Special Values of L-functions for $GL(n)$/CM-field

Pune - Mumbai Number Theory Seminar

July 3^{rd} - 4^{th}, 2020.

(Preprint available on my homepage.)

§1 A Motivating Exercise:

Let F be an imaginary quadratic field of discriminant $-\Delta$.

$\chi : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$ an "algebraic" Hecke character;

$\chi_\infty : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$; $\chi_\infty(z) = (z/\bar{z})^{\ell/2}$, $\ell \in \mathbb{Z}$ $(re^{i\theta} \mapsto e^{i\ell\theta})$

Think of χ_f as a character on group of ideals rel. prime $\mathfrak{f}_\chi$ - cond. of χ.

Define : $L(s, \chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) / N(\mathfrak{a})^s$, $\mathfrak{a}$. runs over int. ideals of F rel. prime to $\mathfrak{f}_\chi$.

Define: $f_\chi(z) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})\, e^{2\pi i N(\mathfrak{a}) z}$

Then $f_\chi \in S_k(N(\mathfrak{f}_\chi)\cdot\Delta, \psi)_{\text{Prim}}$ "primitive cusp form of explicitly describable level $N(\mathfrak{f})\Delta$, weight $k = \ell + 1$, nebentypus ψ - related to χ.

f_χ - "modular form of CM-type."

(i) $L(s, f_\chi) = L(s, \chi)$. (ii) $L(s, f_\chi, \omega_{F/\mathbb{Q}}) = L(s, f_\chi)$.

Shimura: $\boxed{L(m, f_\chi, \xi) \approx (2\pi i)^m \cdot u^\pm(f_\chi) \cdot \mathfrak{g}(\xi)}$ (Math. Ann. 1977)

critical pt. Twisting character periods of f_χ Gauss sum of ξ.

Critical points $= \{1, 2, \ldots, k-1\}$

It's a purely formal exercise to deduce (a result for $GL(1)/F$):

$$\boxed{\begin{gathered} i\, \mathfrak{g}(\omega_{F/\mathbb{Q}}) \cdot \frac{L(m, \chi)}{L(m+1, \chi)} \in \overline{\mathbb{Q}} \\ \text{and } \forall\, \eta \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}) : \\ \eta\left(i \cdot \mathfrak{g}(\omega_{F/\mathbb{Q}}) \cdot \frac{L(m, \chi)}{L(m+1, \chi)} \right) = i\, \mathfrak{g}(\omega_{F/\mathbb{Q}}) \cdot \frac{L(m, {}^\eta\chi)}{L(m+1, {}^\eta\chi)} \end{gathered}}$$

Main Theorem of the talk: Generalize this to $GL(n)$/CM-field.

§2 Notations:

$$\begin{array}{c} F \text{ tot. imag.} \\ {\scriptstyle 2}\big| \\ F^+ \text{ tot. real.} \\ {\scriptstyle r_{F^+}}\big| \\ \mathbb{Q} \end{array}$$

F = CM field = a totally imaginary quadratic ext^n of a totally real field.

$$r_F := [F:\mathbb{Q}] = 2[F^+:\mathbb{Q}] =: 2r_{F^+}$$

Basic data: $G = \mathrm{Res}_{F/\mathbb{Q}}(GL_n/F)$ $\qquad GL_n \supset T_n \supset Z_n$ (center)

$T = \mathrm{Res}_{F/\mathbb{Q}}(T_n/E)$

E - coefficient field $\qquad F \hookrightarrow E$, $E/\mathbb{Q}$ Galois

$\mu \in X^*(T \times E)$ $\qquad \mu = (\mu^\tau)_{\tau: F \to E}$, $\mu^\tau \in X^*(T_n \times_{F,\tau} E)$

$$\mathrm{Hom}(F,E) \longrightarrow \mathrm{Hom}(F^+,E), \quad \tau \longmapsto \tau|_{F^+} = \tau^+$$

$\{\tau, \tau'\} \longmapsto \tau^+$, $\quad \tau'$ = conjugate of τ.

$\mu^\tau = (\mu^\tau_1 \geq \dots \geq \mu^\tau_n)$, $\mu^\tau_i \in \mathbb{Z}$; μ - dominant integral weight.

Assume: μ is pure: $\exists\, w \in \mathbb{Z}$ s.t. $\mu^\tau_j + \mu^{\tau'}_{n-j+1} = w$; $\mu \in X_0^*(T \times E)$

Fact: Only pure weights support cuspidal cohomology.

$\mathcal{M}_{\mu,E}$ = abs. irred., fin-dim, rep^n of $G \times E$ with h.w. μ

$K_f \subset G(\mathbb{A}_f)$ open compact subgroup. (assume: K_f is neat.)

$$G_\infty = G(\mathbb{R}) = GL_n(F \otimes \mathbb{R}) = \prod_{\mathrm{Hom}(F^+,\mathbb{R})} GL_n(\mathbb{C})$$

C_∞ = max. cpct subgroup of $G(\mathbb{R})$ = $\prod U(n)$

$K_\infty = S(\mathbb{R})\, C_\infty = S(\mathbb{R})^\circ C_\infty$, $\quad S \cong \mathbb{G}_m/\mathbb{Q} \hookrightarrow \mathrm{Res}_{F/\mathbb{Q}}(Z_n/F)$

$$S^G_{K_f} = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K_f = GL_n(F) \backslash GL_n(\mathbb{A}_F) / K_\infty K_f$$

$\widetilde{\mathcal{M}}_{\mu,E}$ = sheaf of E-vector spaces on $S^G_{K_f}$.

Basic object of interest:

$$H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\mu,E}) \circlearrowleft \mathcal{H}^G_{K_f}$$

– a suitable Hecke algebra $\mathcal{C}^\infty_c(G_f /\!/ K_f)$.

"Cohomology of arithmetic groups" $\qquad$ "Cohomology of locally symmetric spaces"

Strongly inner cohomology $H^\bullet_{!!}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\mu,E}) \subset H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\mu,E})$

is characterised by the property that for some (& hence any) $\iota : E \to \mathbb{C}$

$$H^\bullet_{!!}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\mu,E}) \otimes_{E,\iota} \mathbb{C} \cong H^\bullet_{cusp}(S^G_{K_f}, \widetilde{\mathcal{M}}_{{}^\iota\mu,\mathbb{C}})$$

$$:= H^\bullet(\mathfrak{g}_\infty, K_\infty ; \mathcal{C}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_f} \otimes M_{{}^\iota\mu,\mathbb{C}})$$

Notation: $\pi_f \in \mathrm{Coh}_{!!}(G, L_n/F, \mu/E)$

π_f - irred. Hecke module appearing in strongly inner cohomology

For any $\iota : E \to \mathbb{C}$, ${}^\iota\pi_f$ = finite part of a cuspidal aut. rep^n ${}^\iota\pi$

We are interested in the critical values of $L(s, {}^\iota\pi)$

Defn: $m \in \frac{n-1}{2} + \mathbb{Z}$ is critical for $L(s, {}^\iota\pi)$ if

$L_\infty(s, {}^\iota\pi)$ and $L_\infty(1-s, {}^\iota\pi^\vee)$

are both finite at $s = m$. (No pole at $s=m$)

Defn: The cuspidal parameters of ${}^\iota\pi$: (determine ${}^\iota\pi_\infty$)

$v \in S_\infty$ = arch. places of F, $v \longleftrightarrow \{z_v, \bar{z}_v\}$ $z_v : F_v \xrightarrow{\sim} \mathbb{C}$

$$\alpha^v = (\alpha_1^v, \ldots, \alpha_n^v) = \left(-\mu_n^{z_v} + \frac{n-1}{2}, -\mu_{n-1}^{z_v} + \frac{n-3}{2}, \ldots, -\mu_1^{z_v} - \frac{(n-1)}{2}\right)$$

$$\beta^v = (\beta_1^v, \ldots, \beta_n^v) = \left(-\mu_1^{\bar{z}_v} - \frac{n-1}{2}, -\mu_2^{\bar{z}_v} - \frac{n-3}{2}, \ldots, -\mu_n^{\bar{z}_v} + \frac{(n-1)}{2}\right)$$

Defn: The cuspidal width of μ :

$$\ell(\mu) = \min\{ |\alpha_j^v - \beta_j^v| : v \in S_\infty, 1 \le j \le n \}$$

Prop^n: Let $\mu \in X_0^*(T \times E)$, $\pi_f \in \mathrm{Coh}_{!!}(GL_n/F, \mu/E)$, $\iota : E \to \mathbb{C}$

The critical set of $L(s, {}^\iota\pi)$:

$$\left\{ m \in \frac{n-1}{2} + \mathbb{Z} \;\middle|\; 1 - \frac{\ell(\mu)}{2} + \frac{w}{2} \le m \le \frac{\ell(\mu)}{2} + \frac{w}{2} \right\}$$

$\mathrm{Crit}(\mu) = \mathrm{Crit}(L(s, {}^\iota\pi))$ = independent of ι; it is a finite set.

§3. The Main Theorem

Let F be a CM-field with max. totally real subfield F^+. ; $r_F = [F:\mathbb{Q}]$.

Let $g(\omega_{F/F^+})$ = Gauss sum of the quadratic Hecke character ω_{F/F^+} of F^+ attached to F via class field theory.

Fix $i = \sqrt{-1}$.

Let $\mu \in X_0^*(T \times E)$, $\pi_f \in \mathrm{Coh}_{!!}(GL_n/F, \mu/E)$.

Assume $\#(\mathrm{Crit}(\mu)) = \ell(\mu) \geqslant 2$.

For $m \in \frac{n-1}{2} + \mathbb{Z}$, suppose m & $m+1$ are in $\mathrm{Crit}(\mu)$. Then:

① If $L(m+1, {}^{\iota}\pi) = 0$ for some ι, then $L(m+1, {}^{\iota}\pi) = 0$ $\forall \iota : E \to \mathbb{C}$.

② $$i^{n r_F/2} \cdot g(\omega_{F/F^+})^n \cdot \frac{L(m, {}^{\iota}\pi)}{L(m+1, {}^{\iota}\pi)} \in \iota(E)$$

③ $\forall \eta \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we have a reciprocity law:

$$\eta\left(i^{n r_F/2} g(\omega_{F/F^+})^n \cdot \frac{L(m, {}^{\iota}\pi)}{L(m+1, {}^{\iota}\pi)}\right) = i^{n r_F/2} g(\omega_{F/F^+})^n \frac{L(m, {}^{\eta\circ\iota}\pi)}{L(m+1, {}^{\eta\circ\iota}\pi)}.$$

$n=1$ follows from Harder, Inventiones '87

$n=2$ " " Hida, Duke '94.

$n=4$, π is a transfer from $GL_2 \times GL_2$, Loïc Grenié (Manuscripta '03)

$n=3$, Gunja Sachdeva's thesis. (JNT '20)

$n=$ general, weak form due to Grobner-Sachdeva (PJM '20)

Above theorem and its proof is independent of all these papers.

§4 Trends in stating rationality results:

- **Shimura:** would start with an object that lives over $\mathbb{C}$, for e.g., a modular form; he would assert that an L-value divided by periods is in $\overline{\mathbb{Q}}$; furthermore it is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant.

- **Deligne:** would start with an object over F with coefficients in E.

$$E \otimes \mathbb{C} = \prod_{\iota : E \to \mathbb{C}} \mathbb{C}.$$ For example, note that we have an array of L-functions: $\{L(s, {}^{\iota}\pi)\}_{\iota : E \to \mathbb{C}}$ takes values in $E \otimes \mathbb{C}$. We may denote that

$$\mathbb{L}(s, \pi) = \{L(s, {}^{\iota}\pi)\}_{\iota : E \to \mathbb{C}}$$

The quatity $i = \sqrt{-1} \in \mathbb{C} \hookrightarrow E \otimes \mathbb{C}$ as $1 \otimes i$

Similarly, the Gauss sum:-

For simplicity, let $\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \longrightarrow \{\pm 1\}$ is a quadratic Dirichlet character, primitive, of conductor $N \geq 1$.

Then $$\mathcal{G}(\chi) = \sum_{a=1}^{N} \left(\chi(a) \otimes e^{2\pi i a/N}\right) = 1 \otimes \sum \chi(a) e^{2\pi i a/N}$$

$\mathbb{C} \hookrightarrow E \otimes \mathbb{C}$ as $z \longmapsto 1 \otimes z$; and diagonally in $\prod_{\iota : E \to \mathbb{C}} \mathbb{C}$.

Deligne's style of stating the above result would be:

$$\boxed{i^{nd} \cdot \mathcal{G}(\omega_{F/F^+})^{n} \cdot \frac{\mathbb{L}(m, \pi)}{\mathbb{L}(m+1, \pi)} \in E,}$$

where $E \hookrightarrow E \otimes \mathbb{C} = \prod_{\iota : E \to \mathbb{C}} \mathbb{C}$ as $e \longmapsto (\iota(e))_{\iota : E \to \mathbb{C}}$

$e \longmapsto e \otimes 1$

§5 An adumbration of proof:

(i) Arthur - Clozel: Consider $AI_F^{F^+}({}^2\pi)$ - repⁿ of $GL_{2n}(\mathbb{A}_{F^+})$.

which may or may not be cuspidal.; put

$${}^2\sigma = AI_F^{F^+}({}^2\pi) \otimes \|\cdot\|_{F^+}^{n/2}$$

${}^2\sigma$ is a symbolic notation.

- $L(s, {}^2\sigma) = L(s + \frac{n}{2}, {}^2\pi)$
- ${}^2\sigma \otimes \omega_{F/F^+} \cong {}^2\sigma$.

(ii) Harder - R (Ann. Math Studies Vol 203) applied to $GL_{2n} \times GL_1$

$\exists$ "relative-period" $\Omega^{\varepsilon'}({}^2\sigma) \in \mathbb{C}^\times$ s.t.

$$\boxed{\frac{1}{\Omega^{\varepsilon'}({}^2\sigma)} \cdot \frac{L(m', {}^2\sigma)}{L(m'+1, {}^2\sigma)} \in \mathbb{Q}(E)}$$

and Galois-equivariant.

ε' = parity of $m' - \frac{(2n+1)}{2}$.

(iii) Relative Periods Vs Betti-Whittaker Periods:

Relative periods: $\Omega^{\varepsilon}({}^2\sigma_f)$ Compare ${}^2\sigma_f \otimes \varepsilon'$ and ${}^2\sigma_f \otimes -\varepsilon'$ in cohomology

B.W periods: $p^{\varepsilon}({}^2\sigma_f)$ compare Whittaker model of ${}^2\sigma_f$ with ${}^2\sigma_f \otimes \varepsilon$

$$\begin{array}{ccl} & \nearrow & H^{n^2}(S^{GL_{2n}/F^+}, \tilde{\mathcal{M}}_\lambda)({}^2\sigma_f \otimes \varepsilon') \\ w(\sigma_f) & & \quad | \\ & \searrow & H^{n^2}(S^{GL_{2n}/F^+}, \tilde{\mathcal{M}}_\lambda)({}^2\sigma_f \otimes -\varepsilon') \end{array}$$

μ determines λ.

Propⁿ:

$$\boxed{\Omega^{\varepsilon'}({}^2\sigma_f) = i^{n^2 r_F/2} \frac{p^{\varepsilon'}({}^2\sigma_f)}{p^{-\varepsilon'}({}^2\sigma_f)}}$$

(iv) Shahidi - R; (IMRN 2007) σ on GL_{2n}/F^+; χ - alg. Hecke char$/F^+$.

$$\boxed{p^{\varepsilon}(\sigma_f \otimes \chi_f) = g(\chi_f)^{2n(2n-1)/2} \cdot p^{\varepsilon \varepsilon_\chi}(\sigma_f)}$$

Take $\chi = \omega_{F/F^+}$

In summary:

$$\frac{L(m, {}^{\iota}\pi)}{L(m+1, {}^{\iota}\pi)} \underset{\text{Aut. Ind.}}{=} \frac{L(m', {}^{\eta}\sigma)}{L(m', {}^{\eta}\sigma)} \underset{(H-R)}{=} \Omega^{\varepsilon'}({}^{\eta}\sigma) = i^{n r_F/2} \frac{p^{\varepsilon'}({}^{\eta}\sigma)}{p^{-\varepsilon'}({}^{\eta}\sigma)} \underset{(S-R)}{=} i^{n r_F/2}\, g(\omega_{F/F^+})^n.$$

§6 Compatibility with Deligne's Conjecture

Deligne's Conjecture (Corvallis) + Proposition (Deligne-R)

Let M_0 be a pure regular motive of rank n over F with coefficients in E. Put $M = \mathrm{Res}_{F/\mathbb{Q}}(M_0)$. Suppose M has no middle Hodge types. Suppose $F = F^+(\sqrt{D})$ for a totally negative D.

Let $\mathbb{L}(s, M) = \{L(s, \iota, M)\}_{\iota : E \to \mathbb{C}}$ be the motivic L-function that takes values in $E \otimes \mathbb{C}$. Suppose m & $m+1$ are critical for $\mathbb{L}(s, M)$. Then:

$$\frac{\mathbb{L}(m, M)}{\mathbb{L}(m+1, M)} = 1 \otimes i^{n r_F/2} \cdot \sqrt{N_{F^+/\mathbb{Q}}(D)}^{\,n}, \quad \text{in } (E \otimes \mathbb{C})/E^{\times}.$$

Lemma: $\boxed{g(\omega_{F/F^+}) =_{\mathbb{Q}^{\times}} \sqrt{N_{F^+/\mathbb{Q}}(D)}.}$

$=_{\mathbb{Q}^{\times}}$ means equality up to $\mathbb{Q}^{\times}$.

Think of this lemma in the spirit of Gauss's famous calculation:

Fact: p - odd prime:

$$\sum \left(\frac{a}{p}\right) e^{2\pi i a/p} = \begin{cases} \sqrt{p}, & p \equiv 1 \ (4) \\ i\sqrt{p}, & p \equiv 3 \ (4) \end{cases}$$

——— X ———.