The Restriction Problem

$$S_n \subset GL_n(\mathbb{C})$$

Given a rep. ρ of $GL_n(\mathbb{C})$, what is $\rho|_{S_n}$?

Example: $\mathrm{Sym}^2 \mathbb{C}^n$

has basis e_i^2, e_{ij} $i<j$

$$\mathrm{Sym}^2 \mathbb{C}^n = V_{(n)}^{\oplus 2} \oplus V_{(n-1,1)}^{\oplus 2} \oplus V_{(n-2,2)}$$

$$W_{\square\square} = V_{\phi}^{\oplus 2} \oplus V_{\square}^{\oplus 2} \oplus V_{\square\square}$$

Interested in poly reps of deg d of $GL_n(\mathbb{C})$

Reps. that occur $\otimes^d \mathbb{C}^n$

$\mathrm{Sym}^d \mathbb{C}^n$, $\Lambda^d \mathbb{C}^n$

$\forall\ \lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$

$W_\lambda(\mathbb{C}^n)$: Weyl module of $GL_n(\mathbb{C})$.

$\Lambda(d,n)$: partitions of d with at most n parts

$\{W_\lambda(\mathbb{C}^n) \mid \lambda \in \Lambda(d,n)\}$ is a complete set of irr. poly. reps. of deg d.

$\Lambda(d)$: all partitions of d.

$\{ V_\mu \mid \mu \in \Lambda(n) \}$

are irred. reps. of S_n.

$$\mathrm{Res}^{GL_n(\mathbb{C})}_{S_n} W_\lambda(\mathbb{C}^n) = \bigoplus_{\mu \in \Lambda(n)} V_\mu^{\oplus r_{\lambda\mu}}$$

$r_{\lambda\mu}$ - restriction coefficients.

Restriction problem: Find a positive combinatorial interpretation of $r_{\lambda\mu}$.

Best model: Littlewood - Richardson problem

$$W_\mu(\mathbb{C}^n) \otimes W_\nu(\mathbb{C}^n) = \bigoplus_\lambda W_\lambda^{\oplus c^\lambda_{\mu\nu}}$$

Littlewood-Richardson coeffs.

Littlewood - Richardson rule (1977, Schützenberger)

$c^\lambda_{\mu\nu}$ = # SSYT of shape λ/μ with Yamanouchi reading word.

= # lattice points in the hive polytope (Knutson-Tao ~2000)

Complexity theory

The computation of $c^{\lambda}_{\mu\nu}$ is

#P-complete.

(Narayanan 2006)

Checking the positivity of $c^{\lambda}_{\mu\nu}$ is in

P (Bürgisser-Ikenmeyer) 2009.

Kronecker problem:

$$V_\mu \otimes V_\nu = \bigoplus_\lambda V_\lambda^{\oplus g^{\lambda}_{\mu\nu}}$$

Kronecker coeffs

Computation is #P-hard

GapP

(Burgisser - Ikenmeyer 2008)

Positivity : NP-hard

(Ikenmeyer - Mulmuley - Walter 2017)

Symmetric functions

$$\operatorname{tr}\left(\rho\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} ; W\right)$$

$= \operatorname{ch}\rho(t_1, \dots, t_n)$ is a symmetric poly. of deg d.

Thm: Two reps. of $GL_n(\mathbb{C})$ are iso $\iff$ chars. are equal.

$$\operatorname{ch}(W_\lambda(\mathbb{C}^n)) = s_\lambda(t_1, \dots, t_n).$$

Schur polynomial

Littlewood-Richardson coeffs:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

Kronecker - coefficients

$$s_\mu * s_\nu = \sum_\lambda g^\lambda_{\mu\nu} s_\lambda$$

internal prod

$$p_\mu * p_\nu = \delta_{\mu\nu} \frac{1}{z_\mu} p_\mu$$

Plethysm:

$\rho : GL_n \longrightarrow GL_m$

$\sigma : GL_m \longrightarrow GL_k$

$\sigma \circ \rho : GL_n \longrightarrow GL_k$

Plethysm is a binary op.

$$(f, g) \longmapsto f[g]$$

such that

$$ch(\sigma \circ \rho) = ch\,\sigma\,[ch\,\rho]$$

If $g = x^{\alpha_1} + x^{\alpha_2} + \cdots$

$$f[g] = f(x^{\alpha_1}, x^{\alpha_2}, \ldots)$$

Example: $g = p_k = x_1^k + x_2^k + \cdots$

$$f[p_k] = f(x_1^k, x_2^k, \ldots)$$

$$p_k[p_\ell] = x_1^{k\ell} + x_2^{k\ell} + \cdots = p_{k\ell}.$$

Plethysm coeffs:

$$W_\mu(W_\nu(\mathbb{C}^n)) = \bigoplus_\lambda W_\lambda(\mathbb{C}^n)^{\oplus p_{\mu\nu}^\lambda}$$

$$s_\mu[s_\nu] = \sum p_{\mu\nu}^\lambda s_\lambda$$

Fischer - Ikenmeyer 2020 arXiv

$p_{\mu\nu}^\lambda$ is NP-hard and #P.

Littlewood (1958) derived a plethystic formula for $r_{\lambda\mu}$:

$$\mathrm{Res}^{GL(n)}_{S_n} W_\lambda(\mathbb{C}^n) = \bigoplus_\mu V_\mu^{\oplus r_{\lambda\mu}}.$$

$$\langle s_\mu[1+h_1+h_2+\dots], s_\lambda \rangle = r_{\lambda\mu}$$

clever manipulations of symmetric fns.

$H = 1+h_1+h_2+\dots$

$$r_{\lambda\mu} := \langle V_\mu, \mathrm{Res}^{GL(n)}_{S_n} W_\lambda(\mathbb{C}^n) \rangle_{S_n}$$

$$\langle \underset{\text{``??}}{s_\mu[H]}, \underset{''}{s_\lambda} \rangle_{GL(n)}.$$

$\mathrm{ch}(\mathrm{Ind}^{GL_n(\mathbb{C})}_{S_n} V_\mu)$ $\mathrm{ch}(W_\lambda(\mathbb{C}^n))$

Construction of $\mathrm{Ind}^{GL_n(\mathbb{C})}_{S_n} V_\mu$

(Narayanan, Paul, __, Srivastava 2020)

$$\mathrm{Ind}^{GL(n)}_{S_n} V = \left\{ f: GL_n(\mathbb{C}) \to V \;\middle|\; \begin{array}{l} f(wg) = \sigma(w) f(g) \\ f \text{ is polynomial} \end{array} \right\}$$

(σ, V) $\qquad g\cdot f(x) = f(xg)$

Thm:

① $\mathrm{Hom}_{GL_n}(\mathrm{Ind}\, V, W) = \mathrm{Hom}_{S_n}(V, \mathrm{Res}\, W)$

② $\mathrm{ch}\, \mathrm{Ind}\, V_\mu = s_\mu[H]$

$\mathrm{ch}\, \mathrm{Ind}\, V = \mathcal{F}(V)\,[H]$

Frobenius char.

$$\mathcal{F}(V) = \frac{1}{n!} \sum_{\omega \in S_n} \chi_V(\omega)\, p_{\text{cycle decomp}(\omega)}$$

Character polynomials approach:

$w \in S_n$

$X_i(\omega) =$ no. of i-cycles in w.

X_i : class fn. on S_n $\forall n$.

$P = \mathbb{C}[X_1, X_2, \ldots]$

$p \in P \rightsquigarrow p(X_1(\omega), X_2(\omega), \ldots)$

class fn. on S_n $\forall n$.

$\mathrm{tr}(\omega; \mathbb{C}^n) = X_1(\omega)$

$\mathrm{tr}(\omega, \mathrm{Sym}^2 \mathbb{C}^n) = X_1 + \binom{X_2}{2}$

Thm (Jan 2020)

① $p \in P$, define

$$\langle p \rangle_n = \frac{1}{n!} \sum_{\omega \in S_n} p(\omega)$$

$\forall$ partition $\alpha = 1^{a_1} 2^{a_2} \dots$

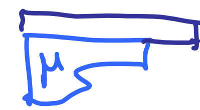
$$\binom{X}{\alpha} := \binom{X_1}{a_1}\binom{X_2}{a_2} \cdots \in P$$

$$\left\langle \binom{X}{\alpha} \right\rangle_n = \begin{cases} \frac{1}{z_\alpha} & \text{if } n \geq |\alpha| \\ 0 & \text{o/w.} \end{cases}$$

② Computation of $S_\lambda \in P$ such that

$$S_\lambda(\omega) = \mathrm{tr}\left(\omega; \mathrm{Res}^{GL(n)}_{S_n} W_\lambda(\mathbb{C}^n)\right) \quad \forall n. \quad \omega \in S_n.$$

③ there are well-known $q_\mu \in P$



$$q_\mu(\omega) = \mathrm{tr}\left(\omega; V_{\mu[n]}\right)$$

$$\mu[n] = (n - |\mu|, \mu_1, \mu_2 \dots)$$

Combining ② & ③ get

$$r_{\lambda\mu} = \langle S_\lambda q_\mu \rangle_n$$