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Equidistribution of best approximations.

Joint w/ Barak Weiss

Basic question: Given $v \in \mathbb{Z}^d$ study

$$\pi^{v}_{v^{\perp}}(\mathbb{Z}^d)$$

$\pi^{u}_{w} : \mathbb{R}^d \to \mathbb{R}^d$
proj w/ kernel u
Image w

Problem: Given a naturally defined sequence $v_k \in \mathbb{Z}^d$, what can be said about the dist of $\left[\pi^{v_k}_{v_k^{\perp}}(\mathbb{Z}^d)\right] \in \mathcal{Z}_{n,d}$

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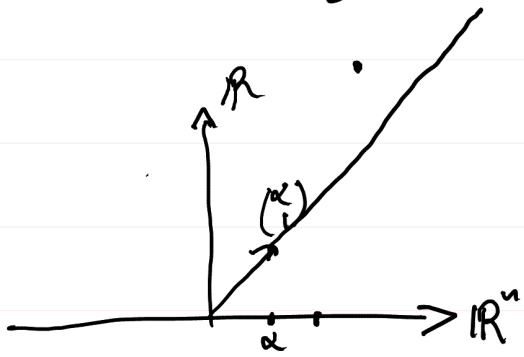
$\boxed{n = d-1}$ $\quad Z_{n,d}$ = space of rank n discrete s.gps of $\mathbb{R}^d$ up to scaling.

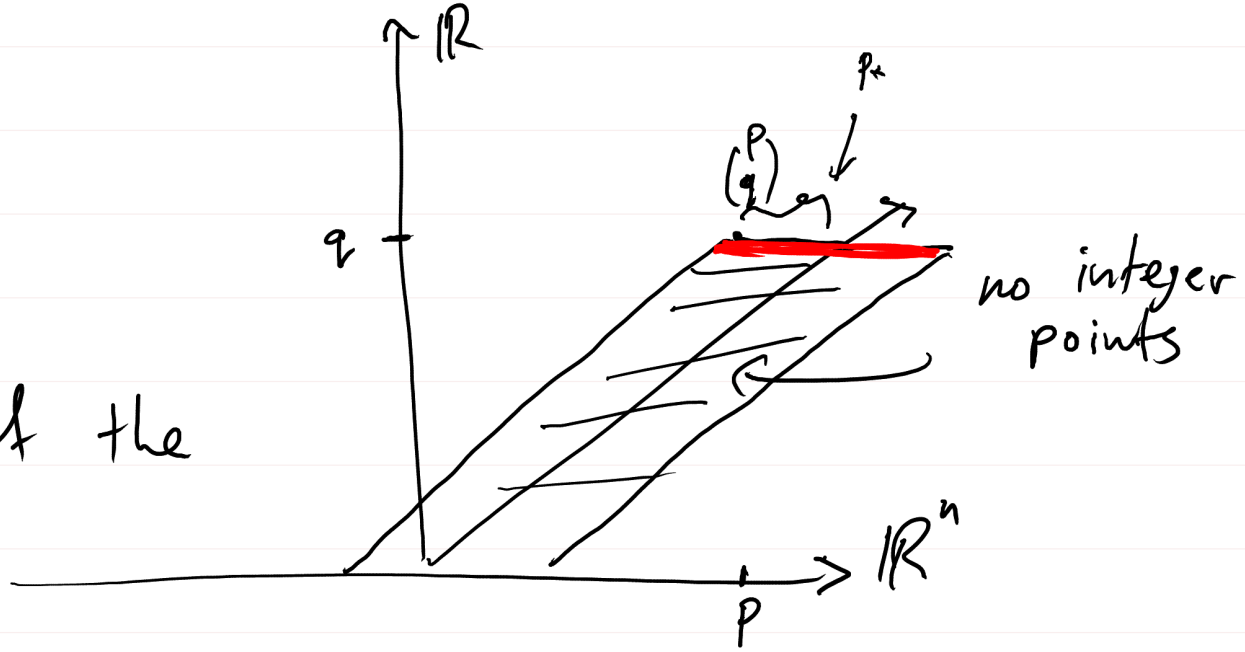


Best approximations: Give $\alpha \in [0,1]^n$

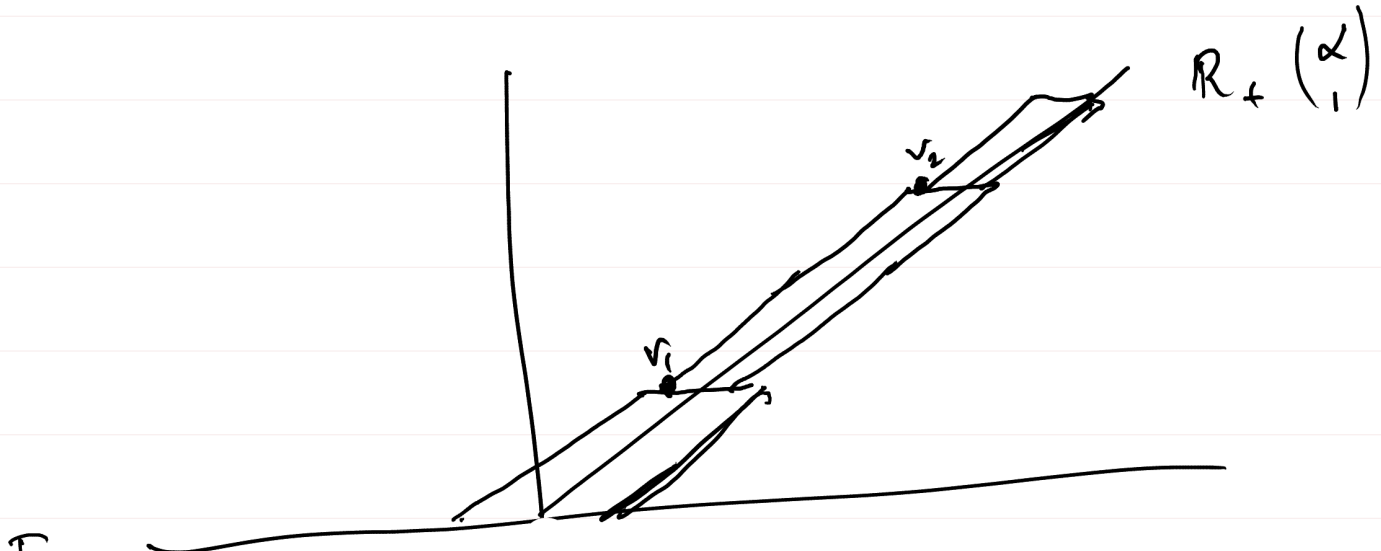
The set of Best apx of α is

$$\left\{ v^{best} = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^d : \forall_{0<q'<q,\ q>0} \forall p' \in \mathbb{Z}^n \right.$$ not true that both

$$\left. \begin{array}{l} \|p' - q'\alpha\| < \|p - q\alpha\| \\ |q'| < |q| \end{array} \right\}$$



Seq of best app is an ordering $v_\kappa = \begin{pmatrix} p_\kappa \\ q_\kappa \end{pmatrix}$ of the set of best app.



For a.e α

Thm: The seq $\left(\dfrac{p_k - q_k^\alpha}{\|p_k - q_k^\alpha\|}, \pi_{V_k^\perp}^{V_k}(\mathbb{Z}^d) \right) \subseteq \mathbb{S}^{n-1} \times \mathcal{Z}_{n,d}$

has a limiting dist $= \nu_{best}$ on $\mathbb{S}^{n-1} \times X_n^\alpha$

space of lattices in $\begin{pmatrix} \alpha \\ 1 \end{pmatrix}^\perp$.

- abs cont w.r.t Haar w/ pos density!
- The proj to $\mathbb{S}^{n-1}$ is uniform
- ν_{best} is not a product measure
- The proj onto X_n^α has density $\to 0$ in the cusp!

Σ-approximations The set of Σ-app is

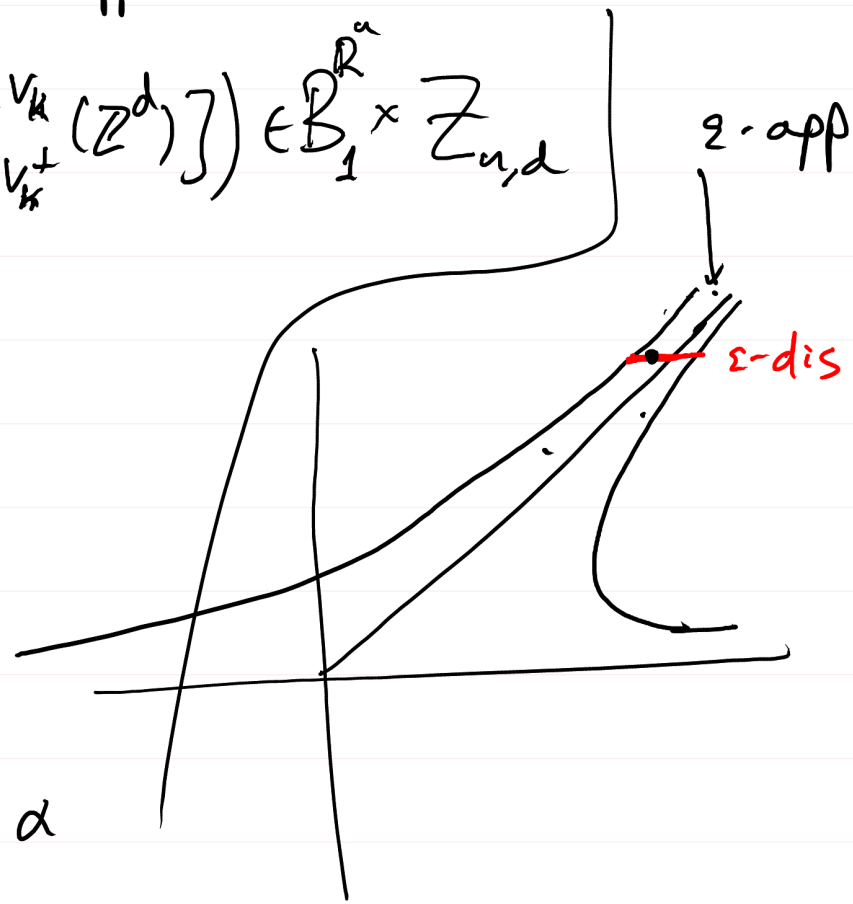
$\forall \Sigma > 0$ $\quad \{ v^\Sigma = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^d : q > 0 \quad q^{1/n} \| p - q\alpha \| < \Sigma \}$

$\rightsquigarrow \{ v_k^\Sigma = \begin{pmatrix} p_k \\ q_k \end{pmatrix} \}$ - seq of Σ- app.

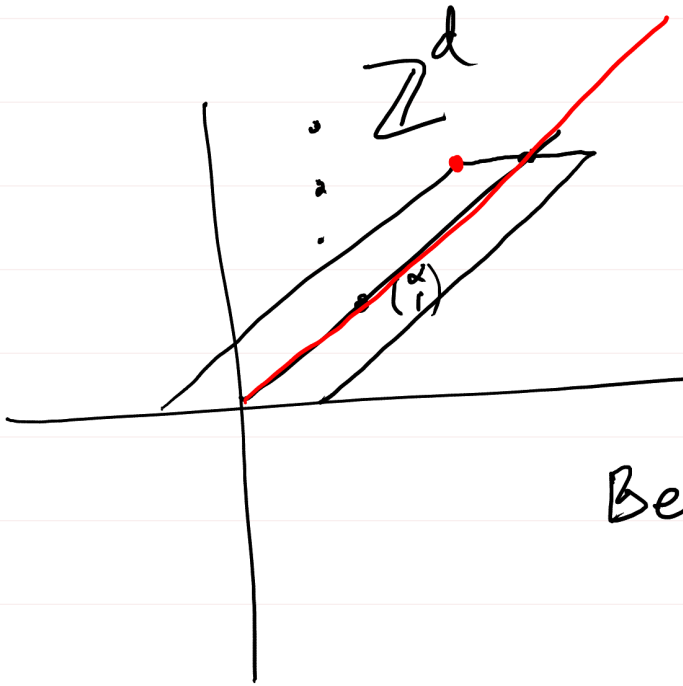
Thm The seq $\left(\frac{p_k - q_k \alpha}{\Sigma \cdot q_k^{-1/n}}, [\pi_{V_k^+}^{V_k}(\mathbb{Z}^d)] \right) \in B_1^{\mathbb{R}^n} \times Z_{n,d}$

equidist (a.s in α) to

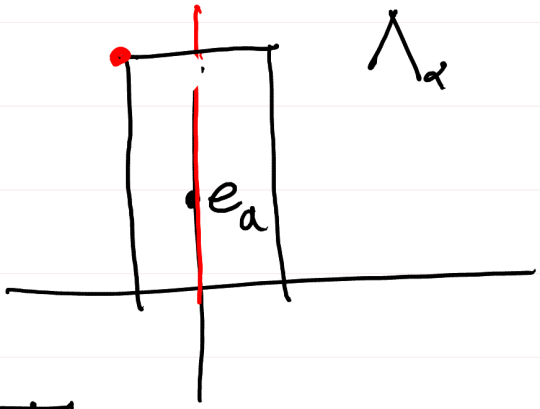
$dvol|_{B_1^{\mathbb{R}^n}} \times (\text{Haar on } X_u^\alpha)$



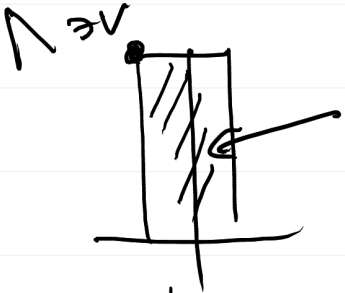
AGT: The directions $\frac{p_k - q_k \alpha}{\| p_k - q_k \alpha \|} \in S^n$ equidist a.s in α



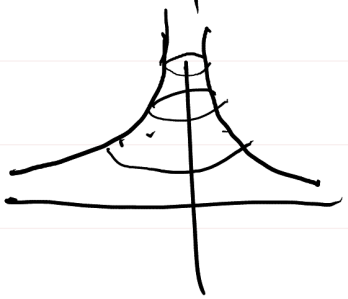
$$\begin{pmatrix} I_n & -\alpha \\ 0 & 1 \end{pmatrix}$$

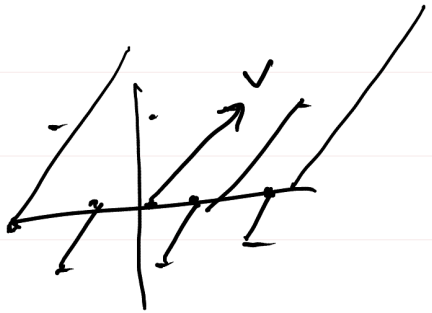


Best app



ε-app





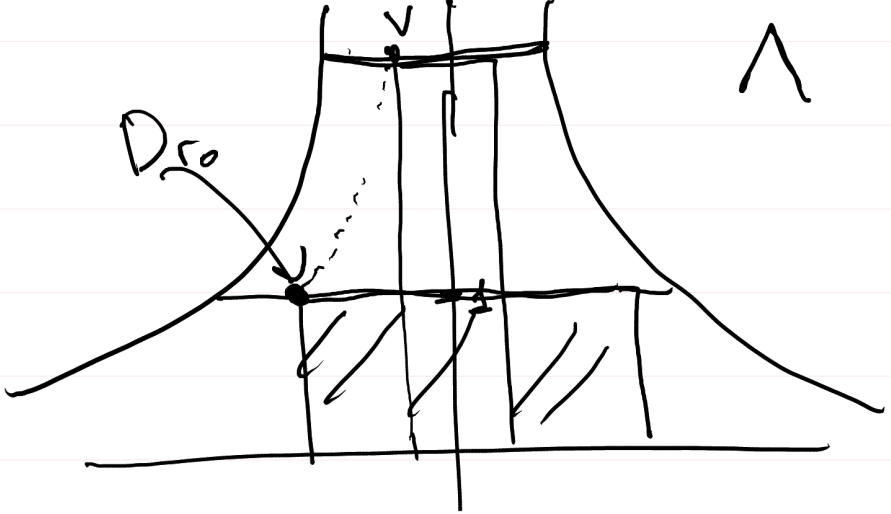
Step 0: replace $\pi_{V_n^\perp}^{V_n}(\mathbb{Z}^d)$ be $\pi_{\mathbb{R}^n}^{V_n}(\mathbb{Z}^d)$

Preduced to study best/ε approximations of vertical line & projections

$\pi_{\mathbb{R}^n}^{V_K}(\Lambda_\alpha)$ $V_K \in \Lambda_\alpha$ best app ε-app

Fix $r_0 \asymp 1$

r_0 - app



apply $a(t) = \begin{pmatrix} e^t I_n & \\ & e^{-nt} \end{pmatrix}$

$$F_{r_0} = \{ \Lambda \in X_d : \Lambda \text{ contains a primitive vector in } D_{r_0} \}$$

$$D_{r_0} = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : \|x\| \leq r_0 \right\}$$

$\|x\|$ in $\mathbb{R}^n$

F_{r_0} is a cross-section for the $a(t)$ flow

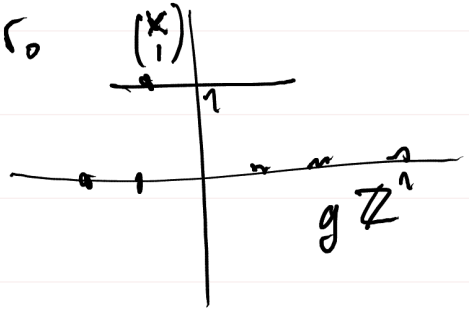
If $a(t)\Lambda$ equidist in X_d

$\Longleftrightarrow$

The seq of hittings to the cross-section equidist w.r.t the cross-section measure

Coordinates on F_{r_0}

$\Lambda \in F_{r_0}$



$\Longleftrightarrow \quad \exists\, g, y$

$$\begin{pmatrix} I_n & x \\ 0 & 1 \end{pmatrix} \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} I & 0 \\ y & 1 \end{pmatrix} \cdot \mathbb{Z}^d$$

Λ