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DIOPHANTINE APPROXIMATION ON TRANSLATION SURFACES

$$\alpha \in \mathbb{R}, p, q \in \mathbb{Z}$$

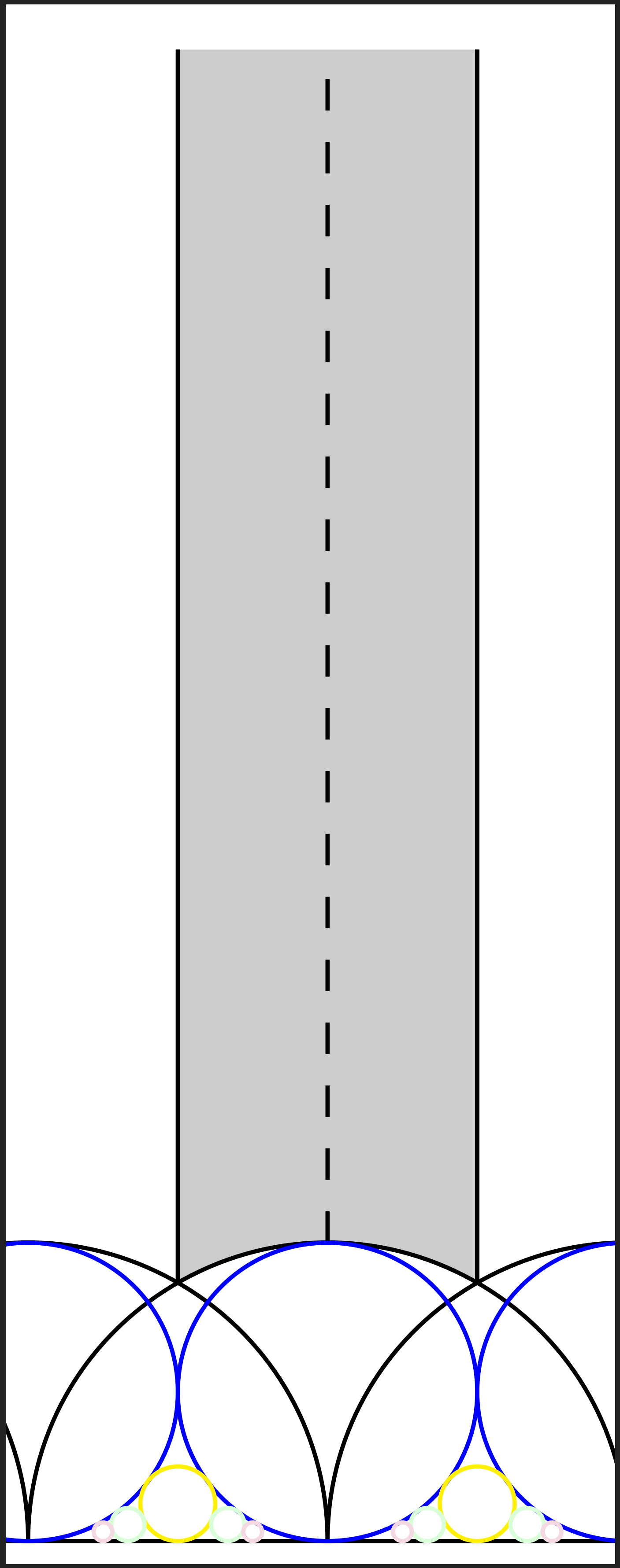
$$\left|\alpha - \frac{p}{q}\right| < \psi(q)$$

Classical Diophantine Approximation

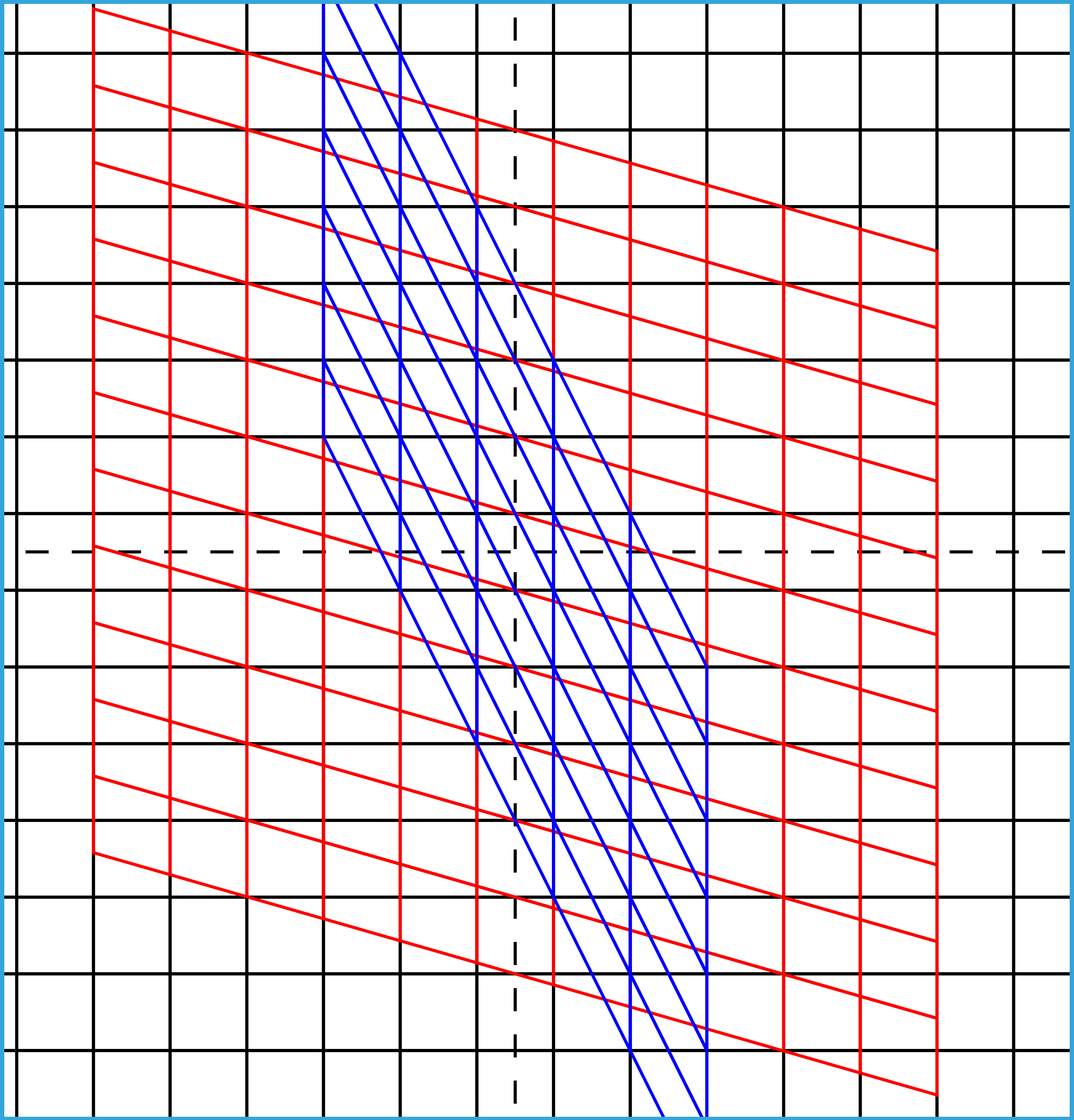
LATTICES

- For a geometer, this is a statement about the geometry of (a family of) lattices.
- In particular, this is a family of shears of the standard integer lattice.

$$\Lambda_\alpha = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2$$



SHEARING



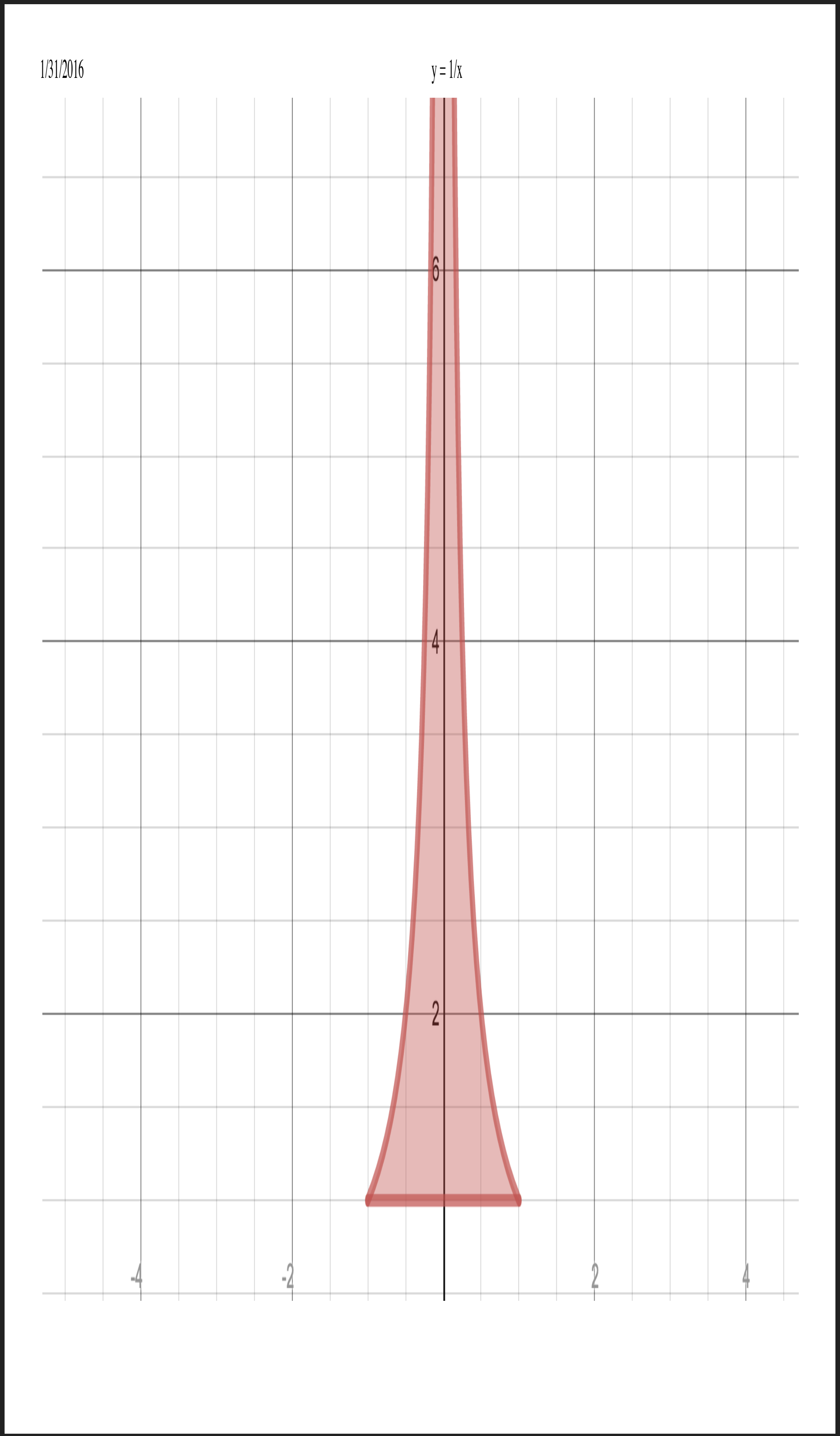
- A typical vector in Λ_α has the form $\begin{pmatrix} p - q\alpha \\ q \end{pmatrix}$
- So asking to solve the inequality

$$\left| \alpha - \frac{p}{q} \right| < \psi(q)$$

- Is the same as finding lattice points

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \Lambda_\alpha \text{ such that } |x| < y\psi(y)$$

CLASSICAL EXAMPLE



$$\psi(y) = \frac{1}{y^2}$$

PIGEONHOLE PRINCIPLE

APPROXIMATIONS AND PROBABILITY

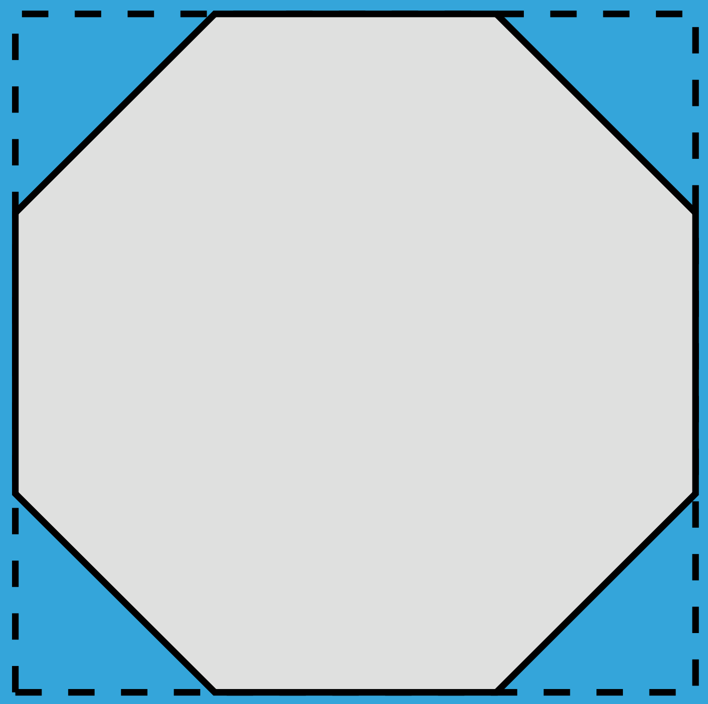
$\text{Fix } \psi, \text{ vary } \alpha$	infinitely many solutions	measure/hdim of set of alpha
$\text{Fix } \alpha, \text{ vary } \psi$	infinitely many solutions	diophantine exponent of alpha
$\psi(y) = A/y$	solutions in range cN < q< dN	limit measure as N grows
$\psi(y) = A/N$	solutions in range cN < q < dN	limit measure as N grows

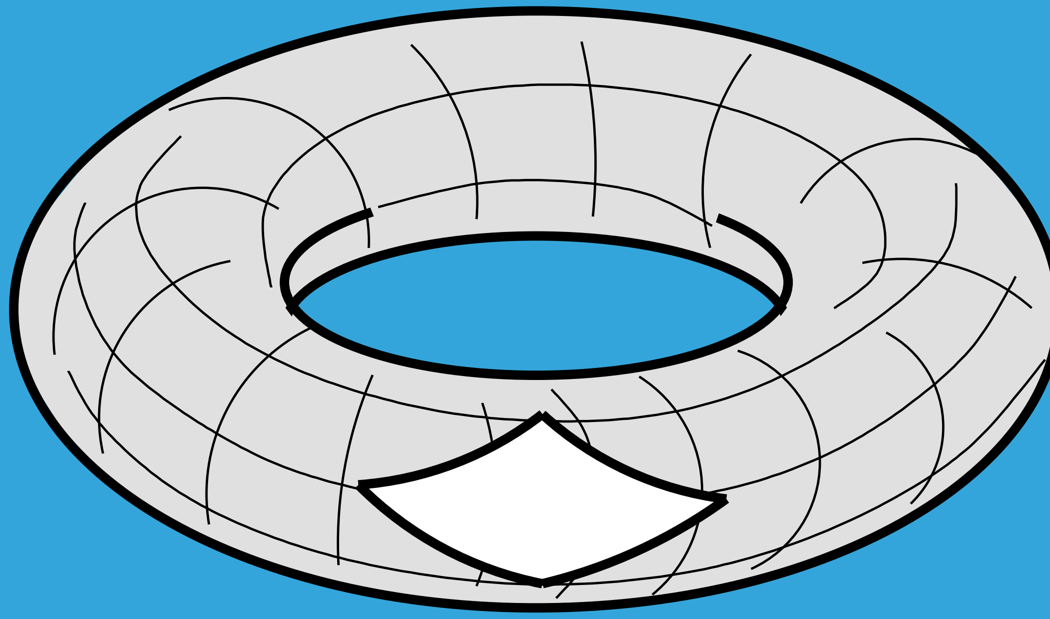
HOW DO WE GENERALIZE?

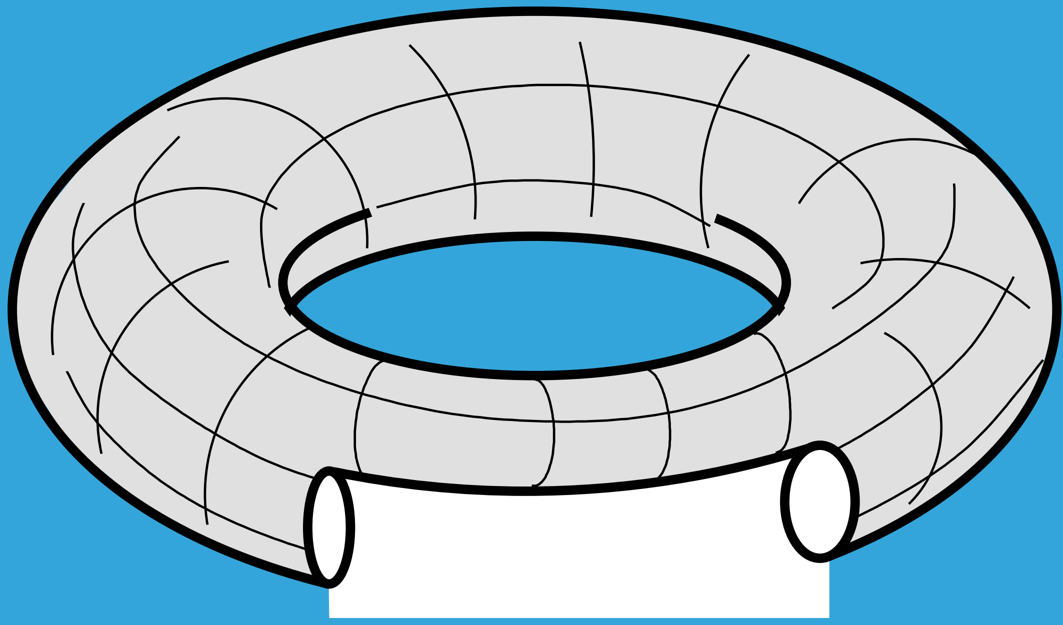
ALSO, HOW DO WE SOLVE?

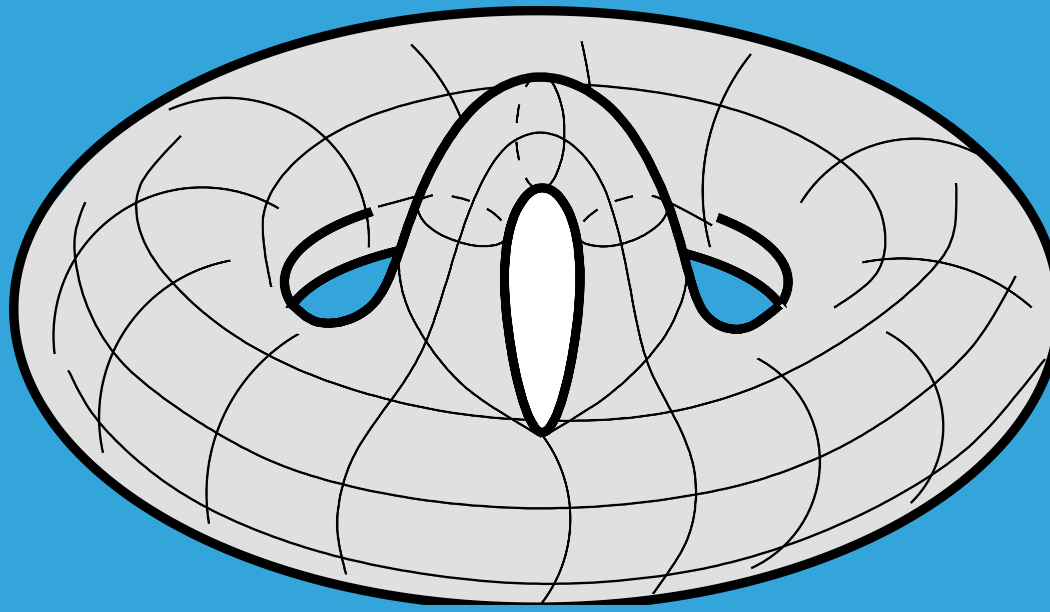
HIGHER DIMENSION/GENUS

- A lattice Λ yields a flat torus $\mathbb{C}/\Lambda$.
- Higher-dimensional Diophantine approximation, related to higher-dimensional lattices (or flat tori).
- Higher genus (flat) surfaces.
- A flat torus is a parallelogram with parallel sides identified by translation.
- A translation surface is a general Euclidean polygon with parallel sides identified by translation.









Cutting Sequences

on the

Double Pentagon

A Mathematical Theorem

from the

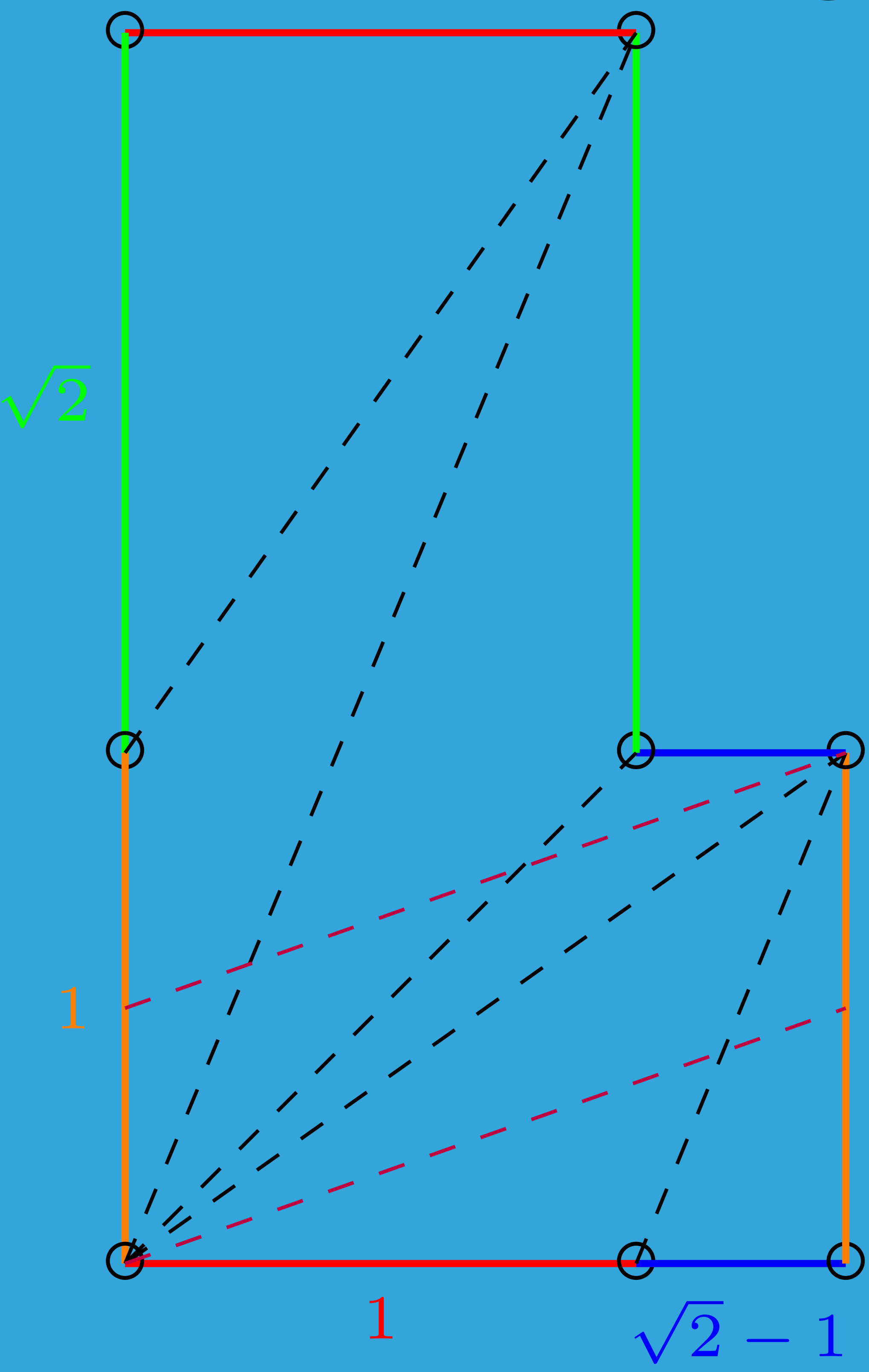
Ph.D. thesis

of

Diana Davis

- Translation surfaces of genus at least 2 have isolated cone type singularities, with angles integer multiples of 2π.
- The order of a singularity is a measure of the excess angle. A singular point has order k if the angle is $2\pi(k+1)$.
- In complex analytic terms, we obtain a Riemann surface X and a holomorphic one-form w. Singular points of order k are zeros of w of order k.

SADDLE CONNECTIONS



a **saddle connection** is a

straight line trajectory connecting two zeros

HOLONOMY VECTORS

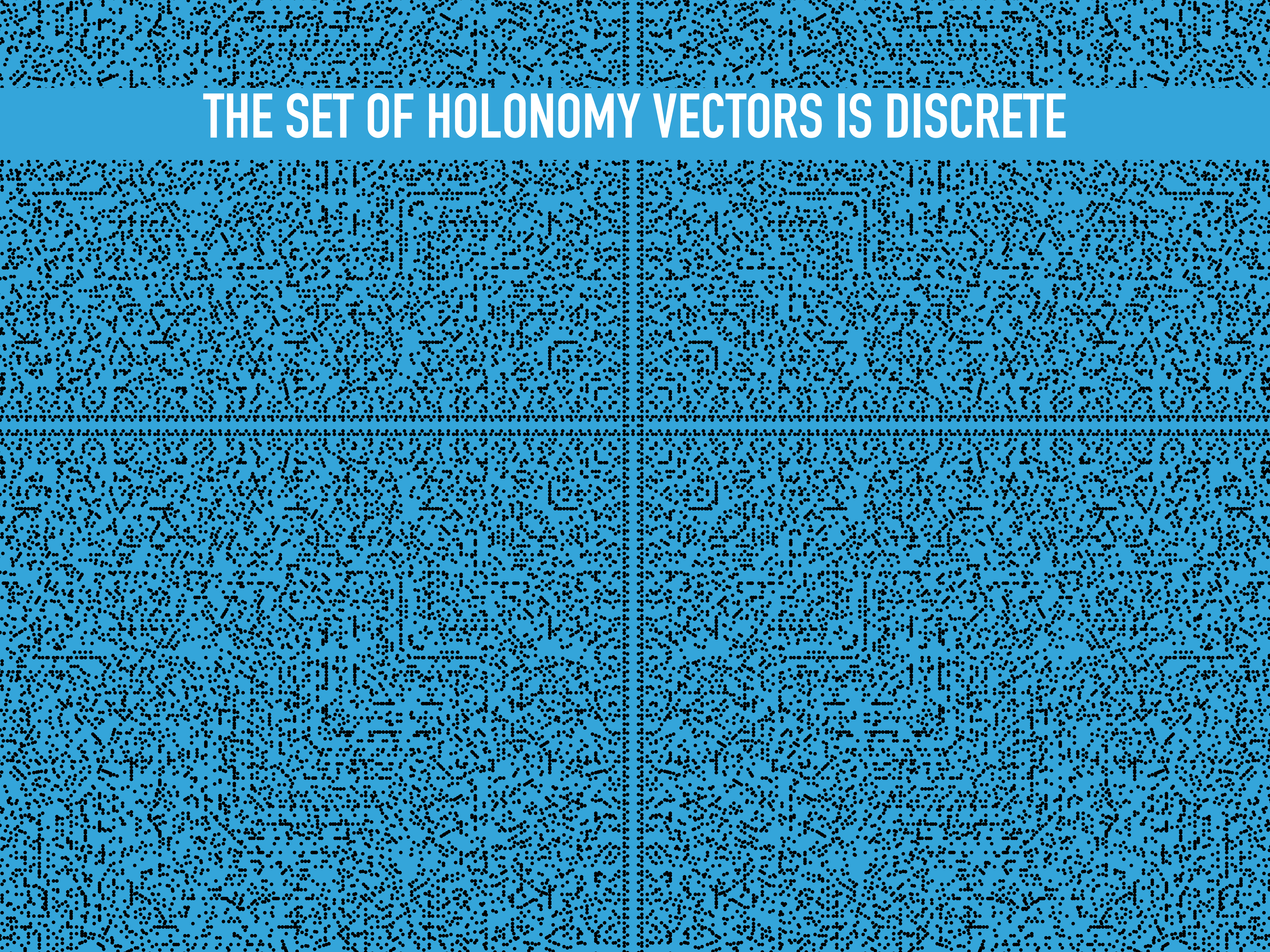
To each saddle connection, we associate a holonomy vector

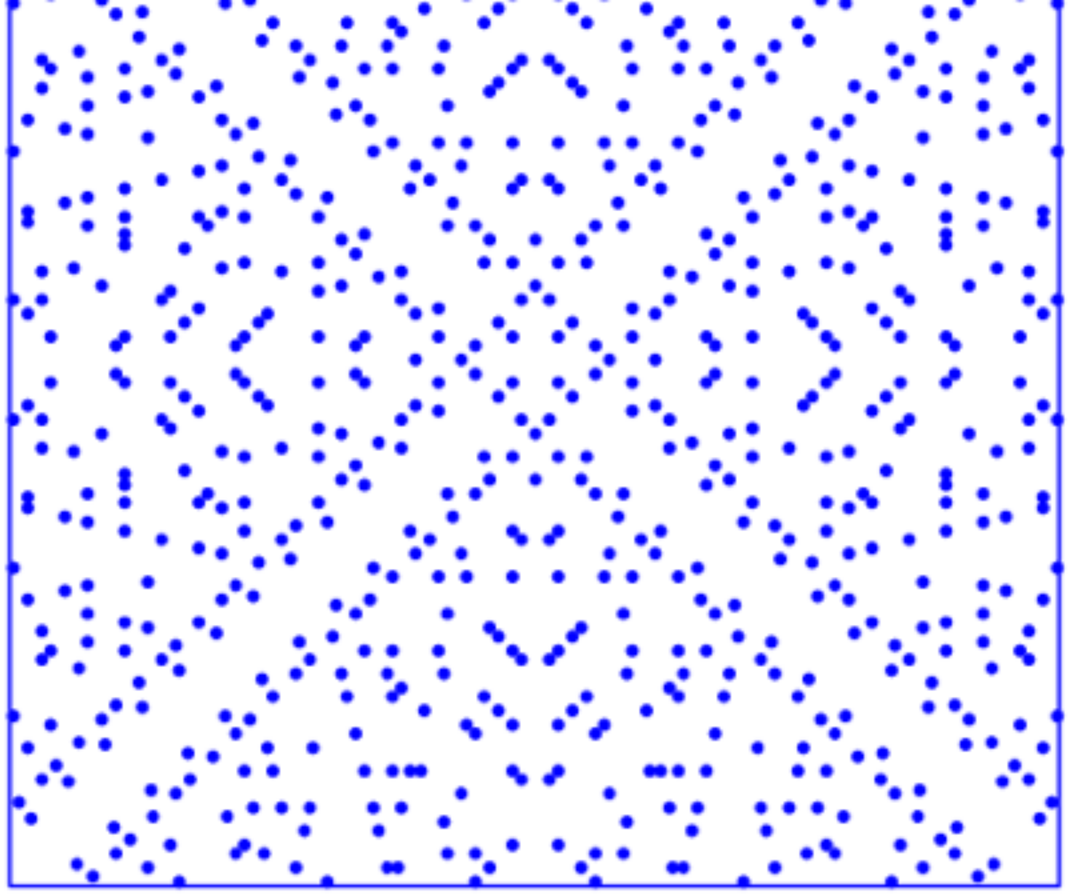
$$\gamma \longmapsto \int_\gamma \omega \in \mathbb{C}$$

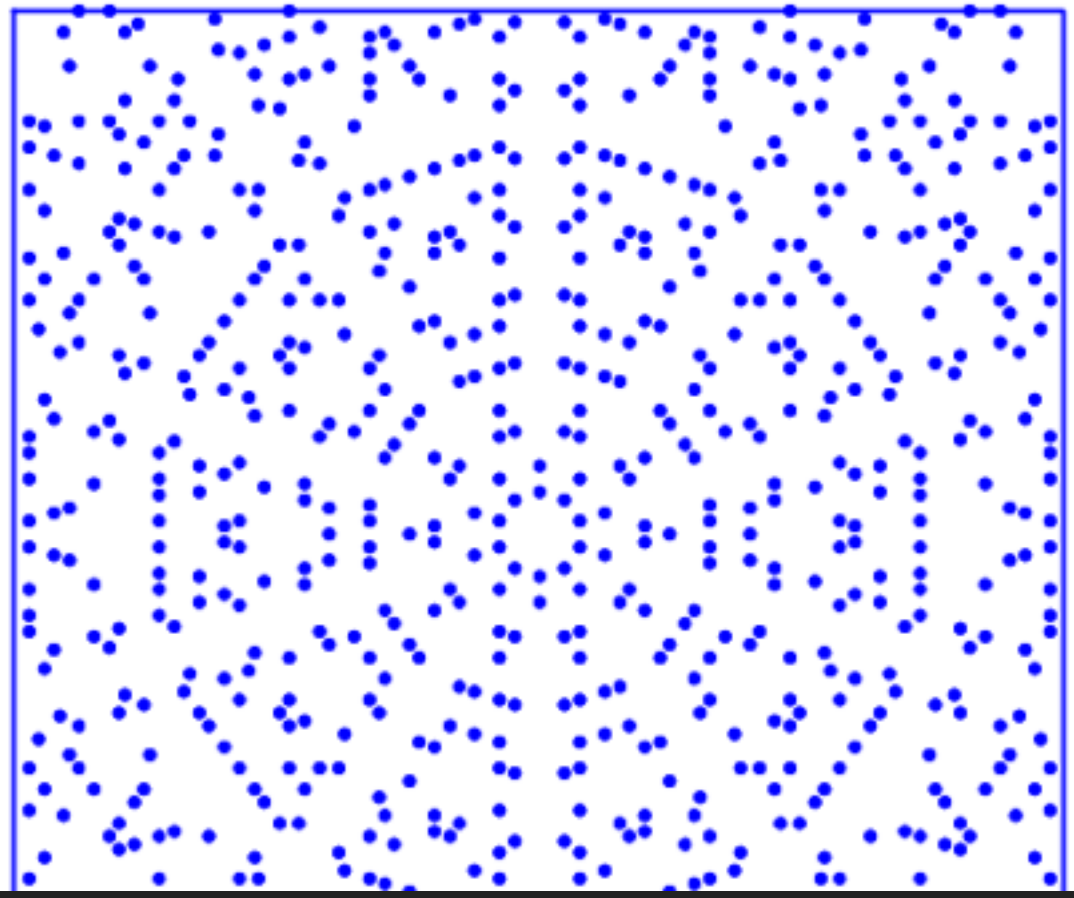
The set of holonomy vectors of the surface

(X, w) is denoted Λ_ω

THE SET OF HOLONOMY VECTORS IS DISCRETE







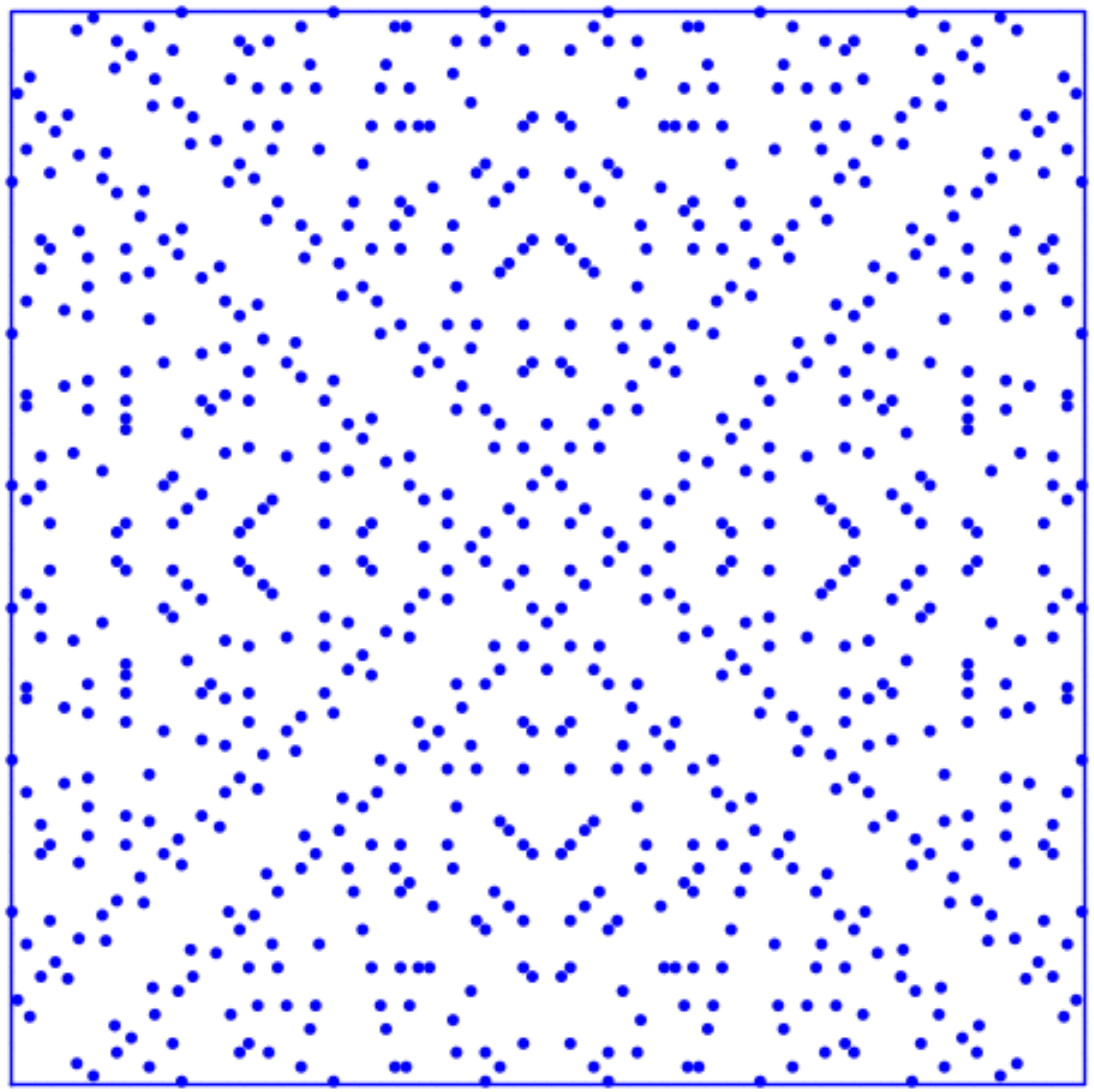
SL(2, R) ACTS ON THE SPACE OF TRANSLATION SURFACES

LINEAR ACTION ON POLYGONS

ERGODIC ABSOLUTELY CONTINUOUS INVARIANT MEASURE

HOLONOMY VECTORS VARY EQUIVARIANTLY

$$\Lambda_{g\omega} = g\Lambda_\omega$$



SADDLE CONNECTION HOLONOMIES HAVE QUADRATIC GROWTH.

Masur, Veech, Eskin-Masur, Vorobets

UNDERSTANDING SADDLE CONNECTIONS

- How well does this set approximate lines?
- Are there analogues of classical Diophantine results?
- What is the (fine scale) distribution of directions?

REVISITING THE CLASSICAL SETTING

- A typical vector in Λ_α has the form $\begin{pmatrix} p - q\alpha \\ q \end{pmatrix}$
- So asking to solve the inequality

$$\left| \alpha - \frac{p}{q} \right| \leq q^{-\nu}$$

- Is the same as finding lattice points

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \Lambda_\alpha \text{ such that } |x| < |y|^{-(\nu-1)}$$

- The diophantine exponent is the supremum of values with infinitely many solutions.

DIOPHANTINE EXPONENT OF A TRANSLATION SURFACE

$$\sup \left\{ \nu : \begin{pmatrix} x \\ y \end{pmatrix} \in \Lambda_\omega \text{ such that } |x| < |y|^{-(\nu-1)} \text{ has infinitely many solutions } \right\}$$

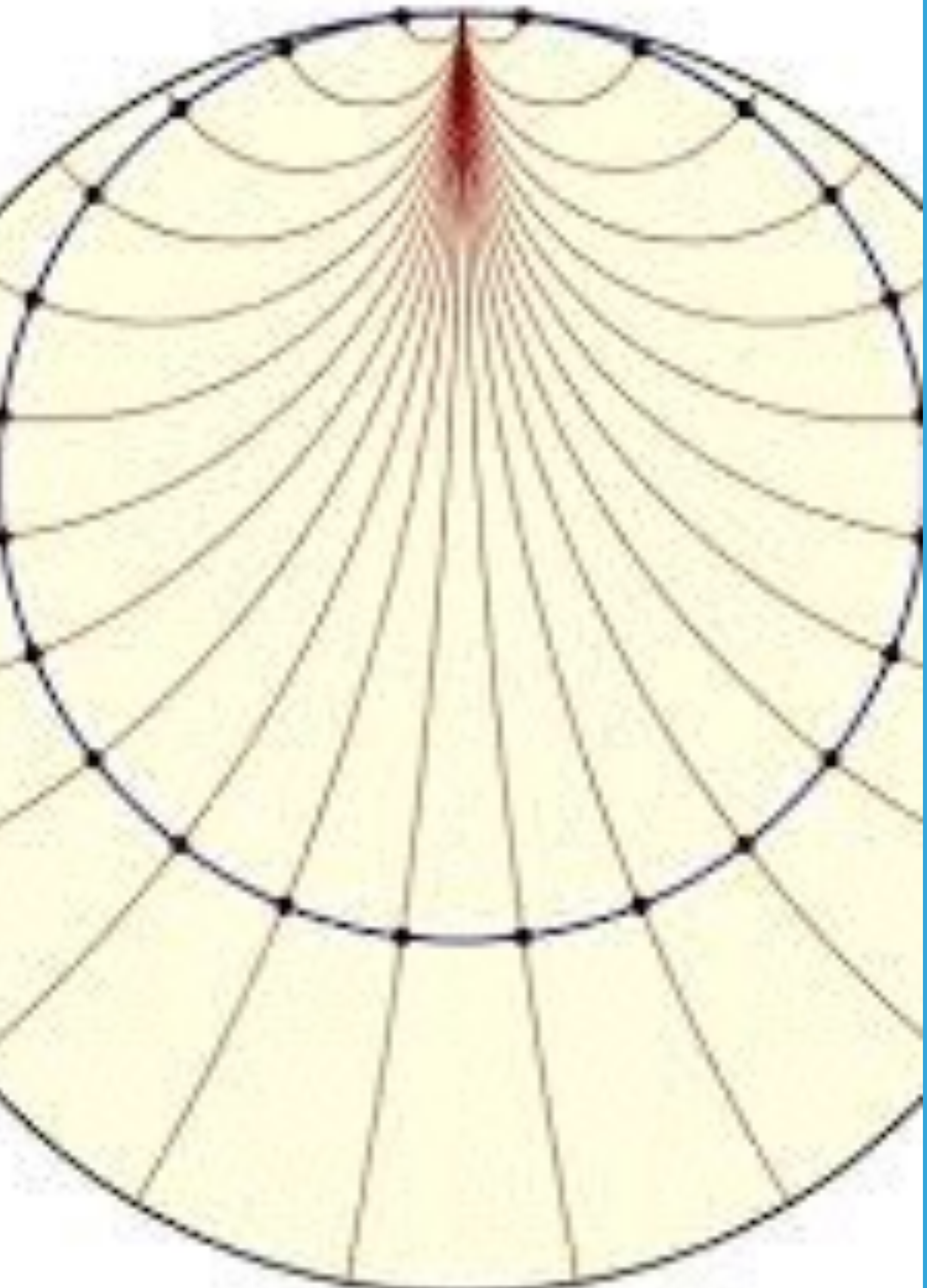
$$:= \mu(\omega)$$

HOW WELL CAN YOU APPROXIMATE THE VERTICAL DIRECTION?

- Space X (moduli space of lattices, translation surfaces), SL(2, R) equivariant assignment of discrete set Λ_x in the plane.
- For x in X,

$$\ell(x) = \min\{\|v\| : v \in \Lambda_x\}$$

$$\alpha(x) = \max\{\|v\|^{-1} : v \in \Lambda_x\}$$



$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$h_s = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$$

For translation surfaces, lattices, and other discrete equivariant assignments

EXPONENTS AND ORBITS

$$\limsup_{t\to\infty} \frac{\log \alpha(g_t x)}{t} = 1 - \frac{1}{\mu(x)}$$

$$\limsup_{|s|\to\infty} \frac{\log \alpha(h_s x)}{\log |s|} = \frac{1}{2} - \frac{1}{\mu(x)}$$

CUSP EXCURSIONS ON PARAMETER SPACES, JLMS

WHEN DO VECTORS GET SHORT?

- A geodesic orbit of a vector is shortest when the components of the vector become equal.
- A horocycle orbit of a vector is shortest when the vector becomes horizontal.
- The arguments generalize to higher dimensions, subspaces, matrices, etc.
- Measure estimates + Borel-Cantelli imply that for almost every translation surface, the diophantine exponent is 2.

DISTRIBUTION OF APPROXIMATES

Given an object x, what is the distribution of vectors v in the associate discrete set satisfying

$$|v_1||v_2| < A$$

$$N < |v_2| < cN$$

Here, A>0, c>1.

- Special case: suppose $h_1 x = x$
- Consider the measure

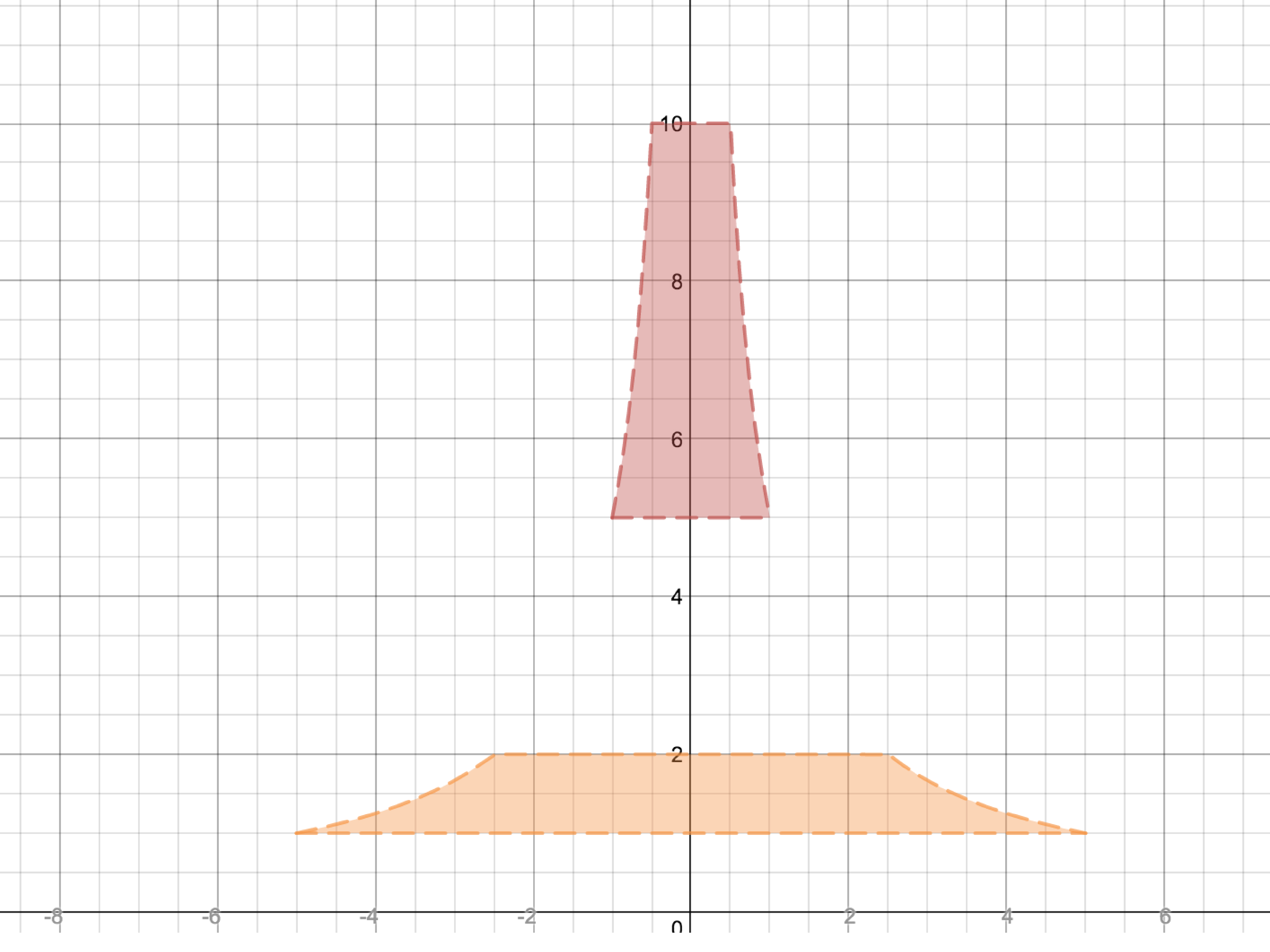
$$\left|\left\{\alpha : \#\left(\Lambda_{h_\alpha x} \cap R_{A,c,N}\right) = k\right\}\right|$$

$$R_{A,c,N} = \left\{(v_1, v_2) \in \mathbb{R}^2 : |v_1||v_2| < A, N < v_2 < CN\right\}$$

- Does this have a limit as N grows?

K=0 STUDIED EXTENSIVELY

PAUL TURAN AND VERA SOS

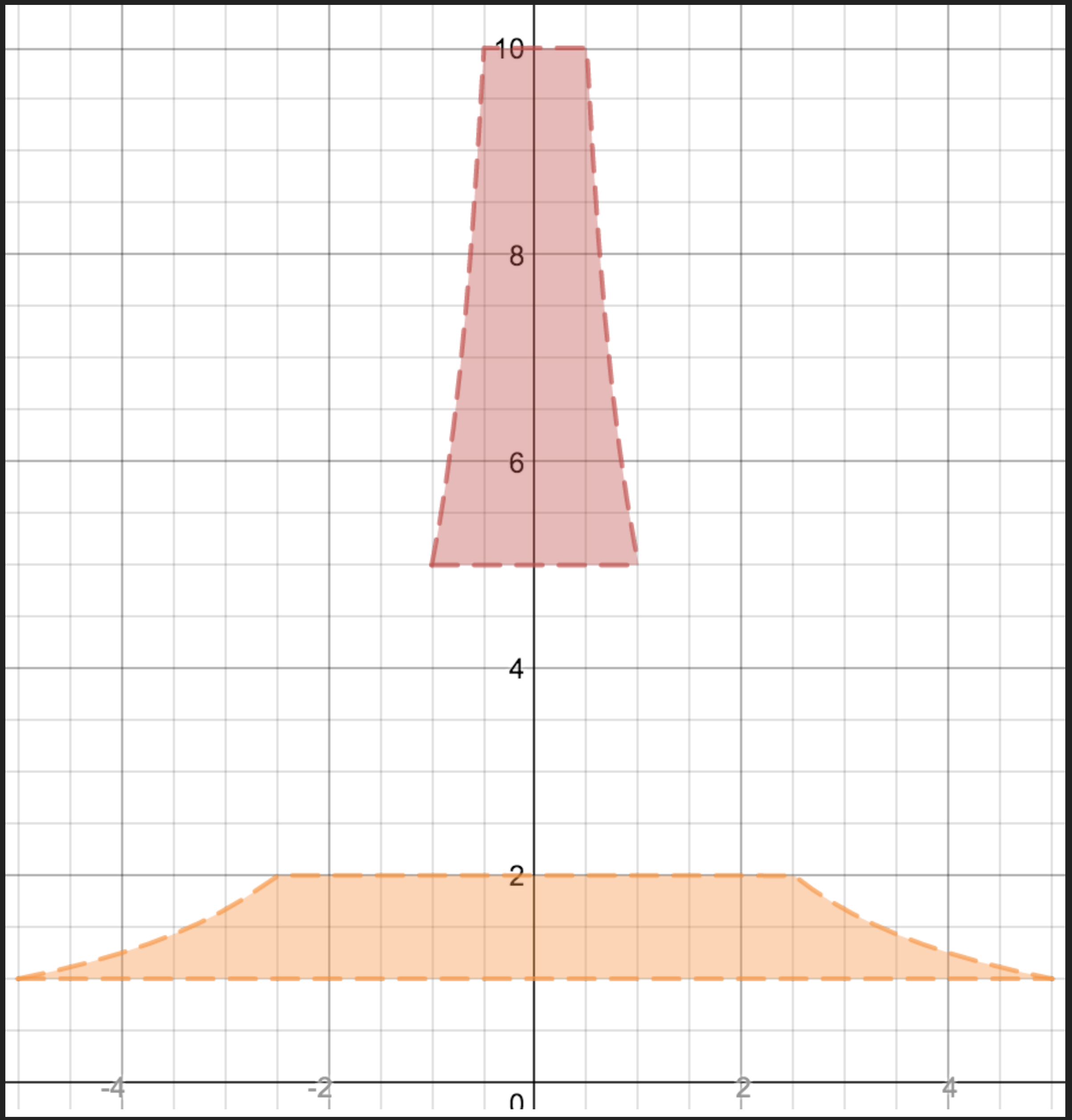


10
8
6
4
2
0
-8
-6
-4
-2
2
4
6

TRANSLATES OF HOROCYCLES

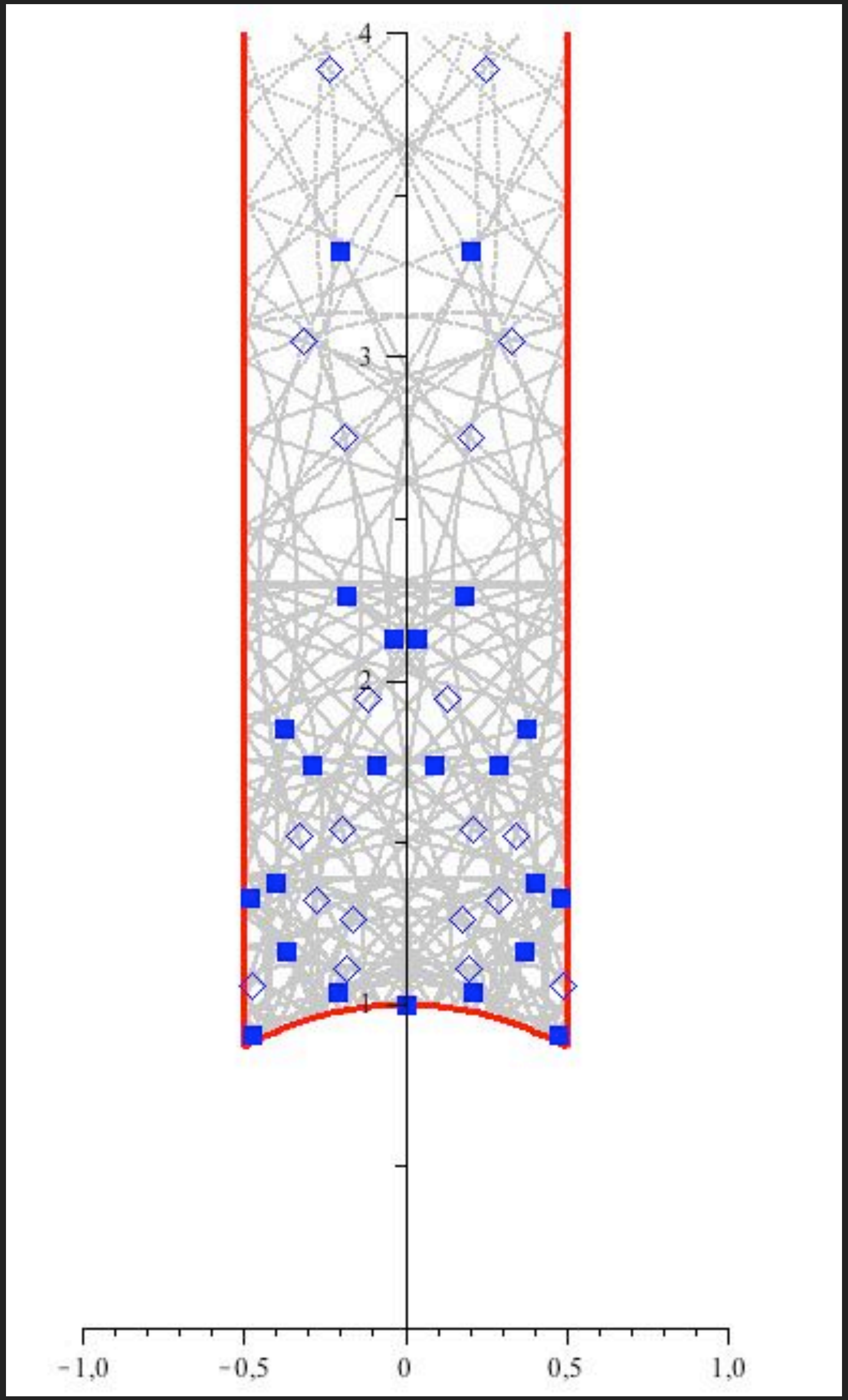
$$g_{\log N} R_{A,c,N} = R_{A,c,1}$$

$$\#\left(\Lambda_{h_\alpha x} \cap R_{A,c,N}\right) =$$
$$\#\left(g_{\log N} \Lambda_{h_\alpha x} R_{A,c,1}\right)$$



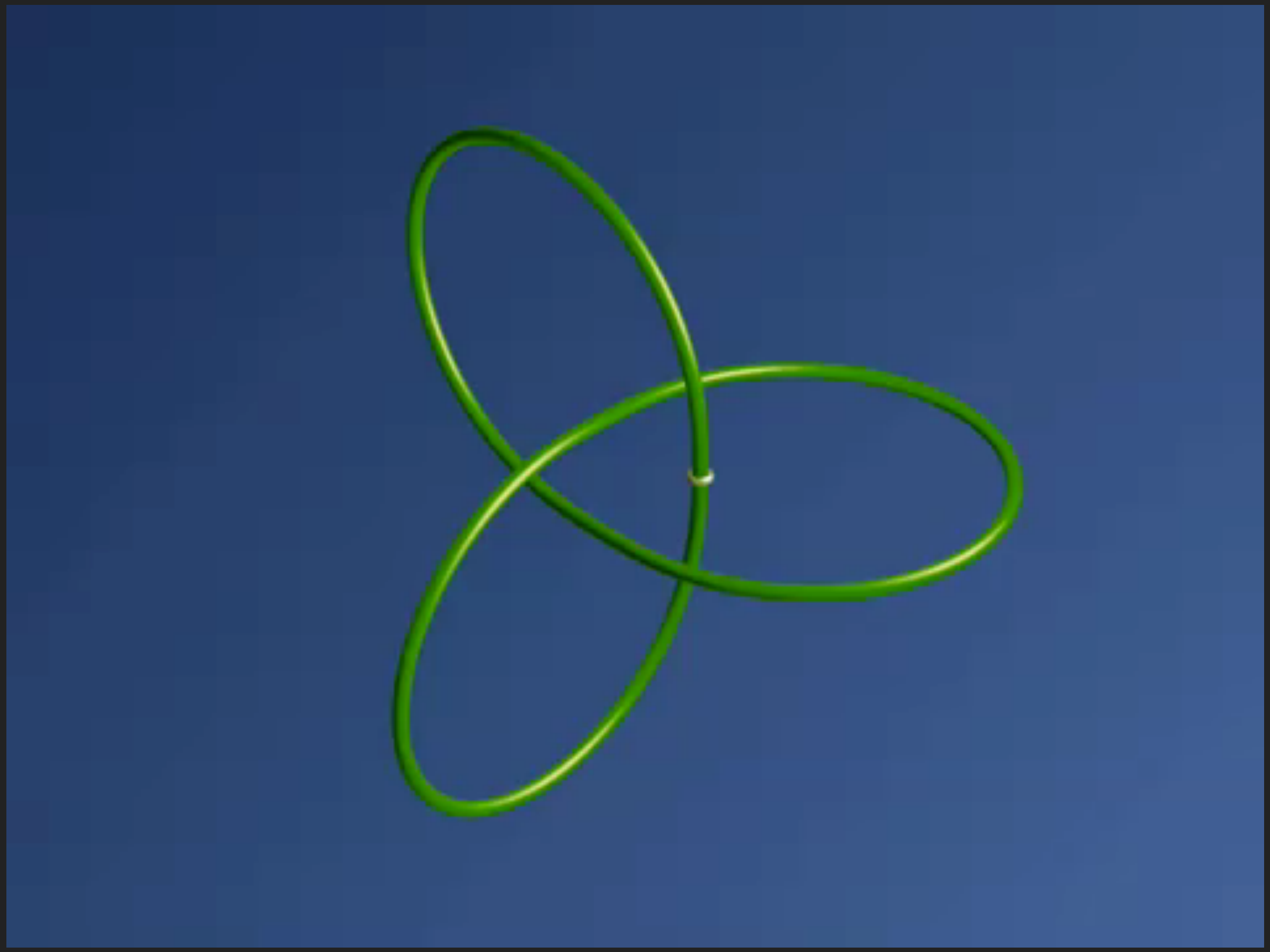
Reduces to limiting distribution of $d\alpha$ on $\{g_{\log N} h_\alpha x : 0 \leq \alpha \leq 1\}$ in X

IN MANY SETTINGS, THESE LONG CLOSED HOROCYCLES EQUIDISTRIBUTE WITH RESPECT TO SOME NATURAL MEASURE.



FOR TRANSLATION SURFACES, UNDERSTANDING THE LIMIT MEASURE IS A DEEP QUESTION.

FOR SPECIAL (LATTICE) SURFACES, REDUCES TO THE HYEPRBOLIC SETTING.



GENERALIZATION TO EQUIVARIANT PROCESSES, APPLICATIONS TO DIOPHANTINE PROBLEMS IN MANY CONTEXTS

A.-Ghosh-Taha, forthcoming

QUESTIONS/CONJECTURES

- Khintchin's theorem for translation surfaces (shrinking target property for geodesic flow on space of translation surfaces).
- Duffin-Schaefer for translation surfaces.
- Applications to approximation in algebraic number fields.