Harmonic measures on Bowditch boundaries of groups hyperbolic relative to virtually nilpotent subgroups

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Main objects

- Relatively hyperbolic groups.
  - Finite free products of finitely generated groups, Hyperbolic groups, geometrically finite Kleinian groups.
  - $\mathbb{Z}^2 * \mathbb{Z}^2$ is *not* a hyperbolic group, but hyperbolic *relative* to two subgroups isomorphic to $\mathbb{Z}^2$.
  - $PSL_2(\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$ is hyperbolic. The COGENT poster shows a different rel. hyp. structure, giving the modular surface.
- The Bowditch boundaries.
  - Top spaces associated to relatively hyperbolic pairs.
- Markov chains on graphs.
  - Green function. Martin boundary. Harmonic measure. Green metric.
  - Drift. Green drift. Entropy.

Hyperbolic metric space (X, ρ)

- Defined using Gromov product

$$(x|y)_z = \frac{1}{2} \cdot (\rho(x,z) + \rho(y,z) - \rho(x,y)).$$

- If geodesic: δ-thin property and

$$|\,(x|y)_z - \mathsf{dist}_\rho(z,[x,y])\,| \lesssim_\delta 1.$$

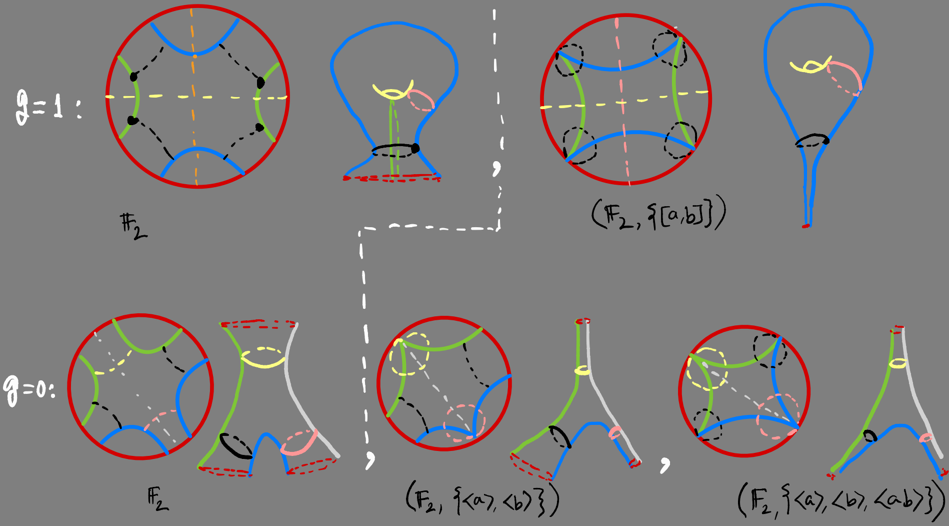
- X proper: ∂X with visual topology and metric

$$\{d_\epsilon \approx e^{-\epsilon\cdot(\cdot|\cdot)_x},\ 0 < \epsilon < \epsilon_0(\delta)\}$$

- Isom(X)"$\longrightarrow$"QS(∂X), (for us QI(X)"$\longleftrightarrow$"QS(∂X))

Relatively hyperbolic group

- Consider the free group $\mathbb{Z} * \mathbb{Z}$ and its actions on $\mathbb{H}^2$.



Some motivation..

- $\Gamma \curvearrowright (X, \rho)$ non-elementary, isometrically, properly discontinuously and say, geom. finitely.
- Busemann function: $z \mapsto \beta_z(\cdot, \cdot)$. $\beta_z(x, y) = \rho(x, z) - \rho(y, z)$.
- Acc. pts. of $\{\beta_{z_n}(x, y)\}_n$ are within bounded distance of each other, but limit need not exist.
- Is there a hyp. metric in QI(ρ) such that the limit exists? (cocycles are useful).
- Does the metric have a well-behaved quasiconformal density? (relation to certain dynamical and 'Diophantine approximation' questions).

Some motivation..

- For Cayley graphs of hyperbolic groups, Blachère-Haïssinsky-Mathieu using work of Ancona on RWs on hyperbolic graphs proposed the use of a Green metric and proved several results on asymptotic objects associated to RWs in Cayley graphs.

- The harmonic measure is by definition a conformal density wrt the Green metric and by an argument of Sullivan the harmonic measure of such walks is Ahlfors-Regular wrt the corresponding visual metric.

- Pushing forward an RW on Γ to a MC on X can not work unless the action is cocompact, since such a Green metric is QI to the word metric under reasonable assumptions on the walk.

Relations between the objects (arXiv:2112.08284v2)

- $(\Gamma, \mathcal{H})$- rel. hyp. $\mathcal{H}$-vir. nil. $\Gamma \curvearrowright X_\Gamma$- cusp uniform.

- $\partial_H(X_\Gamma, \rho_G) \simeq \partial_M(X_\Gamma, P) \simeq \partial(X_\Gamma, \rho_X) \simeq \partial_B(\Gamma, H)$.

- ν- A.R (uniform const.); μ-Doubling (const. varies with basepoint).

- $\lim_{n\to\infty} \frac{1}{n} \cdot \mathbb{E}[\rho_X(Y_0, Y_n)] = l, \quad \lim_{n\to\infty} \frac{1}{n} \cdot \mathbb{E}[\rho_G(Y_0, Y_n)] = l_G,$

  $\lim_{n\to\infty} \frac{1}{n} \cdot \mathbb{E}[-\log p^{(n)}(Y_0, Y_n)] = h;$

  - $h \leqslant l D_\Gamma$.
  - $\dim \nu = \frac{l_G}{\epsilon l} = \frac{h}{\epsilon l}, \quad \dim \mu = \frac{D_\Gamma}{\epsilon}$.

Relatively hyperbolic group

- Γ – group is (nice) rel. hyp. if f.g. and
- if $\exists$ finitely many f.g subgroups $\mathcal{H} := H_1, \ldots, H_s$
- for which:
  - $\exists$ proper, hyperbolic metric space X, such that
  - Γ acts properly discontinuously and isometrically on X, and
- cusp-uniformly:
  - there is a Γ-equivariant collection of disjoint open horoballs on whose complement the action is cocompact and
  - the H_i represent conjugacy classes of corresponding maximal parabolic subgroups.

Bowditch boundary

- If $\Gamma \rightharpoonup X$ cusp-uniformly, the Bowditch boundary $\partial_B(\Gamma, \mathcal{H})$ of the pair $(\Gamma, \mathcal{H})$ is the visual boundary ∂X with the visual topology.
- $\partial_B(\Gamma, \mathcal{H})$ is a QI-invariant for the pair $(\Gamma, \mathcal{H})$.
- Two cusped spaces need not be QI.
- For a QI-inv. metric/QS structure on $\partial_B(\Gamma, \mathcal{H})$, consider cusped-spaces of constant horospherical distortion (Healy–Hruska). All such $(\Gamma, \mathcal{H})$-cuped-spaces are QI equivalent and this gives a metric/QS structure to $\partial_B(\Gamma, \mathcal{H})$. We will mean this space by the Bowditch boundary of $(\Gamma, \mathcal{H})$.
- An example we will discuss is a construction of Groves–Manning.

X-valued Markov chains

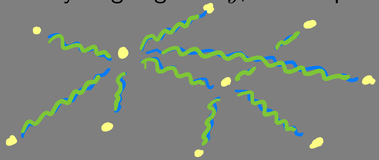
- $X = (V, E)$ is some graph. $X = (X, d, \lambda = \sum_{x \in X} \delta_x)$.
- A real non-negative matrix P with row sum one (stochastic). A probability $\overline{m}$ in X.
- Random variables $\{Y_i\}_{i \in \mathbb{N}_0}$ with values in X.
  - $\mathbb{E}_{\overline{m}}[\chi_{Y_{i+1}^{-1}} \,|\, \sigma(Y_i)]$ and $\sigma(Y_0, \ldots, Y_{i-1})$ are independent.
  - $\mathbb{E}_{\overline{m}}[\chi_{Y_{i+1}^{-1}\{y\}} \,|\, \sigma(Y_i)] = p(Y_i, y)$.
- $\mathbb{P}_{\overline{m}}[Y_n = y] = \sum_{x_0, \ldots, x_{n-1}} \overline{m}(x_0) \cdot \prod_{i=0}^{n-2} p(x_i, x_{i+1}) \cdot p(x_{n-1}, y)$.
- Use Γ to choose a P, when X has a nice Γ-action by automorphisms of X.

X-valued Markov chains

- Example: If μ is a probability on Γ (support generates Γ), then $p(x, gx) = \mu(g)$, defines Γ-invariant Markov chains in X. (Furstenberg)
- Example: Any stochastic matrix P, such that $p(gx, gy) = p(x, y)$, for all $g \in \Gamma$

Green metric 1

- P and 'heat flow'.
  - $\sum_y p^{(n)}(x,y) = 1.$
  - $p^{(n)}(x,y)$ probability of going $x \to y$, in n-steps.



- Green function $(x,y) \mapsto G(x,y)$: Total heat received by $y \in X$ when diffused from a unit of heat sourced at x.
  - $G(x,y) = \sum_{n=0} p^{(n)}(x,y)$. Finiteness $\iff$ Transience.
- Idea: Larger $G(x,y) \iff$ smaller 'distance' between x and y wrt the interactions.
  - Identity: $\frac{G(x,y)}{G(y,y)} \cdot \frac{G(y,z)}{G(z,z)} \leqslant \frac{G(x,z)}{G(z,z)}$
  - Green's metric: $\rho_G(x,y) = -\log \frac{G(x,y)}{G(y,y)}$.
- (P related to ρ_X) $\rightsquigarrow$ (ρ_G related to ρ_X) ?

Martin boundary and Harmonic measure for transient walk

- $K_y(x, z) = \frac{G(x,z)}{G(y,z)}$, for $x, y, z \in X$.
- Given $y \in X$, $z \mapsto K_y(\cdot, z)$ is an embedding from X to $\mathcal{S}_y$, the space of positive superharmonic functions, assuming one at the point y.
- The Martin compactification $X \cup \partial_M X$ is the closure of $X \subset \mathcal{S}_y$ with respect to the compact-open topology in $\mathcal{S}_y$, and the boundary $\partial_M X$ is called the Martin boundary. For a sequence $\{z_n\}_n \subset Y$, the limit $\xi = \lim_n z_n \in \partial_M Y$, is denoted $K(\cdot, \xi) = \lim_n K(\cdot, z_n)$.

Martin boundary and Harmonic measure for transient walk

- Martin representation formula: given $f \geqslant 0$ harmonic, there is a unique Borel measure μ_y^f in $\underline{\partial}_M X$, such that for all $x \in Y$,

$$f(x) = \int_{\underline{\partial}_M Y} K_y(x, \xi)\, d\mu_y^f(\xi).$$

- Martin convergence theorem: for any $x \in X$, for the Markov chain (X, P), $\mathbb{P}_x$-a.e path, $Y_n \xrightarrow{n \to \infty} X_\infty$, where Y_∞ is a $\underline{\partial}_M X$-valued random variable with distribution

$$\nu_x(E) := \mathbb{P}_x[X_\infty \in E] = \int_E K_y(x, \xi)\, d\mu_y^1(\xi),$$

μ_y^1 being the measure corresponding to the constant harmonic function one

Martin boundary and Harmonic measure for transient walks

- For all $x, y, z \in X$,

$$\nu_y = \mu_y^1 \quad \text{and} \quad \frac{d\nu_x}{d\nu_z}(\xi) = \frac{K_y(x,\xi)}{K_y(z,\xi)} = K_z(x,\xi), \quad \text{for } \nu_z\text{-a.e } \xi \in \underline{\partial}_M X,$$

- Markov property/Harmonicity: $\nu_x = \sum_y p(x,y) \cdot \nu_y$.
- Martin boundary is the Busemann boundary of the Green metric: $x \mapsto (z \mapsto \rho_G(x,z) - \rho_G(y,z))$.

Earlier work

- Random walks on groups with hyperbolic properties and on hyperbolic graphs, have been studied extensively.
    - Ancona, Kaimanovich, Ledrappier, LePrince, Gouezel, Gouezel-Lalley...
    - Dussaule, Dussaule-Gekhtman-Gerasimov-Potyagailo, Gekhtman-Gerasimov- Potyagailo-Yang...
    - Kaimanovich, Kaimanovich-Woess, Connell-Muchnik, Maher-Tiozzo, Gekhtman-Tiozzo, Tanaka...
    - Varopoulos...
- The type of our results overlap the most with those in the work of Blachère-Haïssinsky-Mathieu.
- A lot is known but the area is still developing and a lot is not known.

Basic result

- $\mathcal{D}(X)$ is the set of quasiruled quasigeodesic metrics QI to the graph metric.

Theorem

Let Γ be a finitely-generated non-elementary relatively hyperbolic group with finitely-generated parabolic subgroups $\mathcal{P} = \{H_i\}_{i=1}^{l}$. Let $X = X_\Gamma$ be its cusped graph. Then the following are equivalent:

- *there exists a random walk on X with Martin boundary as the Bowditch boundary of Γ such that the harmonic measure is an Ahlfors-regular, Γ-equivariant conformal density associated to a metric in $\mathcal{D}(X)$;*
- *H_i for $i \in \{1, \ldots, l\}$, are virtually nilpotent.*

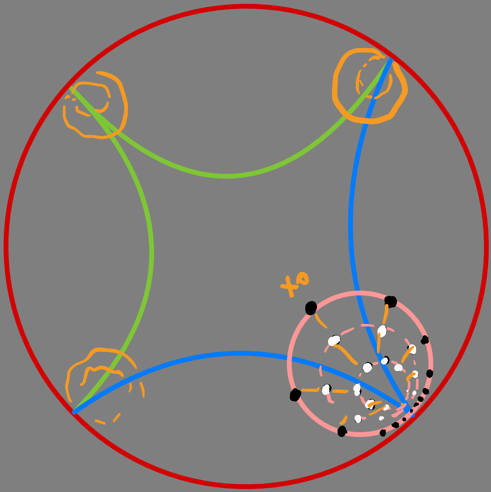
- Next we define the graph and mention steps in the proof.

Groves-Manning graph $X = X_\Gamma$

$$V_\Gamma = (\Gamma \times \{0\}) \bigsqcup_{i=1}^{l} \left(\bigcup_{g \in \Gamma} gH_i \times \mathbb{N} \right).$$

- The edge set $E_\Gamma \subset V_\Gamma \times V_\Gamma$ contains pairs of the following types:
  - $((g,0),(g',0)) \in E_\Gamma$ if and only if (g,g') is an edge in $Cay(\Gamma, S)$,
  - for $n \in \mathbb{N}_0$, $g \in \Gamma$ and $h, h' \in H_i$ for some $i \in \{1, \ldots, l\}$, $((gh,n),(gh',n)) \in E_\Gamma$ if and only if $d_{(\Gamma,S)}(h,h') \leqslant a^n$.
  - for $n \in \mathbb{N}_0$, $g \in \Gamma$ and $h, h' \in H_i$ for some $i \in \{1, \ldots, l\}$, $((gh,n),(gh',n+1)) \in E_\Gamma$ if and only if $d_{(\Gamma,S)}(h,h') \leqslant a^{n+1}$.
- Has to have unbounded and exponentially increasing degree.

Groves-Manning graph



The space (X, ρ_X, m)

- For above X, $(X, \rho_X, \lambda) \longrightarrow (\mathbb{H}^2, \rho_{\mathbb{H}^2}, vol_{\mathbb{H}^2})$. QI but not volume preserving.

- Define an $m : X \to (0, \infty)$, $(X, \rho_X, m) \longrightarrow (\mathbb{H}^2, \rho_{\mathbb{H}^2}, vol_{\mathbb{H}^2})$ is (quasi) vol. preserving.

- Total heat $G(\cdot, y)$ through y can be very small even if it is close to the source in ρ_X, for example if y has many neighbours to share it with.

- We'll make a correction for this.

An X-valued reversible Markov chain (X, P, m)

Write d for the graph distance in $Cay(\Gamma, S)$ and g_H for growth of H. With data as above, define $m : X \to (0, \infty)$ and $P = (p(x, y))_{x,y \in X}$ as follows:

- $m((g, 0)) = m_0$
- $m((g, n)) = \frac{c_n}{g_H(a^n)}$ for $n \in \mathbb{N}$,
- for $g \sim g'$ in $Cay(\Gamma)$,

$$p((g, 0), (g', 0)) = \frac{1}{p} \cdot \frac{1}{\mathrm{deg}(Cay(\Gamma))},$$

- for $g' \in gH$ where $\mathrm{d}(g, g') \leqslant a^{n+1}$ in $Cay(\Gamma)$ and $n \geqslant 0$,

$$p((g, n + 1), (g', n)) = \frac{1}{q_n \cdot g_H(a^{n+1})}.$$

- for $g' \in gH$ where $\mathrm{d}(g, g') \leqslant a^{n+1}$ in $Cay(\Gamma)$ and $n \geqslant 0$,

$$p((g, n), (g', n + 1)) = \frac{1}{p_n \cdot g_H(a^{n+2})}.$$

- for $g' \in gH$, $n \in \mathbb{N}$ the remaining probability for jump starting at (g, n) is distributed among neighbours of (g, n) in $m^{-1}(\{m((g, n))\})$ in any Γ-invariant way (for example, uniformly).

Spectrum of the Markov operator: $\rho(P)$

- Reversible: $m(x) \cdot p(x,y) = m(y) \cdot p(y,x)$.
  - $Pf(x) = \sum_y p(x,y) \cdot f(y)$.
  - $\|P\|_{\ell^2(X)} = \|P_m\|_{\ell^2(X,m)}$, where $f \mapsto f/\sqrt{m}$ from $\ell^2(Y) \to \ell^2(Y,m)$.
  - P_m is symmetric. Then $\rho(P_m) = \|P_m\|_{\ell^2(X,m)}$, but $\rho(P) = \rho(P_m)$.
- Strong Isoperimetric inequality for $(X,P,m) \iff \rho(P) < 1$.
- Strong IS: $\exists \kappa > 0$, such that for all finite sets E,

$$m(E) \leqslant \kappa \cdot \sigma_{P,m}(\partial E).$$

Transition kernel

- Triangle inequality:

$$\frac{G(x,y)}{G(y,y)} \cdot G(y,z) \leqslant G(x,z),$$

  for all x, y, z.

- Ancona inequality used by Ancona to identify the Martin boundary as the visual boundary in suitable hyperbolic graphs:

$$C^r \cdot \frac{G(x,y)}{G(y,y)} \cdot G(y,z) \gtrsim G(x,z),$$

  for $\mathrm{dist}(y, [x,z]) \leqslant r$.

Transition kernel

- Simple computation (reversibility + doubling): Lower bound

$$p^{(\rho_X(x,y)+1)}(x,y) \geqslant m(y) \cdot \lambda^{\rho_X(x,y)},$$

$0 < \lambda < 1.$

- Have $p^{(n)}(x,x) \leqslant \rho(P)^n \to 0.$
  - Need $p^{(n)}(x,x) \lesssim m(x) \cdot \rho(P)^n$.

- $p^{(n)}(x,x)/m(x)$ almost subadditive (doubling + Γ-invariance) $\implies$ $p^{(n)}(x,x) \lesssim m(x) \cdot \rho(P)^n$.

Harnack inequality

- These lead to the Harnack inequalities.

Lemma

The following hold:

- *If* $x, y, z \in X$, *and* $x \neq y$, *then* $G(x,y) \leqslant C^{\rho_X(x,z)} \cdot G(z,y)$.
- *For* $x, y \in X$, $m(x) \cdot G(x,x) \leqslant C^{\rho_X(x,y)} \cdot G(y,x)$
- *For* u *positive,* P*-superharmonic, and* $x, y \in X$, $m(x) \cdot u(x) \leqslant C^{\rho_X(x,y)} \cdot u(y)$.

Corollary

If u *is a non-negative* P*-harmonic function, then there exists* $C \geqslant 1$ *such that* $u(x) \leqslant C^{\rho_X(x,y)} \cdot u(y)$.

Green metric 2

- Modify the Green metric $\rho_G(x,y) = -\log G_m(x,y)$ where
$$G_m(x,y) = \begin{cases} \frac{G(x,y)}{C \cdot m(y)} & x \neq y \\ 1 & x = y. \end{cases}$$
suitable constant $C > 0$.
- Follows from Harnack inequalities and a triangle inquality
$$\frac{G(x,y)}{m(y) \cdot G(y,y)} \cdot \frac{G(y,z)}{G(z,z)} \lesssim \frac{G(x,z)}{G(z,z)}.$$
- Ancona inquality given Harnack and triangle inequalities for G:

  There exist constants $C_0, C \geqslant 1$ such that for $x, y, z \in X$ with $\text{dist}_X(y, [x,z]) \leqslant r$, we have
$$G(x,z) \lesssim C^{2r} \cdot \frac{G(x,y)}{m(y)} \cdot G(y,z), \qquad (1)$$
  for any geodesic $[x,z]$ joining x, z.
- $\Gamma \subset \text{Isom}(X, \rho_G)$.

Martin boundary and conformality of ν

- Both G and G_m give the same Busemann functions (and Busemann boundary) but second ρ_G is hyperbolic and QI to ρ_X (Harnack inequalities).
- Ancona inequality $\implies \partial X = \partial_M X$, as a homeomorphic extension of the embedding $z \mapsto (x \mapsto K(x, z))$.
- Conformality: For the harmonic measures $\{\nu_x\}_{x \in X}$, we get for ν_x-a.e $\xi \in \partial X$,

$$\frac{d\nu_y}{d\nu_x}(\xi) = e^{-\beta_\xi^G(y,x)}.$$

Shadow lemmas

Lemma (Shadow lemma for ν)

There exists $R_\nu > 0$, such that for any $x \in X$, $r \geqslant R_\nu$, so that for all $y \in X$,

$$\nu_x(S_G(x, B_G(y,r))) \approx_r e^{-\rho_G(x,y)}.$$

- $D_\Gamma := \limsup_{n\to\infty} \frac{1}{n} \cdot \log(\#(\Gamma \cap B_X(e,n)))$.

Lemma (Shadow lemma for μ)

There exists $R_\mu > 0$, such that for any $x \in X$, $r \geqslant R_\mu$, so that for all $y \in X$,

$$\mu_x(S(x, B(y,r))) \approx_r e^{(d_H \cdot \log a - D_\Gamma)\cdot \rho_X(y,\Gamma)} \cdot e^{-D_\Gamma \cdot \rho_X(x,y)}.$$

Two immediate consequences

- The total energy through a point in a horoball of higher rank is lower in comparison:
  - the Green function inside a horoball only depends on the rank of the corresponding parabolic subgroup H:

$$|\rho_G(x,y) - d_H \cdot \log a \cdot \rho_X(x,y)| \leqslant C,$$

for some $C = C(X, P, m) > 0$.

- $D_\Gamma > \frac{d_H}{2} \cdot \log a$.

Growth

- $W_G(n,r) =$ set of nearest points in $\Gamma \subset X$ to e, of cosets in an annulus of width r at distance $n \cdot r$.

Lemma

There exists $R_X > 0$ such that, for any $x \in \Gamma$, U any ball in $(\partial X, d_G)$, and $n \in \mathbb{N}$ large

$$\#\{g \in W_G(n, R_X) \mid S_G(x, B_G(\gamma, R_X)) \cap U \neq \varnothing, \} \approx e^{n \cdot R_X} \cdot \nu_x(U),$$

and if U is a ball in $(\partial X, d_X)$, for n large enough,

$$\#\{g \in W_G(n, R_X) \mid S_X(x, B_G(\gamma, R_X)) \cap U \neq \varnothing, \} \approx e^{n \cdot D_\Gamma \cdot R_X} \cdot \mu_x(U),$$

Tracking

Lemma

Let $x \in X$. Let $k \in \mathbb{N}$. Then for any $D > 0$,

$$\mathbb{P}_x[\rho_G(Y_k, \gamma^{x,Y_\infty}) \geqslant D] \leqslant C \cdot e^{-D},$$

for some $C > 0$. In particular, $\mathbb{P}_x$-a.e $\omega \in X^{\mathbb{N}}$,

$$\limsup_{n\to\infty} \frac{\rho_G(Y_n(\omega), \gamma^{x,Y_\infty(\omega)})}{\log n} \leqslant 1.$$

- Same for (ρ_X, μ) (by QI).

Cusp-excursions

Lemma

We have for $x \in X$*,*

$$\nu_x \left(\left\{ \gamma_\infty^x \in \partial X \;\middle|\; \limsup_{t \to \infty} \frac{\rho_G(\gamma^x(t), \Gamma)}{\log t} = 1 \right\} \right) = 1,$$

and

$$\mu_x \left(\left\{ \gamma_\infty^x \in \partial X \;\middle|\; \limsup_{t \to \infty} \frac{\rho_X(\gamma^x(t), \Gamma)}{\log t} = \frac{1}{2 \cdot D_\Gamma - d_H \cdot \log a} \right\} \right) = 1.$$

where γ_x *is any* (X, ρ_X)*-geodesic ray starting at* X *(the first limit is on times where* $\gamma^x(t) \in V_\Gamma$*).*

Ergodic aspects

- $X^{\mathbb{N}_0} \longrightarrow \Gamma \backslash X \times X^{\mathbb{N}}$.
- $\mathbb{P}_m = \sum_{z \in X} m(z) \cdot \mathbb{P}_z \longrightarrow \mathbb{P}'_{\overline{m}} = \sum_{n \in \mathbb{N}_0} m((e,n)) \cdot \mathbb{P}_{(e,n)}$.

Ergodicity..generators

- Pick a connected set of X (containing e; exists by connectedness of the graph) as the set of representatives of $\Gamma\backslash X$, namely $\{e\} \times \mathbb{N}_0$. Then

$$S_X = \{y \in X \mid p(x,y) \neq 0,\ x \in \Gamma\backslash X\}$$

$$= \left\{ (s,n) \in \Gamma \times \mathbb{N}_0 \;\middle|\; \begin{array}{l} s \in B_\Gamma(e,a), \text{ if } n = 0 \text{ and,} \\ s \in B_H(e,a^{n+1}), \text{ if } n \in \mathbb{N}. \end{array} \right\}$$

- For this walk, S_X generates X as a semigroup under

$$(g,n) * (g',n') = (gg',n').$$

- $T(\{(s_1^{(n_1)}, n_1), \ldots, (s_k^{(n_k)}, n_k), \ldots\}) = \{(s_2^{(n_2)}, n_2), \ldots, (s_{k+1}^{(n_{k+1})}, n_{k+1}), \ldots\}.$
- $\mathbb{P}_{\overline{m}}(\{\omega \mid X_i(\omega) = (s_i, n_i), 1 \leqslant i \leqslant k\})$

$$= \sum_{(e,n)\in\Gamma\backslash X} \overline{m}((e,n)) \cdot p((e,n),(s_1^{(n_1)}, n_1)) \cdot \prod_{j=1}^{k-1} p((e,n_j),(s_{j+1}^{(n_{j+1})}, n_{j+1})).$$

- $\overline{m}((e,k)) = \frac{m((e,k))}{\sum_{n\in\mathbb{N}_0} m((e,n))}.$
- This left-shift is mixing.

Ergodicity..two-sided shift

- $(\partial X, \Gamma, \mu)$ and $(\partial X, \Gamma, \nu)$ are ergodic.
- $(\partial X \times \partial X, \Gamma, \mu \times \mu)$, $(\partial X \times \partial X, \Gamma, \nu \times \nu)$ are ergodic.
- $\omega = \{\ldots, (s_{-i}^{-1}, n_{-i}), \ldots (s_{-1}^{-1}, n_{-1}), (e, n_0), (s_1, n_1), \ldots, (s_j, n_j), \ldots\}$,

  $T(\omega) = \{\ldots, (s_{-i}^{-1}, n_{-i}), \ldots (s_{-1}^{-1}, n_{-1}), (s_1^{-1}, n_0), (e, n_1), (s_2, n_2), \ldots (s_j, n_j), \ldots\}$,
- $(\Omega, \tilde{\mathbb{P}}_{\overline{m}}) \longrightarrow (\partial X \times \partial X, \sum_{n \in \mathbb{N}_0} \overline{m}((e, n)) \cdot \nu_{(e,n)} \otimes \nu_{(e,n)})$.
  - Convex combination of the harmonic measures.
  - $\omega \mapsto (Z_{-\infty}(\omega), Z_{\infty}(\omega))$.

Drift, Entropy

Theorem

For $\{Y_i\}_{i\in\mathbb{N}_0}$*, a process corresponding to* (X, P, m) *the following limits exist and are constants* $\mathbb{P}_m$*-a.e* $\omega \in X^{\mathbb{N}}$*:*

$$(\textit{drift}) \qquad \lim_{n\to\infty} \frac{\rho_X(Y_0(\omega), Y_n(\omega))}{n} =: l,$$

$$(\textit{Green drift}) \qquad \lim_{n\to\infty} \frac{\rho_G(Y_0(\omega), Y_n(\omega))}{n} =: l_G,$$

$$(\textit{asymptotic entropy}) \qquad \lim_{n\to\infty} \frac{-\log p^{(n)}(Y_0(\omega), Y_n(\omega))}{n} =: h.$$

Relations

Proposition

- $c \cdot l \leqslant h \leqslant l \cdot D_\Gamma$.
- $h = l \cdot D_\Gamma$ *iff:*
  - $\nu \approx \mu$.
  - $|\rho_G - D_\Gamma \cdot \rho_X| \lesssim 1$.
- $h = l_G = \int_{\Omega_1 \times \partial X} \beta_\xi^G(X_1(\omega)^{-1}, (e, n_{X_2(\omega)}))\, d\nu_{X_2(\omega)}(\xi)\, d\mathbb{P}_{\overline{m}}(\omega)$.

Corollary

If $h = l \cdot D_\Gamma \implies$ *then all the parabolic subgroups have rank* $= D_\Gamma / \log a$.

Dimension

Theorem

We have for the harmonic measure ν_x supported in ∂X,

$$\lim_{r\to 0}\frac{\log(\nu_x(B^X_\infty(\xi,r)))}{\log r}=\frac{l_G}{\epsilon_X\cdot l},\quad \lim_{r\to 0}\frac{\log(\nu_x(B^G_\infty(\xi,r)))}{\log r}=\frac{1}{\epsilon_G},$$

and

$$\lim_{r\to 0}\frac{\log(\nu_x(B^X_\infty(\xi,r)))}{\log r}=\frac{h}{\epsilon_X\cdot l},\quad \lim_{r\to 0}\frac{\log(\nu_x(B^G_\infty(\xi,r)))}{\log r}=\frac{h}{\epsilon_G\cdot l_G}.$$

For the Patterson-Sullivan density μ of (Γ,X), we have

$$\lim_{r\to 0}\frac{\log(\mu_x(B^X_\infty(\xi,r)))}{\log r}=\frac{D_\Gamma}{\epsilon_X}.$$

Thank You!