Chance and Chaos

How to predict the unpredictable

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Tata Institute of Fundamental Research, Mumbai
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Yusuf Hamied Visiting Professorship

Chance and chaos: how to predict the unpredictable

- The butterfly effect

- Chance and randomness: an experiment

- Kneeding dough – a model for chaos?

- How to win the lottery...

The butterfly effect

A butterfly's wing flap in Brazil . . . . sets off a tornado in Texas

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, 139th MEETING

Subject..........................Predictability; Does the Flap of a Butterfly's wings in Brazil Set Off a Tornado in Texas?

Author..........................Edward N. Lorenz, Sc.D.
Professor of Meteorology

Address..........................Massachusetts Institute of Technology
Cambridge, Mass. 02139

Time..............................10:00 a.m., December 29, 1972

Place.............................Sheraton Park Hotel, Wilmington Room

Program..........................AAAS Section on Environmental Sciences
New Approaches to Global Weather: GARP
(The Global Atmospheric Research Program)

Convention Address...............Sheraton Park Hotel

RELEASE TIME
10:00 a.m., December 29

Lest I appear frivolous in even posing the title question, let alone suggesting that it might have an affirmative answer, let me try to place it in proper perspective by offering two propositions.

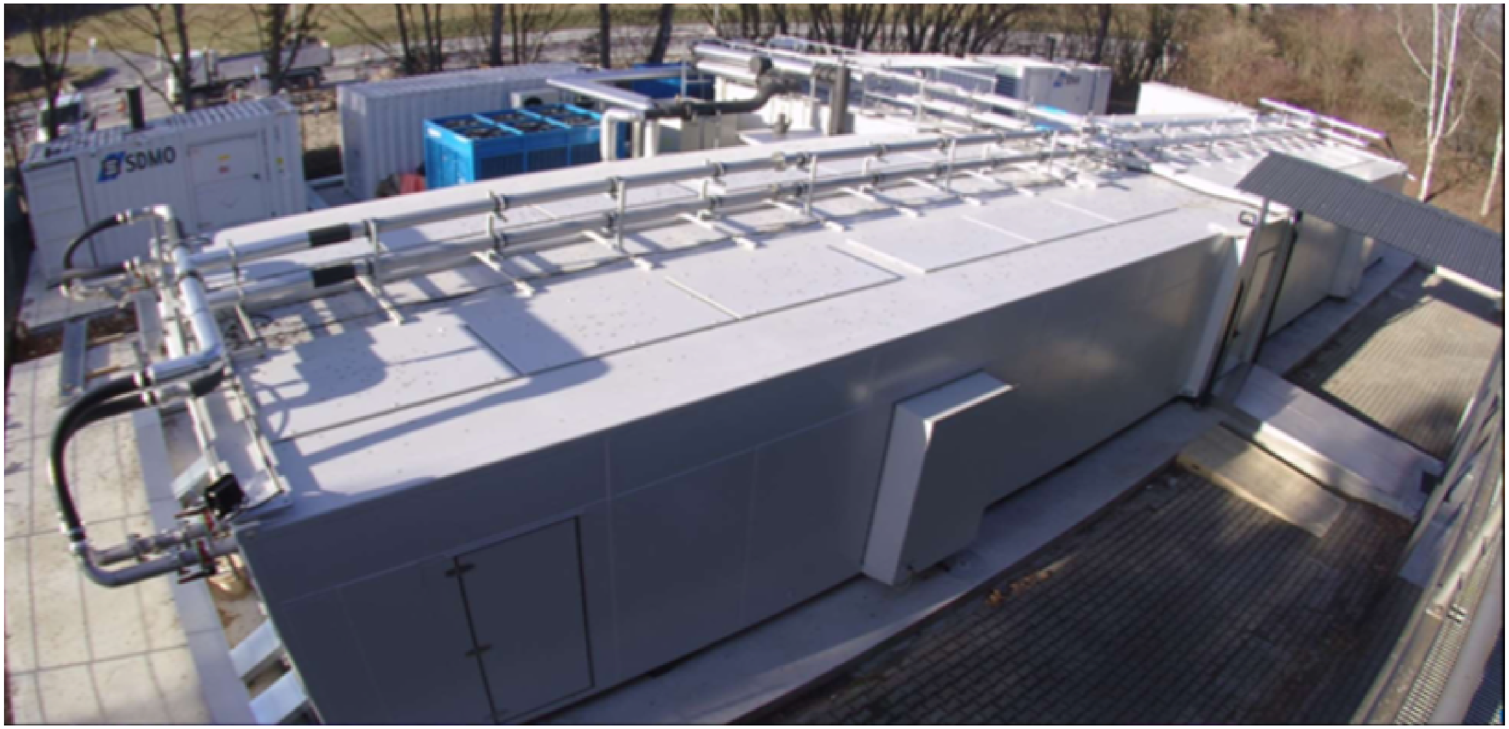
1. If a single flap of a butterfly's wings can be instrumental in generating a tornado, so also can all the previous and subsequent flaps of its wings, as can the flaps of the wings of millions of other butterflies, not to mention the activities of innumerable more powerful creatures, including our own species.

2. If the flap of a butterfly's wings can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado.

More generally, I am proposing that over the years minuscule disturbances neither increase nor decrease the frequency of occurrence of various weather events such as tornados; the most that they may do is to

Edward Lorenz (1917-2008)

Climate modelling and extreme weather prediction



New supercomputer in Bristol, starting service end 2023:
one of world's most energy efficient and low carbon supercomputers

Met Office

University of BRISTOL
Elizabeth Blackwell Institute for Health Research

Cabot Institute for the Environment
University of Bristol
Many minds, one mission

The deterministic universe, chance and chaos

It is a fundamental hypothesis of science that, if we know the state of a system, we can use the laws of physics to predict its future evolution.

A deterministic universe does not explain, however, the many unpredictable, randomly occuring events we experience in our day to day lives.*

In the rest of this lecture we will explore this subject and in particular resolve the apparent paradox of the existence of randomness and chaos in a deterministic universe.

Finally we will learn that chaos can be a useful tool!

*It also cannot explain free will and the existence of evil: but this is for another lecture.

Unpredictability

There are three features that can make a system unpredictable:

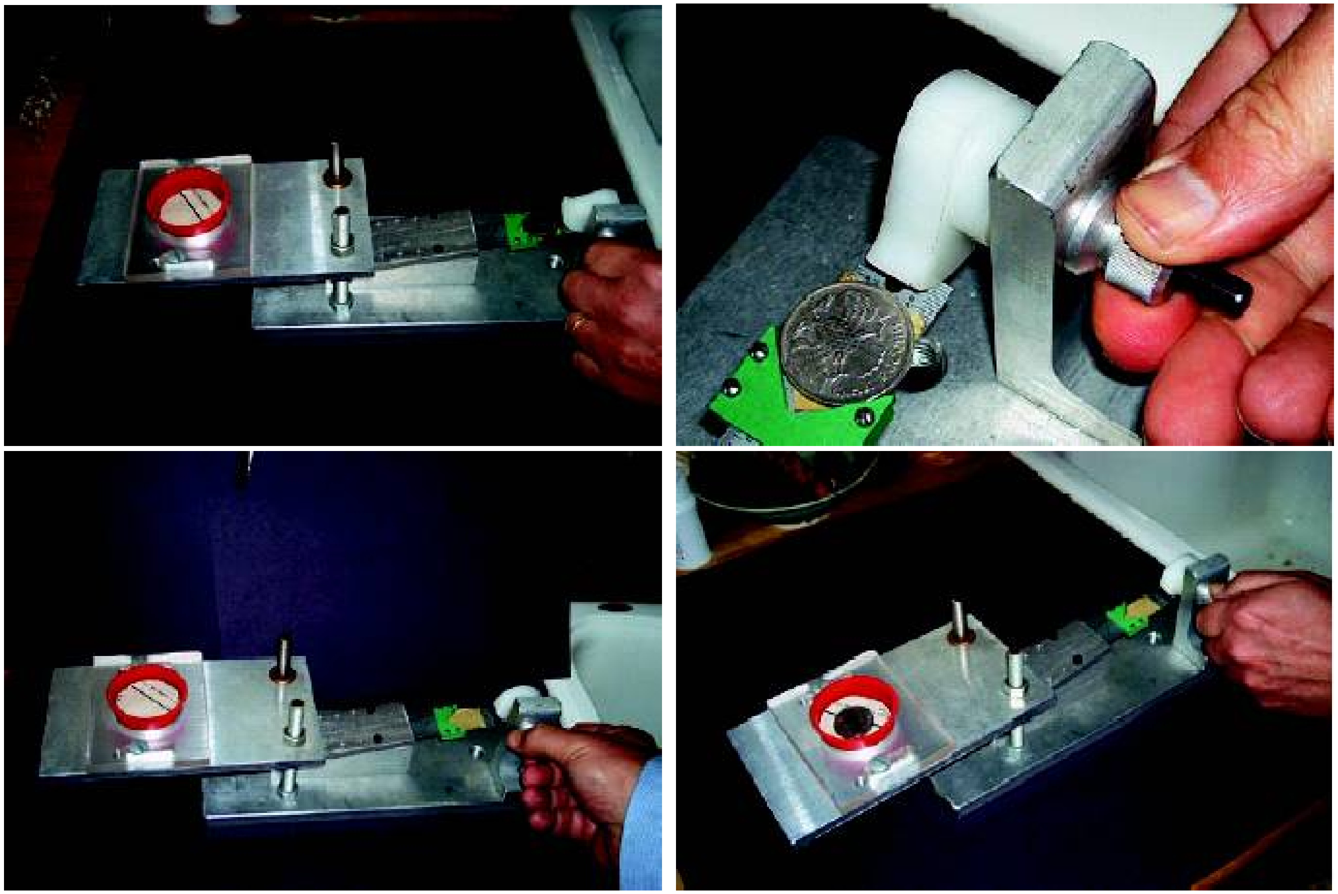
- Absence of an accurate mathematical model
- Lack of sufficiently detailed information on the current state of the system
- Extremely sensitive dependence on initial conditions (“chaos”)

Example: The fair coin toss

A coin toss is one of the most popular ways of generating a random outcome: ***heads*** or ***tails***.

According to Newton's laws of motion, the knowledge of the initial position, velocity and spin of a coin precisely determines the outcome of a coin toss—*head* or *tail*. So where does randomness enter?

The perfect coin tosser



from: P. Diaconis et al., Dynamical Bias in the Coin Toss, SIAM Review '07

For more on this google *Diaconis + numberphile + coin toss*

Let's do an experiment with dice!*

Imagine you are a butterfly in Brazil. Roll the dice and see what you do next

1	Butterfly rests	
2	Butterfly eats	
3	Butterfly sleeps	
4	Butterfly thinks	
5	Butterfly reads a book	
6	Butterfly flaps its wings	

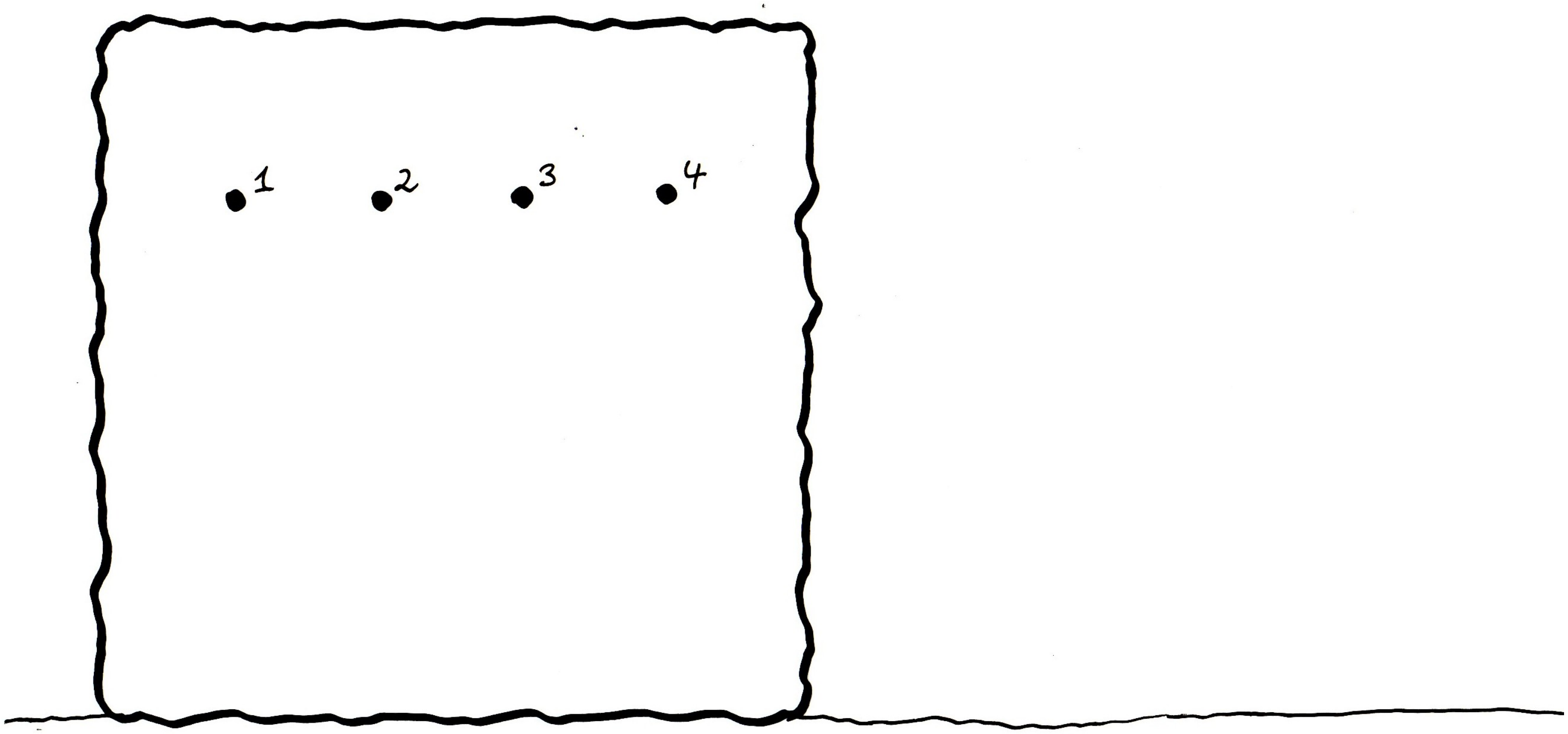
*with assistance from Rashmita Hore and Sundara Narasimham

The law of large numbers

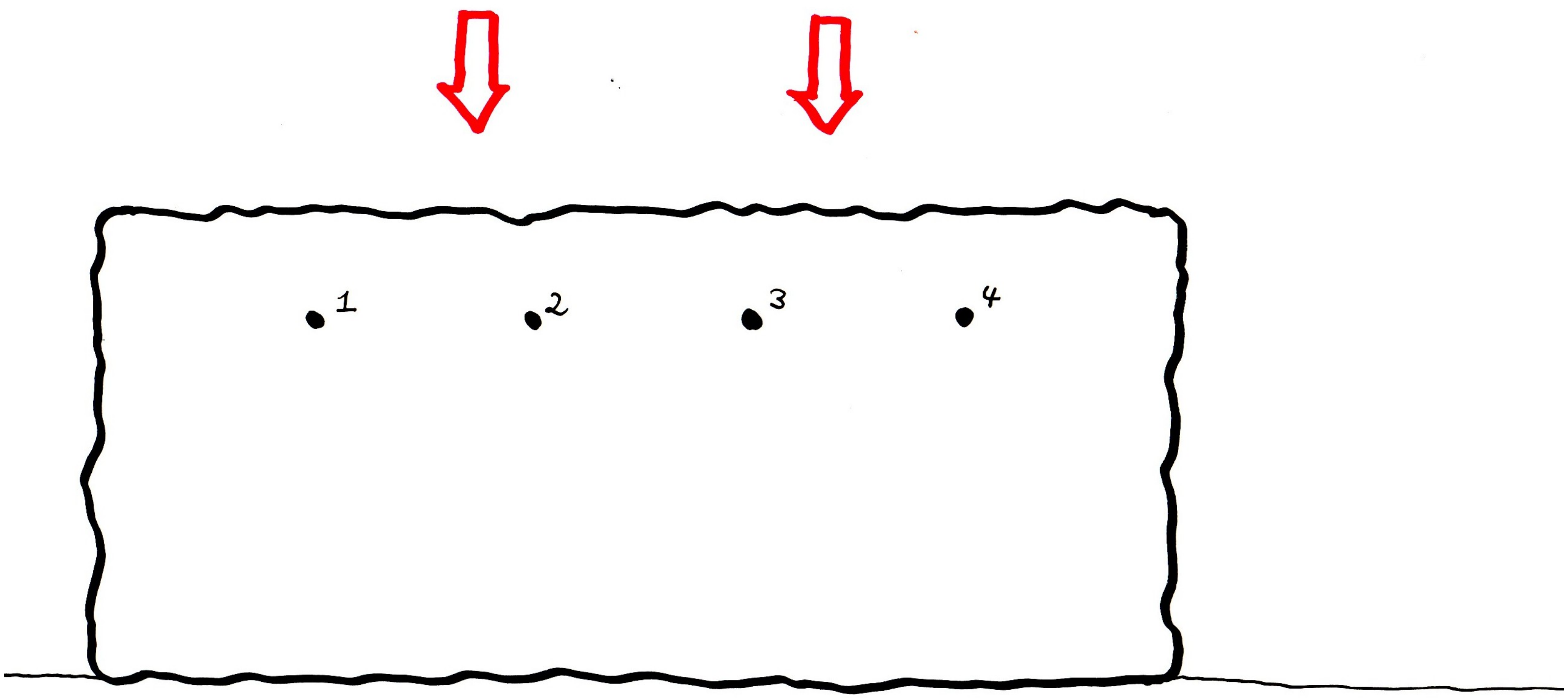
The law of large numbers tells us that with probability tending to one, the outcomes 1, 2, 3, 4, 5 or 6 appear with the same frequency 1/6 if we have a large number of independent dice.

So, up to a tiny error, we can *predict* that there will be 1/6 of all butterflies flapping their wings – even though it is impossible to predict the behaviour of each individual butterfly.

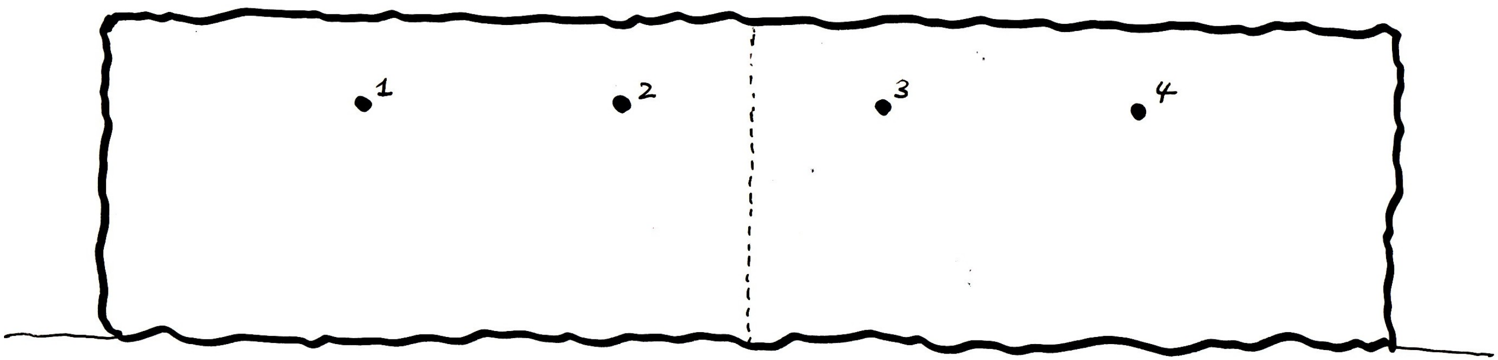
Kneading dough: a simple model for generating randomness



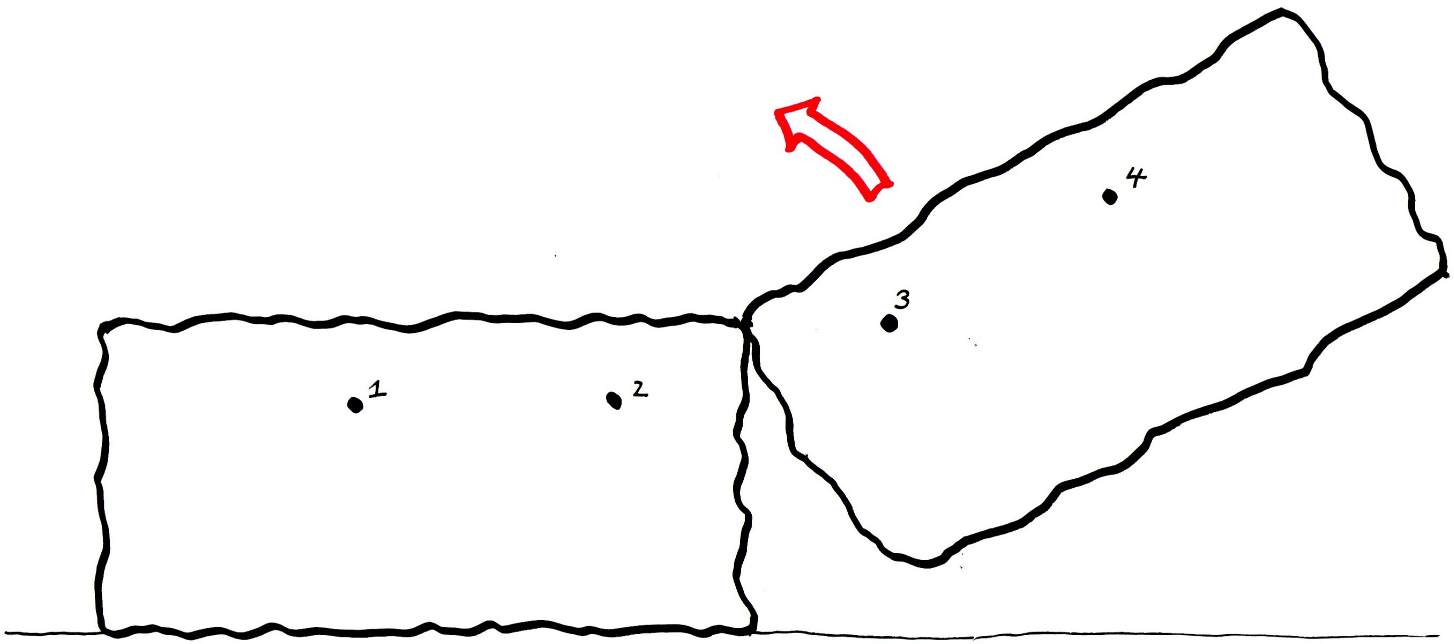
Step 1: Place test raisins in the dough.



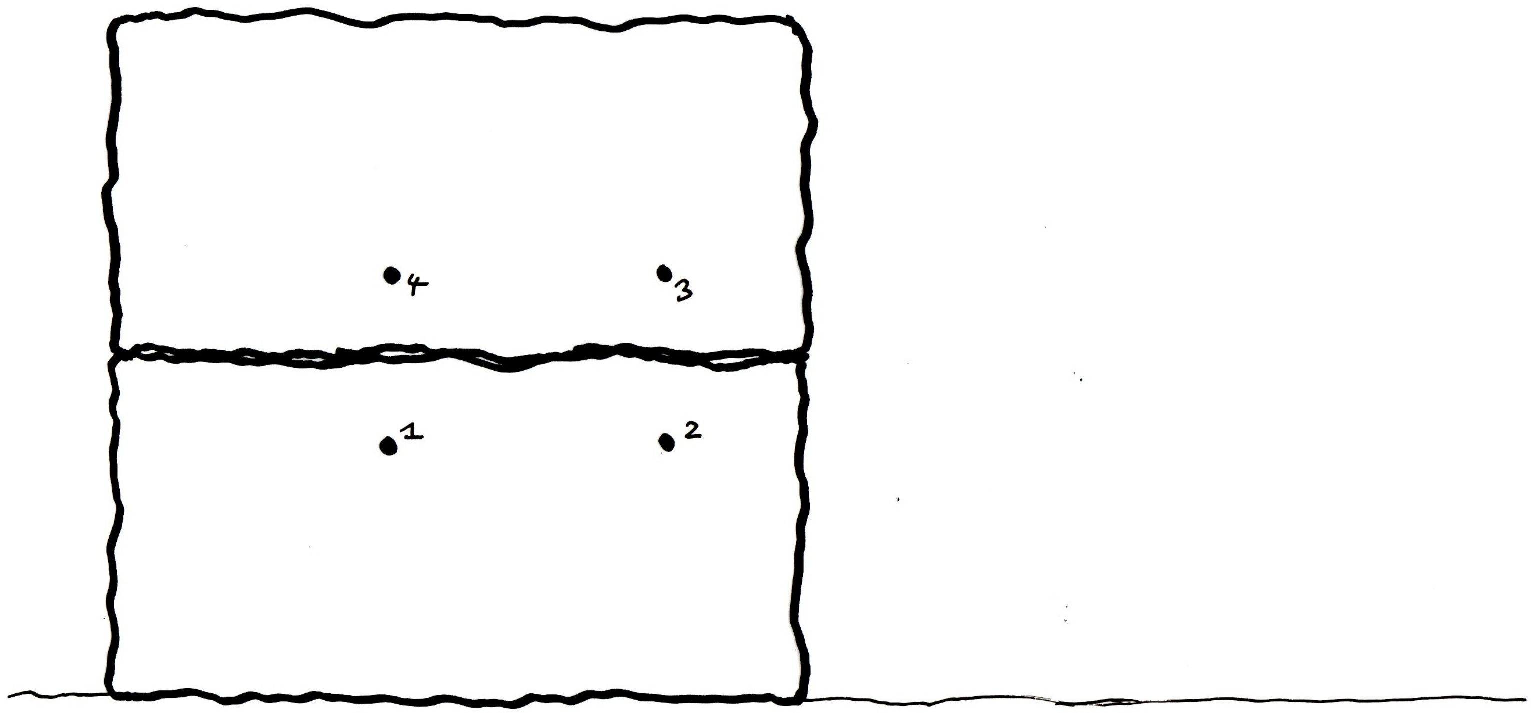
Step 2: Press the dough until . . .



. . . it has half the height. Note that the distance between the raisins has doubled.



Step 3: Fold half of the stretched dough over to obtain original shape.



Step 4: Continue repeating Steps 2 and 3.

Exponential sensitivity, and the amplification of randomness

Due to the doubling of the distance after each iteration, even two very close raisins will move apart rapidly, since

$$2 \times 2 \times 2 \times \cdots \times 2 = 2^n$$

grows exponentially fast in the number of iterations n.

We have thus constructed a system with exponentially sensitive dependence on initial data. But how "random" is it? To see how it compares with a fair coin toss, let us say that we have *heads* if our test raisin is in the top half of the bread loaf, and *tails* when it is in the bottom half.

The following mathematical theorem tells us that, given any ever-so-small inaccuracy in the initial position of the raisin, after a sufficiently large number of iterations, our system produces a fair coin toss.

A mathematical theorem

Assume a test raisin is placed in the center of the bread dough with an accuracy of 1%, relative to the size of the dough sample. Then, after kneading the dough n times,

$$\text{the probability of heads is} \quad \frac{1}{2} \pm \frac{100}{2^n}.$$

n	9	10	11	12
$\frac{1}{2^n}$	0.001953125	0.0009765625	0.00048828125	0.000244140625
$\frac{100}{2^n}$	0.1953125	0.09765625	0.048828125	0.0244140625

That is, kneading dough genrates (up to an exponentially small error) the same random events as an unbiased coin toss.

. . . in summary:

In chaotic systems, a tiny amount of uncertainty in the initial data produces almost perfect randomness after a very short time.

That is, we have ***exponential amplification of randomness***.

How can we turn this observation into a useful tool?

The grand challenge

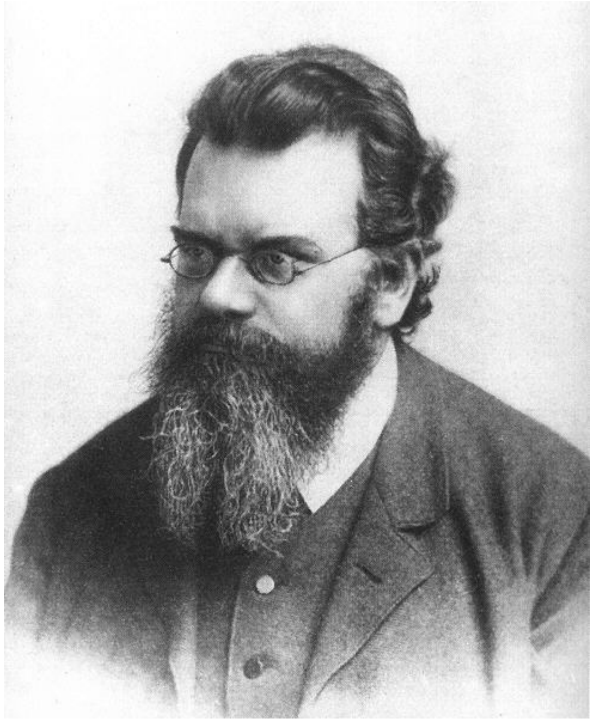
Explain the equations of fluid dynamics such as the Navier-Stokes equations ***from fundamental principles*** e.g. Newton's laws of motion, or quantum mechanics

We do not know (yet) how to do this!

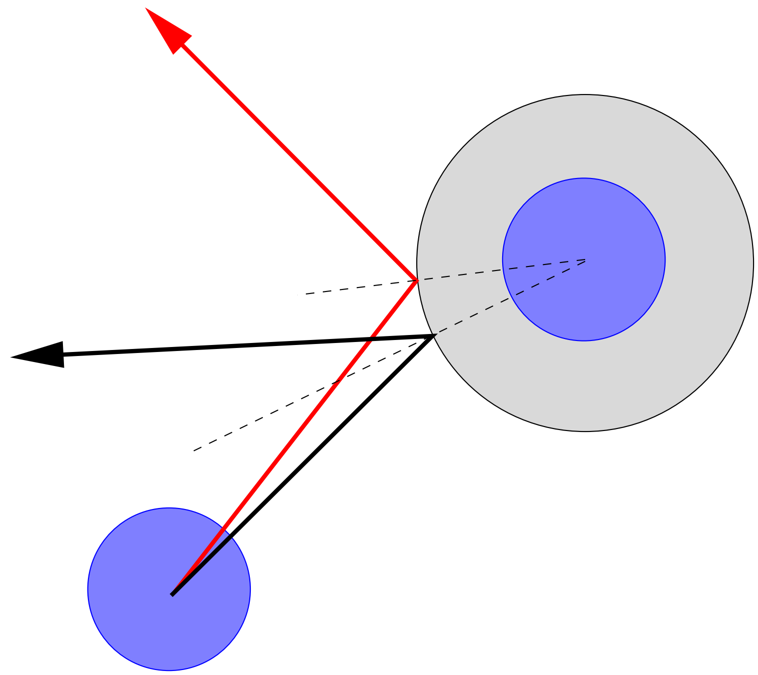
Helpful chaos: Boltzmann's statistical mechanics

Boltzmann proposed to explain the motion of a gas cloud by using the dynamics of microscopic particles—atoms and molecules, whose existence was highly disputed during Boltzmann's lifetime.

In his 1872 paper, Boltzmann postulated the famous *Boltzmann equation*, assuming that the dynamics of the colliding gas molecules is chaotic.



Ludwig Boltzmann (1844-1906)



The Boltzmann gas: The initial angle between trajectories doubles after each collision between two spheres. We have exponentially sensitive dependence on initial conditions!

Helpful chaos: Boltzmann's statistical mechanics

The first rigorous justification of the Boltzmann equation was given by Oscar Lanford in 1975 for the dynamics over very short time intervals. The problem for the more realistic macroscopic time scales is still wide open.

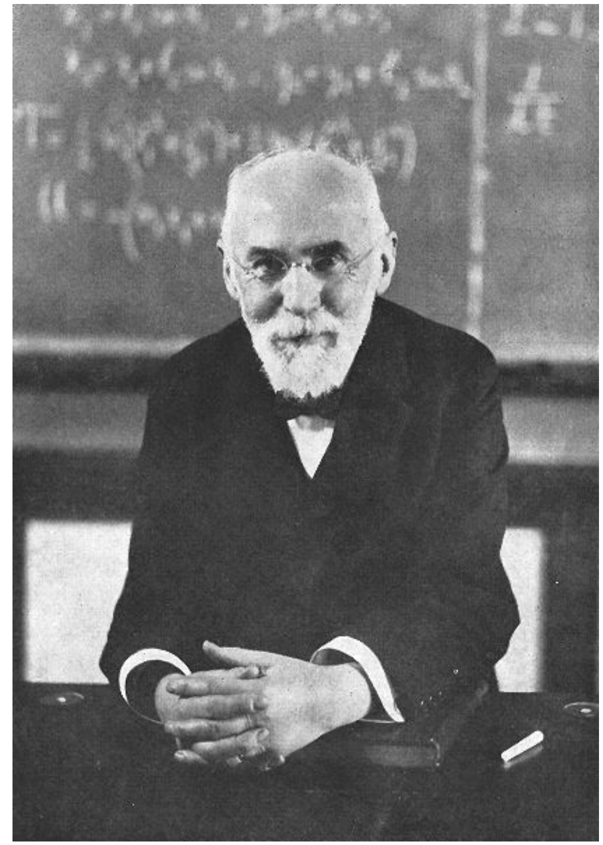
One of the new stars in the field is Laure Saint-Raymond. She and collaborators analysed the dynamics of the Boltzmann gas near equilibrium and also found a derivation of the Navier-Stokes equation from Boltzmann's equation.

Laure Saint-Raymond (1975*)

The Lorentz gas

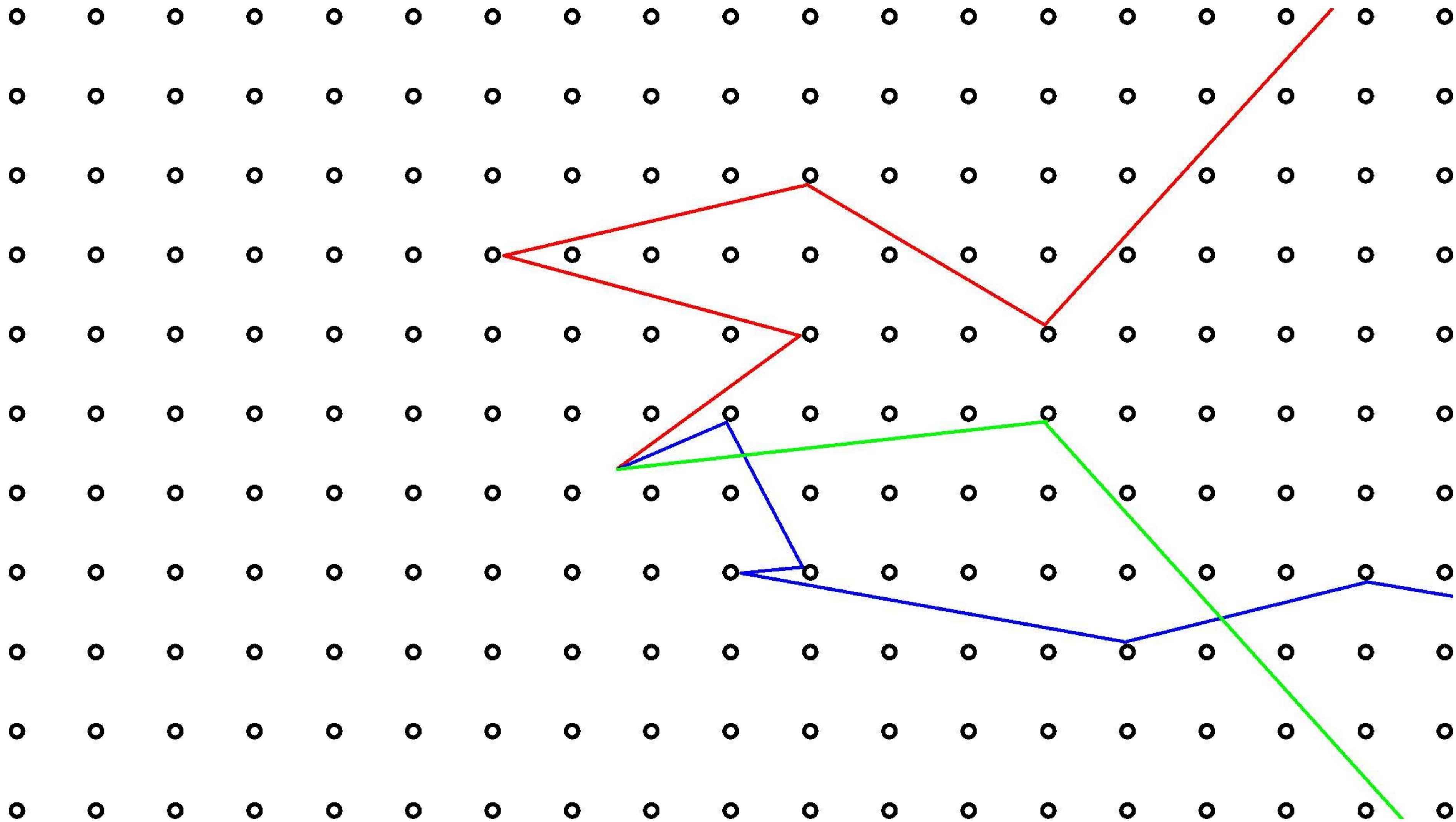
In an attempt to describe the evolution of a dilute electron gas in a metal, Lorentz proposed in 1905 a model, where the heavier atoms are assumed to be fixed, whereas the electrons are interacting with the atoms but not with each other. For simplicity, Lorentz assumed (like Boltzmann) that the atoms can be modeled by elastic spheres.

The Lorentz gas is still one of the iconic models for chaotic transport properties.



Hendrik Lorentz (1853-1928)

Nobel Prize in Physics 1902



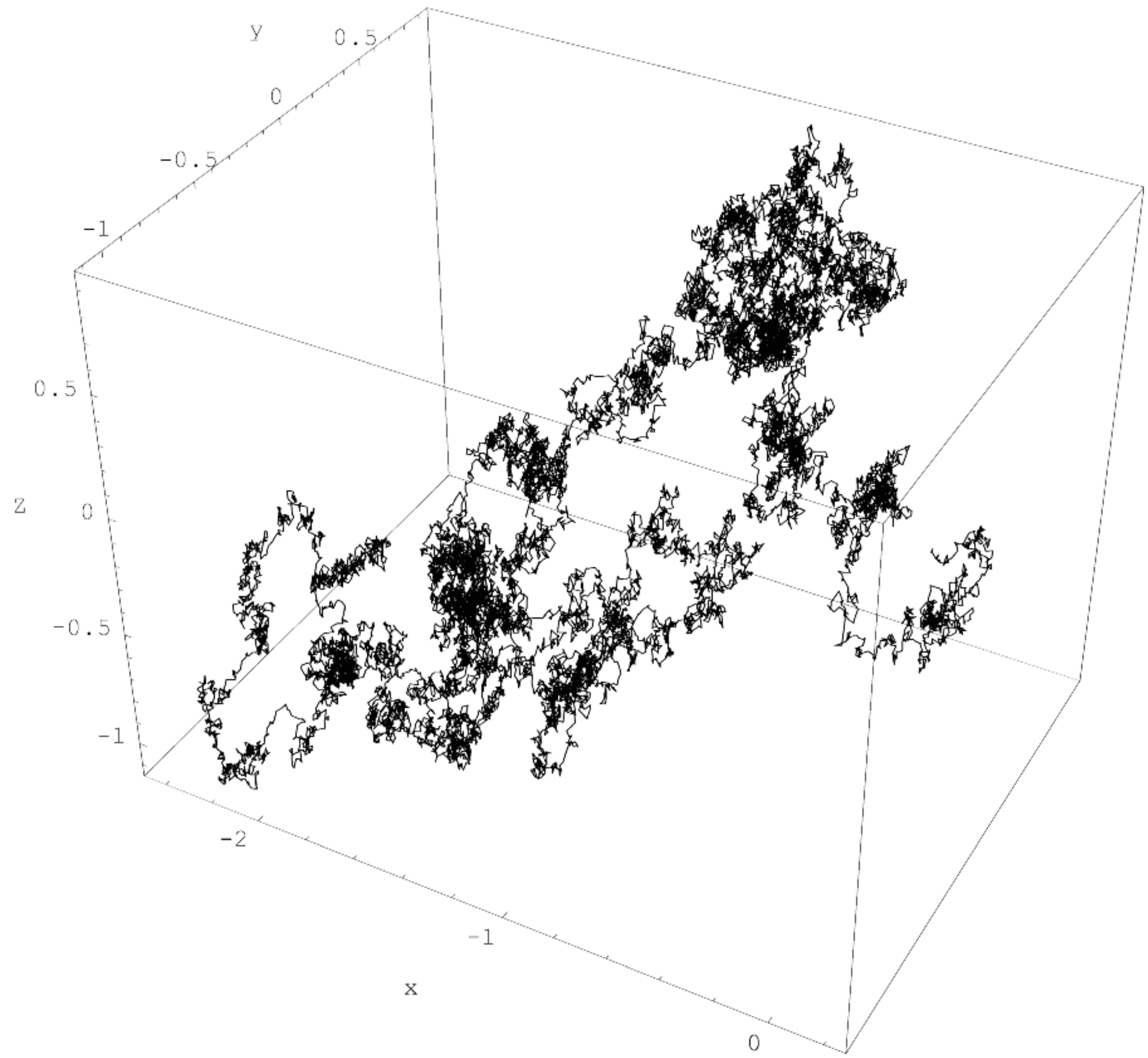
The Lorentz gas in a crystal

The Lorentz gas and Brownian motion

Yakov Sinai (Princeton University) is one of the pioneers in understanding the chaotic properties of the Lorentz gas. He proved in 1980, jointly with Leonid Bunimovich, that *in the limit of long times the dynamics appears as random as Brownian motion*.

Yakov Sinai (1935*)

Abel Prize 2014



A typical Brownian path in three dimensional space.

Visibility in a forest

A basic problem in understanding the Lorentz gas is concerned with the distribution of the *free path length*, which is the distance an electron travels between consecutive collisions.

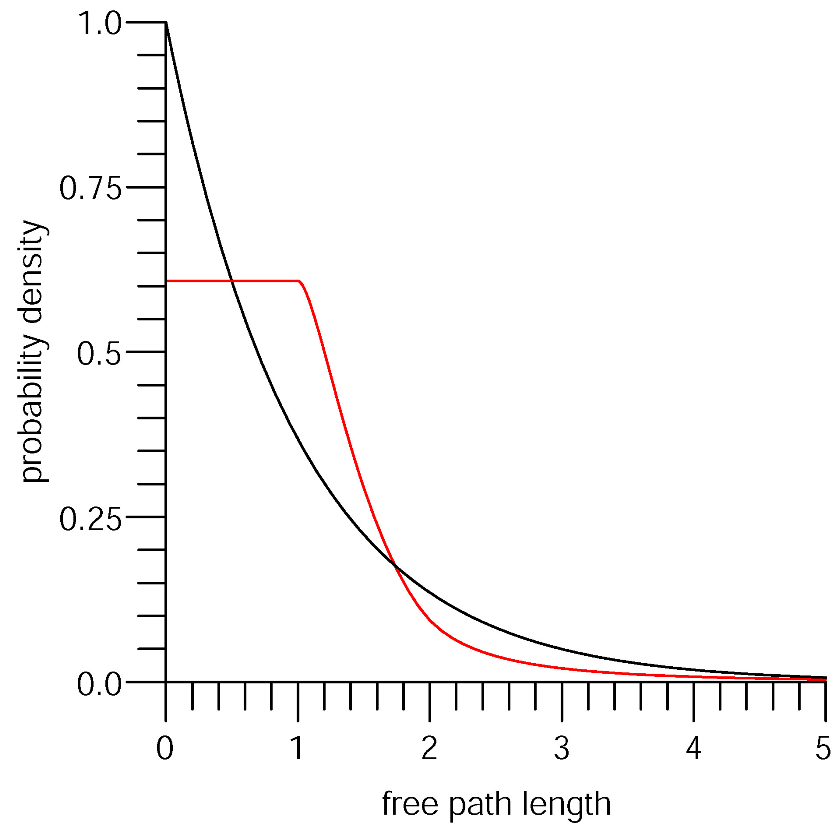
This leads to natural problems in probability theory and number theory, respectively, which, in the two-dimensional case were paraphrased by Pólya as the *problem of visibility in a forest*.

George Pólya (1887-1985)

The distribution of free path lengths

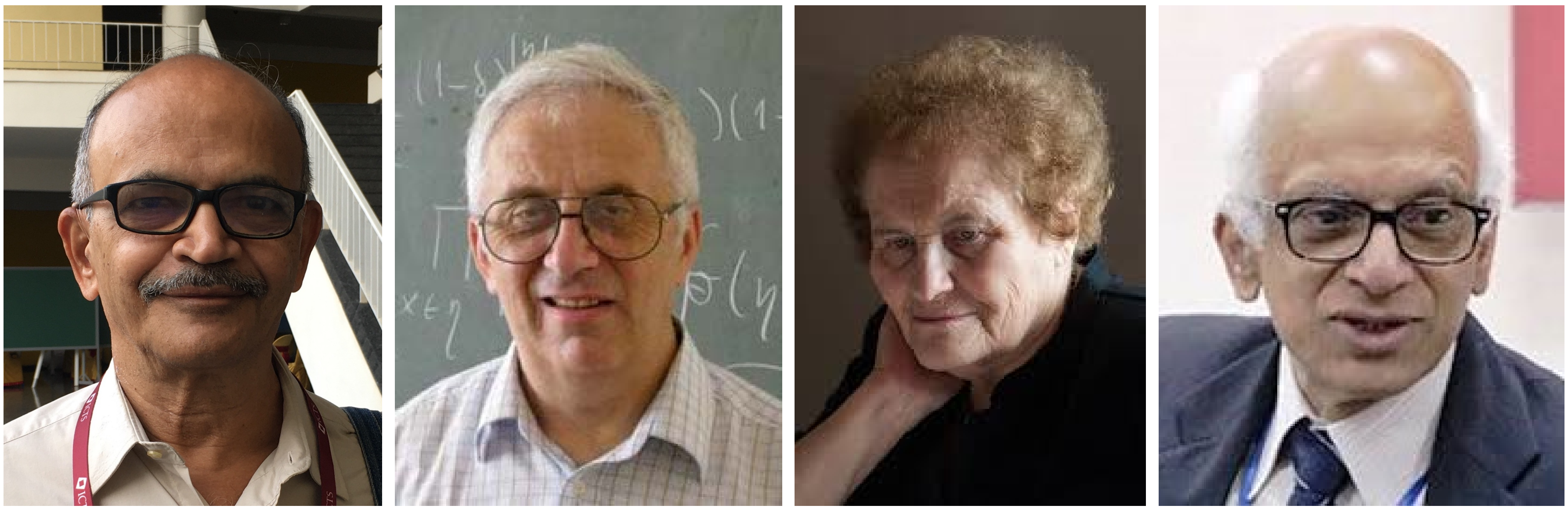
In the case of the Lorentz gas with a random configuration of atoms, the probability density for finding a free path of length x is $\exp(-x)$ as shown in the **black curve**.

The **red curve** represents the distribution of free paths for the Lorentz gas in a crystal.*



*Dahlquist, Nonlinearity 1997; Boca & Zaharescu, Comm. Math. Phys. 2007; Marklof & Strömbergsson, Annals Math. 2010

The underpinning mathematics:
Ergodic theory and dynamical systems on Lie groups



S.G. Dani, G.A. Margulis (Fields Medal 1978), M.E. Ratner, M.S. Raghunathan

The underpinning mathematics:
Ergodic theory and dynamical systems on Lie groups

International Colloquium on Lie Groups and Ergodic Theory at TIFR Mumbai 1996

S.G. Dani, Marina Ratner, Virendra Singh, R. Chidambaram, Hillel Fürstenberg, Anatole Katok, M.S. Raghunathan

Image credit: TIFR Archives

Conclusions

- ***Can the flap of a butterfly's wings in Brazil set of a tornado in Texas?***

- ***Can one predict the unpredictable?***

- ***Can one predict the numbers in the next lottery draw?***

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- ***Can one predict the unpredictable?*** In some sense, yes. In chaotic systems it is virtually impossible to predict the state of a system even after a short time. But we can predict accurately the probability that it will be in a given state after a relatively short time (exponentially fast decay of correlations). The more chaotic the system is, the more accurate the prediction of probabilities.

- ***Can one predict the numbers in the next lottery draw?***

Conclusions

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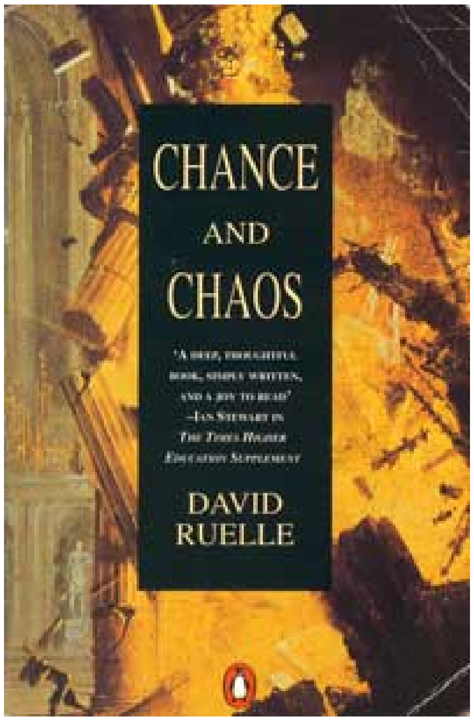
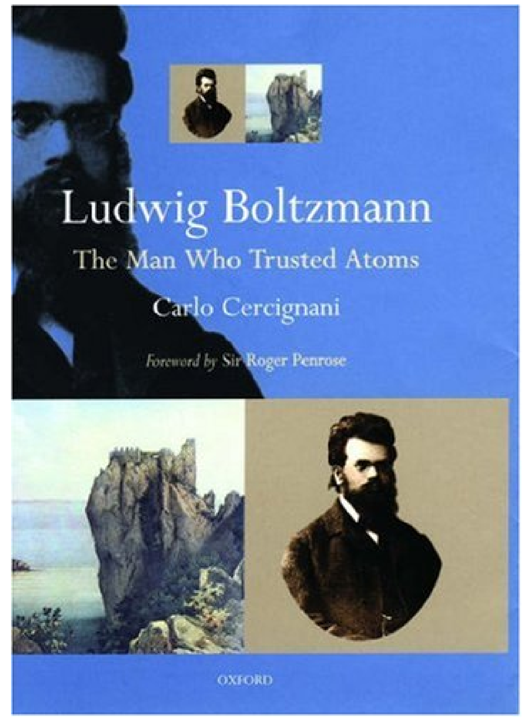
- ***Can one predict the numbers in the next lottery draw?*** No - the lottery balls form a dynamical system that is extremely sensitive to initial conditions, just as the Lorentz gas. So the answer is No. But I can predict who wins in the lottery: those how run it. Here the win is guaranteed!

Recommendations for further reading



Chance and Chaos
by David Ruelle
(Penguin Books 1993)



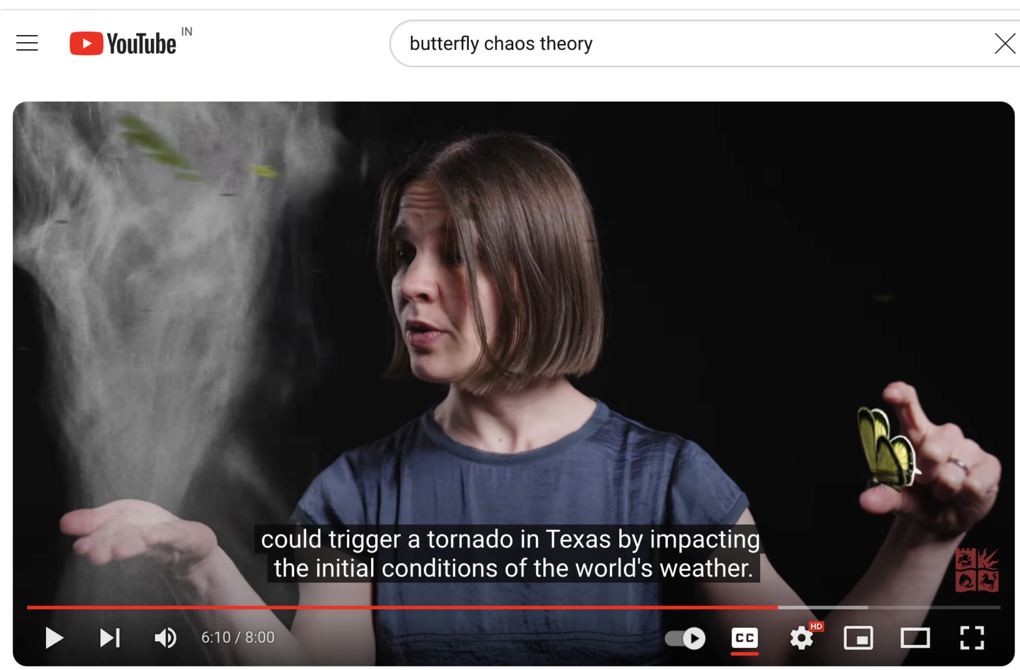
Ludwig Boltzmann—The Man Who Trusted Atoms by Carlo Cercignani
(Oxford University Press 1998)

Thank you

Milind Pilankar and staff from the School of Mathematics as well as the Outreach Team at TIFR Mumbai for organising this event

Rashmita Hore and Sundara Narasimham for their assistance with the experiment

Director Jayaram Chengalur, Professor Mahan Mj and Professor Anish Ghosh for hosting the lecture



Chaos Theory and The Butterfly Effect - Predicting The Unpredictable