Non-admissible irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ in characteristic p

(joint work with E. Ghate)

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Outline

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- Introduction
- Formalism of diagrams
- Diagrams associated to Galois representations
- Infinite-dimensional diagrams and the construction

Smooth representations

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Example

Let $\pi = C^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ = the space of locally constant C-valued functions on $\mathbb{P}^1(\mathbb{Q}_p)$ equipped with a natural action of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Then π is a smooth admissible C-linear representation of G.

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Diagrams of Schneider-Stuhler can be used to construct mod p representations of reductive p-adic groups.

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- $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathcal{O}) \subset G$ a maximal compact subgroup.
- $$K_1 = \begin{pmatrix} 1+\varpi\mathcal{O} & \varpi\mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{pmatrix} \subset I_1 = \begin{pmatrix} 1+\varpi\mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{pmatrix}$$
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- $N = N_G(I)$, a subgroup of G generated by I and $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$.
  Z = Center of G.

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A *diagram* is a triple (D_0, D_1, r) consisting of
- D_0 a smooth representation of KZ over $\overline{\mathbb{F}_p}$ with a trivial action of ϖ,
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In all the diagrams of this talk, D_0 will be a finite dimensional representation of $\mathrm{GL}_2(\mathbb{F}_{p^f})$.

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To sum up, given a diagram (D_0, D_1, r), there is a (non-canonical) smooth G-action on $\mathrm{inj}_K D_0$.

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Fact: Given a weight σ, the space σ^{I_1} has dimension 1; let χ_σ be the corresponding character. If $\chi_\sigma \neq \chi_\sigma^s$ then there is a unique weight σ^s such that $\chi_{\sigma^s} = \chi_\sigma^s$.

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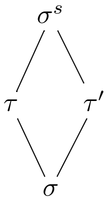
When $f = 2$, $D_0 = \mathrm{Ind}_I^K \chi_\sigma^s$ in Example 2 has 3-step socle filtration with socle and cosocle weights contributing to the space of its I_1-fixed vectors.



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Theorem (Breuil-Paskunas)

When $F = \mathbb{Q}_p$, the above construction gives a bijection between the set of diagrams of type mentioned in Example 1 and the set of smooth admissible irreducible supercuspidal representations of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}_p}$. The inverse map is given by $\pi \mapsto (\langle K \cdot \pi^{I_1} \rangle, \pi^{I_1}, r)$.

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In the process of generalizing the above theorem to representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$, Breuil and Paskunas constructed several interesting families of diagrams and interesting corresponding representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ with fixed socle.

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Theorem (Breuil-Paskunas)

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Diagrams in the above theorem are called *Diamond diagrams*.

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Given a Diamond diagram $(D_0(\rho), D_1(\rho), r)$ *associated to an irreducible* ρ*, the smooth* $\overline{\mathbb{F}_p}$*-representation* π *of* $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ *generated by* $D_0(\rho)$ *inside* $\mathrm{inj}_K D_0(\rho)$ *is admissible irreducible and supercuspidal.*

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His idea was to construct an indecomposable infinite-dimensional diagram from a Diamond diagram $(D_0(\rho), D_1(\rho), r)$ making use of extra characters in $D_1(\rho)$.

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Diamond diagrams and mod p representations

Theorem (Breuil-Paskunas)

Given a Diamond diagram $(D_0(\rho), D_1(\rho), r)$ associated to an irreducible ρ, the smooth $\overline{\mathbb{F}_p}$-representation π of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ generated by $D_0(\rho)$ inside $\mathrm{inj}_K D_0(\rho)$ is admissible irreducible and supercuspidal.

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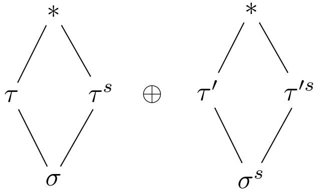

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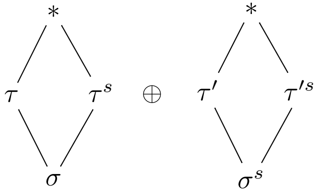

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We use this subdiagram (D_0, D_1, r) to construct indecomposable infinite-dimensional diagrams.

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- Let $\Omega(\infty) = \bigoplus_{\mathbb{Z}} \mathrm{inj}_K D_0$ with component-wise KZ-action. One uses N-action on $\mathrm{inj}_K D_0$ to define an N-action $\Omega(\infty)$ such that $D_1(\infty) \hookrightarrow \Omega(\infty)$ is N-equivariant.

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- The K and N actions on $\Omega(\infty)$ coincide on I and hence glue together to give a smooth G-action on $\Omega(\infty)$.
- Let $\pi \subset \Omega(\infty)$ be the subrepresentation of G generated by $D_0(\infty)$.

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Key Proposition

If σ (or σ^s) at i-th index is in π' then D_{0,σ^s} (resp. $D_{0,\sigma}$) at i-th index is a K-subrepresentation of π'.

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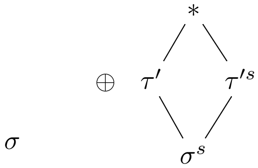

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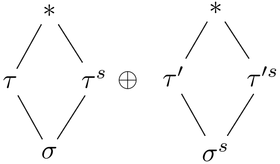

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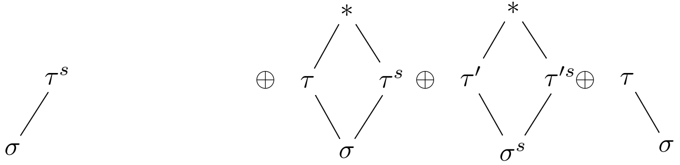

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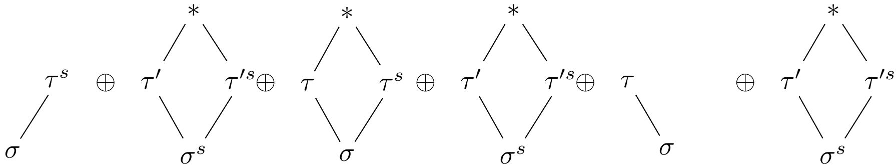

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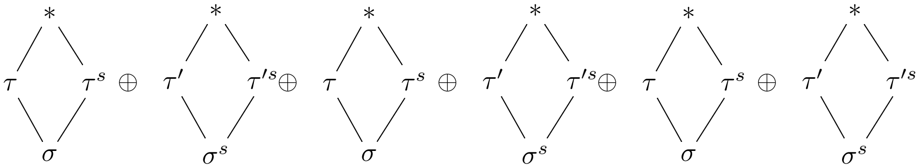

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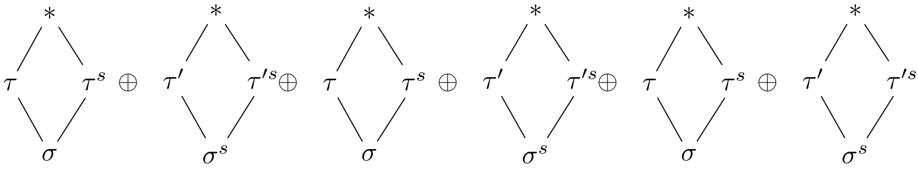

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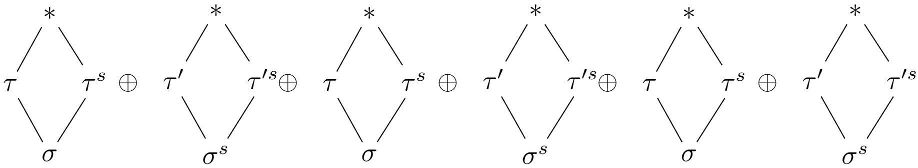

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Remark

- *It is possible to define π over the finite field $\mathbb{F}_{p^2}$.*
- *The $\mathbb{F}_{p^2}$-model of π can be used to produce a smooth irreducible non-admissible representation of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ over any field of characteristic p.*

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- What about connected reductive groups other than GL_2? The formalism of diagrams for the groups of semi-simple rank 1 has been developed by Koziol, Xu, Herzig and Vignéras.
- Does every smooth irreducible representation of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}_p}$ possess a central character? It does when $F = \mathbb{Q}_p$, and also when F is any finite extension of $\mathbb{Q}_p$ but the representations are over algebraically closed uncountable field of characteristic p.

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Thank you!