Cohomological representations of real reductive groups

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G connected reductive algebraic group$/\mathbb{R}$

- $GL(n)$, $SL(n)$, $Sp(2g)$, $SO(p,q)$, $U(a,b)$

$G(\mathbb{R})$ real Lie group (connected if G is simply-connected, but not in general)

- $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $Sp(2g,\mathbb{R})$, $SO(p,q)$, $U(a,b)$

$K \subset G(\mathbb{R})$ is a maximal compact subgroup, θ involution of $\mathfrak{g}_0$ fixed by K

- $O(n)$, $SO(n)$, $U(g)$, $SO(p) \times SO(q)$, $U(a) \times U(b)$, $\quad \theta(X) = -{}^tX$ or $-{}^t\bar{X}$

$\mathfrak{g}_0$ is the Lie algebra of $G(\mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$ Cartan decomposition, $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$

G^u is the compact real form of $G(\mathbb{R})$, i.e. the subgroup of $G(\mathbb{C})$ with Lie algebra $\mathfrak{g}^u = \mathfrak{k} + i\mathfrak{p}$

- $U(n)$, $SU(n)$, $Sp^c(2g)$, $SO(p+q)$, $U(a+b)$

A $(\mathfrak{g}, K)$-*module* is a $\mathbb{C}$-vector space with compatible actions of $\mathfrak{g}$ and K in which every vector is K-finite and which is K-semisimple. It is *admissible* if the K-multiplicities are finite. It has an *infinitesimal character* if the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by a character; the characters of $Z(\mathfrak{g})$ are parametrized by $\mathfrak{h}^*/W$ where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra. (Ex: For $GL(n,\mathbb{R})$ this is $\in \mathbb{C}^n/S_n$.) Irreducible $(\mathfrak{g}, K)$-modules have infinitesimal characters and are admissible.

Given a continuous repn of $G(\mathbb{R})$ on a Hilbert space, taking the smooth and K-finite vectors gives a $(\mathfrak{g}, K)$-module. This functor from repns to $(\mathfrak{g}, K)$-modules reflects isomorphisms. On unitary repns it restricts to an equivalence with unitarizable $(\mathfrak{g}, K)$-modules. In general, this achieves an algebraization of the theory of (unitary) repns of real Lie groups, and henceforth "repn" and "$(\mathfrak{g}, K)$-module" will be used interchangeably. (These results are due to Harish-Chandra.)

If G is a simple p-adic group and V is a smooth repn with $H^*(G,V) \neq 0$ then either $V = \mathbb{C}$ and $H^*(G,\mathbb{C}) = \mathbb{C}[0]$ or V is Steinberg and $H^*(G,V) = \mathbb{C}[-rk]$. Cohomological repns are the real counterpart of this very simple piece of the repn theory of p-adic groups.

DEFINITION: A $(\mathfrak{g}, K)$-module V is *cohomological* if there is a finite-dimensional repn E of $G(\mathbb{C})$ for which $H^*(\mathfrak{g}, K, V \otimes E) = Ext^*_{(\mathfrak{g},K)}(E^*, V) \neq 0$.

The cohomology is computed by the complex $C^i(\mathfrak{g}, K, W) = Hom_K(\wedge^i(\mathfrak{g}/\mathfrak{k}), W)$ with

$$(d\phi)(X_0, \ldots, X_i) = \sum_i (-1)^i X_i \cdot \phi(X_0, \ldots, \widehat{X}_i, \ldots, X_i) + \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_i).$$

Note that $H^0(\mathfrak{g}, K, W) = W^{\mathfrak{g}}$ and H^i are derived functors.

EXAMPLE: For the trivial repn, $(C^*(\mathfrak{g}, K, \mathbb{C}), d)$ is the complex of $G(\mathbb{R})$-invariant differential forms on $G(\mathbb{R})/K$. So $H^*(\mathfrak{g}, K, \mathbb{C}) = H^*(G^u/K)$.

REMARK: $H^*(\mathfrak{g}, K, V \otimes E) = 0$ if infinitesimal characters of V and E^* differ.

REMARK: We will also consider V such that $H^*(\mathfrak{g}, K^0, V \otimes E) \neq 0$, where $K^0 \subset K$ is the identity component. For example, for $GL(2, \mathbb{R})$ both $triv$ and $sgn(det)$ are $(\mathfrak{g}, K^0)$-cohomological, but only $triv$ is $(\mathfrak{g}, K)$-cohomological.

EXAMPLE: $SL(2,\mathbb{R})$. Let $k \geq 1$. Consider the normalized induced repn

$$I_k := Ind_{B(\mathbb{R})}^{G(\mathbb{R})}\, \varepsilon^k \otimes \chi_k$$

where $B(\mathbb{R}) = \{\pm I\} \cdot \left\{ \begin{pmatrix} e^a & \\ & e^{-a} \end{pmatrix} \right\} \cdot \left\{ \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \right\}$ and $\chi_k\left(\begin{pmatrix} e^a & \\ & e^{-a} \end{pmatrix} \right) = e^{ka}$.

Since $I_k|_K = \bigoplus_{n \equiv k+1 (2)} \mathbb{C}\, e^{in\theta}$ we have the picture:

The two extreme boxes are subrepns $D_k^{\pm}$ and the middle one is a quotient F_k (the f.d. repn of dimension k), so we can make (nontrivial) extensions

$$0 \to D_k^{\pm} \to I_k/D_k^{\mp} \to F_k \to 0 \qquad \in \quad Ext^1(F_k^*, D_k^{\pm}).$$

Concretely, since $\mathfrak{g}/\mathfrak{k}$ is the K-type $+2$, we see that $Hom_K(\wedge^* \mathfrak{g}/\mathfrak{k}, D_k^{\pm} \otimes F_l)$ is zero if $l < k$, is the exact complex $\mathbb{C} \to \mathbb{C}$ if $k > l$, and is $\mathbb{C}[-1]$ if $k = l$. Thus $H^*(\mathfrak{g}, K, D_k^{\pm} \otimes F_k) = \mathbb{C}[-1]$ and cohomology is carried by the extreme K-type.

The cohomological repns are $D_k^{\pm}$ and the trivial repn.

EXAMPLE: $GL(2,\mathbb{R})$. There is an irreducible $(\mathfrak{g}, K)$-module D_k trivial on the central $\mathbb{R}_+$ with $D_k|_{SL(2,\mathbb{R})} = D_k^+ \oplus D_k^-$. The $(\mathfrak{g}, K)$-cohomology has rank one while $(\mathfrak{g}, K^0)$-cohomology has rank two. Note that $D_k \otimes sgn \cong D_k$.

EXAMPLE: Discrete series repns are the closed summands of $L^2(G(\mathbb{R}))$. Harish-Chandra showed that they exist if and only if G contains a Cartan which is compact modulo centre. Suppose G is semisimple and simply-connected (so that $G(\mathbb{R})$ is connected) and let $T \subset K$ be a compact Cartan. The discrete series are parametrized by dominant characters of T modulo the action of the Weyl group $W(K,T)$. Thus fixing the infinitesimal character they are $|W(G,T)/W(K,T)|$ in number. Discrete series repns are cohomological: If V is a discrete series repn with the same infinitesimal character as E^*, then

$$H^*(\mathfrak{g}, K, V \otimes E) = \mathbb{C}[-dim(G(\mathbb{R})/K)/2].$$

Thus d.s. are "atomic" for cohomology.

The interest in cohomological repns comes from the following, a version of Hodge theory:

MATSUSHIMA'S FORMULA: For $\Gamma \subset G(\mathbb{R})$ a cocompact arithmetic group

$$H^i(\Gamma\backslash G(\mathbb{R})/K) = \bigoplus_{\pi \subset L^2(\Gamma\backslash G(\mathbb{R}))} m_\Gamma(\pi)\, H^i(\mathfrak{g}, K, \pi)$$

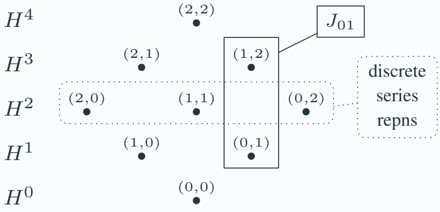
where $m_\Gamma(\pi) = \dim Hom_G(\pi, L^2(\Gamma\backslash G(\mathbb{R}))$.

REMARK: The symmetric $G(\mathbb{R})/K$ has a $G(\mathbb{R})$-invariant complex structure when $Z(K) \cong S^1$, e.g. $Sp(2g), U(p,q), SO(2,n)$. In this case $\mathfrak{g}/\mathfrak{k} = \mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, the decomposition of $C^*(\mathfrak{g}, K, \pi)$ passes to cohomology for π unitary, and the resulting decomposition of $H^*(\Gamma\backslash G(\mathbb{R})/K)$ is the Hodge decomposition.

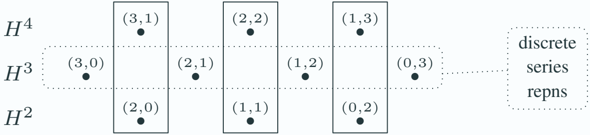
VANISHING THEOREM: If $V \neq \mathbb{C}$ is unitary then $H^*(\mathfrak{g}, K, V \otimes E) = 0$ for $* < rk_{\mathbb{R}}(G)$.

EXAMPLE 2: $SU(1,n)$. For each (i,j) with $i+j \leq n$ there is a unique cohomological repn J_{ij} which has one-dim'l cohomology in each bidegree $(i,j), (i+1,j+1), \ldots, (n-j,n-i)$. If $i+j=n$ these are discrete series repns and $J_{00} = \mathbb{C}$.

It is helpful to think in terms of the Hodge diamond. For $n=2$ it is



EXAMPLE: $Sp(4,\mathbb{R})$. Ignoring the trivial repn and $H^1 = H^5 = 0$, we have



There are eight nontrivial cohomological repns: Four d.s. with $H^3 \neq 0$ and three nontempered repns with cohomology indicated by boxes.

Vogan-Zuckerman classification

Unitary cohomological repns were classified in the early 1980s by work of Parthasarathy, Kumaresan, and Vogan-Zuckerman.

The end result is that they are all constructed by cohomological induction (which can usually be treated as a black box) from unitary characters on certain non-real parabolic subalgebras of $\mathfrak{g}$. (This is formally parallel to Deligne-Lusztig theory.)

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be the complexified Cartan involution given by K.

Definition: A *θ-stable parabolic subalgebra* for G is a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ such that (1) $\theta(\mathfrak{q}) = \mathfrak{q}$ and (2) $\mathfrak{l} = \mathfrak{q} \cap \bar{\mathfrak{q}}$ is a Levi subalgebra.

Notice that $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is not defined over $\mathbb{R}$ unless $\mathfrak{q} = \mathfrak{g}$, i.e. it does not come from $\mathfrak{g}_0$, but $\mathfrak{l}$ is the Lie algebra of a connected reductive subgroup $L \subset G$.

Given a θ-stable $\mathfrak{q}$ and a unitary character $\lambda : \mathfrak{l} \to \mathbb{C}$, cohomological induction produces a unitary $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ which has, for the correct E,

$$\mathrm{H}^*(\mathfrak{g}, K, A_{\mathfrak{q}}(\lambda) \otimes E) = \mathrm{H}^{*-\dim \mathfrak{u}\cap\mathfrak{p}}(\mathfrak{l}, K \cap L, \mathbb{C}) = \mathrm{H}^{*-\dim \mathfrak{u}\cap\mathfrak{p}}(L^u/K \cap L).$$

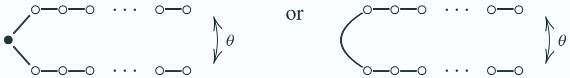
Vogan-Zuckerman (1984): Every unitary cohomological $(\mathfrak{g}, K)$-module is an $A_{\mathfrak{q}}(\lambda)$.

Since θ-stable parabolics are easily enumerated this gives an overparametrization of the unitary cohomological repns. (It can happen that $A_{\mathfrak{q}} \cong A_{\mathfrak{q}'}$ for $\mathfrak{q}$ not K-conjugate to $\mathfrak{q}'$.)

For coefficients: Restrict to $\mathfrak{q}$ s.t. $A_{\mathfrak{q}}(\lambda)$ has correct infinitesimal character.

EXAMPLE: $GL(n, \mathbb{R})$

The Dynkin diagram w.r.t. a fundamental Cartan and suitable choice of θ-stable Borel is:



if n is even or n is odd. We see that a θ-stable parabolic containing the fixed Borel has Levi is of the form $GL(n_0, \mathbb{R}) \times \times_{i=1}^{r} GL(n_i, \mathbb{C})$ where $n_0 + 2\sum_i n_i = n$.

In $GL(n, \mathbb{R})$ any two θ-stable Borels are K-conjugate, so these are all the θ-stable parabolics up to K-conjugacy.

This classification of cohomological repns for $GL(n, \mathbb{R})$ was proved earlier by Speh, and has a different formulation (later).

EX: For $GL(2n, \mathbb{R})$ the repn given by $n + n = 2n$ is the Speh repn.

Recall that over a field with $k = \bar{k}$, triples $G \supset B \supset T$ consisting of a connected reductive group, Borel subgroup, maximal torus are classified by the based root datum:

$$(X^*(T), \Delta = \{\text{simple roots}\}, X_*(T), \check{\Delta} = \{\text{simple coroots}\})$$

The *Langlands dual group* $\widehat{G}$ of G is the connected reductive group with based root datum

$$(X_*(T), \check{\Delta}, X^*(T), \Delta).$$

It comes equipped with a torus $\widehat{T}$ with $X^*(\widehat{T}) = X_*(T), X_*(\widehat{T}) = X^*(T)$ and Borel $\widehat{B}$.

The duality $G \leftrightarrow \widehat{G}$ exchanges groups of type B_n and C_n and respects the other types. Adjoint and simply-connected groups are exchanged.

- $GL(n) \leftrightarrow GL(n), SL(n) \leftrightarrow PGL(n)$
- $Sp(2g) \leftrightarrow SO(2g+1), PGSp(2g) \leftrightarrow Spin(2g+1)$
- $SO(2n) \leftrightarrow SO(2n)$

EXAMPLE: The f.d. complex representations of a complex group G are parametrized by the highest weight in $X^*(T)/W = X_*(\widehat{T})/W = Hom(\mathbb{G}_m, \widehat{G})/\widehat{G}$.
It is more natural to use the infinitesimal character in $\mathfrak{t}^*/W$, given by $\mathbb{C}^* \to \widehat{T}$ by $z \mapsto z^{\lambda + \check{\rho}}$ where $\lambda =$ h.w. of E, $\check{\rho} =$ half-sum of positive coroots for $\widehat{G}$.

For groups over general fields the L-group of G/k incorporates the Galois group (in general) or the Weil group of k (in the local or global cases).

THE L-GROUP

DEFINITION: The Weil group of $\mathbb{C}$ is $W_{\mathbb{C}} = \mathbb{C}^*$. The Weil group of $\mathbb{R}$ is $W_{\mathbb{R}} = \mathbb{C}^* \sqcup j\mathbb{C}^*$ where $j^2 = -1$, $jzj^{-1} = \bar{z}$. Thus $1 \to W_{\mathbb{C}} \to W_{\mathbb{R}} \to Gal(\mathbb{C}/\mathbb{R}) \to 1$. The norm $|\cdot| : W_{\mathbb{R}} \to \mathbb{R}^*$ by $z \mapsto z\bar{z}$ and $j \mapsto 1$ induces $W_{\mathbb{R}}^{ab} = \mathbb{R}^*$.

For G/k, $Gal(\bar{k}/k)$ acts on the based root datum by automorphisms, with the action depending only on the inner form of G. (When $k = \mathbb{R}$ this amounts to an involution of the diagram of $\widehat{G}$, for which there are not too many choices.) This can be lifted to an action by automorphisms on $\widehat{G}$ by choosing a splitting. The Weil group acts through $W_{\mathbb{R}} \to Gal(\mathbb{C}/\mathbb{R})$ and the L-group

$$^L G = W_{\mathbb{R}} \ltimes \widehat{G}$$

is independent, up to isomorphism, of choices. By definition, if G is split then $^L G = W_{\mathbb{R}} \times \widehat{G}$, and $^L G \cong {}^L G^*$ where G^* is the quasisplit inner form of G.

- for the quasisplit nonsplit group $G = SO(n-1, n+1)$ the L-group is $W_{\mathbb{R}} \ltimes SO(2n, \mathbb{C})$ where Galois acts by the nontrivial involution of the D_n diagram
- for $G = U(n,n)$ or $U(n, n+1)$ we have $^L G = W_{\mathbb{R}} \ltimes \widehat{G}$ where one uses the involution $g \mapsto {}^t g^{-1}$ on $\widehat{G} = GL(2n, \mathbb{C})$ or $GL(2n+1, \mathbb{C})$.
- for $p + q = 2n+1$, $^L SO(p,q) = {}^L SO(n, n+1) = W_{\mathbb{R}} \times Sp(2n, \mathbb{C})$
- for $p + q = 2n$, $^L SO(p,q) = {}^L SO(n,n)$ or $^L SO(n-1, n+1)$ depending on whether $p - n$ is even or odd.
- $^L U(a,b) = {}^L U(n,n)$ if $a + b = 2n$ and $^L U(n, n+1)$ if $a + b = 2n+1$.

Forming the L-group is compatible with restriction of scalars: $^L R_{\mathbb{C}/\mathbb{R}} G = W_{\mathbb{R}} \ltimes (\widehat{G} \times \widehat{G})$.

DEFINITION: An *L-parameter* for G is a cts homomorphism $\phi : W_{\mathbb{R}} \to {}^L G$ over $W_{\mathbb{R}}$ s.t.

(1) $\phi(W_{\mathbb{R}})$ consists of semisimple elements and
(2) ϕ is *admissible* for G: If ϕ factors through $W_{\mathbb{R}} \ltimes \widehat{M} \subset {}^L G$ for a Levi $\widehat{M} \subset \widehat{G}$, then the Levi comes from a parabolic of $G/\mathbb{R}$. (The condition is empty for G quasisplit.)

It is *tempered* if $\phi(W_{\mathbb{R}})$ is bounded. Let $\Phi(G) = \{L\text{-parameters for } G\}$.

Note that $\Phi(G) \subset H^1(W_{\mathbb{R}}, \widehat{G})$ (if $\phi(w) = w \ltimes a_w$ then $w \mapsto a_w$ is a 1-cocycle).

LOCAL LANGLANDS/$\mathbb{R}$: There is a surjective mapping with finite fibres (L-packets, singletons for $GL(n, \mathbb{R})$)

$$\left\{\begin{matrix}\text{isomorphism classes of irreducible} \\ \text{admissible } (\mathfrak{g}, K)\text{-modules}\end{matrix}\right\} \longrightarrow \Phi(G)/\widehat{G}$$

under which tempered repns correspond to tempered parameters (+ more).

The theorem follows from: (1) Every repn is the unique irreducible quotient of an induced repn from a tempered repn with a positive character. (2) Every tempered repn is unitarily induced from a discrete series. (3) Harish-Chandra's parametrization of discrete series.

REMARKS: Under $H^1(W_{\mathbb{R}}, Z(\widehat{G})) \twoheadrightarrow Hom_{cts}(G(\mathbb{R}), \mathbb{C}^*)$, twisting parameters corresponds to twisting repns by characters, e.g. $\pi|det|^s \mapsto |\cdot|^s \phi_\pi$ for $GL(n, \mathbb{R})$.

REMARKS: (i) Unitarity does not work well with LL. (ii) $\Phi(G)/\widehat{G}$ has geometry!

REMARK: The infinitesimal character of an L-parameter $\phi : W_{\mathbb{R}} \to {}^L G$ is defined as follows. We may conjugate so that $\phi(\mathbb{C}^*) \subset \widehat{T}$. Any continuous $\mathbb{C}^* \to \widehat{T}$ is given by

$$z \mapsto z^{\lambda} \bar{z}^{\mu} \qquad \text{for } \lambda, \mu \in X_*(\widehat{T})_{\mathbb{C}}, \lambda - \mu \in X_*(\widehat{T}).$$

(Here $z^{\lambda}\bar{z}^{\mu}$ means the point at which a character $\nu \in X^*(\widehat{T})$ takes value $z^{\langle \nu, \lambda \rangle} \bar{z}^{\langle \nu, \mu \rangle}$.) Then $\mu \in X_*(\widehat{T})_{\mathbb{C}} = \mathfrak{t}^*$ is the infinitesimal character of ϕ. With this definition, LL respects infinitesimal characters.

EXAMPLE: Parameters of $GL(n)$ are semisimple repns of $W_{\mathbb{R}}$ of dimension n. The irreducible repns of $W_{\mathbb{R}}$ are of dimension one or two, and include:

- $\omega_{\mathbb{R}} |\cdot|^s$ for $s \in \mathbb{C}^*$, where $\omega_{\mathbb{R}}$ is the sign character of $\mathbb{R}^*$
- for $\left(\frac{z}{\bar{z}}\right)^{1/2}$ denoting the unitary character $z = re^{i\theta} \mapsto e^{i\theta}$ of $\mathbb{C}^*$, we have

$$\sigma_k = Ind_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \left(\frac{z}{\bar{z}}\right)^{k/2}$$

for $k \geq 1$. Explicitly, $z \mapsto diag(\left(\frac{z}{\bar{z}}\right)^{k/2}, \left(\frac{z}{\bar{z}}\right)^{-k/2})$ and $j \mapsto \begin{pmatrix} 0 & (-1)^k \\ 1 & 0 \end{pmatrix}$.

So any sum of such repns with total dimension n labels a unique repn of $GL(n, \mathbb{R})$.

EXAMPLE: σ_k is the parameter of the discrete series D_k for $GL(2, \mathbb{R})$. For k odd the parameter lands in $SL(2, \mathbb{C})$ and the corresponding repn comes from $PGL(2, \mathbb{R})$.

The existence of such a formalism has consequences:

- L-packets of repns are functorial in homomorphisms of the dual group, i.e. given ${}^LG_1 \to {}^LG_2$ can transfer packets of repns from $G_1(\mathbb{R})$ to $G_2(\mathbb{R})$.
- L-packets on G transfer to the quasisplit inner form G^* since $\Phi(G) \subset \Phi(G^*)$
- If G is an inner form of a compact group G^u then a f.d. repn E of G^u transfers to a packet on $G(\mathbb{R})$. This is the packet of discrete series with infinitesimal character equal to that of E.

Under these transfers other invariants (e.g. character values, etc) are related. E.g. this “explains" the formal relation between H-C character formula for d.s. and Weyl’s character formula.

The classification of the unitary dual is a difficult unsolved problem. Fortunately, not all unitary repns are relevant to the discrete spectrum of automorphic forms. To understand the contribution of nontempered repns to the discrete spectrum Arthur introduced A-parameters and A-packets.

DEFINITION: An *A-parameter* is a cts homom $\psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^L G$ over $W_{\mathbb{R}}$ s.t.
(1) $\psi|_{W_{\mathbb{R}}}$ is a tempered L-parameter
(2) $\psi|_{SL_2(\mathbb{C})}$ is algebraic.

Let $\Psi(G)$ be the set of A-parameters. The retraction

$$\Psi(G) \to \Phi(G) \qquad \text{by} \qquad \psi \mapsto \phi_\psi(w) := \psi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right)$$

induces $\Psi(G)/\widehat{G} \hookrightarrow \Phi(G)/\widehat{G}$.

Arthur conjectures that for each $\psi \in \Psi(G)$ there is an A-packet $\Pi(\psi)$ of *unitary* repns s.t.
(1) certain signed combinations of the characters in $\Pi(\psi)$ are stable distributions on G
(2) $\Pi(\phi_\psi) \subset \Pi(\psi)$ for G quasisplit (3) more conditions ...

and the local components of discrete automorphic repns for G should belong to A-packets. Local A-packets may overlap but global A-packets are disjoint.

A-packets for cohomological repns were defined by Adams-Johnson (1987). General A-packets were defined for real groups by Adams-Barbasch-Vogan (1994) (using the geometry of $\Phi(G)/\widehat{G}$) and for many classical groups by Arthur (2013) (using global harmonic analysis). Their agreement has been verified (Arancibia-Mœglin-Renard (2017), others).

We will now describe the parameters of (Adams-Johnson) A-packets of cohomological repns. For simplicity we mainly consider trivial coefficients ($E = \mathbb{C}$), making remarks about the general case.

Let H be a θ-stable fundamental torus, so that $H = TA$ with T maximally compact. We may choose a θ-stable Borel $\mathfrak{b}$ containing $\mathfrak{h}$. The θ-stable parabolic subalgebras $\mathfrak{q} \supset \mathfrak{b}$ are classified by subsets of the simple roots of H in G fixed by B. Since

$$(X^*(H), \Delta_G, X_*(H), \check{\Delta}_G) = (X_*(\widehat{H}), \check{\Delta}_{\widehat{G}}, X^*(\widehat{H}), \Delta_{\widehat{G}})$$

this gives a subset of the simple roots of the dual group and hence a parabolic $\widehat{Q} \supset \widehat{B}$. The condition that $\mathfrak{q}$ is θ-stable translates to the following condition on $\widehat{Q}$:

DEFINITION: A *self-associate parabolic* for G is a parabolic $\widehat{Q} \subset \widehat{G}$ which is conjugate to its opposite parabolic under $j \ltimes \widehat{G} \subset {}^L G$. (Necessarily it is then conjugate by $j \ltimes w_0$ where w_0 is the longest element of $W(\widehat{G}, \widehat{T})$.)

We have a diagram:

$$\begin{array}{ccc} \{\theta\text{-stable parabolic subalgebras}\}/K & \xrightarrow{\mathfrak{q} \mapsto \widehat{Q}} & \left\{\text{self-associate parabolics in } \widehat{G}\right\} \\ \downarrow{\scriptstyle \mathfrak{q} \mapsto A_{\mathfrak{q}}} & & \\ \{\text{cohomological repns}\} & & \end{array}$$

The Adams-Johnson A-packets are $\Pi_{\widehat{Q}} = \{A_{\mathfrak{q}} : \mathfrak{q} \mapsto \widehat{Q}\}$, i.e. they are indexed by self-associate $\widehat{Q} \subset \widehat{G}$. If $K^0 \neq K$ one must twist by characters of $G(\mathbb{R})/G(\mathbb{R})^0 = K/K^0$ to get all $(\mathfrak{g}, K^0)$-cohomological repns.

Recall that a unipotent element in a complex group H is called *regular* if its centralizer has minimal dimension, equal to the rank of H. Regular unipotents exist and are all conjugate. Given a regular unipotent u there is a homomorphism $SL_2(\mathbb{C}) \to H$ taking $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ to u, and this *principal* SL_2 is unique up to conjugacy. (Ex: for $SL(n, \mathbb{C})$ take Sym^{n-1}).

Now with a self-associate parabolic $\widehat{Q}$ we will associate an A-parameter. First we produce the complex parameter $W_{\mathbb{C}} \times SL_2(\mathbb{C}) \to {}^LG$. Standard properties of the principal SL_2 allows us to write down a parameter $\mathbb{C}^* \times SL_2(\mathbb{C}) \to {}^LG$ s.t.

- the infinitesimal character is that of the trivial repn (or any given f.d. repn E in general)
- the image of $\mathbb{C}^*$ lies in $\widehat{T}^{j \ltimes w_0 = -1}$ and its centralizer is the Levi $\widehat{L}$ of $\widehat{Q}$
- $SL_2(\mathbb{C})$ is mapped to a principal SL_2 in $\widehat{L}$

These properties suffice to extend the parameter:

THEOREM 1

The parameter $W_{\mathbb{C}} \times SL_2(\mathbb{C}) \to {}^LG$ defined by a self-associate parabolic in $\widehat{G}$ extends to an A-parameter $W_{\mathbb{R}} \times SL_2(\mathbb{C})$. The possible extensions modulo conjugacy has a transitive action of $H^1(W_{\mathbb{R}}, Z(\widehat{G}))$.

In particular, there is a unique extension if G is simply-connected. This gives:

COROLLARY 2

If G is simply-connected then the A-packets of cohomological repns for trivial coefficients are in bijection with standard self-associate parabolics of the dual group.

EXAMPLES

EXAMPLE: If G has discrete series (i.e. G and G^* are inner forms) then any parabolic is self-associate and the theorem/corollary are well-known.

EXAMPLE: If $G = R_{\mathbb{C}/\mathbb{R}} G'$ is a complex group then self-associate parabolics of the dual group correspond to parabolics of G which come from G'.

EXAMPLE: For $GL(n, \mathbb{R})$ a self-associate parabolic in $GL(n, \mathbb{C})$ is given by an ordered partition $\sum_i n_i = n$ satisfying $(n_1, \ldots, n_r) = (n_r, \ldots, n_1)$. The cohomological A-parameter for this parabolic is given, according to the parity of r, by:

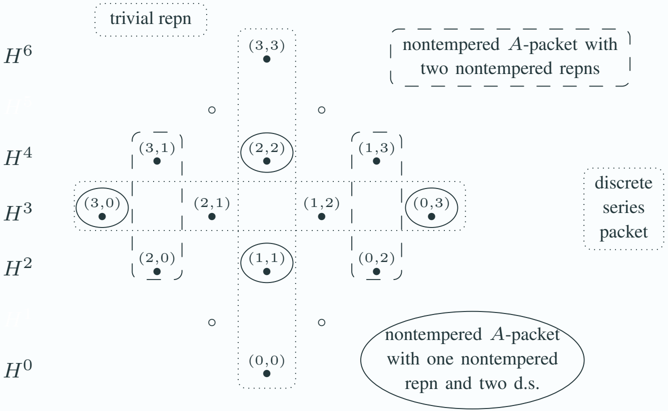
$$\psi = \sum_{i \geq 1}^{r/2} \sigma_{k_i} \otimes [n_i] \qquad \text{or} \qquad \sum_{i \geq 1}^{(r-1)/2} \sigma_{k_i} \otimes [n_i] \; + \; \omega_{\mathbb{R}}^{?} \otimes [n_{(r+1)/2}]$$

where $? \in \{0, 1\}$, $[a]$ denotes the a-dim'l irreducible repn of $SL_2(\mathbb{C})$, and the k_i are determined by the required infinitesimal character. The L-parameters are ϕ_ψ for such ϕ.

The tempered parameters come from $(1, \ldots, 1)$ and is unique if n is even and has two possibilities for n odd.

EXAMPLE: The Speh repn of $GL(2n, \mathbb{R})$ corresponds to the partition $n + n = 2n$. It has A-parameter $\sigma_n \otimes [n]$ and is cohomological for $E = \mathbb{C}$.

EXAMPLE: $Sp(4, \mathbb{R})$. There are four cohomological A-packets:



Theorem 3

If $G(\mathbb{R})$ is connected then there is a unique A-packet consisting of tempered cohomological repns, with parameter given by

$$W_{\mathbb{R}} \xrightarrow{id\times\sigma_1} W_{\mathbb{R}} \times SL_2(\mathbb{C}) \longrightarrow {}^L G$$

where the second comes from a Galois-stable principal $SL_2(\mathbb{C}) \to \widehat{G}$. (In general, one must also allow twisting by characters of K/K^0 of order two, and similarly for parameters.)

If G has discrete series then the tempered packets consist of d.s. repns. In general they are fundamental series. Thus for $GL(2n, \mathbb{R})$ it is a single repn.

For an A-parameter $\psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^L G$ define its tempered companion L-parameter $T(\psi)$ by

$$W_{\mathbb{R}} \xrightarrow{id\times\sigma_1} W_{\mathbb{R}} \times SL_2(\mathbb{C}) \xrightarrow{\psi} {}^L G.$$

(This is the analogue of the tempered companion parameter in the p-adic case given by restriction to the "diagonal SL_2" in $W'_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ (Mœglin).)

Theorem 4

An A-parameter $\psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^L G$ is cohomological if and only if $T(\psi)$ is a tempered cohomological parameter for G.

Now consider $r : (\widehat{G}_1 \supset \widehat{B}_1 \supset \widehat{T}_1) \to (\widehat{G}_2 \supset \widehat{B}_2 \supset \widehat{T}_2)$ that takes a regular unipotent to a regular unipotent. (Ex: $Sym^{n-1} : SL_2(\mathbb{C}) \to GL(n, \mathbb{C})$.)
Then $r_* : X_*(\widehat{T}_1) \to X_*(\widehat{T}_2)$ takes $\check{\rho}_1$ to $\check{\rho}_2$ (where $\check{\rho}$ is the half-sum of positive coroots).
Then functoriality takes the f.d. repns of $G_1(\mathbb{C})$ to f.d. repns of $G_2(\mathbb{C})$, taking the repn with h.w. $\lambda \in X_*(\widehat{T}_1)$ to the one with h.w. $r_*(\lambda)$).

THEOREM 5

Let $r : {}^LG_1 \to {}^LG_2$ be a homomorphism of L-groups such that a regular unipotent in $\widehat{G}_1$ goes to a regular unipotent in $\widehat{G}_2$. Then

(1) $\psi \mapsto r \circ \psi$ *takes cohomological parameters for G_1 to cohomological parameters for G_2*

(2) *Conversely, if ${}^LG_1 \to {}^LG_2$ has abelian kernel and if $r \circ \psi$ is cohomological for G_2 then ψ is cohomological for G_1.*

(Here cohomological means $(\mathfrak{g}, K^0)$-cohomological. If we work with coefficients then we must transfer coefficients as above.)

If r has the regular unipotent property then there is a natural way to transfer parabolics and Levis between $\widehat{G}_1$ and $\widehat{G}_2$, which has the property that $L_1 \leftrightarrow L_2$ if a regular unipotent in L_1 goes to one in L_2, and conversely, if a unipotent in L_1 goes to a regular unipotent in L_2 then it must be regular in L_1. Given this observation the theorem and the characterization of parameters in Theorem 1 the theorem follows easily.

As a corollary, the transfer of a tempered cohomological packet under such an L-homomorphism is always a tempered cohomological packet. Other instances where functoriality takes cohomological packets to cohomological packets also follow.

EXAMPLES: CLASSICAL GROUPS

The inclusions of complex groups $\widehat{G}_1 \subset \widehat{G}_2$ with $\widehat{G}_1$ simple for which a regular unipotent in $\widehat{G}_1$ is regular unipotent in $\widehat{G}_2$ are:

- $Sp(2g, \mathbb{C}) \subset GL(2g, \mathbb{C})$
- $SO(2n+1, \mathbb{C}) \subset GL(2n+1, \mathbb{C})$
- $SO(2n-1, \mathbb{C}) \subset SO(2n, \mathbb{C})$
- $F_4 \subset E_6$, $\quad G_2 \subset Spin(7, \mathbb{C}) \subset SO(8, \mathbb{C})$, $\quad G_2 \subset SO(7, \mathbb{C}) \subset SL(7, \mathbb{C})$
- $\widehat{G}_1$ is a principal $SL_2(\mathbb{C})$ in $\widehat{G}_2$

As a corollary to the previous theorem the parameters of some classical groups can be described in terms of the well-understood parameters of $GL(n)$.

THEOREM 6

(1) Let G be $Sp_{2g}(\mathbb{R})$, or $SO(p,q)$ with $p+q$ odd. Then there is a natural embedding $r : {}^L G \hookrightarrow W_{\mathbb{R}} \times SL(N, \mathbb{C})$ for $N = 2g+1$ or $N = 2p+2q$. An A-parameter ψ for $G(\mathbb{R})$ is cohomological if and only if $r \circ \psi$ is a cohomological A-parameter for $PGL_N(\mathbb{R})$. A cohomological A-parameter for $PGL_N(\mathbb{R})$ is automatically an A-parameter (hence cohomological) for the symplectic group if N is odd.

(2) If $m = 2n$ is even, a cohomological parameter for $GL(2n, \mathbb{R})/\mathbb{R}+$ with values in $GL(2n, \mathbb{C})$ is either symplectic or orthogonal (as a representation of $W_{\mathbb{R}} \times SL_2(\mathbb{C})$). In the first case it is a parameter for $SO(p,q)(\mathbb{R})$ for $p+q = 2n+1$, and in the second case, it is a parameter for $SO(p,q)(\mathbb{R})$ for $p+q = 2n$, with the parity of p fixed by the determinant.

(3) If $G = U(p,q)$, then an A-parameter for G is cohomological if and only if its restriction to $W_{\mathbb{C}}$ is a cohomological A-parameter for $(R_{\mathbb{C}/\mathbb{R}} GL(m))(\mathbb{R}) = GL(m, \mathbb{C})$, $m = p+q$.

The sum of dimensions of cohomology in an A-packet is independent of the A-packet:

THEOREM 7

Let $H = TA$ be a θ-stable fundamental torus with compact part T and split part A. For an A-packet $\Pi = \Pi(\psi)$ of cohomological repns,

$$\sum_{\pi\in\Pi}\sum_{i\geq 0}\dim H^i(\mathfrak{g},K,\pi) = 2^\delta \left|\frac{W(G,H)^\theta}{W(G(\mathbb{R}),H(\mathbb{R}))}\right|$$

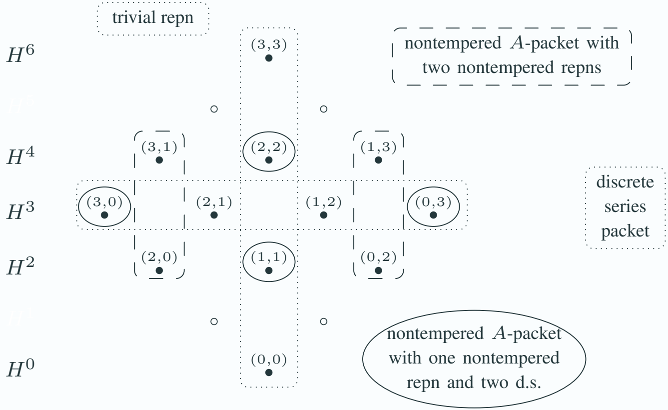
where $\delta = \dim A$ is the discrete series defect*, $W(G,H)^\theta = \{w \in W(G,H) : w\theta = \theta w\}$, and $W(G(\mathbb{R}),H(\mathbb{R}))$ is the subgroup of elements with representatives in $G(\mathbb{R})$.*

The Adams-Johnson parametrization of A-packets and the induction formalism reduces the general case to the case $\Pi = \{triv\}$, which is $\sum_i \dim H^i(G^u/K) = 2^\delta \left|\frac{W(G,H)^\theta}{W(G(\mathbb{R}),H(\mathbb{R}))}\right|$. For this simply compute the Lefschetz trace of $gK \mapsto \theta(tgK) = \theta(tg)K$ on G^u/K where $\overline{\langle t\rangle} = T$ and θ is the global Cartan involution. $\square$

QUESTIONS: Can one explain the 2^δ geometrically? Can one realize this as a global statement?

EXAMPLE

EXAMPLE: $Sp(4,\mathbb{R})$. We have the Hodge diamond



The sum of dimensions is 4.

There is an easy "wrong" answer to the first question given by the reduction to the fundamental Levi $M = Z_G(A)$. Choosing a parabolic P with Levi M gives an embedding $\widehat{M} \subset \widehat{G}$.

For a self-associate parabolic $\widehat{Q} \subset \widehat{G}$ the Adams-Johnson packet $\Pi_{\widehat{Q}}$ for G and the Adams-Johnson packet $\Pi_{\widehat{Q} \cap \widehat{M}}$ for M are in bijection and there is an equality

$$\sum_i \dim H^i(\mathfrak{g}, K, A_{\mathfrak{q}}) = \sum_i \dim H^i(\mathfrak{m}, K \cap M, A_{\mathfrak{q} \cap \mathfrak{m}})$$

for any θ-stable parabolic subalgebra $\mathfrak{q}$ for G. (The inverse of $A_{\mathfrak{q}} \mapsto A_{\mathfrak{q} \cap \mathfrak{m}}$ is given by sending π_M to the Langlands quotient of $\mathrm{Ind}_P^G \pi_M$.) Similarly, well-known facts about Weyl groups in real groups give

$$\frac{W(G,H)^\theta}{W(G(\mathbb{R}), H(\mathbb{R}))} = \frac{W(M,H)}{W(M(\mathbb{R}), H(\mathbb{R}))}.$$

Thus the equality of dimensions for G is equivalent to the one for M, which is

$$\sum_{\pi \in \Pi_{\widehat{Q} \cap \widehat{M}}} \sum_{i \geq 0} \dim H^i(\mathfrak{m}, K \cap M, \pi) = 2^\delta \left| \frac{W(M,H)}{W(M(\mathbb{R}), H(\mathbb{R}))} \right|.$$

In this equality the factor 2^δ is geometric: It comes from the free action of $A^u = A(\mathbb{C}) \cap G^u$ on $M^u / K \cap M$ and $\mathrm{H}^*(M^u / K \cap M) = \wedge^* \mathfrak{a}^* \otimes H^*(A^u \backslash M^u / K \cap M)$. $\square$

But perhaps a better explanation is possible combining the two questions ...