Brief survey of Viazovska's work on sphere packing

TIFR

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Lattice packing of spheres

Definition

A lattice in $\mathbb{R}^d$ is a discrete subgroup of $\mathbb{R}^d$ of rank d. Equivalently, it is the integral span of a basis of $\mathbb{R}^d$.

Lattice packing

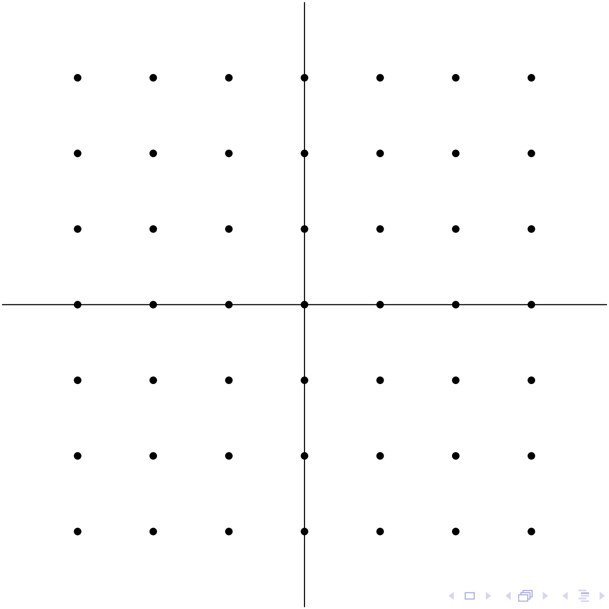
Let Λ be a lattice in $\mathbb{R}^d$. Sphere packing on lattice Λ is putting spheres of radius r centred at each of point of Λ, where

$$r = \frac{1}{2} \min_{x \in \Lambda \setminus \{0\}} |x|$$

Density of lattice packing is

$$\frac{\pi^{d/2} r^d}{\Gamma\left(\frac{d}{2}+1\right)} \cdot \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)}.$$

Example: $\mathbb{Z}^2$ in $\mathbb{R}^2$



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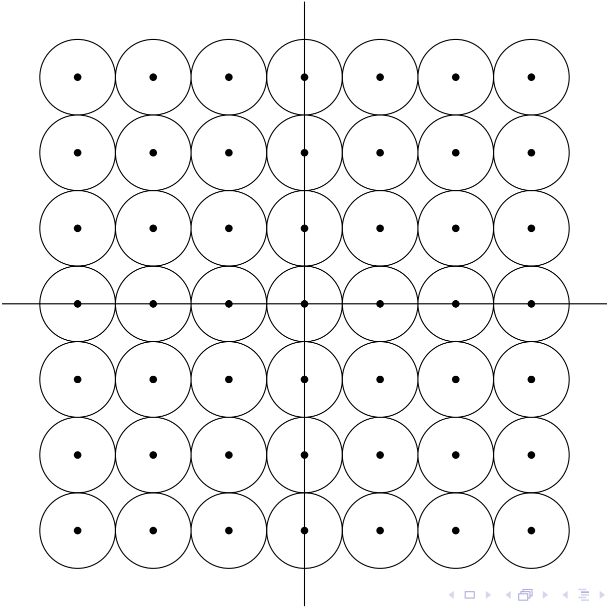
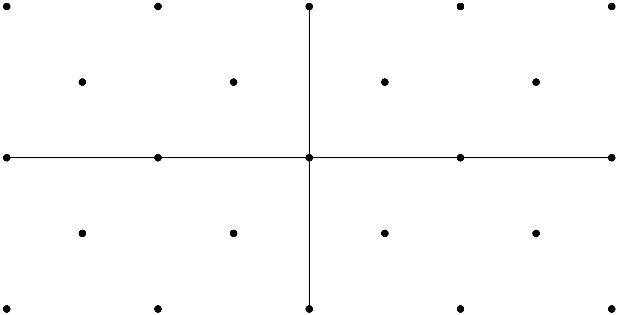


Example: Lattice D_d in $\mathbb{R}^d$

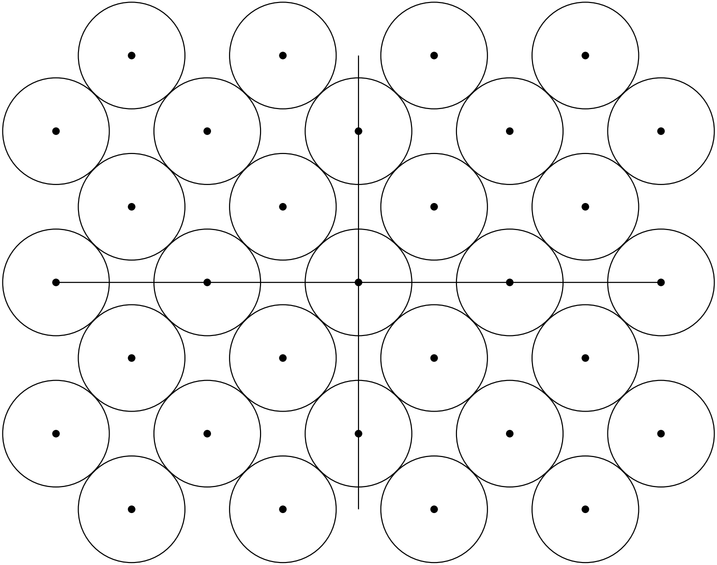
$$D_d = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : x_1 + \cdots + x_d \text{ is even } \}.$$



$\{e_1 + e_i : 1 \leqslant i \leqslant d\}$ is a $\mathbb{Z}$-basis of D_d.

Note that the minimum distance between lattice points in D_d is $\sqrt{2}$.

Example: Lattice D_d in $\mathbb{R}^d$



Example

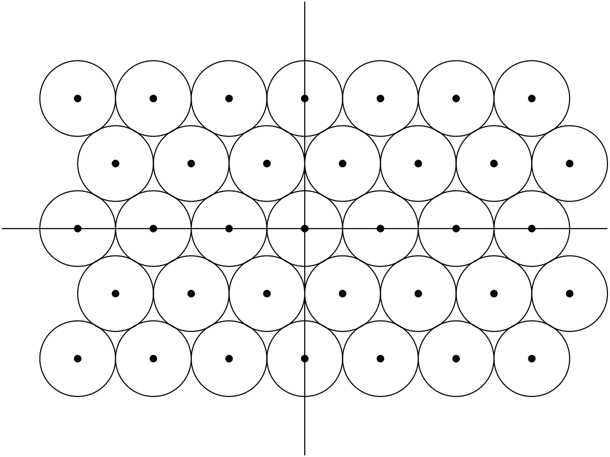
Theorem (Hales)

D_3 gives the densest sphere packing in $\mathbb{R}^3$.

Figure: By Funkdooby - A load of old balls. (wikipedia)

D_4 and D_5 give the densest known sphere packings in $\mathbb{R}^4$ and $\mathbb{R}^5$, respectively.

Densest packing in dimension 2



This is the A_2 lattice in $\mathbb{R}^2$.

The E_8 lattice

When $d \geqslant 6$ there are "holes" in the D_d lattice packing. The hole at $(\frac{1}{2}, \ldots, \frac{1}{2})$ is at a distance of $\sqrt{d/4}$ from lattice points. This hole is exactly enough to fit another sphere (of radius $\sqrt{2}/2$) when $d = 8$.

$$E_8 = D_8 \cup D_8 + \left(\frac{1}{2}, \ldots, \frac{1}{2}\right).$$

$$\left\{\left(\frac{1}{2}, \ldots, \frac{1}{2}\right), e_1 + e_j : 1 \leqslant j \leqslant 7\right\}$$

is a $\mathbb{Z}$-basis for E_8.

E_8 is an integral, even, unimodular lattice (distance between any vectors in E_8 is of the form $\sqrt{2k}$ for some integer $k \geqslant 0$ and $\text{vol}(\mathbb{R}^8/E_8) = 1$).

The packing radius of E_8 is $r = \sqrt{2}/2$.

Hence the density of E_8 packing is $\frac{\pi^4(\sqrt{2}/2)^8}{4!} = \frac{\pi^4}{384} = 0.2536\ldots$.

Magic function from linear programming bounds

As we saw in the previous talk, we need an even Schwartz function $f : \mathbb{R}^8 \to \mathbb{R}$ such that

$$f(x) \leqslant 0 \text{ for all } x \in \mathbb{R}^8 \text{ satisfying } |x| \geqslant \sqrt{2},$$
$$\widehat{f}(y) \geqslant 0 \text{ for all } y \in \mathbb{R}^8,$$
$$f(0) = \hat{f}(0) = 1.$$

Enough to assume that f is "radial" i.e. depends only on $|x|$.

For numerical computations Cohn and Miller used functions of the form

$$f(x) = p(|x|^2)e^{-\pi|x|^2},$$

where p is a well-chosen polynomial.

Modular group

$$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

The group $SL_2(\mathbb{Z})$ acts on $\mathcal{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

For an integer $N \geqslant 1$, define

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Modular forms

Weight k actions of $SL_2(\mathbb{Z})$ on a function $f : \mathcal{H} \to \mathbb{C}$ is defined by

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

A *weight k holomorphic modular form* for $SL_2(\mathbb{Z})$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that

(1) $f|_k\gamma(z) = f(z)$ for all $\gamma \in SL_2(\mathbb{Z})$. e.g.
$f|_k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(z) = f(z+1) = f(z)$.

(2) f is *holomorphic at the cusp* i.e. $f(z)$ has Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}$.

If there are finitely many terms with negative indices, then f is called *weakly holomorphic*.

Modular forms

A *weight k holomorphic modular form* for $\Gamma(N)$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that

(1) $f|_k\gamma = f$ for all $\gamma \in \Gamma(N)$.
e.g. $f|_k \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} = f(z+N) = f(z)$.

(2) f is *holomorphic at cusps* i.e. $f|_k\gamma(z)$ has Fourier expansion
$f|_k\gamma(z) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n z}{N}}$, for all $\gamma \in SL_2(\mathbb{Z})$.

$SL_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence one only needs to check

$$f(z+1) = f(z), \qquad f(-1/z) = z^k f(z).$$

Examples of modular forms

For an even integer $k \geqslant 4$, define

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} (cz+d)^{-k}$$
$$= 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}.$$

Fact: Product of modular forms of weight k and weight ℓ is a modular form of weight $k+\ell$. The graded ring of modular forms for $SL_2(\mathbb{Z})$ is generated by E_4 and E_6.

$$\Theta_{E_8}(z) = \sum_{x\in E_8} e^{\pi i |x|^2 z} = E_4(z).$$

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}$$

satisfies $z^{-2}E_2(-1/z) = E_2(z) - 6i/\pi z$ (*quasimodular form*).

Examples of modular forms

Ramanujan's Delta function

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

is the *unique cusp form* of weight 12 for $SL_2(\mathbb{Z})$. It vanishes at cusp but not at any point in $\mathcal{H}$.

$U(z) = \left(\sum_{n\in\mathbb{Z}} e^{\pi i n z}\right)^4$ is a modular form of weight 2 for $\Gamma(2)$.
Define $W = U|_2 T(z) = U(z+1)$ and $V = U - W$.

The graded ring of $\Gamma(2)$ modular forms is generated by U and W.

Magic functions!

We write $f = f_+ + f_-$, where $f_+ = (f + \widehat{f})/2$ and $f_- = (f - \widehat{f})/2$.
So f_+ and f_- are eigenfunctions for Fourier transform.
Thus we want radial eigenfunctions $f_\pm$ for Fourier transform with single root at $\sqrt{2}$ and double roots at $\sqrt{2n}$, for $n \geqslant 2$.

To illustrate the method, we construct a radial Fourier eigenfunction g, with Fourier eigenvalue -1, and simple roots at $\sqrt{n}$ for all $n \geqslant 3$.

$$g(x) = \frac{1}{2}\int_{-1}^{1} \psi(z) e^{\pi i z |x|^2} dz,$$

where ψ is a holomorphic function on $\mathcal{H}$ and the contour from -1 to 1 is the semicircle centred at 0.

Magic functions!

If $\psi(z+2) = \psi(z)$, then ψ has Fourier expansion $\psi(z) = \sum_{n\in\mathbb{Z}} a_n e^{\pi i n z}$. Thus we get

$$g(\sqrt{n}) = \frac{1}{2}\int_{-1}^{1} \psi(z) e^{\pi i z n} dz = a_{-n}.$$

(Assuming interchanging sum and integral is OK).

As the Fourier transform of $x \mapsto e^{\pi i z|x|^2}$ is $y \mapsto z^{-4} e^{\pi i(-1/z)|y|^2}$, we get

$$\widehat{g}(y) = \frac{1}{2}\int_{-1}^{1} \psi(z) z^{-4} e^{\pi i(-1/z)|y|^2} dz.$$

To get $\widehat{g} = -g$, we must have $z^4\psi(-1/z) = \psi(z)$. In other words, $\psi(z)$ should be a *weakly holomorphic* modular form of weight -2 on $\Gamma = \langle S, T^2\rangle \subset SL_2(\mathbb{Z})$. It suffices to construct a holomorphic modular form $\Delta\psi$ of weight 10 on Γ.

Magic functions!

Put

$$\psi = \frac{-U^5 + 2U^4 W}{\Delta}.$$

This ψ works and we get

$$g(\sqrt{n}) = \begin{cases} -240 & \text{if } n = 0, \\ 8 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geqslant 3. \end{cases}$$

We can rewrite the contour integral defining g as

$$\begin{aligned} g(x) &= \frac{e^{-\pi i |x|^2} - e^{\pi i |x|^2}}{2} \int_0^{i\infty} \psi(u+1) e^{\pi i u |x|^2} du \\ &= \sin(\pi |x|^2) \int_0^{\infty} \psi(it+1) e^{-\pi t |x|^2} dt. \end{aligned}$$

Magic functions!

The last expression "generalises":

$$f_{-}(x) = 4\sin(\pi|x|^2/2)^2 \int_0^\infty \psi(it) e^{-\pi t|x|^2} dt$$

with

$$\psi = \frac{W^3(5U^2 - 5UW + 2W^2)}{\Delta}.$$

$$f_{+}(x) = -4i\sin(\pi|x|^2/2)^2 \int_0^{i\infty} z^{-2}\chi(-1/z) e^{\pi iz|x|^2} dz,$$

with

$$\chi = \frac{(E_2E_4 - E_6)^2}{\Delta}.$$