Distinction, base change, and formal degree

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(joint work with Nadir Matringe)

G - p-adic group

$H < G$

π - irreducible admissible representation of G (over $\mathbb{C}$)

π is distinguished wrt H

if the space of H-invariant linear forms on π is non-zero;

$$\mathrm{Hom}_H(\pi, 1) \neq 0.$$

Thus, $\exists\ 0 \neq \ell : \pi \to \mathbb{C}$ with $\ell(\pi(h)v) = \ell(v)$ for $v \in \pi$, $h \in H$.

For a character χ of H, π is χ-distinguished wrt H if

$$\mathrm{Hom}_H(\pi, \chi) \neq 0.$$

Jacquet's philosophy

Specially interesting case:

$$H < G \text{ is the fixed points of an involution on } G.$$

Expectation ("Jacquet's philosophy"):

H-distinguished representations are lifts from a "related" group G'.

Galois involution:

E/F - quadratic extension of p-adic fields
$\mathbf{G}$ - quasi-split group over F
$G = \mathbf{G}(E)$, $H = \mathbf{G}(F)$

Conjectures due to Dipendra Prasad about:

- characterising π for which $\mathrm{Hom}_H(\pi, 1) \neq 0$.
- computing $\dim \mathrm{Hom}_H(\pi, 1)$.

These are in terms of the Langlands parameters.

$(GL_n(E), GL_n(F))$

The most well-understood case:

$$G = GL_n(E), H = GL_n(F)$$

Some early works:

- Jacquet-Lai (1985)
- Harder-Langlands-Rapoport (1986)
- Yuval Flicker (1988, 1991, 1992, 1993)
- Jeff Hakim (1991)
- Dipendra Prasad (1992, 1999, 2001)
- Flicker-Hakim (1994)

Later works include: Anthony Kable (2004), A-Kable-Tandon (2004), A-Rajan (2005), A (2008). Significant advances were made in the papers of Nadir Matringe (2009, 2010, 2011) and Omer Offen (2011).

Recall that "Jacquet's philosophy" connects distinction for symmetric spaces with base change lifts. In the Galois case, Dipendra Prasad's conjectures give a recipe for the multiplicity of the space of invariant forms, which is in terms of base change (the key is the cardinality of the fiber of the relevant base change map).

In this talk, we present a result, joint with Nadir Matringe, which illustrates the connection between distinction and base change in yet another way.

The result is for the pair $(GL_n(E), GL_n(F))$ and in fact for discrete series representations of $GL_n(E)$ distinguished with respect to $GL_n(F)$.

Our result

Things to keep in mind:

1. $(GL_n(E), GL_n(F))$ is of multiplicity one [Flicker '91]

$$\dim_{\mathbb{C}} \mathrm{Hom}_{GL_n(F)}(\pi, 1) \leq 1$$

2. Two natural $GL_n(F)$-invariant forms [Flicker '88, Kable '04]

So they differ by a constant

3. Flicker-Rallis conjecture [Matringe '11, Mok '15]

Distinction is BC from $U(n, E/F)$

Our result evaluates this proportionality constant and it involves the formal degrees of the base changed and base changing representations. Precise statement towards the end of the talk.

Key ingredients of proof

1. FE for the Asai L-function [Flicker '93, Kable '04]
2. RHS is connected to one linear form [Kable '04]
3. LHS is connected to the other form [A-Kable-Tandon '04]

   This step is subtle and involves a new functional equation [A-Matringe '17] which in turn requires knowledge of the sign of a local (Rankin-Selberg) root number, conjectured in [A '08] and proved by [Matringe-Offen '18].

4. Hiraga-Ichino-Ikeda conjecture [HII '08, Beuzart-Plessis '18]

   Relates "an adjoint gamma value" with the formal degree. We need this for $GL(n)$, proved by [HII '08], and for $U(n, E/F)$, proved by [Beuzart-Plessis '18].

5. Sign of an Asai root number [A '08, BP '18, Shankman '18]

Plan for the rest of the talk

We've introduced the notion of distinction for a pair (G, H). For the pair $(GL_n(E), GL_n(F))$, distinguished (generic) representations are base change lifts from $U(n, E/F)$. We'll now introduce the base change map from $U(n, E/F)$ to $GL_n(E)$ and state the Flicker-Rallis conjecture. This is one of the two inputs for the statement of our result. We'll then introduce the other input: the notion of the formal degree of a discrete series representation. Finally, we'll introduce the two natural invariant linear forms on a $GL_n(F)$-distinguished discrete series representation of $GL_n(E)$, and state our result.

$$U(n) = \{g \in GL_n(E) \mid {}^t g^\sigma J g = J\},$$

where

$$J = \text{anti-diag}((-1)^{n-1}, \ldots, 1)) \;\&\; \mathrm{Gal}(E/F) = \langle \sigma \rangle.$$

Langlands Dual:

$${}^L U(n) = GL_n(\mathbb{C}) \rtimes W_F,$$

where the Weil group acts by projection to Gal(E/F), and $w_\sigma \in W_F \backslash W_E$ acts as the automorphism

$$g \mapsto J\, {}^t g^{-1} J^{-1}.$$

Weil-Deligne Group:

$$W_k' = W_k \times SL_2(\mathbb{C})$$

Base change for $U(n)$

Langlands Parameters:

$$\Phi(U(n)) = \{\phi : W'_F \to {}^L U(n)\}/\sim$$

$$\Phi(GL_n(E)) = \{\rho : W'_E \to GL_n(\mathbb{C})\}/\sim$$

Base Change:

$$\begin{array}{cccccc} \mathrm{BC} & : & \Phi(U(n)) & \to & \Phi(GL_n(E)) \\ & & \phi & \mapsto & \phi|_{W'_E} \end{array}$$

Flicker-Rallis Conjecture:

π - irreducible admissible generic representation of $GL_n(E)$

$$\rho_\pi \in \mathrm{Image(BC)} \iff \begin{cases} \pi \text{ is distinguished} & n \text{ odd} \\ \pi \text{ is } \omega_{E/F}\text{-distinguished} & n \text{ even.} \end{cases}$$

Flicker-Rallis Conjecture

ρ - conjugate self-dual Langlands parameter for $GL_n(E)$

$$\rho : W'_E \to GL_n(\mathbb{C}),\ \rho^\sigma \cong \rho^\vee,\ \rho^\sigma(g) = \rho(w_\sigma^{-1} g w_\sigma)$$

ρ is said to be <u>of parity $\eta(\rho) \in \{\pm 1\}$</u> if there is a non-degenerate bilinear form B on ρ with

- $B(\rho(g)v, \rho^\sigma(g)w) = B(v, w)$,
- $B(v, w) = \eta(\rho) \cdot B(w, \rho(w_\sigma^2)v)$

where $w_\sigma \in W_F \backslash W_E$.

Flicker-Rallis Conjecture

Flicker-Rallis conjecture follows from combining

Theorem (Matringe '11)

An irreducible admissible generic representation of $GL_n(E)$ is distinguished with respect to $GL_n(F)$ if and only if its Langlands parameter is conjugate self-dual of parity $+1$.

and

Theorem (Mok '15)

A Langlands parameter for $GL_n(E)$ is in the image of the base change map from $U(n)$-parameters if and only if it is conjugate self-dual of parity $(-1)^{n-1}$.

Formal degree

Recall the orthogonality relations for matrix coefficients for a finite group. A matrix coefficient of a (unitary) representation π is a function on G given by

$$g \mapsto \langle \pi(g)v, w \rangle$$

where $v, w \in \pi$. If π is irreducible, we have

$$\frac{1}{|G|} \sum_{g \in G} \langle \pi(g)v, v' \rangle \overline{\langle \pi(g)w, w' \rangle} = \frac{1}{\dim \pi} \langle v, w \rangle \overline{\langle v', w' \rangle}.$$

If π is a discrete series representation of a p-adic group, there exists $d_\mu(\pi) \in \mathbb{R}_{>0}$ such that

$$\int_{G/Z} \langle \pi(g)v, v' \rangle \overline{\langle \pi(g)w, w' \rangle} d\mu(g) = \frac{1}{d_\mu(\pi)} \langle v, w \rangle \overline{\langle v', w' \rangle}.$$

This $d_\mu(\pi)$ is the formal degree of π (which depends on the choice of the Haar measure μ).

Formal degree

With respect to a specific choice of measure as in [HII '08], we have:

1. $d(\pi) = \frac{1-q^{-1}}{n} \left| \lim_{s \to 0} \frac{\gamma(s, \pi \times \pi^\vee, \psi)}{1 - q^{-s}} \right|$ [HII '08]
2. $d(\rho) = \frac{1}{2} \left| \gamma(0, \pi, r', \psi_0) \right|$ [Beuzart-Plessis '18]

for $GL(n)$ and $U(n, E/F)$ respectively, where in (2), the gamma factor is the twisted Asai gamma factor.

Two invariant forms

Let ψ be a non-degenerate character of $N(E)/N(F)$. Consider

$$\ell(W) = \int_{N_n(F)\backslash P_n(F)} W(p)dp$$

on the ψ-Whittaker model of π. [Flicker '88].

The form ℓ is always non-zero and clearly $P_n(F)$-invariant. It is $GL_n(F)$-invariant precisely when π is $GL_n(F)$-distinguished.

Now consider

$$\lambda(W) = \int_{F^\times N_n(F)\backslash \mathrm{GL}_n(F)} W(g)dg$$

which is obviously $GL_n(F)$-invariant but this integral is well-defined only when π is a discrete series representation. So assume π is discrete series. [Kable '04]

The form λ is non-zero precisely when π is $GL_n(F)$-distinguished.

Our result

Thus if π is a discrete series representation of $GL_n(E)$ which is distinguished wrt $GL_n(F)$, then by multiplicity one, the forms λ and ℓ differ by a scalar.

Let ρ be the (discrete series) representation of $U(n, E/F)$ that base changes to π (stably or unstably depending on the parity of n). They have their respective formal degrees $d(\rho)$ and $d(\pi)$.

Theorem (A-Matringe '20)

We have

$$\lambda = c \cdot \frac{d(\rho)}{d(\pi)} \cdot \ell,$$

where c is a positive constant that does not depend on the representations ρ and π.