C. S. Seshadri Memorial Lectures
Standard Monomials

Peter Littelmann
Universität zu Köln

August 11, 2020

Seshadri's interpretation of Standard monomial theory started in 1978 with the following article:

Geometry of G/P-I
Theory of Standard Monomials for
Minuscule Representations

By C.S. Seshadri

Introduction. Let G be a semi-simple simply connected algebraic group defined over an algebraically closed field k. Let P be a maximal parabolic subgroup in G and L the ample line bundle on G/P which generates Pic G/P. We say that P is *minuscule* if the Weyl group W (fixing of course a Borel subgroup, maximal torus etc.) acts transitively on the weight vectors of the irreducible G-module $H^0(G/P, L)$ (k assumed to be of characteristic zero. Note then that $H^0(G/P, L)$ is the irreducible G-module associated to a fundamental weight). We can then index these weight vectors by $\{p_\tau\}$, $\tau \in W/W_{i(P)}$ (where $W_{i(P)}$ denotes the Weyl group of the maximal parabolic subgroup $i(P)$ which is the transform of P under the Weyl involution i) so that p_τ is the highest weight vector, when $\tau \equiv$ Identity (mod $W_{i(P)}$). We say that a monomial in $\{p_\tau\}$, say

$$p_{\tau_1} p_{\tau_2} \cdots p_{\tau_m} \in H^0(G/P, L^m)$$

is *standard* if $\tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_m$ (cf. Def. 1). Then the main result of this paper is the following (cf. . Th. 1):

Standard monomials of length m, which are distinct, form a basis of $H^0(G/P, L^m)$, $m \geqslant 0$, *P minuscule*. (*)

It is proved that this result holds also in arbitrary characteristic.

When $G = SL(n)$, every maximal parabolic subgroup is minuscule. In this case, when the base field k is of characteristic zero, (*) is due

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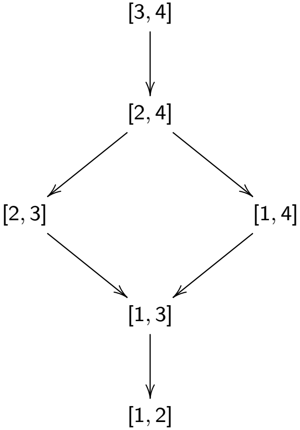
WHAT IS STANDARD MONOMIAL THEORY
A simple example: Grassmann variety $G_{2,4} \hookrightarrow \mathbb{P}(\Lambda^2\mathbb{K}^4)$

Inclusions of Schubert varieties



$\mathbb{K}$ algebraically closed

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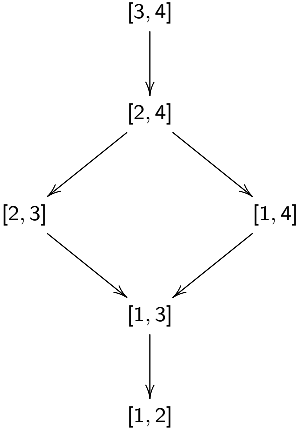

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weight spaces - one dimensional

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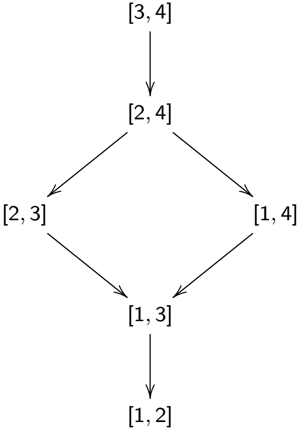

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dual space - Plücker coordinates

$p_{1,2},\ p_{1,3},\ p_{1,4},\ p_{2,3},\ p_{2,4},\ p_{3,4}$

Plücker coordinates generate

homogeneous coordinate ring of $G_{2,4}$.

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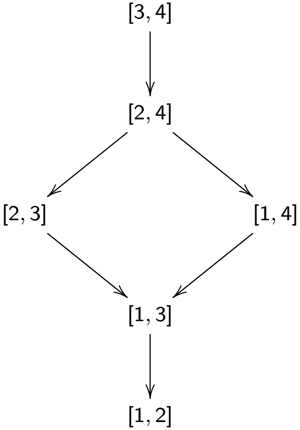

product of Plücker coordinates:

$$p_{i_1,j_1}\cdots p_{i_\ell,j_\ell} = \begin{array}{|c|c|c|}\hline i_1 & \cdots & i_\ell \\ \hline j_1 & \cdots & j_\ell \\ \hline \end{array}$$

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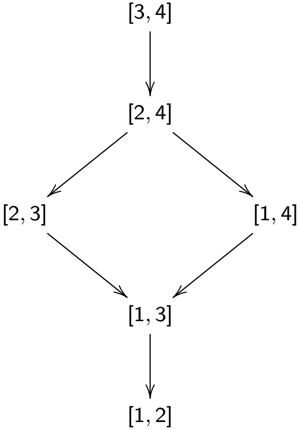

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standard monomial

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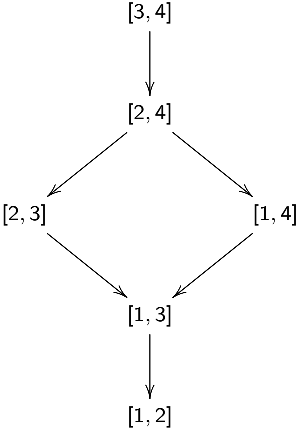

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can be adapted to Grassmann variety $G_{d,n}$.

Grassmann variety $G_{d,n} \hookrightarrow \mathbb{P}(\Lambda^d \mathbb{K}^n)$

Theorem The standard monomials form a basis of the homogeneous coordinate ring of the Grassmann variety $G_{d,n}$.

The basis is compatible with Schubert varieties. I.e. the standard monomials which do not vanish identically on a Schubert variety, they form a basis for the homogeneous coordinate ring of the Schubert variety.

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Char $\mathbb{K} = 0$: Hodge (1943) (inspired by Young).
Char $\mathbb{K} = p > 0$: Musili (1972).

Both proofs use Plücker relations, describing $G_{d,n} \hookrightarrow \mathbb{P}(\Lambda^d \mathbb{K}^n)$

What is Standard monomial theory

Seshadri's approach:

G simply connected, simple algebraic group, P maximal parabolic, ϖ minuscule fundamental weight,

$$G/P \hookrightarrow \mathbb{P}(V(\varpi)), \quad R = \bigoplus_{n \geq 0} H^0(G/P, \mathcal{L}_{n\varpi})$$

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New strategy: no use of global relations

Program

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To do:
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Aims:
Vanishing of higher cohomology

(projective) normality of Schubert varieties

Cohen-Macaulayness of the multicone over Schubert varieties

Singular locus of Schubert varieties

Character formulae

Behaviour of unions and intersections of Schubert varieties

Applications to GIT

⋮

Some comments

- omitted the multicone case for simplicity

Strategy: the term *standard monomial* is used in Groebner theory too, (and there is no contradiction, i.e. for an appropriate choice of a monomials order . . .) but the strategy is different.

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Strategy: the term *standard monomial* is used in Groebner theory too, (and there is no contradiction, i.e. for an appropriate choice of a monomials order ...) but the strategy is different.

Together with his collaborators, Lakshmibai and Musili, Seshadri developed a machinery to construct a standard monomial theory by increasing induction via chains of Schubert varieties.
Suppose $G/Q \hookrightarrow \mathbb{P}(V(\lambda))$, for $\tau \in W/W_Q$ let

$p_\tau \in H^0(G/Q, \mathcal{L}_\lambda)$, of weight $-\tau(\lambda)$
extremal weight vector, generalized Plücker coordinate

multiplication by this section

$$0 \longrightarrow \mathcal{O}_{X(\tau)}(\mathcal{L}_{(n-1)\lambda})) \xrightarrow{\cdot p_\tau} \mathcal{O}_{X(\tau)}(\mathcal{L}_{n\lambda})) \to \mathcal{O}_{H_\tau}(\mathcal{L}_{n\lambda})) \to 0$$

The filtration

This sequence plays a special role:

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Set theoretically, $H_\tau = X(\tau) \cap \{p_\tau = 0\}$ is the union of all codimension one Schubert varieties. If H_τ is reduced, then by passing to the long exact cohomology sequence one sees how the increasing induction works.

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For $n = 1$: filtration of $\mathcal{O}_{X(\tau)}(\mathcal{L}_\lambda) \rightsquigarrow$ sum of structure sheaves $\mathcal{O}_{X(\kappa)}$. The construction leads, step by step, to a basis of $H^0(X(\tau), \mathcal{L}_\lambda)$ and an indexing system for the basis by sequences of Weyl group elements and rational numbers, for example, for all classical type groups, $\mathtt{E}_6, \mathtt{G}_2$.

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In its most general form (even Kac-Moody groups) an indexing system was formulated as a conjecture by Lakshmibai. The conjecture has been proved, and the system is called the set of *Lakshmibai-Seshadri paths*, it turned out to be a special class within a combinatorial tool now called path models for representations.

The filtration

The problem of constructing a standard monomial theory was solved by Seshadri and his collaborators for many cases using this filtration as a tool (for example all classical type groups). The type free proof I gave in 1998 was by different means (quantum groups at a root of unity), but the existence of this filtration is a very important property.

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It leads to very interesting connections of standard monomial theory with equivariant K-theory and representation theory, which were studied by Seshadri and others:

- T-equivariant Pieri-Chevalley formula: $\mathcal{O}_{X(\tau)}(\mathcal{L}_\lambda)$ admits a filtration $\rightarrow$ associated graded $\bigoplus_j \mathcal{O}_{X(\kappa_j)} \otimes \chi_j$; the κ_j, χ_j are determined by Lakshmibai-Seshadri paths; $K_T(G/Q)$ Grothendieck ring of T-equivariant sheaves;
- good filtrations of tensor products; generalization of Littlewood Richardson rule.

Back to the challenge at the beginning of the inductive procedure

To find a basis of $H^0(G/Q, \mathcal{L}_\lambda)$: Today have several bases of representations: canonical basis, dual canonical basis, semicanonical basis, MV-cycle basis, standard monomial basis, ...
But at that time? For $G = SL_n$,...

To find an indexing system: Today one has several combinatorial tools: crystal bases, integral points in polytopes and cones (toric degeneration via a cluster variety structure or Newton-Okounkov theory), path model theory ...

But at that time? For $G = SL_n$,...

Back to the challenge

Only somebody like Seshadri, with a great vision, an incredible geometric intuition and a fair degree of optimism and humour was able to take up this challenge.

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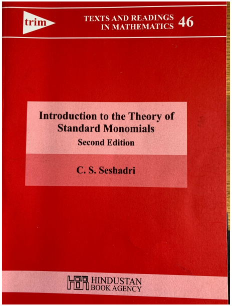
I remember very well his course on Standard Monomial Theory at Brandeis University in 1983/84. I was a visiting phd student for that year, being one of the note writers for this book. I remember with great pleasure the many meetings I had then with Seshadri. He explaining me the mathematics and, with a fine sense of humour, the up and downs in the development of standard monomial theory.



Outlook

Seshadri has had a new look at the approach by Hodge, it is time to have a new look at Seshadri's approach. For example, the indexing system (LS-paths) turns up in other contexts: affine buildings, Brownian motion, affine Grassmannian, NO-theory... Let me mention two connections:

- LS-paths $\leftrightarrow$ MV cycle basis: Weyl group combinatoric translates into a parameterization of an open and dense subset of an MV cycle: natural indexing of another basis; finite dimensional flag variety $\leftrightarrow$ affine Grassmannian;

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- Newton Okounkov theory: chains of Schubert varieties $\rightarrow$ valuations of $R = \bigoplus_{n \geq 0} H^0(G/Q, \mathcal{L}_{n\lambda})$ via successive vanishing multiplicities $\rightarrow$ Newton-Okounkov body: integral points = basis (example: Gelfand-Tsetlin patterns and generalizations by Zhelobenko).

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  Running over all chains + taking minimum $\rightarrow$ quasivaluation, Newton-Okounkov body $\rightarrow$ integral points = LS-paths.

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  Running over all chains + taking minimum $\rightarrow$ quasivaluation, Newton-Okounkov body $\rightarrow$ integral points = LS-paths.

  Standard monomials are a perfect section to the associated filtration.

Outlook by Seshadri

A picture from a meeting in Rome

Outlook by Seshadri

Quote from "Standard Monomial Theory-A historical account"

In retrospect the proof of SMT in G/P-V could be termed K-theoretic. It would indeed by very nice to have a similar proof of the general SMT ....

I have felt that a good understanding of SMT would be via a *cellular Riemann-Roch formula* as the definition of LS paths could be formulated geometrically in terms of the canonical cellular decomposition of G/B. The formulation via B-filtrations and Grothendieck rings seems to provide this approach.

A few References

C. S. **Seshadri**, *Geometry of G/P I*, pp. 207–239, Tata Inst. Fund. Res. Studies in Math., 8, Springer, Berlin-New York, (1978).

V. Lakshmibai; C. S. **Seshadri**, *Geometry of G/P II*, Proc. Indian Acad. Sci. Sect. A 87, no. 2, 1–54, (1978).

V. Lakshmibai; C. Musili; C. S. **Seshadri**, *Geometry of G/P III*, Proc. Indian Acad. Sci. Sect. A Math. Sci. 88, no. 3, 93–177, (1979).

V. Lakshmibai; C. Musili; C. S. **Seshadri**, *Geometry of G/P IV*, Proc. Indian Acad. Sci. Sect. A Math. Sci. 88, no. 4, 279–362, (1979).

V. Lakshmibai; C. S. **Seshadri**, *Geometry of G/P V*, J. Algebra 100, no. 2, 462–557, (1986).

C. S. **Seshadri**, *Introduction to the theory of standard monomials*, Second edition. Texts and Readings in Mathematics, 46. Hindustan Book Agency, New Delhi, (2014).

P. Littelmann; C. S. **Seshadri**, *A Pieri-Chevalley type formula for $K(G/B)$ and standard monomial theory*, pp. 155–176, Progr. Math., 210, Birkhäuser Boston, Boston, MA, (2003).

C. S. **Seshadri**, *Standard Monomial Theory-A historical account*, Collected papers of C. S. Seshadri. Volume 2. Schubert geometry and representation theory. Hindustan Book Agency, New Delhi, (2012).

Some further references

where LS-paths and related objects show up...far from being complete!

N. Bardy-Panse; S. Gaussent; G. Rousseau, *Macdonald's formula for Kac-Moody groups over local fields.* Proc. Lond. Math. Soc. (3) 119 (2019), no. 1, 135–175.

P. Baumann, Pierre; S. Gaussent, *On Mirković-Vilonen cycles and crystal combinatorics*. Represent. Theory 12 (2008), 83–130

P. Biane; P. Bougerol; N. O'Connell, *Littelmann paths and Brownian paths*. Duke Math. J. 130 (2005), no. 1, 127–167.

R. Chirivì, *LS algebras and application to Schubert varieties*. Transform. Groups 5 (2000), no. 3, 245–264.

S. Gaussent; P. Littelmann, *One-skeleton galleries, the path model, and a generalization of Macdonald's formula for Hall-Littlewood polynomials.* Int. Math. Res. Not. IMRN 2012, no. 12, 2649–2707.

S. Gaussent; G. Rousseau, *Spherical Hecke algebras for Kac-Moody groups over local fields*, Ann. of Math. (2) 180 (2014), no. 3, 1051–1087

S. Gaussent; G. Rousseau, *Kac-Moody groups, hovels and Littelmann paths*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 7, 2605–2657.

J. Guilhot, *Admissible subsets and Littelmann paths in affine Kazhdan-Lusztig theory,* Transform. Groups 23 (2018), no. 4, 915–938.

P. Hitzelberger *Kostant convexity for affine buildings*, Forum Math. 22 (2010), no. 5, 959–971

M. Kapovich; J. Millson, *A path model for geodesics in Euclidean buildings and its applications to representation theory*. Groups Geom. Dyn. 2 (2008), no. 3, 405–480

E. Milićević, P. Schwer, A. Thomas, *Affine Deligne-Lusztig varieties and folded galleries governed by chimneys*, arXiv:2006.16288, (2020).

S. Naito, F. Nomoto, D. Sagaki, *Representation-theoretic interpretation of Cherednik-Orr's recursion formula for the specialization of nonsymmetric Macdonald polynomials at* $T = \infty$, Transform. Groups 24 (2019), no. 1, 155–191

K. Naoi, *Weyl modules, Demazure modules and finite crystals for non-simply laced type,* Adv. Math. 229 (2012), no. 2, 875–934.

H. Pittie, A. Ram, *A Pieri-Chevalley formula in the K-theory of a G/B-bundle.* Electron. Res. Announc. Amer. Math. Soc. 5 (1999), 102–107.

A. Ram; M. Yip; *A combinatorial formula for Macdonald polynomials.* Adv. Math. 226 (2011), no. 1, 309–331.