GS 2024, Mathematics: Stage II

Instructions

1. Every claim needs a justification.
2. If a question consists of two parts, (a) and (b), you may use part (a) to solve part (b), even if you have not worked (a) out.
3. Do NOT write your name or affiliation or any personal detail other than application number (of the form GS2024MTHPHDxxxxxx or GS2024MTHIPHxxxxxx) on the paper.
4. There are 10 problems in this paper. Each of these carries 10 points.
5. Use both sides of each sheet for writing your answers.
6. Extra/rough sheets: Two extra sheets have been provided. If these do not suffice, you can ask the invigilator for more sheets.
7. Extra sheet etiquette:
   - On the top of each extra sheet, write clearly which problem is being attempted on that sheet. **Do not do more than one problem on one extra sheet.**
   - Write your application number clearly at the top of each extra sheet.
   - All extra sheet(s) should be stapled onto this answer booklet, whether or not you consider them rough work.
8. If a given sheet contains part of your work on a particular problem, and that work is continued on some other page, indicate this clearly.
9. No books, notes, electronic devices etc. are allowed.
10. $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, \dots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
11. All rings are assumed to be associative and containing a multiplicative identity denoted by 1.

1. Let X denote the set of sequences of 0's and 1's. Define $d : X \times X \to \mathbb{R}$ by $d((x_n), (y_n)) = \sup_{n \in \mathbb{N}} \{(1/2^n)|x_n - y_n|\}$.
   (a) Show that (X, d) is a metric space.
   (b) Show that X is complete with respect to d.
2. Let G be a finite group of square-free order, and let H be a subgroup of G with the following property: for any nontrivial subgroup $K \subseteq G$, the subgroup $H \cap K$ is nontrivial. Show that $H = G$.
3. Consider the real vector space $V = \{p(x) \in \mathbb{R}[x] | \deg p(x) \leq 10\}$. Consider the linear transformations $S, T : V \to V$ defined by

$$S : p(x) \mapsto p(x) + p'(x),$$

$$T : p(x) \mapsto p(x + 1).$$

   Are the linear transformations S and T similar over the real numbers?
4. Let n be a positive integer and let p be a prime number such that $p \equiv 1 \pmod{n}$. Let A be a square matrix with entries in $\mathbb{Z}/p\mathbb{Z}$ such that $A^n = I$. Prove that A is diagonalizable over $\mathbb{Z}/p\mathbb{Z}$.
5. (a) Let $f, g : [0, 1] \to \mathbb{R}$ be continuous functions such that for $a, b \in [0, 1]$, we have

$$f(a) = f(b) \implies g(a) = g(b).$$

   Show that there exists a continuous map $h : f([0, 1]) \to \mathbb{R}$ such that

$$g \equiv h \circ f \quad \text{on} \quad [0, 1].$$

   (b) Conclude that there exists a sequence $\{p_n\}$ of polynomials such that $\{p_n \circ f\}$ converges to g uniformly on $[0, 1]$.
6. Let (X, d) be a nonempty compact metric space. Let $f : X \to X$ be a continuous function such that $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$. Show that f has a unique fixed point.
7. Let R and S be distinct subrings of $\mathbb{Q}$, each with exactly two prime ideals. Show that $1/2$ belongs to at least one of R and S.
8. (a) Show that if n and m are positive integers such that $n \equiv m \pmod{20}$, then we have $n^n \equiv m^m \pmod{10}$.
   (b) Which of the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ occur as the last digit of n^n, for infinitely many positive integers n?

9. Let $f : (0, \infty) \to (0, \infty)$ be a continuously differentiable function: this means that f is differentiable at x for all $x \in (0, \infty)$, and that $f' : (0, \infty) \to \mathbb{R}$ is continuous. Assume that

$$\lim_{x \to \infty} (f'(x) + f(x)) = 0.$$

Show that $\lim_{x \to \infty} f(x) = 0$.
Hint. First show that for all $\varepsilon > 0$ and $M \in (0, \infty)$, there exists $x_0 > M$ such that $f(x_0) < \epsilon$.

10. Suppose that we are given two bags A and B each containing finitely many balls labelled with a number in the set $\{0, \ldots, 10\}$. It is given that on choosing a ball from bag A and a ball from bag B at random, the sum of the numbers on them takes each of the values $0, \ldots, 10$ with probability $\frac{1}{11}$ each.

    (a) Show that one of the bags A and B has all its balls labelled with 0.

    (b) What are the possibilities for the labels on the balls in the other bag?