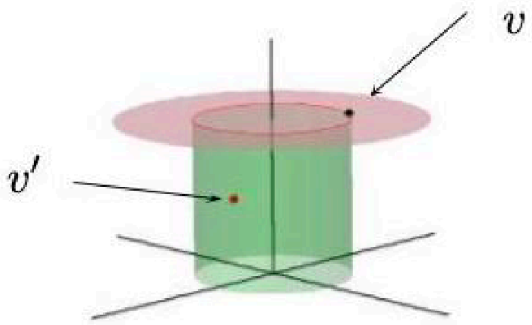
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Geometric and Arithmetic Aspects of Approximation Vectors

COGENT, March 21, 2022

joint with Uri Shapira

(continuation of Goa, 2016!)

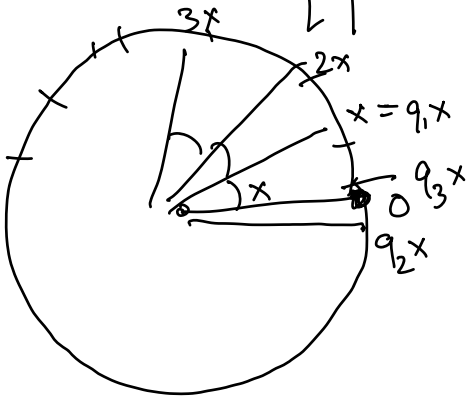
Let $x \in \mathbb{R}^n \setminus \mathbb{Q}^n$. The best approximations are defined recursively. $q_1 = 1$, given $q_1, \dots, q_k \in \mathbb{N}$, $p_1, \dots, p_{k-1} \in \mathbb{Z}$

$\langle x \rangle = \operatorname{dist}(x, \mathbb{Z})$

Then p_k is the integer closest to $q_k x$,

and q_{k+1} is chosen as

$$\min \{ q : \langle qx \rangle < \langle q_k x \rangle \}$$



$\frac{p_k}{q_k} \to x$

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$$

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \dots \cfrac{1}{a_n}}$$

Thm 1 (Moeckel '82) (Statistics of arithmetic of l.a. numbers, $d=1$). Let $a,b\in\mathbb{Z}$, $m\in\mathbb{N}$

There is an explicit (easy to compute) prob.

$P(m,a,b)$ s.t. for a.e. $x\in\mathbb{R}$

$$\frac{1}{k}\#\left\{1\le i\le k:\ \begin{matrix}p_i\equiv a\\ q_i\equiv b\end{matrix}\ (\mathrm{mod}\ m)\right\}\xrightarrow[x\to\infty]{} P(m,a,b)$$

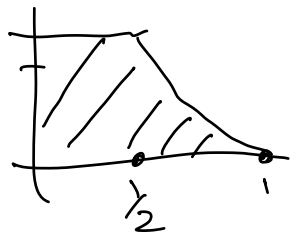
Example: $m=2$

p_i	0	1	1	0
q_i	1	0	1	0
P	1/3	1/3	1/3	0

Thm 2 (Dobkin-Lenstra Conj., proved by Bosma-Jager - Wiedijk '83). Define $|\mathrm{disp}|\left(\frac{p}{q},x\right)=|q(p-qx)|$

$$=\quad q^2\left|x-\frac{p}{q}\right|$$

let $F(t)=\begin{cases}\frac{1}{\log 2} & 0\le t\le\frac12\\ \frac{1}{\log 2}\left(\frac{1}{t}-1\right) & \frac12\le t\le 1\end{cases}$



$$\mu_F((\alpha,\beta))=\int_\alpha^\beta F(t)\,dt$$

Then for a.e. $x\in\mathbb{R}$, $\forall\alpha<\beta$,

$$\frac{1}{k}\#\left\{1\le i\le k:\ |\mathrm{disp}|\left(\frac{p_i}{q_i},x\right)\in(\alpha,\beta)\right\}\to\mu_F(a,b).$$

Cor The expectation of $|\mathrm{disp}|=\frac{1}{4\log 2}\simeq 0.3606\cdots$

Rk In fact thms 1 & 2 hold jointly, e.g. for a.e. x

$$\frac{1}{k}\#\left\{1\le i\le k : \begin{matrix} q_i\equiv 0 \pmod 2 \\ p_i\equiv 1 \end{matrix} \text{ & } |x-\tfrac{p_i}{q_i}|q_i^2\in(\tfrac{1}{3},\tfrac{2}{3})\right\}$$

$$\longrightarrow_{k\to\infty} \frac{1}{3}\int_{\frac{1}{3}}^{\frac{2}{3}} F(t)\,dt$$

Higher dimensions Fix a norm $\|\cdot\|$ on $\mathbb{R}^d$

$\underline{x}\in\mathbb{R}^d$ $\langle \underline{x}\rangle = \operatorname{dist}(\underline{x},\mathbb{Z}^d)$ ← w.r.t. to $\|\cdot\|$.

Define best approximations recursively as before

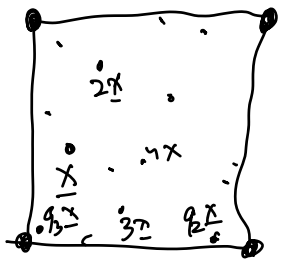
$q_1=1$. Suppose have chosen $q_1,\dots,q_k\in\mathbb{N}$

$\underline{p}_1,\dots,\underline{p}_{k-1}\in\mathbb{Z}^d$

then $\underline{p}_k$ = nearest vector in $\mathbb{Z}^d$ to $q_k\underline{x}$

and $q_{k+1} = \min\{q : \langle q\underline{x}\rangle < \langle q_{k-1}\underline{x}\rangle\}$.

Interpretation, $d=2$



$\mathbb{T}^2 = S^1\times S^1$

Thm 1' (Statistics of best approximations, $d\ge 2$)

Let $m\in\mathbb{N}$, $a\in\mathbb{Z}^d$, $b\in\mathbb{Z}$ $\exists$ $P=P(m,\underline{a},b)$

s.t. for a.e. $\underline{x}$,

$$\frac{1}{k}\#\left\{1\le i\le k:\ \begin{matrix} p_i\equiv a \pmod m \\ q_i\equiv b\end{matrix}\right\}\longrightarrow P.$$

$k\to\infty$

Thm 2' (Statistics of displacements, $d\ge 2$)

Fix $\|\cdot\|$. Define $\mathrm{disp}\left(\frac{p}{q},x\right)=q^d\left(\frac{p}{q}-x\right)\in\mathbb{R}^d$.

There is a measure μ, depending on $\|\cdot\|$,

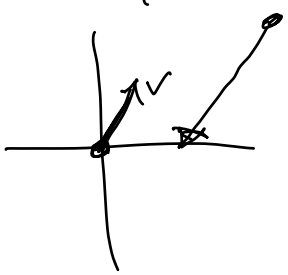
s.t. for a.e. $\underline{x}$, $\frac{1}{k}\sum_{i=1}^{k}\delta_{\mathrm{disp}\left(\frac{p_i}{q_i},x\right)}\xrightarrow[\text{weak}-*]{k\to\infty}\mu$.

Athreya-Ghosh-Tseng JLMS '15 equidistribution of directions of disp, for a related problem.

A further quantity to measure is the shape of the projected lattice. Let $v=\begin{pmatrix}p\\ q\end{pmatrix}\in\mathbb{Z}^{d+1}$

$\pi^v:\mathbb{R}^{d+1}\longrightarrow\mathbb{R}^d=\left\{\begin{pmatrix}y_1\\ \vdots\\ y_d\\ 0\end{pmatrix}\right\}$.

π^v projection with $v\in\ker\pi^v$.



$\Lambda(v)$ = unit covolume rescaling of $\pi^v(\mathbb{Z}^{d+1})$.

$\Lambda(v)\in\mathcal{X}_d=\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ lattices of covolume.

Thm 3 There is a prob measure ν on $\mathcal{X}_d$ s.t. for a.e. Λ, $\frac{1}{K}\sum_{i=1}^{K} \delta_{\Lambda(v_i)} \xrightarrow[K\to\infty]{} \nu$ weak-$*$.

$v_i = \begin{pmatrix} p_i \\ q_i \end{pmatrix}$

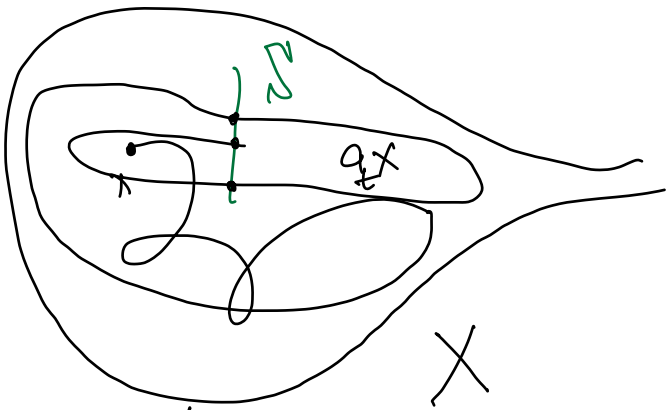
Thm 4 The equidistribution in previous results (namely thms 1', 2', 3') is joint.

Proof ingredients

Reduction to dynamics on $SL_{d+1}(\mathbb{R})/SL_{d+1}(\mathbb{Z}) = \mathcal{X}_{d+1}$

(one more like in Dani correspondence)

A cross section for a flow (Series, Arnoux, Nakada, Nogueira, ..., Cheung-Chevallier)



$T_S : S \to S$

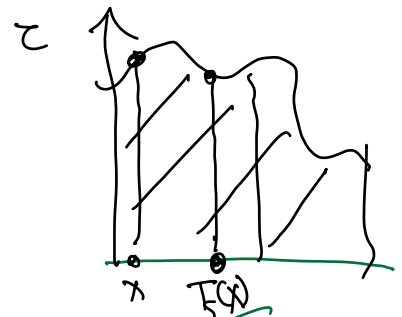
Poincaré map

first return map

S a cross-section

1405

Ambrose-Kakutani: $\left\{ \begin{matrix} a_t\text{-inv. prob} \\ \text{measures on } X \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} T_S \text{ inv. measures} \\ \text{on } S \end{matrix} \right\}$



We will work with $\mathcal{X}_{d+1} = \mathrm{SL}_{d+1}(\mathbb{R})/\mathrm{SL}_{d+1}(\mathbb{Z})$

$a_t = \begin{pmatrix} e^t & & & \\ & \ddots & & \\ & & e^t & \\ & & & e^{-dt} \end{pmatrix}$ $\quad a_t \curvearrowright \mathcal{X}_{d+1} = X$

$a_t \curvearrowright \mathcal{Y}_{d+1}$

$\mathcal{Y}_{d+1} = \mathrm{SL}_{d+1}(\mathbb{A})/\mathrm{SL}_{d+1}(\mathbb{Q})$ $\quad \downarrow$

$a_t \curvearrowright \mathcal{X}_{d+1}$

<u>Thm</u> (1) There is a cross-section $S \subset \mathcal{X}_{d+1}$ for the (a_t)-flow, so that the hitting times $t_1 < t_2 < \dots$ of the trajectory $\Lambda_{\underline{x}} = \begin{pmatrix} 1 & & 0 & \\ & \ddots & & \underline{x} \\ 0 & & 1 & \\ & & & 1 \end{pmatrix} \mathbb{Z}^{d+1}$ satisfy $q_i = \frac{1}{d} \log t_i$

displacement $\mathrm{disp}\left(\frac{p_i}{q_i}, x\right)$ can be recovered from $a_{t_i} \Lambda_{\underline{x}}$, same for the projected lattice.

(2) In the adelic lift

$$\begin{array}{ccc} \tilde{\mathcal{S}} & \subset & \left(\tilde{\Lambda}_x \in \pi^{-1}(\Lambda_x)\right) \mathcal{X}_{d+1} \\ \downarrow & & \downarrow \pi \\ \mathcal{S} & \subset & \mathcal{X}_{d+1} \end{array}$$

define $\tilde{\mathcal{S}} = \pi^{-1}(\mathcal{S})$

the intersections of $a_{t_i} \tilde{\Lambda}_{\underline{x}}$ can recover arithmetic of p_i, q_i.

$$\mathcal{S} = \left\{ \Lambda \in \mathcal{X}_{d+1} : \begin{array}{l} \Lambda \text{ contains a primitive vector} \\ \begin{pmatrix} \underline{z} \\ 1 \end{pmatrix}, \underline{z} \in \mathbb{R}^d, \ \Lambda \cap (B(0, \|\underline{z}\|) \times [0,1]) \\ = \{0, v\} \end{array} \right\}$$