Some selections from the work of June Huh

Piyush Srivastava

Matrices

Consider the set of columns of a matrix

$$A := \begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \dots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

and the sets

$$\mathcal{I} := \{S \subseteq \text{columns}\,(A) \mid S \text{ is linearly independent}\}$$

$$\mathcal{B} := \{B \in \mathcal{I} \mid B \text{ is "maximal"}\}$$ the "bases" of the column space

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Exchange property

$B_1, B_2 \in \mathcal{B}$ and $a_1 \in B_1 \setminus B_2 \implies$ there exists $a_2 \in B_2 \setminus B_1$ s.t. $(B_1 \setminus \{a_1\}) \cup \{a_2\} \in \mathcal{B}$

Matroids: Combinatorial "geometries"

[Nakasawa/Whitney, 1935]

$$M = (\Omega, \mathcal{B})$$

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- Ω is a finite ground set
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- Ω is a finite ground set
- $\mathcal{B}$ (set of "bases") is a collection of subsets of Ω which satisfy the exchange property
  - ...they must therefore all be of the same size!
  - $\mathcal{I}$ ("independent sets") are then all subsets of elements of $\mathcal{B}$
  - $\mathrm{rank}(M)$ is the size of any base

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$B_1, B_2 \in \mathcal{B}$ and $a_1 \in B_1 \setminus B_2 \implies$ there exists $a_2 \in B_2 \setminus B_1$ s.t. $(B_1 \setminus \{a_1\}) \cup \{a_2\} \in \mathcal{B}$

Example: Linear matroids

Consider the set of columns of a matrix (entries in $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{Q}$, or ...)

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Example: Graphic matroids

$$G = (V, E) \text{ is an undirected graph}$$

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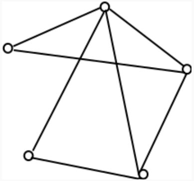
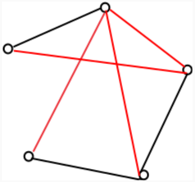
The bases are the spanning trees of G

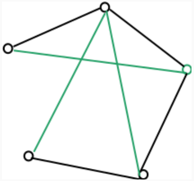
The independent sets are the spanning forests of G



An example graph



An example spanning tree



An example spanning forest

The bases generating polynomial

Let M be a matroid with ground set Ω and bases $\mathcal{B}$

The bases generating polynomial

$$g_M\left((x_e)_{e\in\Omega}\right) := \sum_{B\in\mathcal{B}} \prod_{e\in B} x_e$$

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Special case: the Spanning-Tree polynomial of graph $G = (V, E)$

$$\textsc{Spanning-Tree}_G\left((x_e)_{e\in E}\right) = \sum_{T:\text{ spanning tree of } G} \prod_{e\in T} x_e$$

Theorem

["classical", similar to Kirchoff's matrix-tree theorem]

For any graph $G = (V, E)$, $\textsc{Spanning-Tree}_G$ is upper-half plane "stable", i.e.,

$$\Im(z_e) > 0 \text{ for all } e \in E \implies \textsc{Spanning-Tree}_G\left((z_e)_{e\in E}\right) \neq 0.$$

From geometry (of roots) to matroids

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Theorem [Choe, Oxley, Sokal, and Wagner, 2004]

If a homogenous multilinear polynomial with non-negative coefficients is upper half-plane stable,
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If a homogenous multilinear polynomial with non-negative coefficients is upper half-plane stable,
then the "support" set of the non-zero coefficients of the polynomial must be the bases of a matroid
But there are matroids for which the bases generating polynomial is not upper half-plane stable

Question

So what is the precise geometric characterization of matroids?

Interlude: Geometry and probability

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The generating polynomial of a multinomial distribution being upper half-plane stable has important probabilistic consequences: [Borcea, Brändén, and Liggett, 2009]

Corollary: "Negative association"

If A and B are disjoint subsets of E and T is sampled uniformly at random from the set of spanning trees of G?

$$\mathbb{P}\left[A \cup B \subseteq T\right] \leq \mathbb{P}\left[A \subseteq T\right] \cdot \mathbb{P}\left[B \subseteq T\right]$$

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Question

Can one get such negative dependence properties more generally in the matroid setting?

E.g. with $|S| = |T| = 1$, but we choose a uniformly random spanning *forest*?

[conjectured by Kahn, and Grimmett and Winkler]

Strongly log concave / Lorentzian polynomials

[Gurvits, 2010; Anari, Gharan, and Vinzant, 2018; Brändén and Huh, 2020]

Strictly Lorentzian polynomials

[Brändén and Huh, 2020]

A homogeneous polynomial with positive coefficients is called strictly Lorentzian if all its partial derivatives of degree exactly two have a Hessian[a] with a “Lorentzian” signature, i.e., with exactly one positive eigenvalue

Lorentzian polynomials are “limits” of strictly Lorentzian polynomials

[a] the matrix of second derivatives

Strongly log concave (SLC) (homogeneous) polynomials

[Gurvits, 2010, see also Anari, Gharan, and Vinzant, 2018]

A homogeneous polynomial with non-negative coefficients is called SLC if the logarithms of the polynomial and of all its successive partial derivatives are concave in the positive orthant

These notions have been shown to be equivalent

[Brändén and Huh, 2020]

Back to matroids

Theorem [Brändén and Huh, 2020, see also Anari, Gharan, and Vinzant, 2018]

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"Not-too-positive" correlation for matroids [Brändén and Huh, 2020, see also Huh, Schröter, and Wang, 2021]

Let e and f be fixed distinct elements of the ground set of a matroid M, and B an uniformly randomly chosen base of M. Then,

$$\mathbb{P}\left[e, f \in M\right] \leq \left(2 - \frac{1}{|B|}\right) \cdot \mathbb{P}\left[e \in M\right] \cdot \mathbb{P}\left[f \in M\right]$$

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For polynomials that are not multilinear, a generalization of matroids to multisets (known as "M-convexity") is needed

A toy proof

What do matroids have to do with the Lorentzian signature?

Let us prove the following "base case" of one side of the Lorentzian-matroid equivalence

Theorem [Brändén and Huh, 2020, see also Anari, Gharan, and Vinzant, 2018]

The basis generating polynomial of a matroid of rank 2 (i.e., when all bases have size 2) is Lorentzian (i.e., its Hessian has at most one positive eigenvalue)

Let Ω be the ground set and $\mathcal{B}$ the set of bases (size 2) of the matroid.

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$$g_M(x) := \sum_{\{e,f\}\in\mathcal{B}} x_e x_f = \frac{1}{2} x^T A x,$$

where A is a $\Omega \times \Omega$ symmetric matrix such that

$$A_{e,f} = \begin{cases} 1 \text{ if } \{e,f\} \text{ is a base} \\ 0 \text{ otherwise} \end{cases}$$

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Note that

$$\text{Hessian}(g_M)(x) = \frac{1}{2} A,$$

so we just have to find the eigenvalues of A

The A matrix

$$A_{e,f} = \begin{cases} 1 \text{ if } \{e, f\} & \text{is a base} \\ 0 \text{ otherwise} \end{cases}$$

Note that if $A_{a,b} = 1$ and $A_{c,e} = 1$ (where a, b, c, e are distinct)

then at least one of $A_{a,c}$ and $A_{c,b}$ must be 1!

[because of the base exchange property]

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Thus, if b belongs to *some* base, and if $A_{a,b} = 0$ and $A_{b,c} = 0$, then we must have $A_{a,c} = 0$!

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Thus Ω partitions into sets $S_0, S_1, S_2, \ldots, S_t$ such that

$$A_{e,f} = \begin{cases} 0 \text{ if } e \in S_0 \text{ or } f \in S_0 \text{ or } e, f \in S_i \text{ with } i \geq 1 \\ 1 \text{ if } e \in S_i \text{ and } f \in S_j \text{ for } i \neq j \text{ and } i, j \geq 1 \end{cases}$$

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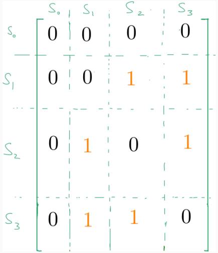
So

$$A = \mathbf{1}_{\Omega \backslash S_0} \mathbf{1}_{\Omega \backslash S_0}^T - \sum_{i=1}^{t} \mathbf{1}_{S_i} \mathbf{1}_{S_i}^T,$$

and this decomposition as a sum of a rank-one positive definite matrix with a negative definite matrix shows that at most one eigenvalue of A can be positive.

The matrix A: a picture



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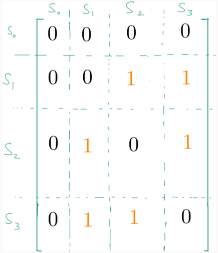

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Back to “reporter mode”

Other successes of the theory: Volume polynomial

Volume polynomial

[Steiner, Minkowski]

If $K_1, K_2, \ldots, K_n$ are convex bodies in $\mathbb{R}^d$, and let $w_1, w_2, \ldots, w_n$ be non-negative coefficients. Then the volume of the Minkowski sum body $w_i K_i$ is given by a polynomial

$$\text{vol}\left(\sum w_i K_i\right) = \sum_{\alpha} \frac{d!}{\alpha!} V(\underbrace{K_1, \ldots, K_1}_{\alpha_1 \text{ times}}, \underbrace{K_2, \ldots, K_2}_{\alpha_2 \text{ times}}, \ldots, \underbrace{K_n, \ldots, K_n}_{\alpha_n \text{ times}}) \prod w_i^{\alpha_i},$$

where α ranges over n-tuples of non-negative integers that sum up to d, and the function V is the so-called mixed volume (which is symmetric and "linear" in its arguments)

- Volume polynomials are SLC (and hence Lorentzian)

already observed to be a consequence of the Alexandrov–Fenchel inequality by Gurvits in [Gurvits, 2010]

Alexandrov-Fenchel for Lorentzian polynomials

Alexandrov-Fenchel inequality

If $K_1, K_2, \ldots, K_d$ are convex bodies in $\mathbb{R}^d$, and V is the mixed volume functions appearing in the volume polynomial, then

$$V(K_1, K_2, K_3, \ldots, K_d)^2 \geq V(K_1, K_1, K_3, \ldots, K_d)V(K_2, K_2, K_3, \ldots, K_d)$$

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- …but not all Lorentzian polynomials are volume polynomials! [Brändén and Huh, 2020]

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- Volume polynomials are Lorentzian
- ...but not all Lorentzian polynomials are volume polynomials! [Brändén and Huh, 2020]

However, the direct analogue of the Alexandrov-Fenchel inequality holds (with the mixed volumes replaced by the corresponding polynomials) for all Lorentzian polynomials [Brändén and Huh, 2020]

Other inequalities: Mason's conjecture(s)

Let M be any matroid, and $\mathcal{I}$ the set of its independent sets

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Ultra log concavity: a strong form of log-concavity

Conjecture: $\frac{I(k)^2}{\binom{n}{k}^2} \geq \frac{I(k-1)}{\binom{n}{k-1}} \frac{I(k+1)}{\binom{n}{k+1}}$ for all k

If $a_i := \mathbb{P}\left[i \text{ heads in } n \text{ independent coin tosses}\right]$

then the sequence (a_i) is ultra log-concave.

[Probably goes back at least to Newton]

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[Proved by Adiprasito, Huh, and Katz, 2018]

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[Proved by Anari, Liu, Gharan, and Vinzant, 2018; Brändén and Huh, 2018]

[A proof in the language of Lorentzian polynomials by Brändén and Huh, 2020]

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Closure under product [Brändén and Huh, 2020]

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- Closure under "MAP" of *stable* polynomials gives a very quick proof of the Heilmann-Lieb theorem that the matching polynomial is real rooted

Other successes of the theory: Algorithms

Mihail-Vazirani conjecture

Let M be a matroid, and let $\mathcal{B}$ be the set of its bases. Consider the following random walk on $\mathcal{B}$:

- If the current state is $B \in \mathcal{B}$, then uniformly at random pick an element e from B, and set $B' = B \setminus \{e\}$.
- Then pick an element f uniformly at random from the set of all elements which can be added to B' to create a base, and set next state to $B' \cup \{f\}$

This random walk is *rapidly mixing*

(The actual conjecture is even stronger and makes further demands on the geometry of the graph of the random walk)

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Let M be a matroid, and let $\mathcal{B}$ be the set of its bases. Consider the following random walk on $\mathcal{B}$:

- If the current state is $B \in \mathcal{B}$, then uniformly at random pick an element e from B, and set $B' = B \setminus \{e\}$.
- Then pick an element f uniformly at random from the set of all elements which can be added to B' to create a base, and set next state to $B' \cup \{f\}$

This random walk is *rapidly mixing*

(The actual conjecture is even stronger and makes further demands on the geometry of the graph of the random walk)

Parallel to the work of Brändén and Huh (2020), Anari, Liu, Oveis-Gharan and Vinzant developed many of the same idea towards a goal of proving this conjecture. This series of work also built upon "high-dimensional expansion" ideas from the work of Dinur, Kaufman, Mass and Oppenheim, and led to a full resolution of this conjecture

Aspects that I couldn't cover

We mostly talked about the beautiful (and elementary![1]) theory of Lorentzian/SLC polynomials, but this ignores many other aspects of June Huh's work. To take a couple of examples:

- Many of the advances described above were inspired by the paper of Adiprasito, Huh, and Katz (2018), which considered a different algebraic view of matroids
- Adiprasito, Huh, and Katz (2018) itself improved a breakthrough result of Huh (2012), which had affirmatively answered a large part of a conjecture due to Reed, Rota, Heron and Welsh, on the log concavity of the coefficients of the chromatic polynomial (and of its generalizations to other matroids)

[1] in the sense of "elementary proof of the prime number theorem"

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Thank you!

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Matroids and "geometry": *rank* and *flats*

Let $\mathcal{M}$ be a matroid with ground set Ω and $\mathcal{I}$ be the set of its independent sets.

Rank

For *any* subset S of Ω, $\text{rank}(S) :=$ size of the largest independent set $\mathcal{I}$ which is a subset of S.

What is rank in a *linear matroid*? *graphic matroid*?

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Flat

$S \subset \Omega$ is a flat if for any $v \notin S$, $\mathrm{rank}(S \cup \{v\}) > \mathrm{rank}(S)$

What are the flats in a *linear matroid*? *graphic matroid*?