Goa, Various Possible Titles

- Invariant Measures for Higher Rank Actions
- Quantitative (Multiple Unipotent) Recurrence and Additional Invariance
- Transportation of Positive Entropy

General Setup for the Talk

Throughout $X = G/\Gamma$, $A < G$ is a diagonalisable subgroup of higher rank ($\dim(A) > 1$), μ is an A-invariant and ergodic measure on X.

Stupid example (Rank one factors):
$X = X_1 \times X_2$ and $A = A_1 \times A_2$ with $\dim A_1 = \dim A_2 = 1$

Assume our space is never of this form, e.g.
by assuming that Γ is irreducible.

We also assume throughout that $h_\mu(a) > 0$ for some $a \in A$.
Better examples will follow ...

Previous results

by Katok=:K, Spatzier, Kalinin, Einsiedler=:E.

The first satisfying result (only assuming positive entropy) was

Theorem (L:=Lindenstrauss)

$X = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})/\Gamma$, $A =$full diagonal subgroup
$\Longrightarrow \mu = m_X$=Haar measure on X.

Previous results

by K, Spatzier, Kalinin, E.

Theorem (L)

$X = SL_2(\mathbb{R})^k/\Gamma$, $A =$*full diagonal subgroup* $\Longrightarrow \mu = m_X$.

Theorem (E,K,L)

$X = SL_3(\mathbb{R})/SL_3(\mathbb{Z})$, $A =$*full diagonal subgroup* $\Longrightarrow \mu = m_X$.

Previous results

Theorem (L)

$X = SL_2(\mathbb{R})^k/\Gamma$, $A =$full diagonal subgroup $\Longrightarrow \mu = m_X$.

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Theorem (E,L)

G semisimple, $A =$full diagonal subgroup in some almost direct factor $\Longrightarrow$ all entropy is explained by rank one factors or unipotent invariance.

Theorem (E,L)

$X = X_1 \times X_2$, $A_1 \cong A_2$, A the diagonal in $A_1 \times A_2$ and μ is a joining $\Longrightarrow \mu$ is algebraic.

New results (E,L, 2015-2016)

To summarise previous results either assumed that A is maximal (at least in some sense) or that μ is special.

New results (E,L, 2015-2016)

From now on we always assume that Γ is an arithmetic and irreducible lattice, and as before that μ has positive entropy for some element of the action.

Theorem (E,L)

$X = SL_2(\mathbb{R})^k/\Gamma$, $A =$rank two subgroup $\Longrightarrow$

- *X is compact and μ is algebraic, or*
- *all of the entropy of μ is explained by unipotent invariance.*

New results (E,L, 2015-2016)

Theorem (E,L)

$X = SL_2(\mathbb{R})^k/\Gamma$, $A =$rank two subgroup $\Longrightarrow$

- *X is compact and μ is algebraic, or*
- *all of the entropy of μ is explained by unipotent invariance.*

Theorem (E,L)

$X = SL_3(\mathbb{R})^2/\Gamma$, $A = A_1 \times A_2$ for two rank one subgroups $A_1, A_2 < SL_3(\mathbb{R}) \Longrightarrow$

- *μ is algebraic, or*
- *all entropy is explained by unipotent invariance, or*
- *μ is supported on a closed orbit of a smaller subgroup and some nontrivial element of A acts trivially on this orbit.*

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These appear in the proof!

To discuss the ideas in the proof we will assume $G = G_1 \times G_2$, $G_j = Sl_2(\mathbb{R})^{10}$, $A_j < G_j$ rank 1, and $A = A_1 \times A_2$.

Furthermore $a \in A_1$ has $h_\mu(a) > 0$ and $b \in A_2 \setminus \{0\}$.

New observation:

Γ irreducible, arithmetic

$$\Longrightarrow h_\mu(b) > 0$$

The proof of this observation is similar to the pf. of positive entropy for AQUE (Bourgain-L).

Suppose $h_\mu(b)=0$. Then the dynamically important Bowen balls for b shrink in measure very slowly:

$$h_\mu(b) \underset{\text{a.s.}}{=} \lim_{n\to\infty} \frac{-\log \mu\left(\bigcap_{k=-n}^{n} b^{-k} B_\varepsilon^X(b^k x)\right)}{2n} = 0$$

$$\Rightarrow \underline{\underline{\mu\left(B_{n,\varepsilon}(x)\right) \gg_\alpha e^{-\alpha n}}}$$

for all $\alpha>0$

"basically does not shrink"

In the (original) AQUE-proof this was lead to a contradiction using all Hecke operators.

- up to complexity (=height) T there are $\frac{T}{\log T}$-many
- hence more than T^β-many
- diagonal flow has up to T only $\log T$ volume
- unipotent flow has up to T at least T^β volume

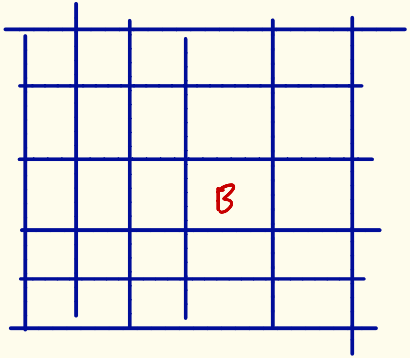
$\Rightarrow$ We want to show e.g.

$$\boxed{\mu\{x\in X \mid d(u_s x, x) > T^{-\gamma} \text{ for all } s \text{ with } |s|\le T\} \to 0 \text{ as } T\to\infty}$$

$$\gamma = \gamma(\beta, \dim G) \approx \frac{\beta}{\dim G}$$

if μ is inv. under U

Boshernitzan "Quantitative Recurrence results", 1993



X covered by $\approx \varepsilon^{-\dim X}$ ε-balls B

Within one $\underline{B}$ look at all points $x \in \underline{B}$ that don't return up to time T under a single m.p. map $\Rightarrow$ measure of that subset $< T^{-\beta}$ (with $\beta = 1$)

$\Rightarrow$ set $T = \varepsilon^{-\frac{(\dim X + 1)}{\beta}}$ & run Borel-Cantelli to obtain that a.e. point has $\forall_{\text{small enough}}\ \varepsilon > 0$ a return within $\varepsilon^{-(\dim X + 1)}$ to the ε-ball.

Recall $a \in A_1 < G_1$ & $h_\mu(a) > 0$

$\Rightarrow \exists\, U < G_1$ that has positive entropy contribution

$\Rightarrow \{u_s \mid |u_s| < T\}$ has volume $\gg T^\beta$ for some $\beta > 0$

Combine with recurrence pf

$\Rightarrow$ for a.e. x $\underset{\text{large enough}}{\forall T}$ $\exists\, u_s, |s| > 1, |u_s| < T$

$u_s \cdot x$ also typical

$d(x, u_s \cdot x) \leq T^{-\gamma}$

$\gamma \approx \frac{\beta}{\dim G}$

Instead of balls we want to use

Bowen balls for b — need very few to cover X (as if $\dim \approx 0$)

→ improves recurrence argument

$\Rightarrow$ for a.e. x $\forall T = e^n$ (large enough) $\exists\, u_s,\ |s| > 1,\ |u_s| < T$

$u_s \cdot x$ typical

$u_s \cdot x \in B_{\kappa n, \varepsilon}(x)$

& we may choose $\kappa > 0$ arbitrarily large

Working a bit harder still we can also find instead of 1 visit exponentially many, say $e^{\frac{\delta}{2}n}$-many, visits of $u_s \cdot x$ to $B_{Kn,\varepsilon}(x)$.

$$u_{s_j} x \sim x \implies \exists\, \gamma_j \in \Gamma \text{ responsible for that}$$

Recall $\left.\begin{array}{l} u \text{ is large in } G_1 \\ \text{but trivial in } G_2 \end{array}\right\} \Rightarrow \gamma$ is large in G_1 & very close to a conjugate of $C_{G_2}(A_2)$ in G_2

Note that the height of γ_j is controlled by the size of u_{s_j} – i.e. by $T = e^n$.

The so obtained lattice elements γ commute (at first only in a factor of G_2) and must be diagonalizable (as they are very close to a diagonal subgp). However, they are too many to fit inside an abelian diagonalizable subgroup $\Rightarrow$ $\lightning$.

Where does the additional invariance come from ?

All of the above can be carried out conditionally over the factor defined by the leafwise measure $\mu_x^{G_b^-}$

$\Rightarrow$ gives two points on a G_b^--orbit with the same G_b^--leafwise measure.