# The mathematics of phase transition : work of Hugo Duminil-Copin

Fields medal symposium 2022

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The man and the mathematician

The mathematics

Some anecdotes

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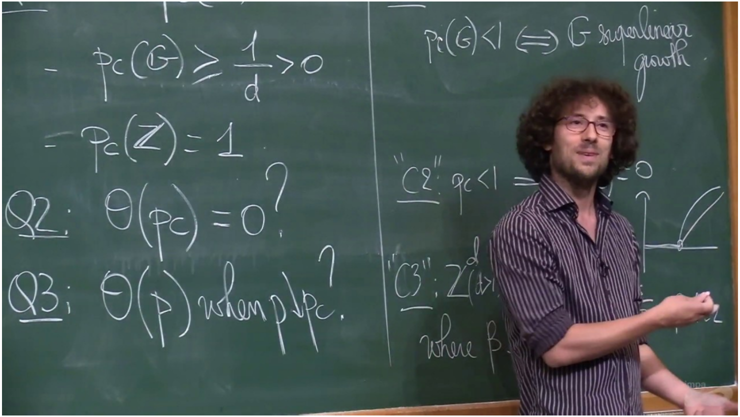
# A snapshot of the mathematician




(Image source: https://www.youtube.com/watch?v=W-4Gp-n2PLM)

# Promising start and a meteoric rise

**2006 – 2008** ÉNS de Paris. Agrégation de mathématiques.

**2006 – 2007** Master. Université Paris XI.

**2003 – 2005** MPSI and MP, Lycée Louis-Le-Grand, Paris.

# Promising start and a meteoric rise

**2016 – present** Permanent Professor, IHÉS.

**2014 – present** Full Professor, Université de Genève.

**2013 – 2014** Assistant Professor, Université de Genève.

**2011 – 2012** Postdoc, Université de Genève.

**2008 – 2011** PhD, Université de Genève (advisor: S. Smirnov).

## Many awards and the ONE

**2022** **Fields medal**.

**2019** Dobrushin prize, Member of Academia Europaea.

**2018** Invited speaker, ICM Rio.

**2017** Loeve Prize, New Horizons Prize in Mathematics, Grand Prix Jacques Herb. de l'Acad. des Sciences.

**2016** Prize of the European Mathematical Society.

**2015** Early Career Award of the International Association of Mathematical Physics, Cours Peccot du Collège de France.

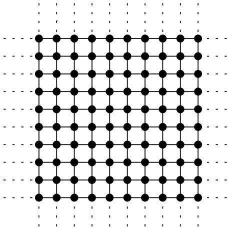
**2012 – 2013** Oberwolfach Prize, Rollo Davidson Prize (with V. Beffara).

## The mathematics of phase transition

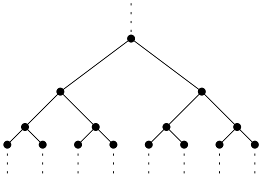
"......solving longstanding problems in the probabilistic theory of phase transitions in statistical physics, especially in dimensions three and four" (Fields medal citation)

" Everything is easier, streamlined. The results are stronger. ... The whole understanding of these physical phenomena has been transformed. ... Basically half of the main open questions were solved by Hugo" (Wendelin Werner commenting on Hugo's contribution to percolation theory)

# What is Percolation?



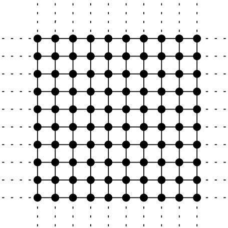
$\mathbb{Z}^2$



$\mathbb{T}_3$

Consider a connected graph $\mathcal{G}$, e.g. the square lattice $\mathbb{Z}^2$ or a tree

# What is Percolation?



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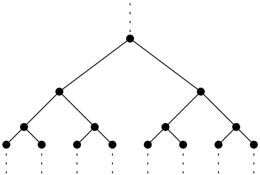
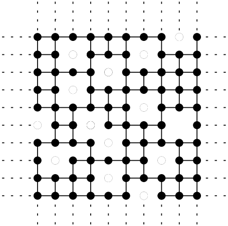


$\mathbb{T}_3$

Let $p$ be a parameter in $[0, 1]$ (the so-called density)

# What is Percolation?



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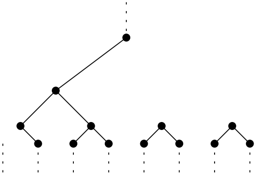
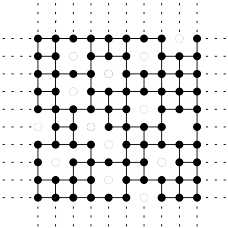


$\mathbb{T}_3$

Declare every site to be open with probability $p$ and closed otherwise

# What is Percolation?



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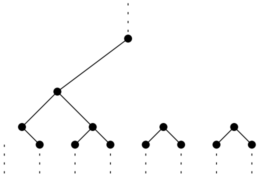
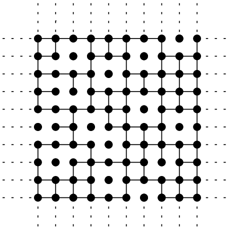


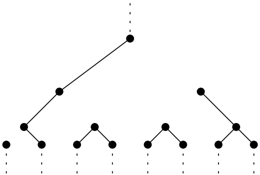
$\mathbb{T}_3$

This model is called site percolation

# What is Percolation?



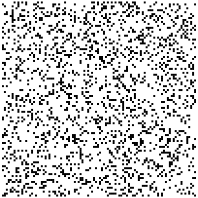
$\mathbb{Z}^2$



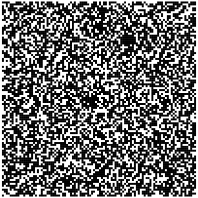
$\mathbb{T}_3$

In a different version (bond percolation) we open or close edges

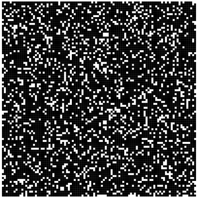
# How does it look like?



(a) $p = 0.2$



(b) $p = 0.6$



(c) $p = 0.8$

# Mathematical representation

We encode a percolation configuration by

$$\omega = (\omega_v : v \in \mathcal{G}) \in \{0,1\}^{\mathcal{G}}$$

where

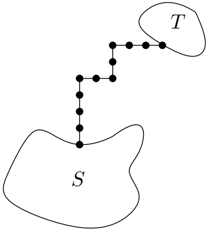
$$\omega_v := \begin{cases} 1 & \text{if } v \text{ is open} \\ 0 & \text{if } v \text{ is closed} \end{cases}$$

# Mathematical representation

The probability measure on the space $\Omega$ of percolation configurations for density $p$ is given by

$$\mathsf{P}_p := \prod_{v \in \mathcal{G}} \left( p\delta_1 + (1-p)\delta_0 \right)$$

# Mathematical representation



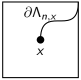

Basic events of interest:

$$\{S \leftrightarrow T\} := \{S, T \subset G \text{ are connected by an open path}\}$$

# The first question: does it percolate?

The quantity to look at is the one-arm probability:

$$\theta_{n,x}(p) := \mathrm{P}_p \Bigg[$$




$$\Bigg] = \mathrm{P}_p[x \leftrightarrow \partial\Lambda_{n,x}]$$

Notice that $\theta_{n,x}(p) \searrow \theta_x(p) := \mathrm{P}_p[x \leftrightarrow \infty]$ as $n \to \infty$

Also notice that $\theta_x(p) > 0$ if and only if $\theta_y(p) > 0$ for all $y \in \mathcal{G}$

# The function $\theta_x(p)$

Can we compute $\theta_{n,x}(p)$ explicitly? Let's make an attempt!

$$\boxed{\theta_{n,x}(p) = \sum_{\omega \in \{0 \longleftrightarrow \partial\Lambda_{n,x}\}} \mathrm{P}_p[\omega]}$$

$$\mathrm{P}_p[\omega] = p^{\sum_{y\in\Lambda_{n,x}} \omega_y}(1-p)^{\sum_{y\in\Lambda_{n,x}} 1-\omega_y} \quad \text{(Easy!)}$$

Evaluate the sum over admissible configurations (VERY difficult!)

# The function $\theta_x(p)$

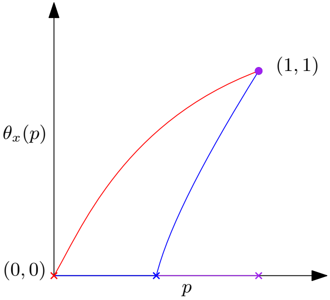
One can compare this with the difficulty of computing the partition function for models in statistical physics (e.g. the Ising model)

In fact this analogy is far from being artificial!

# The function $\theta_x(p)$: the phase diagram

The following properties are not difficult to see from the definition:

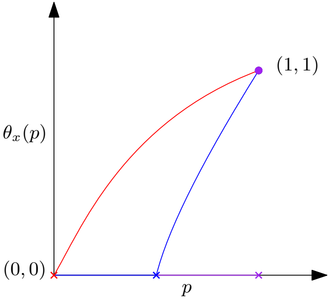
$\theta_x(p)$ is non-decreasing in $p$, $\theta(0) = 0$ and $\theta(1) = 1$



# The function $\theta_x(p)$: the critical density

Therefore there exists a critical parameter $p_c = p_c(\mathcal{G})$ defined as:

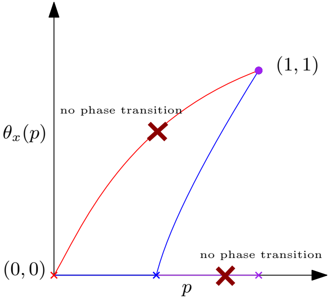
$$p_c := \sup\{p \in [0,1] : \theta(p) = 0\}$$



# The function $\theta_x(p)$: existence of phase transition

Notice that $p_c$ can be *a priori* 0 or 1 (no phase transition)

$$p_c := \sup\{p \in [0,1] : \theta(p) = 0\}$$



# Existence of phase transition

$p_c$ is always positive! In fact, $p_c(\mathcal{G}) \geq \frac{1}{\max_{v \in \mathcal{G}} \deg(v)}$

The proof is essentially an energy-entropy type argument

# Existence of phase transition

$p_c$ is always positive! In fact, $p_c(\mathcal{G}) \geq \frac{1}{\max_{v \in \mathcal{G}} \deg(v)}$

The proof is essentially an energy-entropy type argument

$p_c(\mathbb{Z}^d) < 1$ for all $d \geq 2$ whereas $p_c(\mathbb{Z}) = 1$

Compare with the fact that the Ising model has no phase transition in dimension 1! (Ising'25)

# Existence of phase transition: general graphs

Deriving a generic condition on $\mathcal{G}$ ensuring $p_c(\mathcal{G}) < 1$ is non-trivial

A natural guess would be that some "suitable" notion of dimension of $\mathcal{G}$ is strictly bigger than 1 (Benjamini-Schramm'96)

# Existence of phase transition: general graphs

**Theorem (Duminil-Copin, G., Raoufi, Severo and Yadin 2018)**

*Let $\mathcal{G}$ be a bounded degree, quasi-transitive graph with super-linear growth, then $p_c(\mathcal{G}) < 1$.*

Typical examples include Cayley graphs of finitely generated group

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# Thank you all!