point of $P(V)$ lying over E at which the differential of the map $P(V) \rightarrow \mathbb{P}^7$ is injective. It would follow that the hyper surface in question is normal, since the set of nonstable points is of codim 3 in $SU_X(2)$. Hence by Zariski's Main Theorem, we are through.

5 Computation of differential

But the computation of the differential of the morphism $P(V) \rightarrow SU_X(2)$ turns out to be somewhat hard. The result is

Lemma 5.1. *The differential is injective at every point of $PH^1\left(\xi^{-2}\right)$ corresponding to stable E, if $\xi^2 \notin \theta$.*

Since the bundle V is induced by a map φ of the Jocobain J^1 into the Grassmannian of 4 dimensional subspaces of $\Gamma(J^2, \mathscr{O}(2\theta))$, Lemma 5.1 424
can be proved by computing the differential of this map and a simple computation of the differential of the natural map of the projective bundle associated to the universal subbundle on the Grassmannian, into the projective space. Thus it is easy to see that Lemma 5.1 would follow from the following statement regarding the differential of φ which is a map $H^1(X, \mathscr{O}) \rightarrow \operatorname{Hom}\left(V_\xi \Gamma\left(K^2 \otimes \xi^{-2}\right)\right)$.

Lemma 5.2. *Let $\upsilon \in T_\xi\left(J^1\right)$, $\xi^2 \notin \theta$. (a) if υ does not belong to any 1-dimensional subspace of $H^1(X, \mathscr{O})$ corresponding to a point of X in its canonical imbedding, then the image of υ under the differential of $J^1 \to Gr_4\left(\Gamma\left(J^2, \mathscr{O}(2\theta)\right)\right)$ is an injective map $V_\xi \to \Gamma\left(K^2 \otimes \xi^{-2}\right)$. (b) if υ belongs to such an 1-dimensional sub-space, then the image has 1-dimensional kernel and this, as a point of $P(V_\xi)$, corresponds to a nonstable extension.*

The space V_ξ can also be identified with the kernel of $\Gamma\left(J^2, \mathscr{O}(2\theta)\right) \to \Gamma(X, K^2 \otimes \xi^{-2})$ obtained by imbedding X as $\xi X \in J^2$. In other words, $V_\xi = \Gamma\left(J^2, I_X \otimes \mathscr{O}(2\theta)\right)$, where I_X is the sheaf of ideals of ξX in J^2. From the exact sequence

$$0 \to I_X^2 \to I_X \to N^* \to 0$$

we obtain a natural map $V_\xi \to \Gamma\left(N^* \otimes K^2 \otimes \xi^{-2}\right)$. It can be shown that the required differential factors as follows.

$$\begin{tikzcd}
H^1(X, \mathscr{O}) \arrow[r] \arrow[dr] & \operatorname{Hom}\left(V_\xi \Gamma\left(K^2 \otimes \xi^{-2}\right)\right) \arrow[d] \\
 & \operatorname{Hom}\left(\Gamma\left(N^* \otimes K^2 \otimes \xi^{-2}\right), \Gamma\left(K^2 \otimes \xi^{-2}\right)\right)
\end{tikzcd}$$

425 The vertical map is induced by the map $V_\xi \to \Gamma\left(N^* \otimes K^2 \otimes \xi^{-2}\right)$ mentioned above. The map $H^1(X, \mathscr{O}) \to \operatorname{Hom}\left(\Gamma\left(N^* \otimes K^2 \xi^{-2}\right)\right)$, $\left(\Gamma\left(K^2 \otimes \xi^{-2}\right)\right)$ can be determined explicitly by using the natural inclusion $N^* \to \Gamma(X, K)_X$. In fact, it is the composite of the natural map

$$H^1(X, \mathscr{O}) \to \operatorname{Hom}\left(\Gamma(X, K) \otimes \Gamma\left(K^2 \otimes \xi^{-2}\right), \Gamma\left(K^2 \otimes \xi^{-2}\right)\right)$$

and the map $\operatorname{Hom}\left(\Gamma(X, K) \otimes \Gamma\left(K^2 \otimes \xi^{-2}\right), \Gamma\left(K^2 \otimes \xi^{-2}\right)\right) \to$ $\operatorname{Hom}\left(\Gamma\left(N^* \otimes K^2 \otimes \xi^{-2}\right), \Gamma\left(K^2 \otimes \xi^{-2}\right)\right)$ induced by $N^* \to \Gamma(X, K)_X$. In view of this, Lemma 5.2 will follow from

Lemma 5.3. *The natural map* $V_\xi \to \Gamma\left(N^* \otimes K^2 \otimes \xi^{-2}\right)$ *is an isomorphism if* $\xi^2 \notin \mathrm{Supp}\theta$*. Geometrically speaking, there is no divisor linearly equivalent to* 2θ *which vanishes on* ξX *with multiplicity* 2*,* $f\,\xi^2 \notin \mathrm{Supp}\theta$*.*

The proof that $E \mapsto D_E$ is an imbedding is completed by proving

Lemma 5.4. *Every stable bundle* E *can be obtained as an extension* $0 \to \xi^{-1} \to E \to \xi \to 0$, $\xi \epsilon J^1$ *and* $\xi^2 \notin \mathrm{Supp}\theta$*.*

To prove that the image is a quartic surface, we first observe that the map $\mathbb{P}(V) \to \mathbb{P}^7$ is generically finite and is of degree 8. If τ_V is the tautological hyperplane bundle along the fibres it suffices to show that $[c_1(\tau_V)]^6[P(V)] = 32$. This follows from

Lemma 5.5. *Let* $c_i (i = 1, 2, 3)$ *denote the Chern classes of the bundle* V *and let* $c(\theta) \epsilon H^2\left(J^1, \mathbb{Z}\right)$ *be the cohomology class defined by a* θ*-divisor in* J^1*. Then we have* 426

(1)

$$c_1(V) = -4c(\theta), c_2(V) = 8[c(\theta)]^2 \textit{and}$$
$$c_3(V) = -\frac{16}{3}[c(\theta)]^3$$

(2) $[c_1(\tau_V)]^6[P(V)] = \left(-c_1^3 + 2c_1c_2 - c_3\right)\left[J^1\right].$

References

[1] A. B. Coble. Algebraic geometry and theta functions, *A.M.S Colloquium Publications* X, 1929.

[2] U. V. Desale and S. Ramanan. Classification of vector bundles of rank 2 on hyperelliptic curves, *Inventiones Math.* 38(1976) 161–185.

[3] D. Mumford. Abelian varieties, *T.I.F.R. Studies in Maths.* O.U.P, 1974.

[4] D. Mumford. On the equations defining abelian varieties, *Inventions Math.* 1(1966) 287–354.

[5] M. S. Narasimhan and S. Ramanan Moduli of vector bundles on a compact Riemann surface. Ann. of Math. 89(1969)19–51.

[6] M. Raynaud. Sections des fibrés vectoriels sur une courbe, *Bull. Soc. Math. France* 110(1982)103–125.

427 [7] W. Wirtinger. Untersuchugen über Thetafunktionen, *Teubner*, 1895.

School of Mathematics,
Tata Institute of Fundamental Research,
Homi Bhabha Road, Colada,
Bombay - 400 005.
INDIA.

Compactification of $M(0, 2)$

By M. S. Narasimhan and G. Trautmann

1 Introduction

429 Let $M(0, 2)$ denote the quasi-projective variety of isomorphism classes of stable rank 2 vector bundles on $\mathbb{P}_3(\mathbb{C})$ with $C_1 = 0$, $C_2 = 2$. This variety was investigated in detail by Hartshorne [1]. See also [2]. He proved that $M(0, 2)$ has the structure of a fibre space over the 9-dimensional variety R of reguli (i.e., the space of smooth quadrics in $\mathbb{P}_3$ with a distinguished system of generating lines), the fibre being an open subset of a smooth quadric in $\mathbb{P}_5$ If σ is the smooth conic in the grassmannian $\mathbb{G}$ of lines in $\mathbb{P}_3$ given by the generators of a regulus ρ, then the fibre over ρ consists of the space of smooth conics γ such that σ and γ are Poncelet related, with σ as the inner conic: a triangle can be inscribed in γ which circumscribes σ.

In the present paper, we study the natural compactification of $M(0, 2)$ and the degeneration of bundles to sheaves with singularities. Geometrically, one has to first compactify the fibres over R which is easy and is done by taking all conics γ, smooth or not, which are Poncelet related to σ; the fibre over σ is then a smooth quadric in $\mathbb{P}_5$, which may be called the Poncelet quadric associated with σ. Next, one has to take a **430** good compactification of the space of reguli R. There is a 'naive' compactification of R, namely the ramified 2-sheeted covering of the space of all quadrics in $\mathbb{P}_3$ defined by the space of singular quadrics. This is not the right one to take and has to be 'modified' and we take as the compactification of R the Hilbert scheme $\mathscr{C}(\mathbb{G})$ of all conics contained in the grassmannian $\mathbb{G} = G(2, 4)$. The variety $\mathscr{C}(\mathbb{G})$ is in fact smooth and there is a tautological conic bundle over $\mathscr{C}(\mathbb{G})$. There is then a Poncelet quadric bundle over $\mathscr{C}(\mathbb{G})$ associated with this conic bundle; it is constructed by considering also the space of conics which are Poncelet related to a singular conic, which turns out to be a pair of hyperplanes in $\mathbb{P}^5$ in the case of a pair of lines and a 'doubled' hyperplance in the case of a double line. This Poncelet quadric bundle over $\mathscr{C}(\mathbb{G})$ is the sought for compactification of $M(0,2)$. We prove that this space is the coarse moduli scheme for semi-stable rank 2 sheaves on $\mathbb{P}_3$ with $C_1 = 0$, $C_2 = 2$, $C_3 = 0$ which are limits of stable vector bundles (see the statement of Theorem in § 3). The non-singular points of this compactification correspond exactly to stable sheaves. It should be remarked that the moduli space of semi-stable rank 2 sheaves with $C_1 = 0$, $C_2 = 2$, $C_3 = 0$ is not irreducible.

We proceed to describe the results and methods of proof in more detail.

2 Poncelet pairs and the Poncelet quadric.

431 Two smooth conics (σ, γ) in $\mathbb{P}_2$ are said to be Poncelet related if we can inscribe a triangle in γ which circumscribes σ We have also to deal with the case when σ is degenerate. For this purpose it is convenient to consider the Poncelet relation as one between conics in $\mathbb{P}_2$ and those in the dual projective space $\mathbb{P}_2^*$. For instance, in the case above, if we consider the polar dual $\overset{\vee}{\gamma}$ of γ with respect to the smooth conic σ, the Poncelet relation between (σ, γ) says that there are three points in σ such that the dual triangle has vertices on $\overset{\vee}{\gamma}$. This relation is symmetric.

If W is a 3-dimensional vector space (over $\mathbb{C}$), the Poncelet relation defines a correspondence between $\mathbb{P}\left(S^2(W)\right)$ and $\mathbb{P}\left(S^2(W^*)\right)$ i.e., a correspondence between the space of conics in $\mathbb{P}(W)$ and in $\mathbb{P}(W^*)$. In particular if σ is a conic in $\mathbb{P}(W)$ we get a quadric in $\mathbb{P}\left(S^2(W)\right)$, the Poncelet quadric corresponding to σ. If we denote also by σ a quadratic form on W defining σ, a quadratic form Q on $S^2(W)$ defining the Poncelet quadric associated σ is given by:

$$Q(X,Y,X'.Y') = \frac{1}{}[\sigma(X,X')\sigma(Y,Y') + \sigma(X,Y')\sigma(Y,X') \\ - \sigma(X,Y)\sigma(X',Y')]$$

This construction can be relativised: given a conic bundle $C \to S$, we can construct the Poncelet quadric bundle over S associated to C.

3 Statement of the main theorem.

432 **Theorem.** *Let* $\mathbb{G}$ *be the grassmannian of lines in* $\mathbb{P}_3(\mathbb{C})$, $\mathscr{C}(\mathbb{C})$ *the Hilbert scheme of conics contained in* $\mathbb{G}$ *and* $Q \to \mathscr{C}(\mathbb{G})$ *associated to the tautological conic bundle over* $\mathscr{C}(\mathbb{G})$. *We then have:*

(1) *The variety* Q *is the space of* S*-equivalence classes of semi-stable sheaves on* $\mathbb{P}_3(\mathbb{C})$ *which are limits, under flat deformations, of bundles in* $M(0,2)$

(2) Q *has the 'coarse moduli property' for flat families* $\{F_s\}_{s\in S}$ *of torsion-free semi-stable rank* 2 *sheaves with* $C_1 = 0$, $C_2 = 2$, $C_3 = 0$ *for which the subset of* S, *consisting of those* $s \in S$ *with* F_s *a stable bundle* $\epsilon M(0,2)$ *is dense in* S *where* S *is normal.*

(3) Q *is the normalisation of a component of the Maruyama scheme of all semi-stable torsion free rank* 2 *sheaves on* $\mathbb{P}_3$ *with* $C_1 = 0$, $C_2 = 2$ $C_3 = 0$.

(4) *The non-singular points of* Q *correspond precisely to stable sheaves*.

In the rest of the paper, we give a brief account of some of the ideas involved in the proof of the theorem.

4 Geometric invariant theoretic (G.I.T) description of the Poncelet bundle over the space of conics in $\mathbb{P}_2$

433 In order to relate the Hilbert scheme $\mathscr{C}(\mathbb{G})$ and the associated Poncelet bundle to sheaves on $\mathbb{P}_3$ we will need a geometric invariant theoretic description of these spaces. As a motivation for this description, which will be given in the next section, we give in this section a G.I.T. parametrisation of the Poncelet bundle over the space of conics in $\mathbb{P}_2$.

We first give a G.I.T. parametrisation of conics in $\mathbb{P}_2$. Let W be a 3-dimensional vector space and let $\widetilde{W} = W \oplus W = W \otimes \mathbb{C}^2$ The action of $SL(2, \mathbb{C})$ on $\mathbb{C}^2$ induces an action on $\widetilde{W}$. This in turn gives an action of $SL(2, \mathbb{C})$ on the grassmannian $G_4\left(\widetilde{W}\right)$ of 4 dimensional subspaces of $\widetilde{W}$, which is linearised with respect to the natural polarisation on $G_4\left(\widetilde{W}\right)$. The G.I.T. quotient $\dfrac{G_4\,(W \oplus W)^{ss}}{SL(2)}$ is then identified with the space of conics in $\mathbb{P}_2$ (As usual, the superscript "ss" denotes the set of semi-stable points). In fact, if for a 4-dimensional subspace $\Gamma \subset W \oplus W$, we consider

$$\sigma(\Gamma) = \left\{L \in \mathbb{P}(W) \mid \dim\left[\left(L \otimes \mathbb{C}^2\right) \cap \Gamma\right] \geq 1\right\}$$

one sees that $\sigma(\Gamma) \neq \mathbb{P}(W)$ if and only if Γ is semi-stable and that $\sigma(\Gamma)$ is a conic if Γ is semi-stable. Moreover Γ is stable if and only if $\sigma(\Gamma)$ is a smooth conic.

We next consider the Poncelet correspondence. (See §2). For this,
434 we consider the grassmannian $G_2\left(\widetilde{W}\right)$ of 2-dimensional subspaces of $W \oplus W$ and observe, as above, that $\dfrac{G_2\left(\widetilde{W}\right)^{ss}}{SL(2)}$ can be identified with the space of conics in $\mathbb{P}\,(W^*)$. To obtain the Poncelet correspondence between $\mathbb{P}\left(S^2(W)\right)$ and $\mathbb{P}\left(S^2\,(W^*)\right)$, we consider the flag manifold $F \subset G_2\left(\widetilde{W}\right) \times G_4\left(\widetilde{W}\right)$ consisting of (M, Γ) with $M \subset \Gamma$ and show that, for the action of $SL(2)$ on F, $\dfrac{F^{ss}}{SL(2)}$ is isomorphic to the Poncelet subvariety contained in $\mathbb{P}\left(S^2\,(W)\right) \times \mathbb{P}\left(S^2\,(W^*)\right)$. Moreover the natural map $F^{ss} \to$

$G_4(W)^{ss}$ induces the Poncelet quadric bundle over the space of conics in $\mathbb{P}(W)$.

5 G.I.T. description of the Hilbert scheme of conics in $\mathbb{G}$ and the associated Poncelet bundle.

Let V be a 4-dimensional vector space and $\mathbb{G} = G_2(V) \subset \mathbb{P}\left(\overset{2}{\wedge} V\right)$. We first give a G.I.T. description of the Hilbert scheme of conics in $\mathbb{P}\left(\overset{2}{\wedge} V\right)$. Note that this scheme is a $\mathbb{P}^5$ bundle over the grassmanian $Z = G_3\left(\overset{2}{\wedge} V\right)$ of planes in $\mathbb{P}\left(\overset{2}{\wedge} V\right)$, the fibre over a plane consisting of the space of conics in that plane. As such, in analogy with § 4, this scheme has the following G.I.T. description. Let U be the universal rank 3 bundle over the grassmannian Z; U is a subbundle of the trivial bundle $\overset{2}{\wedge} V \otimes \mathscr{O}_Z$ over Z. Then $SL(2,\mathbb{C})$ operates on the relative grassmannian bundle $G_4(U \oplus U)_2$ and the G.I.T quotient is the Hilbert scheme of conics in $\mathbb{P}\left(\overset{2}{\wedge} V\right)$. We also have an obvious G.I.T. description of the associated Poncelet bundle as in § 4.

To obtain the Hilbert scheme $\mathscr{C}(\mathbb{G})$ of conics in G as a G.I.T. quotient we define a subvariety Y of $G_4(U \oplus U)$ as follows. If $(z,\Gamma) \in G_4(U \oplus U)$, with $z \in Z$, then (z,Γ) defines a quadratic form $\sigma(\Gamma)$ on the 435 fibre U of U at z, and the Plücker quadric G defines a quadratic form $\rho(z)$ on U_z (Here $\rho(z)$ could be zero which means that $\mathbb{P}(U_z) \subset G$ and $\sigma(\Gamma)$ could be zero which means that Γ is not semi-stable). We define Y by the condition that $(z,\Gamma) \in Y$ if and only if $\sigma(\Gamma)$ and $\rho(z)$ are linearly dependent. Let

$$Y^{ss} = Y \cap G_4(U \oplus U)^{ss} \text{ and } Y^{s} = Y \cap G_4(U \oplus U)^{s}.$$

We thee see that the natural map

$\dfrac{Y^{ss}}{SL(2)} \to \mathscr{C}(\mathbb{G})$ induces an isomorphism.

To get the Poncelet bundle over $\mathscr{C}(\mathbb{G})$, we look at the (relative) flag variety

$$X \subset G_4(U \oplus U) \times_Z G_2(U \oplus U)$$

consisting of (z, Γ, M) with $(z, \Gamma) \in Y$ and $M \subset \Gamma$. The morphism $X \to Y$ is a bundle with $G(2,4)$ as fibres. The G.I.T. quotient $\dfrac{X^{ss}}{SL(2)}$ is the Poncelet bundle Q associated to the 'universal' conic bundle on $\mathscr{C}(\mathbb{G})$; moreover, $Q \to \mathscr{C}(\mathbb{G})$ is induced by $X^{ss} \to Y^{ss}$. It turns out that the singular points of Q correspond exactly the non-stable points of X^{ss}.

6 $\mathscr{C}(\mathbb{G})$ as the moduli space of rank 4 'kernel' sheaves on $\mathbb{P}_3$.

436 The connection between the foregoing considerations and sheaves on $\mathbb{P}_3$ will be made by constructing, in the next section, a family of monads paramterised by X^{ss} whose 'cohomology' will give the required rank 2 semi-stable sheaves on $\mathbb{P}_3$.

In this section we outline the construction of (the family of) the two modules (and the 'arrow') on the right side of the monad. These are parameterised actually by Y^{ss} (The space $\mathscr{C}(\mathbb{G})$ is, in some sense, the moduli space of the kernel sheaves of the monads).

For doing this, it is convenient to study a G.I.T. parametrisation of the 'naive' compactification of the space reguli and relate it to $\mathscr{C}(\mathbb{G})$ which is a modification of it. Consider the action of $SL(2)$ on $G = G_2(V \oplus V)$. Then $\dfrac{G_2(V \oplus V)^{ss}}{SL(2)}$ gives the naive compactification. The canonical map $(V^* \oplus V^*) \otimes \mathscr{O}_g \to A^*$, where A is the tautological rank 2 bundle on G may be interpreted as a map

$$\alpha : \left(\overset{3}{\wedge} V \oplus \overset{3}{\wedge} V \right) \otimes \mathscr{O}_G \to \overset{4}{\wedge} V \otimes A^*$$

on observing that $\overset{3}{\wedge} V \cong V^* \otimes \overset{4}{\wedge} V$. Note also that we have a morphism of bundles

$$\left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right) \otimes O_G \to \overset{3}{\wedge} V \otimes A^*$$

The relation between $\mathscr{C}(\mathbb{G})$ and $\dfrac{G_2(V \oplus V)^{ss}}{SL(2)}$ comes from

Lemma. *In* $\Gamma \in G_4(U \oplus U) \subset G_4\left(\overset{2}{\wedge} V \oplus \wedge V\right)$*. Define* $N_\Gamma \subset V \oplus V$ *to be* 437
the subspace of all pairs (x, y), $x, y \in V$*, satisfying*

$$\xi \wedge x + \eta \wedge y = 0 \text{ for any } (\xi, \eta) \in \Gamma.$$

If $\Gamma \in Y^{ss}$*, then* N_Γ *is* 2*-dimensional.*

Using the lemma we define a morphism $Y^{ss} \to G_2(V \oplus V)^{ss}$ (which in turn defines the modification map $\mathscr{C}(\mathbb{G}) \to \dfrac{G_2(V \oplus V)^{ss}}{SL(2)}$. We consider the diagram

$$\begin{tikzcd}
Y^{ss} \arrow[r] \arrow[d, "q"] \arrow[rrr, bend left, "\gamma"] & G_4(U \oplus U) \arrow[r] & G_3\left(\overset{2}{\wedge} V\right) \times G_4\left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right) \arrow[r] & G_4\left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right) \\
G = G_2(V \oplus V) & A \arrow[l] & & C \arrow[u]
\end{tikzcd}$$

where A and C are the tautological subbundles on the grass mannians and

$$\gamma : Y^{ss} \to G_4\left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right)$$

the composite map.

We are now in a position to define the right hand side of the family of monads. Lift the map $\alpha\left(\overset{3}{\wedge} V \oplus \overset{3}{\wedge} V\right) \otimes \mathscr{O}_G \to \overset{4}{\wedge} V \otimes A^*$ (defined above) first to Y^{ss} by the morphism $Y^{ss} \to G$ and then to $\mathbb{P}_3 \times Y^{ss}$ by the 438
projection pr onto Y. We get an epimorphism

$$\left(\overset{3}{\wedge} V \oplus \overset{3}{\wedge} V\right) \otimes \mathscr{O}_{\mathbb{P}^3 \times Y^{ss}} \to \overset{4}{\wedge} V \otimes pr^* q^* A^*$$

Similarly, starting from the Koszul homomorphism

$$\overset{3}{\wedge} V \otimes \mathscr{O}_{\mathbb{P}_3} \to \overset{4}{\wedge} V \otimes \mathscr{O}(1) \text{ on } \mathbb{P}_3,$$

we get an epimorphism

$$\left(\overset{3}{\wedge} V \oplus \overset{3}{\wedge} V\right) \otimes \mathscr{O}_{\mathbb{P}_3 \times Y} \to \left(\overset{4}{\wedge} V \otimes \overset{4}{\wedge} V\right) \otimes \mathscr{O}_{\mathbb{P}_3 \times Y} \qquad (1)$$

Taking the sum of these homomorphisms, we obtain a homomorphism over $\mathbb{P}_3 \times Y^{ss}$:

$$\widetilde{\alpha} : \left(\overset{3}{\wedge} V \oplus \overset{3}{\wedge} V\right) \otimes \mathscr{O}_{\mathbb{P}_3 \times Y} \to \left(\overset{4}{\wedge} V \oplus \overset{4}{\wedge} V\right) \otimes$$
$$\mathscr{O}_{\mathbb{P}_3 \times Y}(1) \oplus \left(\overset{4}{\wedge} V \otimes pr^* q^* A^*\right).$$

This gives the required family of ‘partial’ monads.

Let $\mathscr{N} = \ker \alpha$ and $\mathscr{A} = Im\widetilde{\alpha}$. We also observe that under the natural map

$$pr^*\gamma^* C(-1) \to \left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right) \otimes \mathscr{O}_{\mathbb{P}_3}(-1) \to \left(\overset{3}{\wedge} V \oplus \overset{3}{\wedge} V\right) \otimes \mathscr{O}_{\mathbb{P}_3 \times Y},$$

439 $pr^*\gamma^* C(-1)$ injects into $\mathscr{N}$ and we define $\mathscr{G}$ by the exact sequence

$$(*) \quad 0 \to pr^*\gamma^* C(-1) \to \mathscr{N} \to \mathscr{G} \to 0.$$

We have

Proposition. (1) *The sheaves $\mathscr{N}$, $\mathscr{A}$ and $\mathscr{C}$ are flat over Y^{ss}*

(2) *For any $y = (z, \Gamma) \in Y^{ss}$, if $\mathscr{N}_y$ denotes the restriction of $\mathscr{N}$ to $\mathbb{P}_3 \times y$, then the natural homomorphism*

$$\Gamma \to H^0(\mathbb{P}_3, \mathscr{N}_y(1))(\text{ given by } (*))$$

is an isomorphism. Moreover

$$C_1(\mathscr{N}_y) = -2, C_2(\mathscr{N}_y) = 3, C_3(\mathscr{N}_y) = -4.$$

(3) *The sheavs $\mathscr{N}_y$ are semi-stable in the sense of Gieseker.*

7 Construction of a family of monads and the universal rank 2 sheaf on $\mathbb{P}_3$

We consider the diagram

$$\begin{tikzcd}
 & & & B \arrow[d] \\
X^{ss} \arrow[r] \arrow[d, "p"] \arrow[rr, bend left, "\beta_0"] \arrow[rrr, bend left=40, "\beta"] & Z \times F \arrow[r] \arrow[dr, "id\times p_4"'] & F \arrow[r] \arrow[dr, "p_4"] & G_2\left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right) \\
Y^{ss} \arrow[r] & G_4(U \oplus U) \arrow[r] & Z \times G_4\left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right) \arrow[r] & G_4\left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right) \\
 & & & C \arrow[u]
\end{tikzcd}$$

Here F denotes the flag manifold of pairs (Γ, M) with $M \subset \Gamma$ and p_2, p_4 **440**
are the natural projections. The map $X \to Z \times F$ is induced by

$$\begin{tikzcd}
G_4(U \oplus U) X_Z G_2(U \oplus U) \arrow[r] & Z \times G_4\left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right) \times G_2\left(\overset{2}{\wedge} V \oplus \overset{2}{\wedge} V\right) \\
x \arrow[u, "\leftarrow"'] \arrow[r] & Z \times F \arrow[u]
\end{tikzcd}$$

As before, B and C denote the tautological bundles. From the diagram, we get an exact sequence of vector bundles on F:

$$0 \to P_2^* B \to P_4^* C \to \overset{\vee}{B} \to 0.$$

Lifting via β_0 and $pr : \mathbb{P}_3 \times X^{ss} \to X^{ss}$, We obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & pr^*\beta^*B(-1) \arrow[r, equal] \arrow[d] & pr^*\beta^*B(-1) \arrow[d] & & \\
0 \arrow[r] & pr^*p^*\gamma^*C(-1) \arrow[r] \arrow[d] & P^*\mathscr{N} \arrow[r] \arrow[d] & p^*\mathscr{G} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & pr^*\beta^*B^\vee \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] & p^*\mathscr{G} \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

441 in which $\mathscr{F}$ is defined at the same time as a quotient of $p^*\mathscr{N}$ and as an extension of $p^*\mathscr{G}$ for each point $x = (z, \Gamma, M) \in X$ with $y = p(x) = (z, \Gamma)$, we obtain the induced diagram on the fibre $\mathbb{P}_3 = \mathbb{P}_3 \times x$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M \otimes \mathscr{O}(-1) \arrow[r, equal] \arrow[d] & M \otimes \mathscr{O}(-1) \arrow[d] & & \\
0 \arrow[r] & \Gamma \otimes \mathscr{O}(-1) \arrow[r] \arrow[d] & \mathscr{N}_y \arrow[r] \arrow[d] & \mathscr{G}_y \arrow[r] & 0 \\
0 \arrow[r] & \overset{\vee}{M} \otimes \mathscr{O}(-1) \arrow[r] \arrow[d] & \mathscr{F}_x \arrow[r] \arrow[d] & \mathscr{G}_y \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Proposition. (1) *For $x \in X^{ss}$, the sheaf $\mathscr{F}_x$ is a torsion free semi-stable sheaf with $C_1 = 0$, $C_2 = 2$ and $C_3 = 0$.*

(2) *For* $x, x' \in X^{ss}$, $\mathscr{F}_x$ *and* $\mathscr{F}_x$ *are* S*-equivalent if and only if* x *and* x' *are equivalent in the sense of G.I.T. i.e.* x, x' *have the same image in* $\frac{X^{ss}}{SL(2)}$.

8 Limit Sheaves

The 'limit sheaves' which occur in the compactification can be explicitly described. Such a description is also useful in carrying out some of the proofs. We give below some examples of limit sheaves. 442

Example 1. *Suppose that* σ *is the smooth conic in* G *given by the generators of a given regulus. The fibre over* x *of the compacification consists of, in addition to some elements of* $M(0,2)$, *sheaves which are obtained from the rank* 2 *trivial bundle on* $\mathscr{P}_3$ *or a null correlation bundle* ($C_1 = 0$, $C_2 b1$) *by a Hecke transformation. More precisely, the sheaves* $\mathscr{F}$ *are of the form*

$$
\begin{aligned}
&0 \to \mathscr{F} \to E \to 0_L(1) \to 0\\
&0 \to \mathscr{F} \to 2\mathscr{O} \to 0_{L_1}(1) \oplus 0_{L_2}(1) \to 0\\
&0 \to \mathscr{F} \to 2\mathscr{O} \to \mathscr{R} \to 0,
\end{aligned}
$$

where E is a null correlation bundle, L, L_1, L_2, are lines in $\mathbb{P}_3$ with $L_1 \cap L_2 = \emptyset$ and $\mathscr{R}$ is an extension of the form

$$0 \to 0_L(1) \to \mathscr{R} \to 0_L(1) \to 0.$$

The elements of $M(0,2)$ correspond to smooth conics which are Poncelet related to σ and the other cases correspond to the various subcases of degenerate conics which are Poncelet related to σ. All these sheaves are stable.

Example 2. *Let* S_1 *and* S_2 *be two planes in* $\mathbb{P}_3$ *intersecting along a line and let* p *and* q *be two distinct points on* $S_1 \cap S_2$. *Then considering the lines in* S_1 *passing through* p *and lines in* S_2 *through* q *we get a conic* 443 σ *in* G, *which is a pair of lines. Let* $\mathbb{P}$ *be the plane in* $\mathbb{P}\left(\overset{2}{\wedge} V\right)$ *in which* σ

is contained and let $\mathbb{P}^*$ *be the dual plane. Then* σ *determines two points* e, f *in* $\mathbb{P}^*$ *and the conics* γ *in* $\mathbb{P}^*$ *passing through either* e *or* f *are those which are Poncelet related to* σ. *Take for* γ *a pair of lines in* $\mathbb{P}^*$ *one passing through* e, *the other through* f *and intersecting outside* $e \cup f$. *Then* γ *defines lines* L *and* K *in* $\mathbb{P}_3$ *with* L *(resp* K*) contained in* S_1 *(resp* S_2*) and passing through* p *(resp* q*).*

Corresponding to γ we have in the compactification the sheaf $I_{LUq} \oplus I_{KUp}$, where I_{LIq} (resp I_{KUp}) denotes the ideal sheaf of LUq (resp KUp).

References

[1] R. Hartshorne. Stable vector bundle of rank 2 on $\mathbb{P}_3$, *Math. Ann.*, (238), 229–280 (1978).

[2] A. Hirschowitz and M. S. Narasimhan. Fibres de't Hooft speciaux et applications, in *Enumerative Geometry and Classical Algebraic Geometry*, Birkhauser 1982.

Tata Institute of Fundamental Research
Universitat Kaiserslautern.

The Witt Group Of A Real Surface

By M. Ojanguren and G. Ayoub

THIS IS A short report on some of the results obtained by the first 445
named author in his thesis(Université De Lausanne, 1984). Full details will appear elsewhere.

1 The Witt group of a scheme

Let X be a noetherian scheme with structure sheaf O_X. We assume that 2 is invertible in O_X. A *vector bundle* over X is, by definition, a locally free O_X-module of finite rank. A *quadratic space over* X is a pair (F, f) consisting of a vector bundle F and an isomorphism $f : F \rightarrow \overset{\vee}{F}$ of F into its dual such that $\overset{\vee}{f} = f$. Notice that we identify F with its double dual through the canonical isomorphism. If $X = \mathrm{Spec}A$ is an affine scheme, F can be identified with the projective A-module F of its global sections and f defines on F a symmetric bilinear scalar product $< x, y >= f(x)(y)$. In particular, if F is free with basis $e_1 \ldots, e_r$, f is defined by the invertible symmetric matrix $(< e_i, e_i >)$. Hence, if A is a field, a quadratic space over $\mathrm{Spec}A$ is the same as a non-degenerate symmetric bilinear form over K.

Given two quadratic spaces (F, f) and (G, g) over X, we define their *orthogonal sum* $(F, f) \perp (G, g)$ as $((F \oplus G, f \oplus g)$. An *isometry* of (F, f) into (G, g) is defined as a linear isomorphism $\phi : F \rightarrow G$ such that $\phi g \phi = f$. The set of isometries of (F, f) into itself is the *orthogonal group* $O(F, f)$ of F.

Given a vector bundle P over X we define a quadratic space $H(P)$

over X - *the hyperbolic space* associated to P- by

$$H(P) = \left(P \oplus \overset{\vee}{P} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

446 For any submodule L of F we define its orthogonal $L^{\perp}$ as the kernel of $\overset{\vee}{i} \circ f$, where $\overset{\vee}{i} : F \to L$ is the dual of the inclusion $i : L \to F$ and f is the isomorphism defining the quadratic structure on F. Clearly, the orthogonal of P in $H(P)$ is P itself. This motivates the following generalization of hyperbolic space. A space (F, f) is said to be *metabolic* if F contains a subbundle (not just a submodule!) L that coincides with its orthogonal: $L = L^{\perp}$. In this case, L is called a *lagrangian* of (F, f). If X is affine, a metabolic space with lagrangian L is isometric to $H(L)$, but in general, a metabolic space need not be hyperbolic.

The category of quadratic spaces over X is a category with "product" in the sense of [3, p. 344]. Let $G(X)$ be its Grothendieck group [3, p. 346]. The Witt group of X is the quotient $W(X)$ of $G(X)$ by the subgroup generated by the classes of metabolic spaces. If $X = \mathrm{Spec} A$, we put $W(A) = W(X)$. If A is a field, $W(A)$ is the group of anisotropic quadratic forms over K introduced by E. Witt in 1937 [16]. The Witt group of a scheme has been defined more recently by M. Knebusch. His lectures at the Queen's conference on a quadratic forms are an excellent introduction to the subject [10].

It is easily checked that the tensor product $(F, f) \otimes (G, g) = (F \otimes G f \otimes g)$ induces a ring structure on $W(X)$. We shall use this fact in § 2, but our main interest here is the group structure of $W(X)$.

It is important to observe that, for any morphism of schemes $\phi : Y \to$
447 X and any quadratic space (F, f) over X, the inverse image $(\phi^* F, \phi^* f)$ is a quadratic space over Y. Furthermore, ϕ^* induces a functorial ring homomorphism $W(\phi) : W(X) \to W(Y)$. Hence W is a contravariant functor from the category of the schemes under consideration to the category of commutative rings. In particular, if X is reduced and irreducible, there is a canonical morphism $W(X) \to W(K)$, where K is the field of rational functions of X. And if X is a k-scheme, k a ring, there is a canonical homomorphism $W(k) \to W(X)$.

Examples. If k is a field (of characteristic $\neq 2$) and $\mathbb{A}^n_k$, $\mathbb{P}^n_k$ are, respectively, the n-dimensional affine and the n-dimensional projective space over k, then

$$W\left(\mathbb{A}^n_k\right) = W(k) = W\left(\mathbb{P}^n_k\right).$$

The first equality is due to karoubi [9]. the second one to Arason [1].

In [11], Knebusch asked if the Witt group of a finitely generated 448 R-scheme is finitely generated. For one-dimensional schemes this is indeed the case, as shown by Knebusch himself in the smooth case and by Dietel [5] in general. In his thesis, the first named author has proved the following results.

Main Theorem. *Let X be an affine real surface and X the normalization of X_{red}. If the cokernel of the canonical homomorphism* $\mathrm{Pic} X \to \mathrm{Pic}\ \overline{X}$ *is finitely generated, then the Witt group of X is finitely generated.*

Corollary. *The Witt group of a normal real surface is finitely generated.*

Counter examples. There are real surfaces for which $W(X)$ is not finitely generated. An easy example is the surface of $\mathbb{R}^3$ defined by the equation $z^2 = x^2 f(y)$, where f is a square-free polynomial of degree at least 3.

Most probably, the main theorem can be extended to quasi-projective surfaces.

2 The classical invariants

We describe how the classical invariants of quadratic forms over fields have been extended to quadratic spaces over schemes. They will prove very useful for the study of the Witt group of a real surface.

If a scheme X is the disjoint union of two closed subschemes X_1, X_2, its Witt ring is simply $W(X_1) \times W(X_2)$. Hence we may (and do) assume that X is connected. In particular, every vector bundle over X has a well defined rank.

(a) The rank homomorphism

The rank of a metabolic space is even because a metabolic space is locally hyperbolic. Hence, the parity of the rank of F only depends on the class of (F, f) in $W(X)$ and yields a ring homomorphism

$$\rho : W(X) \to \frac{\mathbb{Z}}{2}$$

called the *rank*.

Example. ρ induces an isomorphism

$$\rho : W(\mathbb{C}) \xrightarrow{\sim} \frac{\mathbb{Z}}{2}$$

449 We denote by $I(X)$ the kernel of ρ: it is the *ideal of even rank spaces*.

(b) The signatures

Let K be a field. An *ordering* of K is a subset P satisfying $P+P \subset P$, $PP \subset P$ and such that K be the disjoint union of P, $-P$ and $\{0\}$.

Let (V, f) be a quadratic space over K. We can choose an orthogonal basis $e_1, \ldots, e_r$ of V. Given any ordering P of K, we define the signature of (V, f) with respect to P as $\sigma_P(V, f) =$ (number of $< e_i, e_i >$ in$P$) $-$ (number of $< e_i, e_i >$ in $-P$). By a well known-theorem of Sylvester, $\sigma_P(V, f)$ does not depend on the choice of the orthogonal basis. The signature of a hyperbolic space being zero, σ_P defines a surjective ring homomorphism

$$\sigma_P : W(K) \to \mathbb{Z}$$

Example. The field $\mathbb{R}$ of real numbers has only one ordering given by P=set of non-zero squares. The homomorphism that it defines is in fact an isomorphism

$$W(\mathbb{R}) \xrightarrow{\sim} \mathbb{Z}.$$

It can be shown that, conversely, every surjective homomorphism $\rho : W(K) \to \mathbb{Z}$ coincides with σ_P for a uniquely defined ordering P of K. We therefore extend the notion of signature to any scheme X by saying that a *signature* of K is surjective ring homomorphism

$$\sigma : W(X) \to \mathbb{Z}.$$

450 Assume now that X is a real quasi-projective variety and let $X(\mathbb{R})$ denote the set of real (closed) points of X. For any $x \in X(\mathbb{R})$, the residue field $k(x)$ is $\mathbb{R}$, hence the canonical homomorphism

$$\sigma_X : W(X) \to W(k(x)) = W(\mathbb{R}) = \mathbb{Z}$$

is a signature of X. Clearly, σ_X only depends on the connected component of $X(\mathbb{R})$ containing x. Since $X(\mathbb{R})$ has only a finite set of connected components, we obtain in this way only a finite number of signatures. A very useful theorem of Knebusch [10] asserts that every signature of X is a σ_x, $x \in X(\mathbb{R})$. In particular, a real variety has only finitely many signatures (and possibly none).

A word of caution may be appropriate. Assume that X is integral and let $\mathbb{R}(X)$ be its field of rational functions. Every signature of $\mathbb{R}(X)$ defines, by composition with $W(X) \to W(\mathbb{R}(X))$, a signature of X. But, in general, not every signature of X arises in this way. For example, if $X = \operatorname{Spec}\left(\dfrac{R[T_1, T_2]}{\left(T_1^2 + T_2^2\right)}\right)$, $X(\mathbb{R})$ consists of one point, hence X has one signature, whereas $\mathbb{R}(X)$ has no ordering because it contains an element of square -1. On the other hand, in spite of the fact that X has only finitely many signatures, $\mathbb{R}(X)$ may have infinitely many. This happens, for instance, if $X = \mathbb{A}^n_{\mathbb{R}}$.

(c) The discriminant

The tensor product of two quadratic spaces of rank 1 is again a quadratic space of rank 1 and the square of such a space is isometric to the "unit space" (O_X, id). Hence the set of isometry classes of rank 1 quadratic spaces has a natural structure of abelian group of exponent 2. 451 We denote it by $Q(X)$.

Let (F, f) be a quadratic space over X, of rank r. The r-th exterior power $\overset{r}{\wedge} f$ maps $\overset{r}{\wedge} F$ to $\overset{r}{\wedge} \overset{\vee}{F}$. We identify $\overset{r}{\wedge} \overset{\vee}{F}$ with $\left(\overset{r}{\wedge} F\right)^{\vee}$, so that $d(F, f) = \left(\overset{r}{\wedge} F, (-1)^{r(r-1)/2} \overset{r}{\wedge} f\right)$ becomes a quadratic space of rank 1. The discriminant of (F, f) is, by definition, the class of $d(F, f)$ in $Q(X)$. The discriminant of a metabolic space is trivial but, in general, d does not define a homomorphism from $W(X)$ to $Q(X)$. Only its restriction to the ideal of even rank spaces gives a group homomorphism

$$\delta : I(X) \to Q(X),$$

which we call the *discriminant*.

Any global function α on X gives rise to a rank one quadratic space (O_X, α). On the other hand, to any rank one quadratic space (I, q) we can associate the isomorphism class of the self-dual line bundle I in Pic$X$. This yields an exact sequence

$$1 \to \frac{\Gamma(X)^{\bullet}}{(\Gamma(X)^{\bullet})^2} \to Q(X) \xrightarrow{\pi}{}_2 \mathrm{Pic}X \to 0,$$

where $\Gamma(X)$ is the ring of global sections of O_X and, for any ring $\wedge$, $\wedge$ denotes its group of units.

Clearly $\pi\delta$ depends only on the linear structure of the quadratic space over X and is a homomorphism on the whole of $W(X)$.

452 **Example.** Let $X = \mathrm{Spec}\left(\dfrac{\mathbb{R}[T_1, T_2]}{\left(T_1^2 + T_2^2 - 1\right)}\right)$. The set of real points of X is a circle, hence X has only one signature σ. It is easy to see that $\mathrm{Pic}X =_2 \mathrm{Pic}X \cong \mathbb{Z}/2$ and it can be shown that

$$\sigma \oplus \pi\delta : W(X) \to \mathbb{Z} \oplus \frac{\mathbb{Z}}{2}$$

is a group isomorphism.

(d) The Clifford invariant

Consider first an affine scheme $X = \mathrm{Spec}A$. As we remarked in § 1, a quadratic space on X is the same as a pair (F, f) where F is a projective A-module of finite type and f a symmetric isomorphism of F into its dual. Let $<, >$ denote the associated bilinear product. The Clifford algebra $C(F, f)$ of (F, f) is the quotient of the tensor algebra $T = A \oplus F \oplus F \oplus F \oplus \ldots$ by its two-sided ideal generated by all elements $x \otimes x - < x, x >, x \in F$. We denote by $i : F \to C(F, f)$ the A-linear map induced by the canonical injection $F \to T$. The pair $(C(F, f), i)$ is uniquely defined by the following universal property: given any A-algebra $\wedge$ and A-linear map ϕ: $F \to \wedge$ such that $\phi(x)^2 =< x, x > 1_\wedge$, there exists a unique homomorphism $\Phi : C(F, f) \to \wedge$ of A-algebras such that $\phi = \Phi \circ i$

The construction of $C(F, f)$ commutes with scalar extensions. Assume now that the rank of F is an even integer $2r$. Then, locally for the étale topology, (F, f) is isometric to $H(A^r)$. A direct computation shows that $C(H(A^r))$. A direct computation shows that $C(H(A^r)) \simeq M_{2r}(A)$; hence, locally for the étale topology, $C(F, f)$ is a matrix algebra. This shows that $C(F, f)$ is an Azumaya algebra.

For a general (non affine) scheme X, the universal property of $C(F, f)$ allows to patch the various $C(F \mid U, fU)$ over a covering consisting 453 of affine open sets U. This patching yields an Azumaya algebra $C(E, f)$ over X. We denote by $\gamma(F, f)$ the class of $C(F, f)$ in the Brauer group of X. Notice that the opposite algebra of $C(F, f)$ satisfies the same universal property as $C(F, f)$ and is therefore canonically isomorphic to $C(F, f)$. From this it follows that $C(F, f) \simeq C(F, f)^{opp}$, and hence $\gamma(E, f) \in_2 Br(X)$.

In general, γ does not define a homomorphism from $W(X)$ to ${}_2Br(X)$, but if we restrict γ to the subgroup $J(X) = \ker(I(X) \xrightarrow{\delta} Q(X))$, we get indeed a group homomorphism

$$\gamma : J(X) \to_2 Br(X).$$

We call γ the *Clifford invariant.*

Example. Let $X = \operatorname{Spec}\left(\dfrac{\mathbb{R}[T_1, T_2, T_3]}{\left(T_1^2 + T_2^2 + T_3^2 - 1\right)}\right)$. Since $X(\mathbb{R})$ is the real sphere, X has only one signature σ. It is well known that $\operatorname{Pic}X = 0$, hence $Q(X) = Q(\mathbb{R}) = \dfrac{\dot{\mathbb{R}}}{\dot{\mathbb{R}}^2} \xrightarrow{\sim} \dfrac{\mathbb{Z}}{2}$. Choose now a real point $x : \operatorname{Spec}\mathbb{R} \to X$ on X. It defines a homomorphism $W(x) : W(X) \to W(R)$. From the commutative diagram

$$\begin{tikzcd}
W(X) \arrow[r, "\sigma"] \arrow[d, "W(x)"'] & \mathbb{Z} \arrow[d, equal] \\
W(\mathbb{R}) \arrow[r, "\sim"] & \mathbb{Z}
\end{tikzcd}$$

and from $Q(X) = Q(\mathbb{R})$ it follows that $\ker\sigma = J(X)$. Hence γ defines a homomorphism $\ker\sigma \rightarrow_2 BrX$. It can be shown that

454 $BrX = \frac{\mathbb{Z}}{2} \oplus \frac{\mathbb{Z}}{2}$, where one copy of $\frac{\mathbb{Z}}{2}$ is represented by the usual (constant) quaternion algebra over X, the other copy being the kernel of $Br(x) : Br(X) \rightarrow Br(\mathbb{R})$. An explicit construction of an algebra representing this kernel shows that γ maps $J(X)$ isomorphically onto $\frac{\mathbb{Z}}{2}$. Hence $W(X) = \mathbb{Z} \oplus \frac{\mathbb{Z}}{2}$.

3 Regular affine surfaces

We assume here that $X = \mathrm{Spec}A$ and that A is a regular 2-dimensional integral affine algebra over $\mathbb{R}$. We want to show that $W(X)$ is a finitely generated group. Let K be the field of fractions of A. We recall a result proved in [13] and, earlier but [4] independently, by W. Pardon.

Theorem 1. *Let A be a regular 2-dimensional domain in which 2 is invertible and K its fields of fractions. The canonical homomorphism $W(A) \rightarrow W(K)$ is injective.*

A similar result holds for 3-dimensional regular domains, but not in dimension 4. It should not be too difficult to prove the analogous statement for smooth quasi-projective surfaces.

We also need the following theorem of Elman and Lam [6].

Theorem 2. *Let K be a field of transcendence degree at most 2 over $\mathbb{R}$. Then every element of $W(K)$ is determined by its classical invariants.*

Let now $X = \mathrm{Spec}A$ be an affine smooth real surface. It clearly suffices to show that $I(X)$ is finitely generated.

Any signature $\sigma : W(K) \to \mathbb{Z}$ restricts to a signature $\sigma_i : W(X) \to \mathbb{Z}$. Since the number of signatures of X is finite, there is a finite num- 455
ber of different σ_i which, together with the discriminant, give rise to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N(X) \arrow[r] \arrow[d] & I(X) \arrow[r, "\pi\sigma_i\times\delta"] \arrow[d] & \mathbb{Z}X\ldots X\mathbb{Z}\times Q(X) \arrow[d] \\
0 \arrow[r] & N(K) \arrow[r] & I(K) \arrow[r, "\pi\sigma\times\delta"'] & \pi\mathbb{Z}\times Q(K)
\end{tikzcd}$$

where the products in the bottom line extend over all the signatures of K.

We show that $Q(X)$ and ${}_2Br(X)$ are finite groups. From the exact sequence of étale sheaves

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{(-)^2} \mathbb{G}_m \to 1,$$

we get a cohomology exact sequence

$$\ldots \to H^i(X,\mathbb{G}_m)^2 \to H^i(X,\mathbb{G}_m) \to H^{i+1}(X,\mu_2) \to H^{i+1}(X,\mathbb{G}_m) \to \ldots$$

For $i = 0$, this is the sequence relating $Q(X)$ to $\dfrac{\Gamma(X)}{(\Gamma(X)^{\cdot})^2}$ and ${}_2\mathrm{Pic}X$, hence $Q(X) = H^1(X,\mu_2)$. For $i = 1$, this sequence shows that ${}_2Br(X)$ is a quotient of $H^2(X,\mu_2)$. Now, for a smooth real quasi-projective variety X, the groups $H^i(X,\mu_2)$ are finite. This can be shown as suggested in [12, p. 244], although the theorem stated there is obviously false for arbitrary ground fields. By the finiteness of $Q(X)$, we are reduced to show that $N(X)$ is finitely generated. In the commutative diagram

$$\begin{tikzcd}
N(X) \arrow[r, "\gamma"] \arrow[d] & {}_2Br(X) \arrow[d] \\
N(K) \arrow[r, "\gamma"] & {}_2Br(K)
\end{tikzcd}$$

the vertical map on the left is injective, by Theorem1 and the bottom 456
map is injective, by Theorem 2; hence the top map is injective as well. Since ${}_2Br(X)$ is finite, $N(X)$ is finite and $W(X)$ is finitely generated.

4 Normal affine surfaces

Assume now that $X = \mathrm{Spec}A$, where A is an integrally closed affine domain over $\mathbb{R}$. As before, let K be the field of fractions of A. In general, for normal surfaces, $W(X) \to W(K)$ is not injective.

Example. $A = \dfrac{[T_1, T_2, T_3]}{\left(T_1^2 + T_2^2 + T_3^2\right)} = \mathbb{R}[t_1, t_2, t_3]$. In this case, $X(\mathbb{R})$ consists of one point, hence X has exactly one signature σ. The quadratic space (A^4, id) represents a non-zero class ξ in $W(X)$ because $\sigma(A^4, id) = 4$. The image of ξ in $W(K)$ is the class of $\left(K^4, id\right)$. Now, in K we have $\left(\dfrac{t_1}{t_3}\right)^2 + \left(\dfrac{t_2}{t_3}\right)^2 = -1$, hence $\left(K^4, id\right)$ is isotropic and splits as $H(K) \perp (V, f)$. But the discriminant of (V, f) must be 1, hence $(V, f) \cong H(K)$ and $\left(K^4, id\right)$ is hyperbolic. This shows that ξ maps to zero in $W(A)$.

Instead of Theorem 1, we shall use the following result, proved in [13].

Theorem 3. *Let A be a normal 2-dimensional domain, K its field of* 457 *fractions. Let ξ be an element of $W(A)$. Assume that ξ is in the kernel of $W(A) \to W(K)$ and also in the kernel of $W(A) \to W\left(\dfrac{A}{m}\right)$ for every singular maximal ideal m of A. Then, $\gamma(\xi) = 0$ implies $\xi = 0$.*

Corollary. *Let A be a normal 2-dimensional affine algebra over $\mathbb{R}$, $\sigma_1, \ldots\ldots \sigma_n$ the signatures of A and $N(X)$ the kernel of the homomorphism*

$$\sigma_1 \times \ldots \times \sigma_n \times \delta : I(X) \to \mathbb{Z} \times \ldots \times \mathbb{Z} \times Q(X)$$

Then, $\ker(N(X) \to W(K)) \subset \ker \mid ({}_2Br(X) \to_2 Br(K))$.

In fact, $N(X) \subset J(X)$, hence γ defines a homomorphism $\gamma : N(X) \to_2 Br(X)$. Since $\sigma_i(\xi) = 0$ for every $\sigma \mid_i$, for any maximal ideal m of A, the image of ξ in $W\left(\dfrac{A}{m}\right) = W(\mathbb{R})$ is zero. On the other hand, if m is not real, $W\left(\dfrac{A}{m}\right) = W(\mathbb{C}) = \dfrac{\mathbb{Z}}{2}$, hence $\xi \in I(X)$ maps to $I\left(\dfrac{A}{m}\right) = 0$

This shows that ξ maps to zero in every $W\left(\frac{A}{m}\right)$. By the theorem above, γ is injective on $\ker(N(X) \to W(K))$.

As in § 3, we first show that $q(X)$ is finite. If $k(= \mathbb{R}$ or $= \mathbb{C})$ is the algebraic closure of $\mathbb{R}$ in K, by [15] the quotient $\frac{A^{\bullet}}{k^{\bullet}}$ is finitely generated. Hence $\frac{A}{(k^{\bullet})^2}$ is also finitely generated and its quotient $\frac{A^{\bullet}}{A^{\bullet 2}}$ is finite. To show that ${}_2\mathrm{Pic}X$ is finite, we consider the open set U of all regular points of X. The group Pic$X$ is a subgroup of the divisor class group of X, which in fact coincides with Pic$U$. Since ${}_2$ Pic$U = H^1(U, \mu_2)$ is finite, ${}_2\mathrm{Pic}X$ is finite. The exact sequence connecting $Q(X)$, $\frac{A^{\bullet}}{A^{\bullet 2}}$ and ${}_2\mathrm{Pic}X$ shows that $Q(X)$ is finite. Proceeding as in §3. we are reduced to showing that $N(X)$ is finitely generated. Let U be any affine open set of X consisting of smooth points. By §3, $W(U)$ is finitely generated, hence it suffices to show that $\ker(N(X) \to W(U))$ is finitely generated. Clearly, this **458**
kernel is contained in $\ker(N(X) \to W(K)) \subset \ker({}_2Br(X) \to_2 Br(K))$. Hence we are reduced to showing that $\ker({}_2Br(X) \to_2 Br(K))$ is finite. According to a result of Grothendieck [7, p.74], this last group is contained in

$$\bigoplus_{x \in S\,2} \left(\frac{C\ell\left(A_x^{bs}\right)}{C\ell(A_x)} \right)$$

where $C\ell$ denotes the divisor class group, S denotes the (finite) set of singular points of X and A_x^{bs} is the strict henselization of the local ring A_x of X at x.

Consider the scalar extension $B = \mathbb{C} \bigoplus_{\mathbb{R}} A$ and put $Y = \mathrm{Spec}B$. Then $A_x^{bs} = B_y^b$, the henselization of B_y, where y is a preimage of x in y. By a general approximation theorem of Hironaka [8, p. 214], $C\ell\left(B_y^{bol}\right) = C\ell\left(\widehat{B}_y\right)$ where B_y^{bol} is the local ring of holomorphic functions of the complex variety associated to Y and $\widehat{B}_y$ is the completion of B_y. The group $C\ell\left(B_y^{hol}\right)$ has been computed by Prill [14] and turns out to be of the form $P \oplus \left(\frac{\mathbb{Q}}{\mathbb{Z}}\right)^m \oplus \mathbb{Q}^c$ where P is a finitely generated group, m is a finite integer and c is either zero or the cardinality of $\mathbb{R}$. On the other hand, $C\ell(A_y)$ is the direct sum of finitely generated group and a divisible group. From these facts, it follows easily that$_2\left(\frac{C\ell\left(B_y^{bs}\right)}{C\ell(A_x)}\right)$ is finite.

5 The general case

We now consider the case of an arbitrary real affine algebra A of dimension 2. Since $W(A)$ is the same as $W(A_{\text{red}})$, we assume that A is reduced. Let K be its total ring of fractions and $\overline{A}$ the integral closure of A in K. 459 Since $\overline{A}$ is a finite product of normal domains, it follows from the results of § 4, that $W(\overline{A})$ is finitely generated. Let $\mathfrak{c}$ denote the conductor of $\overline{A}$ in A. We consider the cartesian diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \overline{A} \arrow[d] \\
\frac{A}{\mathfrak{c}} \arrow[r] & \frac{\overline{A}}{\mathfrak{c}}
\end{tikzcd}$$

and the corresponding Mayer-Vietoris sequence of Grothendieck-Witt groups

$$\ldots \to KO_1\left(\frac{A}{\mathfrak{c}}\right) \to KO(A) \to\to KO\left(\overline{A}\right) \times KO\left(\frac{A}{\mathfrak{c}}\right) \to \ldots .$$

Here, for any scheme, X, $KO(X)$ is the quotient of the Grothendieck group of the category of quadratic spaces over X, modulo the subgroup generated by the difference $(F, f) - H(L)$, where L is a lagrangian of the (metabolic) space (F, f). A reasonable definition of KO_1 seems to be known only for an affine scheme $X = \mathrm{Spec}R : KO_1(X) = KO_1(R) = \frac{O(R)}{[O(R), O(R)]}$, where $O(R)$ denotes the inductive limit of the orthogonal groups $O_{2n}(R)$ of the spaces $H(R^n)$. Associating to every bundle P over X the hyperbolic space $H(P)$ yields an exact sequence

$$K_0(X) \xrightarrow{H} KO(X) \to W(X) \to 0$$

which, combined with the Mayer-Vietoris sequence, gives a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & K_0(A) \arrow[r] \arrow[d] & K_0\left(\overline{A}\right) \arrow[d] & \times & K_0\left(\dfrac{A}{\mathfrak{c}}\right) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & KO(A) \arrow[r] \arrow[d] & KO\left(\overline{A}\right) \arrow[d] & \times & KO\left(\dfrac{A}{\mathfrak{c}}\right) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & W(A) \arrow[r] \arrow[d] & W\left(\overline{A}\right) \arrow[d] & \times & W\left(\dfrac{A}{\mathfrak{c}}\right) \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & & 0 &
\end{tikzcd}$$

A careful analysis of this diagram leads to a proof of the main theorem. **460**
Although the details are rather tricky, the gist of the argument can be sketched as follows: we know that $W\left(\overline{A}\right)$ and $W\left(\frac{A}{\mathfrak{c}}\right)$ are finitely generated; hence we only have to show that the kernel $N(A)$ of $W(A) \to W\left(\overline{A}\right) \times W\left(\frac{A}{\mathfrak{c}}\right)$ is finitely generated. A class in this kernel comes from an element ξ in $KO(A)$ which becomes hyperbolic in $KO\left(\overline{A}\right)$ and in $KO\left(\frac{A}{\mathfrak{c}}\right)$. In general, $\xi \notin M(A) = \ker(KO(A)) \to KO\left(\overline{A}\right) \times KO\left(\frac{A}{\mathfrak{c}}\right)$, but the condition that coker $(\mathrm{Pic} A \to \mathrm{Pic} \overline{A})$ be finitely generated is precisely what is needed to reduce the finite generation of $N(A)$ to that of $M(A)$. To show that $M(A)$ is finitely generated, we consider any $\xi \in M(A)$. By a quadratic analogue of Serre's theorem on projective modules [13], we may assume that ξ is represented by a space (F, f) such that $(F, f) \otimes_A \overline{A} \cong H\left(\overline{A}^2\right)$ and $(F, f) \otimes_A \frac{A}{\mathfrak{c}} \cong H\left(\frac{A}{\mathfrak{c}}^2\right)$. This means that ξ is a "Minor patching" of $\overline{H}(A^2)$ and $H\left(\frac{A}{\mathfrak{c}}^2\right)$ over $\frac{A}{\mathfrak{c}}$ via an isometry of $H\left(\left(\frac{\overline{A}}{\mathfrak{c}}\right)^2\right)$. Denote $\frac{A}{\mathfrak{c}}$ by B and let $S(B)$ be the subgroup of $GL_2(B) \times GL_2(B)$ consisting of the pairs (α, β) with $\det\alpha = \det\beta$. We identify $H(B^2)$ with $(M_2(B), \det)$ and associate to $(\alpha, \beta) \in S(B)$ the

461 isometry sending $\xi \in M_2(B)$ to $\alpha\xi\beta^{-1}$. This gives a group homomorphism $S(B) \to O_4(B)$. It follows from results of Bass [2], that the image of $S(B)$ is a normal subgroup of finite index in $O_4(B)$. On the other, hand, it is easy to check that patching $H\left(\overline{A}^2\right)$ and $H\left(\left(\frac{A}{\mathfrak{c}}\right)^2\right)$ via a matrix coming from $S(B)$ yield a stably hyperbolic space (F, f) over A. Hence, upto Witt equivalence, there are only a finite number of such ξ and $M(A)$ is indeed finitely generated.

Acknowledgement. We thank W. Pardon for suggesting to us that the results of Elman and Lam could be used for our problem.

References

[1] J. Arason, Der Wittring projektiver Räume, *Math, Ann.*, 253 (1980), 205–212.

[2] H. Bass, *Lectures on topics in algebraic K-theory*, Tata Institute of Fundamental Research, Bombay, 1961.

[3] H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968.

[4] J. L. Borges, El tiempo circular, *Obras completas*, Ultramar, Madrid, 1977.

[5] G. Dietel, Wittringe reeller Kurven, I,II, *Commun. Algebra* 11(1983), 2393–2494.

462 [6] R. Elman and T.Y. Lam, Classification theorems for quadratic forms over fields, *Comm. Math. Helv.*, 49(1974), 373–381.

[7] A. Grothendieck, Le groupe de Brauer, I, II, III, in *Dix exposés sur la cohomologie des schémas*, North Holland Amsterdam, 1968.

[8] H. Hironaka, Formal line bundles along exceptional loci, in *Algebraic Geometry*, Bombay Colloquim 1968, Bombay, 1969.

[9] M. Karoubi, Localisation des formes quadratiques II, *Ann.Sci. Ec Norm. Sup.*, 8(1975), 99–155.

[10] M. Knebusch, Symmetric bilinear forms over algebraic varieties, *Queen's Papers in pure and Appl. Math.*, 46(1977), 103–283.

[11] M. Knebusch, Some open problems, *Queen's Papers in pure and Appl. Math.*, 46(1977),

[12] J. S. Milne, *Etale cohomology*, Princeton, 1980.

[13] M. Ojanguren, A splitting theorem for quadratic forms, *Comm. Math. Helv.*, 57(1982), 145–157.

[14] D. Prill, The divisor class group of some rings of holomorphic functions, *Math, Zeit.*, 121 (1971), 58–80.

[15] P. Samuel, A propos du théorème des unités, *Bull. Sc. Math.*, 90(1966), 89–96.

[16] E. Witt, Theorie der quadratischen Former in beliebigen Körpern, **463** *Journal Reine Angew. Math.*, 176(1937), 31–44.

Institut de Mathematiques
Faculté des Sciences
Université de Lausanne
1015 Lausanne - Dorigny
Switzerland.

Anisotropic Quadratic Spaces Over The Plane

By M. Ojanguren, R. Parimala and R. Sridharan

Introduction

Let K be a field of characteristic $\neq 2$. It is well-known that any quadratic **465** space over the affine line $\mathbb{A}^1_K$ is extended from K. It was proved in [7] that any *isotropic* quadratic space over $\mathbb{A}^n_K$ is extended from K for all n. However, it $n \geq 2$, there do exist, in general, anisotropic quadratic spaces over $\mathbb{A}^n_K$ which are not extended from K ([5], [9]). In [8], we constructed positive definite quadratic spaces of arbitrarily large ranks over the real affine plane $\mathbb{A}^2_{\mathbb{R}}$. More generally, if K is any field which admits of an anisotropic quadratic space q_0 of rank $\geqq 3$, then there exist [10] indecomposable quadratic spaces over $\mathbb{A}^2_K$, with q_o as the "form on the fibre".

The aim of this paper is to give a classification of anisotropic quadratic spaces over $\mathbb{A}^2_K$ in terms of linear algebraic data. Our method is to exploit a theorem [6, theorem 2.1], which reduces the classification problem to that of quadratic spaces over the projective plane and then use suitable adaptations of the work of Barth-Hulek ([1], [2], [4]). Using this classification, we show that the set of isometry classes of indecomposable quadratic spaces of second Chern class 4 (see § 2 for definition) over $\mathbb{A}^2_K$ with a given form q_o on the fibre is in bijection with the orbit of an anisotropic conic C_o over K.

For vector bundles over $\mathbb{P}^2$, we follow generally the definitions and noatation of Barth-Hulek [2] and Hulek [4].

1 Classification of Quadratic spaces over $\mathbb{P}^2_K$

466 We recall, in this section, some results of Barth-Hulek regarding the classification of quadratic spaces over $\mathbb{P}^2_K$ in terms of linear algebra ([2], [4]). We remark that though these results are stated by them only for the field of complex numbers, they are valid for any field. Throughout this section, we denote by K a field of characteristic $\neq 2$.

By a *skew-symmetric, pre-stable triple* α we mean a triple $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, $\alpha_i \in M_n(K)$, $\alpha_i^t = -\alpha_i$ such that for any $v \in K^n$, $V \neq 0$, the subspace of K^n spanned by $\{\alpha_0 v, \alpha_1 v, \alpha_2 v\}$ is at least two-dimensional. Two pre-stable triples $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $\beta = (\beta_0, \beta_1, \beta_2)$ are said to be equivalent, if there exists $u \in GL_n(K)$ such that $\alpha_i = u\beta_i u^t$, $0 \leq i \leq 2$. We denote the set of equivalence classes of such triples by $\mathscr{K}(n)$. Given any skew-symmetric, pre-stable triple α, we shall associate to it a quadratic space over $\mathbb{P}^2_K$.

Let $\mathcal{O}$ be the structure sheaf of $\mathbb{P}^2_K$ and let Z_0, Z_1, Z_2 be the homogeneous coordinates, so that $\Gamma(\mathcal{O}(1))$ has Z_0, Z_1, Z_2 as a basis. Let $V = (\Gamma(\mathcal{O}(1))^*$ and let $\{v_0, v_1, v_2\}$ be the basis of V, dual to the basis $\{Z_0, Z_1, Z_2\}$. We define a linear map $A(\alpha) : K^n \otimes V \to K^n \otimes V^*$ by $A(\alpha)(\varphi \otimes v_i) = \alpha_{i+1}(\varphi) \otimes Z_{i\;1} - \alpha_{i-1}(\varphi) \otimes Z_i$, where $i = 0, 1, 2 \pmod 3$.

For the choice of the canonical basis of K^n. and the basis for V and V^* defined above, the matrix of $A(\alpha)$ is given by

$$\begin{pmatrix} 0 & \alpha_2 & -\alpha_1 \\ -\alpha_2 & 0 & \alpha_0 \\ \alpha_1 & -\alpha_0 & 0 \end{pmatrix}$$

467 Let U denote the image of $A(\alpha)$. Let $a : K^n \otimes \mathcal{O}(-1) \to U \otimes \mathcal{O}$ be the composite $K^n \otimes \mathcal{O}(-1) \xrightarrow{1\otimes s} K^n \otimes V \otimes \mathcal{O} \xrightarrow{A(\alpha)\otimes 1} U \otimes \mathcal{O}$, and $c : U \otimes \mathcal{O} \to K^n \otimes \mathcal{O}(1)$, the restriction to $U \otimes \mathcal{O}$ of the map $K^n \otimes V^* \otimes \mathcal{O} \xrightarrow{1\otimes s^t} K^n \otimes \mathcal{O}(1)$, where $s : \mathcal{O}(-1) \to V \otimes \mathcal{O}$ is induced by the multiplication $\Gamma(\mathcal{O}(1)) \otimes \mathcal{O}(-1) \to \mathcal{O}$ and s^t denotes the transpose of s.

We have a self-dual monad ([2, p.10])

$$\begin{tikzcd}
M(\alpha) : K^n \otimes \mathfrak{o}(-1) \arrow[r, "a"] \arrow[d, equal] & U \otimes \mathfrak{o} \arrow[r, "c"] \arrow[d, "\varphi"'] & K^n \otimes \mathfrak{o}(1) \arrow[d, equal] \\
M(\alpha)^* : K^n \otimes \mathfrak{o}(-1) \arrow[r, "c^t"] & U^* \otimes \mathfrak{o} \arrow[r, "a^t"] & K^n \otimes \mathfrak{o}(1),
\end{tikzcd}$$

the map $\varphi : U \xrightarrow{\sim} U^*$ being defined by $\varphi(x) = i^t(y)$, where $i : U \to K^n \otimes V^*$ denotes the canonical inclusion and y is any preimage of x under $A(\alpha)$. Let $\mathscr{E}(\alpha)$ denote the cohomology of the monad $M(\alpha)$. The isomorphism φ of $M(\alpha)$ with $M(\alpha)^*$ yields an isomorphism $q : \mathscr{E}(\alpha) \xrightarrow{\sim} \mathscr{E}(\alpha)^*$. Using the fact that $A(\alpha)$ is symmetric, one verifies that $q^t = q$, so that $\mathscr{E}(\alpha)$ carries a non-singular quadratic structure. We note [4] that $c_1(\mathscr{E}(\alpha)) = 0$, $c_2(\mathscr{E}(\alpha)) = n$, and rank $\mathscr{E}(\alpha) =$ rank $A(\alpha) - 2n$ and $\mathscr{E}(\alpha)$ is s-stable, i.e. $H^o(\mathscr{E}(\alpha)) = 0$.

Let α and α' be equivalent skew-symmetric pre-stable triples and let $u \in GL_n(K)$ be such that $u^t\alpha'_i u = \alpha_i$, $0 \leq i \leq 2$. We then have an isomorphism of the corresponding self-dual monads:

$$\begin{tikzcd}
& K^n \otimes \mathfrak{o} \mid (-1) \arrow[rr, "a"] \arrow[dl, "u\otimes 1"'] \arrow[dd, "Id"' near start] & & U \otimes \mathfrak{o} \arrow[rr, "c"] \arrow[dl, "(u^t)^{-1}\otimes 1"'] \arrow[dd, "\varphi\otimes 1" near start] & & K^n \otimes \mathfrak{o}(1) \arrow[dl, "(u^t)^{-1}\otimes 1"'] \arrow[dd, "Id"'] \\
K^n \otimes \mathfrak{o}(-1) \arrow[rr, "a'"] \arrow[dd, "Id"'] & & U' \otimes \mathfrak{o} \arrow[rr, "c'"] \arrow[dd] & & K^n \otimes \mathfrak{o}(1) \arrow[dd] & \\
& K^n \otimes \mathfrak{o}(-1) \arrow[rr, "c^t"] \arrow[dl, "u\otimes 1"'] & & U^* \otimes \mathfrak{o} \arrow[rr, "a^t"] \arrow[dl, "u\otimes 1"'] & & K^n \otimes \mathfrak{o}(1) \arrow[dl, "(u^t)^{-1}\otimes 1"'] \\
K^n \otimes \mathfrak{o}(-1) \arrow[rr, "c'^t"] & & U'^* \otimes \mathfrak{o} \arrow[rr, "a'^t"] & & K^n \otimes \mathfrak{o}(1) &
\end{tikzcd}$$

where $u : U \to U'$ is induced by the map $u \otimes 1 : K^n \otimes V^* \to K^n \otimes V^*$. **468**
This yields an isometry $u : \mathscr{E}(\alpha) \to \mathscr{E}(\alpha')$, i.e. we have a commutative diagram.

$$\begin{tikzcd}
\mathscr{E}(\alpha) \arrow[r, "u"] \arrow[d, "q"'] & \mathscr{E}(\alpha') \arrow[d, "q"] \\
\mathscr{E}(\alpha)^* & \mathscr{E}(\alpha')^* \arrow[l, "u^t"']
\end{tikzcd}$$

Let $Q(n)$ denote the set of isometry classes of s-stable quadratic spaces $(\mathscr{E}, q)$ over $\mathbb{P}^2_K$ with $c_2(\mathscr{E}) = n$. The assignment $\alpha \to (\mathscr{E}(\alpha), q)$ defines a map $\mathscr{K}(n) \overset{m}{\longrightarrow} Q(n)$.

We first show that m is injective. Suppose $\alpha, \alpha' \in \mathscr{K}(n)$ with $(\mathscr{E}(\alpha), q) \approx (\mathscr{E}(\alpha'), q')$. By definition, $\mathscr{E}(\alpha)$ and $\mathscr{E})(\alpha)$ are the coho-
469 mologies of the self-dual monads $M(\alpha)$, $M(\alpha')$ respectively. In view of ([2, prop. 4]), an isometry $f : \mathscr{E}(\alpha) \to \mathscr{E}(\alpha')$ is induced by a unique isomorphism of monads.

$$\begin{tikzcd}
K^n \otimes \mathfrak{o}(-1) \arrow[r, "a"] \arrow[d, "u\otimes 1"] & U \otimes \mathfrak{o} \arrow[r, "c"] \arrow[d, "\beta"] & K^n \otimes \mathfrak{o}(1) \arrow[d, "(u^t)^{-1}\otimes 1"] \\
K^n \otimes \mathfrak{o}(-1) \arrow[r, "a'"] & U' \otimes \mathfrak{o} \arrow[r, "c'"] & K^n \otimes \mathfrak{o}(1)
\end{tikzcd}$$

We therefore have a commutative diagram

$$\begin{tikzcd}
K^n \otimes V \arrow[r, "A(\alpha)"] \arrow[d, "u\otimes 1"'] & K^n \otimes V^* \arrow[d, "(u^t)^{-1}\otimes 1"] \\
K^n \otimes V^A \arrow[r, "(\alpha')"] & K^n \otimes V^*,
\end{tikzcd}$$

so that $u^t \alpha'_i u = \alpha_i$, $0 \leq i \leq 2$. This $\alpha' \sim \alpha$.

We next show that m is surjective.

Let $(\mathscr{E}, q)$ be a quadratic space over $\mathbb{P}^2_K$ which is s-stable and with $c_2(\mathscr{E}) = n$. Then $\dim\big(H^1(\mathscr{E}(-2))\big) = \dim\big(H^1(\mathscr{E}(-1))\big) = n$. We have multiplication maps

$$\alpha_{Z_i} : H^1(\mathscr{E}(-2)) \to H^1(\mathscr{E}(-1))$$

for $0 \leq i \leq 2$. Let $\theta : H^1(\mathscr{E}(-1)) \to H^1(\mathscr{E}(-2))$ be the composite
470 $H^1(\mathscr{E}(-1)) \xrightarrow{H^1(q(-1))} H^1(\mathscr{E}^*(-1)) \overset{S}{\longrightarrow} H^1(\mathscr{E}(-2))^*$, where s is the Serre-duality map. If $\{e_i\}$ is a basis of $H^1(\mathscr{E}(-1))$ and $\{f_i\}$ a basis of $H^1(\mathscr{E}(-2))$ dual to the basis $\{\theta(e_i)\}$, then α_{Z_i} are represented with respect to these bases by matrices $\alpha_i \in M_n(K)$ with $\alpha_i^t = -\alpha_i$([8, Th.20]). Let $\alpha = (\alpha_i)$. One can show ([4, Proof of theorem 1.5.2]), that $\mathscr{E}$ belongs to the monad

$$H^1(\mathscr{E}(-2)) \otimes \mathfrak{o}(-1) \to H^1(\mathscr{E} \otimes \Omega) \otimes \mathfrak{o} \to H^1(\mathscr{E}(-1)) \otimes \mathfrak{o}(1).$$

If we identify $H^1(\mathscr{E}(-1))$ with $H^1(\mathscr{E}(-2))^*$ through θ and identify both of these spaces with K^n through the choices of the bases described above, then $\mathscr{E}$ belongs to the monad

$$M(\alpha) : K^n \otimes \mathfrak{o}(-1) \xrightarrow{a} U \otimes \mathfrak{o} \xrightarrow{c} K^n \otimes \mathfrak{o}(1),$$

which is self dual with respect to $(\mathscr{E}, q)$. Thus $(\mathscr{E}, q) \xrightarrow{\sim} \mathscr{E}(\alpha)$ and we have the following

Theorem 1.1. *The map* $[\alpha] \mapsto [\mathscr{E}(\alpha)]$ *is a bijection between* $\mathscr{K}(n)$ *and* $Q(n)$.

2 Anisotropic quadratic spaces over the affine plane

By an *anisotropic quadratic space* over *an irreducible scheme*, we mean a quadratic space which is anisotropic over the function field.

The problem of classification of anisotropic quadratic spaces over $\mathbb{A}^2_K$ is equivalent to the problem of classification of anisotropic quadratic spaces over $\mathbb{P}^2_K$ thanks to the following

Theorem 2.1. *([6, Theorem 2.1]), Every quadratic space over* $\mathbb{A}^2_K$ *extends to* $\mathbb{P}^2_K$ *and the extension is unique upto isometry if the space is anisotropic.* **471**

The following lemma describes the type of bundles on $\mathbb{P}^2_K$ one obtains, by extending anisotropic quadratic spaces from A^2_K.

Lemma 2.2. *Let q be an anisotropic quadratic space over* $\mathbb{A}^2_K$ *which does not represent any unit of K. Let* $\mathscr{E}(q)$ *be the extension of q to* $\mathbb{P}^2_K$. *Then*

(i) $\mathscr{E}(q)$ *is s-stable (in the sense of Hulek) i.e.,* $H^\circ(\mathscr{E}(q)) = H^\circ(\mathscr{E}(q)^*) = 0$.

(ii) $c_1(\mathscr{E}(q)) = 0$

(iii) *The bundle $\mathscr{E}(q)$, restricted to every line in $\mathbb{P}^2_K$, defined over K, is trivial.*

Proof. Since $\mathscr{E}(q) \xrightarrow{\sim} \mathscr{E}(q)^*$, to check that $\mathscr{E}(q)$ is s-stable, it is enough to prove that $H^\circ(\mathscr{E}(q)) = 0$. Let s be a global section of $\mathscr{E}(q)$. Then, the map $x \mapsto q(s(x))$ is a global function on $\mathbb{P}^2_K$ and is hence a constant. If this constant is non-zero, then s defines a trivial line sub-bundle which is an orthogonal summand of $\mathscr{E}(q)$. This contradicts (in view of Theorem 2.1) the assumption that q does not represent a unit. If the constant is zero, since q is anisotropic, we have $s(x) = 0$ for all x, implying $s = 0$.

472 (ii) Since $\mathscr{E}(q)$ supports a non-singular quadratic form, $\det \mathscr{E}(q)$ is trivial and hence $c_1(\mathscr{E}(q)) = 0$.

(iii) Let $L \subset \mathbb{P}^2_K$ be a line defined over K. Since $\mathscr{E}(q) \mid L$ supports an anisotropic form, the underlying bundle of $\mathscr{E}(q) \mid L$ is trivial.

Motivated by the above lemma, we define an anisotropic quadratic space q over $\mathbb{A}^2_K$ to the s-stable if it does not represent a unit of K. For any anisotropic quadratic space q over $\mathbb{A}^2_K$, we define its *second Chern class* $c_2(q)$ to be $c_2(\mathscr{E}(q))$

Proposition 2.3. *Let q be an s-stable quadratic space over $\mathbb{A}^2_K$. Then $C_2(q)$ is an even integer.*

Proof. Let $\mathscr{E} = \mathscr{E}(q)$ denote the extension of q to $\mathbb{P}^2_K$. Let Z_i, $0 \le i \le 2$ be the co-ordinates of $\mathbb{P}^2_K$ and $\alpha_i, 0 \le i \le 2$ be the skew symmetric matrices representing the multiplication maps $\alpha_{Z_i} : H^1(\mathscr{E}(-2)) \to H^1(\mathscr{E}(-1))$ with respect to suitable bases, as described in § 1. We recall ([4, § 1.7.1]) that for $\lambda, \mu, \nu \in K$, the bundle $\mathscr{E} \mid Z = \lambda Z_0 + \mu Z_1 + \nu Z_2$ is trivial if and only if $\alpha_{\mathbb{Z}}$ is an isomorphism. By Lemma 2.2 (iii), $\mathscr{E} \mid Z_i$ is trivial and hence α_i are non-singular for $0 \le i \le 2$. Since α_i are skew symmetric and non-singular, it follows that $n = c_2(\mathscr{E})$ is even. In fact, if $K = \mathbb{R}$, the field of real numbers, the following stronger result, which, however, is not needed in what follows, holds.

Proposition 2.4. *For any s-stable quadratic space q over $\mathbb{A}^2_{\mathbb{R}}$, $c_2(q)$ is divisible by* 4.

Proof. Let $\pi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{R}}$ denote the projection. The set of all lines
473 $Z = \lambda Z_o + \mu Z_1 + \nu Z_2$ in $\mathbb{P}^2_{\mathbb{C}}$, such that $\pi^* \mathscr{E}(q) Z$ is not trivial is a curve in

the dual plane, defined by the equation $\det(X_o\alpha_o + X_1\alpha_1 + X_2\alpha_2) = 0$, unless it is the whole plane. The latter possibility does not arise because α_i being non-singular. $\det(X_o\alpha_o + X_1\alpha_1 + X_2\alpha_2) \neq 0$. In view of ([4, § 1.7.1]) $c_2(q) = deg\det(X_o\alpha_o + X_1\alpha_1 + X_2\alpha_2)$. We have $\det(X_o\alpha_0 + X_1\alpha_1 + X_2\alpha_2) = (Pf(X_0\alpha_0 + X_1\alpha_1 + X_2\alpha_2))^2$, where Pf denotes the Pfaffian. Since $\mathscr{E}(q)$ restricted any $\mathbb{R}$ rational line is trivial, the curve $pf(X_o\alpha_o + X_1\alpha_1 + X_2\alpha_2) = 0$ which is defined over $\mathbb{R}$, has no $\mathbb{R}$-rational point. Thus, $Pf(X_o\alpha_o + X_1\alpha_1 + X_2\alpha_2) = 0$ is a curve of even degree, so that $c_2(q) = 2degpf(X_o\alpha_o + X_1\alpha_1 + X_2\alpha_2)$ is divisible by 4.

Let $(\mathscr{E}, q)$ be a quadratic space over $X = \mathbb{A}^2_K$ or $\mathbb{P}^2_K$. Then, there exists a quadratic form q_o over K such that at every point $x \in X$, the quadratic space $(\mathscr{E}_x, q_x)$ which is the fibre at x, is isometric to q_O. This can be deduced if $X = \mathbb{A}^2_K$ from the fact (Karoubi's theorem) that $(\mathscr{E}, q)$ is "stably extended" from a form q_O over K. If $X = \mathbb{P}^2_K$, and if q_1, q_2, q_3 are the forms over K corresponding to the restrictions of $(\mathscr{E}, q)$ to the affine planes, then the connectedness of $\mathbb{P}^2_K$ shows that $q_1 \approx q_2 \approx q_3$ We call q_o the "form on the fibre of $(\mathscr{E}, q)$". Clearly $(\mathscr{E}, q)$ is anisotropic if and only if q_o is anisotropic.

Proposition 2.5. *Let $(\mathscr{E}, q)$ be an s-stable quadratic space over $\mathbb{P}^2_K$ with $\alpha = (\alpha_i)$ as the corresponding skew-symmetric triple. Then $\omega = \alpha_2\alpha_0^{-1}\alpha_1 - \alpha_1\alpha_0^{-1}\alpha_2$ is symmetric and ω modulo its radical is the form on the fibre of $(\mathscr{E}, q)$.*

Proof. Let U be the image of the map $A(\alpha) : K^{3n} \to K^{3n}$, where $A(\alpha)$ is as in § 1. Let $\alpha : K^n \to U$ be defined as $a = \begin{pmatrix} -\alpha_1 \\ \alpha_0 \\ 0 \end{pmatrix}$

Let $c : U \to K^n$ be the restriction of the map $K^{3n} \xrightarrow{(0,0,1)} K^n$; then 474
the fibre of $\mathscr{E}(q)$ over $(0, 0, 1)$ is the homology of the complex

$$K^n \xrightarrow{a} U \xrightarrow{c} K^n$$

with the form on the fibre induced by the map $\varphi : U \to U^*$ defined by $\varphi(x)(z) =< z, y >$, where $A(\alpha)(y) = x$, and $\langle , \rangle$ denotes the canonical inner product on K^{3n}. Let

$$T = \begin{pmatrix} u & u\alpha_1\alpha_0^{-1} & u\alpha_2\alpha_0^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $uu^t = \begin{pmatrix} \widetilde{\omega} & 0 \\ 0 & 0 \end{pmatrix}$ and $\widetilde{\omega}$ is an invertible r by r matrix Transforming K^{3n} by T,we get an isometric space given as the homology of the complex

$$K^n \xrightarrow{\begin{pmatrix} 0 \\ \alpha_0 \\ 0 \end{pmatrix}} K^r \oplus K^n \oplus K^n \xrightarrow{(0,0,1)} K^n (*)$$

Since

$$TA(\alpha)T^t = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 0 & \alpha_0 \\ 0 & \alpha_0 & 0 \end{pmatrix}$$

475 $(T^t)^{-1}\,\varphi T^{-1}$ is represented by the matrix $\begin{pmatrix} \widetilde{\omega}^{-1} & 0 & 0 \\ 0 & 0 & -\alpha_0^{-1} \\ 0 & \alpha_0^{-1} & 0 \end{pmatrix}$ so that the form on the fibre of the complex is given by $\widetilde{\omega}^{-1}$. This proves the Proposition.

We shall now fix an anisotropic form q_o over K of rank $r \geq 2$ and an integer $n \geq r$. Let q be an s-stable quadratic space over $\mathbb{A}^2_K$ with q_o as the form on the fibre and with $c_2(q) = n$. Let (α_i), $0 \leq i \leq 2$ be a (skew-symmetric) triple corresponding to the bundle $\mathscr{E}(q)$. By Proposition 2.5, the form on the fibre of $\mathscr{E}(q)$ is q_1, where $q_1 \perp < 0, \dots, 0 >_{n-r} \overset{\sim}{\longrightarrow} \omega$ Let $u \in GL_n(K)$ be such that $u\omega u^t = q_o \perp < 0, \dots 0 >_{n-r}$. If $\beta_i = u\alpha_i u^t$, $0 \leq i \leq 2$, then $\beta_2\beta_0^{-1}\beta_1 - \beta_1\beta_0^{-1}\beta_2 = q_o \perp < 0, \dots, 0 >_{n-r}$. Thus in the equivalence class of α, there exists a triple β with $\beta_2\beta_0^{-1}\beta_1 - \beta_1\beta_0^{-1}\beta_2 = q_0 \perp < 0, \dots 0 >_{n-r}$.

Let $\mathscr{K}(n, q_0) = \{\alpha = (\alpha_0, \alpha_1, \alpha_2) \mid \alpha_i \in GL_i(K)$, α pre-stable, $\alpha_i^t = -\alpha i$ and $\alpha_2\alpha_0^{-1}\alpha_1 - \alpha_1\alpha_0^{-1}\alpha_2 = q_0 \perp < 0, \dots, 0 >_{n-r}\}$ modulo the equivalence relation $\alpha \sim \beta$ if and only if there exist $u \in GL_n(K)$,

$\lambda \in K^*$ such that $\beta_i = \lambda u^t \alpha_i u$, $0 \leq i \leq 2$. We set $Q(n, q_0)$= set of similitude classes of s-stable quadratic spaces over $\mathbb{A}^2_K$ with q_o as the form on the fibre and $c_2 = n$, similitude being isometry upto a scalar of K^*

As an immediate consequence of Theorem 1.1. we have the following

Theorem 2.6. *The assignment $[q] \mapsto [\alpha(\mathscr{E}(q))]$ gives rise to a bijection between the sets $Q(n, q_0)$ and $\mathscr{K}(n, q_0)$.* **476**

Remark. The condition of pre-stability of α in the definition of $Q(n, q_0)$ can be dropped if n=rank q_0.

3 Quaternion algebras associated to triples of skew symmetric matrices

Definition 3.1. A triple $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, $\alpha_i \in GL_n(K)$ is anisotropic if α is pre-stable, $\alpha_i^t = -\alpha_i$, $0 \leq i \leq 2$ and $\widetilde{\omega}$ is anisotropic, where $\omega = \alpha_2\alpha_0^{-1}\alpha_1 - \alpha_1\alpha_0^{-1}\alpha_2$, and $\widetilde{\omega}$ denotes ω modulo its radical.

We observe that a pre-stable triple α is anisotropic if and only if the corresponding quadratic space $\mathscr{E}(\alpha)$ is anisotropic.

Lemma 3.2. *Let $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ be an anisotropic triple. Then, for λ, μ, $v \in K$, not all zero, $\lambda\alpha_0 + \mu\alpha_1 + v\alpha_2$ is non-singular.*

Proof. The triple α defines an anisotropic quadratic space $\mathscr{E}(\alpha)$ over $\mathbb{P}^2_K$. Since any anisotropic quadratic space over $\mathbb{P}^1_K$ is trivial as a vector bundle, $\mathscr{E}(\alpha)$, restricted to any K rational line Z of $\mathbb{P}^2_K$ is trivial. Hence the corresponding multiplication map $\alpha_Z : H^1(\mathscr{E}(-2)) \to H^1(\mathscr{E}(-1))$ is an isomorphism. For $Z = \lambda Z_0 + \mu Z_1 + vZ_2$, α_Z is represented by the matrix $\lambda\alpha_0 + \mu\alpha_1 + v\alpha_2$ for a suitable choice of bases. Hence, it follows that $\lambda\alpha_0 + \mu\alpha_1 + v\alpha_2$ is non-singular.

Proposition 3.3. *Let $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ be an anisotropic triple with $\alpha_i \in GL_4(K)$, $0 \leq i \leq 2$. Then $\omega = \alpha_2\alpha_0^{-1}\alpha_1\alpha_1\alpha_0^{-1}\alpha_2$ is non-singular and determinant ω is a square.*

Proof. If rank $\omega \leq 2$, since α is pre-stable, it gives rise to an s-stable quadratic space of rank ≤ 2 over $\mathbb{A}^2_K$. This contradicts the fact ([9, 477 Proposition 1.1]) that any rank 2 quadratic space over $\mathbb{A}^2_K$ is extended from K. Hence rank $\omega \geq 3$. Let $U \in GL_4(K)$ be such that $U\omega U^t = \theta < 1, -a, -b, c >$, with $\theta, a, b, c, \in K^*$. Replacing α_i by $U\alpha_i U^t$, we assume, without loss of generality, that

$$\alpha_2\alpha_0^{-1}\alpha_1 - \alpha_1\alpha_0^{-1}\alpha_2 = \theta < 1, -a, -b, c > \qquad (*)$$

Replacing α_i by $\alpha_i + x\alpha_0$, $i = 1, 2$ or α_1 by $\alpha_1 + x\alpha_2$, $x \in K$ neither changes the relation $(*)$ nor the non-singularity of the α_i, in view of Lemma 3.2. We therefore assume that α_i have the following form

$$\alpha_0^{-1} = \begin{pmatrix} \lambda\epsilon & A \\ -A^t & \mu\epsilon \end{pmatrix}, \alpha_1 = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix} \alpha_2 = \begin{pmatrix} 0 & C \\ -C^t & \nu\epsilon \end{pmatrix}$$

where $\lambda, \mu, \nu \in K$, $A \in M_2(K)$, $B, C \in GL_2(K)$ and $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The condition $(*)$ yields:

$$\mu(C \in B^t - B \in C^t) = \theta \begin{pmatrix} 1 & 0 \\ 0 & -a \end{pmatrix} \qquad (1)$$

$$\lambda\left(C^t \in B - B^t \in C\right) + \nu\left(\epsilon A^t B - B^t A\epsilon\right) = \theta \begin{pmatrix} -b & 0 \\ 0 & c \end{pmatrix} \qquad (2)$$

$$CA^tB - BA^tC = \mu\nu B \qquad (3)$$

From (3), we get $CA^t - BA^tCB^{-1} = \mu\nu$. Comparing the traces, we have $2\mu\nu = 0$. From (1), it follows that $\mu \neq 0$ since θ, $a \in K^*$, so that 478 $\nu = 0$. From (2), it follows that $\lambda \neq 0$ since θ, $b \in K^*$. We get from (1) and (2).

$$(\det B)C - (\det C)BC^{-1}B = \theta\mu^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -a \end{pmatrix} \epsilon^{-1} B$$

$$(\det C)BC^{-1}B - (\det B)C = \theta\lambda^{-1} B\epsilon^{-1} \begin{pmatrix} -b & 0 \\ 0 & c \end{pmatrix}$$

Thus

$$0\mu^{-1}\begin{pmatrix}1 & 0\\ 0 & -a\end{pmatrix}\epsilon^{-1}B = -\theta\lambda^{-1}B\epsilon^{-1}\begin{pmatrix}-b & 0\\ 0 & c\end{pmatrix}$$

Comparing the determinants, we obtain $\lambda^2 a = \mu^2 bc$. Since $\lambda, a \in K^*$, it follows that $c \neq 0$, so that ω is non-singular. Further $\det\omega = \theta^4 abc = \theta^4\lambda^{-2}\mu^2(bc)^2$ which is a square.

Lemma 3.4. *Let $q = \theta < 1, -a, -b, ab >$, with $\theta, a, b, \in K^*$.*

Let $\alpha \in M_4(K)$ which satisfies $\alpha + q\alpha^t q^{-1}. = 0$ and $\alpha \cdot q\alpha^t q^{-1} \in K$. Then α has either of the following forms

$$\begin{pmatrix}0 & \mu & x & y\\ a\mu & 0 & -ay & -x\\ xb & yb & 0 & \mu\\ -yab & -xb & a\mu & 0\end{pmatrix} \tag{$*$}$$

$$\begin{pmatrix}0 & \mu & x & y\\ a\mu & 0 & ay & x\\ xb & -yb & 0 & -\mu\\ -yab & xb & -a\mu & 0\end{pmatrix} \tag{$**$}$$

Proof. Let $\eta_a\begin{pmatrix}1 & 0\\ 0 & -a\end{pmatrix}, \alpha = \begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}$ with $\alpha_i \in M_2(K)$. **479**

Then, $\alpha + q\alpha^t q^{-1} = 0$ gives

$$\begin{aligned}\alpha_1\eta_a + \eta_a\alpha_1^t &= 0\\ \alpha_4\eta_a + \eta_a\alpha_4^t &= 0\\ \eta_a\alpha_3^t - b\alpha_2\eta_a &= 0\end{aligned} \tag{1}$$

The fact that $\alpha q\alpha^t q^{-1} \in K$ gives

$$\begin{aligned}\alpha_1^2 + \alpha_2\alpha_3 = \alpha_3\alpha_2 + \alpha_4^2 \in K\\ \alpha_1\alpha_2 + \alpha_2\alpha_4 = \alpha_3\alpha_1 + \alpha_4\alpha_3 = 0\end{aligned} \tag{2}$$

From (1), one obtains $\alpha_1 = \begin{pmatrix}0 & \mu\\ a\mu & 0\end{pmatrix}, \alpha_4 = \begin{pmatrix}0 & 0\\ a0 & 0\end{pmatrix}$ From (2), it follows that $\alpha_1^2 - \alpha_4^2 = \alpha_3\alpha_2 - \alpha_2\alpha_3$. Comparing the trace we get $Tr\left(\alpha_1^2\right) =$

$Tr(\alpha_4^2)$, i.e. $\mu^2 = \theta^2$. According as $\mu = \pm\theta$, we get α to be of the form $(*)$ or $(**)$ respectively.

Theorem 3.5. *Let $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ be an anisotropic triple with $\alpha_i \in GL_4(K)$, $0 \leq i \leq 2 \leq$. Let $\alpha_2\alpha_0^{-1}\alpha_1 - \alpha_1\alpha_0^{-1}\alpha_2 = \omega$*
Then

(i) The K-subalgebra $H(\alpha)$ of $M_4(K)$ generated by $\alpha_1\alpha_0^{-1}$ and $\alpha_2\alpha_0^{-1}$ is a quaternion algebra over K.

480 *(ii) The form ω is non-singular and the involution on $M_4(K)$ given by $x \to \omega x^t \omega^{-1}$ restricts to the canonical involutions on $H(\alpha)$ and $H(\alpha)^c$, where $H(\alpha)^c$ denotes the commutant of $H(\alpha)$ in $M_4(K)$.*

(iii) The reduced norm on $H(\alpha)$ is isometric to ω up to a scalar.

(iv) For any element $x = \lambda + \mu\alpha_1\alpha_0^{-1} + v\alpha_2\alpha_0^{-1} \in H(\alpha)$; $\lambda, \mu, v \in K$, we have $Nrd(x) = Pf(\lambda\alpha_0 + \mu\alpha_1 + v\alpha_2)$.

Proof. By Proposition 3.3, it follows that ω is non-singular and that disc $\omega = 1$. Replacing α_i by $u\alpha_i u^t$ and ω by $u\omega u^t$ changes the algebra $H(\alpha)$ to $uH(\alpha)u^{-1}$. We assume without loss of generality that $\omega = \theta < 1, -a, -b, ab >$. Changing α_i, $i = 1, 2$ to $\alpha_i + x\alpha_0$ or $\alpha_1 + x\alpha_2$, $x \in K$, does not change ω or the algebra $H(\alpha)$, nor does it affect the invertibility of the α_i. we thus assume, following the proof of Proposition 3.2, that α_i have the form

$$\alpha_0^{-1} = \begin{pmatrix} \lambda\epsilon & A \\ -A^t & \mu\epsilon \end{pmatrix} \alpha_1 = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix}$$

where $A \in M_2(K)$, $B, C \in GL_2(K)$, $\lambda, \mu \in K^*$, $\lambda^2 = \mu^2 b^2$ with

$$(\det B)CB^{-1} - (\det C)BC^{-1} = \theta\mu^{-1}\eta_a\epsilon^{-1} \tag{1}$$

$$(\det C)C^{-1}B - (\det B)B^{-1}C = \theta\lambda^{-1}\epsilon^{-1}(-b\eta_a) \tag{2}$$

$$CA^tB = BA^tC. \tag{3}$$

481 Let $\lambda = e\mu b$ where $e = \pm 1$. From (1) and (2), we get

$$\eta_a \epsilon^{-1} B = eB\epsilon^{-1}\eta_a$$
$$\eta_a \epsilon^{-1} C = eC\epsilon^{-1}\eta_a$$

These imply that $B = \begin{pmatrix} x & y \\ -ey & -eax \end{pmatrix}, c = \begin{pmatrix} x' & y' \\ -ey' & -eax' \end{pmatrix}$, $x, y, x', y' \in K$. Let $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ Using (1) and (3), we get $q = -er$, $p = -eas$. Thus $A = \begin{pmatrix} -eas & -er \\ r & s \end{pmatrix}$ Let W denote the K-subspace of $M_4(K)$ spanned by $\{1, \omega\alpha_0^{-1}, \alpha_1\alpha_0^{-1}, \alpha_2\alpha_0^{-1}\}$. Then $W \subset H(\alpha)$, since $w\alpha_0^{-1} = \alpha_2\alpha_0^{-1}\alpha_1\alpha_0^{-1} - \alpha_1\alpha_0^{-1}\alpha_2\alpha_0^{-1} \in H(\alpha)$. We first show that $W = K + X$ or $K + X'$, where $K \subset M_4(K)$ as scalar matrics and X is the set of all matrices of the form $(*)$ and X' is the set of all matrices of the form $(**)$.

$$\omega\alpha_0^{-1} = \theta \begin{pmatrix} \eta_a & 0 \\ 0 & -b\eta_a \end{pmatrix} \begin{pmatrix} e\mu b\epsilon & A \\ -A^t & \mu\epsilon \end{pmatrix}$$
$$= \theta \begin{pmatrix} 0 & e\mu b & -eas & -er \\ e\mu ab & 0 & -ar & -as \\ -eabs & br & 0 & -\mu b \\ eabr & -abs & -\mu ab & 0 \end{pmatrix}$$

$$\alpha_1\alpha_0^{-1} = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix} \begin{pmatrix} e\mu b\epsilon & A \\ -A^t & \mu\epsilon \end{pmatrix}$$

$$= (eaxs + eyr).1 + \begin{pmatrix} 0 & -(xr + ys) & -\mu y & \mu x \\ -a(xr + ys) & 0 & e\mu ax & -e\mu y \\ -\mu by & -e\mu bx & 0 & e(xr + ys) \\ -\mu abx & -e\mu by & ea(xr + ys) & 0 \end{pmatrix}$$

The matrix $\alpha_2\alpha_0^{-1}$ has a form similar to that of $\alpha_1\alpha_0^{-1}$ above, with x, y **482**
replaced by x', y'. These matrices belong to $K + X$. if $e = 1$ and to $K + X'$ if $e = -1$.

We claim that 1, $\omega\alpha_0^{-1} \cdot \alpha_1\alpha_0^{-1}$, $\alpha_2\alpha_0^{-1}$ are linearly independent over K. In fact, if $x_O 1 + x_1\omega\alpha_O^{-1} + x_2\alpha_1\alpha_0^{-1} + x_3\alpha_2\alpha_0^{-1} = 0$, with $x_i \in K$, then $x_0\alpha_0 + x_1\omega + x_2\alpha_1 + x_3\alpha_2 = 0$. Comparing traces, we get $x_1(1 - a - b + ab) = 0$. Since ω is anisotropic, we get $x_1 = 0$. From the forms of α_i, $0 \leq i \leq 2$, we get $x_0 = 0$ and $x_2 B = x_3 C$. If x_2 and x_3 are not both zero, equation (1) is contradicted. Thus $x_i = 0$ for $0 \leq i \leq 3$ and $\dim W = 4$. Since $K + X$ and $K + X'$ are both of dimension 4 over K, it follows that $W = K + X$ or $W = K + X'$ according as $e = \pm 1$. It is easy to check that $K + X$ and $K + X'$ are both K-subalgebras of $M_4(K)$ and hence the inclusion $W \subset H(\alpha) \subset K + X$ or $W \subset H(\alpha) \subset K + X'$ gives that $H(\alpha) = K + X$ or $H(\alpha) = K + X'$. It is also easy to see that $K + X$ and $K + X'$ are quaternion algebras with the canonical involution given by $x \to \omega x^t \omega^{-1}$. The algebra $K + X$ is generated by the two elements

$$i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \end{pmatrix}, j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ b & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{pmatrix}$$

483 with the relations $i^2 = a$, $j^2 = b$, $ij + ji = 0$. Hence its norm with respect to the basis $1, i, j, ij$ is given by $\langle 1, -a, -b, ab \rangle \approx \theta_{\omega}^{-1}$.

One can similarly show that the norm in $K + X'$ is also isometric to $\theta^{-1}\omega$. The commutant of $K + X$ in $M_4(K)$ is checked to be $K + X'$. To complete the proof of the theorem, we need only to check (iv) which follows from the following

Lemma 3.6. *Let α, β be two skew symmetric matrices in $M_{2n}(R)$, where R is any commutative ring with α non-singular. Then, the matrix $\beta\alpha^{-1}$ satisfies the equation $Pf(x\alpha - \beta) = 0$.*

Proof. Given a skew symmetric matrix $\gamma \in M_{2n}(R)$, if $\gamma^* \in M_{2n}(R)$ is defined by

$$\begin{aligned} \gamma^*_{ij} &= (-1)^{i+j-1} Pf(\Gamma_{ij}); i \leq j \\ &= (-1)^{i+j} Pf(\Gamma_{ij}), i > j, \end{aligned}$$

where Γ_{ij} is the skew symmetric matrix obtained from γ by suppressing the i^{th} and j^{th} rows and columns, then $\gamma\gamma^* = \gamma^*\gamma = Pf(\gamma).\ Id$ (see[11], (24)). Thus, if γ is treated as an endomorphism of R^{2n}, then coker γ is annihilated by $Pf(\gamma)$. Given skew symmetric matrices $\alpha, \beta \in M_{2n}(R)$ with α non-singular, then we have an exact sequence [3, p. 630] of $R[x]$-

modules

$$0 \to R[x] \xrightarrow{2nx\alpha-\beta} R[x]^{2n} \xrightarrow{\varphi} R^{2n} \to 0,$$

where φ is defined by $\varphi\left(\sum a_i x^i\right) = \sum \left(\beta\alpha^{-1}\right)^i (a_i)$, $a_i \in R^{2n}$. Here R^{2n} is 484
treated as an $R[x]$ -module by letting x operate as $\beta\alpha^{-1}$. By the above remark, $f(x) = Pf(x\alpha - \beta)$ annihilates R^{2n}, i.e. $f\left(\beta\alpha^{-1}\right) = 0$. This proves the lemma.

4 Anisotropic quadratic spaces with $c_2 \leq 4$

We have already seen (Proposition 2.3) that the second chern class c_2 of an s-stable quadratic space over $\mathbb{A}^2_K$ is even. we denote by q_0 an anisotropic quadratic form over K. Since every quadratic space of rank 2 over $\mathbb{A}^2_K$ is extended from K [9, Prop 1.1], $\mathscr{K}(n, q_0) = \phi$ if rank $q_0 = 2$. In particular, for $c_2 = 2$, $\mathscr{K}(2, q_0) = \phi$. We next consider the case $c_2 = 4$. In view of Proposition 3.3, $\mathscr{K}(4, q_0) = \phi$, unless rank $q_0 = 4$ and disc $q_0 = 1$. In this case, q_0 is, upto a scalar, the norm from a quaternion algebra H_0. Let C_0 be the conic in $\mathbb{P}^2_K$ giving the norm on any three dimensional subspace of H_0. We shall show that $\mathscr{K}(4, q_0)$ is in bijection with the orbit of C_0 under the action of $GL_3(K)$.

Let $q_0 = \theta < 1, -a, -b, ab >$. Let C_0 be the conic in $\mathbb{P}^2_K$ defined by $az_0^2 + bz_1^2 - z_2^2 = 0$. We define a map $c : \mathscr{K}(4, q_0) \xrightarrow{K} \left\{\text{set of conics in} \mathbb{P}^2_K\right\}$ as follows. Let $[\alpha] \in \mathscr{K}(4, q_0)$. Then, $Pf(Z_0\alpha_0 + Z_1\alpha_1 + Z_2\alpha_2)$ is a homogenous polynomial, which is not zero since α_i are non-singular, and is of degree 2. We define $c([\alpha]) : Pf(Z_0\alpha_0 + Z_1\alpha_1 + Z_2\alpha_2) = 0$. The map is well-defined on $\mathscr{K}(4, q_0)$ since if $\beta_i = \lambda u\alpha_i u^t$, $Pf(Z_0\beta_0 + Z_1\beta_1 + Z_2\beta_2) = \lambda^2 \det u.Pf(Z_0\alpha_0 + Z_1\alpha_1 + Z_2\alpha_2)$ which again determines the same conic. 485

Theorem. *The map c induces a bijection between* $\mathscr{K}(4, q_0)$ *and the orbit of the conic* C_0 *under the action of* $GL_3(K)$. *The image of c is precisely* $\dfrac{GL_3(K)}{(subgroup\,fixing C_0)}$.

Proof. Suppose $[\alpha]$, $[\beta] \in \mathscr{K}(4, q_0)$ with $c(\alpha) = c(\beta)$. Then $Pf(Z_0\alpha_0 + Z_1\alpha_1 + Z_2\alpha_2) = \lambda.Pf(Z_0\beta_0 + Z_1\beta_1 + Z_2\beta_2)$, with $\lambda \in K^*$. If $Pf\alpha_0 = a$,

$Pf\beta_0 = b$, then $a^{-1}Pf(Z_0\alpha_0 + Z_1\alpha_1 + Z_2\alpha_2) = b^{-1}Pf(Z_0\beta_0 + Z_1\beta_1 + Z_2\beta_2)$. The map $\lambda + \mu\alpha_1\alpha_0^{-1} + v\alpha_2\alpha_0^{-1} \to \lambda + \mu\beta_1\beta_0^{-1} + v\beta_2\beta_0^{-1}$ gives an isometry from a 3-dimensional subspace of $H(\alpha)$ onto the corresponding subspace of $H(\beta)$ which maps the identity elements onto each other. This can be extended to an isomorphism of $H(\alpha)$ onto $H(\beta)$.

Let $u \in GL_4(K)$ with $uH(\alpha)u^{-1} = H(\beta)$. Then

$$\begin{aligned} u\alpha_1\alpha_0^{-1}u^{-1} &= \beta_1\beta_0^{-1} \\ u\alpha_2\alpha_0^{-1}u^{-1} &= \beta_2\beta_0^{-1} \end{aligned} \tag{1}$$

To show that $u\alpha_i u^t = \lambda\beta_i$, $\lambda \in K^*$, $0 \leq i \leq 2$, it suffices to show that $u\alpha_0 u^t = \lambda\beta_0$. From (1), we get

$$uq_0\alpha_0^{-1}u^{-1} = u\left(\alpha_2\alpha_0^{-1}\alpha_1 - \alpha_1\alpha_0^{-1}\alpha_2\right)\alpha_0^{-1}u^{-1} = \beta_0^{-1} \tag{2}$$

We have the following commutative diagram:

$$\begin{tikzcd} H(\alpha) \arrow[r, "*"] \arrow[d, "\mathrm{Int}u"'] & H(\alpha) \arrow[d, "\mathrm{Int}u"] \\ H(\beta) \arrow[r, "*"] & H(\beta), \end{tikzcd}$$

486 (since int u is an isomorphism of $H(\alpha)$ onto $H(\beta)$, it commutes with the canonical involutions* of these algebras). By theorem 3.5, the involutions are precisely given by $x \mapsto q_0x^tq_0^{-1}$. Thus

$$uq_0x^tq_0^{-1}u^{-1} = q_0u^{t-1}x^tu^tq_0^{-1},\ \forall x \in H(\alpha),$$

i.e.

$$\left(u^tq_0^{-1}uq_0\right)x^t = x^t\left(u^tq_0^{-1}uq_0\right)\ \forall x \in H(\alpha),$$

i.e.

$$x\left(q_0u^tq_0^{-1}u\right) = \left(q_0u^tq_0^{-1}u\right)x\ \forall x \in H(\alpha),$$

i.e.

$$q_0u^tq_0^{-1}u \in H(\alpha)^c \text{ and } \left(q_0u^tq_0^{-1}u\right)^* = q_0u^tq_0^{-1}u.$$

Thus the element $q_0 u^t q_0^{-1} u \in H(\alpha)^c$ is invariant under the canonical involution of $H(\alpha)^c$ and hence is a constant, say $\lambda \in K^*$ so that

$$uq_0 = \lambda q_0 u^{t-1}$$

Substituting this in (2), we get

$$\lambda q_0 u^{t^{-1}} \alpha_0^{-1} u^{-1} = q_0 \beta_0^{-1}, \text{ i.e. } u\alpha_0 u^t = \lambda \beta_0.$$

We next show that for $[\alpha], [\beta] \in \mathscr{K}(4, q_0), C(\alpha) = \lambda u^t C(\beta) u$ for some $u \in GL_3(K)$, $\lambda \in K^*$. Since the norms in $H(\alpha)$ and $H(\beta)$ are isometric to $\theta^{-1} q_0$, $H(\alpha)$ and $H(\beta)$ are isomorphic. Hence the reduced norms restricted to $K + K\alpha_1\alpha_0^{-1} + K\alpha_2\alpha_0^{-1}$ and $K + K\beta_1\beta_0^{-1} + K\beta_2\beta_0^{-1}$ are isometric upto a scalar. Let $u \in GL_3(K)$ be the matrix of transformation of these spaces with respect to the bases $\left(1, \alpha_1\alpha_0^{-1}, \alpha_2\alpha_0^{-1}\right)$
and $\left(1, \beta_1\beta_0^{-1} . \beta_2\beta_0^{-1}\right)$ respectively. Then $uPf\,(Z_0\alpha_0 + Z_1\alpha_1 + Z_2\alpha_2)\,u^t =$ **487**
$\lambda Pf\,(Z_0\beta_0 + Z_1\beta_1 + Z_2\beta_2)$.

Let now $c([\alpha]) = C_1$. Then, the full orbit of C_1 under $GL_3(K)$ is contained in the image of C. In fact, if $C_2 = uC_1u^t$, $u \in GL_3(K)$, and $\beta_1 = \sum_{j=0}^{2} u_{ij}\alpha_j$, $0 \leq i \leq 2$, where $u = (u_{ij})$, then, $c(\beta) = uC_1u^t$. Thus we have shown that the image of c is a full orbit of a conic provided it is non-empty. In fact, if C_1: $4aZ_0^2 + bZ_1^2 - Z_2^2 = 0$, then C_1 is equivalent to C_0 and C_1 is the image of $[\alpha]$ defined by

$$\alpha_0 = \begin{pmatrix} 0 & 2\theta^{-1} & 0 & 0 \\ -2\theta^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\theta^{-1}a^{-1} \\ 0 & 0 & 2\theta^{-1}a^{-1} & 0 \end{pmatrix}$$

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \\ -1 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{pmatrix} \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

References

[1] W. Barth, Moduli of vector bundles on the projective plane, *Invent. Math*. 42, 63–91 (1977). **488**

[2] W. Barth and K. Hulek, Monads and moduli of vector bundles, *Manuscripta Math*. 25, 323–347 (1978)

[3] H. Bass, *Algebraic K-Theory,* Benjamin 1968.

[4] K. Hulek, On the classification of stable rank-r vector bundles over the projective plane, *Vector Bundles and Differential Equations,* Proc. Nice 1979, Progress in Mathematics 7, Brikhauser

[5] M. A. Knus and M. Ojanguren, Modules and quadratic forms over polynomial algebras, *Proc. Amer. Math. Soc*. 66, 223–226(1977)

[6] M. A. Knus, R. Parimala and R. Sridharan, Non-free projective modules over $\mathbb{H}[X, Y]$ and stable bundles over $\mathbb{P}_2(\mathbb{C})$, *Invent. Math.* 65, 13–27 (1981)

[7] M. Ojanguren, Formes quadratiques sur less algébres de polynômes, *C.R.Acad.Sc.* 287, (Serie A) 695–698 (1978)

[8] M. Ojanguren, R. Parimala and R. Sridharan, Indecomposable quadratic bundles of rank $4n$ over the real affine plane, *Invent. Math*. 71, 643–653 (1983)

[9] R. Parimala, Failure of a quadratic analogue of Serre's conjecture, *Amer. J. of Math*. 100, 913–924 (1978)

489 [10] R. Parimala, Indecomposable quadratic spaces over the affine plane, to appear in *Advances in Mathematics*.

[11] E. Von Weber, *Vorlesungen über* das Pfaff'sche Problem, Leipzig, 1900.

** Institut de Mathematiques
Faculté des Sciences
Université de Lausanne
1015 Lausanne — Dorigny
Switzerland

* School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005
India

Unitary Bundles and Restrictions of Stable Sheaves

By A. Ramanathan

1 Introduction

491 This is a report on joint work with V. B. Mehta. Here we will state the main results and give a sketch of their proofs. Full details will appear in [4].

Let X be a smooth projective variety of dimension n over an algebraically closed field k. Let H be an ample line bundle on X. For any coherent sheaf $\mathscr{F}$ on X let $c_1(\mathscr{F})$ denote its first Chern class. By $\deg \mathscr{F}$ we mean the intersection number $c_1(\mathscr{F}) \cdot H^{n-1}$ and by $rk\mathscr{F}$ we mean the rank of the generic fibre of $\mathscr{F}$ over the function field of X. Let V be a torsion free sheaf on X. Mumford has given the following definition of stability for V. For any proper subsheaf $W \subset V$ if we have $\deg \frac{W}{rk}V < \deg \frac{V}{rk}V$ then V is said to be stable. If, on the other hand, only the weak inequality $\deg \frac{W}{rk}W \leq \deg \frac{V}{rk}V$ holds, V is called *semistable*.

For a smooth projective curve C stable bundles and their moduli have been studied in depth by Narasimhan, Ramanan and Seshadri (see [5], [6]).

492 For higher dimensional varieties X, clearly it would be of some use to know how the restriction of semi-stable or stable bundles on X to suitable subvarieties of X behave. In [3], we proved that the restriction of a semistable bundle on X to complete intersection sub-varieties in general position and of suitably high multi-degree remain semistable. The proof

was based on an unpublished manuscript of Mumford. In this paper we extend the method to prove that stable bundles on X also remain stable on such subvarieties. This has some interesting consequences which we describe now.

In [5], it is proved that over the field of complex numbers for a curve C of genus ≥ 2, stable bundles on C of degree zero are precisely those which come from irreducible unitary representations of the fundamental group $\pi_1(C)$. This means that a vector bundle $V \to C$ of rank r is stable if and only if there is an irreducible unitary representation $\rho : \pi_1(C) \to U(r) \subset GL(r)$ such that V is the associated bundle for the principal $\pi_1(C)$-bundle $\widetilde{C} \to C$ corresponding to ρ, where $\widetilde{C}$ is the universal covering of C

For higher dimensions, the right generalisation of this theorem was formulated by Kobayashi [2]. For a vector bundle to come from a unitary representation of the fundamental group it is necessary and sufficient that it admit a hermitian metric whose hermitian connection is flat. Generalising this, Kobayashi conjectured that a vector bundle is stable (more precisely a direct sum of stable bundles) if and only if it admits a hermitian metric for which the associated connection has curvature
493 which satisfies the Einstein-Kahler condition. (See [2] for precise statements). Kobayashi and Lubke proved that if a bundle admits such a metric then it is a direct sum of stable bundles. When all the Chern classes of the bundle vanish, this condition on curvature reduces to flatness, i.e. coming from a unitary representation of the fundamental group.

In [1], Donaldson proved that if $\dim X = 2$ and $V \to X$ a stable vector bundle with $c_1(V) = c_2(V) = 0$ then V comes from an irreducible representation of $\pi_1(X)$. We show here how this result combined with our restriction theorem yields the same result for higher dimensional X.

2 Restriction Theorem

We assume, without loss of generality, that H is very ample. If $s_1, \ldots, s_r$ are sufficiently general elements of $H^0(X, H^{m_r})$ respectively, then their common zeroes defined by $s_1 = \ldots = s_r = 0$ is a complete intersection subvariety of X of codimension r. We denote this subvariety by $Y(s_1 \dots s_r)$ or Y_s.

Theorem 1. *Let V be a stable (resp. semistable) torsion free sheaf on X. Then there exists an integer N (depending on V) such that for all $m \geq N$ and sufficiently general elements $s_i \in H^0\left(X, H^{2^m}\right) i = 1, \dots, r$ the restriction $V \mid Y_s$ is stable (resp. semistable) with respect to $H \mid Y_s$.*

We sketch the proof. If $V \mid Y_s$ is stable, it is easy to see that V is stable. Only the converse needs to be proved. First it is easy to see that we can reduce to the case where Y_s is a curve.

Suppose $V \mid Y_s$ is not stable. Then one can associate a canonical 494
subsheaf W of $V \mid Y_s$ which contradicts stability. If the sheaf W on Y_S can be extended to a sheaf $\widetilde{W}$ on X together with an inclusion $\widetilde{W} \subset V$ we would be through, for then W would contradict the stability of V. To achieve such an extension two arguments due to Mumford are used.

(A) Weil's Lemma. This essentially says that any line bundle on the generic Y_0 comes from a unique line bundle on X_K where K is the function field of the parameter variety $\{H^0(X, H^{2^m})\}^{n-1}$ through which s varies and Y_0 is the corresponding complete intersection variety over K.

(B) A degeneration argument. One can construct a 1-parameter family $C \to S$ of smooth complete intersection curves in X of degree 2^m degenerating to a reducible curve with two smooth components each of degree 2^{m-1}. The sheaf V gives, by pull back, sheaves on the curves of the family and one can compare the degree of instability of V on the curves of deg 2^m with that of the curves of deg 2^{m-1}.

Using (A), one gets a line bundle L_s on X which restricts to det W on Y_S. Using (B), one can take L_S to be L, independent of s. Then one sees that L admits homomorphism $L \to \overset{r}{\wedge} V$ where r is the rank of W. Using further a boundedness argument and the lemma of Enriques-Serveri we show that there is a $\widetilde{W} \to V$ with det $\widetilde{W} = L$ which restricts to $W \to V$ on Y_S. Thus we are led to the contradiction V is not stable. Hence $V \mid Y_S$ must have been stable to begin with.

3 Narasimhan-Seshadri Theorem for Higher Dimensions

495 **Theorem 2.** *Let X be a projective nonsingular variety of dimension n over the field of complex numbers $\mathbb{C}$. Let H be an ample line bundle on X. Let V be a vector bundle on X with $C_1(V) = 0$ and $C_2(V) \cdot H^{n-1} = 0$. Then V comes from an irreducible unitary representation of the fundamental group $\pi_1(X)$ if and only if V is stable with respect to H.*

Proof. Let $C \to X$ be a complete intersection curve. Then $\pi_1(C) \to \pi_1(X)$ is surjective. Hence an irreducible unitary representation of $\pi_1(X)$ gives by composition an irreducible unitary representation of $\pi_1(C)$. Hence a vector bundle on X associated to such a representation of $\pi_1(X)$ gives on restriction to C a stable bundle by the theorem of Narasimhan-Seshadri. Thus V on X itself must be stable (see Theorem 1 above).

Let V be a stable bundle. The set S of all representations $\rho : \pi_1(X) \to GL(r)$ can obviously be parametrised by an algebraic variety. For example, if $a_1, \ldots, a_g$ are the generators of $\pi_1(X)$ with the relations $R_1, \ldots$ then the above set can be identified with the subvariety of the product $GL(r)^g$ satisfying the relations $R_1, \ldots$ Therefore we can find using the lemma of Enriques-Severi (cf.[3]) an N such that for $m \geq N$ for a general complete intersection variety Y in X of degree m the restriction $\mathrm{Hom}_X(V, W) \to \mathrm{Hom}_Y(V \mid Y, W \mid Y)$ is surjective for all $W \in S$.

496 Further using Theorem 1 we can find a smooth complete intersection surface $Y \subset X$ such that $V \mid Y$ is stable. Then by the result of Donaldson there is an irreducible unitary representation $\rho : \pi_1(Y) \to U(r) \subset GL(r)$ giving $V \mid Y$. Since, by Lefchetz, $\pi_1(Y) \overset{\approx}{\longrightarrow} \pi_1(X)$, ρ gives a representation of $\pi_1(X)$ as well and hence a unitary bundle V_ρ on X. Since $Hom\left(V \mid Y, V_\rho \mid Y\right) \neq 0$ we have a nonzero map $\varphi : V \to V_\rho$ on X which must be an isomorphism since the subvariety $\det \varphi = 0$ does not intersect the surface Y.

References

[1] Donaldson, S. K. Anti self dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. Lond. Math. Soc.*, in press (1984).

[2] Kobayashi, S. Curvature and stability of vector bundles, *Proc. Japan Acad*. 58 A (1982), 158–162.

[3] Mehta V. B., and Ramanthan, A. Semistable sheaves on projective varieties and their restrictions to curves, *Math. Ann.* 258(1982), 213–224. **497**

[4] Metha V. B., and Ramanathan, A. Restriction of stable sheaves and representations of the fundamental group, *Invent. Math.,* 77(1984), 163–172.

[5] Narasimhan, M. S., and Seshadri, C.S. Stable and unitary bundles on a compact Riemann surface, *Ann. Math*., 82 (1965), 540–567.

[6] Ramanan, S. Vector bundles over algebraic curves, in *Proceedings of the I.C.M* Helsinki, 1978.

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road,
Bombay - 400 005.

Line Bundles On Schubert Varieties

By C. S. Seshadri

1 Introduction

Recently a gap has been found by V. Kac in the proof of the main results 499 of the important work of Demazure [1] on Schubert varieties in flag varieties associated to a semi-simple algebraic group G. The purpose of this paper is to justify this work of Demazure; in fact, the conjectures stated b. Demazure in his paper, which essentially mean that his main results hold in arbitrary characteristic or even over Spec $\mathbb{Z}$, also follow.

The main contribution of this paper is the proof of the normality of a Schubert variety (in arbitrary characteristic). This, together with a recent beautiful work of V. B. Mehta and A. Ramanathan [4], give the required justification of the work of Demazure, as well as his conjectures over Spec $\mathbb{Z}$.

Recently, A. Joseph [2] has justified "Demazure's character formula" for large dominant weights (in characteristic zero). It can be easily seen that this is equivalent to showing that a Schubert variety 500 is normal (in characteristic zero). Combined with the above cited work of V. B. Mehta and A. Ramanathan, this work of Joseph also leads to a justification of the main results of Demazure [1], but not his conjectures over Spec $\mathbb{Z}$.

When G is a *classical group*, the work of Demazure (as well as his conjectures over Spec $\mathbb{Z}$) is also a consequence of "Standard Monomial Theory" (cf. [3], [5], [6])[1]

---

[1] In the lecture, we spoke only about this. The written version presents a later development.

Addendum (January, 1985)

This paper was written around May 1984 and represents the state of affairs around that time in justifying the work of Demazure [1]. Soon after, S. Ramanan and A. Ramanathan (cf. Projective normality of Flag Varieties and Schubert Varieties” Invent. math., 79(1985), 217–224) proved Theorem 2 of this paper as well as the normality of Schubert varieties by methods related to [4]. Recently, A. Ramanathan (cf. “Schubert varieties are arithmetically cohen-Macaulay”. Invent. Math. 80 (1985) 283–294) has proved the arithmetic Cohen-Macaulay nature of Schubert varieties in *arbitrary characteristic* (cf. Remark 5 of this paper for the case of characteristic zero). The arithmetic Cohen-Macaulay nature of Schubert varieties (for arbitrary characteristic) was known earlier only for special classes of Schubert varieties as a consequence of standard monomial theory (e.g. $G = SL(n)$).

2 Normality of a Schubert variety

501 Let G be a semi-simple, simply-connected, Chevalley group defined over a field k. Let us fix a maximal torus T and a Borel subgroup B containing T. Let $W = \dfrac{N(T)}{T}$ be the Weyl group, $N(T)$ being the normalizer of T. We talk of roots, weights etc. with respect to T, B.

For $w \in W$, let $e(w)$ denote the image in $\dfrac{G}{B}$ of a representative of w in $N(T)$. It is a fixed point for the canonical action of T on $\dfrac{G}{B}$. Let $X(w)$ denote the closure in $\dfrac{G}{B}$ of the “Bruhat cell” $B\,e(w)$ $\left(\text{in the Zariski topology of } \dfrac{G}{B}\right)$, endowed with canonical structure of a reduced subscheme of $\dfrac{G}{B}$. We call $X(w)$ the *Schubert variety* assoicated to $w \in W$. Then we have the following:

Theorem 1. *Every Schubert variety* $X(w)$ *in* $\dfrac{G}{B}$ $(w \in W)$ *is normal.*

Proof. The proof is by decreasing induction on the length

$$l(w) = \dim X(w) \quad \text{of} \quad X(w).$$

If $w = w_0$, the element of maximal length in W; $X(w_0) = \dfrac{G}{B}$ which is smooth, in particular, therefore normal. We assume now that every Schubert variety X, such that $\dim X > l(w)$, is normal.

Let $\tau = s_\alpha w$ be such that α is a simple root and $l(\tau) = l(W) + 1$, so that $X(w) \subset X(\tau)$ and $l(w) = l(\tau) - 1$. By the induction hypothesis, $X(\tau)$ is normal. Let P_α be the minimal parabolic subgroup of G, associated to α. Then one knows that $X(\tau)$ is P_α-stable. Then we have a canonical morphism:

$$\begin{cases} P_\alpha \times X(w) \to X(\tau) \text{ defined by} \\ (p, x) \to p \cdot x \end{cases}$$

Set $Z = P_\alpha \times^B X(w)$ i.e. the set of equivalence classes under the equivalence relation 502

$$(p, x) \sim \left(pb, b^{-1}x\right) \text{ for some } b \in B(p \in P_\alpha, X \in Xw).$$

Then the above morphism goes down to a morphism:

$$\Psi : Z \to X(\tau)$$

It is seen without much difficulty that Ψ is a birational P_α morphism and that we have a canonical morphism:

$$p : Z \to \mathbb{P}^1, \mathbb{P}^1 \simeq \frac{P_\alpha}{B}$$

We note that p is a (locally trivial) fibre space with base $\mathbb{P}^1$ and fibre $X(w)$; in fact, it is the fibre space with fibre $X(w)$, associated to the principal fibration $P_\alpha \to \mathbb{P}^1$ with structure group B (B acts on $X(w)$).

Let us take the normalization morphism $\widetilde{X(w)} \to X(w)$. Observe that we have a natural action of B on $\widetilde{X(w)}$ and that this map is a B-morphism. Set $\widetilde{Z} = P_\alpha \times^B \widetilde{X(w)}$. We denote by $\widetilde{\Psi}$ the canonical morphism:

$$\widetilde{\Psi} : \widetilde{Z} \to X(\tau)$$

Again we have a canonical morphism 503

$$\widetilde{p} : \widetilde{Z} \to \mathbb{P}^1$$

which is a fibre space over $\mathbb{P}^1$ with fibre $\widetilde{X(w)}$. Then we have the following:

(i) $\widetilde{Z} \xrightarrow{v} Z \xrightarrow{\Psi} X(\tau), \widetilde{\Psi} = \overline{\Psi} \circ v$

(ii)
$$\begin{tikzcd}
\widetilde{Z} \arrow[rr, "v"] \arrow[dr, "\widetilde{P}"'] & & Z \arrow[dl, "P"] \\
& \mathbb{P}^1 &
\end{tikzcd} \qquad \widetilde{p} = p \circ v$$

Let $\mathscr{F}$ be an object on $X(w)$ $\left(\text{resp.}\widetilde{X(w)}\right)$, say a line bundle on which B acts consistent with the canonical action of B on $X(w)\left(\text{resp.}\widetilde{X(w)}\right)$. We denote by $\mathscr{F}$ the “associated” object on Z $\left(\text{resp.}\widetilde{Z}\right)$ i.e. $\mathscr{F} = P_\alpha \times^B \mathscr{F}$. For a line bundle M on $\dfrac{G}{B}$, we denote by $M(\tau)$ (resp.$M(w)$) the restriction of M to $X(\tau)$ resp.$X(w)$) and by $\widetilde{M(w)}$ the pullback of $M(w)$ to $\widetilde{X(w)}$. We have then the following

Lemma 1. *(i)* $\Psi^*(M(\tau)) \simeq M(w)^\sharp$

(ii) $\widetilde{\Psi}^*(M(\tau)) \simeq \widetilde{M(w)}^\sharp$

Proof. We observe that Ψ and $\widetilde{\Psi}$ are P_α-morphisms. Now M is a homogenous line bundle on G/B i.e. G acts on M. Hence P_α acts on $M(\tau)$, so that P_α acts on $\Psi^*(M(\tau))\left(\text{resp.}\widetilde{\Psi}^*(M(\tau))\right)$. Suppose now that H is a 504 group, K a subgroup and $\mathscr{G}$ an object on the space H/K on which H acts (consistent with its action on H/K). Let $\mathscr{G}_0$ denote the “fibre” of $\mathscr{G}$ over the point $e \in H/K$ corresponding to the coset K. We see that the isotropy subgroup of H at e, namely K, acts on $\mathscr{G}_0$. If $\mathscr{G}_0^\sharp$ denotes the object over H/K. “associated” to the K-principal fibre space $H \to H/K$, we see easily that $\mathscr{G} \simeq \mathscr{G}_0^\sharp$ Applying this principle to our case, by taking $H = P_\alpha$, $K = B$ and $\mathscr{G} = \Psi^*(M(\tau))\left(\text{resp.}\widetilde{\Psi}^*(M(\tau))\right)$, we find that $\mathscr{G}_0 \simeq M(w)\left(\text{resp.}\widetilde{M(w)}\right)$ and the lemma follows.

Let us fix an ample line bundle L on G/B (say, associated to $\rho = \left(\dfrac{1}{2}\text{sum of positive roots}\right)$. We use also the notation of Serre, say

$\mathscr{O}_{X(\tau)}(m)$ etc. for the restriction of L^m to $X(\tau)$ (or the associated coherent sheaves). Set

$$V_m = H^0(X(w), \mathscr{O}_{X(w)}(m)), \widetilde{V_m} = H^0\left(\widetilde{X(w)}, \mathscr{O}_{\widetilde{X(w)}}(m)\right)$$

where $\mathscr{O}_{\widetilde{x(w)}}(1)$ denotes the pull-back of L to $\widetilde{X(w)}$ (note that it is again ample). Now V_m and $\widetilde{V_m}$ are B modules. We denote by V_m $\left(\text{resp.}\widetilde{V}_m\right)$ the vector bundles on $\mathbb{P}^1$, associated to the principal B-fibration $P_\alpha \to \mathbb{P}^1$ with fibre V_m $\left(\text{resp.}\widetilde{V}_m\right)$. Then we have the following:

Lemma 2. $H^0\left(\mathbb{P}^1, V_m\right) \simeq H^0\left(\mathbb{P}^1, \widetilde{V}_m\right) \simeq H^0(X(\tau), \mathscr{O}_{X(\tau)}(m))$ *for* $m \geq 0$.

Proof. One knows that the morphism Ψ and $\widetilde{\Psi}$ are birational. Since $X(\tau)$ is normal, it follows that

$$\Psi_*(\mathscr{O}_z)\left(\text{resp.}\widetilde{\Psi_*}(\mathscr{O}_{\widetilde{Z}})\right) \simeq \mathscr{O}_{X(\tau)}.$$

Hence

$$\begin{cases} H^0(Z, \Psi^*(L(\tau)^m)) \simeq H^0(X(\tau), (L(\tau)^m)) \\ = H^0(X(\tau), \mathscr{O}_{X(\tau)}(m)) \\ H^0(\widetilde{Z}, \widetilde{\Psi}^*(L(\tau)^m) \simeq H^0(X(\tau), L(\tau)^m) \\ = H^0(X(\tau), \mathscr{O}_{X(\tau)}(m)) \end{cases} \tag{1}$$

We observe now that P_α acts on $p_*(\Psi^*(L(\tau)^m))\left(\text{resp.}\widetilde{p}_*\left(\widetilde{\Psi}^*(L(\tau)^m)\right)\right)$ 505
consistent with its action on $\mathbb{P}^1$, since Ψ and p(resp.$\widetilde{\Psi}$ and $(\widetilde{p})$ are P_α -morphisms. It follows then easily that these coherent sheaves on $\mathbb{P}^1$ are *locally free*; in fact, by Lemma 1 (and an argument similar to its proof), we see that

$$\begin{cases} p_*(\Psi^*(L(\tau)^m)) \simeq V_m \\ \widetilde{p}_*\left(\widetilde{\Psi}^*(L(\tau)^m)\right) \simeq \widetilde{V}_m \end{cases} \tag{2}$$

On the other hand, we have

$$\begin{cases} H^0\left(\mathbb{P}^1, p_*(\Psi^*(L)(\tau)^m)\right) \simeq H^0(Z, \Psi^*(L(\tau)^m)) \\ \left(\text{resp.}H^0\left(\mathbb{P}^1, p_*\left(\widetilde{\Psi}^*(L(\tau)^m)\right)\right) \simeq H^0\left(\widetilde{Z}, \widetilde{\Psi}^*(L(\tau)^m)\right)\right) \end{cases} \tag{3}$$

Now (1), (2) and (3) imply Lemma 2.

Remark 1. For the proof of Theorem 1, it suffices to use Lemma 2 for $m >> 0$

506 Let us now set $C = \mathscr{O}_{\widetilde{X(w)}}/\mathscr{O}_{X(w)}$ a coherent $\mathscr{O}_{X(w)}$ module. We have then the exact sequence of $\mathscr{O}_{X(w)}$ modules:

$$0 \to \mathscr{O}_{X(w)} \to \mathscr{O}_{\widetilde{X(w)}} \to C \to 0$$

Tensoring by $L(w)^m$, we obtain the exact sequence of $\mathscr{O}_{\widetilde{X(w)}}$-modules:

$$0 \to \mathscr{O}_{X(w)}(m) \to \mathscr{O}_{\widetilde{X(w)}}(m) \to C \to 0 \tag{4}$$

(we use the principle that if $\delta : \widetilde{X(w)} \to X(w)$ is the canonical morphism and N a line bundle (or invertible sheaf) on $X(w)$, we have

$$\delta_* \left(\delta^* N \simeq \mathscr{O}_{\widetilde{X(w)}} \otimes_{\mathscr{O}_{X(w)}} N\right)$$

We observe that

$$H^0(X(w), \mathscr{O}_{\widetilde{X(w)}}(m)) \simeq H^0\left(\widetilde{X(w)}, \widetilde{L(w)}^m\right) \simeq \widetilde{V}_m$$
$$\left(\widetilde{L(w)} \text{ pull-back of } L(w) \text{ on } \widetilde{X(w)}\right).$$

Let us now choose m_0 such that for $m \geq m_0$, one has

$$\begin{cases} (a)\ H^1(X(w), \mathscr{O}_{X(w)}(m)) = 0, \text{ and} \\ (b)\ H^0(X(\tau), \mathscr{O}_{X(\tau)}(m)) \to H^0(X(w), \mathscr{O}_{X(w)}(m)) \to 0 \\ \text{ is exact.} \end{cases} \tag{5}$$

507 Then writing the cohomology exact sequence of (4), we get by (5) (a):

$$\begin{cases} 0 \to V_m \to \widetilde{V}_m \to H^0(X(w), C(m)) \to 0 \\ \text{exact for } m \geq m_0. \end{cases} \tag{6}$$

Set

$$W_m = H^0(X(w), C(m))$$

Now $C(m)$ is a coherent $\mathscr{O}_{X(w)}$ -module on which B operates consistent with its action on $X(w)$ (since $\widetilde{X(w)} \to X(w)$ is a B morphism etc.).

Hence W_m is also a B-module. We denote by W_m the vector bundle on $\mathbb{P}^1$, associated to the B module W_m Then we get the following exact sequence of vector bundles

$$0 \to V_m \to \widetilde{V}_m \to W_m \to 0 \tag{7}$$

Now we claim that

$$H^1\left(\mathbb{P}^1, V_m\right) = 0, m \geq m_0. \tag{8}$$

To prove (8), let $E = H^0(X(\tau), \mathscr{O}_{X(\tau)}(m))$. Then the canonical map $E \to V_m$ is *surjective* for $m \geq m_0$ by (5) (b). let $K = \ker(E \to V_m)$. Observe that E is a P_α -module (since $X(\tau)$ is P_α -stable). Hence the associated vector bundle E on $\mathbb{P}^1$ is trivial. Thus we get the following exact sequence of vector bundles on $\mathbb{P}^1$:

$$0 \to K \to E \to V_m \to 0 \tag{9}$$

Since E is trivial, $H^1\left(\mathbb{P}^1, E\right) = 0$. Hence, writing the cohomology exact 508
sequence of (9), we deduce that $H^1\left(\mathbb{P}^1, V_m\right) = 0$ for $m \geq m_0$ (using the fact $H^2\left(\mathbb{P}^1, K\right) = 0$. This proves the assertion (8).

Using the assertion (8), the cohomology exact sequence of (7) leads to the following exact sequence:

$$0 \to H^0\left(\mathbb{P}^1, V_m\right) \to H^0\left(\mathbb{P}^1, V_m\right) \to H^0\left(\mathbb{P}^1, W_m\right) \to 0, m \geq m_0, \tag{10}$$

Now Lemma 2 implies that

$$H^0\left(\mathbb{P}^1, V_m\right) \xrightarrow{\sim} H^0\left(\mathbb{P}^1, \widetilde{\mathbb{V}}_m\right), m \geq 0.$$

Hence, we conclude that

$$H^0\left(\mathbb{P}^1, W_m\right) = 0 \text{ for } m \geq m_0.$$

$$\text{(Recall that } W_m = H^0(X(w), C(m))\text{)).}$$

Observe that

$$C \neq 0 \Leftrightarrow X(w) \text{ is not normal.}$$

Further, if Q is the parabolic subgroup of G generated by B and the minimal parabolic subgroups P_β (β simple) such that P_β leaves $X(w)$-stable, we see that Q leaves $X(w)$ stable and C is a $Q - \mathscr{O}_{X(w)}$ -module. Hence, the required normality of $X(w)$ is a consequence of the following: **509**

Lemma 3. *Let $\mathscr{F}$ be a coherent $Q - \mathscr{O}_{X(w)}$ module. Let $W_m = H^0(X(w), \mathscr{F}(m))$ (note that $\mathscr{F}(m)$ is also a $Q - \mathscr{O}_{X(w)}$-module). Then if $\mathscr{F} \neq 0$, there is a simple root α which moves $X(w)$ (i.e. if $\tau = s_\alpha w, \tau > w$) such that*

$$H^0\left(\mathbb{P}^1, W_m\right) \neq 0 \textit{ for } m >> 0$$

(where W_m is the vector bundle on $\mathbb{P}^1 = P_\alpha/B$, defined as above, associated to W_m).

Proof. Let $J = \operatorname{Ann}\mathscr{F}$ (annihilator of $\mathscr{F}$ - an ideal sheaf of $\mathscr{O}_{X(w)}$). Let $\mathscr{O}_Y = \mathscr{O}_{X(w)/J}$. We observe that J is a Q-sheaf or that Y is a Q-scheme. Then Y_{red} is also a Q-scheme. We observe that $\mathscr{F}$ "lives on" Y i.e. it is the canonical extension of a $Q - \mathscr{O}_Y$-module. We denote this $Q - \mathscr{O}_Y$ -module by the same letter $\mathscr{F}$. Let I be the ideal sheaf of $\mathscr{O}_Y$ such that

$$\mathscr{O}_{Y_{\text{red}}}. = \frac{\mathscr{O}_Y}{I}$$

We observe that I is a $Q - \mathscr{O}_Y$-module and that there exists an integer n such that

$$I^n\mathscr{F} = (0) \text{ and } I^{n-1}\mathscr{F} \neq (0), n \geq 1.$$

510 (note $\mathscr{F} \neq (0)$). If $n = 1$, the above relation means that $I \cdot \mathscr{F} = (0)$. Set $\mathscr{G} = I^{n-1}\mathscr{F}$. Since all the powers of I are again $Q - \mathscr{O}_Y$ sheaves, we see that $\mathscr{G}$ is also a $Q - \mathscr{O}_Y$ sheaf. Further $\mathscr{G} \neq (0)$. We observe that it suffices to prove the lemma for $\mathscr{G}$ for if $W'_m = H^0(X(w), \mathscr{G}(m))$, we have the exact sequence:

$$0 \to H^0(X(w), \mathscr{G}(m)) \to H^0(X(w), \mathscr{F}(m))$$

which gives the exact sequence of vector bundles on $\mathbb{P}^1$.

$$0 \to W'_m \to W_m$$

which, in turn, gives the exact sequence

$$0 \to H^0\left(\mathbb{P}^1, W_m'\right) \to H^0\left(\mathbb{P}^1, W_m\right)$$

Hence

$$H^0\left(\mathbb{P}^1, W_m\right) \neq (0) \Rightarrow H^0\left(\mathbb{P}^1, W_m\right) \neq (0).$$

This proves the above claim that it suffices to prove the lemma for $\mathscr{G}$.

We observe that $I \cdot \mathscr{G} = 0$ i.e. $\mathscr{G}$ lives on Y_{red}. Let $Y_1, \ldots, Y_r$ denote the distinct irreducible components of Y_{red}. These are all Q-stable (Q is connected). We can also suppose that G does not live on Y' such that $Y' \neq Y$ and Y' is the union of some irreducible components of Y (otherwise, we replace Y by the corresponding union etc.). Let I_1 be the ideal sheaf of $\mathscr{O}_{Y_{\text{red}}}$ defining the closed subscheme Y_1 of Y_{red} and I_2 **511** the ideal sheaf of $\mathscr{O}_{Y_{\text{red}}}$ defining the closed subschemes $Y_2 \cup \cdots \cup Y_r$ of Y_{red}. Set $\mathscr{G}_1 = I_2 \cdot \mathscr{G}$. Then $\mathscr{G}_1$ is a $Q - \mathscr{O}_{Y_{\text{red}}}$ subsheaf of $\mathscr{G}$ (the sheaf of sections of $\mathscr{G}_1$ are those of $\mathscr{G}$ which vanish on $Y_2 \cup \cdot \cup Y_r$). Observe that I_1 annihilates $\mathscr{G}_1$, so that $\mathscr{G}_1$ lives on Y_1 $\left(\text{for } \mathscr{O}_{Y_1} = \dfrac{\mathscr{O}_{Y_{\text{red}}}}{I_1}\right)$. Now $\mathscr{G}_1 \neq (0)$ for if $G_1 = I_2\mathscr{G} = (0)$, $\mathscr{G}$ would live on $Y_2 \cup \cdots \cup Y_r$, which is not the case. Thus, by an argument as above, it suffices to prove the lemma for $\mathscr{G}_1$.

Let $\mathscr{G}$ denote the torsion subsheaf of $\mathscr{G}_1$, $\mathscr{G}_1$ being now considered as a $Q - \mathscr{O}_{Y_1}$ -module ($Y_1 - aQ$ stable Schubert subvariety of $X(w)$). We observe that $\mathscr{G}'$ is a $Q - \mathscr{O}_{Y_1}$ -sub-module of $\mathscr{G}_1$. If $\mathscr{G}' \neq (0)$, it suffices to prove the lemma for $\mathscr{G}'$. Observe that the support of $\mathscr{G}'$ is a closed subscheme of Y_1, *properly contained in* Y_1. Repeating the above procedure, we would get a $Q - \mathscr{O}_{Y_2}$ - module $\mathscr{G}_2$, $\mathscr{G}_2 \neq (0)$, such that Y_2 is a Schubert variety *properly contained in* Y_1, and it would suffice to prove the lemma for $\mathscr{G}_2$. If we repeat this procedure and everytime we get that the torsion subsheaves $\mathscr{G}_2'\mathscr{G}_3', \ldots$ etc. are $\neq (0)$, we would get an infinite strictly decreasing chain of Schubert varieties, which would lead to a contradiction. Therefore, at some point the torsion sheaf would be (0).

Thus, as a consequence of the above method, it would suffice to prove the lemma when $\mathscr{F}$ lives on a Schubert subvariety $X(\theta)$ of $X(w)$, such that considered as a sheaf on $X(\theta)$, $\mathscr{F}$ is *torsion free* (in particular

$\neq (0)$). Note that $\mathscr{F}$ is a $Q - \mathscr{O}_{X(\theta)}$-module. Hence $X(\theta)$ is Q-stable.
512 Let α be a simple root which moves $X(\theta)$ i.e. if $\varphi = s_\alpha\theta$, then $\varphi > \theta$. Note that α also moves $X(w)$, for otherwise, P_α leaves $X(w)$ stable i.e. $P_\alpha \subset Q$ and hence P_α leaves $X(\theta)$ stable, which is not the case.

Now by the Borel fixed point theorem, $\mathscr{F}(m_0)$ and a non-zero section $s \in \mathscr{F}(m_0)$ such that the line through s is B-fixed. Now multiplication by s induces an inclusion

$$j : \mathscr{O}_{X(\theta)} \to \mathscr{F}(m_0),$$
$$f \to f \cdot s \cdot$$

Let $\mathscr{G}$ denote the image of $\mathscr{O}_{X(\theta)}$ in $\mathscr{F}(m_0)$. Now $\mathscr{G}$ is a $B - \mathscr{O}_{X(\theta)}$ submodule of $\mathscr{F}(m_0)$. Note that $\mathscr{G}$ need not be B-isomorphic to $\mathscr{O}_{X(\theta)}$ (the action of B on $\mathscr{G}$ differs from that on $\mathscr{O}_{X(\theta)}$ by a character of B, namely the character $\mathcal{X}$ which defines the action of B on the line through s). It would suffice to prove the lemma for $\mathscr{G}$ i.e. if

$$W'_m = H^0(X(\theta), \mathscr{G}(m)) = H^0(X(w), \mathscr{G}(m)),$$

($\mathscr{G}$ being considered canonically as an $\mathscr{O}_{X(w)}$ -module), then $H^0\left(\mathbb{P}^1 . W_m\right)$ $\neq (0)$ for $m >> 0$. Note that the action of B on $\mathscr{G}(m)$ (for all m) differs by the same character $\mathcal{X}$ from the action on $\mathscr{O}_{X(w)}(m)$. Consider the canonical morphism

$$\Psi : Z = P_\alpha \times^B X(\theta) \to X(\varphi) \ (\varphi = s_\alpha\theta)$$

513 We note that Ψ is *birational* (since $\varphi > \theta$). We have again the fibration:

$$p : Z \to \mathbb{P}^1 \left(= \frac{P_\alpha}{B} \right)$$
(fibre type $X(\theta)$)

Consider the line bundle $\mathscr{G}(m)^\sharp$ on Z, which is associated to the $B - \mathscr{O}_{X(\theta)}$ -module $\mathscr{G}(m)$. We see that

$$\mathscr{G}(m)^\sharp = \left(\mathscr{O}_{X(\theta)}(m)^\sharp\right) \otimes N$$

where N is a line bundle on Z (or rather the corresponding sheaf), which comes from $\mathbb{P}^1$ (N is the line bundle on $\mathbb{P}^1 = \dfrac{P_\alpha}{B}$, associated to the character $\mathcal{X}$ of B). We have seen that (by Lemma 1)

$$(\mathscr{O}_{X(\theta)}(m))^\sharp = \Psi^* \left(\mathscr{O}_{X(\phi)}(m)\right).$$

Then we get

$$p_* \left(\mathscr{G}(m)^\sharp\right) = p_* \left(\mathscr{O}_{X(\varphi)}(m)^\sharp\right) \otimes N$$

where we use the same notation N for the line bundle on $\mathbb{P}^1$ as well as its inverse image by p. Setting V_m to be B-module

$$V_m = H^0(X(\theta), \mathscr{O}_{X(\theta)}(m))$$

and V_m the vector bundle on $\mathbb{P}^1$, associated to the principal B-fibration $P_\alpha \to \mathbb{P}^1 = \dfrac{P_\alpha}{B}$, we see that

$$p_* \mathscr{O}_{X(\theta)}(m)^\sharp \cong V_m$$

the proof being as in Lemma 2 (see (2) in the proof of Lemma 2). Hence **514**

$$p_* \left(\mathscr{G}(m)^\sharp\right) \simeq V_m \otimes N.$$

If now

$$W_m = H^0(X(w), \mathscr{G}(m)).$$

We see that

$$p_* \left(\mathscr{G}(m)^\sharp\right) \simeq W_m \simeq V_m \otimes N$$

We see also that

$$H^0 \left(\mathbb{P}^1, W_m\right) \simeq H^0 \left(Z, \mathscr{G}(m)^\sharp\right)$$

Thus to show that

$$H^0 \left(\mathbb{P}^1, W_m\right) \neq (0) \text{ for } m >> 0$$

it suffices to show that

$$H^0 \left(Z, \mathscr{G}(m)^\sharp\right) \neq (0) \text{ for } m >> 0.$$

i.e.

$$H^0\left(Z, \Psi^*\left(\mathscr{O}_{X(\varphi)}(m)\right)\otimes \mathscr{O}_Z N\right) \neq (0) \text{ for } m >> 0.$$

Now 515

$$H^0\left(Z, \Psi^*\left(\mathscr{O}_{X(\varphi)}(m)\right)\otimes_{\mathscr{O}_Z} N\right) = H^0(X(\varphi), \Psi_*(\triangle))$$

where

$$\triangle = \Psi^*\left(\mathscr{O}_{X(\varphi)}(m)\right)\otimes_{\mathscr{O}_Z} N.$$

We find that

$$\Psi_*(\triangle) = \Psi_*(N)\otimes \mathscr{O}_{X(\varphi)}\mathscr{O}_{X(\varphi)}(m).$$

Since Ψ is birational and N is a *line bundle*, we see that $\Psi_*(N) \neq (0)$. Hence by Serre's theorems, we have

$$H^0(X(\varphi), \Psi_*(N)\otimes \mathscr{O}_{X(\varphi)}(m)) \neq (0) \text{ for } m >> 0.$$

This proves $(*)$. Hence Lemma 3. follows and therefore Theorem 1 as well.

3 The work of Demazure.

Proposition 1. Let τ, $w \in W$ be a such that $\tau = s_\alpha w$, α simple and $\tau > w$. Let $Z = P_\alpha \times^B X(w)$ (P_α -minimal parabolic subgroup of G associated to α) and Ψ the birational morphism $\Psi : Z \to X(\tau)$ (as in Theorem 1). Then we have the following:

(i) $R^0\Psi_*(\mathscr{O}_Z) = \mathscr{O}_{X(\tau)}$

(ii) $R^q\Psi_*(\mathscr{O}_Z) = 0, q > 0$

516 (iii) For any line bundle M on $X(\tau)$, we have $H^i(X(\tau), M) \simeq H^i(Z, \Psi^*(M))\ \forall i$.

Proof. As is well-known (iii) is a consequence of (i) and (ii) and (i) follows from the normality of $X(\tau)$ (of Theorem 1). Hence we have only to prove (ii). We fix an ample line bundle L on G/B and denote by $\mathscr{O}_{X(\tau)}(m)$ the restriction of L^m to $X(\tau)$. Set $V_m = H^0(X(w), \mathscr{O}_{X(w)}(m))$

and V_m the bundle on $\mathbb{P}^1 = \dfrac{P_\alpha}{B}$, associated to the principal fibration $p : P_\alpha \to \dfrac{P_\alpha}{B}$. Let p denote the canonical morphism $p : Z \to \mathbb{P}^1$. We claim that

$$R^q\left(p_*\Psi^*\left(\mathscr{O}_{X(\tau)}(m)\right)\right) = 0, q > 0 \text{ and } m >> 0. \tag{1}$$

To see this, we first observe that by Lemma 1

$$\Psi^*\left(\mathscr{O}_{X(\tau)}(m)\right) \simeq \mathscr{O}_{X(w)}(m)^\sharp$$

$\mathscr{O}_{X(w)}(m)^\sharp$ being the line bundle on Z, "associated to" the line bundle $\mathscr{O}_{X(w)}(m)$ on $X(w)\left(\text{ for the fibration } P_\alpha \to \dfrac{P_\alpha}{B}\right)$. Now

$$H^i(X(w), \mathscr{O}_{x(w)}(m)) = 0, i > 0, m >> 0.$$

Now (i) is an immediate consequence of the fact that p is a locally trivial fibre space of fibre type $X(w)$ (in fact, the fibre of p over the point corresponding to the coset B can be canonically identified with $X(w)$). By the usual Leray spectral sequence argument, $(*)$ implies that

$$\begin{cases} H^P\left(\mathbb{P}^1, p_*\Psi^*\left(\mathscr{O}_{X(\tau)}(m)\right)\right) \simeq H^P\left(Z, \Psi^*\left(\mathscr{O}_{X(\tau)}(m)\right)\right) \\ \text{for all } p. \end{cases} \tag{2}$$

Now by Lemma 1 and (8) of Theorem 1, we have 517

$$\text{(a) } p_*\Psi^*\left(\mathscr{O}_{X(\tau)}(m)\right) \simeq V_m, \text{ and}$$
$$\text{(b) } H^i\left(\mathbb{P}^1, V_m\right) = 0, i > 0, m >> 0.$$

Hence by (2), we conclude that

$$H^p\left(Z, \Psi^*\left(\mathscr{O}_{X(\tau)}(m)\right)\right) = 0, p > 0, m >> 0. \tag{3}$$

We claim now that

$$\begin{cases} H^0(X(\tau), \left(R^9\Psi_*(\mathscr{O}_Z)\right) \otimes \mathscr{O}_{X(\tau)}(m)) \simeq \\ H^q(Z, \Psi^*\left(\mathscr{O}_{X(\tau)}(m)\right), q > 0, m >> 0. \end{cases} \tag{4}$$

To prove this consider the Leray spectral sequence

$$\begin{cases} H^p\,(X(\tau), R^q\Psi_*\,(\Psi^*\,(\mathscr{O}_{X(\tau)}(m)))) \\ \Rightarrow H^{p+q}\,(Z, \Psi^*\,(\mathscr{O}_{X(\tau)}(m))) \end{cases} \tag{5}$$

Now we have

$$R^q\Psi_*\,(\Psi^*\,(\mathscr{O}_{X(\tau)}(m))) \simeq (R^q\Psi_*(\mathscr{O}_Z)) \otimes \mathscr{O}_{X(\tau)}(m).$$

518 so that

$$\begin{aligned} &H^p(X(\tau), R^q\Psi_*\,(\Psi^*\,(\mathscr{O}_{X(\tau)}(m))) \simeq \\ &H^p(X(\tau), (R^q\Psi_*(\mathscr{O}_Z)(m)) \end{aligned} \tag{6}$$

One has

$$H^p(X(\tau), (R^q\Psi_*(\mathscr{O}_Z))\,(m)) = 0, p > 0, m >> 0.$$

Hence the spectral sequence (5) degenerates and we get

$$\begin{gathered} H^0(X(\tau), (R^q\Psi_*(\mathscr{O}_Z))\,(m)) \simeq \\ H^q\,(Z, \Psi^*\,(\mathscr{O}_{X(\tau)}(m)))\,, m >> 0. \end{gathered} \tag{7}$$

Using (3), we deduce that

$$H^0(X(\tau), (R^q\Psi_*(\mathscr{O}_Z))\,(m)) = 0, q > 0, m >> 0. \tag{8}$$

Now (8) implies that

$$R^q\Psi_*(\mathscr{O}_Z) = 0, q > 0.$$

This completes the proof of Proposition 1.

Theorem 2. *Let $X(\tau)$, $\tau \in W$, be a Schubert variety in G/B and L a line bundle on G/B associated to a dominant weight. Then we have the following:*

519 *(i) The canonical restriction map*

$$H^0\left(\frac{G}{B}, L\right) \to H^0(X(\tau), L)$$

is surjective, and

(ii) $H^i(X(\tau), L) = 0, i > 0.$

Proof. These follow essentially from the results of V. B. Mehta and A. Ramanathan [4] where they prove the analogues of (i) and (ii) for an *ample* line bundle L on G/P, P a parabolic subgroup of G. To apply their result, if L is as in the above theorem, note that it comes from an ample line bundle L' on G/P for a well determined parabolic subgroup P of G. Let X' denote the image of X in G/P. One knows that the cohomology groups of L' are preserved by pull-back to $\frac{G}{B}$; in fact, by Theorem 1 and proofs similar to that of Prop.1, one can show easily that the cohomology groups of L'/X' are also preserved by pull-back to X. This proves Theorem 2.

Let $\mathbb{Z}[N]$ denote the group ring of the multiplicative group $\exp N$, where

$$\exp N = \{\exp\lambda \mid \lambda \in N\} \text{ and } N = \text{Hom }(T, \mathbb{G}_m).$$

Let $X(w)$ and $X(\tau)$ be Schubert varieties (as in Prop. 1) such that $\tau = S_\alpha w, \tau > w$, α simple. Let L_λ denote the line bundle on G/B, "associated" to a *dominant weight* λ (we adopt the convention that when the base field is of characteristic zero, $H^0(G/B, L_\lambda)$ is the dual of the irreducible 520 module with highest weight λ). Now the "characters" of the T modules $H^0(X(w), L_\lambda)$ and $H^0(X(\tau), L_\lambda)$ are elements of $\mathbb{Z}[N]$ and are denoted respectively by $F(w)$ and $F(\tau)$. Let L_α be the linear operator $L_\alpha : \mathbb{Z}[N] \to \mathbb{Z}[N]$ defined by

$$L_\alpha(\exp\lambda) = \frac{\exp\lambda - \exp(S_\alpha\lambda)}{1 - \exp\alpha}, \lambda \in N$$

Let $M_\sim$ be the operator $M_\alpha : \mathbb{Z}[N] \to \mathbb{Z}[N]$, defined by

$$\begin{cases} M_\alpha(\exp\lambda) = (\exp p).L_\alpha(\exp(\lambda - p)) \\ \rho = \dfrac{1}{2} - \text{ sum of positive roots} \end{cases}$$

Theorem 3. *We have the following "character formula".*

$$M_\alpha(F(w)) = F(\tau) \textit{ with } F \textit{ (identity)} = \exp(-\lambda).$$

Proof. By Theorem 2, the following sequence is exact:

$$H^0(X(\tau), L_\lambda) \to H^0(X(w), L_\lambda) \to 0 \tag{$*$}$$

Set

$$E = H^0(X(\tau), L_\lambda), V = H^0(X(w), L_\lambda). \tag{1}$$

521 We denote by $\mathbb{E}$ and $\mathbb{V}$ the vector bundles on $\mathbb{P}^1 = \dfrac{P_\alpha}{B}$, associated to the principal B-fibration $P_\alpha \to \dfrac{P_\alpha}{B}$ We see that $\mathbb{E}$ is trivial and we have the following exact sequence of vector bundles on $\mathbb{P}^1$:

$$0 \to K \to \mathbb{E} \to V \to 0 \tag{2}$$

Writing the cohomology exact sequence for (2), then we conclude (as for the proof of (8) of Theorem 1) that

$$H^i\left(\mathbb{P}^1, \mathbb{V}\right) = 0, i \geq 1. \tag{3}$$

If Ψ and p denote the canonical maps

$$\Psi : Z \to X(\tau) \text{ and } p : Z \to \mathbb{P}^1 \left(= \frac{P_\alpha}{B}\right)$$

We conclude as in the proof of Lemma 2, that

$$p_* \left(\Psi^*(L_\lambda)\right) \simeq \mathbb{V} \tag{4}$$

Now (4) implies that

$$H^0\left(Z, \Psi^*(L_\lambda)\right) \simeq H^0\left(\mathbb{P}^1, \mathbb{V}\right) \tag{5}$$

and by Prop. 1, we have

$$H^0\left(Z, \Psi^*(L_\lambda)\right) \simeq H^0(X(\tau), L_\lambda)$$

so that we find that

$$H^0\left(\mathbb{P}^1, \mathbb{V}\right) \simeq H^0(X(\tau), L_\lambda) \tag{6}$$

522 In view of (3) and (6), we have:

$$\begin{cases} \text{Char } H^0(X(\tau), L_\lambda) = \text{ Char } X\left(\mathbb{P}^1, \mathbb{V}\right) \\ V = H^0(X(w), L_\lambda) \end{cases} \tag{7}$$

Now it is easily seen that Char $X\left(\mathbb{P}^1, \mathbb{V}\right)$ is obtained by applying the operator M_α to V (essentially the Weyl character formula for $SL(2)$, for more details, see [6]). This proves Theorem 3.

Remark 2. The formula of Theorem 3 is essentially the Demazure character formula (of [1] and [5]). If the dominant weight λ is "sufficiently large". $(*)$ in the proof of Theorem 2 is an immediate consequences of the basic theorems of Serre on projective varieties (for this purpose, we may have to work on $\dfrac{G}{P}$, P being a parabolic setgroup, such that L_λ comes from an ample line bundle on $\dfrac{G}{P}$ etc.). Hence, Demazure's character formula for sufficiently large λ, is a consequence of the normality of Schubert varieties and one does not need Theorem 2. On the other hand, as we mentioned in the introduction, the Demazure character formula for sufficiently large λ, implies the normality of Schubert varieties. The proof is by increasing induction on the dimension of Schubert varieties. With the notations as in Theorem 1, it suffices to show that $X(\tau)$ is normal supposing that $X(w)$ is normal. Then Z is normal. We see easily (as in the above proofs) that for sufficiently large λ (say for $L_{m\lambda}, m >> 0$ L_λ ample on G/B), $H^0\left(Z, \Psi^*(L_{m\lambda})\right)$ is given by the Demazure character formula. Hence, by our hypothesis, we deduce that

$$H^0(X(\tau), L_{m\lambda}) \xrightarrow{\sim} H^0(Z, \Psi^*(L_{m\lambda})), m >> 0.$$

From this we deduce easily that $\Psi_*(\mathscr{O}_Z) \simeq \mathscr{O}_{X(\tau)}$, which implies that 523
$X(\tau)$ is normal, since Z is normal.

Remark 3. Let $G_{\mathbb{Z}}$ denote the semi-simple, simply connected Chevalley group scheme over $\mathbb{Z}$ such that $G = G_{\mathbb{Z}} \times_{\text{Spec } \mathbb{Z}} \text{Spec } k$. We have a Borel subgroup scheme $B_{\mathbb{Z}}$ of $G_{\mathbb{Z}}$, corresponding to B and we have the "flag scheme" $\dfrac{G_{\mathbb{Z}}}{B_{\mathbb{Z}}}$ which behaves well under base change with respect to any field e.g. $\dfrac{G_{\mathbb{Z}}}{B_{\mathbb{Z}}} \times_{\text{Spec } \mathbb{Z}} \text{Spec } k \simeq G/B$. For any $\tau \in W$, we define the Schubert subscheme $X_{\mathbb{Z}}(\tau)$ of $\dfrac{G_{\mathbb{Z}}}{B_{\mathbb{Z}}}$ as the "flat closure" in $\dfrac{G_{\mathbb{Z}}}{B_{\mathbb{Z}}}$ of the Schubert scheme associated to τ in $\dfrac{G_{\mathbb{Z}}}{B_{\mathbb{Z}}} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Q}$. We claim that

$$X_{\mathbb{Z}}(\tau) \times_{\text{Spec } \mathbb{Z}} \text{Spec } k \simeq X(\tau) \tag{1}$$

We refer to (1) as saying that the Schubert scheme $X_{\mathbb{Z}}(\tau)$ "behaves well" under base change with respect to any field. To prove (1) let us denote the following base changes by Spec k as

$$\begin{cases} \overline{X(\tau)} = X_{\mathbb{Z}}(\tau) \times_{\text{Spec } \mathbb{Z}} \text{Spec } k \\ \overline{L}_{\lambda} = L_{\lambda,\mathbb{Z}} \times_{\text{Spec } \mathbb{Z}} \text{Spec } k \end{cases} \tag{2}$$

Then we see easily that

$$\overline{X(\tau)}_{\text{red}} = X(\tau) \tag{3}$$

524 Let now λ be a dominant weight such that the associated line bundle L_{λ} on G/B is very ample. One can also associate to λ a line bundle $L_{\lambda,\mathbb{Z}}$ on $\dfrac{G_{\mathbb{Z}}}{B_{\mathbb{Z}}}$, which is relatively ample with respect to $\mathbb{Z}$. Let us denote by $X_{\mathbb{Q}}(\tau)$ $\left(\text{resp.} L_{\lambda,\mathbb{Q}}\right)$ the base change of $X_{\mathbb{Z}}(\tau)$ (resp.$L_{\lambda,\mathbb{Z}}$) by Spec $\mathbb{Q} \to$ Spec $\mathbb{Z}$. Then since $X_{\mathbb{Z}}(\tau)$ is $\mathbb{Z}$ -flat, we see that

$$\dim H^0\left(\overline{X(\tau)}, \overline{L}_{\lambda}^{m}\right) = \dim H^0\left(X_{\mathbb{Q}}(\tau), L_{\lambda,\mathbb{Q}}^{m}\right), m >> 0. \tag{4}$$

On the other hand, by Theorem 2, we see that the character formula is independent of the base field.

This fact implies that

$$\dim H^0\left(X_{\mathbb{Q}}(\tau), L_{\lambda,\mathbb{Q}}^{m}\right) = \dim H^0\left(X(\tau), L_{\lambda}^{m}\right), m >> 0. \tag{5}$$

Combining (4); (5) we get

$$\dim H^0\left(\overline{X(\tau)}, \overline{L}_{\lambda}^{m}\right) = \dim H^0\left(X(\tau), L_{\lambda}^{m}\right), \text{ for } m >> 0. \tag{6}$$

Since $X(\tau)_{\text{red}} = X(\tau)$, we deduce easily from (6) that $X(\tau) = \overline{X(\tau)}$. This proves (1).

Remark 4. Let λ be a dominant weight and let us use the notations as in Remark 3. Then as a consequence of Remark 3 and the vanishing theorem (see (ii) of Theorem 2), we deduce that

$$H^0\left(X_{\mathbb{Z}}(\tau), L_{\lambda,\mathbb{Z}}\right) \otimes_{\mathbb{Z}} k \simeq H^0(X(\tau), L_\lambda) \tag{1}$$

Further, as a consequence of the exactness of 525

$$H^0(G/B, L_\lambda) \to H^0(X(\tau), L_\lambda) \to 0 \tag{2}$$

(see (1), Theorem 2), we deduce easily that the following sequence is exact.

$$H^0\left(\frac{G_{\mathbb{Z}}}{B_{\mathbb{Z}}}, L_{\lambda,\mathbb{Z}}\right) \to H^0\left(X_{\mathbb{Z}}(\tau), L_{\lambda,\mathbb{Z}}\right) \to 0 \tag{3}$$

Let us now set

a. $V_{\lambda,\mathbb{Z}}(\tau) = \left(H^0\left(X_{\mathbb{Z}}(\tau), L_{\lambda,\mathbb{Z}}\right)\right)^*$ (dual)

b. $V_{\lambda,\mathbb{Z}}(w_0) = V_{\lambda,\mathbb{Z}} = \left(H^0\left(\frac{G_{\mathbb{Z}}}{B_{\mathbb{Z}}}, L_{\lambda,\mathbb{Z}}\right)\right)^*$

c. $V_{\lambda,k}(\tau) = V_{\lambda,\mathbb{Z}} \otimes_{\mathbb{Z}} k = \left(H^0\left(X(\tau), L_\lambda\right)\right)^*$

Then because of (1), (2) and (3), we deduce that

$$V_{\lambda,\mathbb{Z}}(\tau) \text{ is a direct summand in } V_{\lambda,\mathbb{Z}} \tag{4}$$

Now $V_{\lambda,\mathbb{Z}}(\tau)$ has a more concrete description as follows.

Let $V_{\lambda,\mathbb{Q}}$ denote the irreducible $G_{\mathbb{Q}} = G_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Q}$ module with highest weight $w_0(\lambda)$ (w_0= weyl involution). Fix a highest weight vector, say $e \in V_{\lambda,\mathbb{Q}}$. Let U denote the enveloping algebra of the Lie algebra of $G_{\mathbb{Q}}$ and $U_{\mathbb{Z}}$(resp.$U^+_{\mathbb{Z}}$, $U^-_{\mathbb{Z}}$) the $\mathbb{Z}$ a subalgebras generated by $\frac{X_\alpha}{n!}$, α any root (resp. positive root, negative root). We now set

$$V'_{\lambda,\mathbb{Z}} = U_{\mathbb{Z}}e (= U^-_{\mathbb{Z}}e) \tag{5}$$

Now any $\tau \in W$ can be represented by a $\mathbb{Z}$ -valued point of $G_{\mathbb{Z}}$ and 526 hence the element $\tau.\ e$ is determined upto the factor ±1. Let us write $e(\tau)$ for $\tau.e$. Set

$$V'_{\lambda,\mathbb{Z}}(\tau) = U^+_{\mathbb{Z}} e(\tau) \tag{6}$$

Now as a consequence of (3), and Theorem 2, it can be shown easily that (for details, see [6])

$$\begin{aligned} V_{\lambda,\mathbb{Z}} &= V'_{\lambda,\mathbb{Z}} \\ V_{\lambda,\mathbb{Z}}(\tau) &= V'_{\lambda,\mathbb{Z}}(\tau) \end{aligned} \tag{7}$$

Hence $V'_{\lambda,\mathbb{Z}}(\tau)$ is a direct summand in $V'_{\lambda,\mathbb{Z}}$. This was conjectured by Demazure in [1].

Remark 5. One can construct canonical desingularigations of $X(\tau)$. This can be done by either following Demazure [1] or refining the construction of Z inductively as follows. Let $\tau = s_\alpha w$ with α simple and $w < \tau$. Suppose we have constructed a desingularigation

$$\Psi_w : Z(w) \to X(w)$$

such that Z_{id} = point. Then we define

$$Z(\tau) = P_\alpha \times^B Z(w)$$

527 and the morphism $\Psi_\tau : Z(\tau) \to X(\tau)$ in the obvious manner. We note that the morphism Ψ_τ depends upon the choice of a reduced decomposition of τ. By repeating the proof of Prop.1, one deduces easily that

(i) $(\Psi_\tau)_* (\mathscr{O}_{z(\tau)}) = \mathscr{O}_{X(\tau)}$,

(ii) $(R^q\Psi_*) (\mathscr{O}_{Z(\tau)} = 0, q > 0$.

When the base field is of *characteristic zero*, we see that (as in Demazure[1]) (i) and (ii) imply that $Z(\tau)$ has only *rational singularities*, in particular that $Z(\tau)$ in Cohen-Macaulay.

References

[1] M. Demazure. Desingularisation des varieties de Schubert generalisees, *Ann. Sc. Ec. Norm. Sup. t.* 7, 1974, p. 53–58.

[2] A. Joseph. On the Demazure character formula (preprint)

[3] V. Lakshmibai and C. S. Seshadri. Geometry of $G/P - V$ Journal of Algebra, 100(1986), p. 362–457.

[4] V. B. Mehta and A. Ramanathan. Frobenius splitting and cohomology vanishing for Schubert varieties, *Ann. Math*, 122(1985), p. 27–40.

[5] V. Lakshmibai, C. Musili and C. S. Seshadri. Geometry of $G/P - IV$ (Standard monomial theory for classical types), *Proc. Indian Acad. Sci.* 88A(1979), P. 280–362. **528**

[6] C. S. Seshadri. Standard monomial theory and the work of Demazure - Advanced Studies in Pure Mathematics I, 1983, *Algebraic Varieties and Analytic Varieties*, p. 355–384.

School of Mathematics,
Tata Institute of Fundamental Research,
Bombay - 400 005.

Present Address:
The Institute of Mathematical Sciences,
Madras - 600 113.

Zero Cycles On A Singular Surface: An Introduction

By V. Srinivas

0

529 This article consists, for the most part, of an introduction to the paper Srinivas (1985) (which will appear elsewhere), and an outline of the proof of the infinite dimensionality theorem for zero cycles which is proved there. The main object under consideration is the Chow (cohomology) group of zero cycles on a normal quasi-projective surface over an algebraically closed field. After briefly surveying the known results for Chow groups of smooth surfaces, we give the various equivalent definitions of the Chow group of a singular surface, and mention results of Collino, Levine, Pedrini, Weibel and the author giving generalisations to the singular case of the above results for smooth surfaces. Next, we introduce relative K-theory and use it to formulate a conjecture about the structure of the Chow group of a normal surface. We then outline the proof of the infinite dimensionality theorem. Finally, we make some remarks about the related problems of the structure of NK_0 and K_{-1}. We give a condition in terms of Picard groups which implies the non-vanishing of NK_0 and K_{-1} of a normal quasi-projective surface over $\mathbb{C}$; we conjecture that this condition is in fact necessary and sufficient for surfaces over any algebraically closed field k.

1 The smooth case

Let X be a smooth surface over an algebraically closed field k. The 530
Chow group of zero cycles on X is defined by

$$CH^2(X) = \frac{\text{Free abelian group on points of } X}{\text{cycles rationally equivalent to zero}}$$

where the group of cycles rationally equivalent to zero is generated by cycles of the form $(f)_C$, where $C \subset X$ is a curve, f a non-zero rational function on C and $(f)_C$ denotes the divisor of f.

The Grothendieck group of vector bundles $K_0(X)$, which also equals the Grothendieck group of coherent sheaves, has a filtration by codimension of support $\left\{F^i K_0(X)\right\}_{0 \leq i \leq 2}$, where

$F^0/F^1 \simeq \mathbb{Z}, F^1/F^2 \simeq \operatorname{Pic} X$, and F^2 is the subgroup of $K_0(X)$ generated by classes of residue fields of points. We have the cycle map $\Psi : CH^2(X) \to F^2K_0(X)$, and the second Chern class $C_2 : F^2K_0(X) \to CH^2(X)$; by the Riemann-Roch theorem (or directly, since X is a surface) Ψ and $-C_2$ are inverse isomorphisms. Thus $CH^2(X)$ has an alternate description as the sub-group of $K_0(X)$ generated by points of X.

Next, $CH^2(X)$ can be interpreted in terms of algebraic K-theory. Let $\mathscr{K}_{2,X}$ be the Zariski sheaf associated to the pre-sheaf $U \to K_2(\Gamma(U, \mathscr{O}_U))$. Then by a result of Quillen (1973), $\mathscr{K}_{2,X}$ has a flasque resolution

$$0 \to \mathscr{K}_{2,X} \to i_* K_2(k(X)) \xrightarrow{T} \bigoplus_{C:\text{ curves}} (i_C)_* \; k(C)^* \xrightarrow{\partial} \bigoplus_{P:\text{ points}} (i_p)_* \; \mathbb{Z} \to 0$$

531 Here $K_2(k(X))$ is K_2 of the function field $k(X)$, which is given (by a result of Matsumoto - see Milnor (1971)) by

$$K_2(k(X)) = \frac{k(X)^* \otimes_{\mathbb{Z}} k(X)^*}{\langle a \otimes (1-a) | a \in k(X)^*, a \neq 1 \rangle}.$$

$i_* K_2 k(X)$ is the constant sheaf $K_2(k(X))$ on X, and similarly $(i_C)_* k(X)^*$, $(i_p)_* \mathbb{Z}$ are the constant sheaves $k(C)^*$, $\mathbb{Z}$ on C and P respectively. The map T can be explicitly described in terms of the above presentation, while ∂ is the sum of divisor maps

$$k(C)^* \to \bigoplus_{P \epsilon C} \mathbb{Z}[P]$$

This resolution can be used to compute the cohomology of $\mathscr{K}_{2,X}$; in particular, we obtain the formula $H^2(X, \mathscr{K}_{2,X}) = CH^2(X)$, This is the K-theoretic description of $CH^2(X)$.

Suppose now that X is a smooth projective surface over an algebraically closed field. By associating to each zero cycle $\sum n_i P_i$ its degree $\sum n_i$, we obtain a surjection $CH^2(X) \to \mathbb{Z}$; let $A_0(X)$ denote the kernel. If $S^n(X)$ denotes the n^{th} symmetric product of X, so that points of $S^n(X)$ parametrize effective zero cycles of degree n, we have a natural map

$$\begin{gathered} r_n : S^n(X) \times S^n(X) \to A_0(X) \\ (A, B) \to [A] - [B] \end{gathered}$$

Now assume that k is uncountable i.e. a universal domain in the sense of Weil. We have the following definition of Mumford (1968).

Definition. $A_0(X)$ is finite dimensional if some r_n is surjective. If none 532
of the r_n is surjective, we say that $A_0(X)$ is *infinite dimensional*.

Theorem (Mumford (1968)). *Let* $k = \mathbb{C}$, *and suppose* $\Gamma\left(X, \Omega^2_X\right) \neq 0$. *Then* $A_0(X)$ *is infinite dimensional.*

This result is false in characteristic $p > 0$. In arbitrary characteristic, we have

Theorem (Bloch (1976)). *Let k be a universal domain of arbitrary characteristic, and suppose the cycle map*

$$\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^2_{et}(X, \mathbb{Q}_\ell(1))$$

is not surjective ($\ell \neq$ *characteristic of* k). *Then* $A_0(X)$ *is infinite dimensional*

(Bloch conjectures that the converse also holds).

Next, we mention some results of Roitman on zero cycles. We have a natural map

$$\varphi : A_0(X) \to Alb\, X$$

where $Alb\,X$ is the Albanese variety of X; if $k = \mathbb{C}$ this can be described as follows:

$$Alb\,X = \Gamma\left(X, \omega_X^1\right)^* \text{ / image } (H_1(X,\mathbb{Z})), \text{ and}$$

$$\varphi\left(\sum([P_i]-[Q_i])\right) = \left(\omega \to \sum \int_{P_i}^{Q_i} \omega\right), \omega\epsilon\Gamma\left(X, \omega_X^1\right); \int_{P_i}^{Q_i}$$

533 denotes the integral along any path joining P_i and Q_i.

Theorem (Roitman (1972), Roitman (1980)). *(1) If $A_0(X)$ is finite dimensional, then $\varphi : A_0(X) \to Alb\,X$ is an isomorphism.*

(2) $\varphi : A_0(X) \to Alb\,X$ is always an isomorphism on torsion subgroups.

(We note that the p-torsion statement in (2), for p = characteristic k, is due to Milne (1982). Roitman's Proof of (1) has a small gap in characteristic p, which is however easily filled-see Srinivas (1985)).

2 The singular case:

We now see to what extent this theory generalises to singular surfaces. Firstly, the three definitions of the Chow group in the smooth case all admit generalisations to the singular case. Let X be a normal quasi-projective surface. The cycle theoretic definition is

$$CH^2(X) = \frac{\text{Free abelian group on smooth points of } X}{\left\langle (f)_C \mid C \subset X \text{ is a curve }, f\epsilon k(C)^*, \text{ and } C \cap X_{\text{sing}} = \phi \right\rangle}$$

where X_{sing} denotes the singular locus of X, and C runs over *closed curves* in X disjoint from X_{sing}.

The Grothendieck group of vector bundles $K_0(X)$ is now in gen-
534 eral different from the Grothendieck group $G_0(X)$ of coherent sheaves. However $K_0(X)$ is also the Grothendieck group of coherent sheaves $\mathscr{F}$ such that for each $x\epsilon X$ the stalk $\mathscr{F}_x$ has finite projective dimension over $\mathscr{O}_{x,X}$ (since X is quasi-projective, such an $\mathscr{F}$ has a finite

resolution by vector bundles). We have a filtration $F^iK_0(X)_{0\leq i\leq 2}$ with $\dfrac{F^0}{F^1} = \mathbb{Z}$, $\dfrac{F^1}{F^2} = \text{Pic}\, X$. and $F^2K_0(X)$ is the kernel of the determinant map $F^1K_0(X) \to \text{Pic}\, X$. Since any vector bundle is trivial in a neighbourhood of X_{sing}, one checks that $F^2K_0(X)$ is also the sub-group generated by residue classes of smooth points. Thus we have a surjective cycle map $\Psi : CH^2(X) \to F^2K_0(X)$.

Theorem (Collino (1981)). $\Psi : CH^2(X) \to F^2K_0(X)$ *is an isomorphism.*

We note here that Fulton has defined a Chow (homology) group

$$CH_0(X) = \frac{\text{Free abelian group on all points}}{\langle (f)_C \mid C \subset X \text{ any curve }, f \epsilon k(C)^* \rangle}$$

This group is related to $G_0(X)$ is an analogous fashion. The natural map $CH^2(X) \to CH_0(X)$ is always surjective, but in general not injective.

Finally, the K-theoretic definition $H^2(X, \mathscr{K}_{2,X})$ still makes sense, though the resolution obtained in the smooth case is not valid here; however we have:

Theorem. $H^2(X, \mathscr{K}_{2,X}) = CH^2(X)$.

This result is due to Collino (1981) when X has 1 singular point; the general case, along lines similar to Collino's proof, is due independently **535** to Levine (1985) and Perdrini and Weibel (1983) Next, we have a result of Levine generalizing Roitman's theorem on torsion 0-cycles.

Theorem (Levine (1985)). *Let* $X/k = \bar{k}$ *be projective,* $f : Y \to X$ *a resolution of singularities. The composite*

$$A_0(X) \xrightarrow{f^*} A_0(Y) \xrightarrow{\varphi} Alb\, Y$$

is an isomorphism on torsion prime to the characteristic of k.

3

Now suppose X is a normal projective surface over a universal domain k. Let $U = X - X_{\mathrm{sing}}$; then there is a natural map $r_n : S^n(U) \times S^n(U) \to A_0(X)$ where $S^n(U)$ is the *nth* symmetric product, and $r_n(A, B) = [A] - [B]$.

Definition. $A_0(X)$ is *finite dimensional* if r_n is surjective for some n; if none of the r_n are surjective, then $A_0(X)$ is *infinite dimensional*.

We have the following straightforward generalisation of Roitman's finite dimensionality theorem.

Theorem (Srinivas (1985)). *$A_0(X)$ is infinte dimensional $\Leftrightarrow$ $A_0(X) \simeq \mathrm{Alb}\, Y$ where $Y \to X$ is a resolution of singularities.*

As a consequence of this result and the Murthy-Swan cancellation theorem Murthy (1976), we see that there are 2 possible situations for the Chow group over a universal domain:

536 (i) (finite dimensional case) for any affine open set $V \subset X$, every vector bundle on V is the direct sum of a trivial bundle and a line bundle.

(ii) (infinte dimensional case) for an arbitrary neighbourhood V of the singular locus $X_{\mathrm{sing}}, CH^2(V)$ has uncountable rank.

In case (ii), one can show that there exist uncountably many non-isomorphic indecomposable vector bundles of rank 2; this is not immediately clear unless Pic$X$ is finitely generated (see Srinivas (1999)).

We now state the infinite dimensionality theorems proved in Srinivas (1985); the proofs are outlined below.

Theorem. (1) *Let $k = \mathbb{C}$, and suppose X is a normal projective surface with $H^2(X, \mathcal{O}_X) \neq 0$. Then $A_0(X)$ is infinite dimensional.*

(2) *Let k be a universal domain of arbitrary characteristic. Let X/k be a normal, projective surface, $\pi : Y \to X$ a resolution of singularities such that the reduced exceptional divisor $E = \pi^{-1}(X_{\mathrm{sing}})$*

has smooth components and normal crossings. Suppose $\mathrm{Pic}^0 Y \to \mathrm{Pic}^0 E$ *is not surjective. Then* $A_0(X)$ *is infinite dimensional.*

We reformulate the hypothesis in slightly different terms. The interesting case of the theorem is when $A_0(Y)$ is finite. dimensional for a resolution $\pi : Y \to X$. If $X/\mathbb{C}$ is projective, this means $H^2(Y, \mathscr{O}_Y) = 0$ by Mumford's result. Thus the Leray spectral sequence for π yields an exact sequence

$$\begin{tikzcd}
H^1(Y, \mathscr{O}_Y) \arrow[r] & \Gamma\left(X, R^1\pi_*\mathscr{O}_Y\right) \arrow[r] \arrow[d, equal] & H^2(X, \mathscr{O}_X) \arrow[r] \arrow[d, equal, "\backslash" marking] & 0 \\
 & \varprojlim_n H^1(E, \mathscr{O}_{nE}) & 0 &
\end{tikzcd}$$

where nE is the subscheme of Y with ideal sheaf $\mathscr{O}_Y(-nE)$. (the equality follows from the formal function theorem Hartshorne (1977)). Since 537
the maps of the inverse system $\left\{H^1(E, \mathscr{O}_{nE})\right\}_{n\geq 1}$ are all surjective, the hypothesis $H^2(X, \mathscr{O}_X) \neq 0$ is equivalent to the statement that $H^1(Y, \mathscr{O}_Y) \to H^1(E, \mathscr{O}_{nE})$ is not surjective for some $n \geq 1$. The hypothesis in 2) of the Theorem is that $H^1(Y, \mathscr{O}_Y) \to H^1(E, \mathscr{O}_E)$ is not surjective, if $\mathrm{Pic}Y$ is reduced.

4 Examples

(1) $X = \mathrm{Spec}\dfrac{\mathbb{C}[x,y,z]}{(z^2 + x^3 + y^7)}$. Bloch and Murthy (unpublished) showed that there is a surjection $F^2K_0(X) \to \Omega^1_{\mathbb{C}/\mathbb{Z}}$, the module of absolute Kahler differentials of $\mathbb{C}$; since $\Omega^1_{\mathbb{C}/\mathbb{Z}}$ has uncountable rank, this gives a negative answer to a question of Murthy on Grothendieck groups of normal graded rings Bass (1973). If $A = \dfrac{\mathbb{C}[x,y,z]}{(z^2 + x^3 + y^7)}$, then $A \subset B = \mathbb{C}[u, v]$ is birational, where $u = \dfrac{x}{y^2}$, $v = \dfrac{z}{y^3}$. The maximal ideal at the origin in A generates the ideal $\left(v^2 + u^3\right)B$ in B, so that there is a resolution

of singularities $Y \to X$ such that the exceptional divisor in Y contains a rational curve with a cusp. If $\overline{X} \supset X$ is a projective surface where $\overline{X} - X$ consists only of non-singular points, and if $\overline{Y} \to \overline{X}$ is the corresponding resoluton, then $H^1(\overline{Y}, \mathcal{O}_{\overline{Y}})$ vanishes as $\overline{Y}$ is a smooth rational surface, **538** while $H^1(E, \mathcal{O}_E) \neq \mathcal{O}$. Hence from the theorem, $F^2K_0(\overline{X})$ is infinite dimensional, and $F^2K_0(X)$ has uncountable rank.

(ii) **Cones.** Let $C \subset \mathbb{P}^n$ be a projectively normal curve, so that the cone $X \subset \mathbb{P}^{n+1}$ over C is a normal projective surface whose only singularity is the vertex $P \epsilon X$. Let $\pi : Y \to X$ be the blow up of X at P. Then $Y \simeq \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(1))$ is a $\mathbb{P}^1$-bundle over C. If $f : Y \to C$, the exceptional divisor E of π is a reduced curve giving the (unique) section of f with normal bundle $\mathcal{O}_c(-1)$. If $I = \mathcal{O}_Y(-E)$ is the ideal sheaf of E, then $\dfrac{I^j}{I^{j+1}} \cong \mathcal{O}_C(j)$. Thus

(i) $H^1(Y, \mathcal{O}_Y) \to H^1(E, \mathcal{O}_E)$ is an isomorphism

(ii) for some $n > 1$, $H^1(Y, \mathcal{O}_Y) \to H^1(E, \mathcal{O}_{nE})$ is not surjective $\Leftrightarrow$ for some $n > 1$, $H^1\left(\dfrac{E, I}{I^n}\right) \neq 0 \Leftrightarrow H^1\left(\dfrac{E, I}{I^2}\right) \neq 0 \Leftrightarrow H^1(C, \mathcal{O}_C(1)) \neq 0$. Thus $H^2(X, \mathcal{O}_X) \neq 0 \Leftrightarrow H^1(C, \mathcal{O}_C(1)) \neq 0 \Leftrightarrow H^0(C, \omega_C(-1)) \neq 0 \Rightarrow \deg C \leq 2g - 2$. Hence $A_0(X)$ is infinite dimensional if C is a complete intersection of genus ≥ 3 over $\mathbb{C}$ (eg. $C \subset \mathbb{P}^2$ of degree ≥ 4), or C is canonically embedded (see Srinivas (1982))

We remark that if $\deg C \geq 2g + 1$, where $C \subset \mathbb{P}^n$ is embedded via a complete linear system, then C is projectively normal and $A_0(X)$ is finite dimensional. If $C \subset \mathbb{P}^n_k$ is projectively normal, and $k = \overline{k}$ has characteristic $p > 0$, then $A_0(X)$ is always finite dimensional Srinivas (1982); in particular (1) in the Theorem is false if $\mathbb{C}$ is replaced by a universal domain in characteristic $p > 0$.

(iii) Let $X \subset \mathbb{A}^4_{\mathbb{C}}$ be the surface $\begin{cases} x^3 + y^3 + z^3 = 0 \\ w^2 + x^2 + y^2 + 1 = 0 \end{cases}$

539 Then X is the double cover of the cubic cone $Y \subset \mathbb{A}^3$ given by $x^3 + y^3 + z^3 = 0$ branched along a smooth quadric section. Clearly

$X \to Y$ is étale over the vertex of Y, so that X has 2 singular points P, Q each analytically isomorphic to the vertex of Y. Let $X_P \to X, X_Q \to X$ be the blow ups of P, Q respectively.

Claim. $CH^2(X_P) = CH^2(X_Q) = 0$, while $CH^2(X)$ has infinite rank.

First we show that $CH^2(X)$ is infinite dimensional. Let $\overline{X}$ be a projective surface containing X as an open set so that $\overline{X} - X$ consists only of smooth points. Let $Z \to \overline{X}$ be the blow up at P, Q. Then Z is smooth and birationally ruled over $C \subset \mathbb{P}^2$ given by $x^3 + y^3 + z^3 = 0$ such that the exceptional curves E_P, E_Q over P and Q respectively, both map isomorphically to C. Thus $\dim H^1(Z, \mathscr{O}_Z) = 1$, while $\dim H^1(E, \mathscr{O}_E) = \dim H^1(E_P, \mathscr{O}_{E_P}) + \dim H^1(E_Q, \mathscr{O}_{E_Q}) = 2$. Hence $CH^2(\overline{X})$ is infinite dimensional so that $CH^2(\overline{X})$ has uncountable rank.

To show that $CH^2(X_P) = CH^2(X_Q) = 0$, since the involution of X/Y interchanges P, Q it suffices to prove $CH^2(X_P) = 0$. Now $\overline{X} \to \overline{Y}$ is branched along a smooth genus 4 curve ($\overline{Y} \subset \mathbb{P}^3$ the projective cone over C) which is tangent to 6 rulings of $\overline{Y}$ (by Riemann-Hurwitz, for example). Hence $Z \to C$ is a non-minimal ruled surface with 6 reducible fibers, each with 2 components which are exceptional curve of the first kind, and E_P, E_Q are sections. Blowing down the 6 exceptional curves meeting E_P,we obtain a minimal ruled surface over C with 2 disjoint sections, namely the images of E_Q and E_P Since the normal bundle of E_Q does not change under this map, the minimal ruled surface is $\mathbb{P}_C(\mathscr{O}_C \oplus \mathscr{O}_C(1))$ (the normal bundle of E_Q is $\mathscr{O}_C(-1)$) i.e. $\overline{X}_P$ is isomorphic to the blow up of $\overline{Y}$ at 6 points. Hence $A_0(\overline{X}_P)$ is finite **540** dimensional, since $A_0(\overline{Y})$ is (see Srinivas (1982)). Thus $A_0(X_P) = 0$ by an easy argument.

5 Relative K-theory

Let T be a scheme, $S \subset T$ a subscheme. One can define relative K-groups $K_i(T, S)$ for $i > 0$ which fit into a natural long exact sequence (see (Coombes and Srinivas, 1982, appendix))

$$\to K_i(T) \to K_i(S) \to K_{i-1}(T, S) \to K_{i-1}(T) \to \dots$$

Let X be a normal quasi-projective surface, $S \subset X$ a closed sub-scheme supported on X_{sing}; let $Y \to X$ be a resolution of singularities, and $E \subset Y$ a subscheme such that we have a diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & Y \arrow[d] \\
S \arrow[r] & X
\end{tikzcd}$$

Then by naturality we have a diagram of long exact sequences

$$\begin{tikzcd}
K_1(Y) \arrow[r] & K_1(E) \arrow[r] & K_0(Y,E) \arrow[r] & K_0(Y) \arrow[r] & K_0(E) \\
K_1(X) \arrow[r] \arrow[u] & K_1(S) \arrow[r] \arrow[u] & K_0(X,S) \arrow[r] \arrow[u] & K_0(X) \arrow[r] \arrow[u] & K_0(S) \arrow[u]
\end{tikzcd}$$

Further, it is shown (loc. cit.) that points of $X - S$ admit relative cycle classes in $K_0(X,S)$ which map to the usual classes in $K_0(X)$, and a similar claim holds for the pair (Y,E). Let

541 $F^2K_0(X,S)$=subgroups of $K_0(X,S)$ generated by points of $Y - S\,F^2$ $K_0(Y,E)$= subgroup of $K_0(Y,E)$ generated by points of $Y - E$.

We have a diagram with surjective arrows (this is easy if $Y - E \simeq X - S$, but true even otherwise)

$$\begin{tikzcd}
F^2K_0(Y,E) \arrow[r] & F^2K_0(Y) \\
F^2K_0(X,S) \arrow[r] \arrow[u] & F^2K_0(X) \arrow[u]
\end{tikzcd}$$

Claim. $F^2K_0(X,S) \simeq F^2K_0(X)$.

Idea of proof. Must show $(F)_C = 0$ in $F^2K_0(X,S)$ for any $C \subset X$ with $C \cap X_{\text{sing}} = 0$, and $f \epsilon k(C)^*$. One checks that from the definition of the relative K-groups in loc. cit., if $\overline{C}$ is the normalisation of C, then the natural maps $K_i(\overline{C}) \to K_i(X)$ factor through $K_{(}X,S)$; now the vanishing of $(f)_C$ in $K_0(X,S)$ follows from the vanishing of $(f)_{\overline{C}}$ in $K_0(\overline{C})$.

We can now state a conjecture due to Bloch and the author. Let $E \subset Y$ be the reduced exceptional divisor.

Conjecture. (i) $F^2K_0(Y, nE) \to\to F^2K_0(Y, (n-1)E)$ is an isomorphism for all sufficiently large n.

(ii) $F^2K_0(X) \simeq \lim_{\overleftarrow{n}} F^2K_0(Y, nE)$.

In particular $\lim_{\overleftarrow{n}} F^2K_0(Y, nE)$ should be independent of the resolution $Y \to X$ We can verify this in general assuming a form of excision for K_1 for certain 1-dimensional schemes, and in any case can verify this for resolutions where E has smooth components and normal crossings.

As a special case of the conjecture, one expects that if X has only **542**
singularities, then $CH^2(X) \simeq CH^2(Y)$.

A closely related problem to that of computing $F^2K_0(X)$ for singular X is the problem of computing $K_0(\mathscr{C}_R)$, where R is the local ring of a singular point on a surface, $\mathscr{C}_R$ the category of R-modules of finite length and finite projective dimension. This problem is of independent interest to algebraists, but not much computation could be done by algebraic methods. Recently, K-theory has provided some results in this area; in Srinivas (1985) a method was introduced which enabled one to prove that

$$K_0(\mathscr{C}_R) = \mathbb{Z} \oplus N - (\text{ torsion })$$

for some N, where R is a rational double point in characteristic $p > 0$ which occurs on a rational surface, or else R is of type E_8 over an arbitrary algebraically closed field. Some further examples in characteristic $p > 0$ were considered in Coombes and Srinivas (1982), and finally we have the results of Levine (1999) and the author Srinivas (1999) (independently).

Theorem. *Let R be the local ring of a rational double point on a surface $X/\mathbb{C}$. Then $K_0(\mathscr{C}_R) = \mathbb{Z}$.*

Corollary. *Let $X/k = \overline{k}$ be a normal quasi-projective surface with only quotient singularities. If $Y \to X$ is a resolution, then $CH^2(X) \simeq CH^2$ (Y).*

We should point out that inspite of the explicit nature of the formula

$$F^2K_0(X) = \varprojlim_n F^2K_0(Y, nE),$$

543 we do not know a single example where we can prove that this formula is true, when the right side is different from $F^2K_0(Y)$. Thus we have only indirect evidence for the truth of the conjecture.[1]

Levine has made a conjecture related to ours. If X is a normal (quasi-projective) surface, and $P \epsilon X$ the unique singular point, $Y \to X$ a resolution, $Z = Y \times_X \operatorname{Spec} R$, $R = \mathscr{O}_{P,X'}$ and if $E \subset Y$ is the reduced exceptional divisor, then Levine constructs (loc. cit.) an exact sequence

$$H^1(Z, \mathscr{K}_{2,z})/N \xrightarrow{\Phi} K_0(\mathscr{C}_R) \xrightarrow{\Psi} F^1G_0(E) \to 0.$$

Here $G_0(E)$ is the Grothendieck group of coherent sheaves on E, F^1 $G_0(E)$ the subgroup generated by points; $H^1(Z, \mathscr{K}_{2,Z})$ is H^1 of the complex.

$$0 \to K_2(k(Y)) \xrightarrow{T} \bigoplus_{x \epsilon Z^1} k(x)^* \xrightarrow{\partial} \bigoplus_{x \epsilon Z^2} \to 0$$

where Z^i is the set of codimension i points; N is defined to be the subgroup of $H^1(Z, \mathscr{K}_{2,Z})$ generated by

$$\operatorname{Ker}\left(\bigoplus_{\text{exceptionalcurves}} k(x)^* \xrightarrow{\partial} \bigoplus_{z \epsilon Z^2} \mathbb{Z} \right).$$

544 **Conjecture (Levine)[2].** $\phi : H^1(Z, \mathscr{K}_{2,Z})/N \to K_0(\mathscr{C}_R)$ *is injective.*

Levine shows that under the natural map $K_0(\mathscr{C}_R) \to K_0(X)$ we have image $SK_0(\mathscr{C}_R) = \operatorname{Ker}(\Psi : K_0(\mathscr{C}_R) \to F^1G_0(E))$. We conjecture that

$$H^1(Z, \mathscr{K}_{2,Z})/N \to \varprojlim_n H^1(E, \mathscr{K}_{2,nE})/N$$

is injective; this is closely related to our conjectural formula for $F^2K_0(X)$.

---

[1] Some examples exist now.

[2] The author now has a proof of Levine's conjecture [23].

6 Sketch of the proof

In this section we sketch the proof of the infinite dimensionality theorem. We only consider the case $k = \mathbb{C}$; the other case uses similar methods. Let $\pi : Y \to X$ be a resolution of singularities. We may assume that $A_0(Y)$ is finite dimensional, since if $A_0(Y)$ is infinte dimensional then trivially $A_0(X)$ is infinite dimensional. Thus we may assume that for some $n \geq 1$, $H^1(Y, \mathscr{O}_Y) \to H^1(E, \mathscr{O}_{nE})$ is not surjective; fix the smallest such value of n. The idea is to show that $F^2K_0(Y, nE)$ is strictly larger than $F^2K_0(Y)$, so that $F^2K_0(X) \to F^2K_0(Y) \simeq AlbY \times \mathbb{Z}$ is not an isomorphism. In particular, we want $K_0(Y, nE) \to K_0(Y)$ to have a kernel i.e. coker $(K_1(Y) \to K_1(nE))$ should be non-zero. One has a natural decomposition $K_1(T) = \Gamma(T, \mathscr{O}_T^*) \oplus SK_1(T)$ for any "reasonable" scheme T, for a certain subgroup $SK_1(T) \to K_1(T)$; in fact $SK_1(T)$ is the intersection of the kernels of all the maps $K_1(T) \to K_1(\mathscr{O}_{t,T})$ for all points $t \epsilon T$. It turns out that $(SK_1(nE)/\text{ image } SK_1(Y)) \subset K_0(Y, nE)$ contains Ker $\left(F^2K_0(Y, nE) \to F^2K_0(Y)\right)$. Thus we will be mainly interested in the map $SK_1(Y) \to SK_1(nE)$. 545

Suppose E is a smooth curve. We have the localisation sequence of Quillen (1973) for any open set $U \subset E$.

$$K_2(U) \to \bigoplus_{P \epsilon E-U} K_1(k(P)) \to K_1(E) \to K_1(U) \to \bigoplus_{P \epsilon E-U} \mathbb{Z} \to K_0(E)$$
$$\to K_0(U) \to 0$$

Taking the direct limit over all open subsets $U \subset E$, we obtain

$$K_2(k(E)) \to \bigoplus_{P \epsilon E} K_1(k(P)) \to K_1(E) \to K_1(k(E)) \to \bigoplus_{P \epsilon E} \mathbb{Z}$$
$$\to K_0(E) \to \mathbb{Z} \to 0$$

Here $k(E)$ is the function field, $K_1(E) = k(E)^*$, and $\partial : k(E)^* \to \oplus \mathbb{Z}$ is the divisor map. Thus Ker $\partial = \Gamma(E, \mathscr{O}_E^*)$ i.e. we have a presentation (since $K_1(k(P)) = k(P)^*$)

$$K_2(E) \to \bigoplus_{P \epsilon E} k(P)^* \to SK_1(E) \to 0.$$

From the resolution (for a smooth curve E - see Quillen (1973)).

$$0 \to \mathscr{K}_{2,E} \to i_* K_2(k(E)) \to \underset{P \epsilon E}{\oplus} (i_P)_* k(P)^* \to 0$$

546 we thus have an isomorphism $SK_1(E) \simeq H^1(E, \mathscr{K}_{2,E})$.

It turns out that even when E has many components, the scheme (nE) still satisfies $SK_1(nE) \simeq H^1(E, \mathscr{K}_{2.nE})$ by an analogous argument. Further, there exists a surjection $SK_1(Y) \to H^1(Y, \mathscr{K}_{2,Y})$ leading to a diagram

$$\begin{tikzcd}
SK_1(Y) \arrow[r] \arrow[d] & H^1(Y, \mathscr{K}_{2Y}) \arrow[d] \\
SK_1(nE) \arrow[r, "\simeq"] & H^1(E, K_{2,nE})
\end{tikzcd}$$

So we are reduced to considering $\varphi : H^1(Y, \mathscr{K}_{2,Y}) \to H^1(E, \mathscr{K}_{2,nE})$ We state a lemma which is a special case of a result of Bloch (1975), (which has already been implicitly used to say $SK_1(nE) \simeq H^1(E, \mathscr{K}_{2,nE})$; the case $n = 2$ is due to Van der Kallen (1971)

Lemma. *Let R be a local $\mathbb{Q}$-algebra, $S_n = \dfrac{R[t]}{(t^n)}$, $n > 1$. Then there is an exact sequence*

$$0 \to \Omega^1_{R/\mathbb{Z}} \to K_2(S_n) \to K_2(S_{n-1}) \to 0$$

This lemma allows us to compute the "derivative" of φ. Let $Y_{k[\epsilon]}$, $(nE)_{k[\epsilon]}$ denote the products (over Spec$k$) of Y, (nE) respectively with Spec$(k[\epsilon])$, where $k[\epsilon]$ is the ring of dual numbers (i.e. $\epsilon^2 = 0$, $\epsilon \neq 0$).

Then

$$H^1(Y_{k[\epsilon]}, \mathscr{K}_2) = H^1(Y, \mathscr{K}_2) \oplus H^1\left(Y, \Omega^1_{Y/\mathbb{Z}}\right)$$

$$H^1((nE)_{k[\epsilon]}, \mathscr{K}_2) = H^1((nE), \mathscr{K}_2) \oplus H^1\left(E, \Omega^1_{(nE)/\mathbb{Z}}\right)$$

547 Thus the derivative of φ is

$$d\varphi : H^1\left(Y, \Omega^1_{Y/\mathbb{Z}}\right) \to H^1\left(E, \Omega^1_{(nE)/\mathbb{Z}}\right).$$

For any “reasonable” scheme T/k, we have an exact sequence

$$0 \to \Omega^1_{k/\mathbb{Z}} \otimes_k \mathscr{O}_T \to \Omega^1_{T/\mathbb{Z}} \to \Omega^1_{T/k} \to 0$$

Thus we have a diagram

$$\begin{tikzcd}
H^O(Y, \Omega^1_{Y/K}) \arrow[r] \arrow[d] & H^1(Y, \mathscr{O}_Y) \otimes_k \Omega^1_{k/\mathbb{Z}} \arrow[r] \arrow[d] & H^1(Y, \Omega^1_{Y/\mathbb{Z}}) \arrow[r] \arrow[d] & {} \\
H^0(E, \Omega^1_{(nE/k)}) \arrow[r] & H^1(E, \mathscr{O}_{nE}) \otimes_k \Omega^1_{k/\mathbb{Z}} \arrow[r] & H^1(E, \Omega^1_{(nE)/\mathbb{Z}}) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
H^1(Y, \Omega^1_{Y/k}) \arrow[d] & \\
H^1(E, \Omega^1_{(nE)/k}) \arrow[r] & 0
\end{tikzcd}$$

The cohomology groups of $\Omega^1_{y/k}$, $\Omega^1_{(nE)/k}$ are finite dimensional over k, while if k is uncountable, $\Omega^1_{k/\mathbb{Z}}$ has uncountable dimension over k. Since

$$H^1(Y, \mathscr{O}_Y) \to H^1(E, \mathscr{O}_{nE})$$

is not surjective, we see that $H^1\left(Y, \Omega^1_{Y/\mathbb{Z}}\right) \to H^1\left(E, \Omega^1_{(nE)/\mathbb{Z}}\right)$ has an uncountably generated cokernel (in fact the cokernel has uncountable dimension over k).

548 To get global results from the above computations, we let k be an uncountable proper subfield of $\mathbb{C}$ which is algebraically closed, such that X, Y, E are all defined over k. Then there is an embedding of k-algebras $A \subset \mathbb{C}$ where $A = k[t]^n_{(t)}$, the super-script h denoting henselisation. Let $Y_A = Y \times_{\mathrm{Spec} k} \mathrm{Spec} A$, etc.; for $m \geq 1$ let $A_m = A/t^m A$ and $Y_{A_m} = Y \times_{\mathrm{Spec}\ k} \mathrm{Spec} A_m$, etc. We prove the following claims:

(i) $H^1((nE)_A, \mathscr{K}_2) \to H^1((nE)_{A_m}, \mathscr{K}_2)$ is surjective for any $m \geq 1$

(ii) $H^1((nE)_A, \mathscr{K}_2) \to H^1((nE)_K, \mathscr{K}_2)$ is injective, where $K = A[1/t]$ is the quotient field.

(iii) coker $(H^1(Y_A, \mathscr{K}_2) \rightarrow H^1((nE)_A, \mathscr{K}_2)) \otimes \mathbb{Q}$ has uncontable rank $\Rightarrow$ coker $(H^1(Y_K, \mathscr{K}_2) \rightarrow H^1((nE)_K, \mathscr{K}_2)) \otimes \mathbb{Q}$ has uncountable rank (here we use the fact that $A_0(Y)$ is finite dimensional, so that $P_g = 0$; if $q > 0$, we use the fact that over $\mathbb{C}$ all surfaces with $p_g = 0, q > 0$ are classified).

(iv) pass from K to $\mathbb{C}$ by a transfer argument (i.e show that

$$K_0(Y_K, (nE)_K) \otimes \mathbb{Q} \rightarrow K_0(Y_{\mathbb{C}}, (nE)_{\mathbb{C}}) \otimes \mathbb{Q}$$

is injective).

These steps will show that $\mathrm{Ker}(K_0(Y, nE) \rightarrow K_0(Y)) \otimes \mathbb{Q}$ has uncountable rank; we want to get a similar result for F^2K_0. For this, we need some information about the boundary map.

$$\partial : SK_1(nE) \rightarrow K_0(Y, nE)$$

549 in the long exact sequence. If $n = 1$, and E is smooth, this map has relatively simple description. In this case

$$SK_1(E) = \frac{\underset{p \epsilon E}{\oplus} k(P)^*}{\text{image } K_2(k(E))}.$$

If $\sum\limits_{i=1}^{m} (\alpha_i)_{P_i}$ is a class in $SK_1(E)$, let $C \subset Y$ be a smooth curve meeting E transversally at $P_1, \ldots\ldots, P_m$ and let $P_{m+1}, \ldots\ldots, P_r$ be the other points of intersection which may also be taken to be transverse. Let $\alpha_{m+1}, \ldots, \alpha_r$ all equal 1; we can now find a rational function f on C, regular at all the P_i with $f(P_i) = \alpha_i$. Then

$$\partial\left(\sum (\alpha_i)_{P_i}\right) = (f)_C \epsilon F^2 K_0(Y, E).$$

Thus $\partial(SK_1(E)) = \mathrm{Ker}(F^2K_0(Y, E) \rightarrow F^2K_0(Y))$.

In general, we obtain elements of $SK_1(nE)$ mapping to relative 0-cycles in a similar way - if $C \subset Y$ is a smooth curve which is not a component of E, and $S = C \cap (nE)$ as schemes, then we have a diagram of relative K-theory sequences (see Coombes and Srinivas (1982), appendix).

$$\begin{tikzcd}
K_1(C) \arrow[r] \arrow[d] & K_1(S) \arrow[r, "\partial"] \arrow[d] & K_0(C,S) \arrow[r] \arrow[d] & K_0(C) \arrow[d] \\
K_1(Y) \arrow[r] & K_1(nE) \arrow[r, "\partial"] & K_0(Y,nE) \arrow[r] & K_0(Y)
\end{tikzcd}$$

where $K_1(S) = \Gamma(S, \mathscr{O}_S^*)$ since $\dim S = 0$. The map $\partial : K_1(S) \to K_0(C,S)$ is as follows; given $g \epsilon \Gamma(S, \mathscr{O}_S^*)$, let f be a rational function on C, which is a unit near S, such that $f\ |_S = g$. Then $\partial(g) = (f)_C \epsilon K_0(C,S)$. Thus image $(\Gamma(S, \mathscr{O}_S^*) \to SK_1(nE))$ gives elements of $SK_1(nE)$ whose 550 boundaries are algebraic cycles.

If we assume $n > 1$, but E is smooth, then by the lemma of Bloch mentioned earlier we get a sequence (I is the ideal sheaf of E on Y)$H^1\left(E, \Omega^1_{E/\mathbb{Z}} \otimes \dfrac{I^{n-1}}{I^n}\right) \to H^1(E, \mathscr{K}_{2,nE}) \to H^1(E, \mathscr{K}_{2,(n-1)E}) \to 0$. If we choose a curve C as above meeting E transversally, and take $g \epsilon \Gamma\left(S, \mathscr{O}_S^*\right)$ such that $g\ |_{C\cap(n-1)E} = 1$, then it turns out that the class in $SK_1(nE)$ obtained from $g \epsilon K_1(S)$ in fact lifts to $H^1\left(E, \Omega^1_{E/\mathbb{Z}} \otimes I^{n-1}/I^n\right)$. We have a flasque resolution of $\Omega^1_{E/\mathbb{Z}} \otimes I^{n-1}/I^n$ as follows:

$0 \to \Omega^1_{E/\mathbb{Z}} \otimes I^{n-1}/I^n \to i_*\Omega^1_{k(E)/\mathbb{Z}} \otimes I^{n-1}/I^n \to \oplus (i_P)_* A_P \to 0$ where $A_P = \left(\Omega^1_{k(E)/\mathbb{Z}} / \Omega^1_{\mathscr{O},P,E/\mathbb{Z}}\right) \otimes I^{n-1}/I^n$.

Thus

$$\frac{H^1\left(E, \Omega^1_{E/\mathbb{Z}} \otimes I^{n-1}/I^n\right) = (\oplus A_P)}{\text{image } \left(\Omega^1_{k(E)/\mathbb{Z}} \otimes I^{n-1}/I^n\right).}$$

If h is a rational function on Y, which is a unit in the semi-local ring $\mathscr{O}_{S,Y}$, such that $h\ |_S = g$, and if $x \epsilon \mathscr{O}_{S,Y}$ such that $x\mathscr{O}_{S,Y}$ is the ideal of C in $\mathscr{O}_{S,Y}$, then

$$\frac{dx}{x} \otimes (1-h) \epsilon \bigoplus_{P \epsilon S} A_P$$

gives the lift to $H^1\left(E, \Omega^1_{E/\mathbb{Z}} \otimes I^{n-1}/I^n\right)$, where we regard $1-h$ as a local 551 section of I^{n-1}/I^n. Thus to check that $\partial(SK_1(nE)) \subset F^2K_0(Y, nE)$, we have to see if groups like $H^1(E, \Omega^1_{E/\mathbb{Z}} \otimes I^{n-1}/I^n)$ are generated by such "logarithmic" principal parts. We could show this using residues for the quotient group $H^1(E, \Omega^1_{E/L} \otimes I^{n-1}/I^n)$ where L is any field of definition of X. Thus if X, Y, E are defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers, we would be done. In general, the argument is complicated by the fact that E has many components, and L may not equal $\overline{\mathbb{Q}}$.

7 NK_0 and K_{-1}

For any scheme X/k we define

$NK_0(X) =$ coker $(K_0(X) \to K_0(X \times \mathbb{A}^1))K_{-1}(X) =$ coker $\big(K_0(X_{k[t]}) \oplus K_0(X_{k[t^{-1}]}) \to K_0(X_{k[t,t^{-1}]})\big)$

If X is regular, $NK_0(X) = K_{-1}(X) = 0$. Murthy had asked for examples of normal graded rings $A = \oplus A_n$ with $NK_0(A) \neq 0$. ($NK_0(A) = NK_0(\mathrm{Spec} A)$). Weibel (1981) showed that $NK_0(A) \neq 0$ for the local ring described in example 6 in the appendix to Nagata's book "Local rings"; this is a 2 dimensional normal domain which is analytically ramified. Swan (see Weibel (1981)) gave some geometric examples in dimension 3. Weibel and Swan also showed that if $A = \bigoplus_{n \geq 0} A_n$ satisfies

552 $NK_0(A) = 0$, then $K_0(A) = K_0(A_0)$. Thus Bloch-murthy's example $\dfrac{\mathbb{C}[x, y, z]}{(z^2 + x^3 + y^7)}$ has $NK_0 \neq 0$.

Now suppose X is a normal quasi-projective surface over an algebraically closed field k. Let $Y \to X$ be a resolution of singularities such that the reduced exceptional divisor E has smooth components and normal crossings.

Conjecture. The following are equivalent:

(i) $NK_0(X) \neq 0$

(ii) [2] $K_{-1}(X) \neq 0$

---

[2] The conjecture for K_{-1} has to be modified; see Srinivas (1999)

(iii) $\mathrm{Pic}\, nE \to \mathrm{Pic}\, E$ is not an isomorphism for some $n > 1$

(iv) $H^1(E, \mathscr{O}_{nE}) \to H^1(E, \mathscr{O}_E)$ is not an isomorphism for some $n > 1$

(iii) and (iv) are clearly equivalent. If $k = \mathbb{C}$ (or more generally, if L is a field of definition of characteristic 0, and k has transcendence degree ≥ 1 over L) then (iv)$\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii); the details of the proof will appear elsewhere. Finally, if X is the affine cone over $C \subset P^n$, with $H^1(C, \mathscr{O}_C(1)) \neq 0$, then $NK_0(X) \neq 0$ in any characteristic $\neq 2$ (see Coombes and Srinivas (1982)), giving examples of affine surfaces X in characterisitic $P > 0$ such that all vector bundles on X are trivial, while $K_0(X \times \mathbb{A}^1) \neq \mathbb{Z}$. Lastly, if $C \subset \mathbb{P}^n_k$ is projectively normal with $\deg C \geq 2g + 1$, g=genus C, and k has characteristic 0 or $p \geq \dim H^0(C, \mathscr{O}_C(1))$, then $NK_0(X) = 0$.

References

H. Bass. Some problems in“classical” algebraic K-theory *Lecture Notes in Math.* No. 342, Springer-Verlag, New-York (1973). 553

S. Bloch. Lectures on algebraic cycles, *Duke Univ. Math. Ser.* IV, Duke Univ. Press, Durham (1976).

S. Bloch. K_2 of artinian $\mathbb{Q}$-algebras with applications to algebraic cycles, *Comm. Alg.*, 3(1975) 405–426.

A. Collino. Quillen's $\mathscr{K}$-theory and algebraic cycles on almost non-singular varieties, *Ill. J. Math.*, 25(1981) 654–666.

K. R. Coombes and V. Srinivas. Relative K-theory and vector bundles on certain singular varieties, *Invert. Math.*, 70(1982) 1–12.

R. Hartshorne. Algebraic Geometry, *Grad. Texts in Math.* No. 52, Springer-Verlag, New York(1977).

W. Van der Kallen. Le K_2 des nombres duaux, *C. R. Acad. Sci.* Paris 273A(1971), 1204–1207.

M. Levine. Bloch’s formula for singular surfaces, *Topology*, 24(2) (1985) 165–174.

M. Levine. Torsion Zero cycles on singular varieties, *Amer. J. Math.*, 107(1985), 737–757.

554 M. Levine. On the *K*-groups of surface singularities, Preprint.

J. Milne. Zero cycles on algebraic varieties in non-zero characteristic: Rojtman’s theorem, *Comp. Math.*, 47(1982).

J. Milnor. *Introducation to Algebraic K-theory*, Ann. Math. Studies No. 72, Prinecton Univ. Press, Princeton (1971).

D. Mumford. Rational equivalence of 0-cycles on algebraic surfaces, *J.Math. Kyota Univ.*, 9(1968) 195–240.

M. P. Murthy and R. G. Swan. Vector bundles on affine surfaces, *Invert. Math.*, 36(1976) 125–165.

C. Pedrini and C. Weibel. K-theory and Chow groups of singular varieties, to appear in proceedings of the AMS *K*-theory Conference, Boulder (1983).

D. Quillen. Higher Algebraic *K*-theory, *Lecture Notes in Math*. No. 341, Springer-Verlag, New York (1973).

A. A. Roitman. Rational equivalence of 0-cycles, *Math USSR Sbornik*, 18 (1972) 571–588.

A. A. Roitman. The torsion of the group of 0-cycles modulo rational equivalence, *Ann. Math.*, 111(1980) 553–569.

555 V. Srinivas. Vector bundels on the cone over a curve, *Compos. Math.*, 47(1982) 249–269.

V. Srinivas. Zero cycles on a singular surface, I, II, *J. Reine Ang. Math.*, 359(1985) 90–105 and 362 (1985) 4–27.

V. Srinivas. Modules of finite length and Chow groups of surfaces with rational double points, to appear in *Ill. J. Math*.

C. Weibel. *K*-theory and analytic isomorphisms, *Invent Math*., 61(1981) 177.197.

V. Srinivas. Grothendieck groups of polynomial and Laurent polynomial rings, to appear in *Duke Math. J.*

V. Srinivas. Indecomposable projective modules on affine domains, to appear in *Compos. Math.*

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005
INDIA.