The Sphere Packing Problem

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'A Conceptual Breakthrough'

March 14, 2016: Maryna S. Viazovska. The Sphere Packing Problem in Dimension 8, 2016. arXiv:1603.04246.

March 21, 2016: H. Cohn, A. Kumar, S.D. Miller, D. Radchenko, and M. Viazovska. The Sphere Packing Problem in Dimension 24, 2016. arXiv:1603.06518.

Consider a Packing of the Euclidean space $\mathbb{R}^n$ by congruent balls with disjoint interiors.

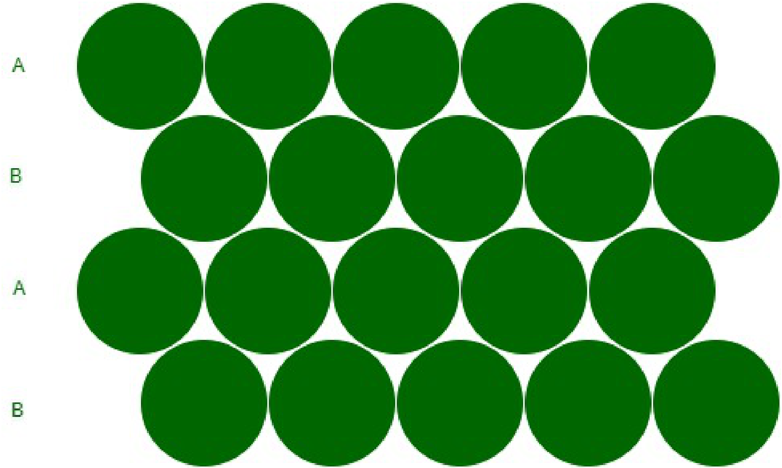
What is the densest packing ?

The Density of Packing refers to the proportion of volume occupied by the spheres in a *large* box

Dimension $n = 1$

In dimension 1, the interval $[-1, 1]$ is a ball of radius 1, centered at 0. And $\mathbb{R} = \cup_{k\in\mathbb{Z}}[k - 1, k + 1]$ is the densest packing.

Sphere Packing in dimension 2



Dimension 2

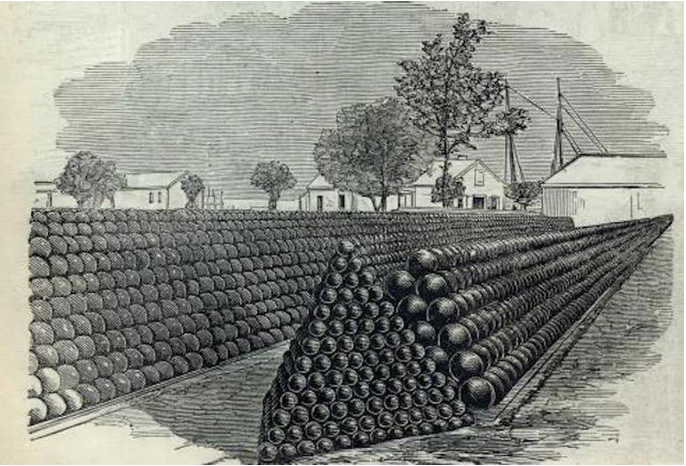
1773: Lagrange proved that among *lattice* packings, the densest packing is hexagonal with packing density $\frac{\pi}{\sqrt{12}}$.

1890 Axel Thue published a proof that this same density is optimal among all packings, but his proof was considered to be incomplete.

1940: L.F. Toth gave the first complete proof for the general case involving all arrangements.

Dimension 3: Another ball game altogether!

The problem actually originated with Cannonballs!



Origin of the problem

Johannes Kepler (1571-1630), conjectured that the face-centered cubic and hexagonal close packing will be the tightest possible.

(Watch : The Best Way to Pack Spheres - Numberphile)

This assertion is since called the **Kepler Conjecture**. It is a part of the 18th problem in Hilbert's list.

History

1831: Gauss proved that the Kepler conjecture is true if the spheres have to be arranged in a regular lattice.

1998-2014: Thomas C. Hales proved Kepler's Conjecture.

Density - Definition

Let $\mathcal{P}$ be a union of congruent balls with disjoint interiors in $\mathbb{R}^n$. The *upper density* of $\mathcal{P}$ is defined as

$$\Delta_{\mathcal{P}} = \limsup_{R\to\infty} \frac{vol(\mathbb{B}^n_R(0) \cap \mathcal{P})}{vol(\mathbb{B}^n_R(0))}$$

And the *sphere packing density* Δ_n is the supremum of upper densities of all sphere packings.

Known results

- $\Delta_1 = 1.$
- $\Delta_2 = \pi/\sqrt{12} = 0.9068.$
- $\Delta_3 = \pi/\sqrt{18} = 0.7404.$

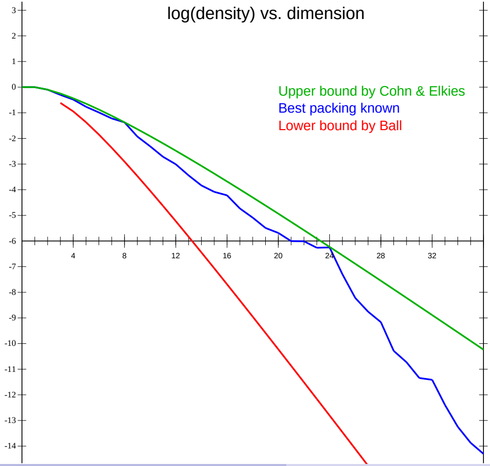
and now,

$$\Delta_8 = \pi^4/384 = 0.254.$$

$$\Delta_{24} = \pi^{12}/12! = 0.00193.$$

Henry Cohn and Noam Elkies

Cohn-Elkies... density upper bounds; graph...dimension 8 and 24.



- Linear Programming Bounds.
- Fourier Analysis, Poisson Summation Formula.
- The Lattice $\mathbb{E}_8$.
- Analytic Number theory, Modular forms.

Fourier Transform

For $f \in L^1(\mathbb{R}^n)$, the Fourier Transform is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x.\xi} dx$$

A beautiful example: The Gaussian.

$$\widehat{e^{-\pi x^2}} = e^{-\pi \xi^2}$$

All functions will be 'nice'

In the following all functions will be assumed to belong to the Schwartz space, i.e. infinitely differentiable, and each function and all its derivatives have *rapid decay*, i.e. faster than any polynomial.

$$\mathcal{S}_{\mathbb{R}^n} = \{f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty \, \forall \text{ multi-indices } \alpha, \, \beta\}$$

Then the Fourier Transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is an isomorphism.

Fourier Series

If f is a 1-periodic function on $\mathbb{R}$, its Fourier coefficients are given by

$$\hat{f}(k) = \int_0^1 f(t)e^{-2\pi ikt}dt$$

Then f has a Fourier series expansion given by,

$$f(t) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ikt}$$

Poisson Summation Formula

For every Schwartz class function f on $\mathbb{R}$, we have,

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{j\in\mathbb{Z}} \hat{f}(j)$$

Proof:

Consider the 1-periodic function

$$f^{\#}(x) = \sum_{k\in\mathbb{Z}} f(x+k)$$

Then $f^{\#}$ has a Fourier series, with Fourier coefficients

$$\widehat{f^{\#}}(j) = \int_0^1 \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi ijx} = \int_{\mathbb{R}} f(x)e^{2\pi ijx}dx = \hat{f}(j)$$

hence

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{j\in\mathbb{Z}} \hat{f}(j)e^{2\pi ijx}$$

Now evauate at $x = 0$.

More generally, for any discrete Lattice Λ in $\mathbb{R}^n$

$$\sum_{k\in\Lambda} f(k) = \frac{1}{vol(\mathbb{R}^n/\Lambda)} \sum_{j\in\Lambda^*} \hat{f}(j)$$

Here $\mathbb{R}^n/\Lambda$ is the fundamental cell of the lattice Λ.

Cohn-Elkies Theorem

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Schwartz class function satisfying:

1. $f(0) = 1 = \hat{f}(0)$ *and* $f(x) \leq 0$ *for all* $\|x\| \geq r$ *for some* $r > 0$.
2. $\hat{f}(\xi) \geq 0$ *for all* $\xi \in \mathbb{R}^n$.

Then the sphere packing density satisfies

$$\Delta_n \leq vol(\mathbb{B}^n_{r/2})$$

Proof

It is enough to prove the theorem for periodic packings. In the general case we can approximate the density Δ_n by the density of periodic packings.

Consider a periodic packing obtained by taking spheres of radius $r/2$ centered at the translates of a lattice Λ by vectors $t_1, t_2, .. t_N$, where r is the minimal distance. Then the density is given by

$$\Delta = \frac{N vol(B_{r/2})}{vol(\mathbb{R}^n/\Lambda)}$$

We will prove that $vol(\mathbb{R}^n/\Lambda) \geq N$. By Poisson summation formula,

$$\sum_{j,k=1}^{N} \sum_{x \in \Lambda} f(t_j - t_k + x) = \frac{1}{vol(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y) \mid \sum_{j=1}^{N} e^{2\pi i <y, t_j>} |^2$$

The left side is less than $Nf(0)$ and the right side is greater than $\frac{N^2}{vol(\mathbb{R}^n/\Lambda)} \hat{f}(0)$, which implies that $vol(\mathbb{R}^n/\Lambda) \geq N$.

Hence the density is at most $vol(B_{r/2})$.

We can assume that the function f in this theorem is a radial function.

For a radial function $\hat{\hat{f}} = f$. Let $f^+ = \frac{f+\hat{f}}{2}$ and $f^- = \frac{f-\hat{f}}{2}$; then f^+ is an eigenfunction of the Fourier transform with eigenvalue $+1$ and f^- with eigenvalue -1

Dimension $n = 8$

In $\mathbb{R}^8$, there is a lattice $\mathbb{E}^8$ given as

$$\mathbb{E}_8 = \{(x_j) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8 : \sum_j x_j \equiv 0 (mod\, 2)\}$$

Facts:

- $\mathbb{E}^{8^*} = \mathbb{E}^8$
- $Vol(\mathbb{R}^8/\mathbb{E}^8) = 1$
- The minimal vector length in $\mathbb{E}^8$ is $\sqrt{2}$. hence the $\mathbb{E}^8$ sphere packing uses balls of radius $r = \frac{\sqrt{2}}{2}$. The $\mathbb{E}^8$ sphere packing has density $Vol(B^8_{1/\sqrt{2})}) = \frac{\pi^4}{4!2^4} = \frac{\pi^4}{384}$
- the vector lengths of elements of $\mathbb{E}^8$ are of the form $\sqrt{2k},\ k \in \mathbb{Z}$.

Search for a magic function

To prove that $\mathbb{E}^8$ gives the densest packing in dimension 8, we need to find a radial Schwartz class function f on $\mathbb{R}^8$ satisfying the following:

1. $f(0) = 1 = \hat{f}(0)$ and $f(x) \leq 0$ for all $\|x\| \geq \sqrt{2}$
2. $\hat{f}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^8$.
3. $f(\sqrt{2k}) = \hat{f}(\sqrt{2k}) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$, and for $k = 2, 3,$ ... these roots must have order 2, so as to avoid any sign changes, and a root of order 1 at $\sqrt{2}$.

   (If f satisfies (1) and (2) above, we get by Poisson summation formula:

   $$1 = f(0) \geq \sum_{x \in \mathbb{E}^8} f(x) = \sum_{x \in \mathbb{E}^8} \hat{f}(x) \geq \hat{f}(0) = 1$$

   This is possible only of f and $\hat{f}$ vanish on $\mathbb{E}^8 \setminus \{0\}$)

Where do we look for such 'magic' functions?

Elkies had this to say:

"We gave many talks and even convened a conference or two to disseminate the problem in the hope that such a function was known or could easily be found if we only know in which mathematical field to look, but found nothing."

... and then Viazovska "pulled a Ramanujan!" using modular forms.

Viazovska

The function f we are looking for is radial, so that $\hat{\hat{f}} = f$. Then we can write $f = f^+ + f^-$, where f^+ and f^- are eigenfunctions of the Fourier transform with eigenvalues ± 1.

There are ways to generate eigenfunctions of the Fourier transform. Viazovska's idea is to use Laplace transforms. Recall that the Gaussian is an eigenfunction.

Let h be a 'nice' function, and define

$$g(x) = \int_0^\infty e^{-t\pi|x|^2} h(t)\, dt$$

Then

$$\hat{g}(\xi) = \int_0^\infty e^{-t\pi|\xi|^2} t^{n/2-2} h(1/t) dt$$

Now if h satisfies $h(1/t) = \epsilon t^{2-n/2} h(t)$, then $\hat{g} = \epsilon g$. Write $h(t) = \phi(it)$, then the condition becomes

$$\phi(-1/z) = z^k \phi(z)$$

with $k = 2 - n/2$.

Modular forms ?!?

In dimension 8 Viazovska puts in an explicit factor to deal with the required zeros of f. The eigenfunctions have the from

$$\sin^2(\pi|x|^2/2)\int_0^\infty h(t)e^{-t\pi|x|^2dt}$$

for some function $h(t)$. What is problematic here is that the roots of the first factor are of order 2 for $|x| = \sqrt{2k}$, $k = 1, 2, ...$ and fourth order at $x = 0$, whereas we need no zero at $x = 0$, and a zero order 1 at $|x| = \sqrt{2}$. The second factor would have to compensate for this. The search then is what choices of h will produce eigenfunctions. Viazovska finds an explicit function for this, which is constructed using modular forms.

Fourier Interpolation on $\mathbb{R}$

Using the integral transform technique, Radchenko and Viazovska constructed an interpolation formula for even Schwartz functions on $\mathbb{R}$

Theorem

There exist even Schwartz functions $a_n : \mathbb{R} \to \mathbb{R}$ such that for every even Schwartz function $f : \mathbb{R} \to \mathbb{R}$, we have

$$f(x) = \sum_{0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{0}^{\infty} \hat{a}_n(x) \hat{f}(\sqrt{n}), \ \forall x \in \mathbb{R}$$

where the right-hand side converges absolutely.

The interpolation bases are constructed explicitly.

References.

- Henry Cohn. A Conceptual Breakthrough in Sphere Packing. 2016.
- Maryna Viazovska. The Sphere packing problem in dimension 8. 2016.