Instructions

1. Every claim needs a justification.
2. Do NOT write your name or affiliation or any personal detail other than application number (of the form GS2023MTHPHDxxxxxx or GS2023MTHIPHxxxxxx) on the paper.
3. There are 10 problems in this paper. All of these carry equal marks.
4. Use both sides of each sheet for writing your answers.
5. Extra/rough sheets: Two extra sheets have been provided. If these do not suffice, you can ask the invigilator for more sheets.
6. Extra sheet etiquette:
   - On the top of each extra sheet, write clearly which problem is being attempted on that sheet. **Do not do more than one problem on one extra sheet.**
   - Write your application number clearly at the top of each extra sheet.
   - All extra sheet(s) should be stapled onto this answer booklet, whether or not you consider them rough work.
7. If a given sheet contains part of your work on a particular problem, and that work is continued on some other page, indicate this clearly.
8. No books, notes, electronic devices etc. are allowed.
9. $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, \dots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
10. All rings are assumed to be associative and containing a multiplicative identity denoted by 1.
11. $\mathbb{C}[\![x]\!]$ denotes the formal power series ring in the variable x over $\mathbb{C}$: its elements are formal sums $\sum_{n=0}^{\infty} a_n x^n$ with $a_n \in \mathbb{C}$ for each n, and its addition and multiplication are given as follows. For $f = \sum_{n=0}^{\infty} a_n x^n, g = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{C}[\![x]\!]$:
$$f + g = \sum_{n=0}^{\infty} (a_n + b_n) x^n, \qquad \text{and} \qquad fg = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} a_i b_{n-i}) x^n.$$
12. For a positive integer n and a field k, $M_n(k)$ denotes the set of $n \times n$ matrices with entries in k, viewed as a vector space of dimension n^2 over k and also as a ring.
13. We give $M_n(\mathbb{C})$ the unique topology for which any $\mathbb{C}$-linear isomorphism $M_n(\mathbb{C}) \to \mathbb{C}^{n^2}$ is a homeomorphism.
14. A map $f : \mathbb{R} \to \mathbb{R}$ is said to be a diffeomorphism if it is a differentiable homeomorphism whose inverse is also differentiable.

1. Let X be a compact subset of $M_n(\mathbb{C})$ and let
$$S = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of a matrix in } X\}.$$
Show that S is compact.

2. Let X denote the set of isomorphism classes of all finite groups. For any finite group G, let $[G]$ denote its class in X. Does there exist an injective function $f : X \to \mathbb{Z}$ such that for every finite group G and a normal subgroup H of G, we have
$$f([G]) = f([H]) \cdot f([G/H])?$$

3. Determine the cardinality of the set
$$\{(a_1, \ldots, a_{10}) \in \{0,1\}^{10} \mid a_i \text{ and } a_{i+1} \text{ are not both 0 for any } 1 \le i \le 9\}.$$

4. Let n be a positive integer and let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a nonzero polynomial. If
$$X = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid f(z) = 0\},$$
then show that $\mathbb{C}^n \setminus X$ is path connected.

5. (a) Show that there is a piecewise linear homeomorphism $f : \mathbb{R} \to \mathbb{R}$ with $f(2x) = 3f(x)$, for all $x \in \mathbb{R}$.

   (b) Does there exist a diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ with $f(2x) = 3f(x)$, for all $x \in \mathbb{R}$?

6. Let R be a commutative ring in which 2 is invertible. Show that the cardinality of the set $\{x \in R \mid x^2 = 1\}$ is the same as the cardinality of the set $\{x \in R \mid x^2 = x\}$.

7. Let $A \in M_n(\mathbb{C})$. Show that the following statements are equivalent.

   (i) A is nilpotent (i.e., $A^m = 0$ for some positive integer m).

   (ii) There exists a $\mathbb{C}$-linear ring homomorphism $\mathbb{C}[\![x]\!] \to M_n(\mathbb{C})$ taking x to A.

8. Let $z \in \mathbb{C}^\times$ and let C_z denote the closure of the cyclic subgroup generated by z in $\mathbb{C}^\times$. Show that $\mathbb{C}^\times / C_z$ is isomorphic as a group to one of the following three groups: $\mathbb{C}^\times$, $\mathbb{R}$, and $S^1 \times S^1$. Here, $\mathbb{R}$ is the additive group of real numbers, $\mathbb{C}^\times$ is the multiplicative group of nonzero complex numbers, and $S^1 \subset \mathbb{C}^\times$ is the subgroup consisting of complex numbers of absolute value one.

9. Let $f_0 : [0,1] \to \mathbb{R}$ be a continuous function and inductively define

$$f_n(x) := \int_0^x f_{n-1}(t)dt,$$

for $n \geq 1$. Assume that for every $x \in [0,1]$, there exists a positive integer n_x such that $f_{n_x}(x) = 0$. Show that there exists a subinterval $[a,b] \subseteq [0,1]$, with $a < b$, such that f_0 is identically zero on $[a,b]$.

10. Let k be a field of characteristic 0 and let n be a positive integer. Show that the following two statements are equivalent.

    (i) If V is a finite dimensional k-vector space and $T : V \to V$ is a k-linear map satisfying

    $$\underbrace{T \circ \cdots \circ T}_{n\text{-times}} = \mathrm{Id},$$

    then V is spanned by eigenvectors for T. Here, $\mathrm{Id} : V \to V$ is the identity linear transformation.

    (ii) The cardinality of the set $\{x \in k \mid x^n = 1\}$ is precisely n.