QUESTION PAPER

GS2020 - PART 2

Question 1. Let k be a field and let $f : M_n(k) \to k$ be a k-linear transformation such that $f(AB) = f(BA)$, for all $A, B \in M_n(k)$. Show that f is a scalar multiple of the trace map.

Question 2. Let R be a finite commutative ring such that $R^\times$ has odd cardinality. Show that the cardinality of R is a power of 2.

Question 3. Let (X, d) be a metric space with the property that $d(x, z) \le \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$. Show that for any $x \in X$, the unit ball

$$B_x = \{y \in X \mid d(x, y) \le 1\}$$

is both open and closed in X.

Question 4. Let k be a field and let $S \subset \mathbb{N}$ be an infinite subset. Let $f \in k[t]$ be a polynomial. Show that there exists a polynomial $g \in k[t]$ such that $f(t)g(t) = \sum_{i \ge 0} a_i t^i$, where $a_i = 0$ for every $i \notin S$.

Question 5. Let $GL_{n+1}(\mathbb{R}) \subset M_{n+1}(\mathbb{R})$ be given the subspace topology and identify $GL_n(\mathbb{R})$ with the subset of $GL_{n+1}(\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

with $A \in GL_n(\mathbb{R})$. Show that $GL_{n+1}(\mathbb{R})/GL_n(\mathbb{R})$ is not compact in the quotient topology.

Question 6. Let H denote the set of homeomorphisms of $\mathbb{R}$ onto itself, viewed as a group under composition. Consider the elements $f_1, f_2 \in H$ defined by $f_1(x) = 2x$ and $f_2(x) = 3x$, for all $x \in \mathbb{R}$. Show that f_1 and f_2 belong to the same conjugacy class in H.

Question 7. Does there exist a sequence $\{a_n\}_n$ in $\mathbb{R}^2$ such that

$$|a_n - a_m| = \sqrt[4]{n - m},$$

for all natural numbers $n \ge m$?

Question 8. Does there exist a nonconstant polynomial $f \in \mathbb{Z}[t]$ such that $|f(n)|$ is a power of some prime number (depending on n), for every $n \in \mathbb{Z}$?
Hint: It may help to first consider the case where the prime number is independent of n.

Question 9. Let P be a convex octagon which can be inscribed in a circle such that four of its sides have length 2 and the other four sides have length 3. Find all possible values of the area of P.

Question 10. Construct a continuous function $f : [0, 1] \to \mathbb{R}$ that cannot be written in the form $g_1 - g_2$, where $g_1, g_2 : [0, 1] \to \mathbb{R}$ are continuous, increasing functions.