Random lattices and their applications in number theory, geometry and statistical mechanics III

Jens Marklof
University of Bristol

Tata Institute for Fundamental Research, Mumbai, February 2023

Outline

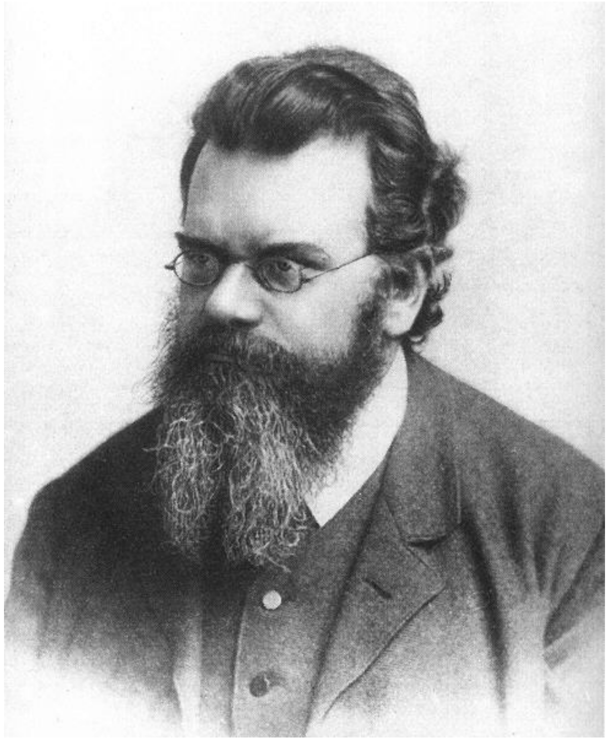
Lecture 1 What are random lattices? Basic features and examples

Lecture 2 Random lattices in number theory and geometry

Lecture 3 Random lattices in statistical mechanics – the Lorentz gas

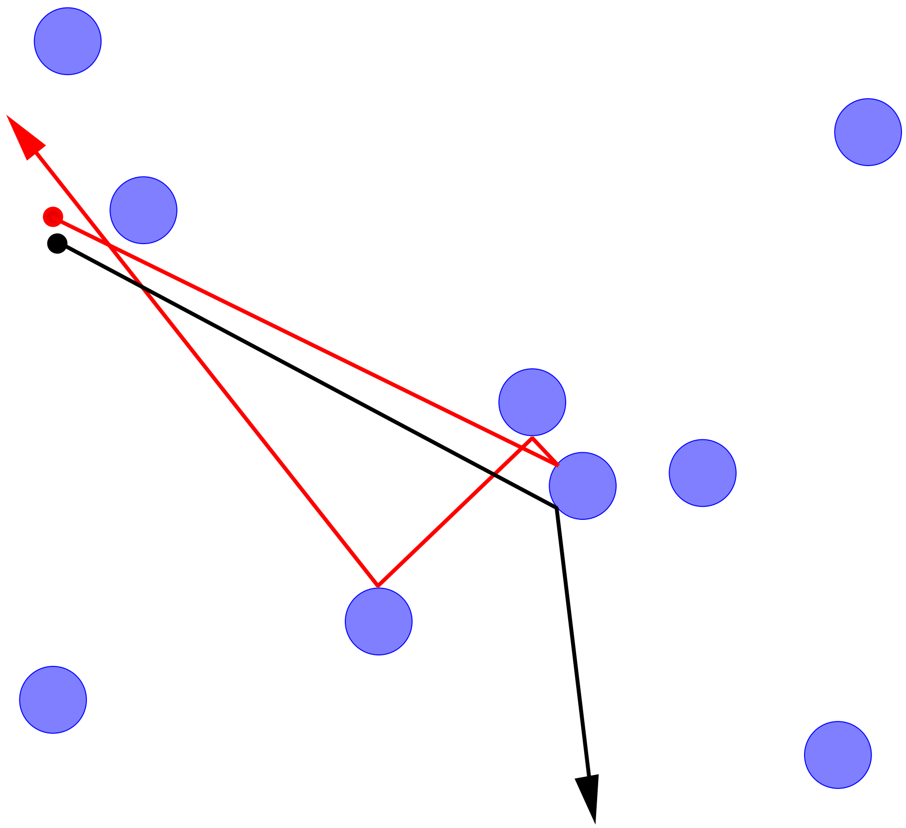
Maxwell and Boltzmann

James Clerk Maxwell (1831-1879)

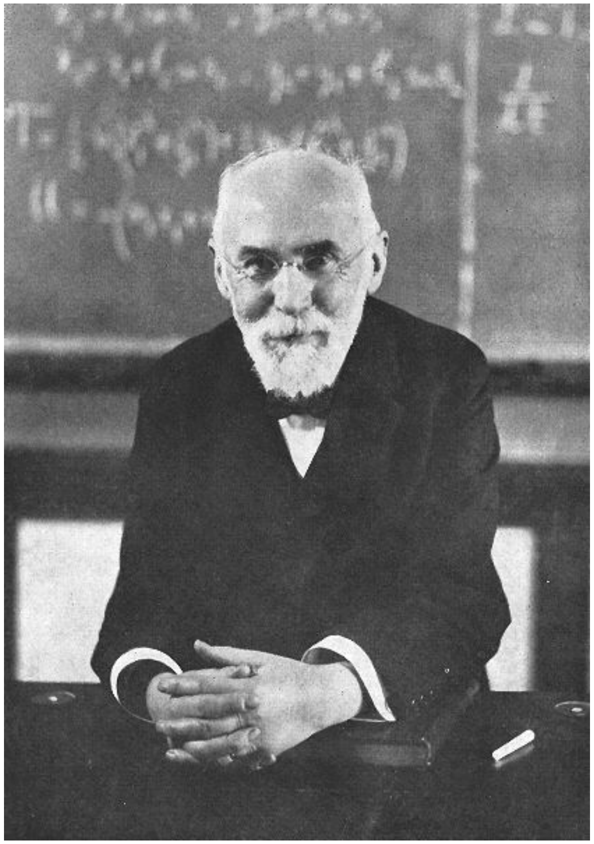


Ludwig Boltzmann (1844-1906)

The Lorentz gas

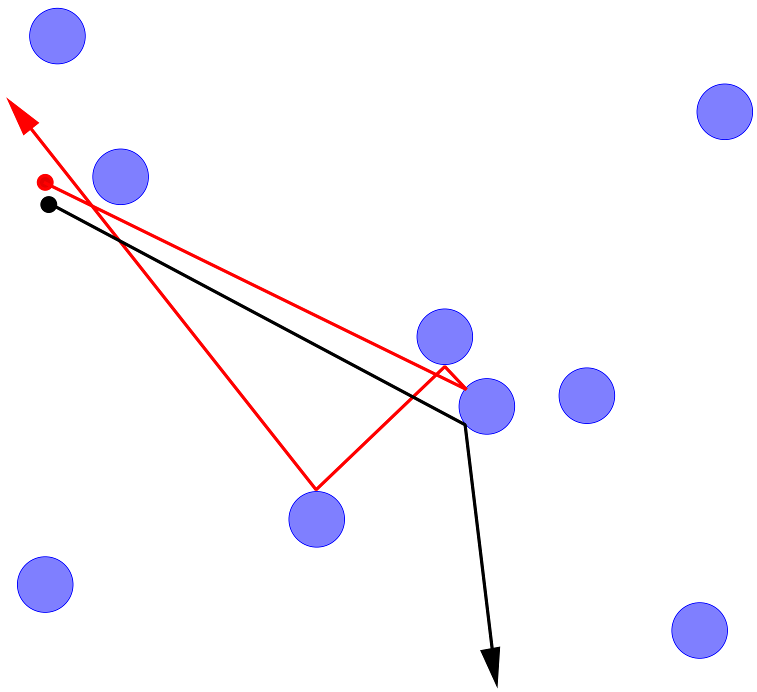


Arch. Neerl. (1905)



Hendrik Lorentz (1853-1928)

The Lorentz gas

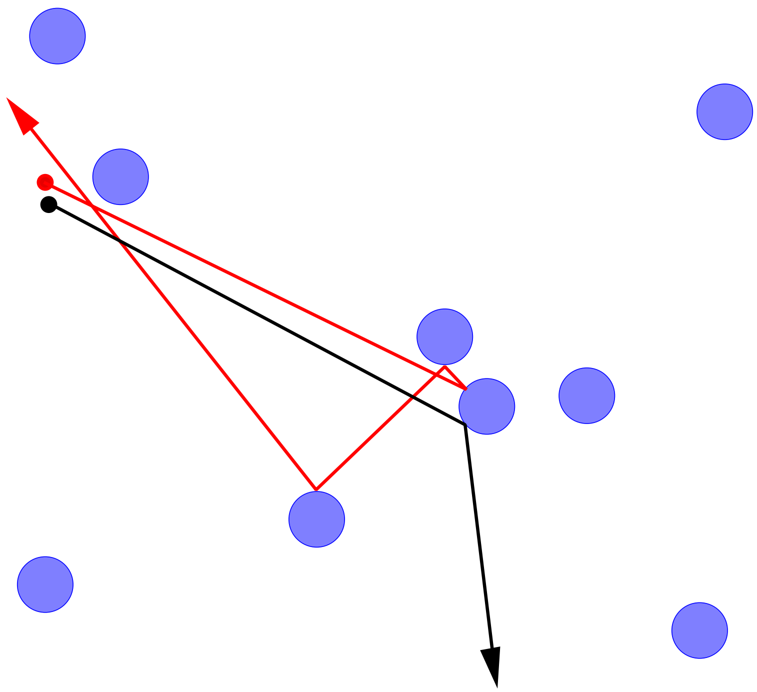


- $\mathcal{P}$ locally finite subset of $\mathbb{R}^d$ with density one, i.e.,

$$\lim_{R\to\infty} \frac{\#(\mathcal{P} \cap R\mathcal{D})}{R^d} = \operatorname{vol} \mathcal{D}$$

  for all bounded sets $\mathcal{D} \subset \mathbb{R}^d$ with $\operatorname{vol} \partial\mathcal{D} = 0$
- scatterers are fixed open balls of radius ρ centered at the points in $\mathcal{P}$

The Lorentz gas



- the particles are assumed to be non-interacting
- each test particle moves with constant velocity $v(t)$ between collisions
- the scattering is specular reflection; we can also treat scattering by compactly supported, spherically symmetric potentials
- we assume w.l.o.g. $\|v(t)\| = 1$

The Boltzmann-Grad (=low-density) limit

- Consider the dynamics in the limit of small scatterer radius ρ
- $\big(\boldsymbol{q}(t), \boldsymbol{v}(t)\big)$ = "microscopic" phase space coordinate at time t
- A volume argument shows that for $\rho \to 0$ the mean free path length (i.e., the average time between consecutive collisions) is asymptotic to

$$\frac{1}{\text{total scattering cross section}} = \frac{1}{\rho^{d-1} \operatorname{vol} B_1^{d-1}}$$

- We thus measure position and time in the "macroscopic" coordinates

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \big(\rho^{d-1}\boldsymbol{q}(\rho^{1-d}t), \boldsymbol{v}(\rho^{1-d}t)\big)$$

- Time evolution of initial data $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$:

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \Phi_\rho^t(\boldsymbol{Q}_0, \boldsymbol{V}_0)$$

The linear Boltzmann equation

- Time evolution of a particle cloud with initial density $f \in \mathsf{L}^1$:

$$f_t^{(\rho)}(\boldsymbol{Q}, \boldsymbol{V}) := f\Big(\Phi_\rho^{-t}(\boldsymbol{Q}, \boldsymbol{V})\Big)$$

In his 1905 paper Lorentz suggested that $f_t^{(\rho)}$ is governed, as $\rho \to 0$, by the linear Boltzmann equation:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}) = \int_{\mathsf{S}_1^{d-1}} \Big[f_t(\boldsymbol{Q}, \boldsymbol{V}') - f_t(\boldsymbol{Q}, \boldsymbol{V})\Big] \sigma(\boldsymbol{V}, \boldsymbol{V}') d\boldsymbol{V}'$$

where $\sigma(\boldsymbol{V}, \boldsymbol{V}')$ is the differential cross section of the individual scatterer.
E.g.: $\sigma(\boldsymbol{V}, \boldsymbol{V}') = \frac{1}{4}\|\boldsymbol{V} - \boldsymbol{V}'\|^{3-d}$ for specular reflection at a hard sphere

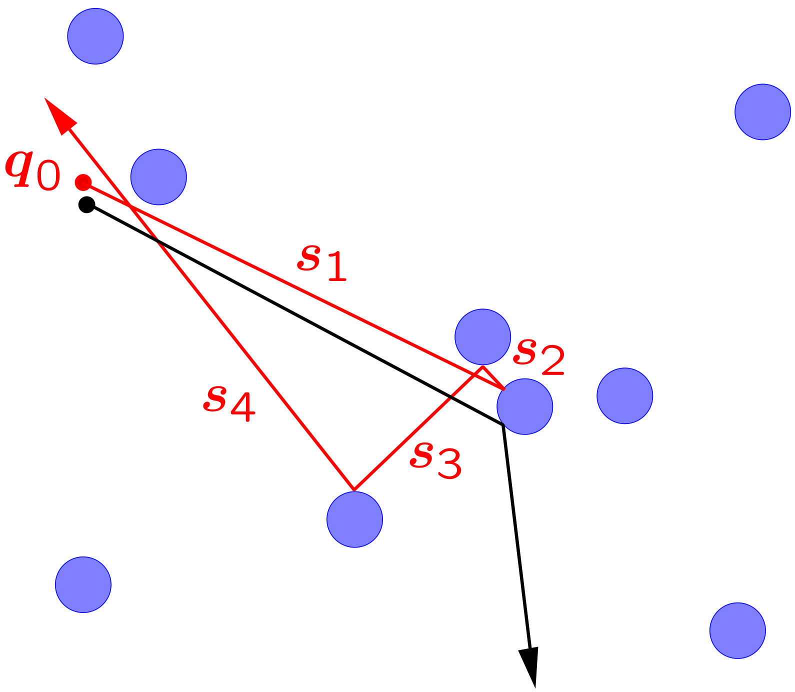
Applications: Neutron transport, radiative transfer, . . .

Main questions:

- What are the random flight processes that emerge in the Boltzmann-Grad limit?

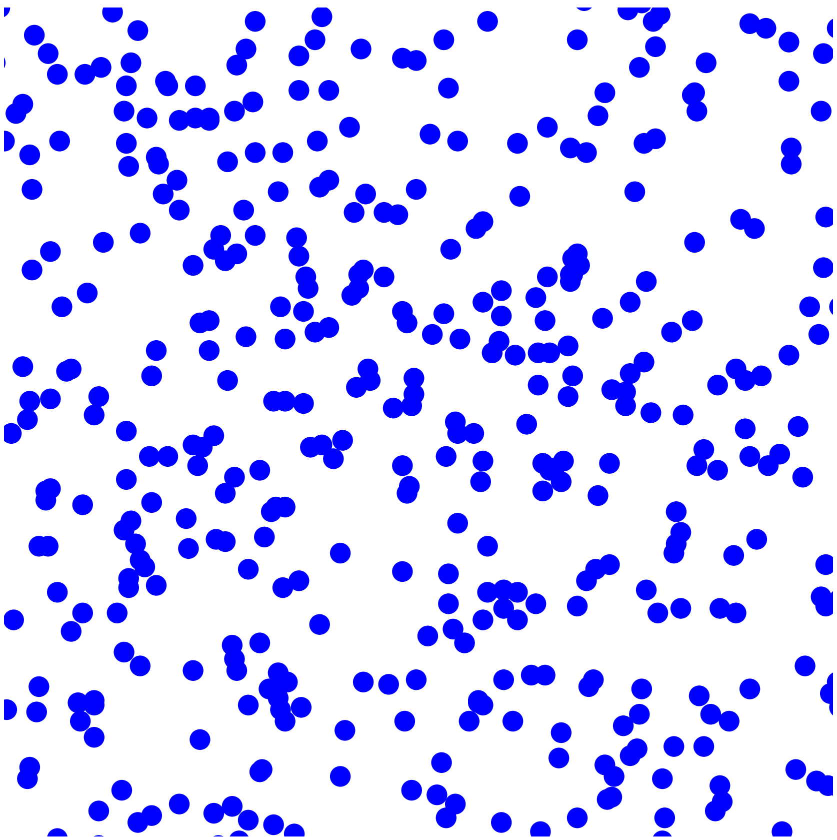
- What are the associated kinetic transport equations?

Key microscopic quantities

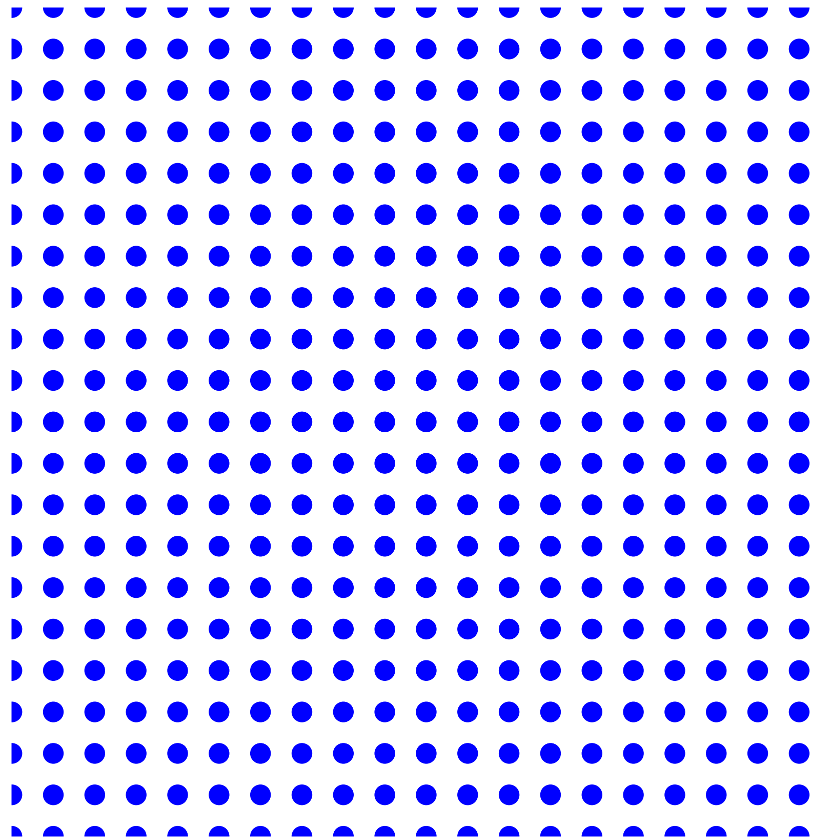


- $\boldsymbol{q}_0, \boldsymbol{v}_0$ initial particle position and velocity ($\|\boldsymbol{v}_0\| = 1$)
- $\tau_1 = \tau_1(\boldsymbol{q}_0, \boldsymbol{v}_0)$ first hitting time
- $\boldsymbol{v}_n = \boldsymbol{v}_n(\boldsymbol{q}_0, \boldsymbol{v}_0)$ velocity after nth collision
- $\tau_{n+1} = \tau_{n+1}(\boldsymbol{q}_0, \boldsymbol{v}_0)$ free path lengths after nth collision
- $\boldsymbol{s}_n = \tau_n \boldsymbol{v}_{n-1}$ travel intinerary
- mean free path $\sim \frac{1}{\rho^{d-1} \operatorname{vol} B_1^{d-1}}$

The Boltzmann-Grad limit



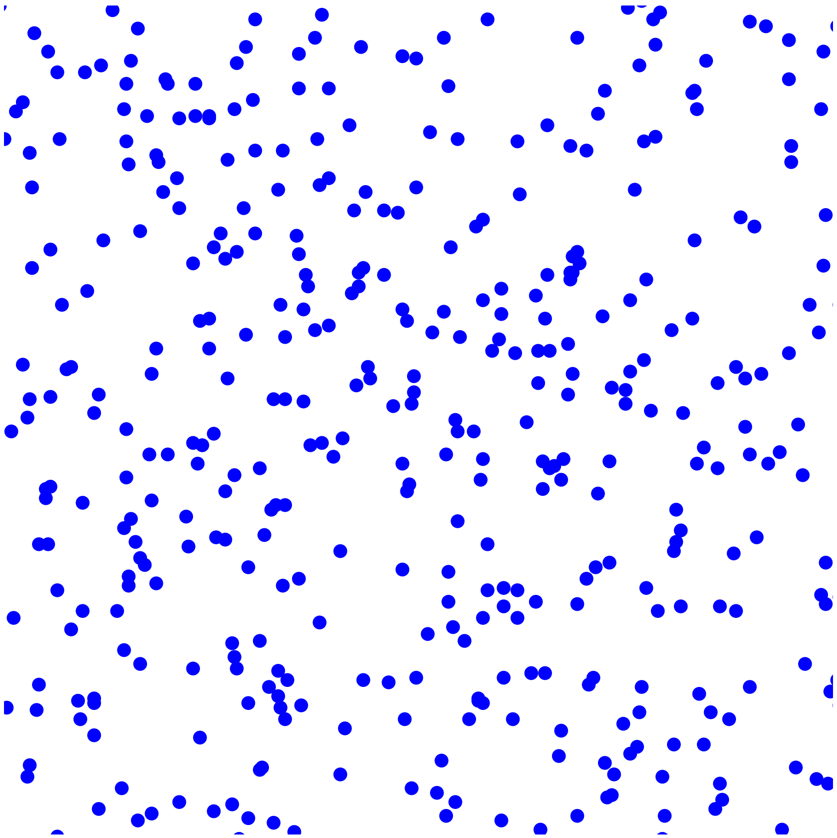
Fixed random scatterer configuration



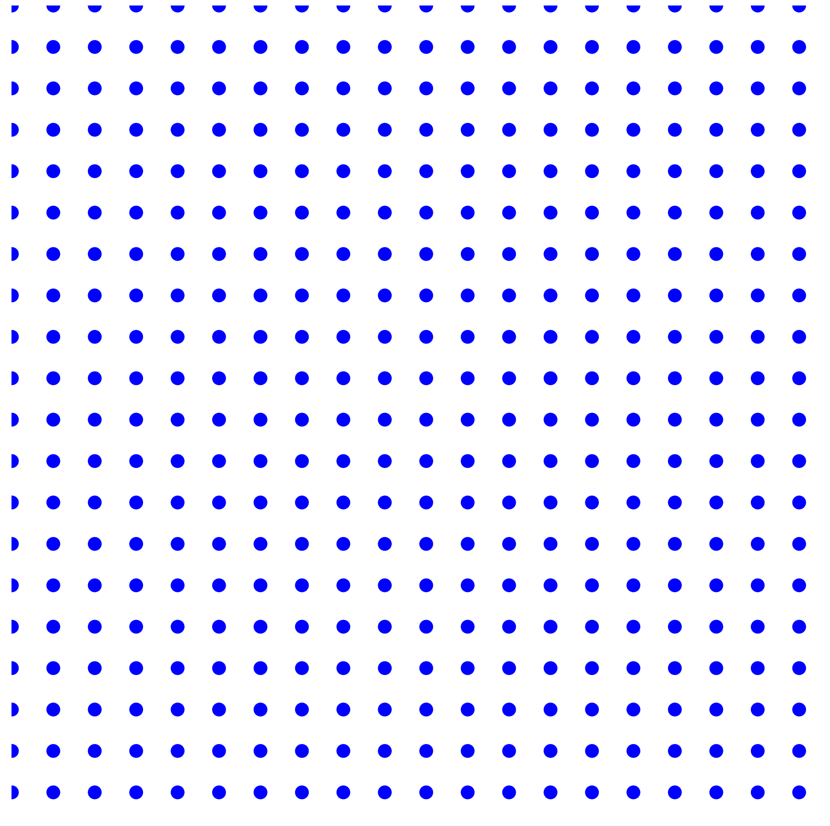
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/4$, mean free path$= \frac{1}{2\rho} = 2$

The Boltzmann-Grad limit



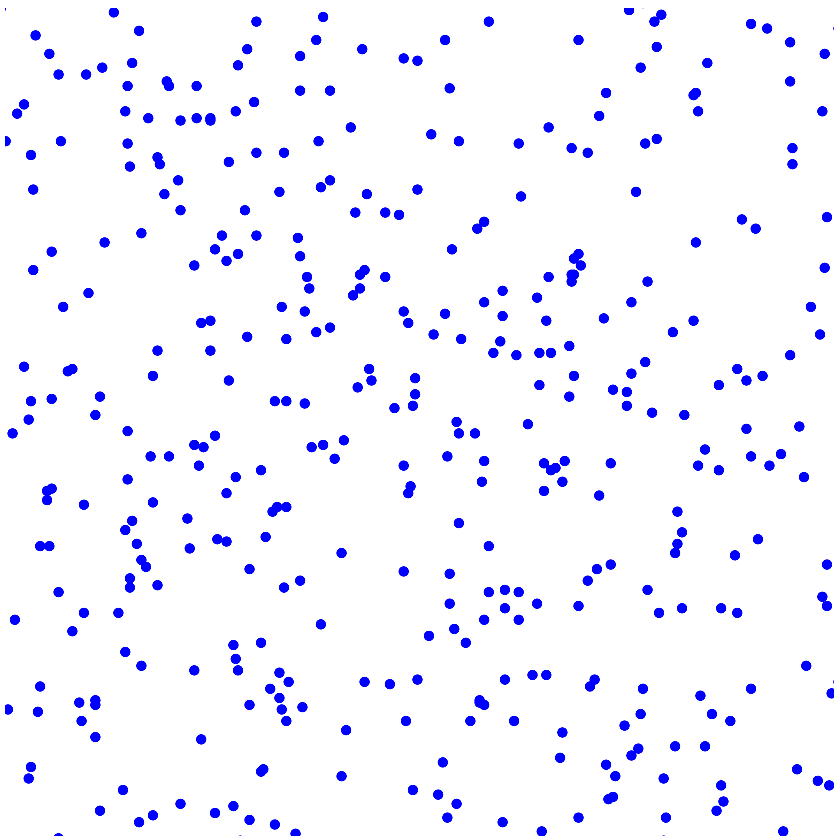
Fixed random scatterer configuration



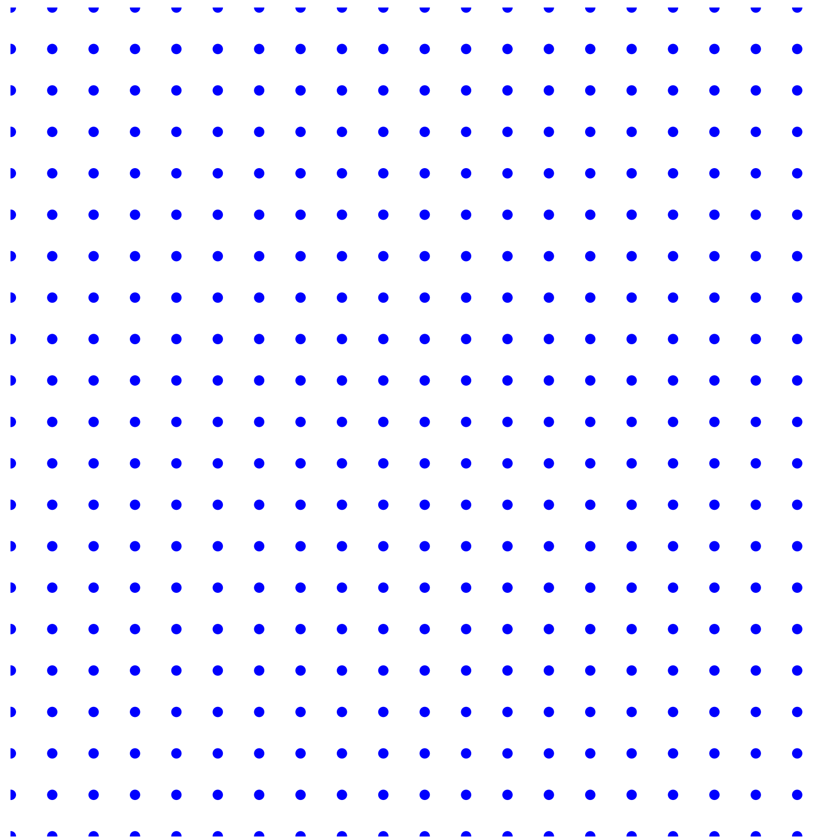
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/6$, mean free path$= \frac{1}{2\rho} = 3$

The Boltzmann-Grad limit



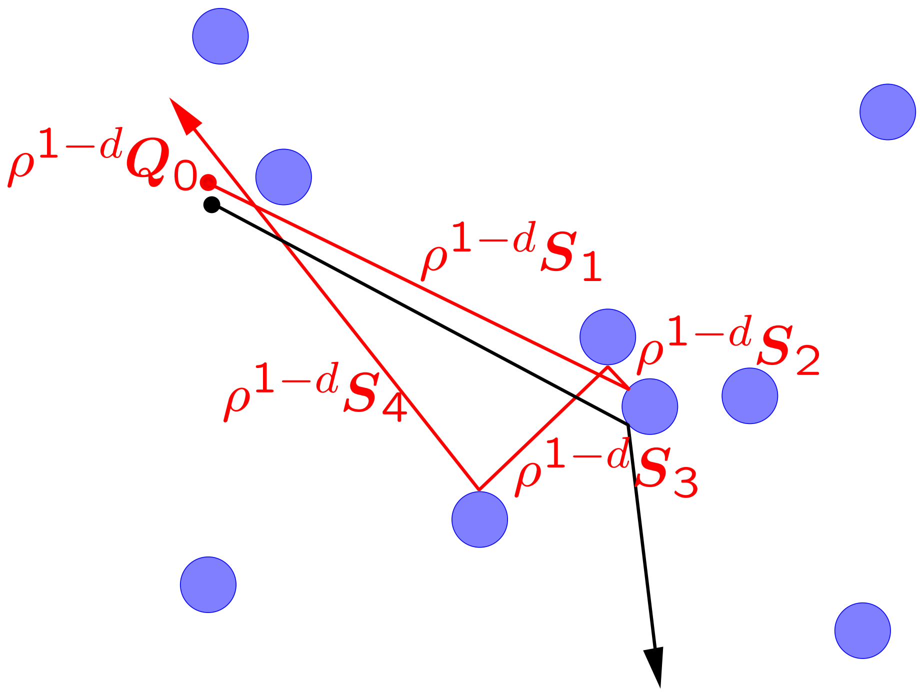
Fixed random scatterer configuration



Periodic scatterer configuration $\mathbb{Z}^2$

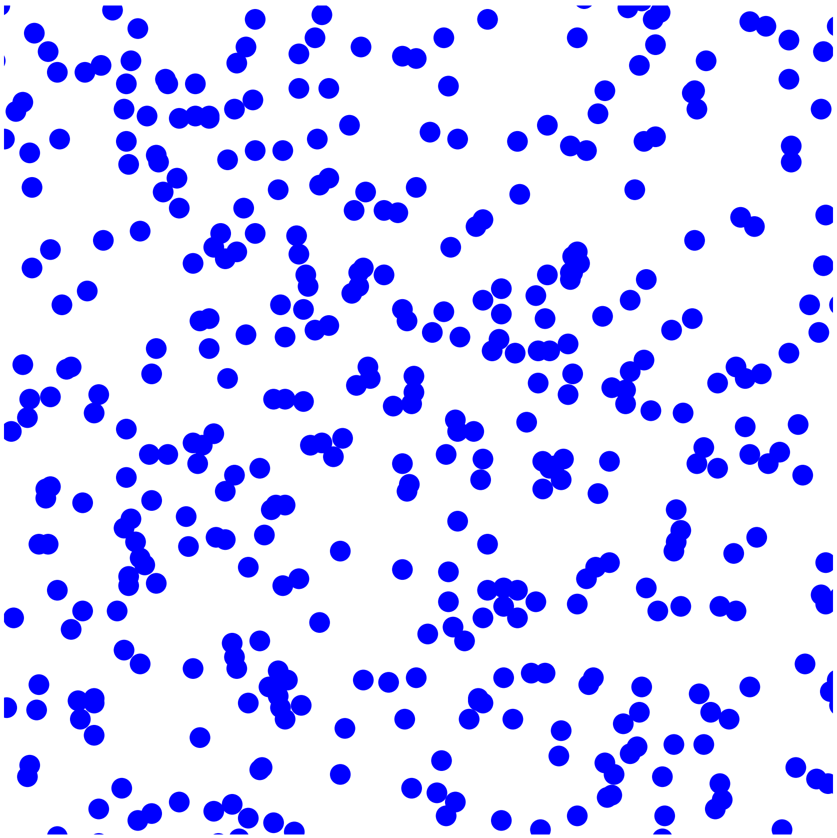
Scattering radius $\rho = 1/8$, mean free path$= \frac{1}{2\rho} = 4$

Key macroscopic quantities

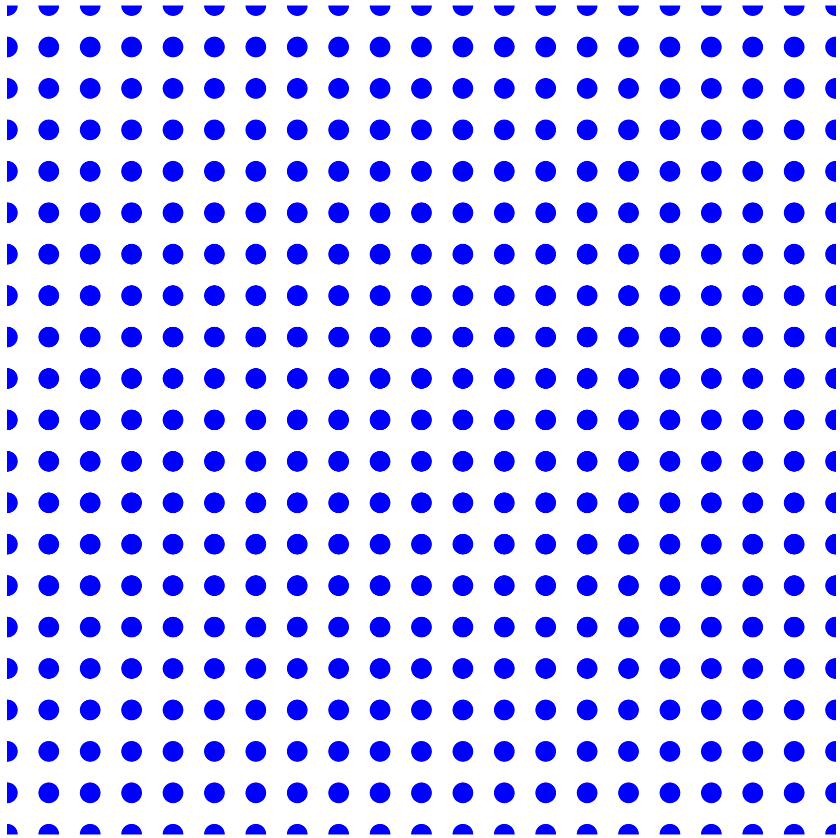


- $\boldsymbol{Q}_0 = \rho^{d-1}\boldsymbol{q}_0, \boldsymbol{V}_0 = \boldsymbol{v}_0$
- $\xi_1 = \rho^{d-1}\tau_1(\rho^{1-d}\boldsymbol{Q}_0, \boldsymbol{V}_0)$
- $\boldsymbol{V}_n = \boldsymbol{v}_n(\rho^{1-d}\boldsymbol{Q}_0, \boldsymbol{V}_0)$
- $\xi_{n+1} = \rho^{d-1}\tau_{n+1}(\rho^{1-d}\boldsymbol{Q}_0, \boldsymbol{V}_0)$
- $\boldsymbol{S}_n = \xi_n \boldsymbol{V}_{n-1} = \rho^{d-1}\boldsymbol{s}$
- (macro) mean free path $\frac{1}{\text{vol}\, B_1^{d-1}}$

The Boltzmann-Grad limit



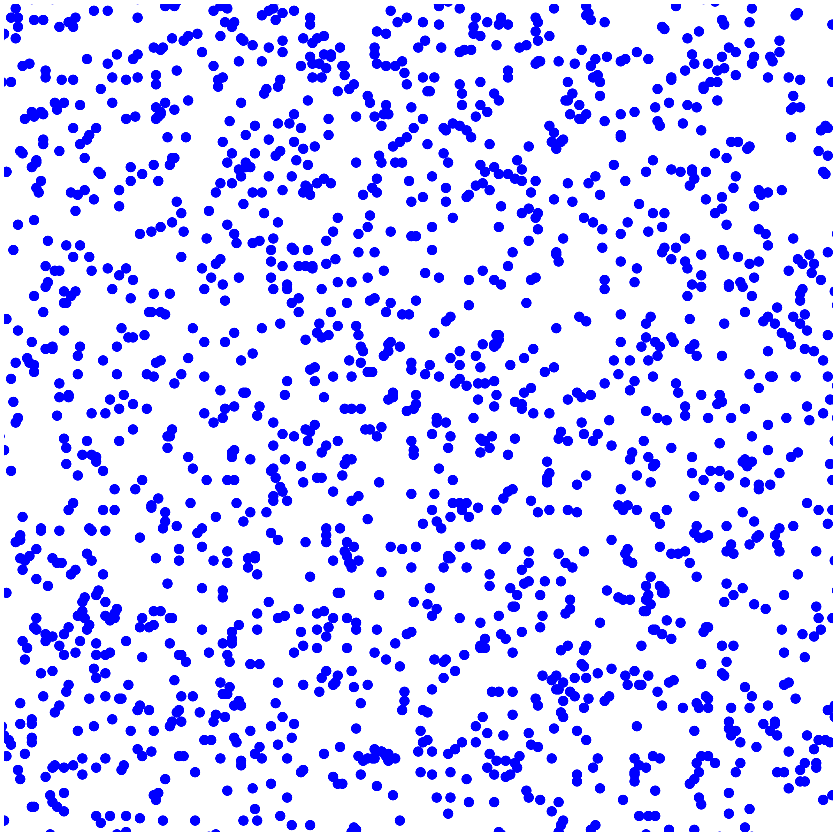
Fixed random scatterer configuration



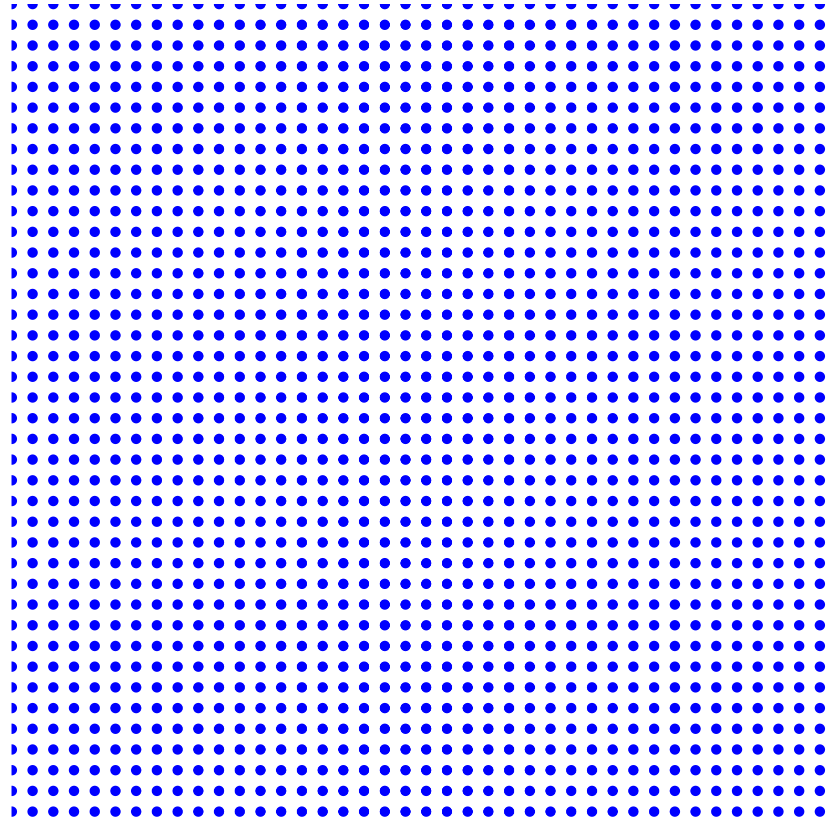
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/4$, mean free path$= \frac{1}{2\rho} = 2$

The Boltzmann-Grad limit



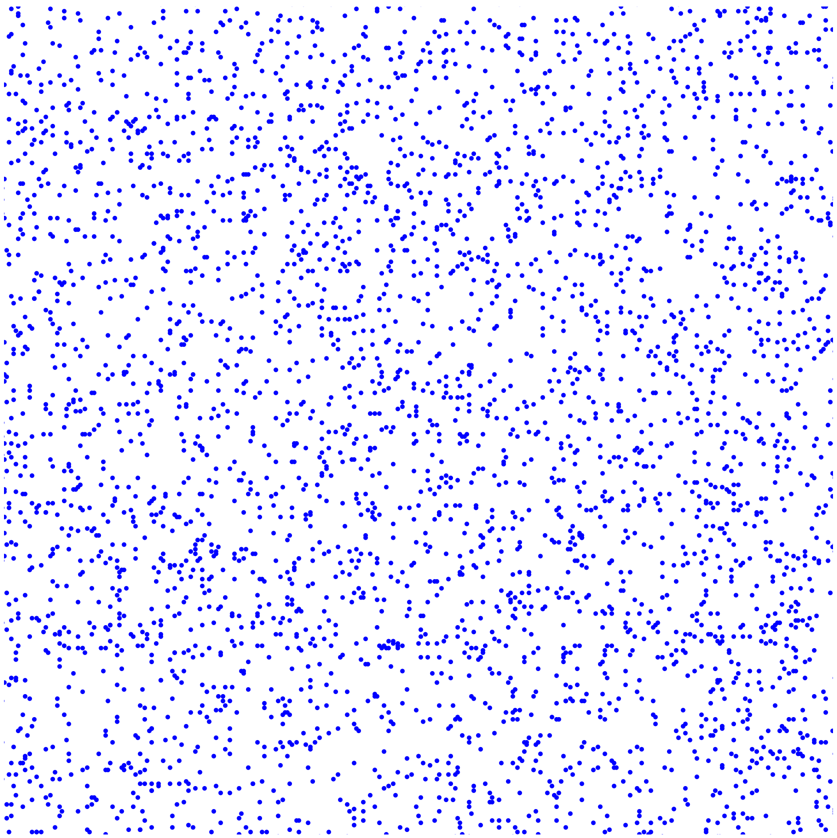
Fixed random scatterer configuration



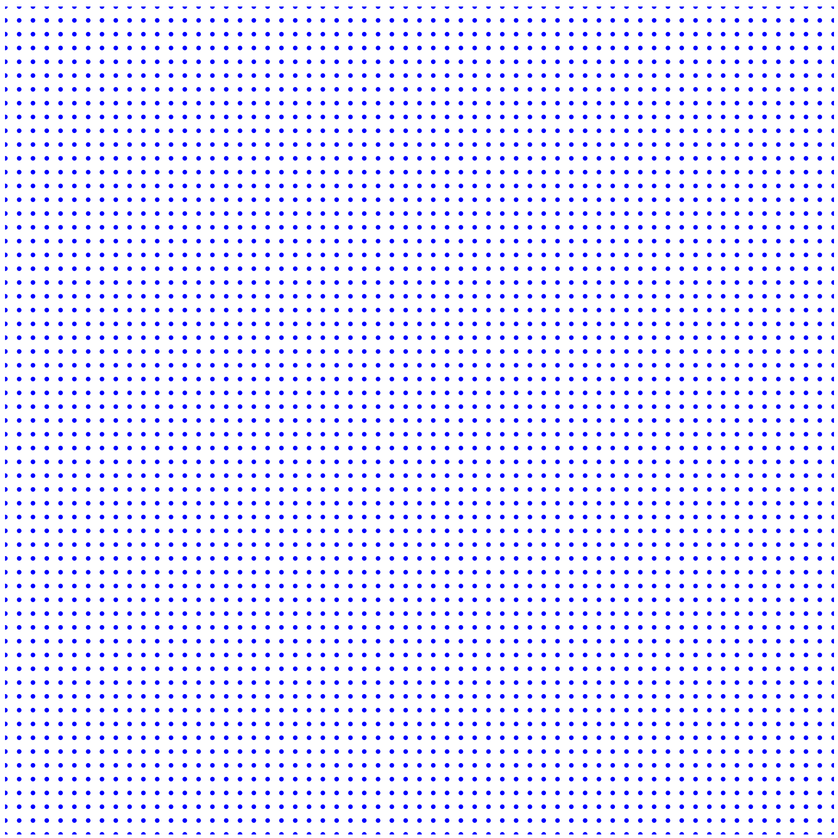
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/4$; $1/2$-zoom: macroscopic mean free path=1

The Boltzmann-Grad limit



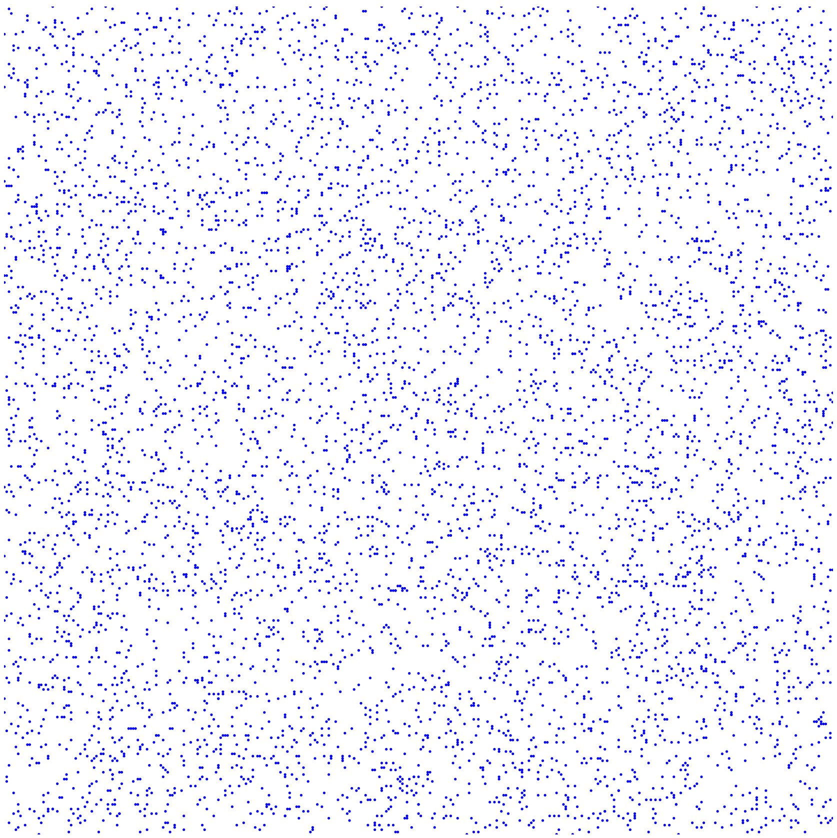
Fixed random scatterer configuration



Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/6$; $1/3$-zoom: macroscopic mean free path=1

The Boltzmann-Grad limit



Fixed random scatterer configuration

Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/8$; $1/4$-zoom: macroscopic mean free path=1

The main result

- $n_t = n_t(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ the number of collisions within time t, i.e.,

$$n_t = \max\left\{n \in \mathbb{Z}_{\geq 0} : T_n \leq t\right\}, \qquad T_n := \sum_{j=1}^{n} \xi_j.$$

- For $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ random w.r.t. $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$,

$$\Theta^{(\rho)} : t \mapsto \Theta^{(\rho)}(t) = \left(\boldsymbol{Q}_0 + \sum_{j=1}^{n_t} \xi_j \boldsymbol{V}_{j-1} + (t - T_{n_t})\boldsymbol{V}_{n_t}, \boldsymbol{V}_{n_t}\right)$$

defines a **random flight process**.

Theorem A
Let $\mathcal{P}$ be admissible. Then, for any $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$, there is a random flight process $\Theta^{(0)}$ with $\mathbb{P}(\xi_j^{(0)} = \infty) = 0$ for all j, such that $\Theta^{(\rho)}$ converges to $\Theta^{(0)}$ in distribution, as $\rho \to 0$.

Outline of proof

The key is to establish the following discrete time analogue of Theorem A.

Theorem B

Let $\mathcal{P}$ be admissible. Then, for any $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$

$$\left\langle \xi_j(\boldsymbol{Q}_0, \boldsymbol{V}_0), \boldsymbol{V}_j(\boldsymbol{Q}_0, \boldsymbol{V}_0) \right\rangle_{j=1}^{\infty}$$

converges in distribution to the random sequence

$$\left\langle \xi_j^{(0)}, \boldsymbol{V}_j^{(0)} \right\rangle_{j=1}^{\infty}$$

(which in general does not form a Markov chain).

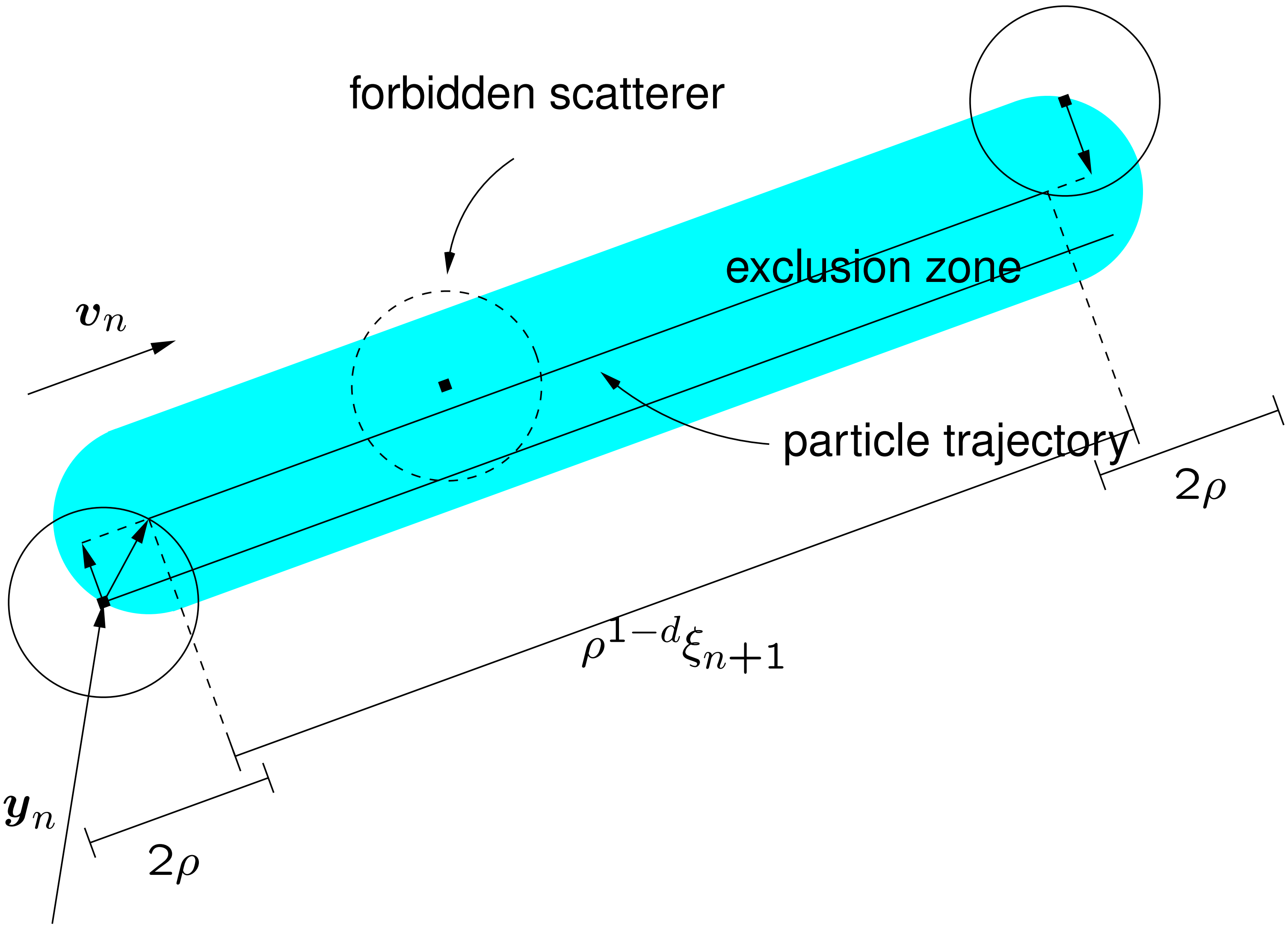
There are three steps:

1. **Rescaling** and spherical equidistribution for each individual inter-collision flight
2. Markovianisation of the limit process through introduction of a **marking** of $\mathcal{P}$
3. **Induction** on the number of inter-collision flights

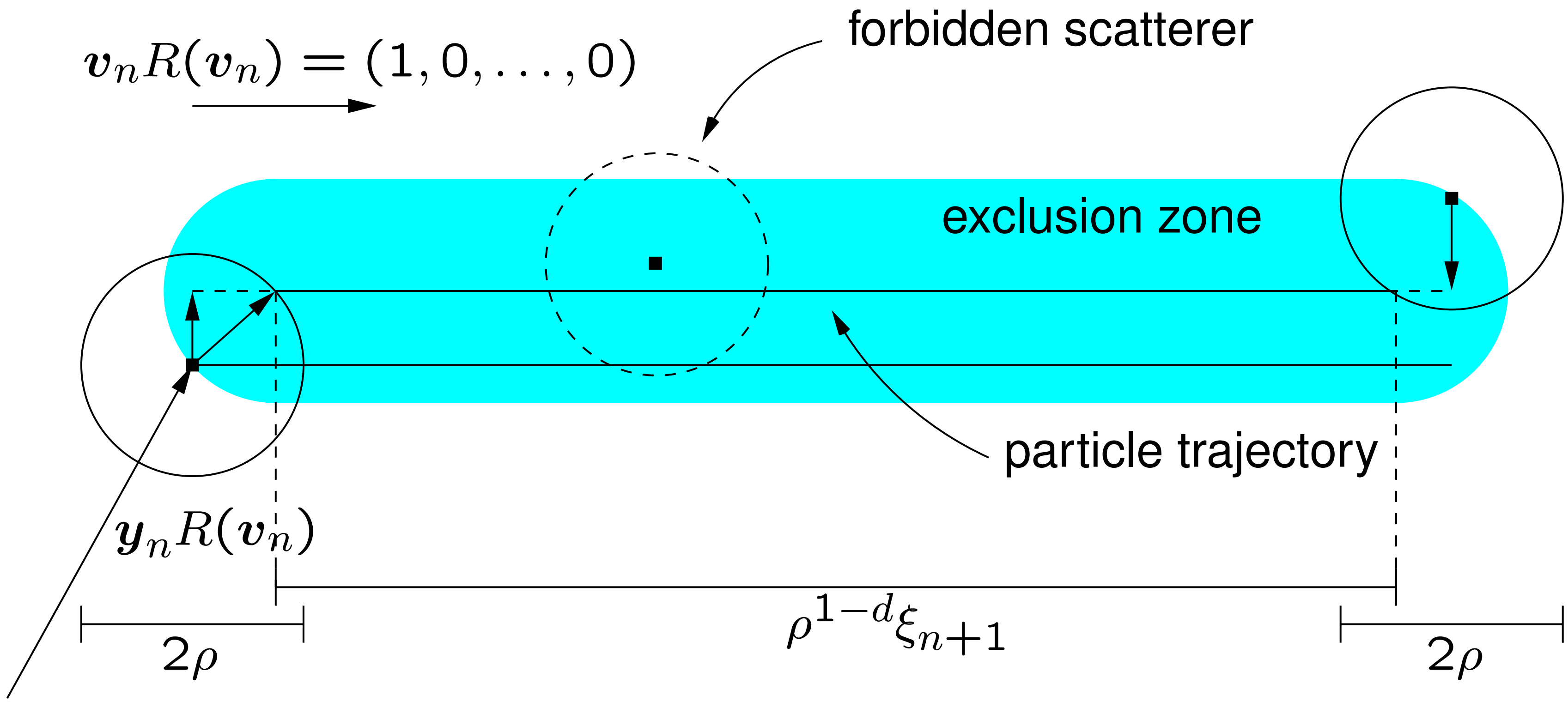
Step 1: Rescaling

Define $R(\boldsymbol{v}) : \mathsf{S}_1^{d-1} \to \mathsf{SO}(d)$ such that $\boldsymbol{v}R(\boldsymbol{v}) = \boldsymbol{e}_1 = (1, 0, \ldots, 0)$ and

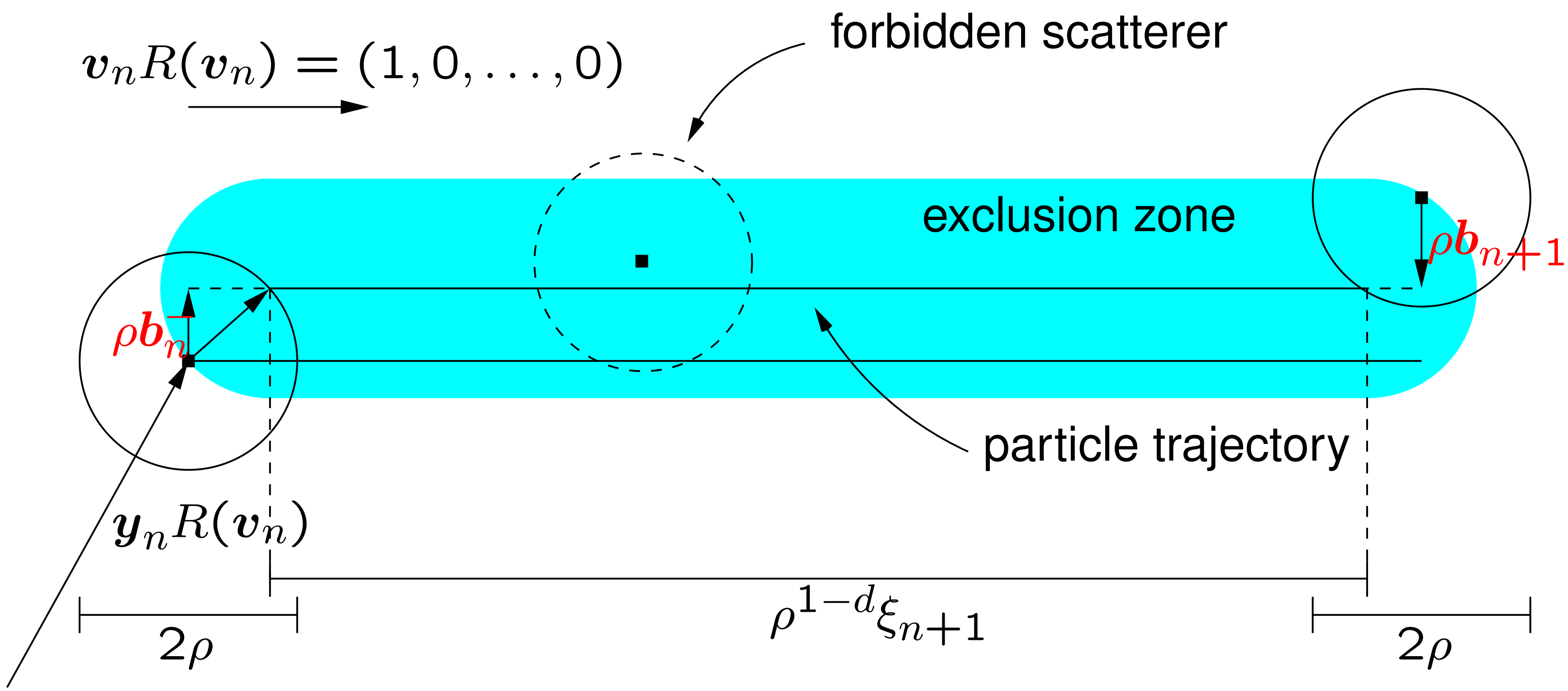
$$D_\rho = \begin{pmatrix} \rho^{d-1} & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{0} & \rho^{-1} 1_{d-1} \end{pmatrix} \in \mathsf{SL}(d, \mathbb{R}).$$



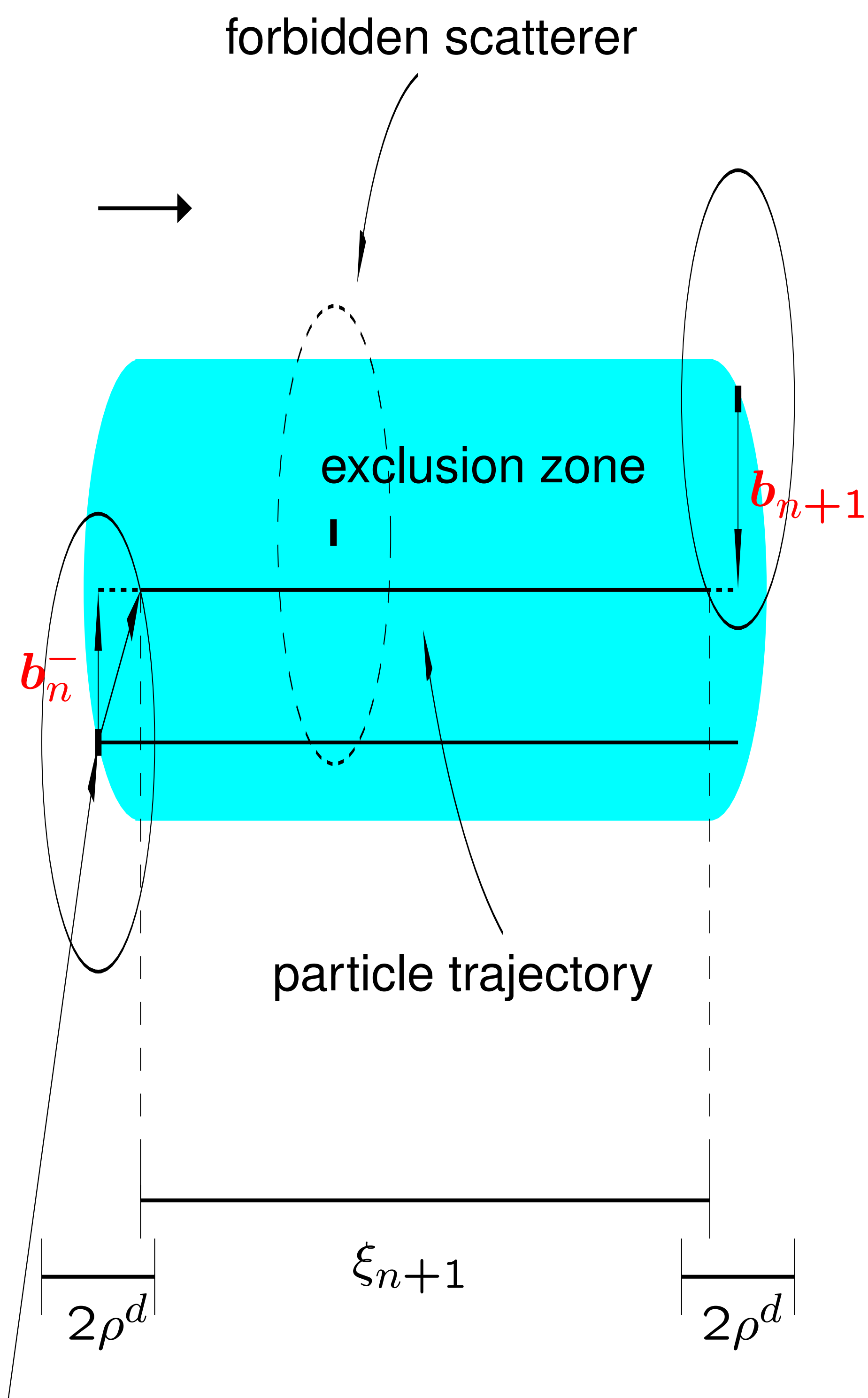
Applying $R(\boldsymbol{v}_n)D_\rho$ to this cylinder orients it along the e_1-axis and makes it well proportioned. First apply $R(\boldsymbol{v}_n)$.



It is important to keep track of the exit parameters $\boldsymbol{b}_n^-$ and impact parameters $\boldsymbol{b}_n$.



Now apply D_ρ.


forbidden scatterer
exclusion zone
b_{n+1}
b_n^-
particle trajectory
ξ_{n+1}
$2\rho^d$
$2\rho^d$

Step 2: Marking

- Under the above rescaling the cylinder converges to a $(\rho, \boldsymbol{v}_n)$-independent cyclinder (with flat caps).

- The point set $\mathcal{P}$ has been replaced by the random point set $(\mathcal{P} - \boldsymbol{y}_n) R(\boldsymbol{v}_n) D_\rho$.

- For $\boldsymbol{y}$ fixed and $\boldsymbol{v}$ random, limit distribution of $(\mathcal{P} - \boldsymbol{y}) R(\boldsymbol{v}) D_\rho$ can in general depend on $\boldsymbol{y} \in \mathcal{P}$. In order to keep track of this, we assign a **mark** to each $\boldsymbol{y}$; we want the space of marks to be nice.

Assumptions on the scatterer configuration $\mathcal{P}$

We say $\mathcal{P}$ is **admissible** if there exists a compact metric space Σ with Borel probability measure m, and map $\varsigma : \mathcal{P} \to \Sigma$ (the marking) such that for

$$\mathcal{X} = \mathbb{R}^d \times \Sigma, \quad \mu_{\mathcal{X}} = \mathrm{vol} \times \mathrm{m}$$
$$\tilde{\mathcal{P}} = \{(\boldsymbol{y}, \varsigma(\boldsymbol{y})) : \boldsymbol{y} \in \mathcal{P})\} \subset \mathcal{X} \qquad \text{(the marked point set)}$$

we have

- **Assumption 1** (density)

$$\lim_{R\to\infty} \frac{\#(\tilde{\mathcal{P}} \cap R\mathcal{D})}{R^d} = \mu_{\mathcal{X}}(\mathcal{D})$$

  for all bounded sets $\mathcal{D} \subset \mathcal{X}$ with $\mu_{\mathcal{X}}(\partial\mathcal{D}) = 0$

- **Assumption 2** (spherical equidistribution) For $\boldsymbol{v}$ random according to λ a.c. w.r.t. vol measure on S_1^{d-1}

$$\tilde{\Xi}_{\rho,\boldsymbol{y}} = (\tilde{\mathcal{P}} - \boldsymbol{y}) R(\boldsymbol{v}) D_\rho \xrightarrow{\mathrm{d}} \tilde{\Xi}_{\varsigma(\boldsymbol{y})} \qquad (\rho \to 0)^*$$

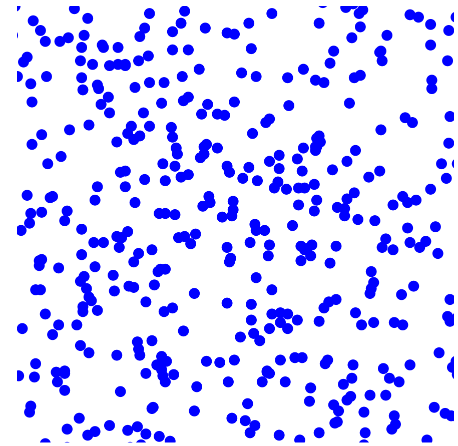
  uniformly for all $\boldsymbol{y} \in \mathcal{P}$ in balls of radius $\asymp \rho^{1-d}$, where $\tilde{\Xi}_\varsigma$ **depends only on** $\varsigma \in \Sigma$

- ... and more

*for $M \in \mathrm{SL}(d,\mathbb{R})$ set $(\boldsymbol{y}, \varsigma(\boldsymbol{y}))M = (\boldsymbol{y}M, \varsigma(\boldsymbol{y}))$

Examples for admissible $\mathcal{P}$

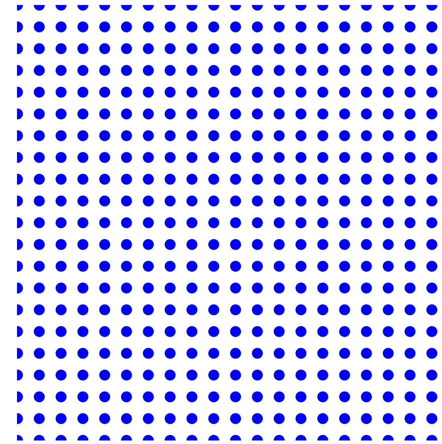
Example 1: $\mathcal{P}$ = a realization of the Poisson process in $\mathbb{R}^d$ with intensity 1, and $\Sigma = \{1\}$; proof that our assumptions satisfied is non-trivial, follows ideas of Boldrighini, Bunimovich and Sinai (J Stat Phys 1983).



Previous results:

- Galavotti (Phys Rev 1969 & report 1972): Poisson distributed hard-sphere scatterer configuration $\mathcal{P}$
- Spohn (Comm Math Phys 1978): extension to more general Fixed random scatterer configurations $\mathcal{P}$ and potentials
- Boldrighini, Bunimovich and Sinai (J Stat Phys 1983): prove convergence for almost every scatterer configuration $\mathcal{P}$ (w.r.t. the Poisson random measure), for hard sphere scatterers only
- Implies CLT for limit process (standard CLT for Markovian random flight process); for intermediate joint Boltzmann-Grad/diffusive scaling see Lutsko and Toth (preprint 2018)

Examples for admissible $\mathcal{P}$



Example 2: $\mathcal{P} = \mathbb{Z}^d$ (or any other Euclidean lattice of co-volume 1) and $\Sigma = \{1\}$ (periodic Lorentz gas); proof uses spherical equidistribution on space of lattices (JM & Strömbergsson, Annals of Math 2010). The limit process is independent of the choice of lattice!

Previous results:

- Caglioti and Golse, Comptes Rendus 2008, J Stat Phys 2010
- JM & Strömbergsson, Nonlinearity 2008, Annals of Math 2010/2011, GAFA 2011

- Polya (Arch Math Phys 1918): "Visibility in a forest" $(d = 2)$
- Dahlquist (Nonlinearity 1997); Boca, Cobeli, Zaharescu (CMP 2000); Caglioti, Golse (CMP 2003); Boca, Gologan, Zaharescu (CMP 2003); Boca, Zaharescu (CMP 2007): Limit distributions for the free path lengths for various sets of initial data $(d = 2)$
- Dumas, Dumas, Golse (J Stat Phys 1997): Asymptotics of mean free path lengths $(d \geq 2)$
- Bourgain, Golse, Wennberg (CMP 1998); Golse, Wennberg (CMP 2000): bounds on possible weak limits $(d \geq 2)$

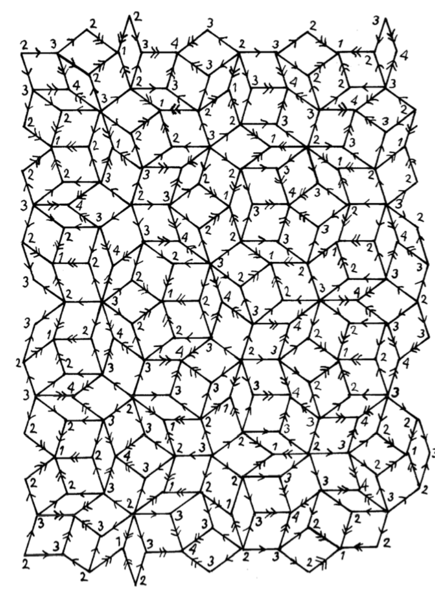
- Boca & Gologan (Annales I Fourier 2009), Boca (NY J Math 2010): honeycomb lattice

Examples for admissible $\mathcal{P}$

Example 3: $\mathcal{P} = \bigcup_{i=1}^{m}(\mathcal{L} + \boldsymbol{\alpha}_i)$ locally finite periodic point set (e.g. the honeycomb/hexagonal lattice), with $\mathcal{L}$ Euclidean lattice of covolume m; $\Sigma = \{1, 2, \ldots, m\}$. Admissible follows from spherical equidistribution, which here is a consequence of Ratner's theorem on $\mathrm{SL}(d,\mathbb{Z}) \ltimes (\mathbb{Z}^d)^k \backslash \mathrm{SL}(d,\mathbb{R}) \ltimes (\mathbb{R}^d)^k$.

Previous results on free path length: Boca & Gologan (Annales I Fourier 2009), Boca (NY J Math 2010)

Example 4: $\mathcal{P} =$ Euclidean cut-and-project set (e.g. the vertex set of a Penrose tiling) and $\Sigma \subset \mathbb{R}^k$ (the internal space in the c&p construction); proof of assumptions uses uses equidistribution of lower dimensional spheres in space of lattices, which is again a consequence of Ratner's theorem (JM & Strömbergsson, CMP 2014; Memoirs AMS in press).



Step 3: Induction $\longrightarrow$ The main theorem

Theorem C

Let $\mathcal{P}$ be admissible. Then, for any $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$, the random process

$$\begin{aligned} &\mathbb{N} \to (\mathbb{R}_{>0} \cup \{+\infty\}) \times \Sigma \times \mathrm{S}_1^{d-1} \\ &j \mapsto \Big(\rho^{d-1}\tau_j(\rho^{1-d}\boldsymbol{q}_0, \boldsymbol{v}_0; \rho), \varsigma_j(\rho^{1-d}\boldsymbol{q}_0, \boldsymbol{v}_0; \rho), \boldsymbol{v}_j(\rho^{1-d}\boldsymbol{q}_0, \boldsymbol{v}_0; \rho)\Big) \end{aligned}$$

converges in distribution to the second-order Markov process

$$j \mapsto \Big(\xi_j, \varsigma_j, \boldsymbol{v}_j\Big),$$

where for any Borel set $A \subset \mathbb{R}_{\geq 0} \times \Sigma \times \mathrm{S}_1^{d-1}$,

$$\mathbb{P}\Big((\xi_1, \varsigma_1, \boldsymbol{v}_1) \in A \,\Big|\, (\boldsymbol{q}_0, \boldsymbol{v}_0)\Big) = \int_A p(\boldsymbol{v}_0; \xi, \varsigma, \boldsymbol{v})\, d\xi\, d\mathrm{m}(\varsigma)\, d\boldsymbol{v},$$

and for $j \geq 2$,

$$\begin{aligned} \mathbb{P}\Big((\xi_j, \varsigma_j, \boldsymbol{v}_j) \in A \,\Big|\, (\boldsymbol{q}_0, \boldsymbol{v}_0),\ \big\langle (\xi_i, \varsigma_i, \boldsymbol{v}_i)\big\rangle_{i=1}^{j-1}\Big) \\ = \int_A p_{\mathbf{0}}(\boldsymbol{v}_{j-2}, \varsigma_{j-1}, \boldsymbol{v}_{j-1}; \xi, \varsigma, \boldsymbol{v})\, d\xi\, d\mathrm{m}(\varsigma)\, d\boldsymbol{v}. \end{aligned}$$

The functions $p, p_{\mathbf{0}}$ depend on $\mathcal{P}$ but are independent of Λ, and for any fixed $\boldsymbol{v}_0, \varsigma, \boldsymbol{v}$ both $p(\boldsymbol{v}_0\,;\,\cdot\,)$ and $p_{\mathbf{0}}(\boldsymbol{v}_0, \varsigma, \boldsymbol{v}\,;\,\cdot\,)$ are probability densities on $\mathbb{R}_{\geq 0} \times \Sigma \times \mathrm{S}_1^{d-1}$. In particular $\mathbb{P}(\xi_j = \infty) = 0$ for all j.

Evolution of densities

Recall: a cloud of particles with initial density $f(\boldsymbol{Q},\boldsymbol{V})$ evolves in time t to

$$[L_\rho^t f](\boldsymbol{Q},\boldsymbol{V}) = f\Big(\Phi_\rho^{-t}(\boldsymbol{Q},\boldsymbol{V})\Big).$$

Theorem D

Let $\mathcal{P}$ be admissible. Then for every $t>0$ there exists a linear operator

$$L^t : \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)) \to \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$$

such that for every $f \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ and any set $\mathcal{A} \subset \mathrm{T}^1(\mathbb{R}^d)$ with boundary of Liouville measure zero,

$$\lim_{\rho\to 0} \int_{\mathcal{A}} [L_\rho^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V} = \int_{\mathcal{A}} [L^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V}.$$

The operator L^t thus describes the macroscopic diffusion of the Lorentz gas in the Boltzmann-Grad limit $\rho\to 0$. (We in fact prove convergence of the Lorentz process to a random flight process.)

Note: The family $\{L^t\}_{t\geq 0}$ does in general *not* form a semigroup.

A generalized linear Boltzmann equation

Consider extended phase space coordinates $(\boldsymbol{Q}, \boldsymbol{V}, \varsigma, \xi, \boldsymbol{V}_+)$:

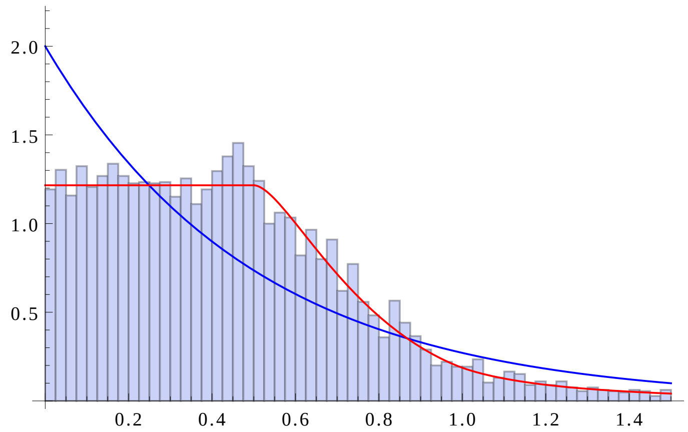
$$(\boldsymbol{Q}, \boldsymbol{V}) \in \mathrm{T}^1(\mathbb{R}^d) \text{ — usual position and momentum}$$
$$\varsigma \in \Sigma \text{ — the mark of current scatterer location}$$
$$\xi \in \mathbb{R}_+ \text{ — flight time until the next scatterer}$$
$$\boldsymbol{V}_+ \in \mathrm{S}_1^{d-1} \text{ — velocity after the next hit}$$

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}} - \frac{\partial}{\partial \xi}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}, \varsigma, \xi, \boldsymbol{V}_+)$$
$$= \int_\Sigma \int_{\mathrm{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}', \varsigma', 0, \boldsymbol{V})\, p_0(\boldsymbol{V}', \varsigma', \boldsymbol{V}, \varsigma, \xi, \boldsymbol{V}_+) d\boldsymbol{V}' d\mathrm{m}(\varsigma').$$

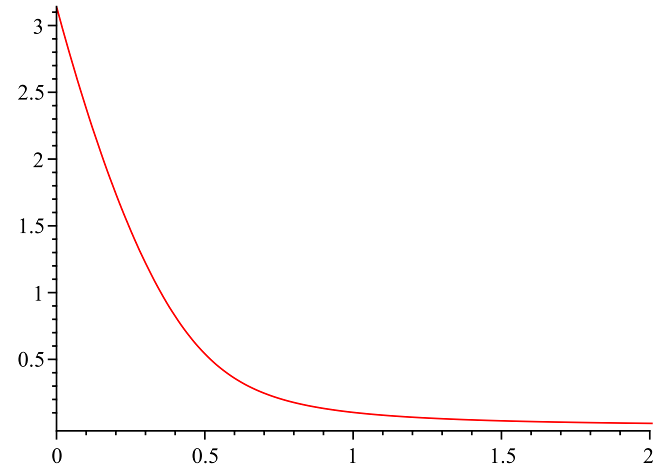
with a collision kernel $p_0(\boldsymbol{V}', \varsigma', \boldsymbol{V}, \varsigma, \xi, \boldsymbol{V}_+)$, which can be expressed as a product of the scattering cross section of an individual scatterer and a certain transition probability for hitting a given point on the next scatterer with mark ς after time ξ, given the present scatterer has mark ς'.

Crystals

The distribution $\Phi(\xi)$ of free path lengths for lattice configurations



$\Phi(\xi)$ in dimension two vs. $2e^{-2\xi}$



$\int_\xi^\infty \Phi(\xi')d\xi'$ in dimension three

Tail asymptotics (JM & Strömbergsson, GAFA 2011):

$$\Phi(\xi) = \frac{2^{2-d}}{d(d+1)\zeta(d)}\xi^{-3} + O\Big(\xi^{-3-\frac{2}{d}}\log\xi\Big) \qquad (\xi \to \infty)$$

$$\Phi(\xi) = \frac{\overline{\sigma}}{\zeta(d)} + O(\xi) \qquad (\xi \to 0)$$

with $\overline{\sigma} = \mathrm{vol}(\mathcal{B}_1^{d-1}) = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)}$.

Application: Superdiffusive central limit theorem

The divergent second moment of the path length distribution leads to $t \log t$ superdiffusion:

Theorem E [JM & B. Toth, CMP 2016]
Let $d \geq 2$ and fix a Euclidean lattice $\mathcal{L} \subset \mathbb{R}^d$ of covolume one. Assume $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ is distributed according to an absolutely continuous Borel probability measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$. Then, for any bounded continuous $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t\to\infty} \lim_{r\to 0} \mathbb{E} f\left(\frac{\boldsymbol{Q}(t) - \boldsymbol{Q}_0}{\Sigma_d \sqrt{t \log t}}\right) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-\frac{1}{2}\|\boldsymbol{x}\|^2} d\boldsymbol{x},$$

with

$$\Sigma_d^2 := \frac{2^{1-d}\overline{\sigma}}{d^2(d+1)\zeta(d)}.$$

For fixed r the analogous result is currently known only in dimension $d = 2$, see Szász & Varjú (J Stat Phys 2007), Chernov & Dolgopyat (Russ. Math Surveys 2009).

Quasicrystals

Cut and project

- $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$, π and π_{int} orthogonal projections onto $\mathbb{R}^d$, $\mathbb{R}^m$

- $\mathcal{L} \subset \mathbb{R}^n$ a lattice of full rank

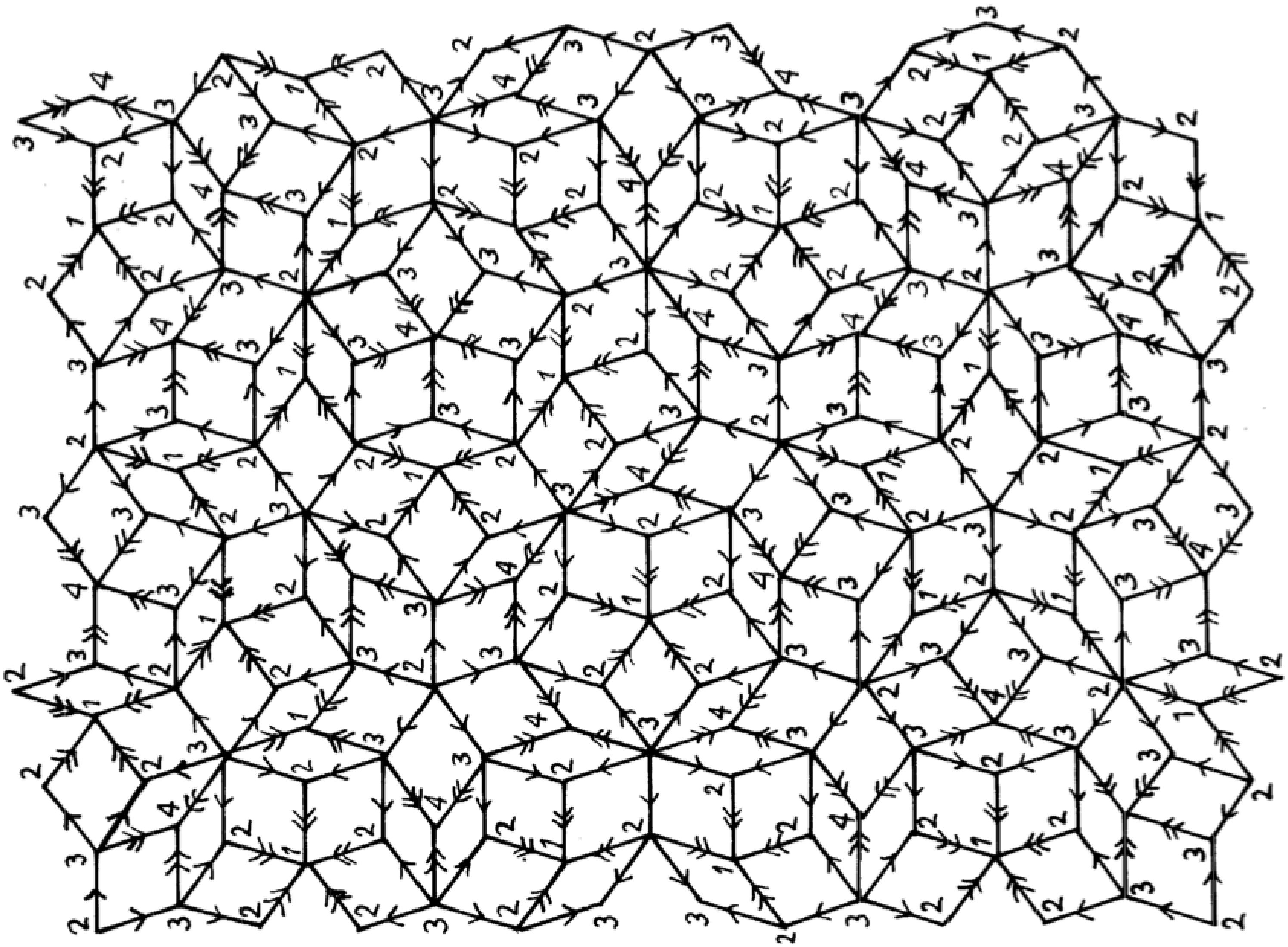
- $\mathcal{A} := \overline{\pi_{\mathrm{int}}(\mathcal{L})}$ is an abelian subgroup of $\mathbb{R}^m$, with Haar measure $\mu_{\mathcal{A}}$

- $\mathcal{W} \subset \mathcal{A}$ a "regular window set"
  (i.e. bounded with non-empty interior, $\mu_{\mathcal{A}}(\partial\mathcal{W}) = 0$)

- $\mathcal{P}(\mathcal{W},\mathcal{L}) = \{\pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathcal{L},\ \pi_{\mathrm{int}}(\boldsymbol{y}) \in \mathcal{W}\} \subset \mathbb{R}^d$
  is called a "regular cut-and-project set"

- $\mathcal{P}(\mathcal{W},\mathcal{L})$ defines the locations of scatterers in our quasicrystal

Example: The Penrose tiling



(from: de Bruijn, Kon Nederl Akad Wetensch Proc Ser A, 1981)

Density

We have the following well known facts:

- $\mathcal{P}(\mathcal{W}, \mathcal{L})$ is a Delone set, i.e., uniformly discrete and relatively dense in $\mathbb{R}^d$

- For any bounded $\mathcal{D} \subset \mathbb{R}^d$ with boundary of Lebesgue measure zero,

$$\lim_{T \to \infty} \frac{\#(\mathcal{P} \cap T\mathcal{D})}{T^d} = c_{\mathcal{L}} \operatorname{vol}(\mathcal{D}) \mu_{\mathcal{A}}(\mathcal{W})$$

(the constant $c_{\mathcal{L}}$ is explicit)

Spherical averages of cut-and-project sets and "random quasicrystals"

- Set $G = \mathsf{SL}(n,\mathbb{R})$, $\Gamma = \mathsf{SL}(n,\mathbb{Z})$, $\mathcal{L} = \mathbb{Z}^n g$ for some $g \in G$.

- Note that for $A \in \mathsf{SL}(d,\mathbb{R})$,

$$\mathcal{P}(\mathcal{W},\mathcal{L})A = \left\{ \pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathbb{Z}^n g \begin{pmatrix} A & 0 \\ 0 & 1_m \end{pmatrix},\ \pi_{\mathsf{int}}(\boldsymbol{y}) \in \mathcal{W} \right\}$$

- This motivates the definition of the embedding of $\mathsf{SL}(d,\mathbb{R})$ in G by the map

$$\varphi_g : \mathsf{SL}(d,\mathbb{R}) \to G, \quad A \mapsto g \begin{pmatrix} A & 0 \\ 0 & 1_m \end{pmatrix} g^{-1}.$$

Spherical averages of cut-and-project sets and "random quasicrystals"

- It follows from Ratner's theorem that there exists a closed connected subgroup H_g ofc G such that
  - $\Gamma \cap H_g$ is a lattice in H_g
  - $\varphi_g(\mathrm{SL}(d,\mathbb{R})) \subset H_g$
  - the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{SL}(d,\mathbb{R}))$ in $\Gamma\backslash G$ is given by $\Gamma\backslash\Gamma H_g$.

- Denote the unique right-H_g invariant probability measure on $\Gamma\backslash\Gamma H_g$ by μ_g.

- Using an equidistribution result for unipotent translates due to N. Shah* based on Ratner's theorem, one can show that† for $\boldsymbol{v}$ random according to any a.c. Borel probability measure on S_1^{d-1},

$$\mathcal{P}(\mathcal{W},\mathcal{L})R(\boldsymbol{v})D_\rho = \{\pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathbb{Z}^n\varphi_g(R(\boldsymbol{v})D_\rho)g,\ \pi_{\mathrm{int}}(\boldsymbol{y}) \in \mathcal{W}\}$$
$$\xrightarrow{\mathrm{d}} \{\pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathbb{Z}^n hg,\ \pi_{\mathrm{int}}(\boldsymbol{y}) \in \mathcal{W}\}$$

where $h \in \Gamma\backslash\Gamma H_g$ is distributed according to μ_g.

*N. Shah, Proc. Indian Acad. Sci. Math. Sci., 1996

†JM & Strömbergsson, Comm. Math. Phys. 2014

Examples

- If $\mathcal{P}(\mathcal{L},\mathcal{W})$, then for almost every $\mathcal{L}$ in the space of lattices, we have

$$H_g = \mathsf{SL}(n,\mathbb{R}), \qquad \Gamma \cap H_g = \mathsf{SL}(n,\mathbb{Z}).$$

- If $\mathcal{P}(\mathcal{L},\mathcal{W})$ is the vertex set of the classical Penrose tiling, we have

$$H_g = \mathsf{SL}(2,\mathbb{R})^2$$

and $\Gamma \cap H_g =$ a congruence subgroup of the Hilbert modular group $\mathsf{SL}(2,\mathcal{O}_K)$, with $\mathcal{O}_K$ the ring of integers of $K = \mathbb{Q}(\sqrt{5})$.

- A complete classification of all H_g that can arise in our context has recently been given by see Ruehr, Smilansky & Weiss (JEMS in press)

Future challenges

- “Classify” point processes that can arise as spherical averages; which of these are $\mathrm{SL}(d, \mathbb{R})$-invariant and/or translation invariant?

- Lorentz gas in force fields; trajectories will be curved

- (Super-) diffusive limits

- Quantum Lorentz gas