1. Let $G$ be a group. Let $H$ be the subgroup of $G$ generated by $\{x^2 \mid x \in G\}$. Let $H'$ be the commutator subgroup of $G$, i.e., the subgroup generated by $\{xyx^{-1}y^{-1} \mid x, y \in G\}$. Show that $H \supset H'$.

2. Let $p$ be a prime number, and let $G$ be a finite group whose order is a power of $p$. Let $F$ be a field of characteristic $p$, and $V$ a nonzero vector space over $F$ equipped with a linear action of $G$.[1] Prove that there exists a nonzero subspace $W \subset V$ such that $G$ acts trivially on $W$.
   **Hint:** One possible strategy involves first proving the result in the special case where $F$ is the finite field $\mathbb{F}_p$ and $V$ is finite dimensional (so that $V$ is a finite set), and using this special case to prove the general case.

3. Find the characteristic polynomial of $A^7$, where $A$ is the real matrix:
$$\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}.$$

4. Let $S$ be the set of all lines in $\mathbb{R}^2$ that pass through the origin and have either rational or infinite slope. Show that the usual action of $\mathrm{SL}_2(\mathbb{Z})$ (the group of $2 \times 2$ integer matrices with determinant 1) on $\mathbb{R}^2$ induces a transitive action of $\mathrm{SL}_2(\mathbb{Z})$ on $S$.

5. Consider the ring $C(0,1)$ of continuous real-valued functions on $(0,1)$. For any $P \in (0,1)$, write $\mathfrak{m}_P$ for the ideal:
$$\mathfrak{m}_P := \{f \in C(0,1) \mid f(P) = 0\}.$$
   (a) Show that $C(0,1)$ contains infinitely many maximal ideals that are not of the form $\mathfrak{m}_P$ for any $P \in (0,1)$.
   (b) Consider the map $\mathbb{R} \to C(0,1)$ which sends $a \in \mathbb{R}$ to the constant function with value $a$. Suppose $\mathfrak{m} \subset C(0,1)$ is a maximal ideal with the property that the composition $\mathbb{R} \to C(0,1) \to C(0,1)/\mathfrak{m}$ is an isomorphism. Show that $\mathfrak{m} = \mathfrak{m}_P$ for some $P \in (0,1)$.

6. Suppose $f$ is a continuous function on $\mathbb{R}$ such that we have:
$$\forall\, x \in \mathbb{R}, \quad f(x) = f(x^2).$$
   Show that $f$ is constant.

[1] In other words, we have an action $G \times V \to V$ of the group $G$ on the set $V$, written $(g, v) \mapsto g \cdot v$, such that for all $g \in G$, the map $v \mapsto g \cdot v$ is an $F$-linear transformation of the $F$-vector space $V$.

7. Let $f$ be a continuous real-valued function on $[0,1]$. Show that:

$$\lim_{n\to\infty}\int_0^1 f(x)\,\sin(nx)\,dx = 0.$$

8. Suppose $f:[0,1]\to[0,1]$ is a nondecreasing function, which may or may not be continuous. Show that there exists $x\in[0,1]$ such that $f(x)=x$.

9. Let $D:\mathbb{C}[x]\to\mathbb{C}[x]$ be the $\mathbb{C}$-linear map given by $Df = df/dx$. Let $e^D:\mathbb{C}[x]\to\mathbb{C}[x]$ be the induced linear map given by:

$$f\mapsto (e^D)(f) = \sum_{n\geq 0}\frac{1}{n!}D^n(f).$$

   (a) Show that the above formula does give a well-defined map, which is linear.

   (b) Compute $\ker(e^D)$ and $\operatorname{coker}(e^D)$.

10. Prove or disprove: any bijective continuous function $(0,1)\to\mathbb{R}$ has a continuous inverse.

11. Prove or disprove: $\mathbb{R}^3\setminus\{(0,0,0)\}$ and $\mathbb{R}^3\setminus B_\varepsilon$ are homeomorphic, where $\varepsilon>0$ is a real number and $B_\varepsilon$ is the closed ball given by $\{(x,y,z)\in\mathbb{R}^3 \mid \sqrt{x^2+y^2+z^2}\leq\varepsilon\}$.

12. Prove or disprove the following assertion. Let $S$ be a subset of $\mathbb{R}$, with the induced order, such that every nonempty subset of $S$ contains a minimum element (in other words, $S$ is well-ordered). Then $S$ is necessarily countable (i.e., empty, finite or countably infinite).

13. Given any two ellipses in $\mathbb{R}^2$, show that there exists a linear (but not necessarily orthogonal) change of coordinates ($x' = ax+by+k, y' = cx+dy+l; ad-bc\neq 0$) so that both the ellipses have axes parallel to the new coordinate axes - namely, under the new coordinate system, each of them can be described by an equation of the form:

$$\frac{(x'-\alpha)^2}{r^2}+\frac{(y'-\beta)^2}{s^2}=1.$$

14. For any $n\geq 1$, let $P_n(x) = \dfrac{d^n}{dx^n}\left(x^n(1-x)^n\right)$. Show that for $m\neq n$:

$$\int_0^1 P_n(x)P_m(x)\,dx = 0.$$